AS5590: Dynamic Soaring - Project Final Review

Dynamic Soaring Trajectory Generation using Optimal Control Indirect Method

Ramkumar RV (AE20D025) &
Izza (AE18B006)

Department of Aerospace Engineering
IIT Madras

June 25, 2021

Dynamic Soaring Trajectory

The dynamic soaring trajectory comprises of the following maneuvers in the wind shear:

- Climbing in the Windward Direction
- Performing a turn from windward to leeward Direction
- Descend in the Leeward Direction
- Perform a turn from leeward to Windward Direction

The equations of motion of the point mass model of the aircraft can be written as

$$\frac{dX}{dt} = \dot{X} = f\left(X, U, t\right) \tag{1}$$

where X is a vector of n state variables, U is a vector of m control variables and t is time

Equations of Motion in the Inertial frame

Dynamic Equations

The point mass equations of motion in the inertial frame are given as:

$$\ddot{x} = \dot{u_g} = -\cos\chi\cos\gamma\left(\frac{D}{m}\right) - (\cos\chi\sin\gamma\cos\mu + \sin\chi\sin\mu)\left(\frac{L}{m}\right) \quad (2)$$

$$\ddot{y} = \dot{v_g} = -\sin\chi\cos\gamma\left(\frac{D}{m}\right) - (\sin\chi\sin\gamma\cos\mu - \cos\chi\sin\mu)\left(\frac{L}{m}\right) \quad (3)$$

$$\ddot{z} = \dot{w_g} = \sin\gamma\left(\frac{D}{m}\right) - \cos\gamma\cos\mu\left(\frac{L}{m}\right) + g; \quad (4)$$

which can also be written as

$$\dot{u_g} = -a_{u1}\frac{D}{m} - a_{u2}\frac{L}{m}\ ;\ \dot{v_g} = -a_{v1}\frac{D}{m} - a_{v2}\frac{L}{m}\ ;\ \dot{w_g} = -a_{w1}\frac{D}{m} - a_{w2}\frac{L}{m} + g; \quad (5)$$

Equations of Motion in the Inertial frame (contd)

Kinematic Equations

The navigation equations are the following:

$$\dot{x} = u_g \tag{6}$$
$$\dot{y} = v_g \tag{7}$$
$$\dot{h} = -w_g \tag{8}$$

Aerodynamic Forces

The Aerodynamic forces are the Drag and Lift which are given as follows:

$$L = C_L \frac{1}{2} \rho V^2 S \tag{9}$$
$$D = C_D \frac{1}{2} \rho V^2 S = (C_{D_0} + K C_L^2) \frac{1}{2} \rho V^2 S \tag{10}$$

On Velocity and Angles

The aerodynamic forces are dependent on the airspeed $\vec{V}$ and the motion of the bird in the inertial frame is dependent on the inertial speed $\vec{V_g} = (u_g, v_g, w_g)^T$ and both are related to each other as follows:

$$\vec{V} = \vec{V_g} - \vec{V_w} \tag{11}$$

where $\vec{V_w} = (-V_w, 0, 0)$ considering only V_w component is only along $x-$ direction and therefore

$$\begin{aligned} \vec{V} &= (u_g + V_w, v_g, w_g)^T \\ V &= \sqrt{(u_g + V_w)^2 + v_g^2 + w_g^2} \end{aligned} \tag{12}$$

The angles γ and χ are defined as follows:

$$\sin\gamma = -\frac{w_g}{V} \quad ; \quad \tan\chi = \frac{v_g}{u_g + V_w} \tag{13}$$

The angle μ defines the bank angle of the lift vector.

On Wind Shear and Boundary conditions

Wind Shear

The wind shear model for the trajectory generation is considered as :

$$V_w = V_{\text{ref}} \left(\frac{h}{h_{\text{ref}}} \right)^p \tag{14}$$

where p is chosen to be 0.143 [Sachs,2005]

Boundary Conditions

Considering an energy neutral trajectory, the following are boundary conditions:

$$u_g(0) = u_g(t_{cyc})\ ;\ v_g(0) = v_g(t_{cyc})\ ;\ w_g(0) = w_g(t_{cyc}) \text{ and } h(0) = h(t_{cyc})$$
$$x(0) = x(t_{cyc})\ \&\ y(0) = y(t_{cyc}) \tag{15}$$

The last two conditions is due to the consideration of periodic trajectory.

