对于下列系统，试判断系统是否是(1)稳定的，(2)因果的，(3)线性的，(4)时不变的，和(5)无记忆的。

(a) $T(x[n]) = g[n]x[n]$, $g[n]$ 已知

(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$

(c) $T(x[n]) = \sum_{k=n-n_0}^{n+n_0} x[k]$

(d) $T(x[n]) = x[n-n_0]$

(e) $T(x[n]) = e^{x[n]}$

(f) $T(x[n]) = ax[n] + b$

(g) $T(x[n]) = x[-n]$

(h) $T(x[n]) = x[n] + 3u[n+1]$

2.1. (a) $T(x[n]) = g[n]x[n]$

- Stable: Let $|x[n]| \le M$ then $|T[x[n]| \le |g[n]|M$. So, it is stable if $|g[n]|$ is bounded.
- Causal: $y_1[n] = g[n]x_1[n]$ and $y_2[n] = g[n]x_2[n]$, so if $x_1[n] = x_2[n]$ for all $n < n_0$, then $y_1[n] = y_2[n]$ for all $n < n_0$, and the system is causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= g[n](ax_1[n] + bx_2[n] \\ &= ag[n]x_1[n] + bg[n]x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  So this is linear.
- Not time-invariant:

$$\begin{aligned} T(x[n-n_0]) &= g[n]x[n-n_0] \\ &\neq y[n-n_0] = g[n-n_0]x[n-n_0] \end{aligned}$$

  which is not TI.
- Memoryless: $y[n] = T(x[n])$ depends only on the n^{th} value of x, so it is memoryless.

(b) $T(x[n]) = \sum_{k=n_0}^{n} x[k]$

- Not Stable: $|x[n]| \le M \to |T(x[n])| \le \sum_{k=n_0}^{n} |x[k]| \le |n - n_0|M$. As $n \to \infty$, $T \to \infty$, so not stable.
- Not Causal: $T(x[n])$ depends on the future values of $x[n]$ when $n < n_0$, so this is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n_0}^{n} ax_1[k] + bx_2[k] \\ &= a\sum_{k=n_0}^{n} x_1[n] + b\sum_{k=n_0}^{n} x_2[n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  The system is linear.
- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= \sum_{k=n_0}^{n} x[k-n_0] \\ &= \sum_{k=0}^{n-n_0} x[k] \\ &\neq y[n-n_0] = \sum_{k=n_0}^{n-n_0} x[k] \end{aligned}$$

  The system is not TI.
- Not Memoryless: Values of $y[n]$ depend on past values for $n > n_0$, so this is not memoryless.

(c) $T(x[n]) \sum_{k=n-n_0}^{n+n_0} x[k]$

- Stable: $|T(x[n])| \le \sum_{k=n-n_0}^{n+n_0} |x[k]| \le \sum_{k=n-n_0}^{n+n_0} x[k]M \le |2n_0 + 1|M$ for $|x[n]| \le M$, so it is stable.
- Not Causal: $T(x[n])$ depends on future values of $x[n]$, so it is not causal.

(d) $T(x[n]) = x[n - n_0]$

- Stable: $|T(x[n])| = |x[n - n_0]| \le M$ if $|x[n] \le M$, so stable.
- Causality: If $n_0 \ge 0$, this is causal, otherwise it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[n - n_0] + bx_2[n - n_0] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is linear.
- TI: $T(x[n - n_d] = x[n - n_0 - n_d] = y[n - n_d]$. This is TI.
- Not memoryless: Unless $n_0 = 0$, this is not memoryless.

(e) $T(x[n]) = e^{x[n]}$

- Stable: $|x[n]| \le M$, $|T(x[n])| = |e^{x[n]}| \le e^{|x[n]|} \le e^M$, this is stable.
- Causal: It doesn't use future values of $x[n]$, so it causal.
- Not linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= e^{ax_1[n] + bx_2[n]} \\ &= e^{ax_1[n]} e^{bx_2[n]} \\ &\ne aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

  This is not linear.
- TI: $T(x[n - n_0]) = e^{x[n - n_0]} = y[n - n_0]$, so this is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of x only, so it is memoryless.

