Definitions of ψ-Functions Available in Robustbase

Manuel Koller

March 3, 2013

Contents

Preamble

Unless otherwise stated, the following definitions of functions are given by Maronna et~al. (2006). Their definitions differ sometimes slightly from the ones stated here. We prefer to use another way of standardizing the functions. To avoid confusion, we first define ψ- and ρ-functions.

A *ψ-function* is defined as a function ψ such that

- ψ is odd,

- $\psi(x) > 0$ for $0 < x < \sup\{\tilde{x} : \psi(\tilde{x}) > 0\}$.

- It's derivative ψ' is 1 at 0, i.e., $\psi'(0) = 1$.

A *ρ-function* is defined as the following definite integral of a ψ-function

$$\rho(x) = \int_0^x \psi(x)dx \ .$$

A ψ-function is called *redescending* if $\psi(x) = 0$ for some $x > 0$. Corresponding to a redescending ψ-function, we define a function $\tilde{\rho}$, that is standardized such that the maximum value it attains is one. In formulas $\tilde{\rho}(x) = \rho(x)/\rho(\infty)$. $\tilde{\rho}$ is a ρ-function as defined in Maronna et~al. (2006).

1 Bisquare

The bisquare family of functions is defined as,

$$\tilde{\rho}(x) = \begin{cases} 1 - [1 - (x/k)^2]^3 & \text{if } |x| \leq k \\ 1 & \text{if } |x| > k \end{cases},$$

with derivative $\tilde{\rho}'(x) = 6\psi(x)/k^2$ where,

$$\psi(x) = x \left[1 - \left(\frac{x}{k} \right)^2 \right]^2 I_{\{|x| \leq k\}} .$$

The constant for 95% efficiency of the regression estimator is 4.685 and the constant for a breakdown point of 0.5 of the S-estimator is 1.548.
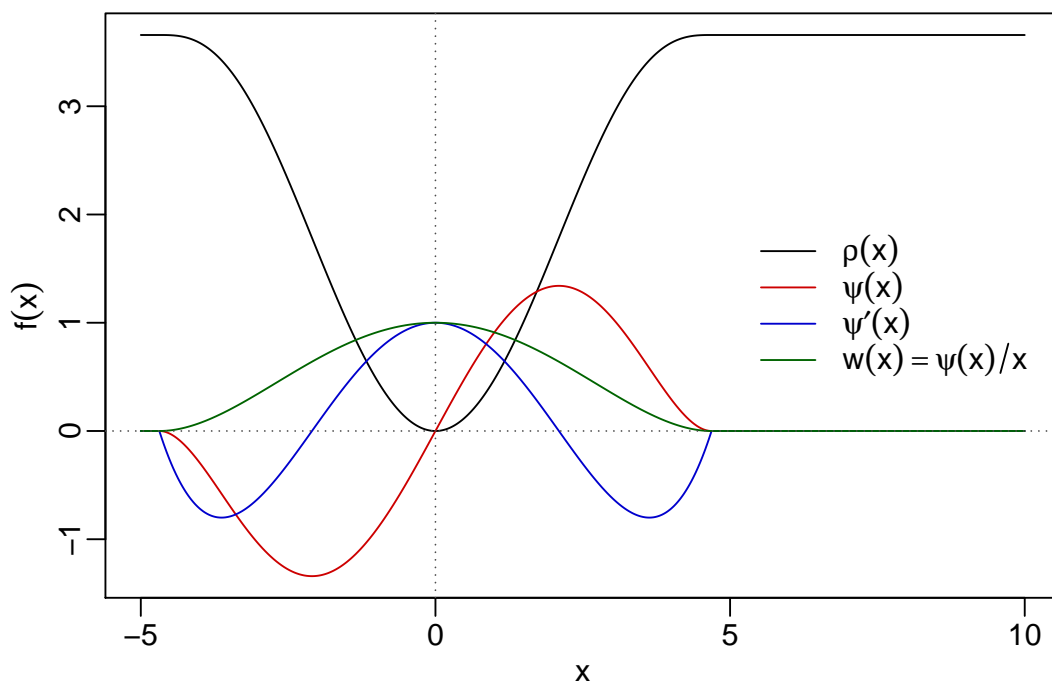

Figure 1: Bisquare family functions using tuning parameter $k = 4.685$.

