Definitions of ψ-Functions Available in Robustbase

Manuel Koller and Martin Mächler

April 24, 2014

Contents

Preamble

Unless otherwise stated, the following definitions of functions are given by Maronna et al. (2006, p. 31), however our definitions differ sometimes slightly from theirs, as we prefer a different way of *standardizing* the functions. To avoid confusion, we first define ψ- and ρ-functions.

Definition 1 *A ψ-function is a piecewise continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that*

1. *ψ is odd, i.e., $\psi(-x) = -\psi(x) \,\forall x$,*

2. *$\psi(x) \geq 0$ for $x \geq 0$, and $\psi(x) > 0$ for $0 < x < x_r := \sup\{\tilde{x} : \psi(\tilde{x}) > 0\}$ ($x_r > 0$, possibly $x_r = \infty$).*

3* *Its slope is 1 at 0, i.e., $\psi'(0) = 1$.*

Note that '3*' is not strictly required mathematically, but we use it for standardization in those cases where ψ is continuous at 0. Then, it also follows (from 1.) that $\psi(0) = 0$, and we require $\psi(0) = 0$ also for the case where ψ is discontinuous in 0, as it is, e.g., for the M-estimator defining the median.

Definition 2 *A ρ-function can be represented by the following integral of a ψ-function,*

$$\rho(x) = \int_0^x \psi(u)du \,, \tag{1}$$

which entails that $\rho(0) = 0$ and ρ is an even function.

A ψ-function is called *redescending* if $\psi(x) = 0$ for all $x \geq x_r$ for $x_r < \infty$, and x_r is often called *rejection point*. Corresponding to a redescending ψ-function, we define the function $\tilde{\rho}$, a version of ρ standardized such as to attain maximum value one. Formally,

$$\tilde{\rho}(x) = \rho(x)/\rho(\infty). \tag{2}$$

Note that $\rho(\infty) = \rho(x_r) \equiv \rho(x) \ \forall \, |x| >= x_r$. $\tilde{\rho}$ is a ρ-function as defined in Maronna et al. (2006) and has been called χ function in other contexts. For example, in package `robustbase`, `Mchi(x, *)` computes $\tilde{\rho}(x)$, whereas `Mpsi(x, *, deriv=-1)` computes $\rho(x)$, both according to the above definitions.

Note that this definition does require a finite rejection point x_r. Consequently, e.g., the score function $s(x) = -f'(x)/f(x)$ for the Cauchy $(= t_1)$ distribution, which is $s(x) = 2x/(1 + x^2)$ and hence non-monotone and "re descends" to 0 for $x \to \pm\infty$, and $\psi_C(x) := s(x)/2$ also fulfills $\psi_C{}'(0) = 1$, but it has $x_r = \infty$ and hence $\psi_C()$ is *not* a redescending *psi*-function in our sense.

1 Monotone ψ-Functions

Montone ψ-functions lead to convex ρ-functions such that the corresponding M-estimators are defined uniquely.

Historically, the "Huber function" has been the first ψ-function, proposed by Peter Huber in Huber (1964).

1.1 Huber

The family of Huber functions is defined as,

$$\rho_k(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } |x| \leq k \\ k(|x| - \frac{k}{2}) & \text{if } |x| > k \end{cases},$$

$$\psi_k(x) = \begin{cases} x & \text{if } |x| \leq k \\ k \ \text{sign}(x) & \text{if } |x| > k \end{cases}.$$

