Chapter II

SPECIAL RELATIVITY

There are limitations on motion that are missed by the Galilean description. The first limitation we discover is the existence of a maximal speed in nature. The maximum speed implies many fascinating results: it leads to observer-varying time and length intervals, to an intimate relation between mass and energy, and to the existence of event horizons. We explore them now.

5. MAXIMUM SPEED, OBSERVERS AT REST, AND MOTION OF LIGHT

> “Fama nihil est celerius.*”

Light is indispensable for a precise description of motion. To check whether a line or a path of motion is straight, we must look along it. In other words, we use light to define straightness. How do we decide whether a plane is flat? We look across it,** again using light. How do we measure length to high precision? With light. How do we measure time to high precision? With light: once it was light from the Sun that was Page 1154 used; nowadays it is light from caesium atoms.

In other words, light is important because it is the standard for *undisturbed motion*. Physics would have evolved much more rapidly if, at some earlier time, light propagation had been recognized as the ideal example of motion.

But is light really a phenomenon of motion? This was already known in ancient Greece, from a simple daily phenomenon, the *shadow*. Shadows prove that light is a moving entity, emanating from the light source, and moving in straight lines.*** The obvious conclusion that light takes a certain amount of time to travel from the source to the surface

* ‘Nothing is faster than rumour.’ This common sentence is a simplified version of Virgil’s phrase: *fama, malum qua non aliud velocius ullum.* ‘Rumour, the evil faster than all.’ From the *Aeneid*, book IV, verses 173 and 174.

** Note that looking along the plane from all sides is not sufficient for this: a surface that a light beam touches right along its length in *all* directions does not need to be flat. Can you give an example? One needs other Challenge 540 n methods to check flatness with light. Can you specify one?

*** Whenever a source produces shadows, the emitted entities are called *rays* or *radiation*. Apart from light, other examples of radiation discovered through shadows were *infrared rays* and *ultraviolet rays*, which emanate from most light sources together with visible light, and *cathode rays*, which were found to be to the motion of a new particle, the *electron*. Shadows also led to the discovery of *X-rays*, which again turned out to be a version of light, with high frequency. *Channel rays* were also discovered via their shadows; they turn out to be travelling ionized atoms. The three types of radioactivity, namely *α-rays* (helium nuclei), *β-rays*

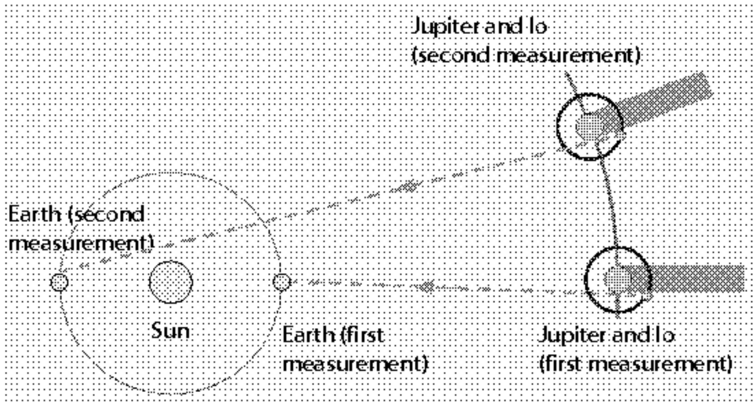

FIGURE 132 Rømer's method of measuring the speed of light

Ref. 231 showing the shadow had already been reached by the Greek thinker Empedocles (*c*. 490 to *c*. 430 bce).

We can confirm this result with a different, equally simple, but subtle argument. Speed can be measured. Therefore the *perfect* speed, which is used as the implicit measurement standard, must have a finite value. An infinite velocity standard would not allow measurements at all. In nature, the lightest entities move with the highest speed. Light, which is indeed light, is an obvious candidate for motion with perfect but finite speed. We will confirm this in a minute. Challenge 541 n

A finite speed of light means that whatever we see is a message from the *past*. When we see the stars, the Sun or a loved one, we always see an image of the past. In a sense, nature prevents us from enjoying the present – we must therefore learn to enjoy the past.

The speed of light is high; therefore it was not measured until 1676, even though many, including Galileo, had tried to do so earlier. The first measurement method was worked out by the Danish astronomer Ole Rømer* when he was studying the orbits of Io and the other moons of Jupiter. He obtained an incorrect value for the speed of light because he used the wrong value for their distance from Earth. However, this was quickly corrected by his peers, including Newton himself. You might try to deduce his method from Figure 132. Since that time it has been known that light takes a bit more than 8 minutes to travel from the Sun to the Earth. This was confirmed in a beautiful way fifty years later, in 1726, by the astronomer James Bradley. Being English, Bradley thought of the 'rain method' to measure the speed of light. Challenge 542 n Page 191 Ref. 232

How can we measure the speed of falling rain? We walk rapidly with an umbrella, measure the angle α at which the rain appears to fall, and then measure our own velocity

(again electrons), and *γ-rays* (high-energy X-rays) also produce shadows. All these discoveries were made between 1890 and 1910: those were the 'ray days' of physics.

* Ole (Olaf) Rømer (1644 Aarhus – 1710 Copenhagen), Danish astronomer. He was the teacher of the Dauphin in Paris, at the time of Louis XIV. The idea of measuring the speed of light in this way was due to the Italian astronomer Givanni Cassini, whose assistant Rømer had been. Rømer continued his measurements until 1681, when Rømer had to leave France, like all protestants (such as Christiaan Huygens), so that his work was interrupted. Back in Denmark, a fire destroyed all his measurement notes. As a result, he was not able to continue improving the precision of his method. Later he became an important administrator and reformer of the Danish state.

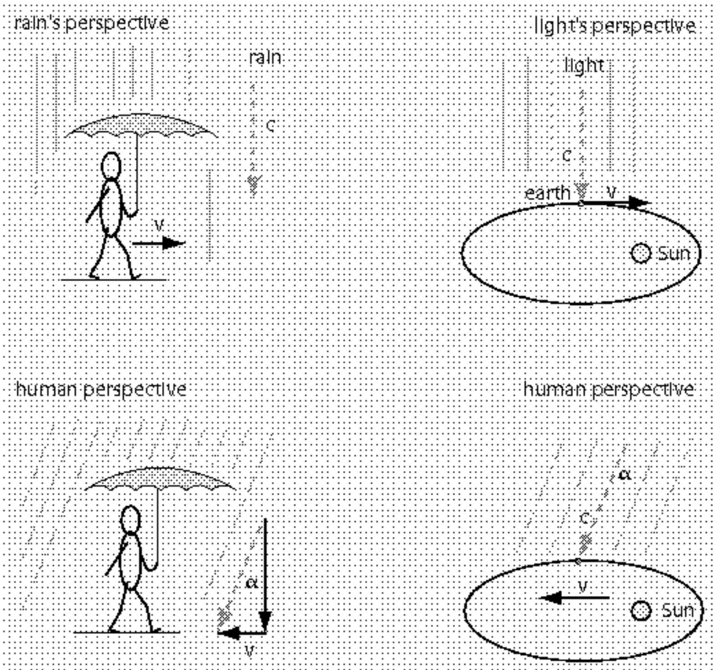

FIGURE 133 The rain method of measuring the speed of light

v. As shown in Figure 133, the speed c of the rain is then given by

$$c = v/\tan\alpha\ . \tag{100}$$

The same measurement can be made for light; we just need to measure the angle at which the light from a star above Earth's orbit arrives at the Earth. Because the Earth is moving relative to the Sun and thus to the star, the angle is not a right one. This effect is called the *aberration* of light; the angle is found most easily by comparing measurements made six months apart. The value of the angle is 20.5″; nowadays it can be measured with a precision of five decimal digits. Given that the speed of the Earth around the Sun is $v = 2\pi R/T = 29.7\,\mathrm{km/s}$, the speed of light must therefore be $c = 3.00 \cdot 10^8\,\mathrm{m/s}$.* This is

* Umbrellas were not common in Britain in 1726; they became fashionable later, after being introduced from China. The umbrella part of the story is made up. In reality, Bradley had his idea while sailing on the Thames, when he noted that on a moving ship the apparent wind has a different direction from that on land. He had observed 50 stars for many years, notably Gamma Draconis, and during that time he had been puzzled by the *sign* of the aberration, which was *opposite* to the effect he was looking for, namely the star parallax. Both the parallax and the aberration for a star above the ecliptic make them describe a small ellipse in the course of an Earth year, though with different rotation senses. Can you see why?

Challenge 543 n

By the way, it follows from special relativity that the formula (100) is wrong, and that the correct formula is $c = v/\sin\alpha$; can you see why?

Challenge 544 n

To determine the speed of the Earth, we first have to determine its distance from the Sun. The simplest method is the one by the Greek thinker Aristarchos of Samos (*c.* 310 to *c.* 230 BCE). We measure the angle between the Moon and the Sun at the moment when the Moon is precisely half full. The cosine of that angle gives the ratio between the distance to the Moon (determined, for example, by the methods of page 117) and the distance to the Sun. The explanation is left as a puzzle for the reader.