2 Hampel

Let $C = \frac{1}{2}a^2 + a(b-a) + \frac{1}{2}a(c-b)$. The Hampel family of functions (Hampel et˜al., 1986) is defined as,

$$\tilde{\rho}(x) = \begin{cases} \frac{1}{2}x^2/C & |x| \leq a \\ \left(\frac{1}{2}a^2 + (|x|-a)a\right)/C & a < |x| \leq b \\ \left(\frac{1}{2}a^2 + a(b-a) + \frac{1}{2}a(|x|-b)\left(1 + \frac{c-|x|}{c-b}\right)\right)/C & b < |x| \leq c \\ 1 & c < |x| \end{cases},$$

$$\psi(x) = \begin{cases} x & |x| \leq a \\ \text{sgn}(x)a & a < |x| \leq b \\ a\frac{c-|x|}{c-b}\text{sgn}(x) & b < |x| \leq c \\ 0 & c < |x| \end{cases}.$$

The Hampel ψ function is chosen to have a slope of 1 in the center and a slope of $1/2$ in the redescending part. That implies the following relation between the three parameters a, b and c,

$$c = 2a + b.$$

We use $a = 1.5$, $b = 3.5$ and $c = 8$. These three parameters are then multiplied by k to get the desired efficiency / breakdown point.

The constant for 95% efficiency of the regression estimator is 0.901 and the constant for a breakdown point of 0.5 of the S-estimator is 0.212.
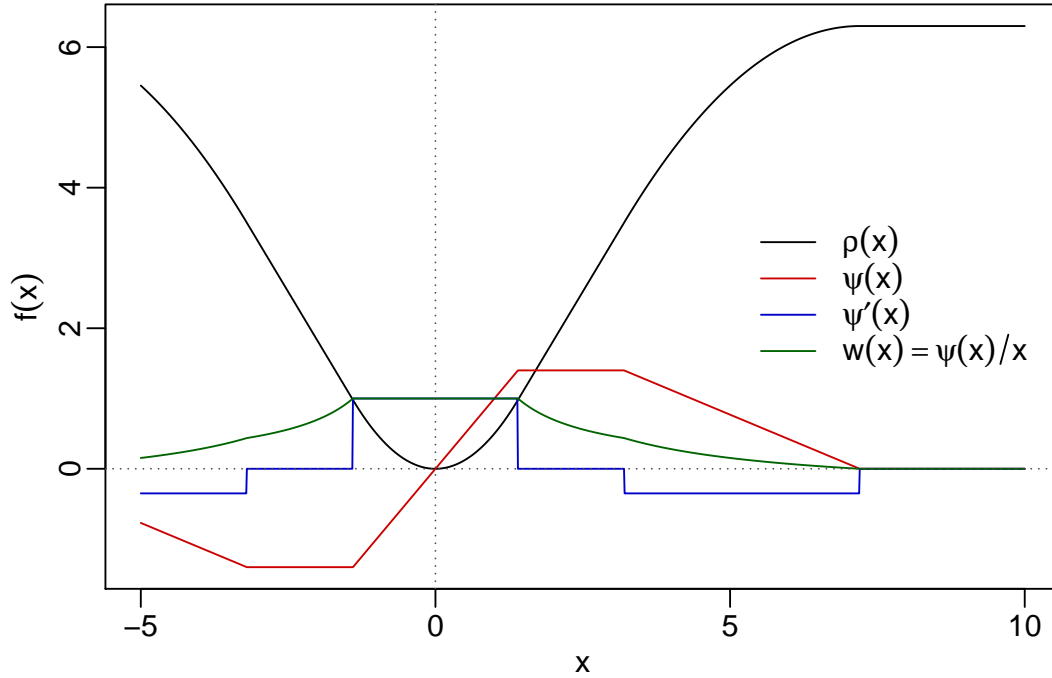

Figure 2: Hampel family of functions using tuning parameters $0.902 \cdot (1.5, 3.5, 9)$.

