Probability Theory

Tutorial 1

1. If there are 30 people in a room (assume that no one is born on February 29 and that any day has the same chance of being anyone's birthday)
   (a) What is the probability of at least two of them to have the same birthday? .
   (b) What is the probability that exactly two people in the room have the same birthday?
2. Compute the probability that 10 married couples are seated at random at a round table, then no wife sits next to her husband.
3. A coin of radius r is tossed onto an infinite chessboard, where each square has a side length of 1. What is the probability that the coin lands entirely within the boundaries of a single square?

Problem 1. *If there are 30 people in a room (assume that no one is born on February 29 and that any day has the same chance of being anyone's birthday)>*

1. *What is the probability of at least two of them to have the same birthday? .*

2. *What is the probability that exactly two people in the room have the same birthday?*

Solution: *We need to compute the cardinality of the set of all possible birthdays where at least two are the same. It is easier and equivalent (why?) to compute the cardinality of the set where no two birthdays are the same. As there cannot be any repetition, this is an example of an 'ordering of length 30 of elements of* $\{1, \ldots, 365\}$

First, we compute the probability that no two people have the same birthday, corresponding to event B*. As discussed above,* B *is an ordering of length* 30 *(*$k = 30$*) from a set of cardinality* 365 *(*$n = 365$*), so*

$$P(B) = \frac{|B|}{|\Omega|} = \frac{365 \times \cdots \times (365 - 30 + 1)}{365^{30}} \approx 0.29$$

The event that at least two people have the same birthday is the complement of event B*, so the probability that at least two out of the 30 people have the same birthday will be close to 71%.*

Now we consider a slightly different question: what is the probability that exactly two people in the room have the same birthday? To construct such an example, we would need to

1. *Choose the two people that have the same birthday.*

2. *Choose a day for their birthday.*

3. *Choose a day for everyone else's birthday, so that no other birthdays are the same.*

In how many ways can we choose the two people that have the same birthday? We need to choose two numbers from $C = \{1, \ldots, 30\}$ *– this will be a sequence of length 2 with no repetition, so their cardinality is* $\binom{30}{2}$*. Now, back to our question: we have computed the number of ways we can pick the two people with the same birthday. There are 365 ways to choose their birthday. For the remaining 28 people, there will be* $364 \times \cdots \times (365 - 28)$ *ways of picking their birthdays since they all need to be different. So, the total number of ways of selecting an outcome in the event 'exactly two people have the same birthday' is*

$$\binom{30}{2} 365 \times 364 \times \cdots \times (365 - 28).$$

Finally, to compute the probability of exactly two people having the same birthday, we need to divide by the cardinality of all possible birthday combinations given by 365^{30}*, which gives approximately 0.38 or 38%.*

Problem 2. *Compute the probability that 10 married couples are seated at random at a round table, then no wife sits next to her husband.*

Solution: *If we let* E_i*,* $i = 1, 2, \ldots, 10$ *denote the event that the i-th couple sits next to each other, it follows that the desired probability is* $1 - P\left(\bigcup_{i=1}^{10} E_i\right)$*. Now, from Theorem 2,*

$$P\left(\bigcup_{i=1}^{10} E_i\right) = \sum_{i=1}^{10} P(E_i) - \sum_{i_1 < i_2} P(E_{i_1} E_{i_2}) + \cdots + (-1)^{n+1} \sum_{i_1 < i_2 < \cdots < i_n} P(E_{i_1} E_{i_2} \cdots E_{i_n}) + \cdots - P(E_1 E_2 \cdots E_{10})$$

