MATLAB and Its Application in Engineering

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Linear programming-Matlab Linprog Function

Standard format

$$\min Z = c_1 x_1 + \ldots + c_n x_n$$

$$lb = \begin{pmatrix} lx_1 & lx_2 & . & . & . & lx_n \end{pmatrix}$$

$$ub = \begin{pmatrix} ux_1 & ux_2 & . & . & . & ux_n \end{pmatrix}$$

$$\min c^T X$$

$$s.t. \begin{cases} AX \le b \\ A_{eq} X = b_{eq} \\ lb \le X \le ub \end{cases}$$

$$s.t. \begin{cases} a_{11}x_1 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + \ldots + a_{2n}x_n \le b_2 \\ . \\ . \\ . \\ a_{m1}x_1 + \ldots + a_{mn}x_n \le b_m \\ a_{11}^{eq}x_1 + \ldots + a_{1n}^{eq}x_n = b_1^{eq} \\ a_{21}^{eq}x_1 + \ldots + a_{2n}^{eq}x_n = b_2^{eq} \\ . \\ . \\ . \\ a_{h1}^{eq}x_1 + \ldots + a_{hn}^{eq}x_n = b_h^{eq} \\ lx_1 \ge x_1 \ge ux_1 \quad lx_2 \ge x_2 \ge ux_2 \ldots lx_n \ge x_n \ge ux_n \end{cases}$$

Linear programming-Matlab Linprog Function

$$[x, fval, exitflag] = linprog(c, A, b, Aeq, beq, lb, ub)$$

1 linprog converged to a solution X.

0 Maximum number of iterations reached.

-2 No feasible point found.

-3 Problem is unbounded.

-4 NaN value encountered during execution of algorithm.

-5 Both primal and dual problems are infeasible.

-7 Magnitude of search direction became too small; no further progress can be made. The problem is ill-posed or badly conditioned.

Application: Resource Allocation

A company produces three types of doors and one type of window. Production of these doors and window are x1,x2,x3 and x4 respectively. Raw material requirements and profit of producing these doors and window are listed in the table below.

Market research shows that the production of x1 should be larger than 40, production of x2 should be larger than 130, production of x3 should be larger than 30, and production of x4 should be less than 10.

Application: Resource Allocation

Available resources are: 1500 unit of wood, 1000 unit of glass, 800 unit of labor.

Make a production plan for the company to maximize the profit

	Wood	Glass	Labor	Profit
Door-x1	5	2	3	12
Door-x2	1	3	2	5
Door-x3	9	4	5	15
Window-x4	12	1	10	10

Application: Resource Allocation

$$\min Z = c^T x$$

$$s.t.\begin{cases} Ax \le b \\ A_{eq}x = b_{eq} \\ lb \le x \le ub \end{cases}$$

$$c = \begin{pmatrix} -1.2 \\ -5 \\ -15 \\ -10 \end{pmatrix} \qquad A = \begin{pmatrix} 5 & 1 & 9 & 12 \\ 2 & 3 & 4 & 1 \\ 3 & 2 & 5 & 10 \end{pmatrix} \qquad b = \begin{pmatrix} 1500 \\ 1000 \\ 800 \end{pmatrix}$$

$$lb = \begin{pmatrix} 40 \\ 130 \\ 30 \\ 0 \end{pmatrix} \qquad ub = \begin{pmatrix} inf \\ inf \\ inf \\ 10 \end{pmatrix}$$

Application: Production plan

```
>> c=[-1.2;-5;-15;-10];
>> A=[5,1,9,12;2,3,4,1;3,2,5,10];
>> b=[1500;1000;800];
>> lb=[40;130;30;0];
>> ub=[inf;inf;inf;10];
>> [x,fval,exitflag]=linprog(c,A,b,[],[],lb,ub)

Optimization terminated.


x =
   40.0000
  130.0000
   84.0000
    0.0000


fval =  -1.9580e+03                exitflag =     1
```

Application: Waste water purification

Plant A and B locates along the Heipu River(黑浦江). A is in the upstream, and produces 30K m3 of waste water everyday. B is in the downstream and produces 20K m3 of waste water everyday. At the site of plant B, there is a branch river merges into the Heipu River.

The daily average discharge of Heipu river is 7.50M m3, and daily average discharge of the branch river is 2.5M m3. It is known that the waste water from plant A will be naturally purified when it flows from A to B. The environment protection department requires the percentage of waste water should no larger than 0.2% of daily discharge。

A costs 1200yuan to process every 10K m3 of its waste waster, whereas B costs 85yuan to process every 10K m3 of its waste waster.

Application: Waste water purification

Assume A processes x1 (10K m3) of as the waste water everyday;
B processes x2 (10K m3) of as the waste water everyday;

$$\min Z = 1200x_1 + 850x_2$$

Percentage of waste water in the river between A and B

$$\frac{3 - x_1}{750} \leq \frac{2}{1000} \qquad x_1 \geq 1.5$$

Percentage of waste water in the river after B

$$\frac{[0.75(3 - x_1) + (2 - x_2)]}{750 + 250} \leq \frac{2}{1000} \qquad 3x_1 + 4x_2 \geq 9$$

Limits of daily waste water process capability

$$x_1 \leq 3$$
$$x_2 \leq 2$$

Application: Waste water purification

$$\min Z = 1200x_1 + 850x_2$$

$$s.t.\begin{cases} 3x_1 + 4x_2 \geq 9 \\ 1.5 \leq x_1 \leq 3 \\ 0 \leq x_2 \leq 2 \end{cases}$$

```
>> c=[1200;850];
>> A=[-3 -4];
>> b=-9;
>> Aeq=[];
>> beq=[];
>> lb=[1.5;0];
>> ub=[3;2];
>> [x,fval,exitflag]=linprog(c,A,b,[],[],lb,ub)
Optimization terminated.
x =
    1.5000
    1.1250
fval =   2.7563e+03        exitflag =  1
```

Application: work schedule

China Western plans to recruit more flight attendants due to the increase of flights number.