The constant k for 95% efficiency of the regression estimator is 1.345.

```
> plot(huberPsi, x., ylim=c(-1.4, 5), leg.loc="topright", main=FALSE)
```
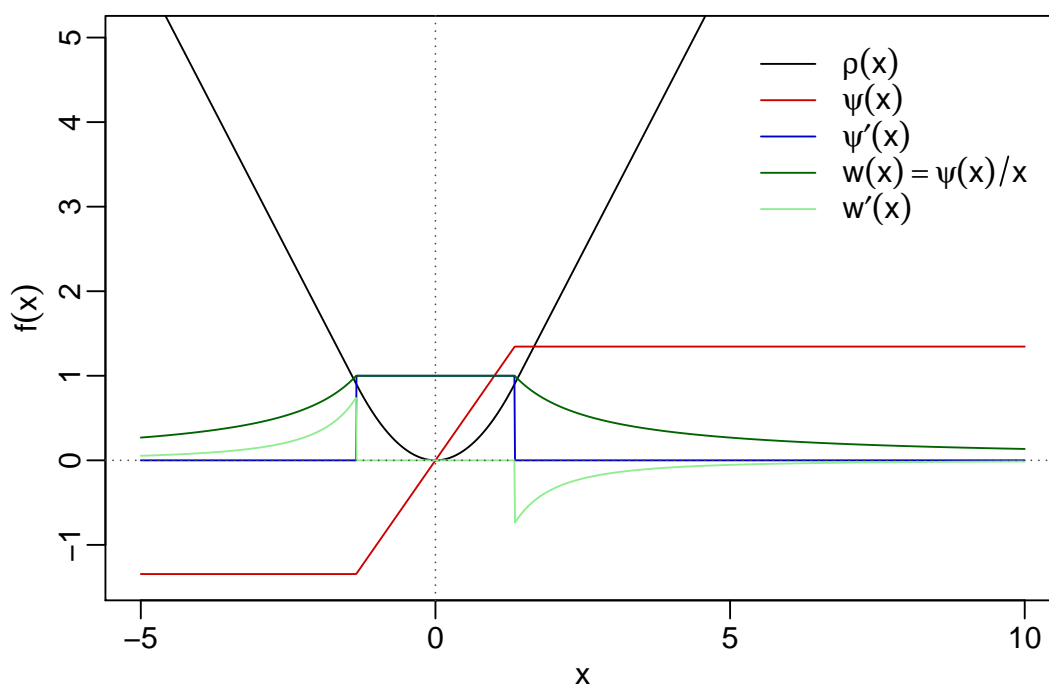

Figure 1: Huber family of functions using tuning parameter $k = 1.345$.

2 Redescenders

For the MM-estimators and their generalizations available via `lmrob()`, the ψ-functions are all redescending, i.e., with finite "rejection point" $x_r = \sup\{t; \psi(t) > 0\} < \infty$. From `lmrob`, the psi functions are available via `lmrob.control`,

```
> names(.Mpsi.tuning.defaults)
```

```
[1] "huber"    "bisquare" "welsh"    "ggw"      "lqq"
[6] "optimal"  "hampel"
```

and their ψ, ρ, ψ', and weight function $w(x) := \psi(x)/x$, are all computed efficiently via C code, and are defined and visualized in the following subsections.

3

2.1 Bisquare

Tukey's bisquare (aka "biweight") family of functions is defined as,

$$\tilde{\rho}_k(x) = \begin{cases} 1 - \left(1 - (x/k)^2\right)^3 & \text{if } |x| \le k \\ 1 & \text{if } |x| > k \end{cases},$$

with derivative $\tilde{\rho}'_k(x) = 6\psi_k(x)/k^2$ where,

$$\psi_k(x) = x \left(1 - \left(\frac{x}{k}\right)^2\right)^2 \cdot I_{\{|x| \le k\}}.$$

