Probability Theory

Tutorial 3

1. We throw a coin an infinite number of times. In each throw, the probability of getting a head is equal to $p \in (0, 1]$. Prove that with probability 1, a pattern of 100 heads in a row will appear infinitely many times.

2. We throw a coin an infinite number of times. In each throw, the probability of getting a head is $p \neq \frac{1}{2}$. For $n = 2, 4, 6, \ldots$, consider the events A_n: in the first n throws, the number of heads and tails is the same. Prove that with probability 1, only finitely many of the events $A_2, A_4, \ldots, A_{2k}, \ldots$ occur.

3. Suppose $P\left(\left[0, \frac{8}{4+n}\right]\right) = \frac{2+e^{-n}}{6}$ for all $n = 1, 2, 3, \ldots$. What must $P(\{0\})$ be?

4. Suppose $P([0, 1]) = 1$, but $P\left(\left[\frac{1}{n}, 1\right]\right) = 0$ for all $n = 1, 2, 3, \ldots$. What must $P(\{0\})$ be?

Problem 1

We throw a coin an infinite number of times. In each throw, the probability of getting a head is equal to $p \in (0, 1]$. Prove that with probability 1, a pattern of 100 heads in a row will appear infinitely many times.

Proof

This proof relies fundamentally on the Second Borel-Cantelli Lemma for independent events.

Second Borel-Cantelli Lemma (for Independent Events): Let $A_1, A_2, \ldots$ be a sequence of independent events. Then, the probability that infinitely many of the events occur, denoted $P(A_k \text{ i.o.})$, is 1 if and only if the sum of their probabilities diverges:

$$P(A_k \text{ i.o.}) = 1 \iff \sum_{k=1}^{\infty} P(A_k) = \infty$$

The event of interest is the appearance of 100 consecutive heads. To apply the Borel-Cantelli Lemma, we must construct an infinite sequence of mutually independent events.

]We partition the infinite sequence of coin throws into non-overlapping blocks, each of length $m = 100$. For $k \in \{0, 1, 2, \ldots\}$, let A_k be the event that the k-th block consists entirely of heads. That is, A_k occurs if the throws from position $100k+1$ up to position $100k+100$ are all heads. Since all individual coin throws are mutually independent, and the blocks A_k and A_j for $j \neq k$ occupy completely distinct sets of throws, the sequence of events $A_0, A_1, A_2, \ldots$ is mutually independent.

The probability of getting a head in a single throw is $P(\text{Head}) = p$. Since A_k is the event of 100 independent heads in a row:

$$P(A_k) = p \cdot p \cdot \ldots \cdot p \quad (100 \text{ times}) = p^{100}$$

We now compute the sum of the probabilities of these independent events:

$$\sum_{k=0}^{\infty} P(A_k) = \sum_{k=0}^{\infty} p^{100}$$

Since $p \in (0, 1]$, the smallest possible value for p^{100} is $\lim_{p \to 0^+} p^{100} = 0$, but since p is strictly greater than 0, $p^{100} > 0$. The sum is an infinite series where every term is the same positive constant $c = p^{100}$:

$$\sum_{k=0}^{\infty} p^{100} = p^{100} + p^{100} + p^{100} + \ldots = \infty$$

The sum diverges.

Since the events A_k are independent and $\sum_{k=0}^{\infty} P(A_k) = \infty$, the Second Borel-Cantelli Lemma guarantees that the probability of the sequence of events A_k occurring infinitely often is 1:

$$P(\{A_k \text{ i.o.}\}) = 1$$

Let A be the event that "a pattern of 100 heads in a row appears infinitely many times." The event $\{A_k \text{ i.o.}\}$ means that the pattern appears infinitely many times only within the predetermined disjoint blocks. Since every occurrence of the pattern within a disjoint block is also an occurrence of the general pattern A, we have the set containment:

$$\{A_k \text{ i.o.}\} \subseteq A$$

Because $\{A_k \text{ i.o.}\}$ is a subset of A, its probability provides a lower bound for $P(A)$:

$$P(A) \geq P(\{A_k \text{ i.o.}\}) = 1$$

Since probability cannot exceed 1, we must have $P(A) = 1$.

Therefore, with probability 1, a pattern of 100 heads in a row will appear infinitely many times.

Problem 2

We throw a coin an infinite number of times. In each throw, the probability of getting a head is $p \neq \frac{1}{2}$. For $n = 2, 4, 6, \ldots$, consider the events A_n: in the first n throws, the number of heads and tails is the same. Prove that with probability 1, only finitely many of the events $A_2, A_4, \ldots, A_{2k}, \ldots$ occur.

Solution

This problem is solved using the First Borel-Cantelli Lemma. This lemma provides a simple condition for an event to occur only a finite number of times (finitely often, or f.o.).

First Borel-Cantelli Lemma: Let $A_1, A_2, \ldots$ be any sequence of events. If the sum of their probabilities converges, then the probability that infinitely many of the events occur is zero:

$$\sum_{k=1}^{\infty} P(A_k) < \infty \implies P(\{A_k \text{ occurs infinitely often (i.o.)}\}) = 0$$

The conclusion $P(\{A_k \text{ i.o.}\}) = 0$ is equivalent to $P(\{A_k \text{ occurs finitely often (f.o.)}\}) = 1$.