Formulation

The performance index / objective function for sustaining the dynamic soaring is to minimize the required wind shear ($V_{W\,ref}$)

$$
\begin{aligned}
J =& \min_{(U,X,t_{cyc})} \; V_W ref \\
\text{subject to } \dot{X} &= f(X, U, t) \\
U_{min} &\leq U \leq U_{max} \\
X(0) &= X(t_{cyc})
\end{aligned}
\tag{16}
$$

where $X = (u_g, v_g, w_g, x_g, y_g, h)^T$ and $U = (C_L, \mu)^T$
The optimal control problem is to determine the control history, the state trajectory including the initial conditions and the cycle time which minimize the shear wind strength at the same time satisfying the dynamic equations, boundary conditions and the control constraints.

Formulation Contd

Introducing Hamiltonian $H = \lambda^T f$, where λ is the Lagrangian Multipliers, the necessary conditions for the optimality is arrived. Following are the necessary conditions:

$$\frac{dX}{dt} = \frac{\partial H}{\partial \lambda} = f(X, U, t) \quad ; \quad \frac{d\lambda}{dt} = -\left(\frac{\partial H}{\partial X}\right) \quad ; \quad \frac{\partial H}{\partial U} = 0 \quad ; \tag{17}$$

State Vector

$$X = [u_g, v_g, w_g, x_g, y_g, h]^T \tag{18}$$

Adjoint Variables

$$\lambda = [\lambda_u, \lambda_v, \lambda_w, \lambda_x, \lambda_y, \lambda_h]^T \tag{19}$$

Formulation Contd

$$f = \left[\left(-a_{u1}\frac{D}{m} - a_{u2}\frac{L}{m} \right), \left(-a_{v1}\frac{D}{m} - a_{v2}\frac{L}{m} \right), \left(-a_{w1}\frac{D}{m} - a_{w2}\frac{L}{m} \right), \right.$$

$$\left. u_g, v_g, -w_g \right]^T$$

and therefore Hamiltonian becomes

$$\begin{aligned} H = \lambda_u \left(-a_{u1}\frac{D}{m} - a_{u2}\frac{L}{m} \right) + \lambda_v \left(-a_{v1}\frac{D}{m} - a_{v2}\frac{L}{m} \right) \\ + \lambda_w \left(g - a_{w1}\frac{D}{m} - a_{w2}\frac{L}{m} \right) + \lambda_x u_g + \lambda_y v_g - \lambda_h h \end{aligned}$$

Formulation Contd

$$H = \left[\frac{\lambda_u}{m}(-a_{u1}D - a_{u2}L) + \frac{\lambda_v}{m}(-a_{v1}D - a_{v2}L)\right.$$
$$+\frac{\lambda_w}{m}(mg - a_{w1}D - a_{w2}L) + \lambda_x u_g + \lambda_y v_g - \lambda_h h$$

Costate Equations

$$\begin{aligned}\frac{d\lambda_u}{dt} = -\frac{\partial H}{\partial u_g} = \frac{\partial}{\partial u_g}(a_{u1}D + a_{u2}L)\frac{\lambda_u}{m} + \frac{\partial}{\partial u_g}(a_{v1}D + a_{v2}L)\frac{\lambda_v}{m} \\ + \frac{\partial}{\partial u_g}(a_{w1}D + a_{w2}L)\frac{\lambda_w}{m} - \lambda_x\end{aligned} \tag{20}$$

Formulation Contd

Costate Equations

$$\frac{d\lambda_v}{dt} = -\frac{\partial H}{\partial v_g} = \frac{\partial}{\partial v_g}(a_{u1}D + a_{u2}L)\frac{\lambda_u}{m} + \frac{\partial}{\partial v_g}(a_{v1}D + a_{v2}L)\frac{\lambda_v}{m} + \frac{\partial}{\partial v_g}(a_{w1}D + a_{w2}L)\frac{\lambda_w}{m} - \lambda_y \tag{21}$$

$$\frac{d\lambda_w}{dt} = -\frac{\partial H}{\partial w_g} = \frac{\partial}{\partial w_g}(a_{u1}D + a_{u2}L)\frac{\lambda_u}{m} + \frac{\partial}{\partial w_g}(a_{v1}D + a_{v2}L)\frac{\lambda_v}{m} + \frac{\partial}{\partial w_g}(a_{w1}D + a_{w2}L)\frac{\lambda_w}{m} - \lambda_h \tag{22}$$

$$\frac{d\lambda_x}{dt} = 0; \quad \frac{d\lambda_y}{dt} = 0 \tag{23}$$

Formulation Contd

Costate Equations

$$\frac{d\lambda_h}{dt} = -\frac{\partial H}{\partial h} = \Bigg[\frac{\partial}{\partial V_w}(a_{u1}D + a_{u2}L)\frac{\lambda_u}{m} + \frac{\partial}{\partial V_w}(a_{v1}D + a_{v2}L)\frac{\lambda_v}{m} + \frac{\partial}{\partial V_w}(a_{w1}D + a_{w2}L)\frac{\lambda_w}{m}\Bigg]\frac{dV_w}{dh} \tag{24}$$