Challenge 545 n

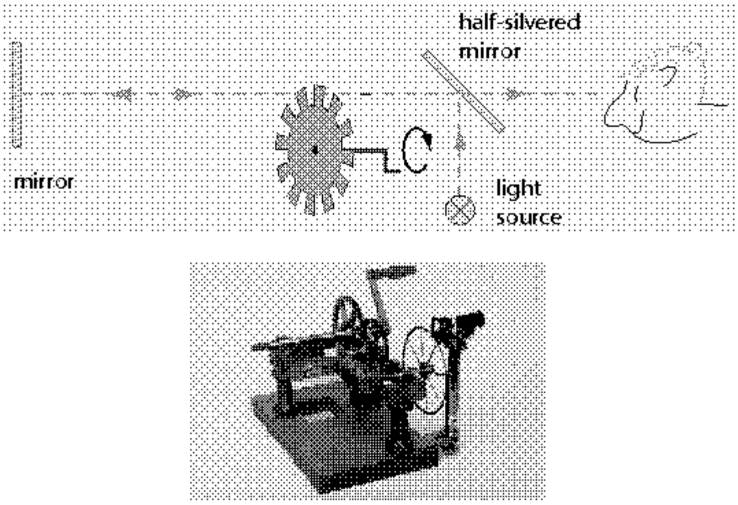

FIGURE 134 Fizeau's set-up to measure the speed of light (© AG Didaktik und Geschichte der Physik, Universität Oldenburg)

an astonishing value, especially when compared with the highest speed ever achieved by a man-made object, namely the Voyager satellites, which travel at 52 Mm/h = 14 km/s, with the growth of children, about 3 nm/s, or with the growth of stalagmites in caves, about 0.3 pm/s. We begin to realize why measurement of the speed of light is a science in its own right.

The first *precise* measurement of the speed of light was made in 1849 by the French physicist Hippolyte Fizeau (1819–1896). His value was only 5 % greater than the modern one. He sent a beam of light towards a distant mirror and measured the time the light took to come back. How did Fizeau measure the time without any electric device? In fact, Page 57 he used the same ideas that are used to measure bullet speeds; part of the answer is given Challenge 547 n in Figure 134. (How far away does the mirror have to be?) A modern reconstruction of Ref. 235 his experiment by Jan Frercks has achieved a precision of 2 %. Today, the experiment is much simpler; in the chapter on electrodynamics we will discover how to measure the Page 560 speed of light using two standard UNIX or Linux computers connected by a cable.

The speed of light is so high that it is even difficult to prove that it is *finite*. Perhaps the most beautiful way to prove this is to photograph a light pulse flying across one's field of view, in the same way as one can photograph a car driving by or a bullet flying through

---

The angle in question is almost a right angle (which would yield an infinite distance), and good instru- Ref. 233 ments are needed to measure it with precision, as Hipparchos noted in an extensive discussion of the problem around 130 BCE. Precise measurement of the angle became possible only in the late seventeenth century, when it was found to be 89.86°, giving a distance ratio of about 400. Today, thanks to radar measurements Page [illegible] of planets, the distance to the Sun is known with the incredible precision of 30 metres. Moon distance vari- Challenge 546 n ations can even be measured to the nearest centimetre; can you guess how this is achieved?

Ref. 234 Aristarchos also determined the radius of the Sun and of the Moon as multiples of those of the Earth. Aristarchos was a remarkable thinker: he was the first to propose the heliocentric system, and perhaps the first to propose that stars were other, faraway suns. For these ideas, several of his contemporaries proposed that he should be condemned to death for impiety. When the Polish monk and astronomer Nicolaus Copernicus (1473–1543) again proposed the heliocentric system two thousand years later, he did not mention Aristarchus, even though he got the idea from him.

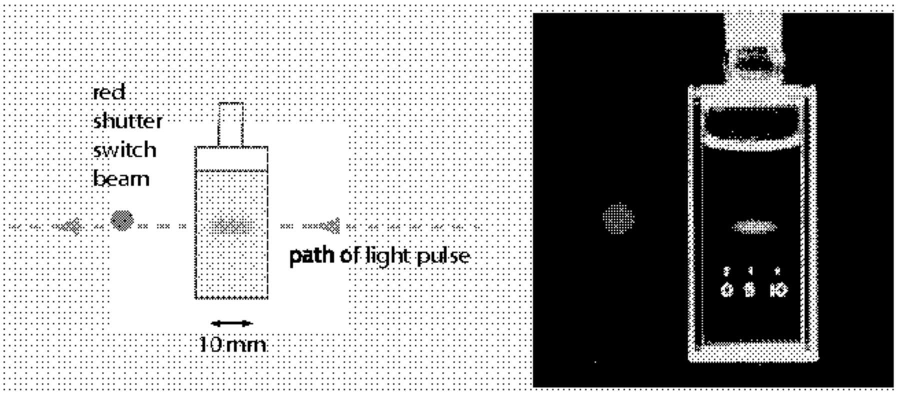

FIGURE 135 A photograph of a light pulse moving from right to left through a bottle with milky water, marked in millimetres (© Tom Mattick)

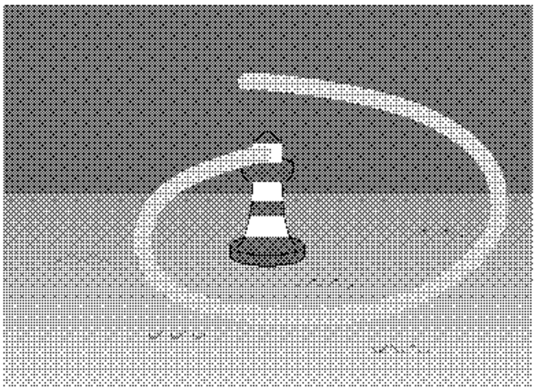

FIGURE 136 A consequence of the finiteness of the speed of light

Ref. 255 the air. Figure 135 shows the first such photograph, produced in 1971 with a standard off-the-shelf reflex camera, a very fast shutter invented by the photographers, and, most noteworthy, not a single piece of electronic equipment. (How fast does such a shutter have to be? How would you build such a shutter? And how would you make sure it opened at the right instant?) Challenge 548 n

A finite speed of light also implies that a rapidly rotating light beam behaves as shown as in Figure 136. In everyday life, the high speed of light and the slow rotation of lighthouses make the effect barely noticeable.

In short, light moves extremely rapidly. It is much faster than lightning, as you might like to check yourself. A century of increasingly precise measurements of the speed have culminated in the modern value Challenge 549 n

$$c = 299\,792\,458 \text{ m/s}. \qquad (101)$$

In fact, this value has now been fixed *exactly*, by definition, and the metre has been defined in terms of c. Table 35 gives a summary of what is known today about the motion of light. Two surprising properties were discovered in the late nineteenth century. They form the

TABLE 35 Properties of the motion of light

Observations about light
Light can move through vacuum.
Light transports energy.
Light has momentum: it can hit bodies.
Light has angular momentum: it can rotate bodies.
Light moves across other light undisturbed.
Light in vacuum always moves faster than any material body does.
The speed of light, its true signal speed, is the forerunner speed. Page 579
In vacuum its value is 299 792 458 m/s.
The proper speed of light is infinite. Page 297
Shadows can move without any speed limit.
Light moves in a straight line when far from matter.
High-intensity light is a wave.
Light beams are approximations when the wavelength is neglected.
In matter, both the forerunner speed and the energy speed of light are lower than in vacuum.
In matter, the group velocity of light pulses can be zero, positive, negative or infinite.

Ref. 237 basis of special relativity.

Can one play tennis using a laser pulse as the ball and mirrors as rackets?

> Et nihil est celerius annis.*
>
> Ovid, *Metamorphoses*.

We all know that in order to throw a stone as far as possible, we run as we throw it; we know instinctively that in that case the stone's speed with respect to the ground is higher. However, to the initial astonishment of everybody, experiments show that light emitted from a moving lamp has the same speed as light emitted from a resting one. Light (in vacuum) is never faster than light; all light beams have the same speed. Many specially Ref. 238 designed experiments have confirmed this result to high precision. The speed of light can be measured with a precision of better than 1 m/s; but even for lamp speeds of more than Challenge 550 n 290 000 000 m/s no differences have been found. (Can you guess what lamps were used?)

In everyday life, we know that a stone arrives more rapidly if we run towards it. Again, for light no difference has been measured. All experiments show that the velocity of light has the *same value* for all observers, even if they are moving with respect to each other or with respect to the light source. The speed of light is indeed the ideal, perfect measurement standard.**

* 'Nothing is faster than the years.' Book X, verse 520.

** An equivalent alternative term for the speed of light is 'radar speed' or 'radio speed'; we will see below Page 563 why this is the case.

The speed of light is also not far from the speed of neutrinos. This was shown most spectacularly by the

Ref. 241 There is also a second set of experimental evidence for the constancy of the speed of light. Every electromagnetic device, such as an electric toothbrush, shows that the speed of light is Page 536 constant. We will discover that magnetic fields would not result from electric currents, as they do every day in every motor and in every loudspeaker, if the speed of light were not constant. This was actually how the constancy was first deduced, by several researchers. Only after understanding this, did the German–Swiss physicist Albert Einstein* show that the constancy is also in agreement with the motion of bodies, as we will do in this section. The connection between electric toothbrushes and re- Page 536 lativity will be described in the chapter on electrodynamics.** In simple terms, if the speed of light were not constant, observers would be able to move at the speed of light. Since light is a wave, such observers would see a wave standing still. However, electromagnetism forbids the such a phenomenon. Therefore, observers cannot reach the speed of light.