Let A_{2k} be the event that in the first $2k$ throws, the number of heads equals the number of tails. This requires exactly k heads and k tails. Since each throw is an independent Bernoulli trial with success probability p (Head) and failure probability $(1-p)$ (Tail), the number of ways to get k heads in $2k$ trials is $\binom{2k}{k}$.

$$P(A_{2k}) = \binom{2k}{k} p^k (1-p)^k$$

We must check the convergence of the series $\sum_{k=1}^{\infty} P(A_{2k})$. We use the Ratio Test.

Let $a_k = P(A_{2k})$. We compute the limit of the ratio $\frac{a_{k+1}}{a_k}$ as $k \to \infty$.

$$\frac{P(A_{2(k+1)})}{P(A_{2k})} = \frac{\binom{2k+2}{k+1} p^{k+1}(1-p)^{k+1}}{\binom{2k}{k} p^k (1-p)^k}$$

We simplify the combinatorial term:

$$\frac{\binom{2k+2}{k+1}}{\binom{2k}{k}} = \frac{\frac{(2k+2)!}{(k+1)!(k+1)!}}{\frac{(2k)!}{k!k!}} = \frac{(2k+2)!}{(2k)!} \cdot \frac{(k!)^2}{((k+1)!)^2}$$

$$= (2k+2)(2k+1) \cdot \frac{1}{(k+1)^2} = \frac{2(k+1)(2k+1)}{(k+1)^2} = \frac{2(2k+1)}{k+1}$$

Now, we compute the limit L:

$$L = \lim_{k \to \infty} \left[p(1-p) \cdot \frac{2(2k+1)}{k+1} \right]$$

$$L = p(1-p) \cdot \lim_{k \to \infty} \left[\frac{4k+2}{k+1} \right] = p(1-p) \cdot 4$$

$$L = 4p(1-p)$$

The Ratio Test states that the series converges if $L < 1$. We analyze the condition $4p(1-p) < 1$:

$$4p - 4p^2 < 1$$

$$0 < 4p^2 - 4p + 1$$

The right side is a perfect square:

$$0 < (2p-1)^2$$

This inequality holds true for all values of p except when $2p - 1 = 0$, which occurs when $p = 1/2$. Since the problem explicitly states that $p \neq 1/2$, the limit L satisfies the condition:

$$L = 4p(1 - p) < 1$$

Therefore, the series converges:

$$\sum_{k=1}^{\infty} P(A_{2k}) < \infty$$

By the First Borel-Cantelli Lemma, since the sum of the probabilities of the events A_{2k} converges, the probability that infinitely many of these events occur is zero:

$$P(\{A_{2k} \text{ i.o.}\}) = 0$$

Consequently, the probability that only a finite number of these events occur is 1:

$$P(\{A_{2k} \text{ f.o.}\}) = 1$$

This proves that with probability 1, the number of heads and tails will only be equal in the first $2k$ throws for a finite number of indices k.

Problem 3

Suppose $P\left(\left[0, \frac{8}{4+n}\right]\right) = \frac{2+e^{-n}}{6}$ for all $n = 1, 2, 3, \ldots$. What must $P(\{0\})$ be?

Solution

We define the sequence of events $A_n = \left[0, \frac{8}{4+n}\right]$. This is a **decreasing** sequence of sets since $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$.

The limit of the sets is the intersection:

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \left[0, \frac{8}{4+n}\right] = \{0\}$$

By the **Continuity of Probability for Decreasing Events**:

$$P\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$$

Substituting the given probability function:

$$P(\{0\}) = \lim_{n\to\infty} P\left(\left[0, \frac{8}{4+n}\right]\right) = \lim_{n\to\infty} \left(\frac{2+e^{-n}}{6}\right)$$

Since $\lim_{n\to\infty} e^{-n} = 0$:

$$P(\{0\}) = \frac{2+0}{6} = \frac{1}{3}$$

Problem 4

Suppose $P([0,1]) = 1$, but $P\left(\left[\frac{1}{n}, 1\right]\right) = 0$ for all $n = 1, 2, 3, \ldots$. What must $P(\{0\})$ be?

Solution

Let $B_n = \left[\frac{1}{n}, 1\right]$. This is an **increasing** sequence of sets: $B_1 \subseteq B_2 \subseteq B_3 \subseteq \ldots$.

1. **Calculate $P((0,1])$:** The limit of the sets is the union:

$$\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = (0,1]$$

   By the **Continuity of Probability for Increasing Events**:

$$P((0,1]) = P\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} P(B_n)$$

   Since $P(B_n) = 0$ for all n:

$$P((0,1]) = \lim_{n\to\infty} 0 = 0$$

2. **Use Additivity:** The sample space $[0,1]$ can be written as the union of two disjoint sets:

$$[0,1] = \{0\} \cup (0,1]$$

   By the additivity axiom:

$$P([0,1]) = P(\{0\}) + P((0,1])$$

3. **Solve for $P(\{0\})$:**

$$1 = P(\{0\}) + 0$$

$$P(\{0\}) = 1$$