Considering the periodicity properties,

$$\lambda_u(0) = \lambda_u(t_{cyc}), \quad \lambda_v(0) = \lambda_v(t_{cyc}), \quad \lambda_w(0) = \lambda_w(t_{cyc}) \tag{25}$$

In addition,

$$\lambda_x(t_{cyc}) = 0, \quad \lambda_y(t_{cyc}) = 0, \quad \lambda_h(t_{cyc}) = -1 \tag{26}$$

Formulation Contd

Optimal control Equation

$$\frac{\partial H}{\partial U} = 0 \implies \frac{\partial H}{\partial C_L} = 0 \quad \text{and} \quad \frac{\partial H}{\partial \mu} = 0 \tag{27}$$

Optimal Bank Angle Control

$$\tan(\mu)_{opt} = \frac{\lambda_u \sin\chi - \lambda_v \cos\chi}{\lambda_u \sin\gamma\cos\chi + \lambda_v \sin\gamma\sin\chi + \lambda_w \cos\gamma} \tag{28}$$

Optimal Lift Coefficient

$$(C_L)_{opt} = -\frac{1}{2k}\left[\frac{\text{Num}}{\text{Den}}\right] \tag{29}$$

$$\text{Num} = [\lambda_u(\cos\mu\sin\gamma\cos\chi - \sin\mu\sin\chi) + \lambda_v(\cos\mu\sin\gamma\sin\chi - \sin\mu\cos\chi)$$

$$+ \lambda_w \cos\mu\cos\gamma]$$

$$\text{Den} = [\lambda_u \cos\gamma\cos\chi + \lambda_v \cos\gamma\sin\chi - \lambda_w \sin\gamma]$$

Costate Equations Simplified

Given that:

$$\sin\gamma = \frac{-w_g}{V} = \frac{-w_g}{\sqrt{(u_g + V_w)^2 + v_g^2 + w_g^2}} \tag{30}$$

the partial derivatives have been found as follows:

Table: Derivatives of $\sin\gamma$ and $\cos\gamma$

$\frac{\partial(\sin\gamma)}{\partial u_g} = \frac{w_g(u_g+V_w)}{V^3}$	$\frac{\partial(\cos\gamma)}{\partial u_g} = \frac{w_g^2(u_g+V_w)}{V^3\sqrt{V^2-w_g^2}}$
$\frac{\partial(\sin\gamma)}{\partial v_g} = \frac{w_g v_g}{V^3}$	$\frac{\partial(\cos\gamma)}{\partial v_g} = \frac{w_g^2 v_g}{V^3\sqrt{V^2-w_g^2}}$
$\frac{\partial(\sin\gamma)}{\partial w_g} = -\frac{(V^2-w_g^2)}{V^3}$	$\frac{\partial(\cos\gamma)}{\partial w_g} = \frac{-w_g\sqrt{V^2-w_g^2}}{V^3}$
$\frac{\partial(\sin\gamma)}{\partial V_w} = \frac{w_g(u_g+V_w)}{V^3}$	$\frac{\partial(\cos\gamma)}{\partial V_w} = \frac{w_g^2(u_g+V_w)}{V^3\sqrt{V^2-w_g^2}}$

Simplification Continued

Given that :

$$\tan\chi = \frac{v_g}{(u_g + V_w)} \tag{31}$$

the partial derivatives have been found as follows:

