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Some global properties of a pair of coupled maps: Quasi-symmetry, periodicity, and synchronicity - F.E.Udwadia , N.Raju

Some global properties of a pair of coupled maps: Quasi-symmetry, periodicity, and synchronicity -

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Motivation

This Paper written by two students of the University of Southern California explains several Global Properties of a pair of Coupled Maps. We have looked at 1D Logistic and Exponential Maps in this course and through this paper the Authors are seen to investigate what all Properties will be shown if we couple two such Maps.

Quasi-symmetry, Periodicity and Synchronicity are studied through this paper. Several numerical Proofs of the Analytical Results are obtained.

Mathematical Principles

Symmetry

Symmetry in Physics is a Transformation that leaves a system fundamentally unchanged after the Transformation. Fundamental Laws in Physics will remain symmetrical whenever or wherever we are in the world. Concepts are studied in Symmetry of Time Translation and Spatial Translations in Physics. If the world was not guided by symmetry then there would be chaos which means everything is unpredictable and not repeating. Quasi- symmetry just shows how close the system's Global Behavior can approach Symmetry.

Periodicity

Periodicity is as the name suggests when a system's motion repeats in equal Intervals of time, when we can view a certain pattern in its Dynamics we can call it Periodic.

Synchronicity

Synchronicity is seen when we see that any two maps are becoming identical.

We have studied one-dimensional Discrete Time Map as follows:

$$x_{n+1} = f(x_n) \tag{1}$$

Logistic and Exponential Maps

Logistic Map is of the form

$$x_{n+1} = rx(1-x)$$

Exponential Map is of the form

$$x_{n+1} = xe^{r(1-x)}$$

where r is the Growth Parameter. It can be plotted as: Two dimensional Discrete- time system

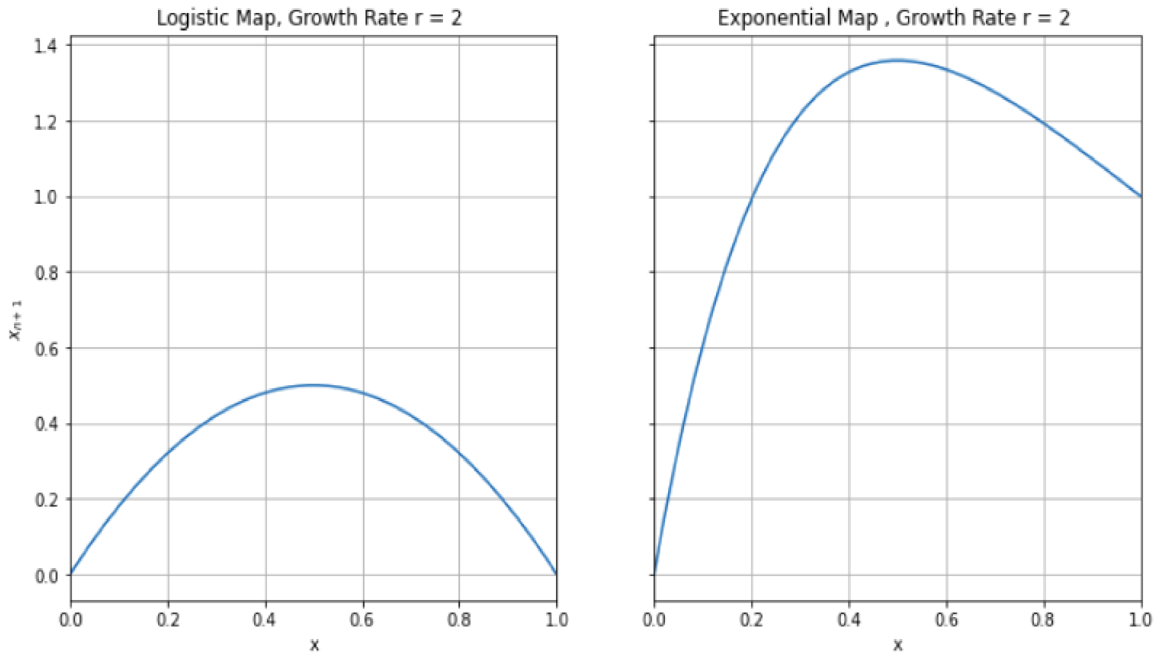

Figure 1: One Dimensional Discrete Map

is obtained by coupling two 1D Maps by a coupling parameter of d. d ranges from 0 to 1 with 1 resulting in no coupling and less than 1 having coupling.

$$\begin{aligned} x_{n+1} &= df(x_n) + (1-d)f(y_n) \\ y_{n+1} &= (1-d)f(x_n) + df(y_n) \end{aligned} \tag{2}$$

We need to study the Global Behavior of these Coupled Maps:

$$(x_{n+1}, y_{n+1}) = M(d) \circ (x_n, y_n) \tag{3}$$

Lyapunov Exponents of these systems are also studied.

Analytical Results

Result 1

For an orbit $(\hat{x_n}, \hat{y_n})$ where n = 0 ,1, 2, ... corresponding to a parameter d there is another orbit such that $(\hat{y_n}, \hat{x_n})$ where n = 0 ,1, 2, ...