Table below lists the minimum flight attendants requirement for different time slot and the corresponding cost.

Time slot	Shift-1	Shift-2	Shift-3	Shift-4	Shift-5	**minimum flight attendants**
6:00-8:00	V					48
8:00-10:00	V	V				79
10:00-12:00	V	V				65
12:00-14:00		V	V			87
14:00-16:00		V	V			64
16:00-18:00			V	V		73
18:00-20:00			V	V		82
20:00-22:00				V		43
22:00-24:00				V	V	52
24:00-6:00					V	15
Cost/staff	170	160	175	180	195	

Application: work schedule

Let xi as the flight attendants number for the i-th shift. i=1， … 5

$$\min Z = 170x_1 + 160x_2 + 175x_3 + 180x_4 + 195x_5$$

$$s.t.\begin{cases} x_1 \geq 48 \\ x_1 + x_2 \geq 79 \\ \cancel{x_1 + x_2 \geq 65} \\ x_1 + x_2 + x_3 \geq 87 \\ x_2 + x_3 \geq 64 \\ \cancel{x_3 + x_4 \geq 73} \\ x_3 + x_4 \geq 82 \\ x_4 \geq 43 \\ x_4 + x_5 \geq 52 \\ x_5 \geq 15 \end{cases}$$

$$x_i \geq 0 \quad i = 1, \dots, 5$$

Application: work schedule

```
>> c=[170;160;175;180;195];
>> A=[-1 0 0 0 0;-1 -1 0 0 0;-1 -1 -1 0 0;0 -1 -1 0 0;0 0 -1 -1 0; 0 0 0 -1 0;0 0 0 -1 -1; 0 0 0 0 -1];
>> b=[-48; -79; -87; -64; -82; -43; -52; -15];
>> [x,fval,exitflag]=linprog(c,A,b)

Optimization terminated.

x =
   48.0000
   31.0000
   39.0000
   43.0000
   15.0000
fval =   3.0610e+04             exitflag =  1
```

Application: transportation

A type of product is made in m factories Ai (i=1，2….m), the production of factory Ai is ai.

The product has n markets Bj (j=1，2….n), the sales of market Bj is bj. The transportation cost from Ai to Bj is cij.

If the production equals to sales of the product, determine the shipping xij for transporting products from factory Ai to Bj

Application: transportation

Total cost $$Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Nonnegative constrains

$$x_{ij} \geq 0 \quad i = 1,2,\dots,m \qquad j = 1,2,\dots,n$$

Production equals to sales

$$\begin{cases} \sum_{j=1}^{n} x_{ij} = a_i & i = 1,2,\dots,m \\ \sum_{i=1}^{m} x_{ij} = b_j & j = 1,2,\dots,n \end{cases}$$

Application: transportation

	variables													
	x_{11}	x_{12}	$\cdots$	x_{1n}	x_{21}	x_{22}	$\cdots$	x_{2n}	$\cdots$	x_{m1}	x_{m2}	$\cdots$	x_{mn}	
A_{eq}	1		$\cdots$	1										$\left.\begin{matrix} supply \\ constrain \end{matrix}\right\}(m\ row)$
					1	1	$\cdots$	1						
									$\ddots$					
										1	1	$\cdots$	1	
	1				1					1				$\left.\begin{matrix} sales \\ constrain \end{matrix}\right\}(n\ coloumn)$
		1				1					1			
			$\ddots$				$\ddots$					$\ddots$		
				1				1					1	

Application: transportation

$$x = (x_{11} \;\; x_{12} \;\; \cdots \;\; x_{1n} \;\; x_{21} \;\; x_{22} \;\; \cdots \;\; x_{2n} \cdots \;\; x_{m1} \;\; x_{m2} \cdots \;\; x_{mn})^T$$

$$c = (c_{11} \;\; c_{12} \;\; \cdots \;\; c_{1n} \;\; c_{21} \;\; c_{22} \;\; \cdots \;\; c_{2n} \cdots \;\; c_{m1} \;\; c_{m2} \cdots \;\; c_{mn})$$

$$A_{eq} = \begin{pmatrix}
1 & 1 & \cdots & 1 & & & & & & & & & \\
 & & & & 1 & 1 & \cdots & 1 & & & & & \\
 & & & & & & & & \ddots & & & & \\
 & & & & & & & & & 1 & 1 & \cdots & 1 \\
1 & & & & 1 & & & & & 1 & & & \\
 & 1 & & & & 1 & & & & & 1 & & \\
 & & \ddots & & & & \ddots & & & & & \ddots & \\
 & & & 1 & & & & 1 & & & & & 1
\end{pmatrix}$$

$$\min Z = c^T x$$

$$s.t. \begin{cases} A_{eq} x = b_{eq} \\ x \geq 0 \end{cases}$$

Application: transportation

m factories

n markets

m row

m column

$$A_{eq} = \begin{pmatrix} O & Z & \cdots & Z \\ Z & O & \cdots & Z \\ \vdots & \vdots & & \vdots \\ Z & Z & \cdots & O \\ I & I & \cdots & I \end{pmatrix}$$

O: nX1 n-dimention unit row vector   ones(1,n)

Z: nX1 n-dimention zero row vector   zeros(1, n)

I: nXn   n-order identity matrix   eye(n)

