Probability Theory

Tutorial 4

1. Let the sample space be $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let $\mathcal{A}$ be the σ-algebra generated by the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, denoted as $\mathcal{A} = \sigma(A, B)$.

   (a) List all sets in $\mathcal{A}$.

   (b) Is the function $X(\omega)$ defined as:

   $$X(\omega) = \begin{cases} 2 & \text{if } \omega \in \{1, 2, 3, 4\} \\ 7 & \text{if } \omega \in \{5, 6\} \end{cases}$$

   a random variable over the measurable space $(\Omega, \mathcal{A})$?

   (c) Give an example of a function on Ω that is not a random variable over $(\Omega, \mathcal{A})$.

   (d) Show that there exists a probability measure P on $(\Omega, \mathcal{A})$ such that $P(A)$ is zero or one for all $A \in \mathcal{A}$, yet P is not a point mass.

2. Let X be a random variable with $P(X > 0) > 0$. Prove that there is a $\delta > 0$ such that $P(X \geq \delta) > 0$.

3. A random variable X has a distribution function $F_X(x)$ given by:

   $$F_X(x) = \begin{cases} 0 & x < 2 \\ 1/2 & 2 \leq x < 4 \\ 3/4 & 4 \leq x < 6 \\ 1 & x \geq 6 \end{cases}$$

   Calculate $P(X = 2)$, $P(X = 4)$, and $P(X = 6)$.

4. Let $\Omega = [0, 3]$ with probability uniform and $X(\omega)$ be the random variable given by:

   $$X(\omega) = \begin{cases} \omega + 1 & \text{si } 0 \leq \omega \leq 1 \\ 2 & \text{si } 1 < \omega \leq 1.5 \\ 3 & \text{si } 1.5 < \omega < 2 \\ -\omega + 4 & \text{si } 2 \leq \omega \leq 3 \end{cases}$$

   Find $F_X(t)$, the distribution function of X.

Problem 1

Let the sample space be $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let $\mathcal{A}$ be the σ-algebra generated by the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, denoted as $\mathcal{A} = \sigma(A, B)$.

Solution

1. List all sets in $\mathcal{A}$.
2. Is the function $X(\omega)$ defined as:

$$X(\omega) = \begin{cases} 2 & \text{if } \omega \in \{1, 2, 3, 4\} \\ 7 & \text{if } \omega \in \{5, 6\} \end{cases}$$

   a random variable over the measurable space $(\Omega, \mathcal{A})$?

3. Give an example of a function on Ω that is not a random variable over $(\Omega, \mathcal{A})$.
4. Show that there exists a probability measure P on $(\Omega, \mathcal{A})$ such that $P(A)$ is zero or one for all $A \in \mathcal{A}$, yet P is not a point mass.

Let the sample space be $\Omega = \{1, 2, 3, 4, 5, 6\}$, and let $\mathcal{A} = \sigma(\{1, 2, 3, 4\}, \{3, 4, 5, 6\})$ be the σ-algebra generated by the sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

a) List all sets in $\mathcal{A}$

The σ-algebra $\mathcal{A}$ consists of all possible unions of its **atoms** (the smallest non-empty sets in $\mathcal{A}$), found by taking intersections of the generating sets and their complements:

- $C_1 = A \cap B^c = \{1, 2, 3, 4\} \cap \{1, 2\} = \{1, 2\}$
- $C_2 = A \cap B = \{1, 2, 3, 4\} \cap \{3, 4, 5, 6\} = \{3, 4\}$
- $C_3 = A^c \cap B = \{5, 6\} \cap \{3, 4, 5, 6\} = \{5, 6\}$

Since there are $k = 3$ non-empty atoms, $\mathcal{A}$ contains $2^3 = 8$ sets.

$$\mathcal{A} = \{\emptyset, \{1, 2\}, \{3, 4\}, \{5, 6\}, \{1, 2, 3, 4\}, \{3, 4, 5, 6\}, \{1, 2, 5, 6\}, \{1, 2, 3, 4, 5, 6\}\}$$

(Note: $\{1, 2, 3, 4\} = A$, $\{3, 4, 5, 6\} = B$, $\{1, 2, 5, 6\} = (A \cap B)^c$).

b) Is the function $X(\omega)$ a random variable over $(\Omega, \mathcal{A})$?

The function is:

$$X(\omega) = \begin{cases} 2 & \text{if } \omega \in \{1, 2, 3, 4\} \\ 7 & \text{if } \omega \in \{5, 6\} \end{cases}$$

X is a random variable if $X^{-1}(E) \in \mathcal{A}$ for every Borel set E. We check the preimages of the values in the range $\{2, 7\}$:

1. $X^{-1}(\{2\}) = \{1, 2, 3, 4\} = A$. Since $A \in \mathcal{A}$.
2. $X^{-1}(\{7\}) = \{5, 6\} = C_3$. Since $C_3 \in \mathcal{A}$.