3 Huber

The family of Huber functions is defined as,

$$\rho(x) = \begin{cases} x^2 & \text{if } |x| \leq k \\ 2k|x| - k^2 & \text{if } |x| > k \end{cases},$$

$$\psi(x) = \begin{cases} x & \text{if } |x| \leq k \\ -k & \text{if } x < -k \\ k & \text{if } x > k \end{cases}.$$

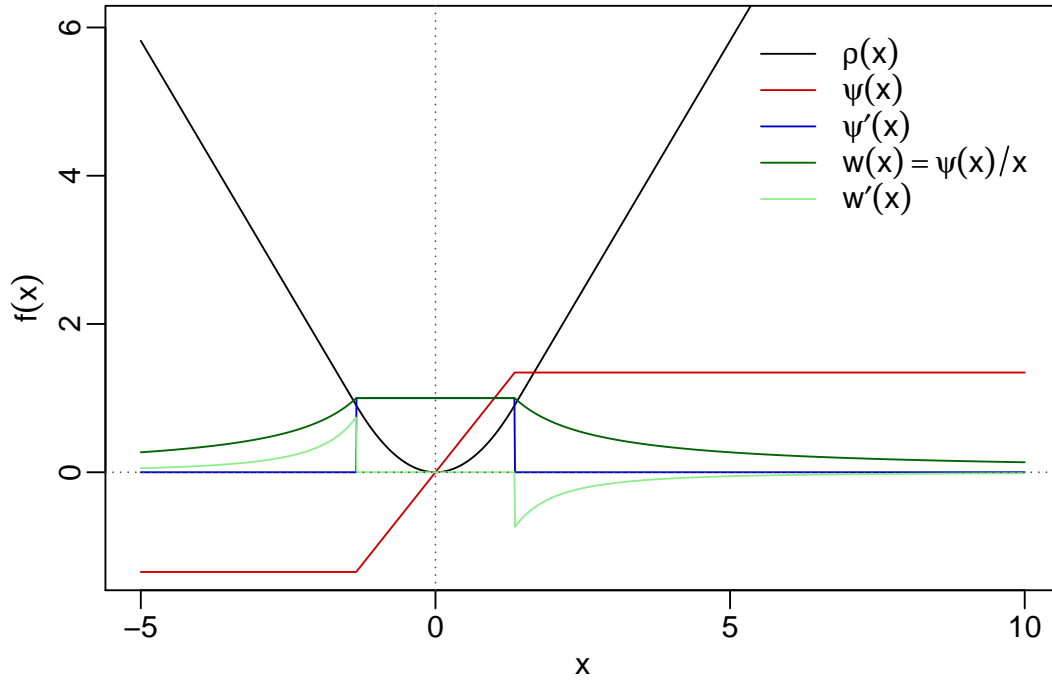The constant for 95% efficiency of the regression estimator is 1.345.

Figure 3: Huber family of functions using tuning parameter $k = 1.345$.

4 GGW

The Generalized Gauss-Weight function, or *ggw* for short, is a generalization of the Welsh ψ-function. In Koller and Stahel (2011) it is defined as,

$$\psi(x, a, b, c) = \begin{cases} x & |x| \leq c \\ \exp\left(-\frac{1}{2} \frac{(|x|-c)^b}{a}\right) x & |x| > c, \end{cases}$$