Albert Einstein

In summary, the velocity v of any physical system in nature (i.e., any localized mass or energy) is bound by

$$v \leqslant c \,. \tag{102}$$

This relation is the basis of special relativity; in fact, the full theory of special relativity is contained in it. Einstein often regretted that the theory was called 'Relativitätstheorie' or 'theory of relativity'; he preferred the name 'Invarianztheorie' or 'theory of invariance', Ref. 244 but was not able to change the name.

---

observation of a supernova in 1987, when the flash and the neutrino pulse arrived a 12 seconds apart. (It is not known whether the difference is due to speed differences or to a different starting point of the two flashes.) What is the first digit for which the two speed values could differ, knowing that the supernova was Challenge 551 n $1.7 \cdot 10^5$ light years away?

Experiments also show that the speed of light is the same in all directions of space, to at least 21 digits of Ref. 239 precision. Other data, taken from gamma ray bursts, show that the speed of light is independent of frequency, Ref. 240 to at least 20 digits of precision.

* Albert Einstein (b. 1879 Ulm, d. 1955 Princeton); one of the greatest physicists ever. He published three important papers in 1905, one about Brownian motion, one about special relativity, and one about the idea of light quanta. Each paper was worth a Nobel Prize, but he was awarded the prize only for the last one. Also in 1905, he proved the famous formula $E_0 = mc^2$ (published in early 1906), possibly triggered by an idea of Page 343 Olinto de Pretto. Although Einstein was one of the founders of quantum theory, he later turned against it. His famous discussions with his friend Niels Bohr nevertheless helped to clarify the field in its most counter-intuitive aspects. He explained the Einstein–de Haas effect which proves that magnetism is due to motion inside materials. In 1915 and 1916, he published his highest achievement: the general theory of relativity, one of the most beautiful and remarkable works of science.

Being Jewish and famous, Einstein was a favourite target of attacks and discrimination by the National Socialist movement; in 1933 he emigrated to the USA. He was not only a great physicist, but also a great Ref. 242 thinker; his collection of thoughts about topics outside physics are worth reading.

Anyone interested in emulating Einstein should know that he published many papers, and that many of them were wrong; he would then correct the results in subsequent papers, and then do so again. This happened so frequently that he made fun of himself about it. Einstein realizes the famous definition of a genius as a person who makes the largest possible number of mistakes in the shortest possible time.

** For information about the influences of relativity on machine design, see the interesting textbook by Van Ref. 243 Bladel.

The constancy of the speed of light is in complete contrast with Galilean mechanics, and proves that the latter is *wrong* at high velocities. At low velocities the description remains good, because the error is small. But if we want a description valid at *all* velocities, we have to discard Galilean mechanics. For example, when we play tennis we use the fact that by hitting the ball in the right way, we can increase or decrease its speed. But with light this is impossible. Even if we take an aeroplane and fly after a light beam, it still moves away with the same speed. Light does not behave like cars. If we accelerate a bus we are driving, the cars on the other side of the road pass by with higher and higher speeds. For light, this is *not* so: light always passes by with the *same* speed.*

Why is this result almost unbelievable, even though the measurements show it unambiguously? Take two observers O and Ω (pronounced 'omega') moving with relative velocity v, such as two cars on opposite sides of the street. Imagine that at the moment they pass each other, a light flash is emitted by a lamp in O. The light flash moves through positions $x(t)$ for O and through positions $\xi(\tau)$ (pronounced 'xi of tau') for Ω. Since the speed of light is the same for both, we have

$$\frac{x}{t} = c = \frac{\xi}{\tau} . \tag{103}$$

However, in the situation described, we obviously have $x \neq \xi$. In other words, the constancy of the speed of light implies that $t \neq \tau$, i.e. that *time is different for observers moving relative to each other*. Time is thus not unique. This surprising result, which has been confirmed by many experiments, was first stated clearly in 1905 by Albert Einstein. Though many others knew about the invariance of c, only the young Einstein had the courage to say that time is observer-dependent, and to face the consequences. Let us do so as well.

Challenge 552 e

Ref. 240

Already in 1895, the discussion of viewpoint invariance had been called the *theory of relativity* by Henri Poincaré.** Einstein called the description of motion without gravity the theory of *special relativity*, and the description of motion with gravity the theory of *general relativity*. Both fields are full of fascinating and counter-intuitive results. In particular, they show that everyday Galilean physics is wrong at high speeds.

Ref. 241

The speed of light is a limit speed. We stress that we are not talking of the situation where a particle moves faster than the speed of light *in matter*, but still slower than the speed of light *in vacuum*. Moving faster than the speed of light in matter is possible. If the particle is charged, this situation gives rise to the so-called *Čerenkov radiation*. It corresponds to the V-shaped wave created by a motor boat on the sea or the cone-shaped shock wave around an aeroplane moving faster than the speed of sound. Čerenkov radiation is regularly observed; for example it is the cause of the blue glow of the water in nuclear reactors. Incidentally, the speed of light in matter can be quite low: in the centre of the Sun,

* Indeed, even with the current measurement precision of $2 \cdot 10^{-13}$, we cannot discern any changes of the speed of light with the speed of the observer.

Ref. 239

** Henri Poincaré (1854–1912), important French mathematician and physicist. Poincaré was one of the most productive men of his time, advancing relativity, quantum theory, and many parts of mathematics.

The most beautiful and simple introduction to relativity is still that given by Albert Einstein himself, for example in *Über die spezielle und allgemeine Relativitätstheorie*, Vieweg, 1997, or in *The Meaning of Relativity*, Methuen, London, 1951. It has taken a century for books almost as beautiful to appear, such as the text by Taylor and Wheeler.

Ref. 246, Ref. 247

the speed of light is estimated to be only around 10 km/year, and even in the laboratory, for some materials, it has been found to be as low as 0.3 m/s. In the following, when we use the term 'speed of light', we mean the speed of light in vacuum. The speed of light in air in smaller than that in vacuum only by a fraction of a percent, so that in most cases, the difference can be neglected.

Ref. 248, Ref. 249

Special relativity in a few lines

The speed of light is constant for all observers. We can thus deduce all relations between what two different observers measure with the help of Figure 137. It shows two observers moving with constant speed against each other in space-time, with the first sending a light flash to the second, from where it is reflected back to the first. Since light speed is constant, light is the only way to compare time and space coordinates for two distant observers. Two distant clocks (like two distant metre bars) can only be compared, or synchronized, using light or radio flashes. Since light speed is constant, light paths are parallel in such diagrams.

Ref. 250

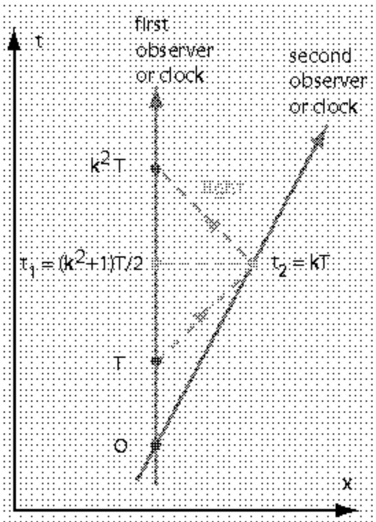

FIGURE 137 A drawing containing most of special relativity

A constant relative speed between two observers implies that a constant factor k relates the time coordinates of events. (Why is the relation linear?) If a flash starts at a time T as measured for the first observer, it arrives at the second at time kT, and then back again at the first at time $k^2 T$. The drawing shows that

Challenge 553 n
Challenge 554 n

$$k = \sqrt{\frac{c+v}{c-v}} \quad \text{or} \quad \frac{v}{c} = \frac{k^2-1}{k^2+1} \,. \tag{104}$$

This factor will appear again in the Doppler effect.*

Page [illegible]

The figure also shows that the time coordinate t_1 assigned by the first observer to the moment in which the light is reflected is different from the coordinate t_2 assigned by the second observer. Time is indeed different for two observers in relative motion. Figure 138 illustrates the result.

The *time dilation factor* between the two time coordinates is found from Figure 137 by comparing the values t_1 and t_2; it is given by

$$\frac{t_1}{t_2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma(v) \,. \tag{105}$$

Time intervals for a moving observer are *shorter* by this factor γ; the time dilation factor is always larger than 1. In other words, *moving clocks go slower*. For everyday speeds the

* The explanation of relativity using the factor k is often called *k-calculus*.

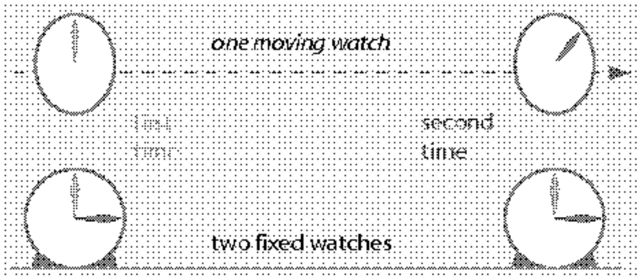

FIGURE 138 Moving clocks go slow

Challenge 555 e effect is tiny. That is why we do not detect time differences in everyday life. Nevertheless, Galilean physics is not correct for speeds near that of light. The same factor γ also appears in the formula $E = \gamma mc^2$, which we will deduce below. Expression (304) or (305) is the only piece of mathematics needed in special relativity: all other results derive from it.