Since the preimages of all values in the range belong to $\mathcal{A}$, **YES, X is a random variable** over $(\Omega, \mathcal{A})$.

c) Give an example of a function on Ω that is not a random variable over $(\Omega, \mathcal{A})$

A function Y is not a random variable if it distinguishes between elements within the same atom. The set $\{1\}$ is not in $\mathcal{A}$ because $\mathcal{A}$ cannot separate $\omega = 1$ from $\omega = 2$.

Define the function $Y(\omega)$ as:

$$Y(\omega) = \begin{cases} 10 & \text{if } \omega = 1 \\ 0 & \text{if } \omega \in \{2, 3, 4, 5, 6\} \end{cases}$$

The preimage of 10 is:

$$Y^{-1}(\{10\}) = \{1\}$$

Since $\{1\} \notin \mathcal{A}$, the condition for Y to be a random variable fails.

Therefore, **Y is not a random variable** over $(\Omega, \mathcal{A})$.

d) Show that there exists a probability P on $(\Omega, \mathcal{A})$ such that $P(A) \in \{0, 1\}$ for all $A \in \mathcal{A}$, yet P is not a point mass

A probability measure P is a **point mass** if $P(\{\omega_0\}) = 1$ for some single ω_0. To satisfy the conditions, P must be non-point mass on Ω but trivial (0 or 1) on $\mathcal{A}$. This is achieved by concentrating all mass on a single atom containing multiple points.

Construction of the Measure P: We concentrate all probability mass on the atom $C_1 = \{1, 2\}$, and distribute it equally between the two points:

$$P(\{\omega\}) = \begin{cases} 1/2 & \text{if } \omega = 1 \\ 1/2 & \text{if } \omega = 2 \\ 0 & \text{if } \omega \in \{3, 4, 5, 6\} \end{cases}$$

Condition 1: P is NOT a Point Mass Since $P(\{\omega\}) = 1/2 < 1$ for every $\omega \in \Omega$, $\boldsymbol{P}$ **is not a point mass**.

Condition 2: $P(A) \in \{0, 1\}$ for all $A \in \mathcal{A}$ We calculate the probability of the atoms:

- $P(C_1) = P(\{1\}) + P(\{2\}) = 1/2 + 1/2 = 1$
- $P(C_2) = 0$; $P(C_3) = 0$.

Any set $A \in \mathcal{A}$ is a union of the atoms.

- If A contains the atom $C_1 = \{1, 2\}$, then $P(A) = 1$.
- If A does not contain C_1, then $P(A) = 0$.

Thus, for all $A \in \mathcal{A}$, $P(A)$ is either **zero or one**. The coarseness of $\mathcal{A}$ prevents it from resolving the non-deterministic nature of P on Ω.

Problem 2

Let X be a random variable with $P(X > 0) > 0$. Prove that there is a $\delta > 0$ such that $P(X \geq \delta) > 0$.

Solution

Method 1: Put $A = \{X > 0\}$ and $A_n = \{X \geq 1/n\}$ for all $n \in \mathbb{N}$. Then, $A_n \uparrow A$ (or A_n converges to A from below), and using Continuity of Probabilities, we have $\lim_{n\to\infty} P(A_n) = P(A)$. Given that $P(A) > 0$, it follows that there must exist some $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \geq N$, we have $P(A_n) > 0$. In particular, $P(A_N) > 0$. We can therefore choose $\delta = 1/N > 0$. This completes the proof.

Method 2: Put $A = \{X > 0\}$ and $A_n = \{X \geq 1/n\}$ for all $n \in \mathbb{N}$. Then, $A = \bigcup_{n=1}^{\infty} A_n$. If for every $n \in \mathbb{N}$, we had $P(A_n) = 0$, then using a fundamental property of probability (countable subadditivity), we would have $P(A) = P(\bigcup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} P(A_n) = \sum_{n=1}^{\infty} 0 = 0$. This result, $P(A) = 0$, is a contradiction to our initial assumption that $P(X > 0) > 0$. Therefore, there must exist at least one $N \in \mathbb{N}$ such that $P(A_N) > 0$. We can then choose $\delta = 1/N > 0$. This concludes the proof.

Problem 3

A random variable X has a distribution function $F_X(x)$ given by:

$$F_X(x) = \begin{cases} 0 & x < 2 \\ 1/2 & 2 \le x < 4 \\ 3/4 & 4 \le x < 6 \\ 1 & x \ge 6 \end{cases}$$

Calculate $P(X = 2)$, $P(X = 4)$, and $P(X = 6)$.

Solution

The point probability at a point t is calculated by the jump in the distribution function: $P(X = t) = F_X(t) - \lim_{x \to t^-} F_X(x)$.

1. **For $t = 2$:**
$$P(X = 2) = F_X(2) - \lim_{x \to 2^-} F_X(x) = \frac{1}{2} - 0 = \mathbf{\frac{1}{2}}$$
2. **For $t = 4$:**
$$P(X = 4) = F_X(4) - \lim_{x \to 4^-} F_X(x) = \frac{3}{4} - \frac{1}{2} = \frac{3}{4} - \frac{2}{4} = \mathbf{\frac{1}{4}}$$
3. **For $t = 6$:**
$$P(X = 6) = F_X(6) - \lim_{x \to 6^-} F_X(x) = 1 - \frac{3}{4} = \mathbf{\frac{1}{4}}$$

Problem 4

Let $\Omega = [0,3]$ with probability uniform and $X(\omega)$ be the random variable given by:

$$X(\omega) = \begin{cases} \omega + 1 & \text{si } 0 \le \omega \le 1 \\ 2 & \text{si } 1 < \omega \le 1.5 \\ 3 & \text{si } 1.5 < \omega < 2 \\ -\omega + 4 & \text{si } 2 \le \omega \le 3 \end{cases}$$

Find $F_X(t)$, the distribution function of X.