The constants for 95% efficiency of the regression estimator are $a = 1.387$, $b = 1.5$ and $c = 1.063$. The constants for a breakdown point of 0.5 of the S-estimator are $a = 0.204$, $b = 1.5$ and $c = 0.296$.
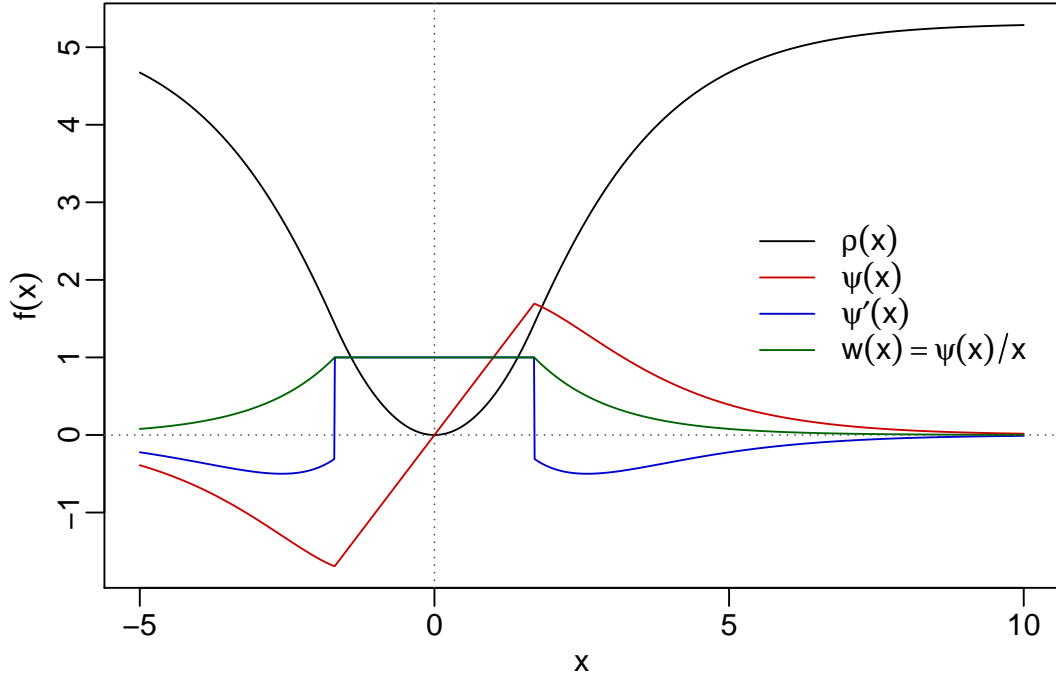

Figure 4: GGW family of functions using tuning parameters $a = 1.387$, $b = 1.5$ and $c = 1.063$.

5 LQQ

The "linear quadratic quadratic"ψ-function, or *lqq* for short, was proposed by Koller and Stahel (2011). It is defined as,

$$\psi(x) = \begin{cases} x & |x| \leq c \\ \text{sign}(x)\left(|x| - \frac{s}{2b}\left(|x| - c\right)^2\right) & c < |x| \leq b + c \\ \text{sign}(x)\left(c + b - \frac{bs}{2} + \frac{s-1}{a}\left(\frac{1}{2}\tilde{x}^2 - a\tilde{x}\right)\right) & b + c < |x| \leq a + b + c \\ 0 & \text{otherwise,} \end{cases}$$

where $\tilde{x} = |x| - b - c$ and $a = (bs - 2b - 2c)/(1 - s)$. The parameter c determines the width of the central identity part. The sharpness of the bend is adjusted by b while the maximal rate of descent is controlled by s ($s = 1 - |\min_x \psi'(x)|$). The length a of the final descent to 0 is determined by b, c and s.