If a light flash is sent forward starting from the second observer and reflected back, he will make the same statement: for him, the first clock is moving, and also for him, the moving clock goes slower. *Each of the observers observes that the other clock goes slower.* The situation is similar to that of two men comparing the number of steps between two identical ladders that are not parallel. A man on either ladder will always observe that the steps of the *other* ladder are shorter. For another analogy, take two people moving away from each other: each of them notes that the other gets smaller as their distance increases.

Naturally, many people have tried to find arguments to avoid the strange conclusion that time differs from observer to observer. But none have succeeded, and experimental results confirm this conclusion. Let us have a look at some of them.

Acceleration of light and the Doppler effect

Light *can* be accelerated. Every mirror does this! We will see in the chapter on electromagnetism that matter also has the power to *bend* light, and thus to accelerate it. However, it
Page 557 will turn out that all these methods only change the *direction* of propagation; none has the power to change the *speed* of light in a vacuum. In short, light is an example of a motion
Challenge 556 n that cannot be stopped. There are only a few other such examples. Can you name one?

What would happen if we could accelerate light to higher speeds? For this to be possible, light would have to be made of particles with non-vanishing mass. Physicists call such particles *massive* particles. If light had mass, it would be necessary to distinguish the 'massless energy speed' c from the speed of light c_L, which would be lower and would depend on the kinetic energy of those massive particles. The speed of light would not be constant, but the massless energy speed would still be so. Massive light particles could be captured, stopped and stored in a box. Such boxes would make electric illumination unnecessary; it would be sufficient to store some daylight in them and release the light, slowly, during the following night, maybe after giving it a push to speed it up.*

Physicists have tested the possibility of massive light in quite some detail. Observations
Ref. 251, Ref. 252 now put any possible mass of light (particles) at less than $1.3 \cdot 10^{-52}$ kg from terrestrial

* Incidentally, massive light would also have *longitudinal* polarization modes. This is in contrast to observations, which show that light is polarized exclusively *transversally* to the propagation direction.

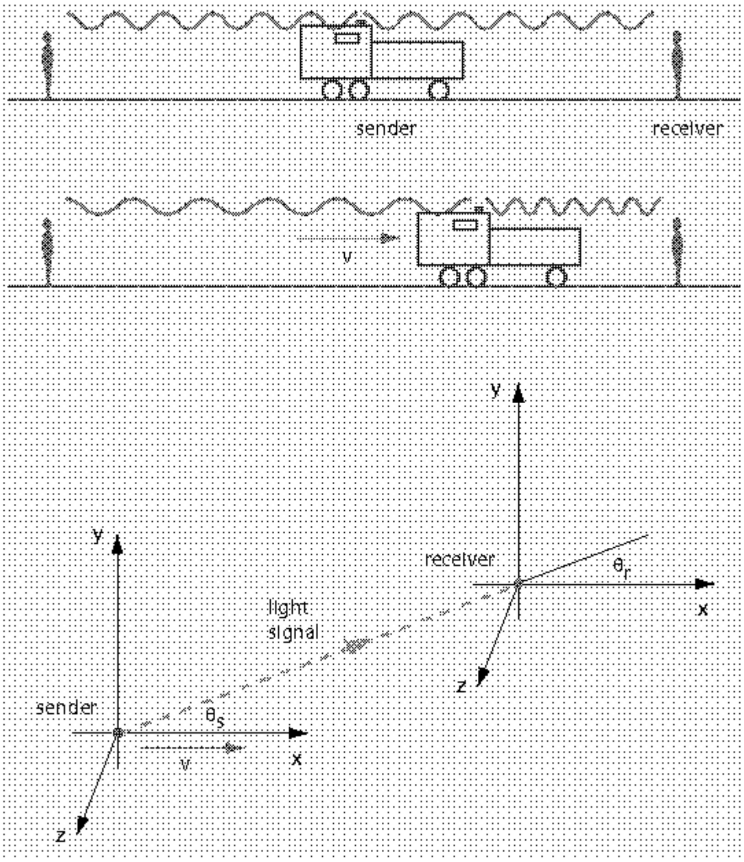

FIGURE 139 The set-up for the observation of the Doppler effect

experiments, and at less than $4 \cdot 10^{-62}$ kg from astrophysical arguments (which are a bit less strict). In other words, light is not heavy, light is light.

But what happens when light hits a *moving* mirror? If the speed of light does not change, something else must. The situation is akin to that of a light source moving with respect to the receiver: the receiver will observe a *different colour* from that observed by the sender. This is called the *Doppler effect*. Christian Doppler* was the first to study the frequency shift in the case of sound waves – the well-known change in whistle tone between approaching and departing trains – and to extend the concept to the case of light waves. As we will see later on, light is (also) a wave, and its colour is determined by its frequency, or equivalently, by its wavelength λ. Like the tone change for moving trains, Doppler realized that a moving light source produces a colour at the receiver that is different from the colour at the source. Simple geometry, and the conservation of the number of maxima and minima, leads to the result

Challenge 557 e

* Christian Andreas Doppler (b. 1803 Salzburg, d. 1853 Venezia), Austrian physicist. Doppler studied the effect named after him for sound and light. In 1842 he predicted (correctly) that one day we would be able to use the effect to measure the motion of distant stars by looking at their colours.

$$\frac{\lambda_r}{\lambda_s} = \frac{1}{\sqrt{1 - v^2/c^2}} \left(1 - \frac{v}{c} \cos\theta_r\right) = \gamma \left(1 - \frac{v}{c} \cos\theta_r\right) . \tag{106}$$

The variables v and θ_r in this expression are defined in Figure 139. Light from an approaching source is thus blue-shifted, whereas light from a departing source is red-shifted. The first observation of the Doppler effect for light was made by Johannes Stark* in 1905, who studied the light emitted by moving atoms. All subsequent experiments confirmed the calculated colour shift within measurement errors; the latest checks have found agreement to within two parts per million. In contrast to sound waves, a colour change is also found when the motion is *transverse* to the light signal. Thus, a yellow rod in rapid motion across the field of view will have a blue leading edge and a red trailing edge prior to the closest approach to the observer. The colours result from a combination of the longitudinal (first-order) Doppler shift and the transverse (second-order) Doppler shift. At a particular angle $\theta_{\text{unshifted}}$ the colours will be the same. (How does the wavelength change in the purely transverse case? What is the expression for $\theta_{\text{unshifted}}$ in terms of v?)

Ref. 255
Challenge 558 n

The colour shift is used in many applications. Almost all solid bodies are mirrors for radio waves. Many buildings have doors that open automatically when one approaches. A little sensor above the door detects the approaching person. It usually does this by measuring the Doppler effect of radio waves emitted by the sensor and reflected by the approaching person. (We will see later that radio waves and light are manifestations of the same phenomenon.) So the doors open whenever something moves towards them. Police radar also uses the Doppler effect, this time to measure the speed of cars.**

Page 563

The Doppler effect also makes it possible to measure the velocity of light sources. Indeed, it is commonly used to measure the speed of distant stars. In these cases, the Doppler shift is often characterized by the *red-shift number* z, defined with the help of wavelength λ or frequency F by

$$z = \frac{\Delta\lambda}{\lambda} = \frac{f_S}{f_R} - 1 = \sqrt{\frac{c+v}{c-v}} - 1 . \tag{107}$$

Can you imagine how the number z is determined? Typical values for z for light sources in the sky range from -0.1 to 3.5, but higher values, up to more than 10, have also been found. Can you determine the corresponding speeds? How can they be so high?

Challenge 560 n
Challenge 561 n

In summary, whenever one tries to change the *speed* of light, one only manages to change its *colour*. That is the Doppler effect.

We know from classical physics that when light passes a large mass, such as a star, it is deflected. Does this deflection lead to a Doppler shift?

Page 133
Challenge 562 n

* Johannes Stark (1874–1957), discovered in 1905 the optical Doppler effect in channel rays, and in 1913 the splitting of spectral lines in electrical fields, nowadays called the Stark effect. For these two discoveries he received the 1919 Nobel Prize for physics. He left his professorship in 1922 and later turned into a full-blown National Socialist. A member of the NSDAP from 1930 onwards, he became known for aggressively criticizing other people's statements about nature purely for ideological reasons; he became rightly despised by the academic community all over the world.

** At what speed does a red traffic light appear green?

Challenge 559 n

The difference between light and sound

The Doppler effect for light is much more important than the Doppler effect for sound. Even if the speed of light were not yet known to be constant, this effect alone would *prove* that time is different for observers moving relative to each other. Why? Time is what we read from our watch. In order to determine whether another watch is synchronized with our own one, we look at both watches. In short, we need to use light signals to synchron-
Ref. 254 ize clocks. Now, any change in the colour of light moving from one observer to another necessarily implies that their watches run differently, and thus that time is *different* for the two of them. One way to see this is to note that also a light source is a clock – 'ticking' very rapidly. So if two observers see different colours from the same source, they measure different numbers of oscillations for the same clock. In other words, time is different for observers moving against each other. Indeed, equation (104) implies that the whole of relativity follows from the full Doppler effect for light. (Can you confirm that the connection between observer-dependent frequencies and observer-dependent time breaks
Challenge 563 n down in the case of the Doppler effect for *sound*?)

Why does the behaviour of light imply special relativity, while that of sound in air does not? The answer is that light is a limit for the motion of energy. Experience shows that there are supersonic aeroplanes, but there are no superluminal rockets. In other words, the limit $v \leqslant c$ is valid only if c is the speed of light, not if c is the speed of sound in air.

