Contents

1 Geometry and Complex Arithmetic

1.1 Introduction

Complex numbers form the set $\mathbb{C}$ and have form $a+ib$ where $a, b \in \mathbb{R}$ and $i^2 = -1$. Initially regarded with suspicion, confusion, even hostility.

Geometric interpretation discovered independently and simultaneously by Gauss, Wessel, Argand. The number $a+ib$ can be represented by a vector from the origin to the point (a, b). Then $\mathbb{C}$ is called the complex plane. There was now a way of making sense of complex numbers; the floodgates of innovation were about to open.

Sum of two complex numbers is given by the parallelogram rule of ordinary vector addition.

Multiplication The length of AB is the product of the lengths of A and B, and the angle of AB is the sum of the angles of A and B.

Symbolic form of addition and multiplication were first discovered by Bombelli. A general cubic can always be reduced to $x^3 = 3px + 2q$. Finding x is tantamount to finding the intersection between a line and x^3. This can always be solved with the remarkable formula $x = \sqrt[3]{q+\sqrt{q^2-p^3}} + \sqrt[3]{q-\sqrt{q^2-p^3}}$. Obviously, there should always be a point of intersection, but what if $p^3 > q^2$? This question gave rise to symbolic addition and multiplication of complex numbers and a practical use for $\mathbb{C}$.

Symbolic addition $(a+i\tilde{a})+(b+i\tilde{b}) = (a+b)+i(\tilde{a}+\tilde{b})$

Symbolic multiplication $(a+i\tilde{a})(b+i\tilde{b}) = (a\tilde{a}-b\tilde{b})+i(a\tilde{b}+b\tilde{a})$

Modulus is the length of $z = \|z\|$.

Argument of z is the angle $\theta = \arg(z) = \tan^{-1}(y/x)$.

Real part of z is its x-coordinate $\mathrm{Re}(z)$.

Imaginary part of z is its y-coordinate $\mathrm{Im}(z)$.

Complex conjugate of z is the reflection of z in the real axis, $\bar{z}$.

Geometric Shorthand (outdated, Euler's formula preferred) a complex number can be represented as $R\angle\phi$.

Geometric multiplication $(R\angle\phi)(r\angle\theta) = (Rr)\angle(\phi+\theta)$

Geometric inverse $\dfrac{1}{r\angle\theta} = (1/r)\angle(-\theta)$

Symbolic inverse $\dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2}$

1.1.1 Equivalence of Symbolic and Geometric Arithmetic

Easy to show addition rules are equivalent.

Geometrically, multiplication by a complex number $A = R\angle\phi$ is a rotation of the plane through angle ϕ and an expansion by factor R.

Proof: the symbolic multiplication rule will be derived from the geometric multiplication rule and then vice versa. (i) Using geometric multiplication, observe that $i^2 = -1$.

Figure [5] shows the lightly shaded being transformed into the darkly shaded shapes by being multiplied by $2\angle\frac{\pi}{3}$. (ii) Rotations and expansions preserve parallelograms, so $A(B+C) = AB + AC$. See Figure [5]. □

Figure [6a] demonstrates the transformation $z \mapsto iz$. It is just a rotation by $\frac{\pi}{2}$.

Figure [6b] is transformed into figure [6c] by multiplying by $4+3i$. It can be interpreted as $z \mapsto Az = (4+3i)z = 4z + 3(iz) = 4z + 3(z \text{ rotated by } \frac{\pi}{2})$. ■

1.2 Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

1.3 Applications

1.3.1 Trigonometry

1.3.2 Geometry

1.3.3 Calculus

1.3.4 Algebra

1.4 Transformations and Euclidean Geometry

2 Complex Functions as Transformations

2.1 Polynomials

2.2 Power Series

2.3 The Exponential Function

2.4 Cosine and Sine

2.5 Multifunctions

2.6 Logarithm Function

2.7 Averaging Over Circles

3 Differentiation: The Amplitwist Concept

3.1

3.2