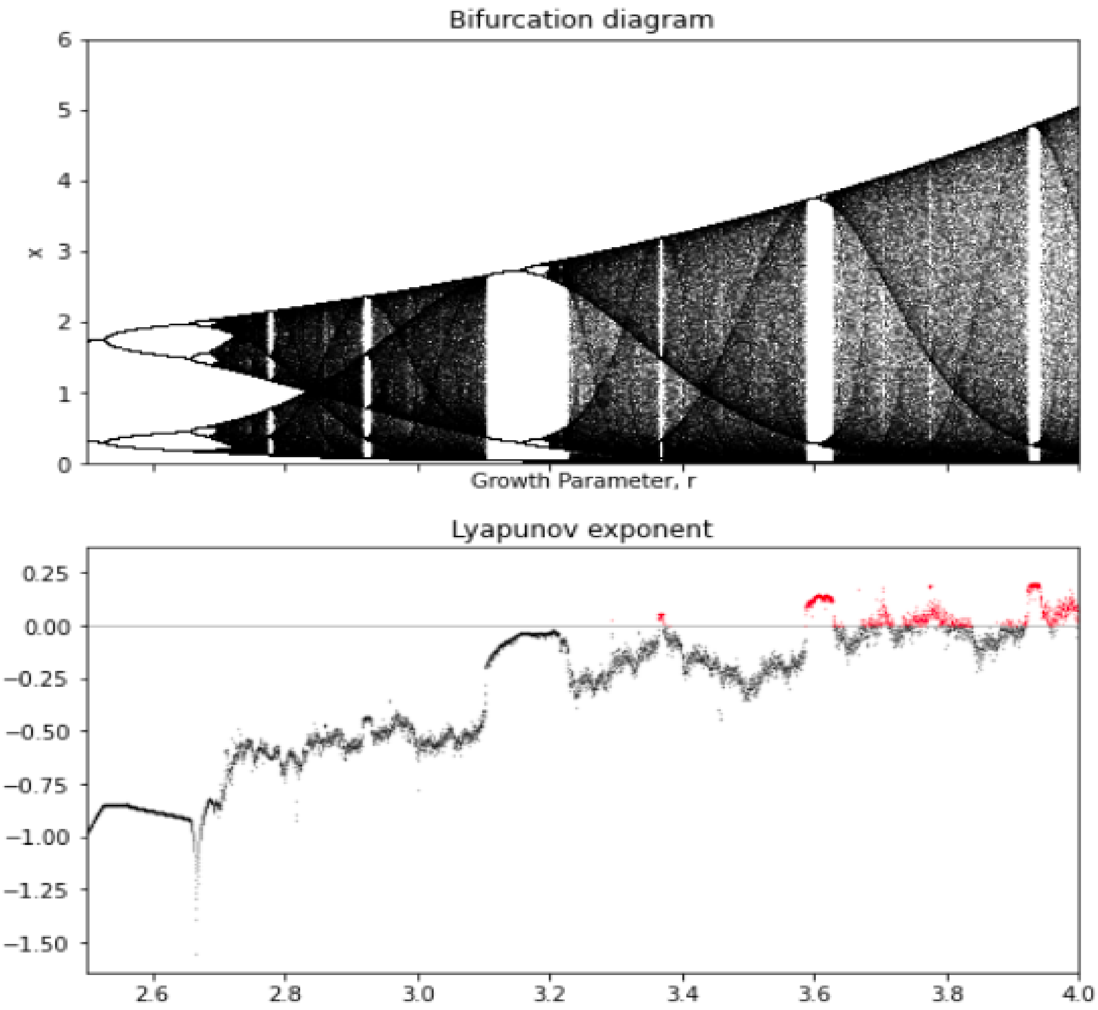

Figure 2: Exponential Bifurcation and Lyapunov Exponent Stability

Proof

On interchanging x_n and y_n in Equation 2 we get:

$$df(y_n) + (1-d)f(x_n) = y_{n+1}$$
$$(1-d)f(y_n) + df(x_n) = x_{n+1}$$

, if $(x_n, y_n) \implies (y_n, x_n)$ we will have $(x_{n+1}, y_{n+1}) \implies (y_{n+1}, x_{n+1})$. Extending this it can be said that suppose the orbits are periodic then for every n - periodic Orbit of the map described as $(\hat{x_n}, \hat{y_n})$, there exists another n- periodic orbit described as $(\hat{y_n}, \hat{x_n})$.

Result 2

Every second Orbit of $(\hat{x_n}, \hat{y_n})$ for a Map of d = $\frac{1}{2}$ + d_0 and a Map of d = $\frac{1}{2}$ - d_0 with $0 \leq d_0 \leq \frac{1}{2}$ will have the same coordinates.

