[REVISION] Algebraic Manipulations and Exponents

a) Factoring Out, Compound Fractions, and Simplifications

Factoring Out a Common Term. Factoring out means writing an expression as a product of a common factor and a simpler expression.

$$ax + ay = a(x + y), \qquad n^2 + 4n = n(n + 4)$$

This is especially useful when simplifying limits or fractions.

Simplifying Fractions. When the numerator and denominator share a common factor, it can be cancelled (if nonzero):

$$\frac{3x^2 + 6x}{3x} = \frac{3x(x + 2)}{3x} = x + 2.$$

Always ensure that the cancelled term is not zero to avoid invalid simplifications.

Compound Fractions. To simplify a compound fraction such as

$$\frac{\frac{a}{b}}{\frac{c}{d}},$$

multiply the numerator by the reciprocal of the denominator:

$$\frac{\frac{a}{b}}{\frac{c}{d}} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Example:

$$\frac{n^2 + 4n}{10n - 5} = \frac{n(n + 4)}{n(10 - 5/n)} = \frac{n + 4}{10 - 5/n}.$$

A: Factoring Out, Compound Fractions, Simplifications

a) Simplify: $\dfrac{6x^2 + 9x}{3x}$.

b) Factor and simplify: $n^2 + 5n - 6$.

c) Simplify the compound fraction: $\frac{\frac{2}{3}}{\frac{4}{9}}$.

d) Factor out the greatest common divisor: $12a^3b^2 - 18a^2b$.

e) Simplify: $\dfrac{n^2 + 4n}{n(10 - 5/n)}$ and state for which n the simplification is valid.

f) Simplify: $\dfrac{x^2 - 9}{x^2 - 6x + 9}$.

g) Multiply and simplify: $\left(\dfrac{2x}{3y}\right)\left(\dfrac{9y^2}{4x^2}\right)$.

h) Simplify: $\dfrac{(n + 2)^2 - (n - 3)^2}{n}$.

i) Reduce the fraction: $\dfrac{4x^3y - 8x^2y^2}{4x^2y}$.

j) Simplify by factoring: $\dfrac{3t^2 - 12}{6t}$.

k) Simplify the complex fraction: $\dfrac{\frac{a}{b} + \frac{c}{d}}{\frac{a}{b} - \frac{c}{d}}$ (give common-denominator form).

l) Factor by grouping: $x^3 + 2x^2 - x - 2$.

m) Simplify: $\dfrac{n^2 + 4n + 7}{(n-2)^2 - (n+3)^2}$ and determine whether the limit as $n \to \infty$ is finite or infinite (no need to compute numeric limit if infinite).

n) Simplify and cancel common factors where allowed: $\dfrac{(3x-6)(x+1)}{3(x-2)}$.

o) Rewrite with a single fraction: $\dfrac{1}{x} + \dfrac{1}{x+1}$.

p) Simplify: $\dfrac{x^2 - 4}{x - 2}$ (state any removable singularities).

q) Simplify: $\dfrac{(2n+4)(n-1)}{2(n+2)}$.

r) Factor completely: $8y^3 - 2y$.

s) Simplify the rational expression: $\dfrac{n^3 - 27}{n^2 + 3n + 9}$.

t) Given $\dfrac{p}{q} = \dfrac{6}{15}$, reduce and rewrite as lowest terms; then multiply numerator and denominator by 7 and simplify.

b) Working with Powers and Exponents

Basic Laws of Exponents. For any real numbers a, b and exponents m, n:

$$a^m \cdot a^n = a^{m+n}, \qquad \frac{a^m}{a^n} = a^{m-n}, \qquad (a^m)^n = a^{mn}.$$

If $a, b > 0$:

$$(ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Negative and Fractional Exponents.

$$a^{-n} = \frac{1}{a^n}, \qquad a^{1/n} = \sqrt[n]{a}.$$

Example:

$$(16)^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8.$$

B: Working with Powers, Exponents, and Roots

a) Simplify: $a^3 \cdot a^5$.

b) Simplify: $\dfrac{b^7}{b^2}$.

c) Simplify: $(x^2)^4$.

d) Write as a single power: $(2a)(2a^2)(2a^3)$.

e) Simplify and express without negative exponents: x^{-3}.

f) Evaluate: $16^{3/4}$.

g) Simplify: $\left(\dfrac{3^2}{3^{-1}}\right)$.

h) Simplify: $(ab)^4$ and then expand to show powers of a and b.

i) Simplify: $\dfrac{(x^3y^{-2})^2}{x^{-1}y}$.

j) Solve for x: $x^{2/3} = 9$ (give real solutions).

k) Compare growth: which grows faster as $n \to \infty$, n^2 or 2^n? Explain briefly.

l) Simplify: $\sqrt[3]{x^6}$ and state when sign issues matter.

m) Simplify: $\left(\dfrac{4}{9}\right)^{-3/2}$.

n) Simplify using exponent rules: $\dfrac{(2x^3)^2}{4x}$.

o) Express as radical: $x^{3/2}$ and simplify for $x = 16$.

p) Simplify: $\left(x^{-1}y^2\right)^{-2}$.

q) Evaluate and simplify: $\dfrac{(27)^{2/3}}{3^{-1}}$.

r) Prove or simplify: $a^m a^n = a^{m+n}$ by example with integers m, n.

s) Simplify and rationalize if needed: $\dfrac{1}{\sqrt{x}}$ (write without a radical in the denominator).

t) Simplify the expression for large n: $\dfrac{n^5 + 2n^3}{n^4}$ and describe leading behaviour as $n \to \infty$ (calculate the limit).

Common Mistakes to Avoid

- $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
- $(a+b)^2 \neq a^2 + b^2$ (correct: $(a+b)^2 = a^2 + 2ab + b^2$)
- $\frac{a+b}{c+d} \neq \frac{a}{c} + \frac{b}{d}$
- $a^{m+n} \neq a^m + a^n$ (correct: $a^{m+n} = a^m \cdot a^n$)
- $(ab)^n \neq a^n + b^n$ (correct: $(ab)^n = a^n b^n$)
- $(a/b)^n \neq a^n/b$ (correct: $(a/b)^n = a^n/b^n$)
- $a^{-n} \neq -a^n$ (correct: $a^{-n} = 1/a^n$)
- $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$
- When cancelling, do not cancel terms—only factors. For example, $\frac{x+2}{x+3} \neq 1$ even though both have x.
- Remember: you can cancel x in $\frac{2x}{3x}$, but not in $\frac{x+2}{x+3}$.
- $(a+b)/(a) \neq 1+b$ (only valid if you factor properly).
- Forgetting domain restrictions: after simplifying $\frac{x^2-4}{x-2} = x+2$, note that $x \neq 2$.
- Taking even roots of negative numbers (e.g. $\sqrt{-4}$) is not real.
- Mixing addition and multiplication rules for exponents incorrectly.
- Dropping parentheses: $-a^2$ means $-(a^2)$, not $(-a)^2$.
- Incorrect cancellation across addition or subtraction: $\frac{x^2+2x}{x} = x+2$, not $x+2x$.
- Confusing $\frac{1}{a+b}$ with $\frac{1}{a} + \frac{1}{b}$.
- Forgetting to distribute exponents: $(3x^2)^3 = 27x^6$, not $3x^6$.
- Forgetting that $(a^m)^n = a^{mn}$, not a^{m+n}.
- Ignoring sign issues: $\sqrt{x^2} = |x|$, not x.