Ex. 1. Compute the limit

$$\lim_{n\to+\infty} \sqrt{n^2+5n+3}-\sqrt{n^2-5n-9}\,.$$

Solution:

If we evaluate the limit directly, we obtain an indeterminate form $\infty-\infty$ (difference of infinities). To solve this, we use trick 3 (i.e., multiplying by the conjugate in the form of 1), giving

$$\begin{aligned}\lim_{n\ \to+\infty} \sqrt{n^2+5n+3}-\sqrt{n^2-5n-9} = \lim_{n\to+\infty} & \sqrt{n^2+5n+3}-\sqrt{n^2-5n-9} \\ & \cdot \frac{\sqrt{n^2+5n+3}+\sqrt{n^2-5n-9}}{\sqrt{n^2+5n+3}+\sqrt{n^2-5n-9}}\,.\end{aligned}$$

For the numerator we use the difference of squares a^2-b^2, then

$$\lim_{n\ \to+\infty} \frac{(n^2+5n+3)-(n^2-5n-9)}{\sqrt{n^2+5n+3}+\sqrt{n^2-5n-9}} = \lim_{n\to+\infty} \frac{10n+12}{\sqrt{n^2+5n+3}+\sqrt{n^2-5n-9}}\,.$$

Here we still obtain ∞/∞, so we use trick 1, which yields

$$\lim_{n\to+\infty} \frac{n}{n} \frac{10+\frac{12}{n}}{\sqrt{1+\frac{5}{n}+\frac{3}{n^2}}+\sqrt{1-\frac{5}{n}-\frac{9}{n^2}}} \overset{\text{Arithmetics}}{=} \frac{10}{1+1} = 5\,.$$

Ex. 2. Differentiate the function and find D_f and $D_{f'}$.

$$f(x) = \mathrm{e}^{2x^2-3x} + \frac{x^2+4}{3x}$$

Solution:

The function consists of two terms (an exponential and a rational function), so the domain is the real numbers except zero (i.e., $\mathbb{R}\setminus\{0\}$). The derivative of a sum is the sum of derivatives. We differentiate the exponential as a composite function and the rational term using the quotient rule.

$$\begin{aligned}f'(x) &= \mathrm{e}^{2x^2-3x}(4x-3) + \frac{2x\cdot 3x-(x^2+4)\cdot 3}{9x^2} \\ &= \mathrm{e}^{2x^2-3x}(4x-3) + \frac{6x^2-3x^2-12}{9x^2} \\ &= \mathrm{e}^{2x^2-3x}(4x-3) + \frac{x^2-4}{3x^2}\end{aligned}$$

The domain of f' is also the real numbers except zero, i.e., $\mathbb{R}\setminus\{0\}$.

Ex. 3. For the hyperbola given by

$$f(x) = \frac{3x+1}{x+2}$$

find the equation of the tangent line to the graph at the point $x_0=-3$. Sketch the hyperbola including this tangent. Also mark the intersections of the hyperbola and of the tangent with the coordinate axes.

Solution:

The equation of the tangent is given by

$$y(x) = f(x_0) + f'(x_0)(x-x_0) = \underbrace{f'(x_0)}_{\text{slope}}\,x + \underbrace{f(x_0)-f'(x_0)x_0}_{\text{constant term}}\,.$$

Compute the derivative of $f(x)$:

$$f'(x) = \frac{3 \cdot (x+2) - (3x+1) \cdot 1}{(x+2)^2} = \frac{3x + 6 - 3x - 1}{(x+2)^2} = \frac{5}{(x+2)^2}\,.$$