To compute $P(E_{i_1} E_{i_2} \cdots E_{i_n})$*, we first note that there are* $19!$ *ways of arranging 20 people around a round table. (Why?) The number of arrangements that result in a specified set of* n *men sitting next to their wives can most easily be obtained by first thinking of each of the* n *married couples as being single entities. If this were the case, then we would need to arrange* $20 - n$ *entities around a round table, and there are clearly* $(20 - n - 1)!$ *such arrangements. Finally, since each of the* n *married couples can be arranged next to each other in one of two possible ways, it follows that there* *are* $2^n(20-n-1)!$ *arrangements that result in a specified set of* n *men each sitting next to their wives. Therefore,*

$$P(E_{i_1}E_{i_2}\cdots E_{i_n}) = \frac{2^n(20-n-1)!}{19!}$$

To compute the full union probability, we must consider all possible combinations of couples at each step of the Inclusion-Exclusion formula. The term $\binom{10}{n}$ *accounts for the number of ways to choose* n *couples out of the 10. For example:*

- *The term* $\binom{10}{1}$ *represents the number of ways to choose exactly one couple to sit together. We choose one couple out of ten, which is* $\binom{10}{1} = 10$ *ways. Each of these single-couple events has a probability of* $\frac{2^1(20-1-1)!}{19!} = \frac{2\cdot 18!}{19!}$.
- *The term* $\binom{10}{2}$ *represents the number of ways to choose exactly two couples to sit together. We choose two couples out of ten, which is* $\binom{10}{2} = 45$ *ways.*
- *This pattern continues for all* n *couples.*

Thus, the probability that at least one married couple sits together is:

$$\binom{10}{1}\frac{2^1(18)!}{19!} - \binom{10}{2}\frac{2^2(17)!}{19!} + \binom{10}{3}\frac{2^3(16)!}{19!} - \cdots - \binom{10}{10}\frac{2^{10}9!}{19!} \approx .6605$$

Finally, the desired probability that no wife sits next to her husband is the complement:

$$1 - P\left(\bigcup_{i=1}^{10} E_i\right) \approx 1 - .6605 = .3395$$

Problem 3. *The Coin on the Infinite Chessboard : A coin of radius* r *is tossed onto an infinite chessboard, where each square has a side length of* 1*. What is the probability that the coin lands entirely within the boundaries of a single square?*

First, we define the Sample Space (Ω)
This problem can be solved by considering a single, representative 1×1 *square. The position of the coin is determined by the coordinates of its center,* (X, Y)*. Assuming the center is uniformly distributed over the square:*

$$\Omega = \{(X, Y) \mid 0 \le X \le 1, \quad 0 \le Y \le 1\}$$

The measure (area) of the sample space is:

$$|\Omega| = 1 \times 1 = 1$$

Now, we define the Favorable Event (A)
The event A *that the coin lands entirely within the square occurs if and only if the center of the coin,* (X, Y)*, is at a distance of at least* r *from all four boundaries of the* 1×1 *square.*
This condition restricts the center's coordinates to the following range:

- *The center must be at least* r *units away from the left edge* $(X = 0)$ *and the right edge* $(X = 1)$*:*

$$r \le X \le 1 - r$$

- *The center must be at least* r *units away from the bottom edge* $(Y = 0)$ *and the top edge* $(Y = 1)$*:*

$$r \le Y \le 1 - r$$

The region of the favorable event A is a central square with a reduced side length $(1-2r)$. The area of the favorable event $|A|$ is:

$$|A| = ((1-r)-r) \times ((1-r)-r) = (1-2r)^2$$

The probability $P(A)$ is the ratio of the favorable area to the total sample space area:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{(1-2r)^2}{1} = (1-2r)^2$$

Assumes a specific radius, $r = \frac{1}{3}$. Substituting this value into the general formula:

$$|A| = \left(1 - 2 \cdot \frac{1}{3}\right)^2 = \left(\frac{3-2}{3}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

Thus, the final probability is:

$$P(A) = \frac{1}{9}$$

Tutorial 2

1. A student buys 2 apples, 3 bananas and 5 coconuts. Every day (fir three days) the student chooses a fruit uniformly at random and eats it.

   (a) What is the probability that the student eats a coconut in day 1 and a banana in day 2?

   (b) What is the probability that on the third day the student will eat the last apple?

   (c) What is the probability that the student eats a coconut on day 2?

2. Assume there are two boxes: Box I and Box II. Box I contains w_1 white balls and b_1 black balls. Box II contains w_2 white balls and b_2 black balls. In the experiment, we first choose a box and then a ball from the chosen box.