Application: transportation

A candy company has 3 factories A1, A2 and A3，their daily production is 7t, 4t and 9t respectively. The candies will be sold to four markets B1, B2, B3 and B4, and their daily sales are 3t, 6t, 5t and 6t respectively. The transportation cost from Ai to Bj is cij as listed in the table below

Factories	Market-B_1	Market-B_2	Market-B_3	Market-B_4
A_1	3	11	3	10
A_2	1	9	2	8
A_3	7	4	10	5

$$\mathrm{c} = (3,11,3,10,1,9,2,8,7,4,10,5)^T$$

$$x = (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34})^T$$

b_{eq}=[7;4;9; 3;6;5;6];

Application: transportation

```
>> c=[3;11;3;10;1;9;2;8;7;4;10;5];
>> O=ones(1,4);
>> Z=zeros(1,4);
>> I=eye(4);
>> Aeq=[O Z Z;Z O Z;Z Z O;I I I];
>> beq=[7;4;9;3;6;5;6];
>> lb=zeros(1,3*4);
>> ub=[];
>> A=[];
>> b=[];
>> x0=[];
>> options=optimset('Algorithm','Simplex');
```

Application: transportation

```
>> [x,fval,exitflag]=linprog(c,A,b,Aeq,beq,lb,ub,x0,options)
Optimization terminated.
x = 2
    0
    5
    0
    1
    0
    0
    3
    0
    6
    0
    3
fval =   85                    exitflag =    1
```

Application: transportation

Production>Sales

$$\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$$

m factories

n markets

$$\min Z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$s.t. \begin{cases} \sum_{j=1}^{n} x_{ij} \le a_i & i = 1,2,\dots,m \\ \sum_{i=1}^{m} x_{ij} = b_j & j = 1,2,\dots,n \\ x_{ij} \ge 0 \quad i = 1,2,\dots,m & j = 1,2,\dots,n \end{cases}$$

Supply constrain

Sales constrain

$$\begin{gathered} \mathrm{minZ} = c^T x \\ Ax \le b \\ A_{eq} x = b_{eq} \\ x \ge 0 \end{gathered}$$

Application: transportation

$$x = (x_{11} \;\; x_{12} \;\; \cdots \; x_{1n} \;\; x_{21} \;\; x_{22} \;\; \cdots \; x_{2n} \; \cdots \; x_{m1} \;\; x_{m2} \;\; \cdots \;\; x_{mn} \;)^T$$
$$c = (c_{11} \;\; c_{12} \;\; \cdots c_{1n} \;\; c_{21} \;\; c_{22} \;\; \cdots c_{2n} \; \cdots \; c_{m1} \;\; c_{m2} \;\; \cdots \;\; c_{mn} \;)^T$$
$$b = (a_1, a_2, \cdots, a_m)$$
$$b_{eq} = (b_1, b_2, \cdots, b_n)^T$$

$$A = \left(\begin{array}{cccc|cccc|ccccc} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ & & & & & & \vdots & & & & & & \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 1 & \cdots & 1 \end{array}\right)$$

m × n

$$A_{\mathrm{eq}} = \left(\begin{array}{cccc|cccc|ccccc} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ & & & & & & \vdots & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right)$$

n × m

Application: transportation

A candy company has 3 factories A1, A2 and A3， their daily production is 7t, 5t and 7t respectively. The candies will be sold to four markets B1, B2, B3 and B4, and their daily sales are 2t, 3t, 4t and 6t respectively. The transportation cost from Ai to Bj is cij as listed in the table below

Factories	Market-B_1	Market-B_2	Market-B_3	Market-B_4
A_1	2	11	3	4
A_2	10	3	5	9
A_3	7	8	1	2

Application: transportation

$$c = (2,11,3,4,10,3,5,9,7,8,1,2)^T$$

$$x = (x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34})^T$$

$$A = \begin{pmatrix} O & Z & Z \\ Z & O & Z \\ Z & Z & O \\ & & \end{pmatrix}$$

$$A_{eq} = (I \quad I \quad I)$$

Application: transportation

```
>> c=[2;11;3;4;10;3;5;9;7;8;1;2];
>> O=ones(1,4);
>> Z=zeros(1,4);
>> I=eye(4);
>> A=[O Z Z;Z O Z;Z Z O];
>> Aeq=[I I I];
>> beq=[2;3;4;6];
>> b=[7;5;7];
>> lb=zeros(1,3*4);
>> ub=[];
>> x0=[];
>> options=optimset('Algorithm','Simplex');
```

Application: transportation

```
>> [x,fval,exitflag]=linprog(c,A,b,Aeq,beq,lb,ub,x0,options)
Optimization terminated.
x =
    2
    0
    0
    3
    0
    3
    0
    0
    0
    0
    4
    3

fval =    35                exitflag =     1
```

Application: transportation

Square Express (方通物流) needs to delivery three kinds of items by using one van.

Item	Weight/t	Volume/m^3	Value/10k yuan
1	1	2	4
2	2	3	5
3	1.5	1.5	3.5

The maximum load of the van is 8t，15m3. Determine the number of items that can be delivered by the van so as to maximize the total value of the item.