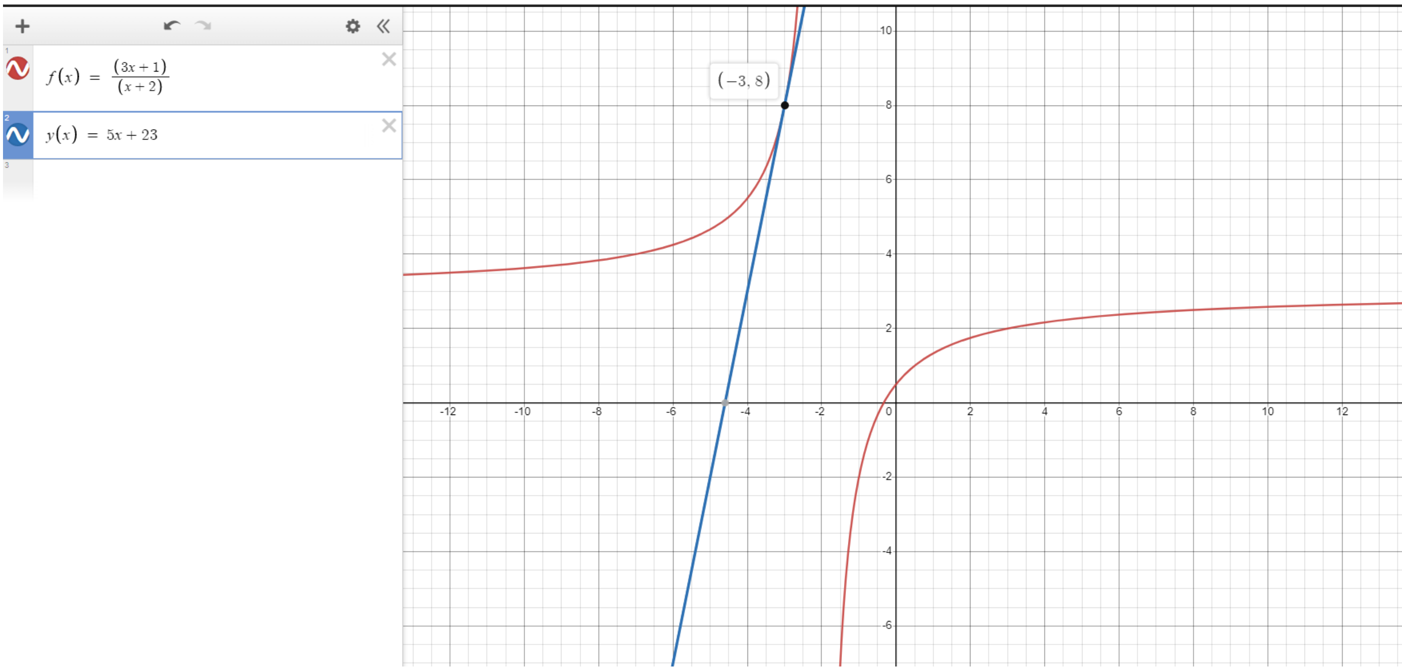
Now we evaluate $f'(-3) = 5$ and $f(-3) = 8$, so the tangent line is

$$y(x) = 5x + 8 - 5 \cdot (-3) = 5x + 23\,.$$

The intercepts of this tangent are $P_y = [0, 23]$ and $P_x = [-23/5, 0]$. For the hyperbola, the intercepts are $P_y = [0, 1/2]$ and $P_x = [-1/3, 0]$.
To sketch the hyperbola, we convert it into centered form:

$$(3x+1) : (x+2) = 3 - \frac{5}{x+2} = 3 + \frac{-5}{x-(-2)}\,.$$

The center is $[-2, 3]$, and since $k = -5 < 0$, the branches lie in the 2nd and 4th quadrants.



Ex. 4. Analyze the behavior of the function

$$f(x) = -x^3 + 3x^2 + 9x + 5\,.$$

Solution:

1. The studied function is a cubic polynomial. For polynomials, the domain is all real numbers, $D_f = \mathbb{R}$. Now let us examine the parity:

$$\begin{aligned} f(-x) &= -(-x)^3 + 3(-x)^2 + 9(-x) + 5 \\ &= x^3 + 3x^2 - 9x + 5\,. \end{aligned}$$

   We see that $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$, so the function is neither even nor odd. We determine the range at the end (from the graph), since inverting a cubic is beyond our scope.

2. Compute intersections with axes. For the x-axis, set $y = 0$:

$$0 = -x^3 + 3x^2 + 9x + 5\,.$$

   This is a cubic equation—solvable by Cardano's formulas (which we do not know), or we guess a root. After some attempts we observe that $x = -1$ is a root, so we can write

$$-x^3 + 3x^2 + 9x + 5 = (x+1)P_2(x)$$

   The polynomial $P_2(x)$ is obtained by

$$P_2(x) = \frac{-x^3 + 3x^2 + 9x + 5}{x+1}\,,$$

$$(-x^3 + 3x^2 + 9x + 5) : (x+1) = -x^2 + 4x + 5\,.$$

   Now we find the roots of $P_2(x) = -x^2 + 4x + 5$ using the discriminant:

$$x_{1,2} = \frac{-4 \pm \sqrt{16+20}}{-2} = \begin{cases} x_1 = -1 \\ x_2 = 5 \end{cases}$$

   We see that $x_{1,3} = -1$ is a double root and the third root is $x_2 = 5$, hence $P_x = [-1, 0]$ and $P_x = [5, 0]$. These roots divide the real line into $(-\infty, -1)$, $(-1, 5)$, and $(5, \infty)$. For the y-axis, set $x = 0$: $P_y = [0, 5]$. Checking signs of the function in the intervals:

$$\begin{aligned} (-\infty, -1) &: + \\ (-1, 5) &: + \\ (5, +\infty) &: - \end{aligned}$$

3. Compute limits at $\pm\infty$:

$$\lim_{x\to+\infty} -x^3 + 3x^2 + 9x + 5 = \lim_{x\to+\infty} -x^3\left(1 + \frac{3}{x} + \frac{9}{x^2} + \frac{5}{x^3}\right) \overset{\text{Arithmetics}}{=} -\infty$$
$$\lim_{x\to-\infty} -x^3 + 3x^2 + 9x + 5 = \lim_{x\to-\infty} -x^3\left(1 + \frac{3}{x} + \frac{9}{x^2} + \frac{5}{x^3}\right) \overset{\text{Arithmetics}}{=} +\infty\,.$$

   Note the signs: for the second limit, $-(-\infty)^3 = +\infty$.

