Probability Theory

Tutorial 6

1. In a certain human population, the Cephalic Index I (skull width expressed as a percentage of its length) is normally distributed among individuals. There is 58% with $I \leq 75$, 38% with $75 \leq I \leq 80$, and 4% with $I > 80$. Find the density function of the index, $f_I(i)$, and the probability $P(78 \leq I \leq 82)$.

2. Alice and Peter agreed to meet at 8 PM to go to the movies. Since they are not punctual, their arrival times X and Y can be assumed to be independent random variables with a uniform distribution between 8 PM and 9 PM.

   (a) What is the joint density of X and Y?

   (b) What is the probability that both arrive between 8:15 PM and 8:45 PM?

   (c) If both are willing to wait no more than 10 minutes for the other after they arrive, what is the probability that they miss each other?

Problem 1

In a certain human population, the Cephalic Index I (skull width expressed as a percentage of its length) is normally distributed among individuals. There is 58% with $I \leq 75$, 38% with $75 \leq I \leq 80$, and 4% with $I > 80$. Find the density function of the index, $f_I(i)$, and the probability $P(78 \leq I \leq 82)$.

Solution

The Cephalic Index I follows a Normal distribution: $I \sim N(\mu, \sigma^2)$. We need to find the mean (μ) and the standard deviation (σ). Let $Z = \frac{I-\mu}{\sigma}$ be the standard Normal variable $Z \sim N(0,1)$, with CDF $\Phi(z)$.

1. Finding μ and σ: We use the given cumulative probabilities:

1. $P(I \leq 75) = 0.58 \implies \Phi\left(\frac{75-\mu}{\sigma}\right) = 0.58$
2. $P(I \leq 80) = P(I \leq 75) + P(75 \leq I \leq 80) = 0.58 + 0.38 = 0.96$

   Using the standard Normal table (or calculator):

   - $\Phi(z_1) = 0.58 \implies z_1 \approx 0.20$
   - $\Phi(z_2) = 0.96 \implies z_2 \approx 1.75$

   This yields a system of linear equations:

$$\begin{aligned} &(1) \quad 75 - \mu = 0.20\sigma \\ &(2) \quad 80 - \mu = 1.75\sigma \end{aligned}$$

Subtracting equation (1) from equation (2):

$$(80-\mu) - (75-\mu) = 1.75\sigma - 0.20\sigma \implies 5 = 1.55\sigma$$

$$\sigma = \frac{5}{1.55} \approx 3.226$$

Substituting σ back into equation (1):

$$75 - \mu = 0.20(3.226) \implies 75 - \mu \approx 0.645$$

$$\mu = 75 - 0.645 \approx 74.355$$

2. The Density Function $f_I(i)$: The PDF of I is:

$$f_I(i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{i-\mu}{\sigma}\right)^2}$$

$$f_I(i) = \frac{1}{\sqrt{2\pi}(3.226)} e^{-\frac{1}{2}\left(\frac{i-74.355}{3.226}\right)^2}$$

3. Calculating $P(78 \leq I \leq 82)$: We standardize the values:

$$Z_{78} = \frac{78 - 74.355}{3.226} \approx \frac{3.645}{3.226} \approx 1.13$$

$$Z_{82} = \frac{82 - 74.355}{3.226} \approx \frac{7.645}{3.226} \approx 2.37$$

$$P(78 \leq I \leq 82) = P(1.13 \leq Z \leq 2.37) = \Phi(2.37) - \Phi(1.13)$$

Using the Z-table values:

$$P(78 \leq I \leq 82) \approx 0.9911 - 0.8708 = 0.1203$$

Problem 2

Alice and Peter agreed to meet at 8 PM to go to the movies. Since they are not punctual, their arrival times X and Y can be assumed to be independent random variables with a uniform distribution between 8 PM and 9 PM.

1. What is the joint density of X and Y?
2. What is the probability that both arrive between 8:15 PM and 8:45 PM?
3. If both are willing to wait no more than 10 minutes for the other after they arrive, what is the probability that they miss each other?

Solution

Let X and Y be the arrival times in minutes after 8:00 PM. Thus, $X, Y \sim U(0, 60)$. The individual PDF is $f_X(x) = f_Y(y) = \frac{1}{60}$ for $x, y \in [0, 60]$, and 0 otherwise.

a) What is the joint density of X and Y? Since X and Y are independent, the joint density is the product of the marginal densities:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{60} \cdot \frac{1}{60} = \frac{1}{3600}$$

for $0 \le x \le 60$ and $0 \le y \le 60$. The density is 0 otherwise. The sample space is a square of area $60 \times 60 = 3600$.

b) What is the probability that both arrive between 8:15 PM and 8:45 PM? In minutes after 8:00 PM, this corresponds to $15 \le X \le 45$ and $15 \le Y \le 45$. The region of integration R is a square defined by $R = [15, 45] \times [15, 45]$. The area of this region is $A(R) = (45 - 15) \times (45 - 15) = 30 \times 30 = 900$. The probability is:

$$P(15 \le X \le 45, 15 \le Y \le 45) = \iint_R f_{X,Y}(x, y)\, dxdy = \frac{1}{3600} \cdot A(R) = \frac{900}{3600} = \frac{1}{4} = 0.25$$

c) If both are willing to wait no more than 10 minutes for the other after arriving, what is the probability that they miss each other? They miss each other if the difference between their arrival times is greater than 10 minutes. We are looking for $P(\text{Miss}) = P(|X - Y| > 10)$.

It is easier to calculate the probability that they **meet** ($P(\text{Meet})$), which is the complement:

$$P(\text{Meet}) = P(|X - Y| \le 10) \implies P(\text{Meet}) = P(-10 \le X - Y \le 10)$$

The region where they meet is $M = \{(x, y) : Y - 10 \le X \le Y + 10\}$ within the 60×60 square.

The probability is given by $\frac{\text{Area}(M)}{\text{Total Area}}$. The total area is 3600. The area where they **miss** ($|X - Y| > 10$) consists of two triangles, T_1 and T_2, in the corners of the square.

- T_1: $X - Y > 10$ (or $Y < X - 10$). Vertices are $(10, 0), (60, 0), (60, 50)$. Base $= 50$, Height $= 50$. $\text{Area}(T_1) = \frac{1}{2}(50^2) = 1250$.
- T_2: $Y - X > 10$ (or $Y > X + 10$). Vertices are $(0, 10), (0, 60), (50, 60)$. Base $= 50$, Height $= 50$. $\text{Area}(T_2) = \frac{1}{2}(50^2) = 1250$.

The total area where they miss is $A(\text{Miss}) = 1250 + 1250 = 2500$.

The probability that they miss each other is:

$$P(\text{Miss}) = \frac{A(\text{Miss})}{\text{Total Area}} = \frac{2500}{3600} = \frac{25}{36} \approx 0.6944$$