However, there is at least one system in nature where the speed of sound is indeed a limit speed for energy: the speed of sound is the limit speed for the motion of *dislocations*
Page 337 in crystalline solids. (We discuss this in detail later on.) As a result, the theory of special relativity is also valid for such dislocations, provided that the speed of light is replaced everywhere by the speed of sound! Dislocations obey the Lorentz transformations, show
Ref. 255 length contraction, and obey the famous energy formula $E = \gamma m c^2$. In all these effects the speed of sound c plays the same role for dislocations as the speed of light plays for general physical systems.

If special relativity is based on the statement that nothing can move faster than light, this statement needs to be carefully checked.

Can one shoot faster than one's shadow?

“ Quid celerius umbra?* ”

for Lucky Luke to achieve the feat shown in Figure 140, his bullet has to move faster than
Challenge 564 e the speed of light. (What about his hand?) In order to emulate Lucky Luke, we could take the largest practical amount of energy available, taking it directly from an electrical power station, and accelerate the lightest 'bullets' that can be handled, namely electrons. This experiment is carried out daily in particle accelerators such as the Large Electron Positron ring, the LEP, of 27 km circumference, located partly in France and partly in Switzerland, near Geneva. There, 40 MW of electrical power (the same amount used by a small city) accelerates electrons and positrons to energies of over 16 nJ (104.5 GeV) each, and their
Ref. 256 speed is measured. The result is shown in Figure 141: even with these impressive means

* 'What is faster than the shadow?' A motto often found on sundials.

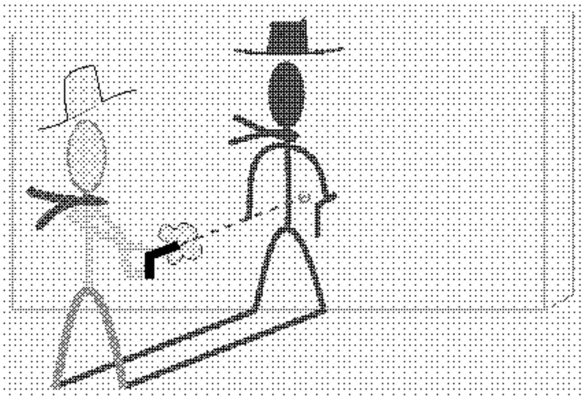

FIGURE 140 Lucky Luke

it is impossible to make electrons move more rapidly than light. (Can you imagine a way to measure energy and speed separately?) Challenge 565 e The speed–energy relation of Figure 141 is a consequence of the maximum speed, and is deduced below. Page 515 These and many similar observations thus show that there is a *limit* to the velocity of objects. Bodies (and radiation) cannot move at velocities higher than the speed of light.* The accuracy of Galilean mechanics was taken for granted for more than three centuries, so that nobody ever thought of checking it; but when this was finally done, as in Figure 141, it was found to be wrong.

The people most unhappy with this limit are computer engineers: if the speed limit were higher, it would be possible to make faster microprocessors and thus faster computers; this would allow, for example, more rapid progress towards the construction of computers that understand and use language.

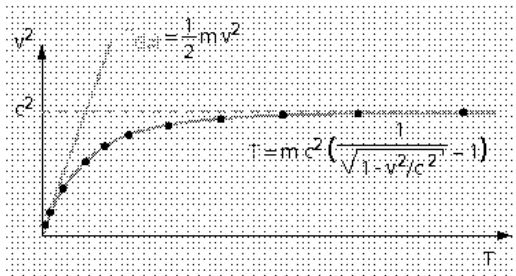

FIGURE 141 Experimental values (dots) for the electron velocity v as function of their kinetic energy T, compared with the prediction of Galilean physics (blue) and that of special relativity (red)

The existence of a limit speed runs counter to Galilean mechanics. In fact, it means that for velocities near that of light, say about 15 000 km/s or more, the expression $mv^2/2$ is *not* equal to the kinetic energy T of the particle. In fact, such high speeds are rather common: many families have an example in their home. Just calculate the speed of electrons inside a television, given that the transformer inside produces 30 kV. Challenge 566 n

* There are still people who refuse to accept these results, as well as the ensuing theory of relativity. Every physicist should enjoy the experience, at least once in his life, of conversing with one of these men. (Strangely, no woman has yet been reported as belonging to this group of people.) This can be done, for example, via the internet, in the sci.physics.relativity newsgroup. See also the http://www.crank.net website. Ref. 257 Crackpots are a fascinating lot, especially since they teach the importance of *precision* in language and in reasoning, which they all, without exception, neglect. Encounters with several of them provided the inspiration for this chapter.

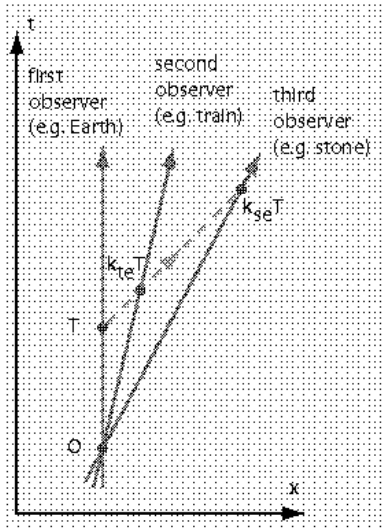

FIGURE 142 How to deduce the composition of velocities

The observation of speed of light as a *limit* speed for objects is easily seen to be a consequence of its *constancy*. Bodies that can be at rest in one frame of reference obviously move more slowly than the maximum velocity (light) in that frame. Now, if something moves more slowly than something else for *one* observer, it does so for all other observers as well. (Trying to imagine a world in which this would not be so is interesting: funny things would happen, such as things interpenetrating each other.) Since the speed of light is the same for all observers, no object can move faster than light, for every observer.

Challenge 567 d

We follow that the maximum speed is the speed of *massless* entities. Electromagnetic waves, including light, are the only known entities that can travel at the maximum speed. Gravitational waves are also predicted to achieve maximum speed. Though the speed of neutrinos cannot be distinguished experimentally from the maximum speed, recent experiments suggest that they do have a tiny mass.

Ref. 253

Conversely, if a phenomenon exists whose speed is the limit speed for one observer, then this limit speed must necessarily be the same for all observers. Is the connection between limit property and observer invariance generally valid in nature?

Challenge 568 e
Challenge 569 r

The composition of velocities

If the speed of light is a limit, no attempt to exceed it can succeed. This implies that when velocities are composed, as when one throws a stone while running, the values cannot simply be added. If a train is travelling at velocity v_{te} relative to the Earth, and somebody throws a stone inside it with velocity v_{st} relative to the train in the same direction, it is usually assumed as evident that the velocity of the stone relative to the Earth is given by $v_{se} = v_{st} + v_{te}$. In fact, both reasoning and measurement show a different result.

The existence of a maximum speed, together with Figure 142, implies that the k-factors must satisfy $k_{se} = k_{st} k_{te}$.* Then we only have to insert the relation (114) between each

* By taking the (natural) logarithm of this equation, one can define a quantity, the *rapidity*, that measures

Challenge 570 e k-factor and the respective speed to get

$$v_{se} = \frac{v_{st} + v_{te}}{1 + v_{st} v_{te}/c^2} \; . \tag{108}$$

Challenge 571 e This is called the *velocity composition formula*. The result is never larger than c and is always smaller than the naive sum of the velocities.* Expression (108) has been confirmed Page 313, page 596 by all of the millions of cases for which it has been checked. You may check that it reduces Ref. 252 to the naive sum for everyday life values.

Observers and the principle of special relativity

Special relativity is built on a simple principle:

▷ *The maximum speed of energy transport is the same for all observers.*

Ref. 250 Or, as Hendrik Lorentz** liked to say, the equivalent:

▷ *The speed v of a physical system is bound by*

$$v \leqslant c \tag{109}$$

for all observers, where c is the speed of light.

This independence of the speed of light from the observer was checked with high preci- Ref. 251 sion by Michelson and Morley*** in the years from 1887 onwards. It has been confirmed in all subsequent experiments; the most precise to date, which achieved a precision of Ref. 252 10^{-14} is shown in Figure 143.

In fact, special relativity is also confirmed by all the precision experiments that were performed *before* it was formulated. You can even confirm it yourself at home. The way Page 565 to do this is shown in the section on electrodynamics.

The existence of a limit speed has several interesting consequences. To explore them, let us keep the rest of Galilean physics intact.**** The limit speed is the speed of light. It is constant for all observers. This constancy implies:

---

the speed and is additive.

Ref. 259 * One can also deduce the Lorentz transformation directly from this expression.

** Hendrik Antoon Lorentz (b. 1853 Arnhem, d. 1928 Haarlem) was, together with Boltzmann and Kelvin, one of the most important physicists of his time. He deduced the so-called Lorentz transformation and the Lorentz contraction from Maxwell's equation for the electrodynamic field. He was the first to understand, long before quantum theory confirmed the idea, that Maxwell's equations for the vacuum also describe matter and all its properties, as long as moving charged point particles – the electrons – are included. He showed this in particular for the dispersion of light, for the Zeeman effect, for the Hall effect and for the Faraday effect. He gave the correct description of the Lorentz force. In 1902, he received the physics Nobel Prize, together with Pieter Zeeman. Outside physics, he was active in the internationalization of scientific collaborations. He was also instrumental in the creation of the largest human-made structures on Earth: the polders of the Zuyder Zee.