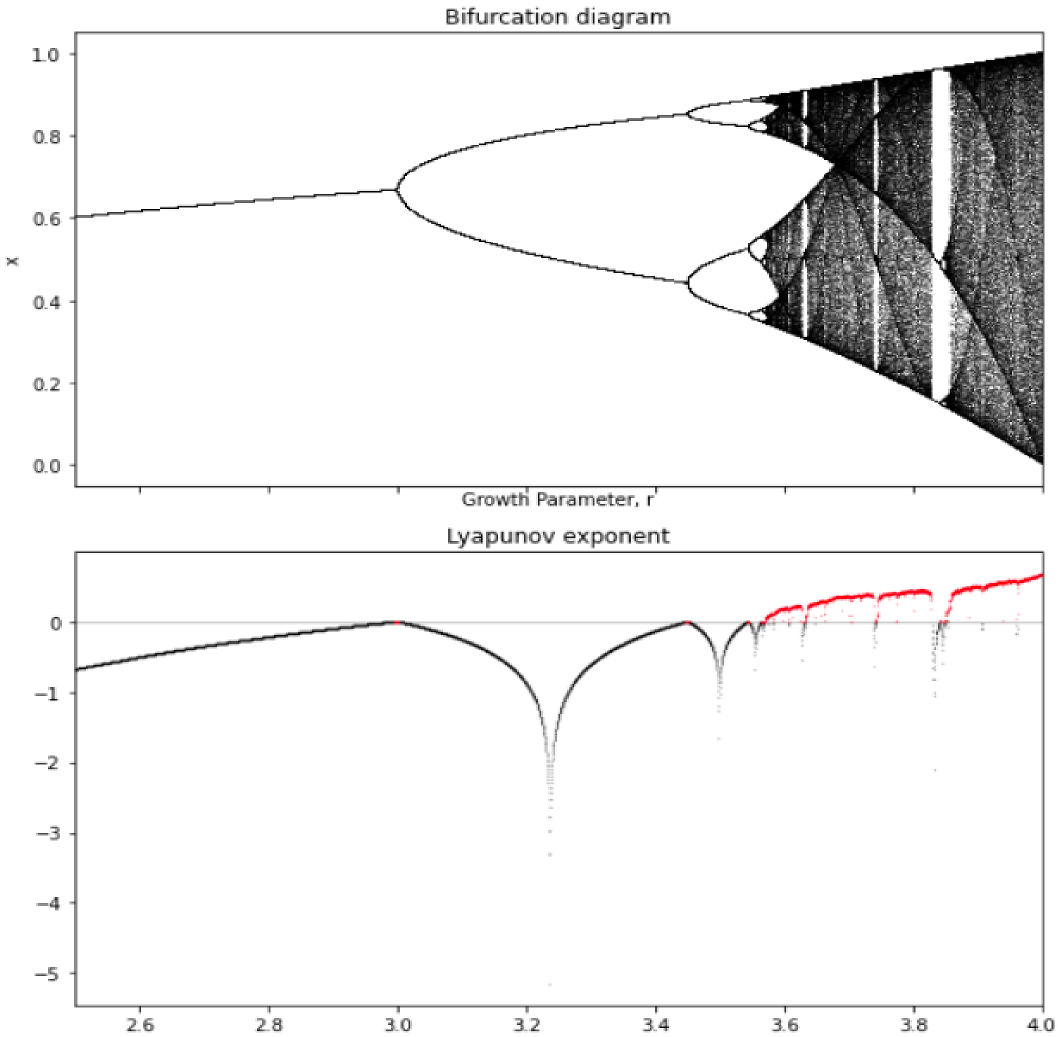

Figure 3: Logistic Bifurcation and Lyapunov Exponent Stability

Proof

$$
\begin{aligned}
M(\frac{1}{2} - d_0) \circ (\hat{x_n}, \hat{y_n}) &= (\hat{x_{n+1}}, \hat{y_{n+1}}) \\
M(\frac{1}{2} - d_0) \circ (\hat{x_{n+1}}, \hat{y_{n+1}}) &= (\hat{x_{n+2}}, \hat{y_{n+2}}) \\
M(\frac{1}{2} + d_0) \circ (\hat{x_n}, \hat{y_n}) &= (\hat{y_{n+1}}, \hat{x_{n+1}}) \\
M(\frac{1}{2} + d_0) \circ (\hat{y_{n+1}}, \hat{x_{n+1}}) &= (\hat{x_{n+2}}, \hat{y_{n+2}})
\end{aligned} \tag{4}
$$

For any 2n period Orbit of d = $\frac{1}{2}$ - d_0 starting from a point (x_0, y_0) , then there will be a Map of d = $\frac{1}{2}$ + d_0 which starts from the same point. Also it is seen that starting from the same point , if a Map with d = $\frac{1}{2}$ - d_0 has a Period of 2n - 1 then the Map with d = $\frac{1}{2}$ + d_0 will have a period 2(2n - 1).

Result 3

Orbits that begin on the diagonal in (x,y) phase space will lie entirely on the diagonal. Its dynamics is similar to that of 1D map.

Proof

$$\begin{aligned}
df(x_n) + (1-d)f(x_n) &= x_{n+1} \\
f(x_n) &= x_{n+1} \\
(1-d)f(x_n) + df(x_n) &= y_{n+1} \\
f(x_n) &= y_{n+1} \\
&= x_{n+1}
\end{aligned}$$

(x_n, x_n) is the fixed point of a 2D Map. Stability characteristics are same. Also its seen that if any two Nonlinear maps are coupled, the periodic behavior in the separate 1D Maps can be seen in the coupled one as well for all values of d.

Result 4

For d = $\frac{1}{2}$ the orbit of the map starting with (x_0, y_0) will consist of points of the form (x_n, x_n) after the every iteration as it is redistributing the sum of $f(x_n)$ and $f(y_n)$.

Proof

$$\begin{aligned}
\frac{1}{2}f(x_n) + \frac{1}{2}f(y_n) &= x_{n+1} \\
\frac{1}{2}f(y_n) + \frac{1}{2}f(x_n) &= y_{n+1} \\
&= x_{n+1}
\end{aligned}$$

Result 5

Each Lyapunov exponent of the map with d = $\frac{1}{2} - d_0$, $0 \leq d_0 \leq \frac{1}{2}$, starting from some (x_0, y_0) is the same as that of the map with d = $\frac{1}{2} + d_0$ starting from the same point (x_0, y_0).

Proof

We use Result 2 here. Lyapunov Exponent is given as :

$$\lambda_i = \lim_{n \to \infty} \frac{\log_e[|\mu_i|]}{n}$$

Jacobian Matrix of the Map is written as:

$$J_n[M(d)]_{(x_n, y_n)} = \begin{bmatrix} df'(x_n) & (1-d)f'(y_n) \\ (1-d)f'(x_n) & df'(y_n) \end{bmatrix} \tag{5}$$

μ_i are the eigen values of Product of the Jacobian Matrices taken at each Iteration. On solving for a Map of d = $\frac{1}{2} + d_0$ and Map of d = $\frac{1}{2} + d_0$. The dynamics of the coupled system is symmetric about d = $\frac{1}{2}$.

Result 6

When d = $\frac{1}{2}$, one Lyapunov exponent tends and to $-\infty$. Suppose d = $\frac{1}{2}$, the Jacobian is:

$$J_n[M(d)]_{(x_n, y_n)} = \begin{bmatrix} \frac{1}{2}f'(x_n) & \frac{1}{2}f'(y_n) \\ \frac{1}{2}f'(x_n) & \frac{1}{2}f'(y_n) \end{bmatrix} \tag{6}$$

For this one, determinant = 0 and hence the exponent tends to infinity.

These results can be called Universal as they do not depend on the kind of function (f) that explains the dynamics.