The constants for 95% efficiency of the regression estimator are $b = 1.473$, $c = 0.982$ and $s = 1.5$. The constants for a breakdown point of 0.5 of the S-estimator are $b = 0.402$, $c = 0.268$ and $s = 1.5$.
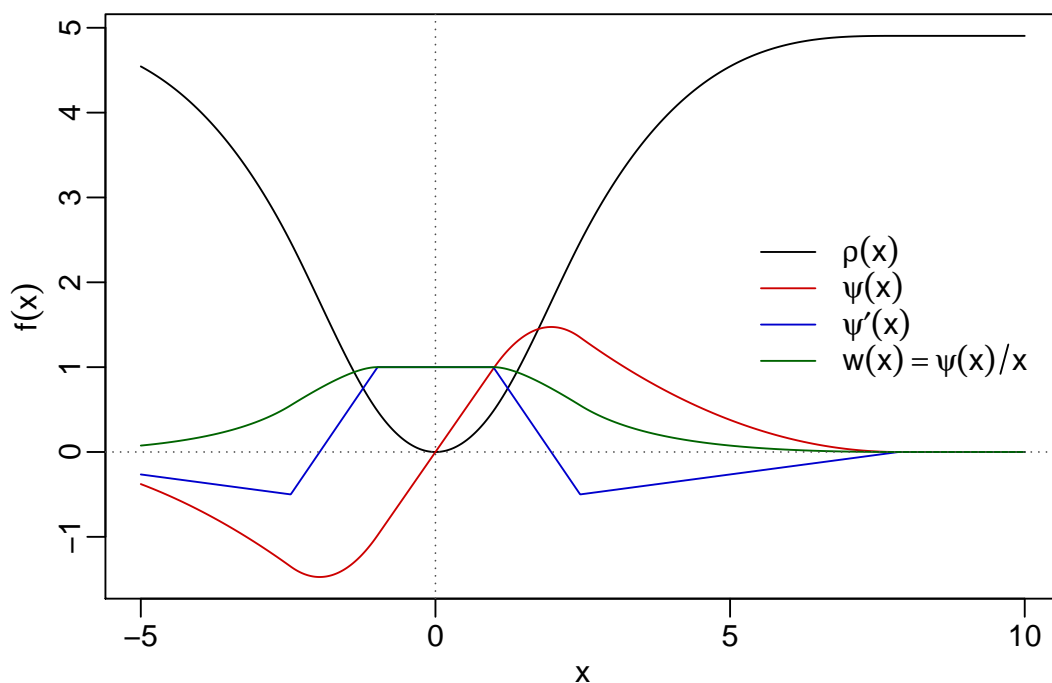

Figure 5: LQQ family of functions using tuning parameters $b = 1.473$, $c = 0.982$ and $s = 1.5$.

6 Optimal

The optimal ψ function as given by Maronna et~al. (2006, Section~5.9.1),

$$\psi(x) = \operatorname{sgn}(x) \left(-\frac{\varphi'(|x|) + c}{\varphi(|x|)} \right)^+,$$

where φ is the standard normal density, c is a constant and $t^+ = \max(t, 0)$ denotes the positive part of t.

The constant for 95% efficiency of the regression estimator is 1.060 and the constant for a breakdown point of 0.5 of the S-estimator is 0.405.
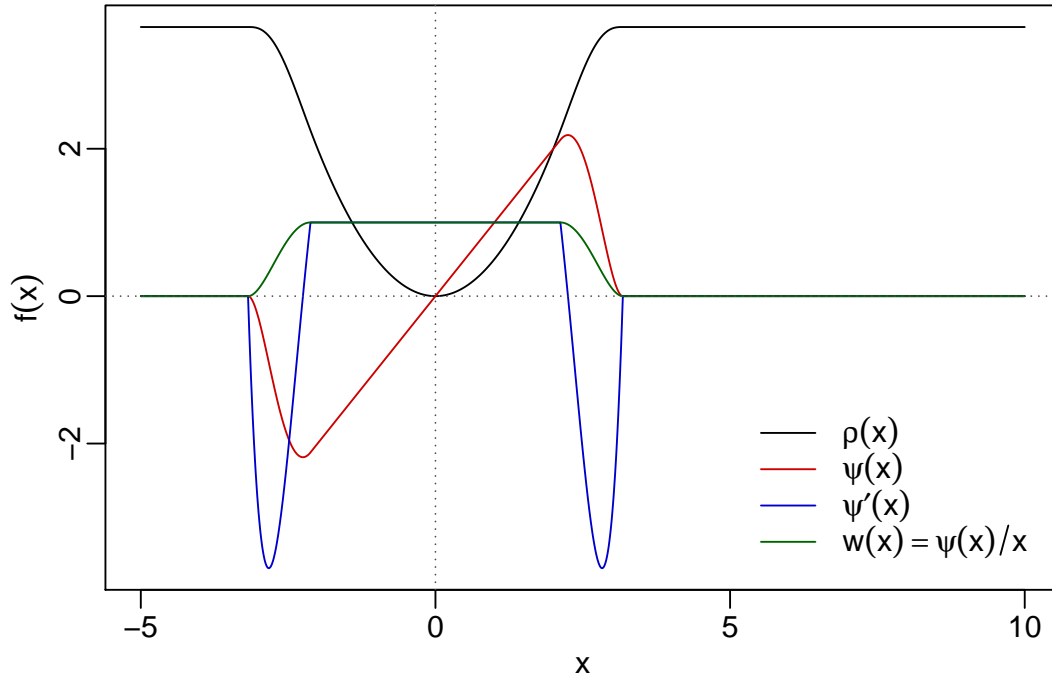

Figure 6: Optimal family of functions using tuning parameter $c = 1.06$.