*** Albert Abraham Michelson (b. 1852 Strelno, d. 1931 Pasadena), Prussian–Polish–US-American physicist, awarded the Nobel Prize in physics in 1907. Michelson called the set-up he devised an *interferometer*, a term still in use today. Edward William Morley (1838–1923), US-American chemist, was Michelson's friend and long-time collaborator.

Page 36 **** This point is essential. For example, Galilean physics states that only *relative* motion is physical. Galilean physics also excludes various mathematically possible ways to realize a constant light speed that would con-

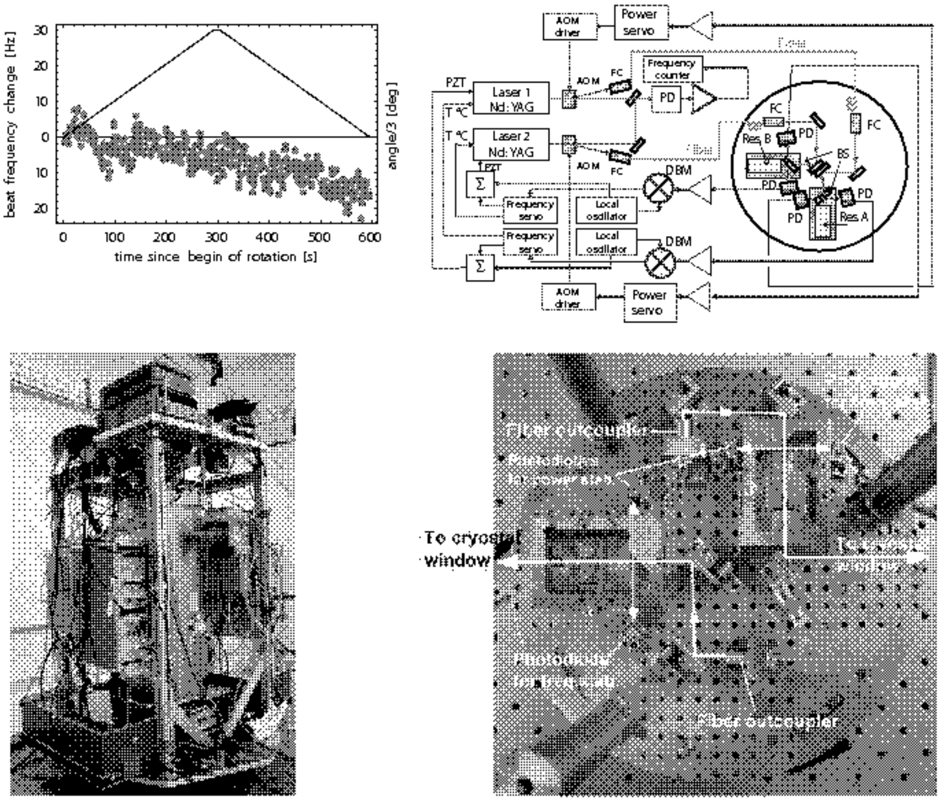

FIGURE 143 The result, the schematics and the cryostat set-up for the most precise Michelson–Morley experiment performed to date (© Stephan Schiller)

— In a closed free-floating room, there is no way to tell the speed of the room.
— There is no notion of absolute rest (or space): rest (like space) is an observer-dependent concept.*
— Time depends on the observer; time is not absolute.

More interesting and specific conclusions can be drawn when two additional conditions are assumed. First, we study situations where gravitation can be neglected. (If this not the case, we need *general* relativity to describe the system.) Secondly, we also assume that the data about the bodies under study – their speed, their position, etc. – can be gathered without disturbing them. (If this not the case, we need *quantum theory* to describe the system.)

To deduce the *precise* way in which the different time intervals and lengths measured by two observers are related to each other, we take an additional simplifying step. We start

---

tradict everyday life.

Einstein's original 1905 paper starts from two principles: the constancy of the speed of light and the equivalence of all inertial observers. The latter principle had already been stated in 1632 by Galileo; only the constancy of the speed of light was new. Despite this fact, the new theory was named – by Poincaré – after

Ref. 243 the old principle, instead of calling in 'invariance theory', as Einstein would have preferred.

Challenge 572 n * Can you give the argument leading to this deduction?

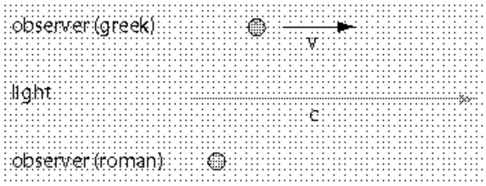

FIGURE 144 Two inertial observers and a beam of light

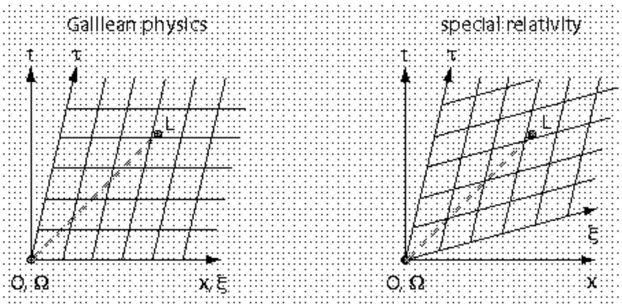

FIGURE 145 Space-time diagrams for light seen from two different observers using coordinates (t, x) and (τ, ξ)

with a situation where no interaction plays a role. In other words, we start with *relativistic kinematics* of bodies moving without disturbance.

If an undisturbed body is observed to travel along a straight line with a constant velocity (or to stay at rest), one calls the observer *inertial*, and the coordinates used by the observer an *inertial frame of reference*. Every inertial observer is itself in undisturbed motion. Examples of inertial observers (or frames) thus include – in *two* dimensions – those moving on a frictionless ice surface or on the floor inside a smoothly running train or ship; for a full example – in all *three* spatial dimensions – we can take a cosmonaut travelling in a space-ship as long as the engine is switched off. Inertial observers in three dimensions might also be called *free-floating* observers. They are thus not so common. Challenge 573 e Non-inertial observers are much more common. Can you confirm this? Inertial observers are the most simple ones, and they form a special set:

— Any two inertial observers move with constant velocity relative to each other (as longs as gravity plays no role, as assumed above).
— All inertial observers are equivalent: they describe the world with the same equations. Because it implies the lack of absolute space and time, this statement was called the *principle of relativity* by Henri Poincaré. However, the *essence* of relativity is the existence of a limit speed.

To see how measured length and space intervals change from one observer to the other, we assume two inertial observers, a Roman one using coordinates x, y, z and t, and a Greek one using coordinates ξ, υ, ζ and τ,* that move with velocity v relative to each other. The axes are chosen in such a way that the velocity points in the x-direction. The

* They are read as 'xi', 'upsilon', 'zeta' and 'tau'. The names, correspondences and pronunciations of all Greek letters are explained in Appendix A.

constancy of the speed of light in any direction for any two observers means that for the motion of light the coordinate differentials are related by

$$0 = (c\mathrm{d}t)^2 - (\mathrm{d}x)^2 - (\mathrm{d}y)^2 - (\mathrm{d}z)^2 = (c\mathrm{d}\tau)^2 - (\mathrm{d}\xi)^2 - (\mathrm{d}\upsilon)^2 - (\mathrm{d}\zeta)^2 \,. \qquad (110)$$

Assume also that a flash lamp at rest for the Greek observer, thus with $\mathrm{d}\xi = 0$, produces two flashes separated by a time interval $\mathrm{d}\tau$. For the Roman observer, the flash lamp moves with speed v, so that $\mathrm{d}x = v\mathrm{d}t$. Inserting this into the previous expression, and assuming linearity and speed direction independence for the general case, we find that intervals are related by

Challenge 574 e

$$\begin{aligned}
\mathrm{d}t &= \gamma(\mathrm{d}\tau + v\mathrm{d}\xi/c^2) = \frac{\mathrm{d}\tau + v\mathrm{d}\xi/c^2}{\sqrt{1-v^2/c^2}} \qquad \text{with} \quad v = \mathrm{d}x/\mathrm{d}t \\
\mathrm{d}x &= \gamma(\mathrm{d}\xi + v\mathrm{d}\tau) = \frac{\mathrm{d}\xi + v\mathrm{d}\tau}{\sqrt{1-v^2/c^2}} \\
\mathrm{d}y &= \mathrm{d}\upsilon \\
\mathrm{d}z &= \mathrm{d}\zeta \;. \qquad (111)
\end{aligned}$$

These expressions describe how length and time intervals measured by different observers are related. At relative speeds v that are small compared to the velocity of light, such as occur in everyday life, the time intervals are essentially equal; the *stretch factor* or *relativistic correction* or *relativistic contraction* γ is then equal to 1 for all practical purposes. However, for velocities *near* that of light the measurements of the two observers give different values. In these cases, space and time *mix*, as shown in Figure 145.

The expressions (111) are also strange in another respect. When two observers look at each other, each of them claims to measure shorter intervals than the other. In other words, special relativity shows that the grass on the other side of the fence is always *shorter* – if one rides along beside the fence on a bicycle and if the grass is inclined. We explore this bizarre result in more detail shortly.

Challenge 575 n

The stretch factor γ is equal to 1 for most practical purposes in everyday life. The largest value humans have ever achieved is about $2 \cdot 10^5$; the largest observed value in nature is about 10^{12}. Can you imagine where they occur?