Numerical Results

First Result is that we can show how by coupling two chaotic systems we can arrive at complete equilibrium. Coupling can also synchronise non linear systems. For certain values of d we observe :

- Periodic Behavior arises in Coupled Logistic Maps
- Periodic Behavior arises in Coupled Exponential Maps
- Periodic Behaviour arises in Lyapunov Exponent Plots

Chaotic Response based on the Initial Condition

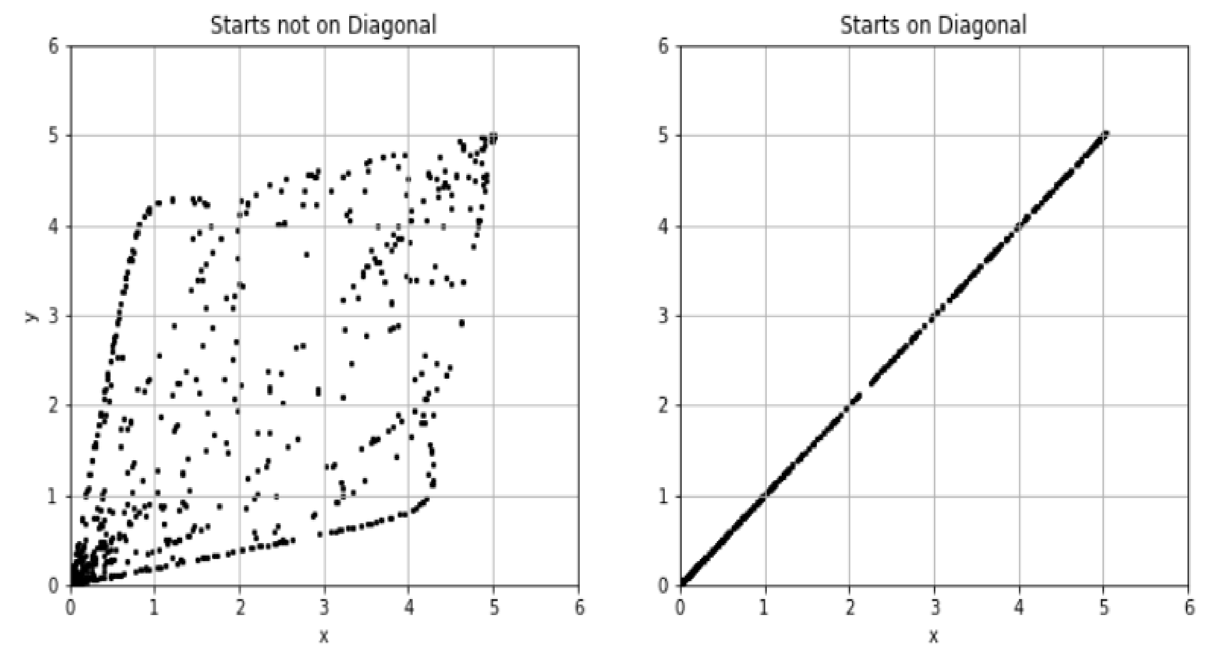

Figure 4: Chaotic Response for different Initial Conditions - Exponential Map

Inference

- For an initial condition which is not on the diagonal, the chaotic Response is spread over a region.
- For an Initial Condition which is on the Diagonal, the choatic Response will be limited to x = y , a 1D line.
- As the Dimension reduces, it can be called as a form of stabilisation. If the dimension again reduces to 0 , then it can be called Periodic.

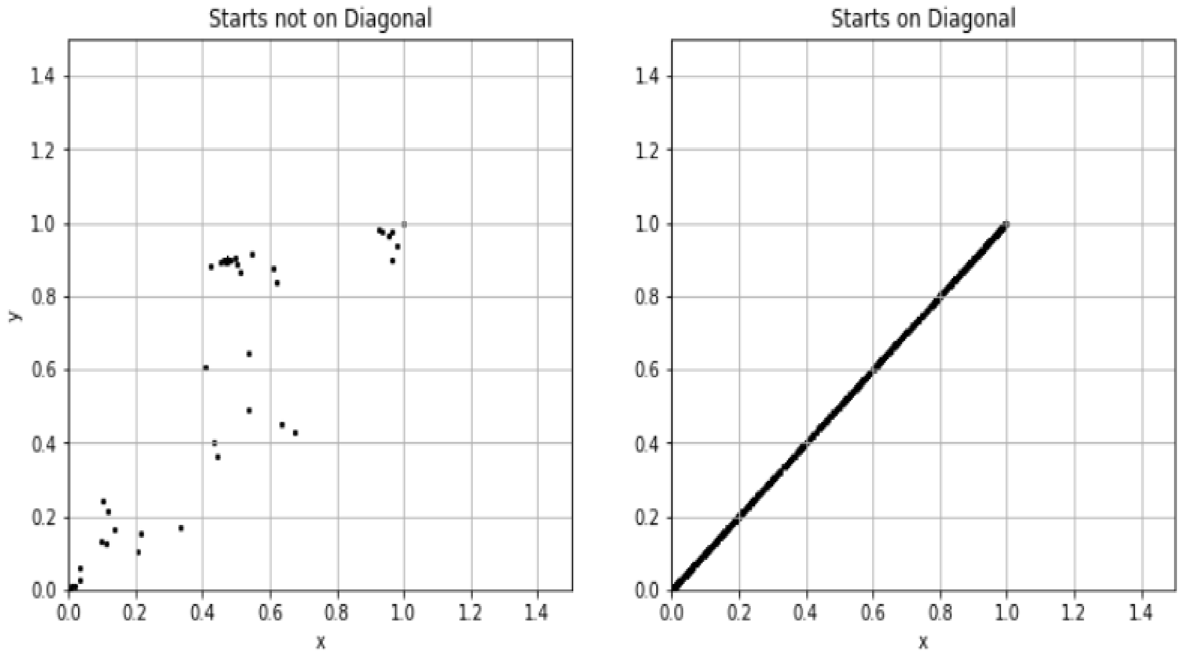

Figure 5: Chaotic Response for different Initial Conditions - Logistic Map

7 Zone Dynamics

Coupled maps can be divided into 7 Zones , where the transition from one to another will be marked by a Tangent or Hopf-like Bifurcation with respect to d. The 7 zone dynamics is explained by a series of bifurcations that are quasi-symmetric about d = $\frac{1}{2}$. Dynamics for d = $\frac{1}{2} - k$ and for d = $\frac{1}{2} + k$ where $0 \leq k \leq \frac{1}{2}$. It is seen that in areas where x-y = 0 is the region or the range of d over which the two systems become synchronous or identical.