(f) $T(x[n]) = ax[n] + b$

- Stable: $|T(x[n])| = |ax[n] + b| \le a|M| + |b|$, which is stable for finite a and b.
- Causal: This doesn't use future values of $x[n]$, so it is causal.
- Not linear:

$$\begin{aligned} T(cx_1[n] + dx_2[n]) &= acx_1[n] + adx_2[n] + b \\ &\ne cT(x_1[n]) + dT(x_2[n]) \end{aligned}$$

  This is not linear.

- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= \sum_{k=n-n_0}^{n+n_0} ax_1[k] + bx_2[k] \\ &= a \sum_{k=n-n_0}^{n+n_0} x_1[k] + b \sum_{k=n-n_0}^{n+n_0} x_2[k] = aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

This is linear.

- TI:

$$\begin{aligned} T(x[n-n_0] &= \sum_{k=n-n_0}^{n+n_0} x[k-n_0] \\ &= \sum_{k=n-n_0}^{n} x[k] \\ &= y[n-n_0] \end{aligned}$$

This is TI.

- Not memoryless: The values of $y[n]$ depend on $2n_0$ other values of x, not memoryless.

- TI: $T(x[n-n_0]) = ax[n-n_0] + b = y[n-n_0]$. It is TI.
- Memoryless: $y[n]$ depends on the n^{th} value of $x[n]$ only, so it is memoryless.

(g) $T(x[n]) = x[-n]$

- Stable: $|T(x[n])| \leq |x[-n]| \leq M$, so it is stable.
- Not causal: For $n < 0$, it depends on the future value of $x[n]$, so it is not causal.
- Linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[-n] + bx_2x[-n] \\ &= aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

This is linear.

- Not TI:

$$\begin{aligned} T(x[n-n_0]) &= x[-n-n_0] \\ &\neq y[n-n_0] = x[-n+n_0] \end{aligned}$$

This is not TI.

- Not memoryless: For $n \neq 0$, it depends on a value of x other than the n^{th} value, so it is not memoryless.

(h) $T(x[n]) = x[n] + u[n+1]$

- Stable: $|T(x[n])| \leq M + 3$ for $n \geq -1$ and $|T(x[n])| \leq M$ for $n < -1$, so it is stable.
- Causal: Since it doesn't use future values of $x[n]$, it is causal.
- Not linear:

$$\begin{aligned} T(ax_1[n] + bx_2[n]) &= ax_1[n] + bx_2[n] + 3u[n+1] \\ &\neq aT(x_1[n]) + bT(x_2[n]) \end{aligned}$$

This is not linear.

- Not TI:

$$\begin{aligned} T(x[n-n_0] &= x[n-n_0] + 3u[n+1] \\ &= y[n-n_0] \\ &= x[n-n_0] + 3u[n-n_0+1] \end{aligned}$$

This is not TI.

- Memoryless: $y[n]$ depends on the n^{th} value of x only, so this is memoryless.

2.4 有线性常系数差分方程如下：

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n-1]$$

当 $x[n]=\delta[n]$ 和 $y[n]=0, n<0$，求 $y[n], n\geqslant 0$。

2.5 一个因果线性时不变系统由下列差分方程描述：

$$y[n]-5y[n-1]+6y[n-2]=2x[n-1]$$

(a) 求系统的齐次响应，也即在 $x[n]=0$ 时，对全部 n 可能的输出。

(b) 求系统的单位脉冲响应。

(c) 求系统的阶跃响应。

2.6 (a) 一线性时不变系统，其输入输出满足如下差分方程：

$$y[n]-\frac{1}{2}y[n]=x[n]+2x[n-1]+x[n-2]$$

求其频率响应 $H(e^{j\omega})$。

(b) 有一系统，其频率响应为

$$H(e^{j\omega})=\frac{1-\frac{1}{2}e^{-j\omega}+e^{-j3\omega}}{1+\frac{1}{2}e^{-j\omega}+\frac{3}{4}e^{-j2\omega}}$$

