Probability Theory

Tutorial 2

1. A student buys 2 apples, 3 bananas and 5 coconuts. Every day (for three days) the student chooses a fruit uniformly at random and eats it.

   (a) What is the probability that the student eats a coconut in day 1 and a banana in day 2?

   (b) What is the probability that on the third day the student will eat the last apple?

   (c) What is the probability that the student eats a coconut on day 2?

2. Assume there are two boxes: Box I and Box II. Box I contains w_1 white balls and b_1 black balls. Box II contains w_2 white balls and b_2 black balls. In the experiment, we first choose a box and then a ball from the chosen box.

   (a) What is the probability that a chosen ball is white?

   (b) Assume that a chosen ball is white. What is the probability that it was taken from Box I.

3. If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of a false alarm (a false indication of aircraft presence), and the probability of a missed detection (nothing registers, even though an aircraft is present)?

Problem 1. *A student buys 2 apples, 3 bananas and 5 coconuts. Every day (fir three days) the student chooses a fruit uniformly at random and eats it.*

1. *What is the probability that the student eats a coconut in day 1 and a banana in day 2?*
2. *What is the probability that on the third day the student will eat the last apple?*
3. *What is the probability that the student eats a coconut on day 2?*

Solution: *The sample space is the set of all triplets that can be constructed with the available fruits, with each outcome corresponding to the fruit eaten on each day. Since by the end of the three days we have full information, so the event space is the power set of the sample space. We define the events*

$A_i = \{$*the student eats an apple on day* $i\}$,
$B_i = \{$*the student eats a banana on day* $i\}$
$C_i = \{$*the student eats a coconut on day* $i\}$.

1. The event 'the student eats a coconut in day 1 and a banana in day 2' corresponds to the event $C_1 \cap B_2$*. Note that the way information about the probability is encoded is through conditional probabilities: the statement 'every day the student chooses a fruit uniformly at random and eats it' can be interpreted as the conditional probability of choosing any of the remaining fruits uniformly at random, so we know that* $P(B_2|C_1) = \frac{3}{9}$.

It follows from the definition of conditional probability that $P(C_1 \cap B_2) = P(B_2|C_1)P(C_1) = \frac{3}{9} \cdot \frac{5}{10} = \frac{1}{6}$.

Writing the probability of an intersection of two events as a product of a conditional probability and a probability is called the 'multiplication rule' and can be extended to intersections of more than two events. For example, let us consider the following question.

2. Since there are exactly two apples, that means that the student will eat the first apple on either day 1 or day 2. So, if A *is the event 'student eats last apple on the third day', we can write* $A = (A_1 \cap A_2^c \cap A_3) \cup (A_1^c \cap A_2 \cap A_3)$.

Notice that the events $A_1 \cap A_2^c \cap A_3$ *and* $A_1^c \cap A_2 \cap A_3$ *are disjoint, therefore* $P(A) = P(A_1 \cap A_2^c \cap A_3) + P(A_1^c \cap A_2 \cap A_3) = P(A_1)P(A_2^c|A_1)P(A_3|A_1 \cap A_2^c) + P(A_1^c)P(A_2|A_1^c)P(A_3|A_1^c \cap A_2)$ $= \frac{2}{10} \cdot \frac{8}{9} \cdot \frac{1}{8} + \frac{8}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} = \frac{1}{45} + \frac{1}{45} = \frac{2}{45}$*, by using the multiplication rule twice.*

3. To compute the probability, we need to condition on what happened in day 1, but going through all possible options. In this case, there are two options that affect the computation of the conditional probability: whether the student also had a coconut on day 1 (event C_1*) or not (event* C_1^c*).*

Where is this formula coming from? We write $C_2 = (C_2 \cap C_1) \cup (C_2 \cap C_1^c)$*. So, from finite additivity, it follows that* $P(C_2) = P(C_2 \cap C_1) + P(C_2 \cap C_1^c)$.

By applying the multiplication rule to the conditional probabilities above, we get the formula which is a specific example of the law of total probabilities.