Application: transportation

Let xi as the loading number for the i-th item

$$Z = 4x_1 + 5x_2 + 3.5x_3$$

$$x_1 + 2x_2 + 1.5x_3 \leq 8$$

$$2x_1 + 3x_2 + 1.5x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

$$\max Z = 4x_1 + 5x_2 + 3.5x_3$$

$$s.t. \begin{cases} x_1 + 2x_2 + 1.5x_3 \leq 8 \\ 2x_1 + 3x_2 + 1.5x_3 \leq 15 \\ x_1,\ x_2, x_3 \geq 0 \end{cases}$$

Application: transportation

```
>> c=[-4;-5;-3.5];
>> A=[1 2 1.5;2 3 1.5];
>> b=[8;15];
>> Aeq=[];
>> beq=[];
>> lb=[0;0];
>> ub=[];
>> [x,fval]=linprog(c,A,b,Aeq,beq,lb,ub)

x =
    7.0000
    0.0000
    0.6667
fval =   -30.3333              exitflag =  1
```

Integer programming

$$\max Z = x_1 + 5x_2$$

$$s.t.\begin{cases} x_1 + 10x_2 \leq 20 \\ x_2 \leq 2 \\ x_1, x_2 \geq 0 \end{cases}$$

x1=2; x2=9/5

x1=0; x2=2

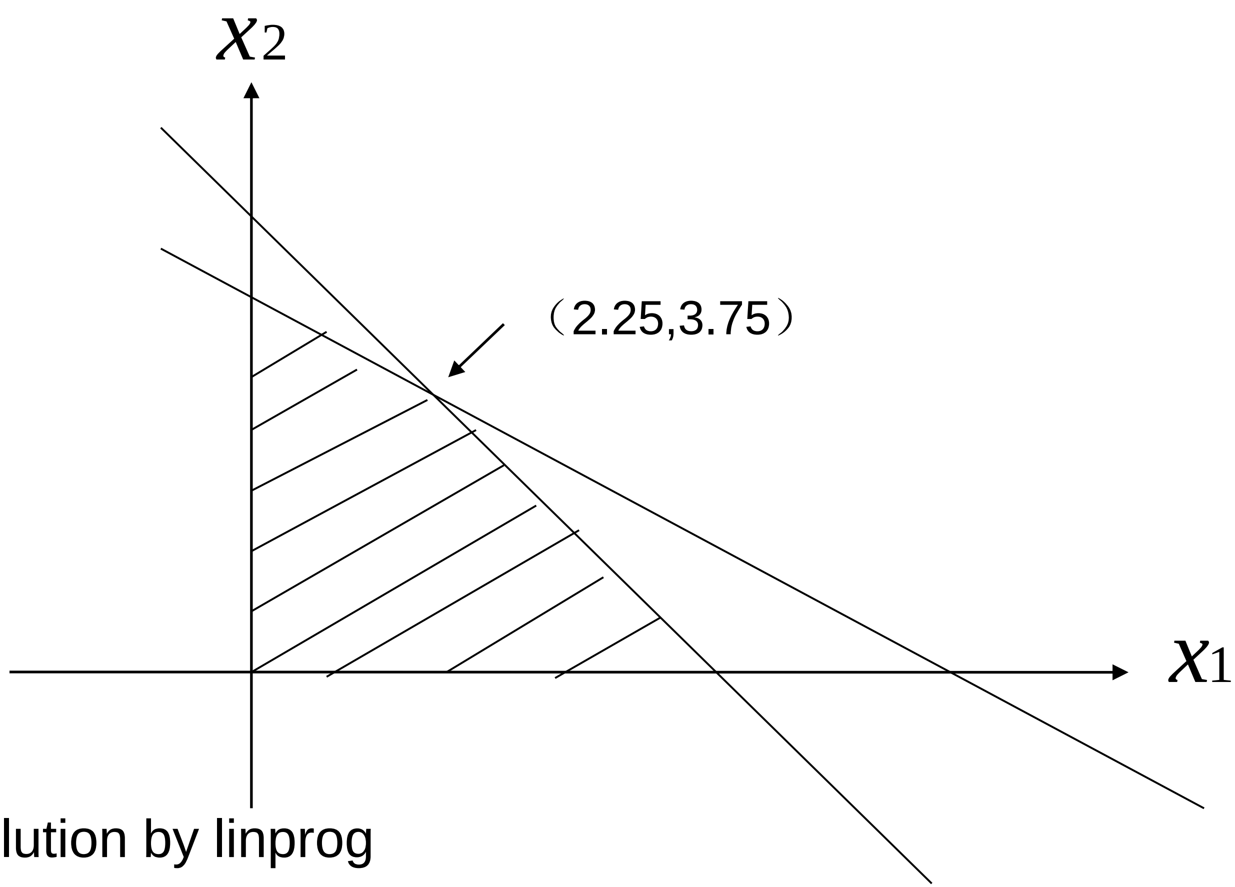
Method A: find optimal solution with linear programming, and round the result.

Method B: Cutting planes

Integer programming

$$\max Z = 5x_1 + 8x_2$$

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1, x_2 \ge 0 \end{cases}$$



Step 1：find optimal solution by linprog

x1=2.25, x2=3.75

Integer programming

$$\max Z = 5x_1 + 8x_2$$

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ \\ x_1, x_2 \ge 0 \end{cases}$$

Step 2：set boundary

Optimal solution from LP

Z=5×2.25+8×3.75=41.25，

Set it as the upper boundary for IP

Pick x1=1,x2=2→Z=21

As lower boundary for IP

$$21 \le Z^* \le 41.25$$

Integer programming

$$\max Z = 5x_1 + 8x_2$$

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ \\ x_1, x_2 \ge 0 \end{cases}$$

Step 3： new planes

Take any original optimal solution

x1=2.25

$$x_1 \le 2$$

$$x_1 \ge 3$$

LP1

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$$

LP2

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \ge 3 \\ x_1, x_2 \ge 0 \end{cases}$$