Inferences

- Dynamics in Zones 5 , 6 and 7 are similar but not identical to 3 , 2, 1.
- Zone 1 and 7 are quasisymmetric Equivalent and they have Complex Dynamics. (Chaotic + Periodic Orbits)
- Zone 2 and 6 are quasisymmetric Equivalent and they have Periodic Behavior.
- Zone 3 and 5 are quasisymmetric Equivalent and they have Chaotic Behavior.
- Zone 4 is chaotic but synchronous, x is very close to y, thus stable.
- The Hopf bifurcation from zone 2 to zone 1, causes the trajectory to follow a single closed curve
- The Hopf bifurcation from zone 6 to 7 causes the trajectory to follow two closed curves
- The transition from zone 2 to 3 and that from zone 6 to 5 are both through a tangent bifurcation leading to an explosive kind of chaos.
- Hopf Bifurcation will mark the transition from Zone 2 to 1 or from Zone 6 to 7.

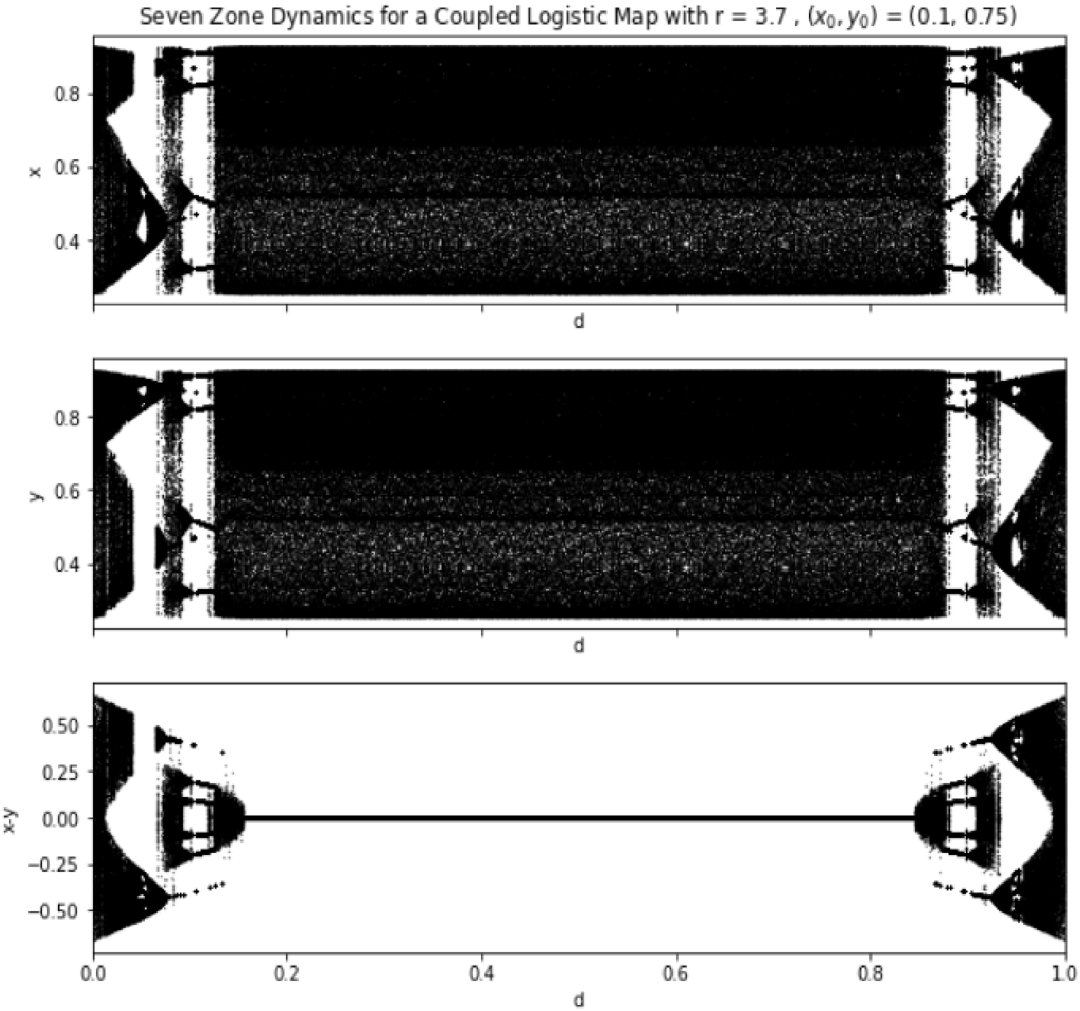

Figure 6: 7 Zone Dynamics (x-y) vs d for Exponential Maps

- Left figure shows the closed loop trajectory caused by a bifurcation of the one-period orbit from Zone 2 to Zone 1.
- Right Figure shows Hopf Bifurcation happening here for a two period Orbit from zone 6 to zone 7. This orbit alternates between the two closed loops. The loops are symmetric about the line x = y.
- Transition from zone 2 to 3 and from 6 to 5 are both about a tangent bifurcation which can lead to chaos.
- Zone 4 will be synchronous

Stability threshold for Zone 4 (Map is synchronous)

Theory

To get the range of d values for which Zone 4 is stable, we see the following:

$$ln|2d - 1| + \lambda = 0$$

where λ is the Largest Lyapunov Exponent. It can be seen in the following diagram. The is the boundary in which Zone 4 is stable.