   (a) What is the probability that a chosen ball is white?

   (b) Assume that a chosen ball is white. What is the probability that it was taken from Box I.

3. If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of a false alarm (a false indication of aircraft presence), and the probability of a missed detection (nothing registers, even though an aircraft is present)?

Problem 4. *A student buys 2 apples, 3 bananas and 5 coconuts. Every day (fir three days) the student chooses a fruit uniformly at random and eats it.*

1. *What is the probability that the student eats a coconut in day 1 and a banana in day 2?*
2. *What is the probability that on the third day the student will eat the last apple?*
3. *What is the probability that the student eats a coconut on day 2?*

Solution: *The sample space is the set of all triplets that can be constructed with the available fruits, with each outcome corresponding to the fruit eaten on each day. Since by the end of the three days we have full information, so the event space is the power set of the sample space. We define the events*

$A_i = \{$*the student eats an apple on day i*$\}$,
$B_i = \{$*the student eats a banana on day i*$\}$
$C_i = \{$*the student eats a coconut on day i*$\}$.

1. The event 'the student eats a coconut in day 1 and a banana in day 2' corresponds to the event $C_1 \cap B_2$. *Note that the way information about the probability is encoded is through conditional probabilities: the statement 'every day the student chooses a fruit uniformly at random and eats it' can be interpreted as the conditional probability of choosing any of the remaining fruits uniformly at random, so we know that* $P(B_2|C_1) = \frac{3}{9}$.

It follows from the definition of conditional probability that $P(C_1 \cap B_2) = P(B_2|C_1)P(C_1) = \frac{3}{9} \cdot \frac{5}{10} = \frac{1}{6}$.

Writing the probability of an intersection of two events as a product of a conditional probability and a probability is called the 'multiplication rule' and can be extended to intersections of more than two events. For example, let us consider the following question.

2. Since there are exactly two apples, that means that the student will eat the first apple on either day 1 or day 2. So, if A is the event 'student eats last apple on the third day', we can write $A = (A_1 \cap A_2^c \cap A_3) \cup (A_1^c \cap A_2 \cap A_3)$.

Notice that the events $A_1 \cap A_2^c \cap A_3$ *and* $A_1^c \cap A_2 \cap A_3$ *are disjoint, therefore* $P(A) = P(A_1 \cap A_2^c \cap A_3) + P(A_1^c \cap A_2 \cap A_3) = P(A_1)P(A_2^c|A_1)P(A_3|A_1 \cap A_2^c) + P(A_1^c)P(A_2|A_1^c)P(A_3|A_1^c \cap A_2)$ $= \frac{2}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} + \frac{8}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} = \frac{1}{45} + \frac{1}{45} = \frac{2}{45}$, *by using the multiplication rule twice.*

3. To compute the probability, we need to condition on what happened in day 1, but going through all possible options. In this case, there are two options that affect the computation of the conditional probability: whether the student also had a coconut on day 1 (event C_1*) or not (event* C_1^c*).*

Where is this formula coming from? We write $C_2 = (C_2 \cap C_1) \cup (C_2 \cap C_1^c)$. *So, from finite additivity, it follows that* $P(C_2) = P(C_2 \cap C_1) + P(C_2 \cap C_1^c)$.

By applying the multiplication rule to the conditional probabilities above, we get the formula which is a specific example of the law of total probabilities.

So

$P(C_2) = P(C_2|C_1) \cdot P(C_1) + P(C_2|C_1^c) \cdot P(C_1^c) = \frac{4}{9} \cdot \frac{5}{10} + \frac{5}{9} \cdot \frac{5}{10} = \frac{1}{2}$.