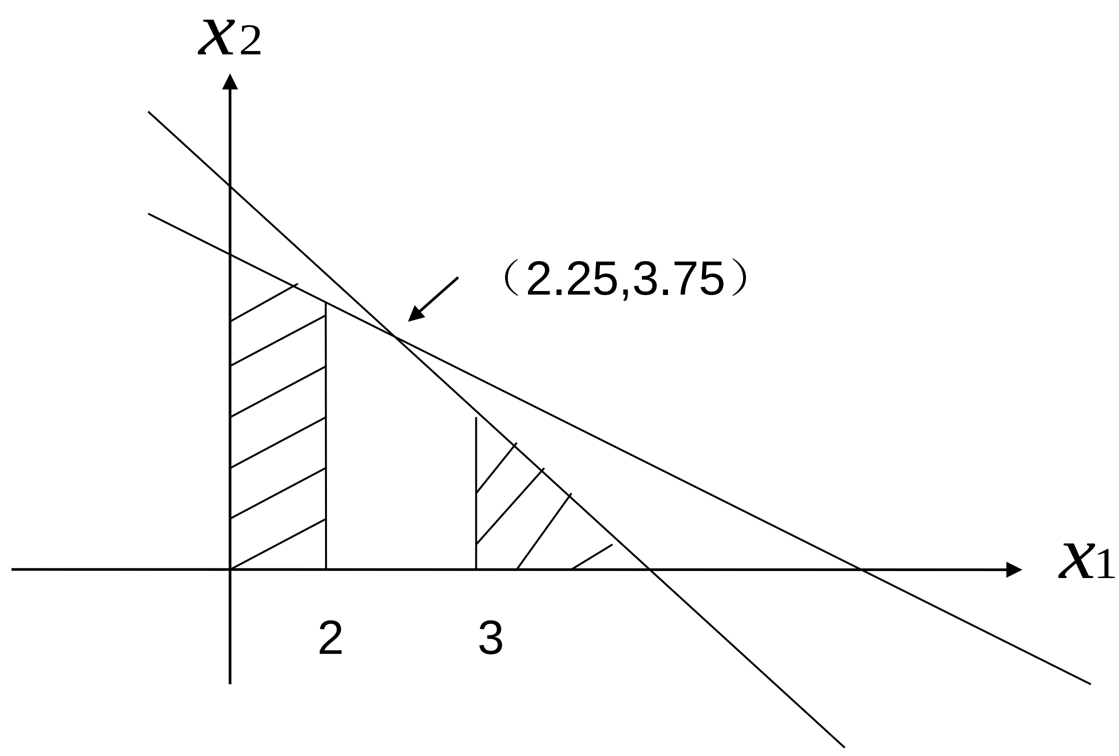
Integer programming

LP1

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_1, x_2 \ge 0 \end{cases}$$

LP2

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \ge 3 \\ x_1, x_2 \ge 0 \end{cases}$$