The constant k for 95% efficiency of the regression estimator is 4.685 and the constant for a breakdown point of 0.5 of the S-estimator is 1.548. Note that the *exact* default tuning constants for M- and MM- estimation in `robustbase` are available via `.Mpsi.tuning.default()` and `.Mchi.tuning.default()`, respectively, e.g., here,

```
> print(c(k.M = .Mpsi.tuning.default("bisquare"),
+         k.S = .Mchi.tuning.default("bisquare")), digits = 10)

     k.M       k.S
4.685061 1.547640
```
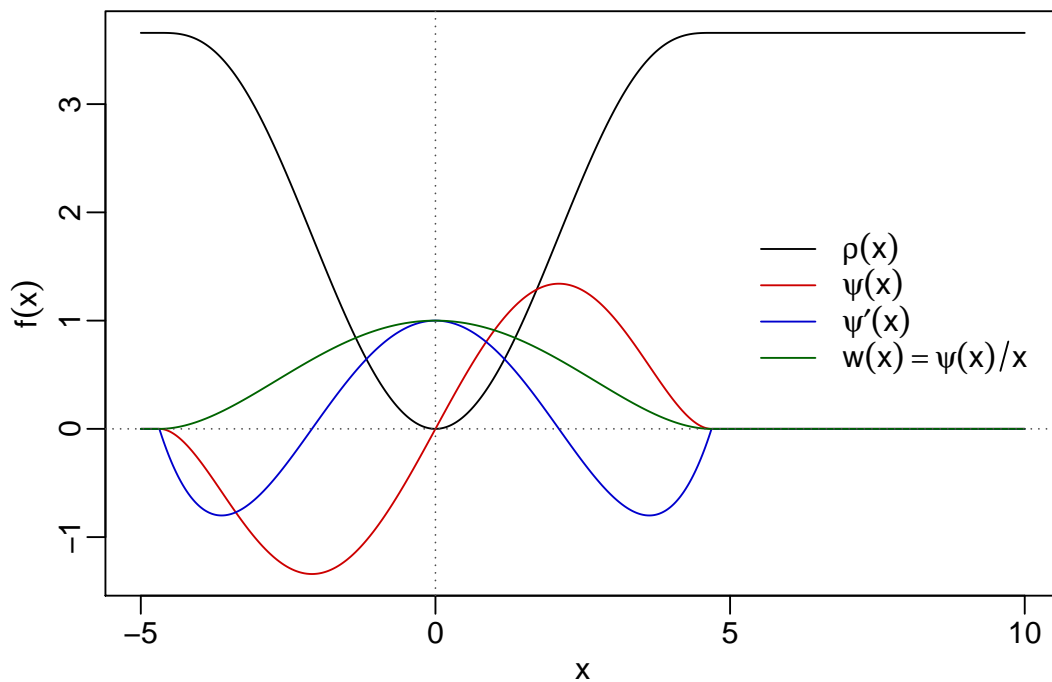

Figure 2: Bisquare family functions using tuning parameter $k = 4.685$.

4

2.2 Hampel

The Hampel family of functions (Hampel et al., 1986) is defined as,

$$\tilde{\rho}_{a,b,r}(x) = \begin{cases} \frac{1}{2}x^2/C & |x| \le a \\ \left(\frac{1}{2}a^2 + a(|x|-a)\right)/C & a < |x| \le b \\ \frac{a}{2}\left(2b - a + (|x|-b)\left(1 + \frac{r-|x|}{r-b}\right)\right)/C & b < |x| \le r \\ 1 & r < |x| \end{cases},$$

$$\psi_{a,b,r}(x) = \begin{cases} x & |x| \le a \\ a \ \mathrm{sign}(x) & a < |x| \le b \\ a \ \mathrm{sign}(x)\frac{r-|x|}{r-b} & b < |x| \le r \\ 0 & r < |x| \end{cases},$$

where $C := \rho(\infty) = \rho(r) = \frac{a}{2}\left(2b - a + (r-b)\right) = \frac{a}{2}(b - a + r)$.

As per our standardization, ψ has slope 1 in the center. The slope of the redescending part ($x \in [b, r]$) is $-a/(r-b)$. If it is set to $-\frac{1}{2}$, as recommended sometimes, one has

$$r = 2a + b.$$

Here however, we restrict ourselves to $a = 1.5k$, $b = 3.5k$, and $r = 8k$, hence a redescending slope of $-\frac{1}{3}$, and vary k to get the desired efficiency or breakdown point.