7 Welsh

The Welsh ψ function is defined as (REFERENCE MISSING),

$$\tilde{\rho}(x, k) = 1 - \exp\left(-\frac{1}{2}\left(\frac{x}{k}\right)^2\right)$$

$$\psi(x, k) = k^2 \tilde{\rho}'(x, k) = x \exp\left(-\frac{1}{2}\left(\frac{x}{k}\right)^2\right)$$

$$\psi'(x, k) = \left(1 - \left(\frac{x}{k}\right)^2\right) \exp\left(-\frac{1}{2}\left(\frac{x}{k}\right)^2\right)$$

The constant for 95% efficiency of the regression estimator is 2.11 and the constant for a breakdown point of 0.5 of the S-estimator is 0.577.
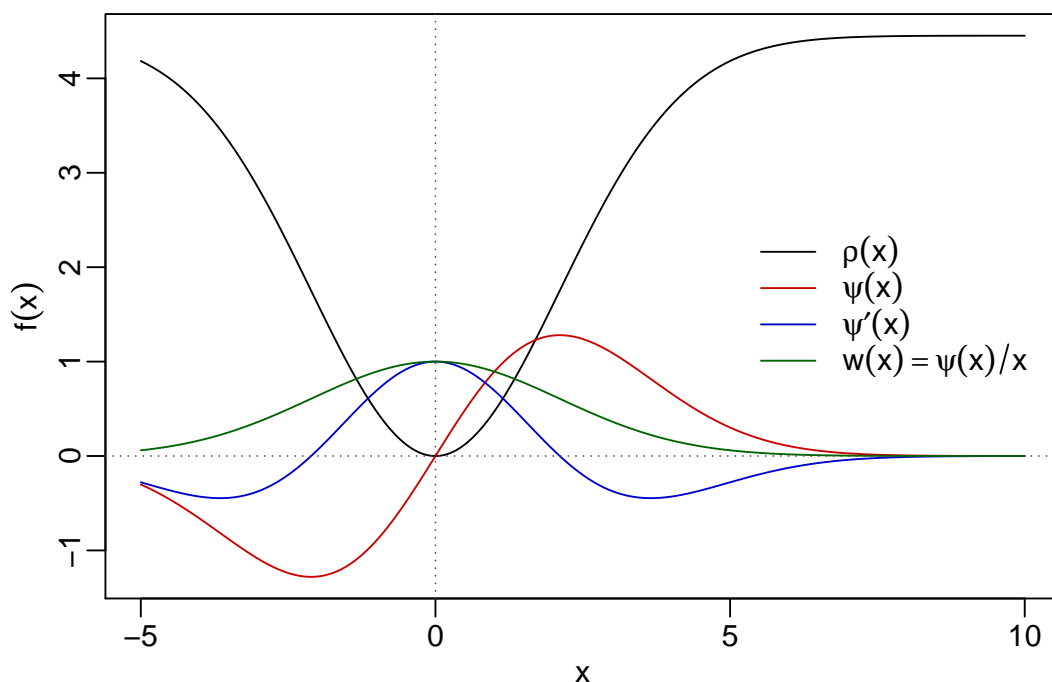

Figure 7: Welsh family of functions using tuning parameter $k = 2.11$.

References

Hampel, F., E.~Ronchetti, P.~Rousseeuw, and W.~Stahel (1986). *Robust Statistics: The Approach Based on Influence Functions*. N.Y.: Wiley.

Koller, M. and W.~A. Stahel (2011). Sharpening wald-type inference in robust regression for small samples. *Computational Statistics & Data Analysis 55*(8), 2504–2515.

Maronna, R.~A., R.~D. Martin, and V.~J. Yohai (2006). *Robust Statistics, Theory and Methods*. N.Y.: Wiley.