Needham - Visual Complex Analysis Notes

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1 Mathematical Background

1.1 Sets

A *set* is a collection of objects. The empty set contains no elements. The universal set is the *universe* of your problem. A subset $A \subseteq B$ means that every element of A is an element of B. A *proper subset* $A \subset B$ means that there is at least one element in B that is not in A.

1.2 Intersection and Union

1.3 Differences

1.4 Complements

1.5 De Morgan's Laws

$$\overline{\bigcup A_i} = \bigcap \overline{A_i}$$

and

$$\overline{\bigcap A_i} = \bigcup \overline{A_i}\,.$$

1.6 Maps

We have

$$\varphi : A \to B$$

where A is the codomain and B is the domain.

1.7 Image and Inverse Image

$$\varphi(X) = \{y \in B \,|\, y = \varphi(x) \text{ for some } x \in X\}$$

and

$$\varphi^{-1}(Y) = \{x \in A \,|\, \varphi(x) \in Y\} \,.(X \subset A, Y \subset B)\,.$$

1.1 Definition. The *image* (or range) of a function; $Im\varphi = \varphi(A)$.

1.8 Injective, Surjective, and Bijective Maps

1.9 Permutation

A *permutation* is a bijection of a finite set.

1.10 Cardinality

$|A| = |B|$ if there exists a bijection $\varphi : A \to B$.

1.11 Finite Sets

A set X is *finite* if $|X||\mathbb{Z}_n| = n$. Then, $|X| = n$.

1.12 Countably Infinite Sets

X is *countable infinite* if $|X| = |\mathbb{Z}_+|$. X is *countable* if it is either finite or countably infinite.

X is *uncountable* if it is not countable.

1.13 Composition

$\varphi : A \to B, \Psi : B \to C$. Then $\Psi \circ \varphi : A \to C$, $(\Psi \circ \varphi)(x) = \Psi(\varphi(x))$.

1.14 Inverse Maps

Only bijective maps have inverses.

1.15 Binary Relations

A *binary relation* is a map

$$* : A \times A \to A$$

It is called *associative* if order of parentheses don't matter. It is called *commutative* if order of elements doesn't matter.

1.16 Equivalence Relations

An *equivalence relation* is a relation that is reflexive ($a \sim a$), symmetric ($a \sim b \implies b \sim A$), and transitive (if $a \sim b$ and $b \sim c$, then $a \sim c$).

1.17 Equivalence Classes

Let A be a set. Then $[a] = \{x \in A \mid x \sim a\}$. $[a]$ is a representative of a class. A *partition* of a set X such that each subset is nonempty, disjoint, and all the subsets cover the entire set. It is then a collection of *nonempty*, *disjoint* subsets of X that covers X.

The set of equivalence classes forms a partition.

1.18 Ordered Set

An ordered set is a set with an *order*.

Notation: $\{a, b, c\}$ - unordered; (a, b, c) - ordered

1.19 Cartesian Product

1.20 Groups and Subgroups

1.21 Cyclic Groups

1.22 Permutation Groups

2 Linear Algebra in One Lecture

2.1 Fields

An *associative ring* is a set R woth two binary operations, addition and multiplication that are both *associative* and satisfy 1) distributivity , 2) has an additive identity 3) every element has an additive inverse.

If R has multiplicative identity called 1, R is called an *associative ring with unit*. If every nonzero element has multiplicative inverse, then R is a *division ring*. If multiplication in a division ring is commutative, then R is a called a *field*.

2.2 Linear Algebra

(see my notes for 454 and for Commutative Algebra as needed)

2.1 Definition. A *vector space over a field* $\mathbb{F}$ is a set V with an associative binary operation

$$+ : V \times V \to V$$

called *addition of vectors* and an operation

$$\cdot : \mathbb{F} \times V \to V$$

called *scalar multiplication* ($(u, v) \mapsto u+v$, and$(a, v) \mapsto a \cdot v$) such that every element has

1. additive inverse
2. addition and multiplication are associative
3. multiplicative identity
4. distributivity of multiplication over addition.

2.2 Definition. Let $B \subset V$ be a subset of V. Then

$$\operatorname{span} B = \{\sum_{i=1}^{m} a_i v_i \,|\, a_i \in \mathbb{F}, v_i \in B\}$$

is the set of all linear combinations of vectors in B.

2.3 Proposition. $\operatorname{span} B$ is a vector subspace.

Proof. □

2.4 Definition. A subset $B \subset V$ is called a *spanning set* of V if it *spans* V, i.e. $\operatorname{span} B = V$.

Other topics to fill out as needed from linear algebra:

1. linear independence
2. basis
3. dimension of a finite dimensional vector space
4. maps (homomorphisms)
5. kernels
6. kernel is a vector subspace
7. nullity
8. images
9. rank
10. quotient spaces

2.5 Theorem. (Rank-Nullity Theorem)

$$dim(Ker(T)) + dim(Im(T)) = dim(V)$$

and

$$null(T) + rank(T) = dim(V)$$