The constant k for 95% efficiency of the regression estimator is 0.902 (0.9016085, to be exact) and the one for a breakdown point of 0.5 of the S-estimator is 0.212 (i.e., 0.2119163).
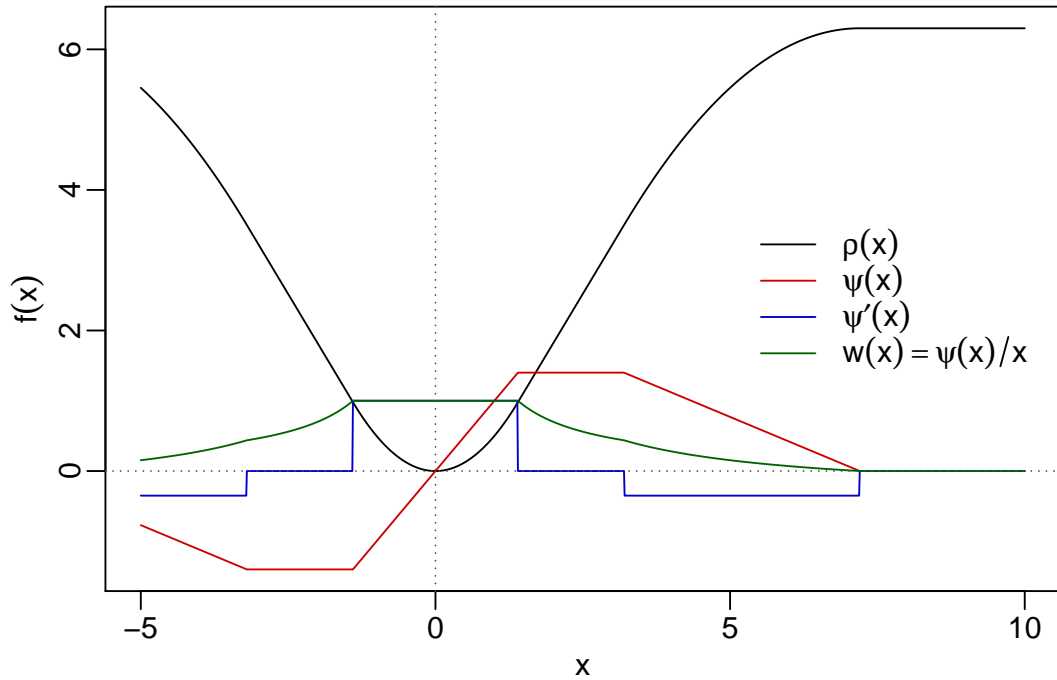

Figure 3: Hampel family of functions using tuning parameters $0.902 \cdot (1.5, 3.5, 8)$.

2.3 GGW

The Generalized Gauss-Weight function, or *ggw* for short, is a generalization of the Welsh ψ-function (below). In Koller and Stahel (2011) it is defined as,

$$\psi_{a,b,c}(x) = \begin{cases} x & |x| \leq c \\ \exp\left(-\frac{1}{2}\frac{(|x|-c)^b}{a}\right) x & |x| > c, \end{cases}.$$