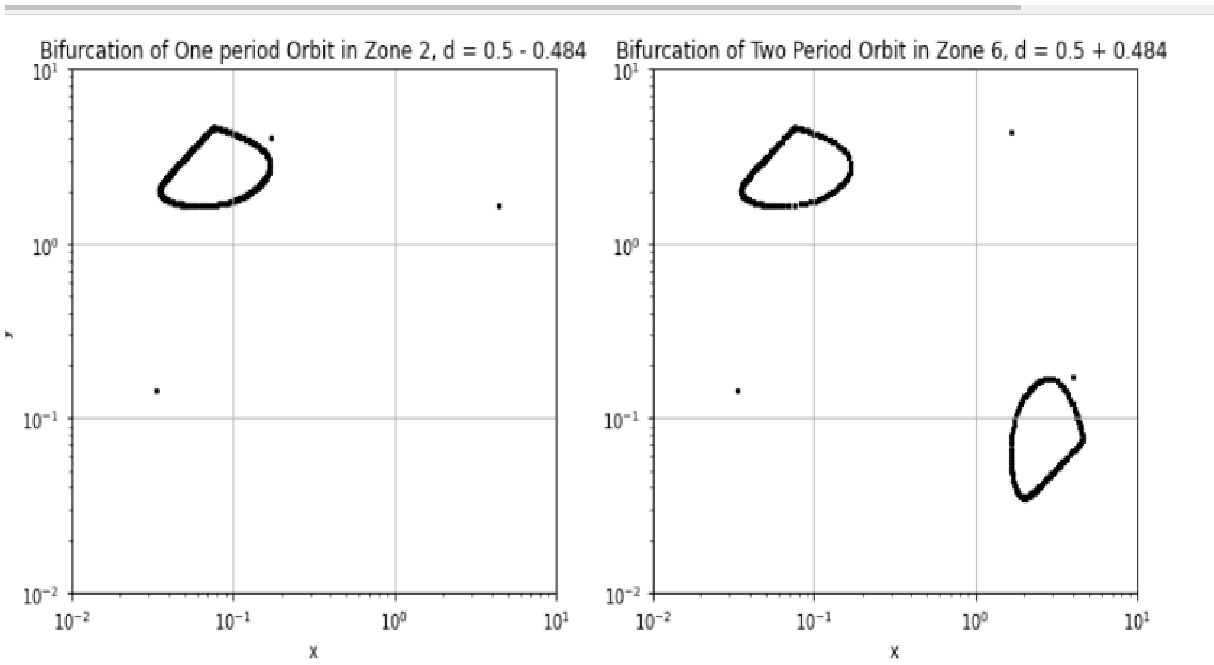

Figure 7: Hopf Bifurcation

Is it Universal?

To see the 7 Zone Dynamics' universality we look at the Logistics map as well. This indicates that this seven-zone dynamics may very well be universal for coupled maps wherein each map has negative Schwartzian curvature. Random Coupling of chaotic systems will most likely result in either a quasi- or complete stabilization as there are either Periodic or Synchronous Orbits mainly.

Exponential Maps

Synchronicity and Periodic Behavior For a random d what is the Probability of finding a periodic or synchronous trajectories. (Chaotic)

Result

The random coupling of two such chaotic units will most likely result in either a quasi- or a complete stabilization.

Analysis using Lyapunov Exponent

Initial Condition not on diagonal

There are some value of d where both exponents are negative. Hence the behavior in that region is stable. This shows the periodic Orbits.

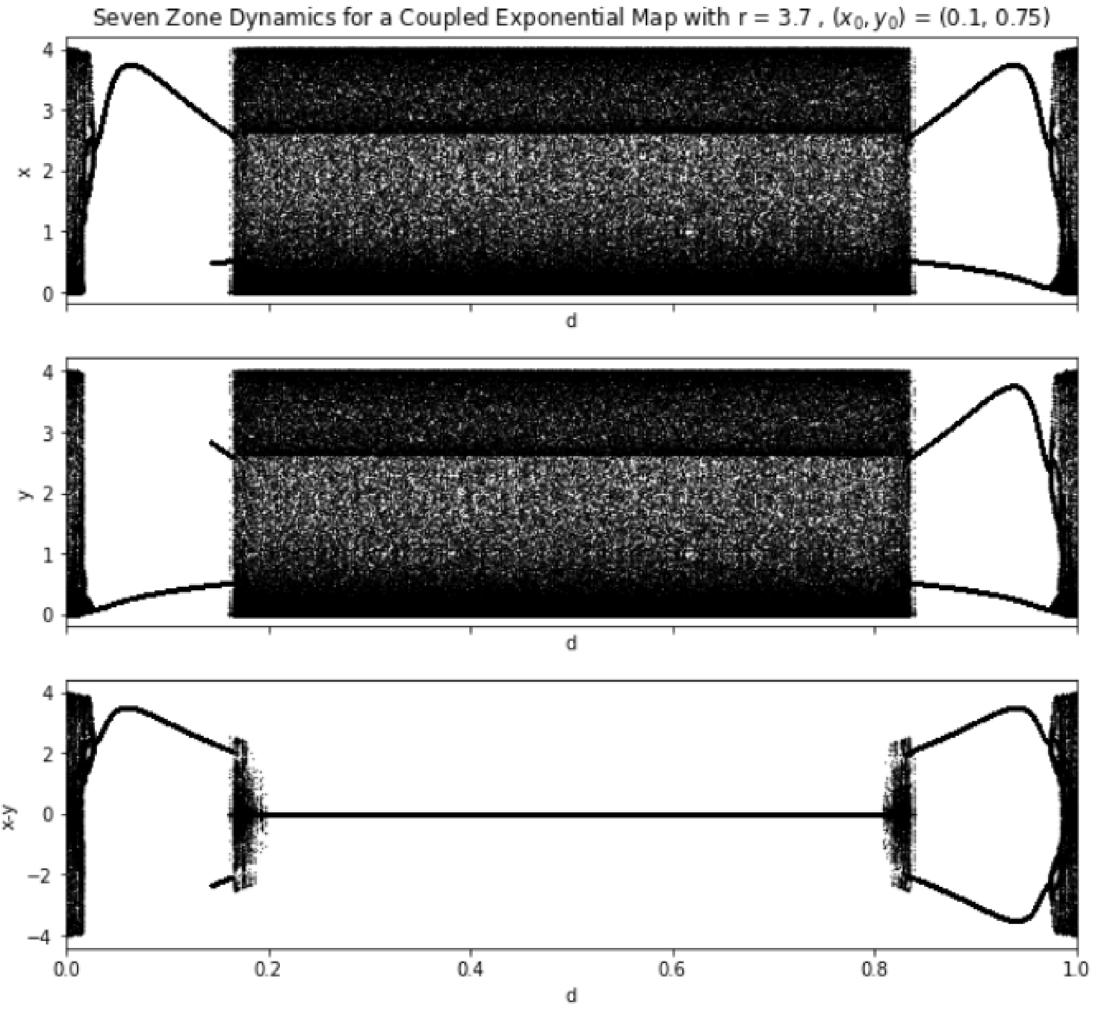

Figure 8: (x-y) vs d for Logistic Maps

Initial Condition on Diagonal

At least one exponent is positive such that the diagonal is chaotic.