Challenge 576 n

Once we know how space and time *intervals* change, we can easily deduce how *coordinates* change. Figures 144 and 145 show that the x coordinate of an event L is the sum of two intervals: the ξ coordinate and the length of the distance between the two origins. In other words, we have

$$\xi = \gamma(x - vt) \quad \text{and} \quad v = \frac{\mathrm{d}x}{\mathrm{d}t} \,. \qquad (112)$$

Using the invariance of the space-time interval, we get

$$\tau = \gamma(t - xv/c^2) \,. \qquad (113)$$

Henri Poincaré called these two relations the *Lorentz transformations of space and time*

after their discoverer, the Dutch physicist Hendrik Antoon Lorentz.* In one of the most
Ref. 263 beautiful discoveries of physics, in 1892 and 1904, Lorentz deduced these relations from
Page 546 the equations of electrodynamics, where they had been lying, waiting to be discovered, since 1865.** In that year James Clerk Maxwell had published the equations in order to describe everything electric and magnetic. However, it was Einstein who first understood that t and τ, as well as x and ξ, are equally correct and thus equally valid descriptions of space and time.

The Lorentz transformation describes the change of viewpoint from one inertial frame to a second, moving one. This change of viewpoint is called a (Lorentz) *boost*. The formulae (112) and (113) for the boost are central to the theories of relativity, both special and general. In fact, the mathematics of special relativity will not get more difficult than that: if you know what a square root is, you can study special relativity in all its beauty.

Many alternative formulae for boosts have been explored, such as expressions in which
Ref. 264 the relative acceleration of the two observers is included, as well as the relative velocity. However, they had all to be discarded after comparing their predictions with experimental results. Before we have a look at such experiments, we continue with a few logical deductions from the boost relations.

What is space-time?

> “Von Stund’ an sollen Raum für sich und Zeit für sich völlig zu Schatten herabsinken und nur noch eine Art Union der beiden soll Selbstständigkeit bewaren.***
>
> Hermann Minkowski.”

The Lorentz transformations tell us something important: that space and time are two aspects of the same basic entity. They ‘mix’ in different ways for different observers. This fact is commonly expressed by stating that time is the *fourth dimension*. This makes sense because the common basic entity – called *space-time* – can be defined as the set of all events, events being described by four coordinates in time and space, and because the set
Challenge 577 n of all events has the properties of a manifold.**** (Can you confirm this?)

In other words, the existence of a maximum speed in nature forces us to introduce a space-time manifold for the description of nature. In the theory of special relativity, the space-time manifold is characterized by a simple property: the *space-time interval* $\mathrm{d}i$
Ref. 265 between two nearby events, defined as

$$\mathrm{d}i^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 = c^2\mathrm{d}t^2\left(1 - \frac{v^2}{c^2}\right) , \qquad (114)$$

* For information about Hendrik Antoon Lorentz, see page 290.

** The same discovery had been published first in 1887 by the German physicist Woldemar Voigt (1850–1919); Voigt – pronounced ‘Fohgt’ – was also the discoverer of the Voigt effect and the Voigt tensor. Independently, in 1889, the Irishman George F. Fitzgerald also found the result.

*** ‘Henceforth space by itself and time by itself shall completely fade into shadows and only a kind of union of the two shall preserve autonomy.’ This famous statement was the starting sentence of Minkowski’s 1908 talk at the meeting of the Gesellschaft für Naturforscher und Ärzte.

Page 1014 **** The term ‘manifold’ is defined in Appendix D.

is independent of the (inertial) observer. Such a space-time is also called Minkowski space-time, after Hermann Minkowski* the teacher of Albert Einstein; he was the first, in 1904, to define the concept of space-time and to understand its usefulness and importance.

The space-time interval di of equation (114) has a simple interpretation. It is the time measured by an observer moving from event (t, x) to event $(t + \mathrm{d}t, x + \mathrm{d}x)$, the so-called *proper time*, multiplied by c. If we neglect the factor c, we could simply call it wristwatch time.

We *live* in a Minkowski space-time, so to speak. Minkowski space-time exists independently of things. And even though coordinate systems can be different from observer to observer, the underlying entity, space-time, is still *unique*, even though space and time by themselves are not.

How does Minkowski space-time differ from Galilean space-time, the combination of everyday space and time? Both space-times are manifolds, i.e. continuum sets of points, both have one temporal and three spatial dimensions, and both manifolds have the topology of the punctured sphere. (Can you confirm this?) Both manifolds are flat, i.e. free of curvature. In both cases, space is what is measured with a metre rule or with a light ray, and time is what is read from a clock. In both cases, space-time is fundamental; it is and remains the *background* and the *container* of things and events.

Challenge 578 n

The central difference, in fact the only one, is that Minkowski space-time, in contrast to the Galilean case, *mixes* space and time, and in particular, does so differently for observers with different speeds, as shown in Figure 115. That is why time is an observer-dependent concept.

The maximum speed in nature thus forces us to describe motion with space-time. That is interesting, because in space-time, speaking in simple terms, *motion does not exist*. Motion exists only in space. In space-time, nothing moves. For each point particle, space-time contains a world-line. In other words, instead of asking why motion exists, we can equivalently ask why space-time is criss-crossed by world-lines. At this point, we are still far from answering either question. What we can do is to explore *how* motion takes place.

Can we travel to the past? – Time and causality

We know that time is different for different observers. Does time nevertheless order events in sequences? The answer given by relativity is a clear 'yes and no'. Certain sets of events are not naturally ordered by time; others sets are. This is best seen in a space-time diagram.

Clearly, two events can be placed in sequence only if one event is the *cause* of the other. But this connection can only apply if the events exchange energy (e.g. through a signal). In other words, a relation of cause and effect between two events implies that energy or signals can travel from one event to the other; therefore, the speed connecting the two events must not be larger than the speed of light. Figure 116 shows that event E at the origin of the coordinate system can only be influenced by events in quadrant IV (the *past light cone*, when all space dimensions are included), and can itself influence only events

* Hermann Minkowski (1864–1909), German mathematician. He had developed similar ideas to Einstein, but the latter was faster. Minkowski then developed the concept of space-time. Minkowski died suddenly at the age of 44.

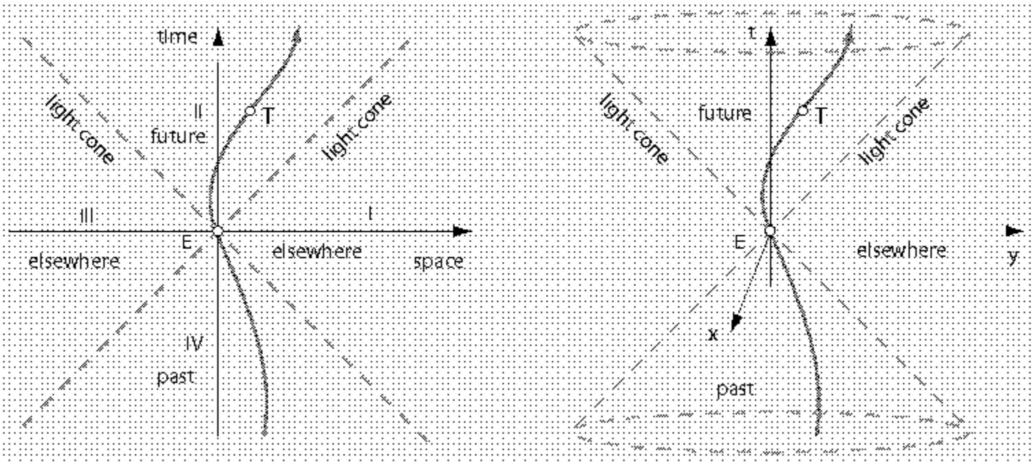

FIGURE 146 A space-time diagram for a moving object T seen from an inertial observer O in the case of one and two spatial dimensions

in quadrant II (the *future light cone*). Events in quadrants I and III neither influence nor are influenced by event E. The light cone defines the boundary between events that *can* be ordered with respect to their origin – namely those inside the cone – and those that *cannot* – those outside the cones, happening elsewhere for all observers. (Some people call all the events happening elsewhere the *present*.) So, time orders events only *partially*. For example, for two events that are not causally connected, their temporal order (or their simultaneity) depends on the observer!

In particular, the past light cone gives the complete set of events that can influence what happens at the origin. One says that the origin is *causally connected* only to the past light cone. This statement reflects the fact that any influence involves transport of energy, and thus cannot travel faster than the speed of light. Note that causal connection is an invariant concept: all observers agree on whether or not it applies to two given events.
Challenge 579 n Can you confirm this?

A vector inside the light cone is called *timelike*; one on the light cone is called *lightlike* or *null*; and one outside the cone is called *spacelike*. For example, the *world-line* of an observer, i.e. the set of all events that make up its past and future history, consists of timelike events only. Time is the fourth dimension; it expands space to space-time and thus 'completes' space-time. This is the relevance of the fourth dimension to special relativity, no more and no less.

Special relativity thus teaches us that causality and time can be defined *only* because light cones exist. If transport of energy at speeds faster than that of light did exist, time could not be defined. Causality, i.e. the possibility of (partially) ordering events for all observers, is due to the existence of a maximal speed.

If the speed of light could be surpassed in some way, the future could influence the past.
Challenge 580 n Can you confirm this? In such situations, one would observe *acausal* effects. However, there is an everyday phenomenon which tells that the speed of light is indeed maximal: our memory. If the future could influence the past, we would also be able to *remember* the future. To put it in another way, if the future could influence the past, the second principle

of thermodynamics would not be valid and our memory would not work.* No other data from everyday life or from experiments provide any evidence that the future can influence the past. In other words, *time travel to the past is impossible.* How the situation changes in quantum theory will be revealed later on. Interestingly, time travel to the future *is* possible, as we will see shortly.