Table: Derivatives of $\sin\chi$ and $\cos\chi$

$\frac{\partial(\sin\chi)}{\partial u_g} = \frac{-(u_g+V_w)v_g}{(V^2-w_g^2)\sqrt{V^2-w_g^2}}$	$\frac{\partial(\cos\chi)}{\partial u_g} = \frac{v_g^2}{(V^2-w_g^2)\sqrt{V^2-w_g^2}}$
$\frac{\partial(\sin\chi)}{\partial v_g} = \frac{(u_g+V_w)^2}{(V^2-w_g^2)\sqrt{V^2-w_g^2}}$	$\frac{\partial(\cos\chi)}{\partial v_g} = \frac{-(u_g+V_w)v_g}{(V^2-w_g^2)\sqrt{V^2-w_g^2}}$
$\frac{\partial(\sin\chi)}{\partial w_g} = 0.0$	$\frac{\partial(\cos\chi)}{\partial w_g} = 0.0$
$\frac{\partial(\sin\chi)}{\partial V_w} = \frac{-(u_g+V_w)v_g}{(V^2-w_g^2)\sqrt{V^2-w_g^2}}$	$\frac{\partial(\cos\chi)}{\partial V_w} = \frac{v_g^2}{(V^2-w_g^2)\sqrt{V^2-w_g^2}}$

These equations are then used to get the partial derivatives of a_{u1} , a_{u2}, a_{v1}, a_{v2}, a_{w1} and a_{w2}.

Continued

From Lift and Drag we have:

Table: Derivatives of L and D

$\frac{\partial L}{\partial u_g} = C_L \rho (u_g + V_w) S$	$\frac{\partial D}{\partial u_g} = C_D \rho (u_g + V_w) S$
$\frac{\partial L}{\partial v_g} = C_L \rho v_g S$	$\frac{\partial D}{\partial v_g} = C_D \rho v_g S$
$\frac{\partial L}{\partial w_g} = C_L \rho w_g S$	$\frac{\partial D}{\partial w_g} = C_D \rho w_g S$
$\frac{\partial L}{\partial V_w} = C_L \rho (u_g + V_w) S$	$\frac{\partial D}{\partial V_w} = C_D \rho (u_g + V_w) S$

Continued

$\frac{\partial a_{u1}}{\partial u_g}$	$\frac{w_g^2(u_g+V_w)^2}{(V^2-w_g^2)V^3}+\frac{v_g^2}{V(V^2-w_g^2)}$
$\frac{\partial a_{u1}}{\partial v_g}$	$\frac{(u_g+V_w)w_g^2 v_g}{V^3(V^2-w_g^2)}-\frac{(u_g+V_w)v_g}{V(V^2-w_g^2)}$
$\frac{\partial a_{u1}}{\partial w_g}$	$\frac{-w_g(u_g+V_w)}{V^3}$
$\frac{\partial a_{u1}}{\partial V_w}$	$\frac{w_g^2(u_g+V_w)^2}{(V^2-w_g^2)V^3}+\frac{v_g^2}{V(V^2-w_g^2)}$

Table: Partial Derivatives of a_{u1}

$\frac{\partial a_{u2}}{\partial u_g}$	$\cos\mu\left(\frac{w_g(u_g+V_w)^2}{V^3(\sqrt{V^2-w_g^2})}-\frac{w_g v_g^2}{V(V^2-w_g^2)^{3/2}}\right)-\sin\mu\frac{(u_g+V_w)v_g}{(V^2-w_g^2)^{3/2}}$
$\frac{\partial a_{u2}}{\partial v_g}$	$\cos\mu\left(\frac{w_g v_g(u_g+V_w)}{V^3(\sqrt{V^2-w_g^2})}+\frac{w_g(u_g+V_w)v_g}{V(V^2-w_g^2)^{3/2}}\right)+\sin\mu\frac{(u_g+V_w)^2}{(V^2-w_g^2)^{3/2}}$
$\frac{\partial a_{u2}}{\partial w_g}$	$\cos\mu\left(-\frac{\sqrt{V^2-w_g^2}(u_g+V_w)}{V^3}\right)$
$\frac{\partial a_{u2}}{\partial V_w}$	$\cos\mu\left(\frac{w_g(u_g+V_w)^2}{V^3(\sqrt{V^2-w_g^2})}-\frac{w_g v_g^2}{V(V^2-w_g^2)^{3/2}}\right)-\sin\mu\frac{(u_g+V_w)v_g}{(V^2-w_g^2)^{3/2}}$

Table: Partial Derivatives of a_{u2}

Continued

$\frac{\partial a_{v1}}{\partial u_g}$	$\frac{v_g w_g^2 (u_g + V_w)}{(V^2 - w_g^2) V^3} - \frac{(u_g + V_w) v_g}{V (V^2 - w_g^2)}$
$\frac{\partial a_{v1}}{\partial v_g}$	$\frac{{v_g}^2 w_g^2}{V^3 (V^2 - w_g^2)} + \frac{(u_g + V_w)^2}{V (V^2 - w_g^2)}$
$\frac{\partial a_{v1}}{\partial w_g}$	$\frac{-w_g v_g}{V^3}$
$\frac{\partial a_{v1}}{\partial V_w}$	$\frac{v_g w_g^2 (u_g + V_w)}{V^3 (V^2 - w_g^2)} - \frac{(u_g + V_w) v_g}{V (V^2 - w_g^2)}$