In both cases, both the Lyapunov exponents are symmetric about d = $\frac{1}{2}$. But Dynamics is not fully symmeteric as the periodicity of orbits can be different.

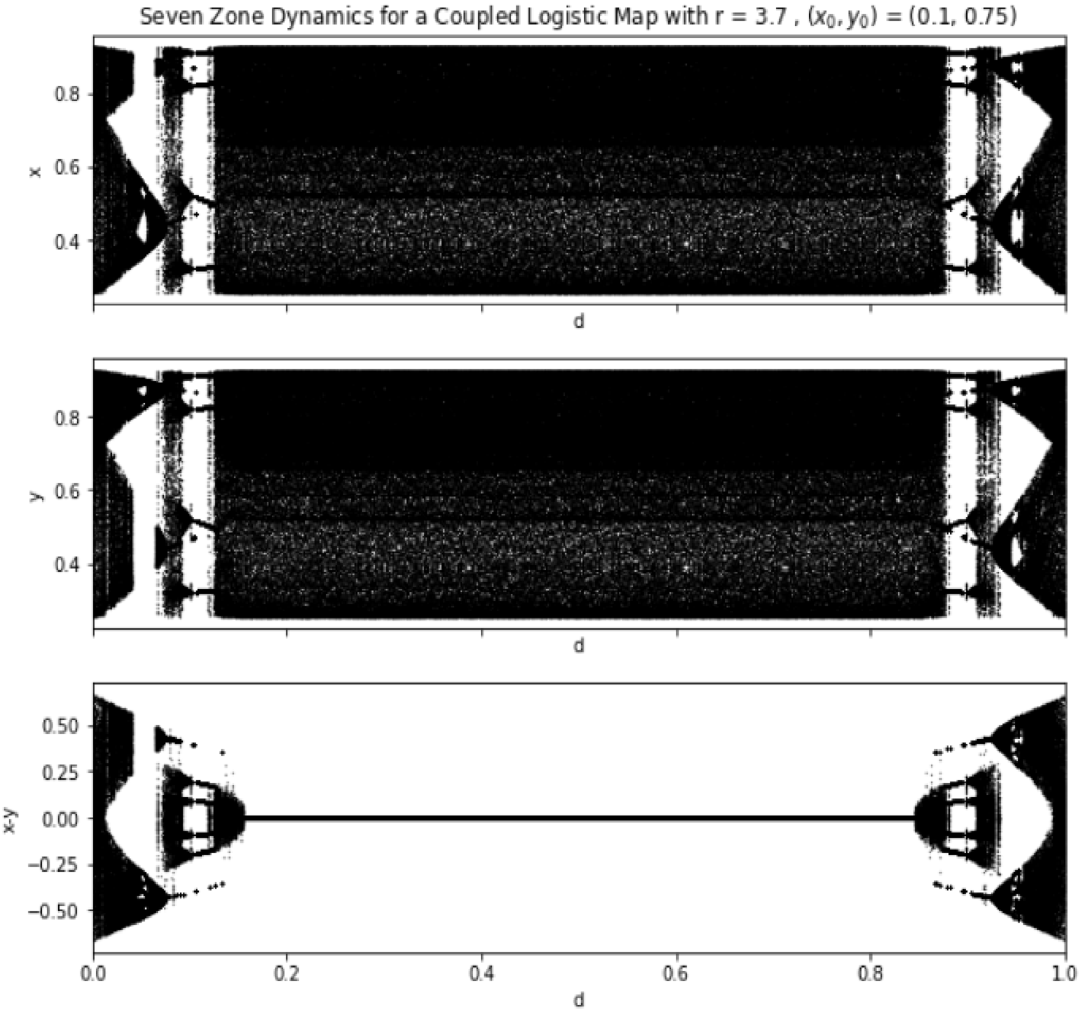

Figure 9: 7 zone Dynamics of Coupled Logistics Map

Summary

This paper mainly discussed the Global Properties of Coupled Maps. At first several Analytical Results were explained and mathematics behind it was understood. Later we can see them proved and justified using Plots. Numerical validation of the Analytics Results are seen. Python codes were written to visualise the same. The major points to take away from this paper is the following:

- Quasi-symmetry on Dynamics and Lyapunov Exponents about the coupling parameter d = $\frac{1}{2}$.
- Coupling can show either Periodic or Synchronous (but chaotic) behavior on Dynamics of Coupled Map.
- Two chaotic Systems can be coupled to bring about stability.
- Looked more into the Seven Zone Dynamics of the Coupled Maps.

Appendices

Logistic and Exponential Maps

```
import numpy as np
import matplotlib.pyplot as plt
import math

def exponential(r,x):
    return x * math.exp(r * (1-x))

def logistic(r,x):
    return (r * x * (1-x))

def plot(title1, title2, xlabel1, ylabel, xlabel2, xlim, ylim):
    ax1.set_title(title1)
    ax2.set_title(title2)
    ax1.grid()
    ax2.grid()
    ax1.set_xlabel(xlabel1)
    ax1.set_ylabel(ylabel)
    ax2.set_xlabel(xlabel2)
    ax1.set_ylim(xlim[0],xlim[1])
    ax2.set_ylim(ylim[0], ylim[1])
    ax1.set_xlim(xlim[0],xlim[1])
    ax2.set_xlim(ylim[0], ylim[1])

log = np.vectorize(logistic)
exp = np.vectorize(exponential)

x = np.linspace(0, 1)

fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 6), sharey=True)

ax1.grid()
ax1.set_xlim(0, 1)
ax1.plot(x, log(2,x))
ax1.set_title("Logistic Map, Growth Rate r = 2 ")
ax1.set_ylabel(r"$x_{n+1}$")
ax1.set_xlabel("x")

ax2.grid()
ax2.set_xlim(0, 1)
ax2.plot(x, exp(2,x))
ax2.set_xlabel("x")
ax2.set_title("Exponential Map , Growth Rate r = 2 ")
```