Solve LP1 by linprog

x1=2, x2=3.8889

Z=41.1111

Solve LP2 by linprog

x1=3, x2=3
Z=39

Integer programming

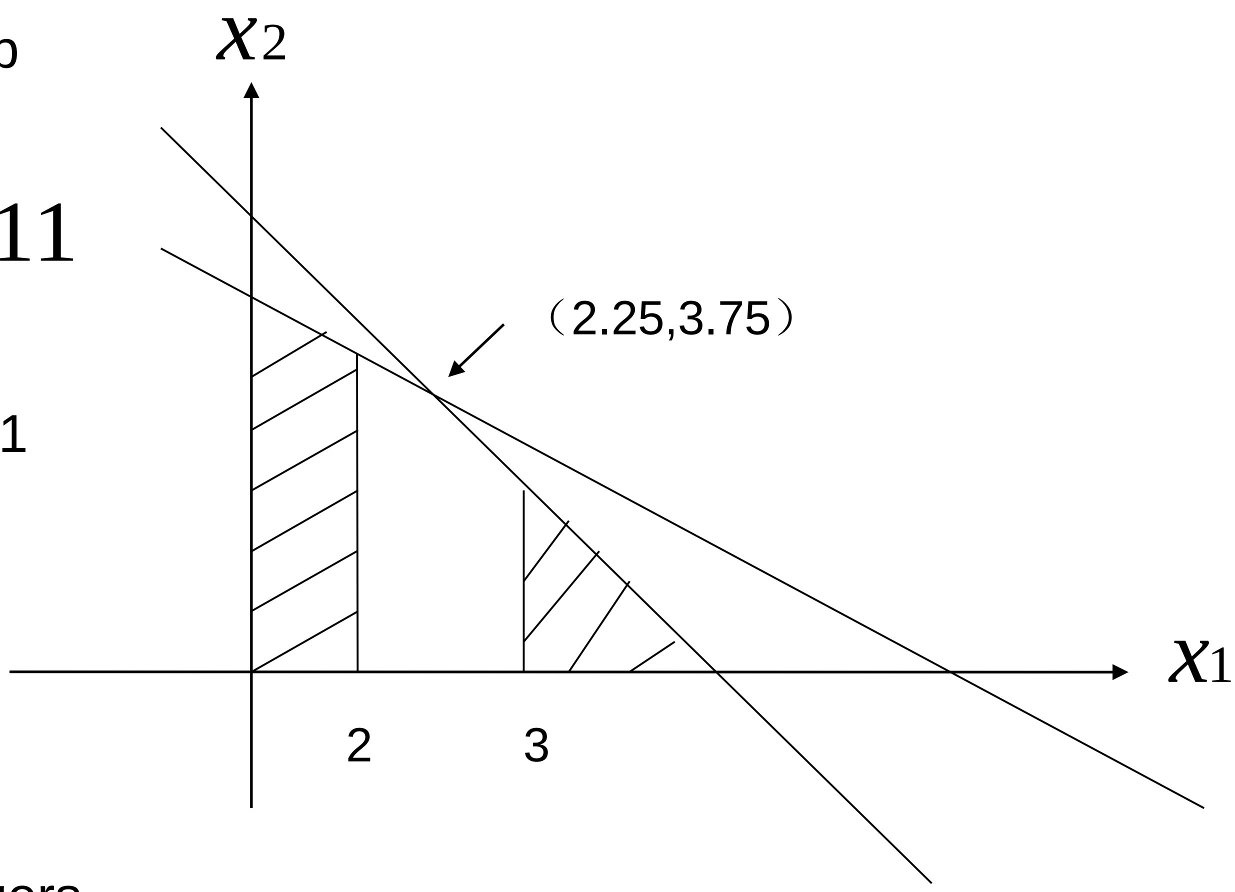
Step 4： modify lb and ub

$$39 \le Z^* \le 41.1111$$

Create new plane for LP1

$$x_2 \le 3$$

$$x_2 \ge 4$$

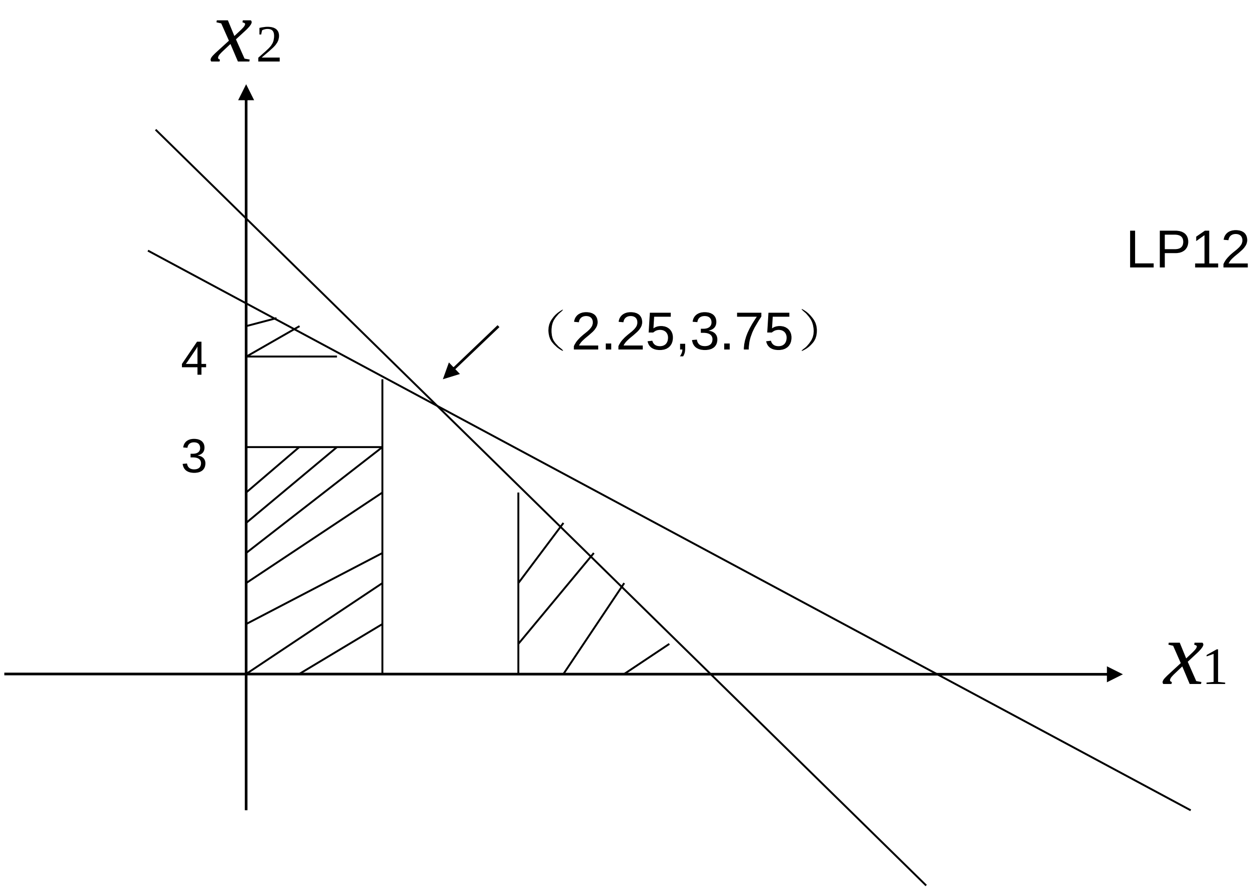


Solution of LP2 are integers

Integer programming

Step 5：New planes for LP1

$$\max Z = 5x_1 + 8x_2$$



LP11

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_2 \le 3 \\ x_1, x_2 \ge 0 \end{cases}$$

LP12

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_2 \le 4 \\ x_1, x_2 \ge 0 \end{cases}$$