写出表征该系统的差分方程。

2.4. The difference equation:

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n-1]$$

To solve, we take the Fourier transform of both sides.

$$Y(e^{j\omega})-\frac{3}{4}Y(e^{j\omega})e^{-j\omega}+\frac{1}{8}Y(e^{j\omega})e^{-j2\omega}=2\cdot X(e^{j\omega})e^{-j\omega}$$

The system function is given by:

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{2e^{-j\omega}}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}} \end{aligned}$$

The impulse response (for $x[n]=\delta[n]$) is the inverse Fourier transform of $H(e^{j\omega})$.

$$H(e^{j\omega})=\frac{-8}{1+\frac{1}{4}e^{-j\omega}}+\frac{8}{1-\frac{1}{2}e^{-j\omega}}$$

Thus,

$$h[n]=-8(\frac{1}{4})^n u[n]+8(\frac{1}{2})^n u[n]. \quad = y[n]$$

2.5. (a) The homogeneous difference equation:

$$y[n] - 5y[n-1] + 6y[n-2] = 0$$

Taking the Z-transform,

$$1 - 5z^{-1} + 6z^{-2} = 0$$
$$(1 - 2z^{-1})(1 - 3z^{-1}) = 0.$$

The homogeneous solution is of the form

$$y_h[n] = A_1(2)^n + A_2(3)^n.$$

(b) We take the z-transform of both sides:

$$Y(z)[1 - 5z^{-1} + 6z^{-2}] = 2z^{-1}X(z)$$

Thus, the system function is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \\ &= \frac{2z^{-1}}{1 - 5z^{-1} + 6z^{-2}} \\ &= \frac{-2}{1 - 2z^{-1}} + \frac{2}{1 - 3z^{-1}}, \end{aligned}$$

where the region of convergence is outside the outermost pole, because the system is causal. Hence the ROC is $|z| > 3$. Taking the inverse z-transform, the impulse response is

$$h[n] = -2(2)^n u[n] + 2(3)^n u[n].$$

2.6. (a) The difference equation:

$$y[n] - \frac{1}{2}y[n-1] = x[n] + 2x[n-1] + x[n-2]$$

Taking the Fourier transform of both sides,

$$Y(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega}] = X(e^{j\omega})[1 + 2e^{-j\omega} + e^{-j2\omega}].$$

Hence, the frequency response is

$$\begin{aligned} H(e^{j\omega}) &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \\ &= \frac{1 + 2e^{-j\omega} + e^{-j2\omega}}{1 - \frac{1}{2}e^{-j\omega}}. \end{aligned}$$

(b) A system with frequency response:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}}{1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}} \\ &= \frac{Y(e^{j\omega})}{X(e^{j\omega})} \end{aligned}$$

cross multiplying,

$$Y(e^{j\omega})[1 + \frac{1}{2}e^{-j\omega} + \frac{3}{4}e^{-j2\omega}] = X(e^{j\omega})[1 - \frac{1}{2}e^{-j\omega} + e^{-j3\omega}],$$

and the inverse transform gives

$$y[n] + \frac{1}{2}y[n-1] + \frac{3}{4}y[n-2] = x[n] - \frac{1}{2}x[n-1] + x[n-3].$$

(c) Let $x[n] = u[n]$ (unit step), then

$$X(z) = \frac{1}{1 - z^{-1}}$$

and

$$\begin{aligned} Y(z) &= X(z) \cdot H(z) \\ &= \frac{2z^{-1}}{(1 - z^{-1})(1 - 2z^{-1})(1 - 3z^{-1})}. \end{aligned}$$