Logistic and Exponential Map - Bifurcation Diagram and Lyapunov Exponent

```
def twoDlogistic(r,x,y, d):
    return d * logistic(r,x) + (1-d)*logistic(r,y), (1-d)*logistic(r,x)+
    d*logistic(r,y)

def twoDexponential(r,x,y,d):
    return d * exponential(r,x) + (1-d)*exponential(r,y), (1-d)*exponential(r,x)+
    d*exponential(r,y)

log2D = np.vectorize(twoDlogistic)
exp2D = np.vectorize(twoDexponential)

iterations = 1000
last = 100

x = 1e-5 * np.ones(n)

lyapunov = np.zeros(n)
one      = np.ones(n)

fig, (ax1, ax2) = plt.subplots(2, 1, figsize=(8, 9), sharex=True)

for i in range(iterations):
    x = exp(r, x)

    #lyapunov += np.log(abs(r - 2 * r * x))
    lyapunov += np.log(abs(np.exp(r * (one-x)) * (r * x * (x-one)/(np.exp(1)*one))
                + one ))

    if i >= (iterations - last):
        ax1.plot(r, x, ',k', alpha=.25)

ax1.set_xlabel('Growth Parameter, r')
ax1.set_ylabel('x')
ax1.set_ylim(0,6)
ax1.set_xlim(2.5, 4)
ax1.set_title("Bifurcation diagram")

ax2.axhline(0, color='k', lw=.5, alpha=.5)

ax2.plot(r[lyapunov < 0], lyapunov[lyapunov < 0] / iterations,'.k'
          , alpha=.5, ms=.5)

ax2.plot(r[lyapunov >= 0], lyapunov[lyapunov >= 0] / iterations, '.r'
          , alpha=.5, ms=.5)

ax2.set_title("Lyapunov exponent")
```

Phase Plot

```
iterations = 1000
last = 100

# Starts not on the Diagonal
x1 = 1e-5
y1 = 5e-5

# Starts on the Diagonal
x2 = 1e-5
y2 = 1e-5

d1 = 0.16
d2 = 0.21

fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 5), sharex=True)

r = 4

x1_data_plot=[]
y1_data_plot=[]

x2_data_plot=[]
y2_data_plot=[]

for i in range(iterations):

    x1_data_plot.append(x1)
    y1_data_plot.append(y1)

    x2_data_plot.append(x2)
    y2_data_plot.append(y2)

    x1, y1 = twoDexponential(r, x1, y1 , d1)
    #x1, y1 = twoDlogistic(r, x1, y1 , d1)
    x2, y2 = twoDexponential(r, x2, y2 , d2)
    #x2, y2 = twoDlogistic(r, x2, y2 , d2)

ax1.scatter(x1_data_plot, y1_data_plot, s = 4.5, facecolor ='black')
ax2.scatter(x2_data_plot, y2_data_plot, s = 4.5, facecolor ='black')

plot("Starts not on Diagonal", "Starts on Diagonal","x", "y", "x", [0,6], [0,6])
```

Seven Zone Dynamics

```
n = 5000
iterations = 1000
last = 100
```

```
x = 0.1 * np.ones(n)
y = 0.75 * np.ones(n)

lyapunov = np.zeros(n)

d = np.linspace(0, 1, n)

fig, (ax1, ax2, ax3) = plt.subplots(3, 1, figsize=(10, 9), sharex=True)

r = 3.7 * np.ones(n)

for i in range(iterations):
    #x, y = exp2D(r, x, y , d)
    x, y = log2D(r, x, y , d)

    if i >= (iterations - last):
        ax1.scatter(d, x,  s = 0.05, facecolor = 'black')
        ax2.scatter(d, y,  s = 0.05, facecolor = 'black')
        ax3.scatter(d, x-y,  s = 0.05, facecolor = 'black')

ax1.set_xlabel('d')
ax1.set_xlim(0,1)
ax1.set_ylabel('x')

ax2.set_xlabel('d')
ax2.set_xlim(0,1)
ax2.set_ylabel('y')

ax3.set_xlabel('d')
ax3.set_xlim(0,1)
ax3.set_ylabel('x-y')

ax1.set_title(r"Seven Zone Dynamics for a Coupled Exponential Map with r = 3.7
, ($x_0,y_0$) = (0.1, 0.75)")
```

Transitions between Zones

```
iterations = 1000
last = 100

x1 = 1e-5
y1 = 5e-5

x2 = 1e-5
y2 = 5e-5

d1 = 0.5 - 0.484
d2 = 0.5 + 0.484
```

```
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 5), sharex=True)
r = 4

x1_data_plot=[]
y1_data_plot=[]

x2_data_plot=[]
y2_data_plot=[]

for i in range(iterations):

    x1_data_plot.append(x1)
    y1_data_plot.append(y1)

    x2_data_plot.append(x2)
    y2_data_plot.append(y2)

    x1, y1 = twoDexponential(r, x1, y1 , d1)
    x2, y2 = twoDexponential(r, x2, y2 , d2)

ax1.scatter(x1_data_plot, y1_data_plot, s = 4.5, facecolor ='black')
ax2.scatter(x2_data_plot, y2_data_plot, s = 4.5, facecolor ='black')

plot("Bifurcation of One period Orbit in Zone 2, d = 0.5 - 0.484" ,
"Bifurcation of Two Period Orbit in Zone 6, d = 0.5 + 0.484 ", "x", "y",
"x",[1e-2, 1e1],[1e-2, 1e1] )

ax1.set_yscale('log')
ax1.set_xscale('log')
ax2.set_yscale('log')
ax2.set_xscale('log')
```

References

Some global properties of a pair of coupled maps: Quasi-symmetry, periodicity, and synchronicity - F.E. Udwadia, N. RajuSome global properties of a pair of coupled maps: Quasi-symmetry, periodicity, and synchronicity - F.E. Udwadia, N. Raju