The constants for 95% efficiency of the regression estimator are $a = 1.387$, $b = 1.5$ and $c = 1.063$. The constants for a breakdown point of 0.5 of the S-estimator are $a = 0.204$, $b = 1.5$ and $c = 0.296$.
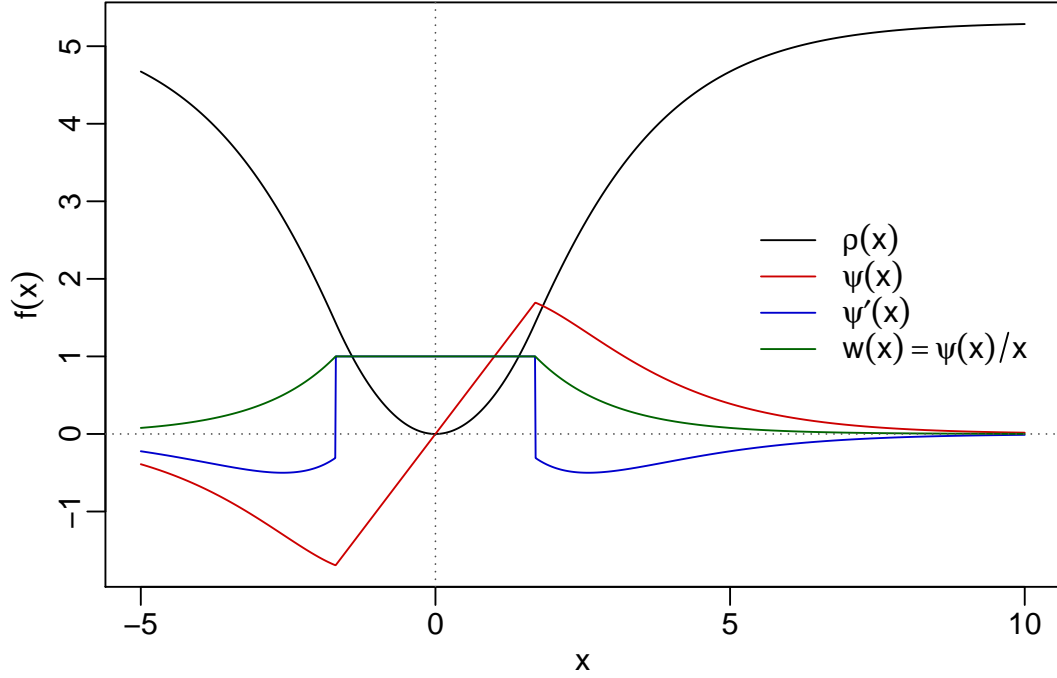

Figure 4: GGW family of functions using tuning parameters $a = 1.387$, $b = 1.5$ and $c = 1.063$.

2.4 LQQ

The "linear quadratic quadratic" ψ-function, or *lqq* for short, was proposed by Koller and Stahel (2011). It is defined as,

$$
\psi_{b,c,s}(x) = \begin{cases} x & |x| \leq c \\ \text{sign}(x)\left(|x| - \frac{s}{2b}\left(|x| - c\right)^2\right) & c < |x| \leq b + c \\ \text{sign}(x)\left(c + b - \frac{bs}{2} + \frac{s-1}{a}\left(\frac{1}{2}\tilde{x}^2 - a\tilde{x}\right)\right) & b + c < |x| \leq a + b + c \\ 0 & \text{otherwise,} \end{cases}
$$

where $\tilde{x} = |x| - b - c$ and $a = (bs - 2b - 2c)/(1 - s)$. The parameter c determines the width of the central identity part. The sharpness of the bend is adjusted by b while the maximal rate of descent is controlled by s ($s = 1 - |\min_x \psi'(x)|$). The length a of the final descent to 0 is determined by b, c and s.

The constants for 95% efficiency of the regression estimator are $b = 1.473$, $c = 0.982$ and $s = 1.5$. The constants for a breakdown point of 0.5 of the S-estimator are $b = 0.402$, $c = 0.268$ and $s = 1.5$.
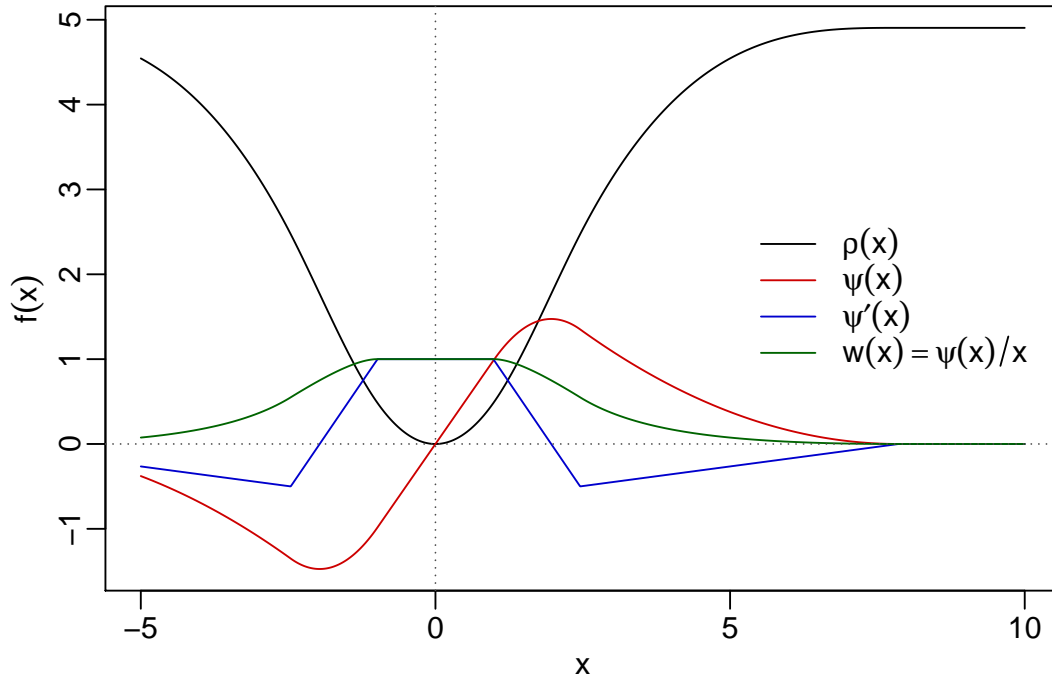

