mid term exam.

답안

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최용석

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\begin{aligned}
\vec{L} &= \sum_i (\vec{r}_{cm} + \vec{r_i}') \times m_i(\vec{v}_{cm} + \vec{v_i}') \\
&= \sum_i (\vec{r}_{cm} \times m_i \vec{v}_{cm}) + \sum_i (\vec{r}_{cm} \times m_i \vec{v_i}') + \sum_i (\vec{r_i}' \times m_i \vec{v}_{cm}) + \sum_i (\vec{r_i}' \times m_i \vec{v_i}') \\
&= \vec{r}_{cm} \times \left(\sum_i m_i \right) \vec{v}_{cm} + \vec{r}_{cm} \times \sum_i m_i \vec{v_i}' + \left(\sum_i m_i \vec{r_i}' \right) \times \vec{v}_{cm} + \sum_i (\vec{r_i}' \times m_i \vec{v_i}')
\end{aligned}
$$

$$\sum_i m_i \vec{r_i}' = \sum_i m_i (\vec{r_i} - \vec{r}_{cm}) = \sum_i m_i \vec{r_i} - m\vec{r}_{cm} = 0$$

$$\sum_i m_i \vec{v_i}' = \sum_i m_i \vec{v_i} - m\vec{v}_{cm} = 0$$

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \sum_i \vec{r_i}' \times m_i \vec{v_i}'$$

12.

$$I_1 = \frac{1}{12} M a^2$$

$$I_2 = \frac{1}{12} M b^2$$

$$I_3 = \frac{1}{12} M (a^2 + b^2)$$

13. .

$$\vec{L} = \sum_i m_i(\vec{r_i} \times \vec{v_i}) = \sum_i \{\vec{r_i} \times (\vec{\omega} \times \vec{r_i})\}$$

$$\vec{\omega} \times \vec{r_1} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{j} \\ 3 & -2 & 4 \\ 1 & -1 & 1 \end{pmatrix} = 2\hat{i} + \hat{j} - \hat{k}$$

$$\vec{r_1} \times (\vec{\omega} \times \vec{r_1}) = \begin{pmatrix} \hat{i} & \hat{j} & \hat{j} \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} = 3\hat{j} + 3\hat{k}$$

$$\vec{\omega} \times \vec{r_2} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{j} \\ 3 & -2 & 4 \\ 2 & 0 & 2 \end{pmatrix} = -4\hat{i} + 2\hat{j} + 4\hat{k}$$

$$\vec{r_2} \times (\vec{\omega} \times \vec{r_2}) = \begin{pmatrix} \hat{i} & \hat{j} & \hat{j} \\ 2 & 0 & 2 \\ -4 & 2 & 4 \end{pmatrix} = -4\hat{i} - 16\hat{j} + 4\hat{k}$$

$$\vec{\omega} \times \vec{r_3} = \begin{pmatrix} \hat{i} & \hat{j} & \hat{j} \\ 3 & -2 & 4 \\ -1 & 1 & 0 \end{pmatrix} = -4\hat{i} - 4\hat{j} + \hat{k}$$

$$\vec{r_3} \times (\vec{\omega} \times \vec{r_3}) = \begin{pmatrix} \hat{i} & \hat{j} & \hat{j} \\ -1 & 1 & 0 \\ -4 & -4 & 1 \end{pmatrix} = \hat{i} + \hat{j} + 8\hat{k}$$

$$\vec{L} = (-4+4)\hat{i} + (6-16+4)\hat{j} + (6+4+32)\hat{k} = -6\hat{j} + 42\hat{k}$$

14.

$$\begin{aligned} I_{xx} &= \sum_i m_i(y_i^2 + z_i^2) \\ &= 2(1+1) + 1(0+4) + 4(1+0) \\ &= 12 \end{aligned}$$

$$\begin{aligned} I_{yy} &= \sum_i m_i(z_i^2 + x_i^2) \\ &= 2(1+1) + 1(4+4) + 4(0+1) \\ &= 16 \end{aligned}$$

$$\begin{aligned} I_{zz} &= \sum_i m_i(x_i^2 + y_i^2) \\ &= 2(1+1) + 1(4+0) + 4(1+1) \\ &= 16 \end{aligned}$$