Curiosities of special relativity

Faster than light: how far can we travel?

How far away from Earth can we travel, given that the trip should not last more than a lifetime, say 80 years, and given that we are allowed to use a rocket whose speed can approach the speed of light as closely as desired? Given the time t we are prepared to spend in a rocket, given the speed v of the rocket and assuming optimistically that it can accelerate and decelerate in a negligible amount of time, the distance d we can move away
Challenge 581 e is given by

$$d = \frac{vt}{\sqrt{1-v^2/c^2}} . \tag{115}$$

The distance d is larger than ct already for $v > 0.71c$, and, if v is chosen large enough, it increases beyond all bounds! In other words, relativity does *not* limit the distance we can travel in a lifetime, and not even the distance we can travel in a single second. We could, in principle, roam the entire universe in less than a second. In situations such as these it makes sense to introduce the concept of *proper velocity* w, defined as

$$w = d/t = \frac{v}{\sqrt{1-v^2/c^2}} = \gamma\, v . \tag{116}$$

As we have just seen, proper velocity is *not* limited by the speed of light; in fact the proper velocity of light itself is infinite.**

Synchronization and time travel – can a mother stay younger than her own daughter?

A maximum speed implies that time is is different for different observers moving relative to each other. So we have to be careful about how we synchronize clocks that are far apart, even if they are at rest with respect to each other in an inertial reference frame. For example, if we have two similar watches showing the same time, and if we carry one of them for a walk and back, they will show different times afterwards. This experiment has

* Another related result is slowly becoming common knowledge. Even if space-time had a nontrivial shape, such as a cylindrical topology with closed time-like curves, one still would not be able to travel into the
Ref. 266 past, in contrast to what many science fiction novels suggest. This is made clear by Stephen Blau in a recent pedagogical paper.

** Using proper velocity, the relation given in equation (109) for the superposition of two velocities $\mathbf{w}_a = \gamma_a \mathbf{v}_a$
Challenge 582 e and $\mathbf{w}_b = \gamma_b \mathbf{v}_b$ simplifies to

$$w_{s\parallel} = \gamma_a \gamma_b (v_a + v_{b\parallel}) \quad \text{and} \quad w_{s\perp} = w_{b\perp} , \tag{117}$$

where the signs $\parallel$ and $\perp$ designate the component in the direction of and the component perpendicular to
Ref. 267 $\mathbf{v}_a$, respectively. One can in fact express all of special relativity in terms of 'proper' quantities.

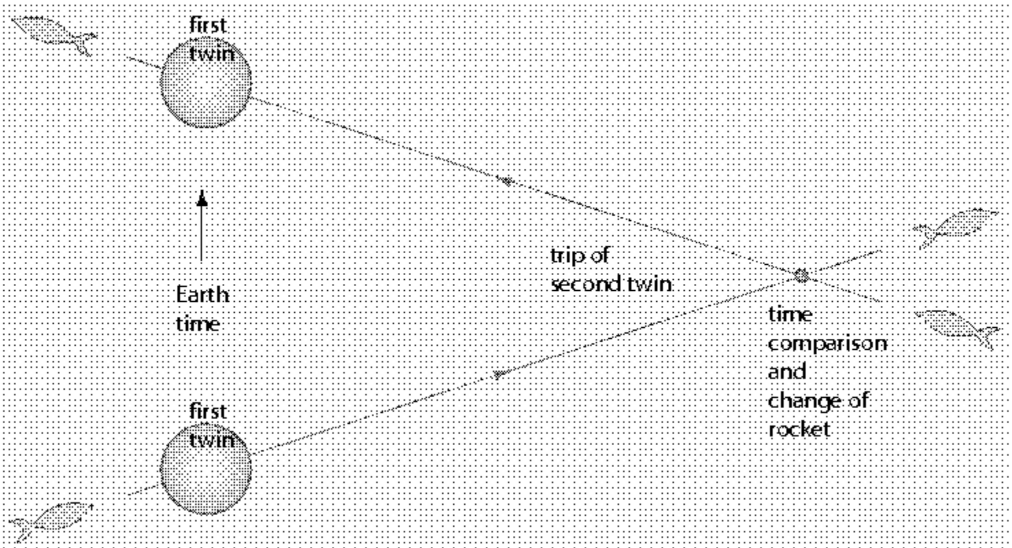

FIGURE 147 The twin paradox

Ref. 268, Ref. 269 actually been performed several times and has fully confirmed the prediction of special relativity. The time difference for a person or a watch in an aeroplane travelling around the Earth once, at about 900 km/h, is of the order of 100 ns – not very noticeable in everyday life. In fact, the delay is easily calculated from the expression

$$\frac{t}{t'} = \gamma \, . \tag{118}$$

Human bodies are clocks; they show the elapsed time, usually called *age*, by various changes in their shape, weight, hair colour, etc. If a person goes on a long and fast trip, on her return she will have aged *less* than a second person who stayed at her (inertial) home.

The most famous illustration of this is the famous *twin paradox* (or *clock paradox*). An adventurous twin jumps on a relativistic rocket that leaves Earth and travels for many years. Far from Earth, he jumps on another relativistic rocket going the other way and returns to Earth. The trip is illustrated in Figure 147. At his arrival, he notes that his twin brother on Earth is much older than himself. Can you explain this result, especially the asymmetry between the two brothers? This result has also been confirmed in many experiments. Ref. 270

Special relativity thus confirms, in a surprising fashion, the well-known observation that those who travel a lot remain younger. The price of the retained youth is, however, that everything around one changes very much more quickly than if one is at rest with the environment.

The twin paradox can also be seen as a confirmation of the possibility of time travel to the future. With the help of a fast rocket that comes back to its starting point, we can arrive at local times that we would never have reached within our lifetime by staying home. Alas, we can *never* return to the past.*

Ref. 271 * There are even special books on time travel, such as the well researched text by Nahin. Note that the concept

One of the simplest experiments confirming the prolonged youth of fast travellers involves the counting of muons. Muons are particles that are continuously formed in the upper atmosphere by cosmic radiation. Muons *at rest* (with respect to the measuring clock) have a finite half-life of 2.2 μs (or, at the speed of light, 660 m). After this amount of time, half of the muons have decayed. This half-life can be measured using simple muon counters. In addition, there exist special counters that only count muons travelling within a certain speed range, say from $0.9950c$ to $0.9954c$. One can put one of these special counters on top of a mountain and put another in the valley below, as shown in Figure 148. The first time this experiment was performed, the height difference was 1.9 km. Flying 1.9 km through the atmosphere at the mentioned speed takes about 6.4 μs. With the half-life just given, a naive calculation finds that only about 13% of the muons observed at the top should arrive at the lower site. However, it is observed that about 82% of the muons arrive below. The reason for this result is the relativistic time dilation. Indeed, at the mentioned speed, muons experience a time difference of only 0.62 μs during the travel from the mountain top to the valley. This shorter time yields a much lower number of lost muons than would be the case without time dilation; moreover, the measured percentage confirms the value of the predicted time dilation factor γ within experimental errors, as you may want to check. A similar effect is seen when relativistic muons are produced in accelerators.

Page 337
Ref. 272
Challenge 583 n
Challenge 584 n

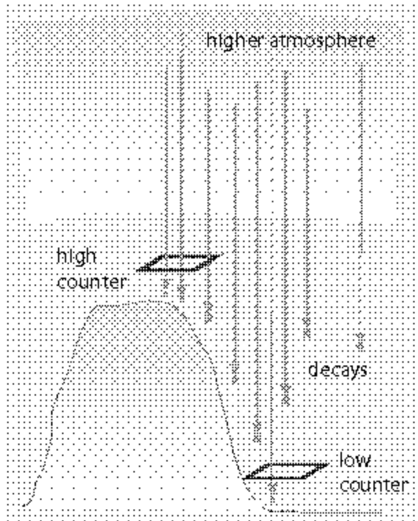

FIGURE 148 More muons than expected arrive at the ground because fast travel keeps them young

Half-life dilation has also been found for many other decaying systems, such as pions, hydrogen atoms, neon atoms and various nuclei, always confirming the predictions of special relativity. Since all bodies in nature are made of particles, the 'youth effect' of high speeds (usually called 'time dilation') applies to bodies of all sizes; indeed, it has not only been observed for particles, but also for lasers, radio transmitters and clocks.

Ref. 252

If motion leads to time dilation, a clock on the Equator, constantly running around the Earth, should go slower than one at the poles. However, this prediction, which was made by Einstein himself, is incorrect. The centrifugal acceleration leads to a reduction in gravitational acceleration that exactly cancels the increase due to the velocity. This story serves as a reminder to be careful when applying special relativity in situations involving gravity. Special relativity is only applicable when space-time is flat, not when gravity is present.

Ref. 273

In short, a mother *can* stay younger than her daughter. We can also conclude that we cannot synchronize clocks at rest with respect to each other simply by walking, clock in hand, from one place to another. The correct way to do so is to exchange light signals. Can you describe how?

Challenge 585 n

of time travel has to be clearly defined; otherwise one has no answer to the clerk who calls his office chair a time machine, as sitting on it allows him to get to the future.