Figure 5: LQQ family of functions using tuning parameters $b = 1.473$, $c = 0.982$ and $s = 1.5$.

2.5 Optimal

The optimal ψ function as given by Maronna et al. (2006, Section 5.9.1),

$$\psi_c(x) = \text{sign}(x)\left(-\frac{\varphi'(|x|) + c}{\varphi(|x|)}\right)_+,$$

where φ is the standard normal density, c is a constant and $t_+ := \max(t, 0)$ denotes the positive part of t.

The constant for 95% efficiency of the regression estimator is 1.060 and the constant for a breakdown point of 0.5 of the S-estimator is 0.405.
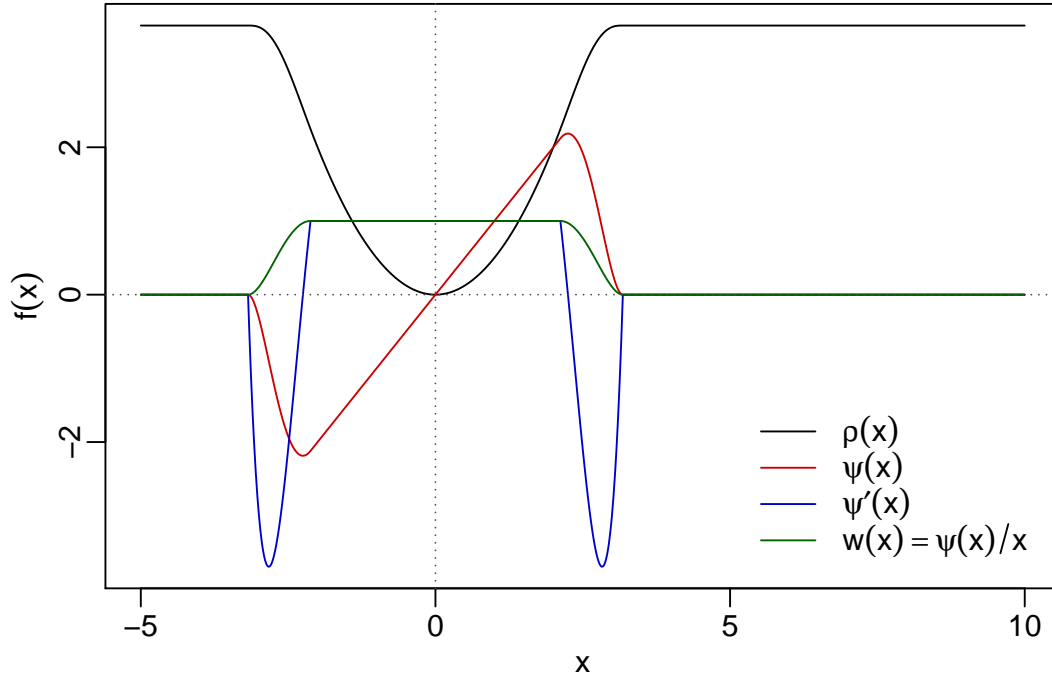

Figure 6: 'Optimal' family of functions using tuning parameter $c = 1.06$.