Problem 5. *Assume there are two boxes: Box I and Box II. Box I contains* w_1 *white balls and* b_1 *black balls. Box II contains* w_2 *white balls and* b_2 *black balls. In the experiment, we first choose a box and then a ball from the chosen box.*

1. *What is the probability that a chosen ball is white?*
2. *Assume that a chosen ball is white. What is the probability that it was taken from Box I.*

Solution:

A - an event that a chosen ball is white. B_1 *- an event that Box I was chosen.* B_2 *- an event that Box II was chosen.* $B_1 \cap B_2 = \emptyset, B_1 \cup B_2 = \Omega$ *hence* $\{B_1, B_2\}$ *is partition of* Ω. $P(B_1) = P(B_2) = \frac{1}{2}$, $P(A|B_1) = \frac{w_1}{w_1+b_1}$, $P(A|B_2) = \frac{w_2}{w_2+b_2}$.

1. $P(A) = P(A|B_1) \cdot P(B_1) + P(A|B_2)P(B_2) = \frac{1}{2}\frac{w_1}{w_1+b_1} + \frac{1}{2}\frac{w_2}{w_2+b_2}$.

2. $P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{\frac{w_1}{w_1+b_1}\frac{1}{2}}{\frac{w_1}{w_1+b_1}\frac{1}{2}+\frac{w_2}{w_2+b_2}\frac{1}{2}}$.

Problem 6. *If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of a false alarm (a false indication of aircraft presence), and the probability of a missed detection (nothing registers, even though an aircraft is present)?*

Solution: *A sequential representation of the sample space is appropriate here, as shown in the figure below.*

Let A and B be the events:

- $A = \{$*an aircraft is present*$\}$
- $B = \{$*the radar registers an aircraft presence*$\}$

and consider also their complements:

- $A^c = \{$*an aircraft is not present*$\}$
- $B^c = \{$*the radar does not register an aircraft presence*$\}$

The given probabilities are recorded along the corresponding branches of the tree describing the sample space. Each event of interest corresponds to a leaf of the tree, and its probability is equal to the product of the probabilities associated with the branches in a path from the root to the corresponding leaf.

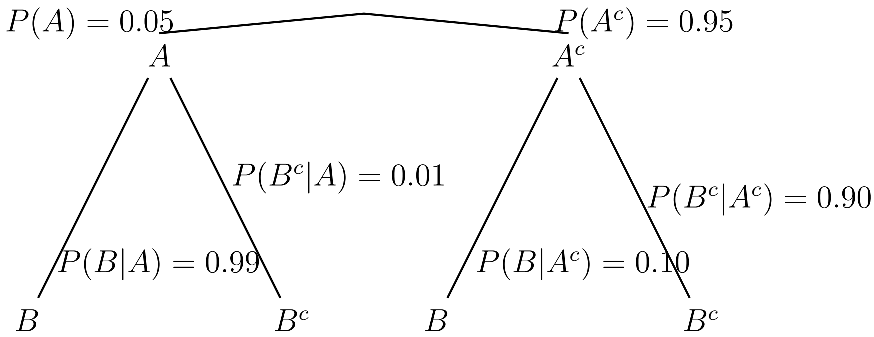

Figure 1: Sequential representation of the sample space for the radar problem.

The desired probabilities of false alarm and missed detection are:

- *The probability of a false alarm is the probability of the radar registering an aircraft when none is present. This corresponds to the event $A^c \cap B$.*

$$P(\textit{false alarm}) = P(A^c \cap B) = P(A^c)P(B|A^c) = 0.95 \cdot 0.10 = 0.095$$

- *The probability of a missed detection is the probability of the radar not registering an aircraft when one is present. This corresponds to the event $A \cap B^c$.*

$$P(\textit{missed detection}) = P(A \cap B^c) = P(A)P(B^c|A) = 0.05 \cdot 0.01 = 0.0005$$

Tutorial 3

1. Let X be a random variable with $P(X > 0) > 0$. Prove that there is a $\delta > 0$ such that $P(X \geq \delta) > 0$.

Problem 7. *Let X be a random variable with $P(X > 0) > 0$. Prove that there is a $\delta > 0$ such that $P(X \geq \delta) > 0$.*

Solution.

