Limit of a function

a) $f(x) = \frac{x^2-x-2}{6-2x}$.
First, we determine the domain

$$D_f = \mathbb{R} \setminus \{3\}.$$

Now we compute the limits at the boundary points of the domain. Start with infinities:

$$\begin{aligned}\lim_{x \to +\infty} \frac{x^2-x-2}{6-2x} &= \lim_{x\to+\infty} \frac{x}{x} \frac{x-1-2/x}{6/x-2} \\ &\overset{\text{Limit arithmetics}}{=} \frac{\lim_{x\to+\infty}(x-1-2/x)}{\lim_{x\to+\infty}(6/x-2)} \\ &= +\infty/(-2) \\ &= -\infty\,,\end{aligned}$$

$$\begin{aligned}\lim_{x \to -\infty} \frac{x^2-x-2}{6-2x} &= \lim_{x\to-\infty} \frac{x}{x} \frac{x-1-2/x}{6/x-2} \\ &\overset{\text{Limit arithmetics}}{=} \frac{\lim_{x\to-\infty}(x-1-2/x)}{\lim_{x\to-\infty}(6/x-2)} \\ &= -\infty/(-2) \\ &= +\infty\,.\end{aligned}$$

Next, compute one-sided limits for $x = 3$:

$$\lim_{x\to3^+} \frac{x^2-x-2}{6-2x} = \frac{4}{0^-} = -\infty\,,$$
$$\lim_{x\to3^-} \frac{x^2-x-2}{6-2x} = \frac{4}{0^+} = +\infty\,.$$

Intercepts can be found in a standard way. For the y-intercept, let $x = 0$:

$$P_y = [0, -1/3].$$

For the x-intercepts, set $y = 0$, so we solve

$$x^2 - x - 2 = 0,$$

which is a quadratic equation with roots $x_1 = -1$ and $x_2 = 2$. The two intercepts with the x-axis are therefore:

$$P_{x1} = [-1, 0], \qquad P_{x2} = [2, 0].$$

Derivatives

a) $f(x) = \sqrt{x^4 + 5x^2}$
We differentiate as a composite function (the outer function is the square root and the inner function is the polynomial $x^4 + 5x^2$). Therefore,

$$f'(x) = \frac{1}{2\sqrt{x^4+5x^2}} \cdot (4x^3 + 10x) = \frac{2x^3+5x}{\sqrt{x^4+5x^2}}.$$

b) $f(x) = \mathrm{e}^{3-5x}(x^4 - 3x + 4)$
We differentiate as a product of two functions: an exponential (which is itself composite with the inner function $3 - 5x$) and the polynomial $x^4 - 3x + 4$. Thus,

$$\begin{aligned}f'(x) &= \left(\mathrm{e}^{3-5x}\right)'(x^4-3x+4) + \mathrm{e}^{3-5x}(x^4-3x+4)' \\ &= \mathrm{e}^{3-5x}(-5)(x^4-3x+4) + \mathrm{e}^{3-5x}(4x^3-3) \\ &= \mathrm{e}^{3-5x}(-5x^4+15x-20+4x^3-3) \\ &= \mathrm{e}^{3-5x}(-5x^4+4x^3+15x-23).\end{aligned}$$

c) $f(x) = (3x^2 + 5x)^3$
We differentiate as a composite function (outer function is the cube, inner function is the polynomial $3x^2 + 5x$). Thus,

$$f'(x) = 3(3x^2+5x)^2 \cdot (6x+5).$$

It is correct to leave the result in this simplified factored form. If we expand:

$$(3x^2+5x)^2 = 9x^4 + 30x^3 + 25x^2,$$

so

$$\begin{aligned} f'(x) &= (9x^4 + 30x^3 + 25x^2)(6x+5) \\ &= 54x^5 + 45x^4 + 180x^4 + 150x^3 + 150x^3 + 125x^2 \\ &= 54x^5 + 225x^4 + 300x^3 + 125x^2. \end{aligned}$$

d) $f(x) = \frac{\ln x}{2x^2+5x}$
This is a quotient of two functions $f_1(x) = \ln x$ and $f_2(x) = 2x^2 + 5x$, so by the quotient rule:

$$f'(x) = \frac{\frac{1}{x}(2x^2+5x) - (4x+5)\ln x}{(2x^2+5x)^2}.$$

We simplify:

$$\frac{1}{x}(2x^2+5x) = 2x+5,$$

and

$$(2x^2+5x)^2 = 4x^4 + 20x^3 + 25x^2.$$

Final result:

$$f'(x) = \frac{2x+5-(4x+5)\ln x}{4x^4+20x^3+25x^2}.$$