So

$P(C_2) = P(C_2|C_1) \cdot P(C_1) + P(C_2|C_1^c) \cdot P(C_1^c) = \frac{4}{9} \cdot \frac{5}{10} + \frac{5}{9} \cdot \frac{5}{10} = \frac{1}{2}$.

Problem 2. *Assume there are two boxes: Box I and Box II. Box I contains* w_1 *white balls and* b_1 *black balls. Box II contains* w_2 *white balls and* b_2 *black balls. In the experiment, we first choose a box and then a ball from the chosen box.*

1. *What is the probability that a chosen ball is white?*
2. *Assume that a chosen ball is white. What is the probability that it was taken from Box I.*

Solution:

A *- an event that a chosen ball is white.* B_1 *- an event that Box I was chosen.* B_2 *- an event that Box II was chosen.* $B_1 \cap B_2 = \emptyset, B_1 \cup B_2 = \Omega$ *hence* $\{B_1, B_2\}$ *is partition of* Ω. $P(B_1) = P(B_2) = \frac{1}{2}$, $P(A|B_1) = \frac{w_1}{w_1+b_1}$, $P(A|B_2) = \frac{w_2}{w_2+b_2}$.

1. $P(A) = P(A|B_1) \cdot P(B_1) + P(A|B_2)P(B_2) = \frac{1}{2}\frac{w_1}{w_1+b_1} + \frac{1}{2}\frac{w_2}{w_2+b_2}$.

2. $P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{\frac{w_1}{w_1+b_1}\frac{1}{2}}{\frac{w_1}{w_1+b_1}\frac{1}{2}+\frac{w_2}{w_2+b_2}\frac{1}{2}}$.

Problem 3. *If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of a false alarm (a false indication of aircraft presence), and the probability of a missed detection (nothing registers, even though an aircraft is present)?*

Solution: *A sequential representation of the sample space is appropriate here, as shown in the figure below.*

Let A and B be the events:

- $A = \{$*an aircraft is present*$\}$
- $B = \{$*the radar registers an aircraft presence*$\}$

and consider also their complements:

- $A^c = \{$*an aircraft is not present*$\}$
- $B^c = \{$*the radar does not register an aircraft presence*$\}$

The given probabilities are recorded along the corresponding branches of the tree describing the sample space. Each event of interest corresponds to a leaf of the tree, and its probability is equal to the product of the probabilities associated with the branches in a path from the root to the corresponding leaf.

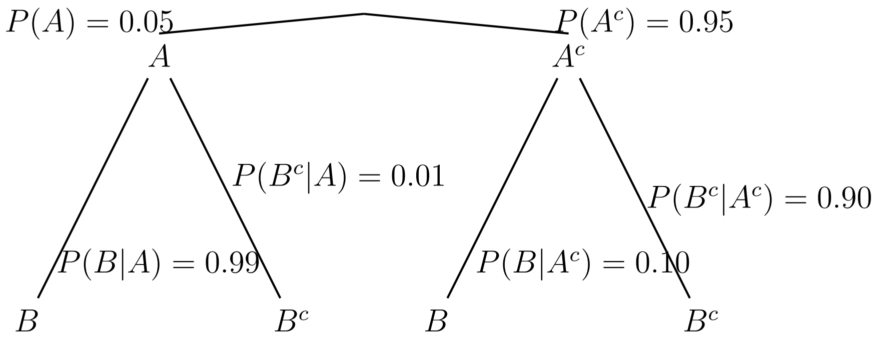

Figure 1: Sequential representation of the sample space for the radar problem.

The desired probabilities of false alarm and missed detection are:

- *The probability of a false alarm is the probability of the radar registering an aircraft when none is present. This corresponds to the event $A^c \cap B$.*

$$P(\textit{false alarm}) = P(A^c \cap B) = P(A^c)P(B|A^c) = 0.95 \cdot 0.10 = 0.095$$

- *The probability of a missed detection is the probability of the radar not registering an aircraft when one is present. This corresponds to the event $A \cap B^c$.*

$$P(\textit{missed detection}) = P(A \cap B^c) = P(A)P(B^c|A) = 0.05 \cdot 0.01 = 0.0005$$