Method 1: Put $A = \{X > 0\}$ and $A_n = \{X \geq 1/n\}$ for all $n \in \mathbb{N}$. Then, $A_n \uparrow A$ (or A_n converges to A from below), and using Proposition 3.3.1 (Continuity of Probabilities), we have $\lim_{n\to\infty} P(A_n) = P(A)$. Given that $P(A) > 0$, it follows that there must exist some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have $P(A_n) > 0$. In particular, $P(A_N) > 0$. We can therefore choose $\delta = 1/N > 0$. This completes the proof.

Method 2: Put $A = \{X > 0\}$ and $A_n = \{X \geq 1/n\}$ for all $n \in \mathbb{N}$. Then, $A = \bigcup_{n=1}^{\infty} A_n$. If for every $n \in \mathbb{N}$, we had $P(A_n) = 0$, then using a fundamental property of probability (countable subadditivity), we would have $P(A) = P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} 0 = 0$. This result, $P(A) = 0$, is a contradiction to our initial assumption that $P(X > 0) > 0$. Therefore, there must exist at least one $N \in \mathbb{N}$ such that $P(A_N) > 0$. We can then choose $\delta = 1/N > 0$. This concludes the proof.

Problem 8. *ver ejemplo 28 y 29 de convergencias. Problema 1 y 2*

Tutorial 1 Adam

Problema 3: Rompimiento de un Palo (The Stick Problem)

Un palo de 1 metro de longitud se rompió en dos puntos. ¿Cuál es la probabilidad de que con las tres piezas resultantes se pueda construir un triángulo?
Sea L_1, L_2, L_3 las longitudes de las tres piezas, tal que $L_1 + L_2 + L_3 = 1$. El espacio de parámetros inicial es $\mathbf{L} = (L_1, L_2) \in [0, 1] \times [0, 1]$.
El espacio muestral Ω relevante (donde $L_3 \geq 0$) es:

$$\Omega = \{(L_1, L_2) \mid L_1 \geq 0, L_2 \geq 0, L_1 + L_2 \leq 1\}$$

El área del espacio muestral es $|\Omega|[cite_start] = \frac{1}{2}$.
Las **condiciones de triángulo** son:

1. $L_1 + L_2 > L_3$
2. $L_1 + L_3 > L_2$
3. $L_2 + L_3 > L_1$

Sustituyendo $L_3 = 1 - L_1 - L_2$[cite: 5]:

1. $L_1 + L_2 > 1 - L_1 - L_2 \implies 2L_1 + 2L_2 > 1 \implies L_1 + L_2 > \frac{1}{2}$ [cite: 16]
2. $L_1 + (1 - L_1 - L_2) > L_2 \implies 1 - L_2 > L_2 \implies L_2 < \frac{1}{2}$ [cite: 17]
3. $L_2 + (1 - L_1 - L_2) > L_1 \implies 1 - L_1 > L_1 \implies L_1 < \frac{1}{2}$ [cite: 17]

La región favorable A está definida por las tres condiciones y $L_1, L_2 > 0$. La región A es un triángulo dentro de Ω con vértices $(\frac{1}{2}, 0)$, $(0, \frac{1}{2})$, y $(\frac{1}{2}, \frac{1}{2})$ que se obtiene del cuadrado $(\frac{1}{2}, \frac{1}{2})$.

$$|A| = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

La probabilidad $P(A)$ es[cite: 18]:

$$P(A) = \frac{|A|}{|\Omega|} = \frac{1/8}{1/2} = \frac{1}{4}$$

Problema 4: La Moneda en el Tablero de Ajedrez (Coin on Chessboard)

Una moneda de radio r es lanzada sobre un tablero de ajedrez infinito cuyos lados tienen longitud 1. ¿Cuál es la probabilidad de que la moneda quede completamente dentro de uno de los cuadrados? El evento A ocurre si el centro de la moneda cae en una región interior del cuadrado 1×1 que esté a una distancia de al meno r de cualquier borde. El cuadrado 1×1 se reduce a un cuadrado central de $(1-2r) \times (1-2r)$. Asumiendo $r = \frac{1}{3}$ (el valor se infiere del diagrama , el área favorable es[cite: 27, 28]:

$$|A| = \left(1 - 2 \cdot \frac{1}{3}\right)^2 = \left(\frac{1}{3}\right)^2 = \frac{1}{9}$$