Table: Partial Derivatives of a_{v1}

$\frac{\partial a_{v2}}{\partial u_g}$	$\cos\mu \left(\frac{v_g w_g (u_g + V_w)}{V^3 \sqrt{V^2 - w_g^2}} + \frac{w_g (u_g + V_w) v_g}{V (V^2 - w_g^2)^{3/2}} \right) + \sin\mu \frac{v_g^2}{(V^2 - w_g^2)^{3/2}}$
$\frac{\partial a_{v2}}{\partial v_g}$	$\cos\mu \left(\frac{v_g^2 w_g}{V^3 \sqrt{V^2 - w_g^2}} - \frac{w_g (u_g + V_w)^2}{V (V^2 - w_g^2)^{3/2}} \right) - \sin\mu \frac{(u_g + V_w) v_g}{(V^2 - w_g^2)^{3/2}}$
$\frac{\partial a_{v2}}{\partial w_g}$	$\cos\mu \left(- \frac{v_g \sqrt{V^2 - w_g^2}}{V^3} \right)$
$\frac{\partial a_{v2}}{\partial V_w}$	$\cos\mu \left(\frac{v_g w_g (u_g + V_w)}{\sqrt{V^2 - w_g^2} V^3} + \frac{w_g (u_g + V_w) v_g}{V (V^2 - w_g^2)^{3/2}} \right) + \sin\mu \frac{v_g^2}{(V^2 - w_g^2)^{3/2}}$

Table: Partial Derivatives of a_{v2}

Continued

Table: Partial Derivatives of a_{w1} and a_{w2}

$\frac{\partial a_{w1}}{\partial u_g}$	$-\frac{w_g(u_g+V_w)}{V^3}$
$\frac{\partial a_{w1}}{\partial v_g}$	$-\frac{w_g v_g}{V^3}$
$\frac{\partial a_{w1}}{\partial w_g}$	$\frac{(V^2-w_g^2)}{V^3}$
$\frac{\partial a_{w1}}{\partial V_w}$	$-\frac{w_g(u_g+V_w)}{V^3}$

$\frac{\partial a_{w2}}{\partial u_g}$	$\cos\mu \frac{w_g^2(u_g+V_w)}{V^3\sqrt{V^2-w_g^2}}$
$\frac{\partial a_{w2}}{\partial v_g}$	$\cos\mu \frac{w_g^2 v_g}{V^3\sqrt{V^2-w_g^2}}$
$\frac{\partial a_{w2}}{\partial w_g}$	$\cos\mu \frac{-w_g\sqrt{V^2-w_g^2}}{V^3}$
$\frac{\partial a_{w2}}{\partial V_w}$	$\cos\mu \frac{w_g^2(u_g+V_w)}{V^3\sqrt{V^2-w_g^2}}$

Costate Equations

$\dot{\lambda} = -\frac{\partial H}{\partial x}$

$$\frac{d\lambda_u}{dt} = -\frac{\partial H}{\partial u_g} = \left[\frac{\partial}{\partial u_g}(a_{u1})D + a_{u1}\frac{\partial D}{\partial u_g} + \frac{\partial}{\partial u_g}(a_{u2})L + a_{u2}\frac{\partial L}{\partial u_g}\right]\frac{\lambda_u}{m}$$
$$+\left[\frac{\partial}{\partial u_g}(a_{v1})D + a_{v1}\frac{\partial D}{\partial u_g} + \frac{\partial}{\partial u_g}(a_{v2})L + a_{v2}\frac{\partial L}{\partial u_g}\right]\frac{\lambda_v}{m}$$
$$+\left[\frac{\partial}{\partial u_g}(a_{w1})D + a_{w1}\frac{\partial D}{\partial u_g} + \frac{\partial}{\partial u_g}(a_{w2})L + a_{w2}\frac{\partial L}{\partial u_g}\right]\frac{\lambda_v}{m} - \lambda_x \quad (32)$$

Similarly all the co-state dynamic equations are derived.