Integer programming

Step 6：solve LP11, LP12

Solve LP11 by linprog

x1=2, x2=3
Z=34
34<Z_min=39

Cut LP11

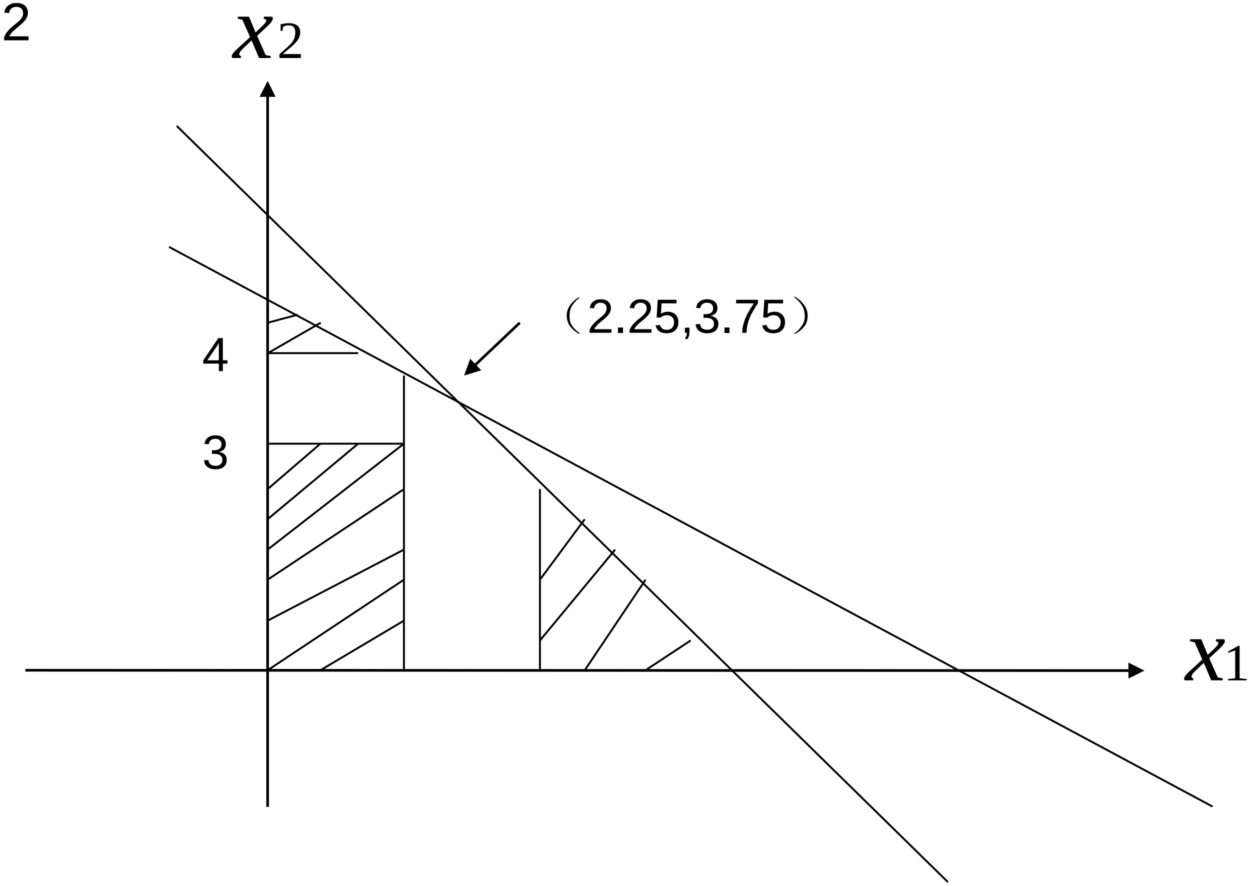
Solve LP12 by linprog

x1=1.8, x2=4

Z=41

41>Z_min=39

Non-integer solution, continue to create new planes for LP12



Integer programming

Step 7：New planes for LP12

$$x_1 \le 1$$

$$\max Z = 5x_1 + 8x_2$$

$$x_1 \ge 2$$

LP121

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_2 \ge 4 \\ x_1 \ge 1 \\ x_1, x_2 \ge 0 \end{cases}$$

LP122

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_2 \ge 4 \\ x_1 \ge 2 \\ x_1, x_2 \ge 0 \end{cases}$$

Integer programming

Step 8：solve LP121,LP122

Solve LP121 by linprog

x1=1, x2=4.4444

$$x_2 \le 4$$

$$x_2 \ge 5$$

Z=40.5556

40.5556>Z_min=39

Non-integer solution, continue to create new planes for LP121

Solve LP122 by linprog

No optimal solution, cut the plane

Integer programming

Step 9：create new planes for LP121

$$\max Z = 5x_1 + 8x_2$$

$$x_2 \le 4$$

$$x_2 \ge 5$$

LP1211

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_2 \ge 4 \\ x_1 \le 1 \\ x_2 \le 4 \\ x_1, x_2 \ge 0 \end{cases}$$

LP1212

$$s.t.\begin{cases} x_1 + x_2 \le 6 \\ 5x_1 + 9x_2 \le 45 \\ x_1 \le 2 \\ x_2 \ge 4 \\ x_1 \le 1 \\ x_2 \le 5 \\ x_1, x_2 \ge 0 \end{cases}$$

Integer programming

Step 10：solve LP1211， LP1212

Solve LP1211 by linprog

x1=1, x2=4

Z=37

37<Z_min=39

Less than lower boundary,

discard the plane

Solve LP1212 by linprog

x1=0, x2=5

Z=40

40>Z_min=39

Integer programming

Integer programming

1.Solve LP by linprog

2.If optimal solution of LP are integers, stop

3.If the optimal solution x0 of LP are non-integers, set Z0 as the ub

4.Use any feasible solution as the new lb $\bar{z}$

$$\underline{z} \le z^{*} \le \bar{z}$$

5.Take any non-integer solution, e.g.xr=br, construct new constrains and create new planes LP1,LP2.

$$x_r \ge [b_r]$$
$$x_r \le [b_r]+1$$

Integer programming

6. Solve LP1 and LP2 by linprog
7. If new optimal solutions are integers, and the corresponding object function value is larger than lb, then update lb
8. If new optimal solutions are non-integers, and the corresponding object function value is less than lb, then discard the plane.
9. Repeat step 5-8，until find optimal integer solution

Integer programming

IP1 function

[x,val,status]=IP1(c,A,b,Aeq,beq,lb,ub,M,e)