La probabilidad $P(A)$, dado que el área total es $1 \times 1 = 1$, es[cite: 29]:

$$P(A) = \frac{|A|}{1} = \frac{1}{9}$$

Problema 1: El Desorden de los Impermeables (The Raincoat Problem)

Suponga n estudiantes con n impermeables. Después de clase, cada estudiante toma un impermeable al azar[cite: 30, 31]. Calcule la probabilidad de que al menos un impermeable termine con su dueño original.
El espacio muestral Ω es el conjunto de todas las permutaciones S_n[cite: 34, 38].

$$|\Omega|[cite_start] = n![cite : 40]$$

Sea A_i el evento donde el estudiante i obtiene el abrigo i[cite: 41, 43]. El evento de interés es $A = \bigcup_{i=1}^{n} A_i$ (al menos un dueño original recupera su abrigo)[cite: 45].
Usando la fórmula de Inclusión-Exclusión[cite: 45]:

$$P(A) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{k_1 < \cdots < k_j} P(A_{k_1} \cap \cdots \cap A_{k_j})$$

La intersección de j eventos, $|A_{k_1} \cap \cdots \cap A_{k_j}|$, es el número de permutaciones que fijan j elementos, que es $(n-j)!$[cite: 47].

$$P(A_{k_1} \cap \cdots \cap A_{k_j}) = \frac{|A_{k_1} \cap \cdots \cap A_{k_j}|}{|\Omega|} = \frac{(n-j)!}{n!}[cite : 47]$$

El número de formas de elegir j elementos de n es $\binom{n}{j}$. La suma de las probabilidades de todas las intersecciones de j eventos es:

$$\sum_{k_1 < \cdots < k_j} P(A_{k_1} \cap \cdots \cap A_{k_j}) = \binom{n}{j} \frac{(n-j)!}{n!} = \frac{n!}{j!(n-j)!} \frac{(n-j)!}{n!} = \frac{1}{j!}$$

Sustituyendo en la fórmula de Inclusión-Exclusión[cite: 50]:

$$P(A) = \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j!} = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots \pm \frac{1}{n!}$$

Cuando n es grande $(n \to \infty)$[cite: 51, 52]:

$$P(A) \xrightarrow{n \to \infty} 1 - e^{-1} \approx 0.632[cite : 52]$$

Problema: La Fila de la Taquilla (The Ticket Office Line)

$2n$ clientes, n con billetes de \$5 y n con billetes de \$10. El boleto cuesta 5 y la taquilla comienza sin dinero Cuál es la probabilidad de que nadie tenga que esperar el cambio (evento A)?
Modelamos a los clientes como una secuencia de ± 1 de longitud $2n$, con n unos (+\$5) y n menos unos (−\$5).

$$|\Omega|[cite_start] = \binom{2n}{n}$$

El evento A ocurre si la suma parcial de la secuencia nunca es negativa (el dinero de la taquilla nunca cae por debajo de cero)[cite: 57]. Esto es un problema clásico de **Caminos de Lattice** (Lattice Paths).
Número de caminos que cumplen la condición (Teorema del Voto)[cite: 41, 42]:

$$|A|[cite_start] = \binom{2n}{n} - \binom{2n}{n+1}[cite: 104]$$

El término $\binom{2n}{n+1}$ es el número de caminos con $(n-1)$ subidas y $(n+1)$ bajadas, que corresponden al evento de que al menos una persona tiene que esperar el cambio[cite: 103, 96].
Probabilidad $P(A)$[cite: 104]:

$$P(A) = \frac{\binom{2n}{n} - \binom{2n}{n+1}}{\binom{2n}{n}} = 1 - \frac{\binom{2n}{n+1}}{\binom{2n}{n}}$$

$$P(A) = 1 - \frac{n}{n+1} = \frac{1}{n+1}[cite: 104]$$