Solution

The sample space is $\Omega = [0,3]$ with a uniform probability measure.

- Total Length: $|\Omega| = 3$.
- Probability of any event $A \subseteq \Omega$: $P(A) = \frac{|A|}{3}$.

The random variable $X(\omega)$ is defined piecewise over Ω:

$$X(\omega) = \begin{cases} \omega + 1 & \text{if } 0 \le \omega \le 1 \quad (\text{Interval } I_1) \\ 2 & \text{if } 1 < \omega \le 1.5 \quad (\text{Interval } I_2) \\ 3 & \text{if } 1.5 < \omega < 2 \quad (\text{Interval } I_3) \\ -\omega + 4 & \text{if } 2 \le \omega \le 3 \quad (\text{Interval } I_4) \end{cases}$$

We analyze the range of X and the probability mass contributed by each interval I_i:

1. **Interval I_1 (Continuous):** $\omega \in [0,1]$. Length $= 1$. $P(I_1) = 1/3$. $X(\omega) = \omega + 1$ covers the continuous range $[\mathbf{1}, \mathbf{2}]$.
2. **Interval I_2 (Point Mass at 2):** $\omega \in (1, 1.5]$. Length $= 0.5$. This contributes a **discrete mass** at $X = 2$: $P(X = 2 \text{ from } I_2) = P(I_2) = \frac{0.5}{3} = \mathbf{1/6}$.
3. **Interval I_3 (Point Mass at 3):** $\omega \in (1.5, 2)$. Length $= 0.5$. This contributes a **discrete mass** at $X = 3$: $P(X = 3) = P(I_3) = \frac{0.5}{3} = \mathbf{1/6}$.
4. **Interval I_4 (Continuous):** $\omega \in [2,3]$. Length $= 1$. $P(I_4) = 1/3$. $X(\omega) = -\omega + 4$ covers the continuous range $[\mathbf{1}, \mathbf{2}]$ (since $X(2) = 2$ and $X(3) = 1$).

The overall range of X is $[\mathbf{1}, \mathbf{2}] \cup \{\mathbf{3}\}$. The minimum value is 1, and the maximum is 3.
We find $F_X(t) = P(X \le t)$ by summing the continuous and discrete probabilities up to t.

Case 1: $t < 1$

Since 1 is the minimum value X can take, $P(X \le t) = 0$.

$$F_X(t) = 0$$

Case 2: $1 \le t < 2$

Probability accumulates from the two continuous segments, I_1 and I_4.

- **Contribution from I_1 ($X = \omega + 1$):** $X \le t \implies \omega + 1 \le t \implies \omega \le t - 1$. Since $t \in [1, 2)$, $t - 1 \in [0, 1)$, so $\omega \in [0, t - 1]$.

$$P(I_1 \text{ part}) = P(0 \le \omega \le t - 1) = \frac{t-1}{3}$$

- **Contribution from I_4 ($X = -\omega + 4$):** $X \le t \implies -\omega + 4 \le t \implies \omega \ge 4 - t$. Since $t \in [1, 2)$, $4 - t \in [2, 3)$, so $\omega \in [4 - t, 3]$.

$$P(I_4 \text{ part}) = P(4 - t \le \omega \le 3) = \frac{3 - (4 - t)}{3} = \frac{t-1}{3}$$

The total accumulated probability is:

$$F_X(t) = \frac{t-1}{3} + \frac{t-1}{3} = \mathbf{\frac{2(t-1)}{3}}$$

Case 3: $2 \le t < 3$

The CDF must include the total continuous mass plus the discrete mass at $X = 2$.

- Continuous mass up to 2: 2/3.
- Discrete mass at $X = 2$ (from I_2): 1/6.

For $t \in [2, 3)$, the CDF is constant at the new accumulated value:

$$F_X(t) = \frac{2}{3} + \frac{1}{6} = \frac{4}{6} + \frac{1}{6} = \mathbf{5/6}$$

Case 4: $t \ge 3$

The CDF must include the remaining discrete mass at $X = 3$.

- Mass accumulated before $t = 3$: 5/6.
- Discrete mass at $X = 3$ (from I_3): 1/6.

The total accumulated probability is:

$$F_X(t) = \frac{5}{6} + \frac{1}{6} = 1$$

$$F_X(t) = \begin{cases} 0 & \text{if } t < 1 \\ \frac{2(t-1)}{3} & \text{if } 1 \le t < 2 \\ 5/6 & \text{if } 2 \le t < 3 \\ 1 & \text{if } t \ge 3 \end{cases}$$