This summary covers the key concepts and examples from Section 11.3 of the textbook.

The Integral Test provides a method to test an infinite series for convergence by comparing it with an improper integral.

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1,\infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x)dx$ is convergent. In other words: (i) If $\int_1^{\infty} f(x)dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent. (ii) If $\int_1^{\infty} f(x)dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: It is not necessary that f be always decreasing. It is sufficient if f is ultimately decreasing, i.e., decreasing for x larger than some number N.

Test the series $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ for convergence or divergence.

SOLUTION: The function $f(x) = \frac{1}{x^2+1}$ is continuous, positive, and decreasing on $[1,\infty)$. We evaluate the integral:

$$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}dx = \lim_{t\to\infty} [\tan^{-1} x]_1^t$$
$$= \lim_{t\to\infty} (\tan^{-1} t - \frac{\pi}{4}) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

Since the integral is convergent, the series is convergent by the Integral Test.

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called the **p-series**.

1. Convergence of a p-series The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \le 1$.

(a) The series $\sum_{n=1}^{\infty} \frac{1}{n^3}$ is convergent because it is a p-series with $p = 3 > 1$. (b) The series $\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is divergent because it is a p-series with $p = 1/3 < 1$.

Determine whether the series $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converges or diverges.

SOLUTION: The function $f(x) = \frac{\ln x}{x}$ is positive and continuous for $x > 1$. To check if it's decreasing, we find the derivative:

$$f'(x) = \frac{(1/x)x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

$f'(x) < 0$ when $\ln x > 1$, or $x > e$. Thus, f is decreasing for $x > e$. We can apply the Integral Test:

$$\int_1^{\infty} \frac{\ln x}{x}dx = \lim_{t\to\infty} \left[\frac{(\ln x)^2}{2}\right]_1^t = \lim_{t\to\infty} \frac{(\ln t)^2}{2} = \infty$$

The integral is divergent, so the series is divergent.

—

For a convergent series $\sum a_n$, we can approximate its sum s with a partial sum s_n. The error in this approximation is the remainder, $R_n = s - s_n$.

2. Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \ge n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x)dx \le R_n \le \int_n^{\infty} f(x)dx$$

(a) Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error. (b) How many terms are required to ensure the sum is accurate to within 0.0005?

SOLUTION: First, we compute the integral: $\int_n^\infty \frac{1}{x^3} dx = \frac{1}{2n^2}$.

(a) The sum of the first 10 terms is $s_{10} \approx 1.1975$. The remainder estimate gives the error:

$$R_{10} \leq \int_{10}^\infty \frac{1}{x^3} dx = \frac{1}{2(10)^2} = \frac{1}{200} = 0.005$$

The error is at most 0.005.

(b) We need $R_n < 0.0005$. Since $R_n \leq \frac{1}{2n^2}$, we set:

$$\frac{1}{2n^2} < 0.0005 \implies n^2 > \frac{1}{0.001} = 1000 \implies n > \sqrt{1000} \approx 31.6$$

We need 32 terms to ensure accuracy to within 0.0005.

We can also get a better estimate for the sum s using the following inequality: **3.**

$$s_n + \int_{n+1}^\infty f(x) dx \leq s \leq s_n + \int_n^\infty f(x) dx$$

Use the inequality above with $n = 10$ to estimate the sum of the series $\sum \frac{1}{n^3}$.

SOLUTION: Using $s_{10} \approx 1.197532$, we have:

$$s_{10} + \int_{11}^\infty \frac{1}{x^3} dx \leq s \leq s_{10} + \int_{10}^\infty \frac{1}{x^3} dx$$

$$1.197532 + \frac{1}{2(11)^2} \leq s \leq 1.197532 + \frac{1}{2(10)^2}$$

$$1.197532 + 0.004132 \leq s \leq 1.197532 + 0.005$$

$$1.201664 \leq s \leq 1.202532$$

If we approximate s by the midpoint of this interval, $s \approx 1.2021$, the error is less than 0.0005.