$$\begin{aligned} I_{xy} = I_{yx} &= -\sum_i m_i x_i y_i \\ &= -2(-1) - 1(0) - 4(-1) \\ &= 6 \end{aligned}$$

$$\begin{aligned} I_{yz} = I_{zy} &= -\sum_i m_i y_i z_i \\ &= -2(-1) - 1(0) - 4(0) \\ &= 2 \end{aligned}$$

$$\begin{aligned} I_{zx} = I_{xz} &= -\sum_i m_i z_i x_i \\ &= -2(1) - 1(4) - 4(0) \\ &= -6 \end{aligned}$$

$$\tilde{I} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} = \begin{pmatrix} 12 & 6 & -6 \\ 6 & 16 & 2 \\ -6 & 2 & 16 \end{pmatrix}$$

15.

$$\begin{aligned}
I_{xx} &= \int_0^a \int_0^a \sigma(y^2+z^2)dydz \\
&= \int_0^a \sigma\left(\frac{a^3}{3}+z^2a\right)dz \\
&= \sigma\left(\frac{a^4}{3}+\frac{a^4}{3}\right) \\
&= \sigma a^2\left(\frac{2}{3}a^2\right) \\
&= \frac{2}{3}Ma^2 \\
&= I_{yy} \\
&= I_{zz}
\end{aligned}$$

$$\begin{aligned}
I_{xy} &= -\int_0^a \int_0^a \sigma xydxdy \\
&= -\sigma\int_0^a \frac{a^2}{2}ydy \\
&= -\sigma\frac{a^4}{4} \\
&= -\frac{1}{4}Ma^2 \\
&= I_{yz} \\
&= I_{zx}
\end{aligned}$$

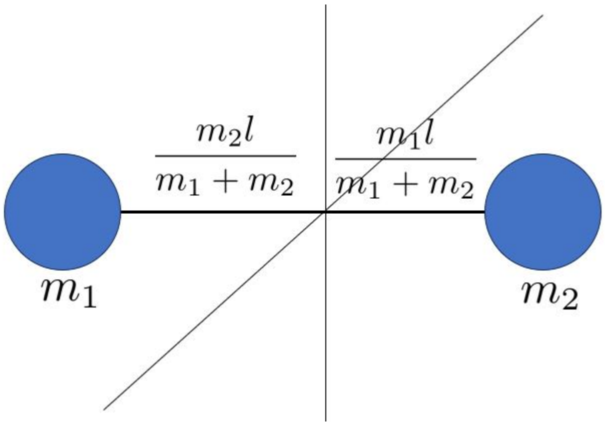
16. (a)

$$
\begin{aligned}
\vec{L} &= \tilde{I}\vec{\omega} = Ma^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \\
&= \frac{1}{12}Ma^2 \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \\
&= \frac{1}{12}Ma^2 \begin{pmatrix} 10 \\ 43 \\ -45 \end{pmatrix} \\
&= \frac{1}{12}Ma^2(10\hat{\imath} + 43\hat{\jmath} - 45\hat{k})
\end{aligned}
$$

(b)

$$
\begin{aligned}
T_{rot} &= \frac{1}{2}\tilde{\vec{\omega}}\tilde{I}\vec{\omega} \\
&= \frac{1}{2}\begin{pmatrix} 2 & 5 & -3 \end{pmatrix} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \frac{1}{12}Ma^2 \\
&= \frac{1}{24}Ma^2 \begin{pmatrix} 10 & 43 & -45 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} \\
&= \frac{1}{24}Ma^2(20 + 215 + 135) \\
&= \frac{185}{12}Ma^2
\end{aligned}
$$

17. 질량중심으로부터 거리는 각각 그림과 같다.



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\begin{aligned}
I_1 &= m_1 \cdot \left(\frac{m_2 l}{m_1 + m_2}\right)^2 + m_2 \cdot \left(\frac{m_1 l}{m_1 + m_2}\right)^2 \\
&= \frac{m_1 m_2 (m_1 + m_2)}{(m_1 + m_2)^2} l^2 \\
&= \frac{m_1 m_2}{m_1 + m_2} l^2 \\
&= I_2 \\
I_3 &= 0
\end{aligned}
$$