Indirect Method Solution Procedure

Procedure

The Indirect Method solution procedure adopted is as follows:

1. Divide the interval $[t_0, t_f]$ into N equally spaced points.
2. Choose an initial Guess for the control (Both C_L and μ) at all the N points
3. Select an Initial Values of the states
4. Solve the state equations using ODE-45 from $[t_0, t_f]$
5. Solve the costate equations using ODE-45 from $[t_f, t_0]$ using the states at all the points for the entire time in the reverse order.
6. Update the control using the control equation.
7. Repeat the process, until the variation in the control is converged. Resulting control is the optimized control history.

Characteristics and Initial Conditions Considered

Aircraft

mass	9.0 kg
S	0.65 m^2
CD0	0.033
K	0.019
h_{ref}	10 m
V_{wref}	8.6 m/s
p	0.143

Table: Characteristics of Aircraft

The Initial conditions for the state vector and conditions of the control vector (final)

$$x_0 = [-15 \quad -20 \quad -15 \quad 15 \quad 15 \quad 0.5]$$

$$\lambda_f = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1]$$

Remarks

- The state dynamics equations of motion in the inertial frame is derived.
- The equations for the costate dynamics is derived.
- The state vector is time marched using ODE45 with the initial conditions and guessed control.
- Using the state and the control, the costate is time marched in the reverse order from tf to t0.
- Control vector is not updated - to be performed.
- Further work to be performed to obtain the initial conditions, minimum wind shear for energy neutral DS trajectory, optimal control history by implementing proper switching conditions for the state (height) and control constraints.

Various Plots

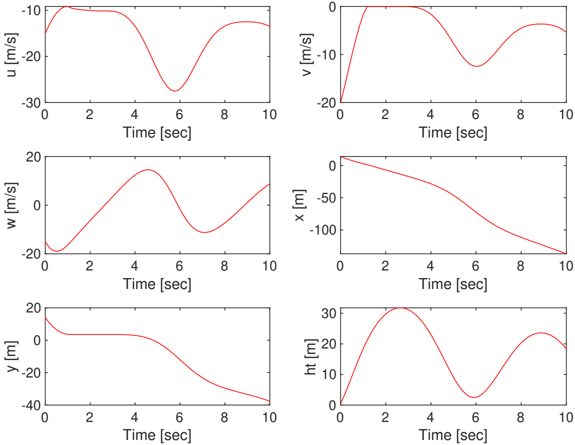

Figure: States Trajectory

Various Plots contd.

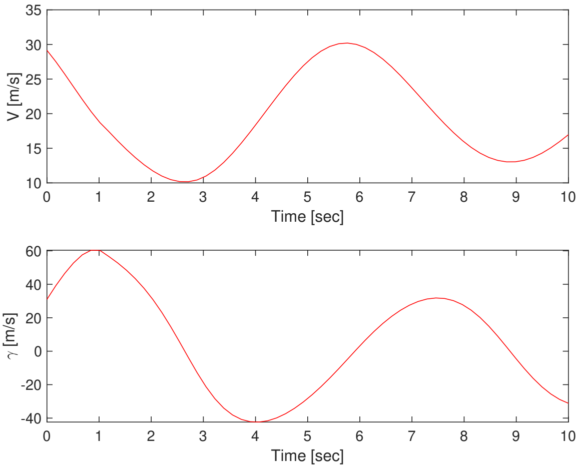

Figure: Velocity and Flight Path Angle

Various Plots contd.

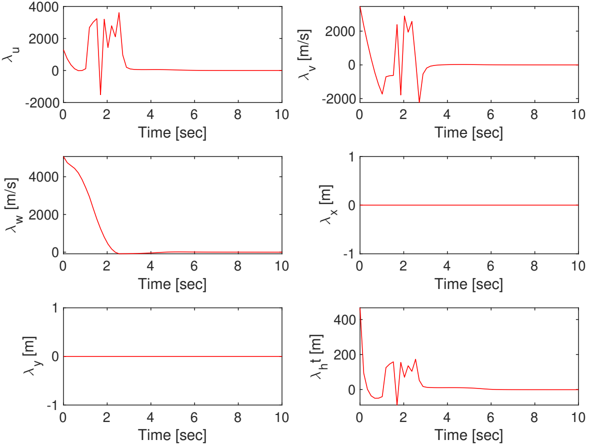

Figure: Adjoint Variable Trajectory

References

Application of Optimal Control Theory to Dynamic Soaring of Seabirds
Sachs, G. and Bussotti, P.
Variational Analysis and Applications pp 975-994.

Optimal Control Systems
D.Subbaram Naidu, 2003

Thank you