4. Compute the first derivative:

$$f'(x) = -3x^2 + 6x + 9\,.$$

The domain of the derivative is also $\mathbb{R}$.

5. Find critical points solving

$$f'(x) = 0 \quad \rightarrow \quad (-3x^2 + 6x + 9) = 0\,.$$

Discriminant gives

$$x_{1,2} = \frac{-6 \pm \sqrt{36+108}}{-6} = \begin{cases} x_1 = \frac{-6+12}{-6} = -1 \\ x_2 = \frac{-6-12}{-6} = 3 \end{cases}$$

6. Now we analyze monotonicity. The points $x = -1, 3$ divide the real axis into $(-\infty, -1)$, $(-1, 3)$, and $(3, \infty)$. The signs of f':

$$\begin{aligned} (-\infty, -1) &: - \\ (-1, 3) &: + \\ (3, \infty) &: - \end{aligned}$$

The function increases on $(-1, 3)$ and decreases on $(-\infty, -1)$ and $(3, \infty)$. Since f' changes sign at both critical points, these are local extrema: at $x = -1$ a local minimum (decreasing $\rightarrow$ increasing) and at $x = 3$ a local maximum (increasing $\rightarrow$ decreasing). There are no global extrema due to divergence at $\pm\infty$.

7. Asymptotes. There are no vertical asymptotes (no holes in the domain). Limits at $\pm\infty$ diverge, so no horizontal asymptotes. For oblique asymptotes $y = kx + q$, compute

$$k = \lim_{x\to\pm\infty} \frac{-x^3 + 3x^2 + 9x + 5}{x} = \lim_{x\to\pm\infty} -x^2 + 3x + 9 = -\infty\,.$$

Since the slope is infinite, there are no oblique asymptotes.

8. Curvature: Compute second derivative:

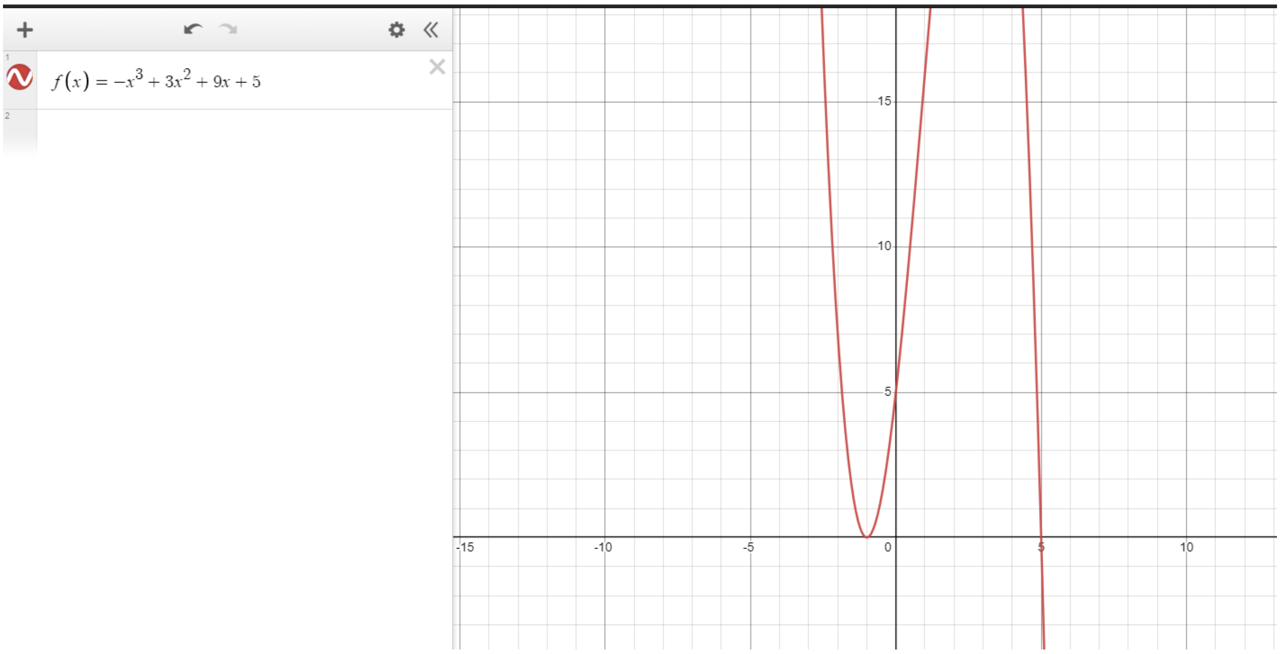
$$f''(x) = -6x + 6\,.$$

Inflection point from $-6x + 6 = 0$ gives $x = 1$. This divides the real line into $(-\infty, 1)$ and $(1, \infty)$:

$$\begin{aligned} (-\infty, 1) &: + \\ (1, \infty) &: - \end{aligned}$$

The function is convex on $(-\infty, 1)$ and concave on $(1, \infty)$.

9. The graph on the next page also shows that the range is all real numbers, $H_f = \mathbb{R}$.


$f(x) = -x^3 + 3x^2 + 9x + 5$
15
10
5
-15
-10
-5
0
5
10