M: index of the integer variables

e: round error

$$\max Z = 17x_1 + 12x_2$$

$$s.t.\begin{cases} 10x_1 + 7x_2 \leq 40 \\ x_1 + x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases}$$

```
M=[1,2];
e=2-24;
c=[-17,-12];
A=[10 7; 1 1];
B=[40; 5]
lb=[0 0];
ub=[inf inf];
[v,val,status]=IP1(c,A,B,[],[],lb,ub,M,e)
```

0-1planning

A company wants to build one or two new plants, possible sites are city A and B． It may also need to build one warehouse

(unit：million)

决策	决策的问题	决策变量	总收益	所需资金
1	是否在甲地设新厂	x_1	9	6
2	是否在乙地设新厂	x_2	5	3
3	是否在甲地设新仓库	x_3	6	5
4	是否在乙地设新仓库	x_4	4	2
			可用资本	10

0-1 planning

$$x_i = \begin{cases} 1 \\ 0 \end{cases} \qquad i = 1,2,3,4$$

Profit $$Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

Capital constrain $$6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10$$

Warehouse constrain $$x_3 + x_4 \le 1$$

$$x_3 \le x_1$$

$$x_4 \le x_2$$

0-1 planning

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_i = 0\text{或}1,\quad i = 1,2,3,4 \end{cases}$$

0-1 planning

Step 1：create new planes

Let x1=0，BP1

$$\max Z = 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_i = 0或1,\quad i = 2,3,4 \end{cases}$$

0-1 planning

Step 1：create new planes

Let x1=1，BP2

$$\max Z = 9 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 3x_2 + 5x_3 + 2x_4 \le 4 \\ x_3 + x_4 \le 1 \\ x_3 \le 1 \\ -x_2 + x_4 \le 0 \\ x_i = 0\text{或}1,\quad i = 2,3,4 \end{cases}$$

0-1 planning

Step 2：set boundaries

Replace the binary constrain with x1<=0; 0<=xi<=1（i=2,3,4）

Solve BP1 with linprog

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_1 \le 0 \\ 0 \le x_i \le 1, \quad i = 2,3,4 \end{cases}$$

x=(0 1 0 1)

Z=9

0-1 planning

Step 2：set boundaries

Replace the binary constrain with x1>=1; 0<=xi<=1（i=2,3,4）

Solve BP2 with linprog

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ -x_1 \le -1 \\ 0 \le x_i \le 1, \quad i = 2,3,4 \end{cases}$$

x=(1 0.8 0 0.8)

Z=16.2

0-1 planning

Step 3：create new planes

$$9 \le Z \le 16.2$$

Assign x1=1 Let x2=0，BP21

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_1 \ge 1 \\ x_2 \le 0 \\ 0 \le x_i \le 1, \quad i = 3,4 \end{cases}$$

x=(1 0 0.8 0)

Z=13.8

0-1 planning

Step 3：create new planes

Assign x1=1 　　　Let x2=1，BP22

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_1 \ge 1 \\ x_2 \ge 1 \\ 0 \le x_i \le 1,\quad i = 3,4 \end{cases}$$

x=(1 1 0 0.5)

Z=16

0-1 planning

Step 4：discard and create new planes

None of the solution of BP21，BP22 are binary

Z value of BP22 is larger than BP21

Assign x1=1，x2=1，Let x3=0，BP221

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_1 \ge 1 \\ x_2 \ge 1 \\ x_3 \le 0 \\ 0 \le x_i \le 1, \quad i = 4 \end{cases}$$

x=(1 1 0 0.5)

Z=16

0-1 planning

Step 4：discard and create new planes

1. None of the solution of BP21，BP22 are binary
2. Z value of BP22 is larger than BP21

Assign x1=1，x2=1，Let x3=1，BP222

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_1 \ge 1 \\ x_2 \ge 1 \\ x_3 \ge 1 \\ 0 \le x_i \le 1, \quad i = 4 \end{cases}$$

No feasible solution

Discard BP222

0-1 planning

Step 5：create new planes

Assign x1=1，x2=1，x3=0，let x4=0，BP2211

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_1 \ge 1 \\ x_2 \ge 1 \\ x_3 \le 0 \\ x_4 \le 0 \end{cases}$$

x=(1 1 0 0)

Z=14

0-1 planning

Step 5：create new planes

Assign x1=1，x2=1，x3=0，let x4=1，BP2212

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_1 \ge 1 \\ x_2 \ge 1 \\ x_3 \le 0 \\ x_4 \ge 1 \end{cases}$$

No feasible solution

0-1 planning

Solution of BP2211 x=(1 1 0 0)

Corresponding objective function Z=14

Solution of BP1 x=(0 1 0 1)

Corresponding objective function Z=9

0-1 planning-bintprog

$$\max Z = 3x_1 - x_2 + 5x_3$$

$$s.t.\begin{cases} x_1 + 2x_2 - x_3 \le 2 \\ x_1 + 4x_2 + x_3 \le 4 \\ x_1 + x_2 \le 3 \\ 4x_1 + x_3 \le 6 \\ x_1, x_2, x_3 = 0 or 1 \end{cases}$$

$$\max Z = 9x_1 + 5x_2 + 6x_3 + 4x_4$$

$$s.t.\begin{cases} 6x_1 + 3x_2 + 5x_3 + 2x_4 \le 10 \\ x_3 + x_4 \le 1 \\ -x_1 + x_3 \le 0 \\ -x_2 + x_4 \le 0 \\ x_i = 0 or 1, \quad i = 1,2,3,4 \end{cases}$$

0-1 planning-bintprog

$$\min Z = c^T x$$

$$Ax \le b$$

x仅取0，1

$$\min Z = c^T x$$

$$s.t.\begin{cases} Ax \le b \\ A_{eq}x = b_{eq} \end{cases}$$

[x,fval]=bintprog(c,A,b)

[x,fval]= bintprog(c,A,b,Aeq,beq)