2.6 Welsh

The Welsh ψ function is defined as,

$$\tilde{\rho}_k(x) = 1 - \exp\bigl(-\left(x/k\right)^2/2\bigr)$$
$$\psi_k(x) = k^2 \tilde{\rho}'_k(x) = x \exp\bigl(-\left(x/k\right)^2/2\bigr)$$
$$\psi'_k(x) = \bigl(1 - \left(x/k\right)^2\bigr) \exp\bigl(-\left(x/k\right)^2/2\bigr)$$

The constant k for 95% efficiency of the regression estimator is 2.11 and the constant for a breakdown point of 0.5 of the S-estimator is 0.577.
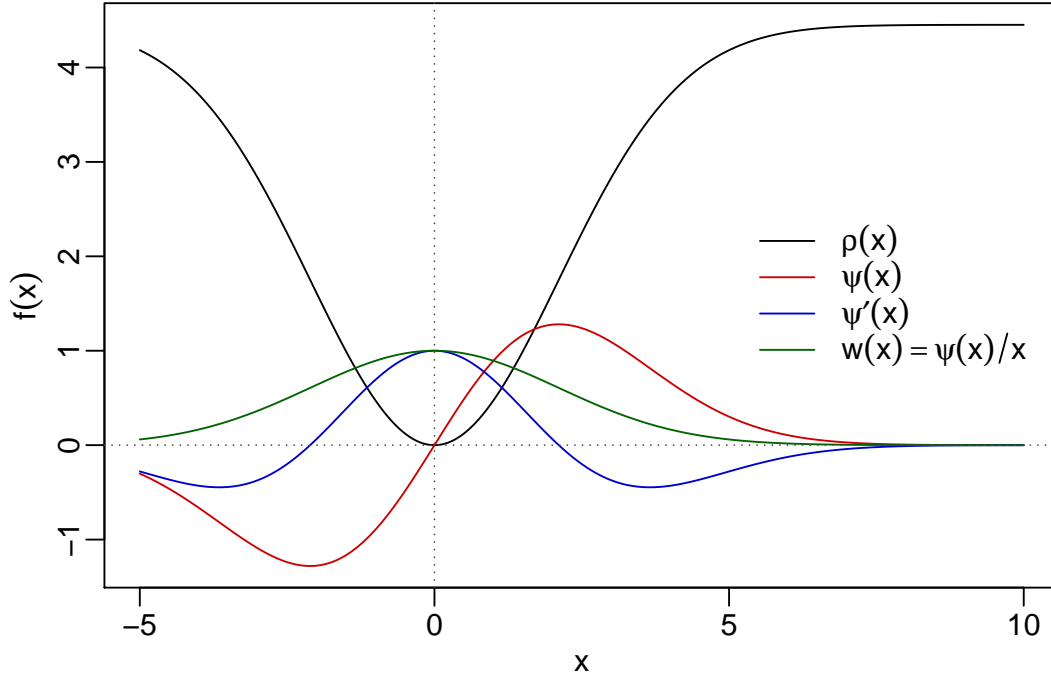

Figure 7: Welsh family of functions using tuning parameter $k = 2.11$.

References

Hampel, F., E. Ronchetti, P. Rousseeuw, and W. Stahel (1986). *Robust Statistics: The Approach Based on Influence Functions.* N.Y.: Wiley.

Huber, P. J. (1964). Robust estimation of a location parameter. *Ann. Math. Statist. 35*, 73–101.

Koller, M. and W. A. Stahel (2011). Sharpening wald-type inference in robust regression for small samples. *Computational Statistics & Data Analysis 55*(8), 2504–2515.

Maronna, R. A., R. D. Martin, and V. J. Yohai (2006). *Robust Statistics, Theory and Methods.* N.Y.: Wiley.