Partial fraction expansion yields

$$Y(z) = \frac{1}{1 - z^{-1}} - \frac{4}{1 - 2z^{-1}} + \frac{3}{1 - 3z^{-1}}.$$

The inverse transform yields:

$$y[n] = u[n] - 4(2)^n u[n] + 3(3)^n u[n].$$

一个因果的 LTI 系统有单位脉冲响应 $h[n]$，其 z 变换是

$$H(z) = \frac{1 + z^{-1}}{\left(1 - \frac{1}{2} z^{-1}\right)\left(1 + \frac{1}{4} z^{-1}\right)}$$

(a) $H(z)$的收敛域是什么？

(b) 系统是稳定的吗？请解释。

(c) 某一输入 $x[n]$，产生的输出为

$$y[n] = -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1]$$

求 $x[n]$的 z 变换 $X(z)$。

(d) 求系统单位脉冲响应 $h[n]$。

3.9.

$$H(z) = \frac{1 + z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 + \frac{1}{4}z^{-1})} = \frac{2}{(1 - \frac{1}{2}z^{-1})} - \frac{1}{(1 + \frac{1}{4}z^{-1})}$$

(a) $h[n]$ causal $\Rightarrow$ ROC outside $|z| = \frac{1}{2} \Rightarrow |z| > \frac{1}{2}$.

(b) ROC includes $|z| = 1$ $\Rightarrow$ stable.

(c)

$$\begin{aligned} y[n] &= -\frac{1}{3}\left(-\frac{1}{4}\right)^n u[n] - \frac{4}{3}(2)^n u[-n-1] \\ Y(z) &= \frac{-\frac{1}{3}}{1 + \frac{1}{4}z^{-1}} + \frac{\frac{4}{3}}{1 - 2z^{-1}} \\ &= \frac{1 + z^{-1}}{(1 + \frac{1}{4}z^{-1})(1 - 2z^{-1})} \qquad \frac{1}{4} < |z| < 2 \\ X(z) &= \frac{Y(z)}{H(z)} = \frac{(1 - \frac{1}{2}z^{-1})}{(1 - 2z^{-1})} \qquad |z| < 2 \\ x[n] &= -(2)^n u[-n-1] + \frac{1}{2}(2)^{n-1} u[-n] \end{aligned}$$

(d)

$$h[n] = 2\left(\frac{1}{2}\right)^n u[n] - \left(-\frac{1}{4}\right)^n u[n]$$

$$
\begin{aligned}
(1)\ w_1[n] &= x[n] + w_2[n-1] \\
(2)\ w_2[n] &= \frac{1}{4}y[n] + w_3[n-1] - \frac{1}{5}x[n] \\
(3)\ w_3[n] &= -\frac{5}{24}y[n] - \frac{1}{12}y[n-1] \\
(4)\quad y[n] &= w_1[n].
\end{aligned}
$$

Combining the above equations, we get:

$$y[n] - \frac{1}{4}y[n-1] + \frac{5}{24}y[n-2] + \frac{1}{12}y[n-3] = x[n] - \frac{1}{5}x[n-1].$$

Taking the Z-transform of this equation and combining terms, we get the following transfer function:

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z^{-3}}$$

which is equal to the initial transfer function.

6.23 考虑一因果线性时不变系统，其系统函数为

$$H(z) = \frac{1 + \frac{1}{5}z^{-1}}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})(1 + \frac{1}{4}z^{-1})}$$

(a) 对下列每种形式画出系统实现的信号流图：

(i) 直接Ⅰ型；

(ii) 直接Ⅱ型；

(iii) 用一阶和二阶直接Ⅱ型节的级联型；

(iv) 用一阶和二阶直接Ⅱ型节的并联型；

(v) 转置直接Ⅱ型。

(b) 对于上面(a)中(v)的流图，写出差分方程，并证明该系统有正确的系统函数。

6.23. Causal LTI system with system function:

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{(1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2})(1 + \frac{1}{4}z^{-1})}.$$

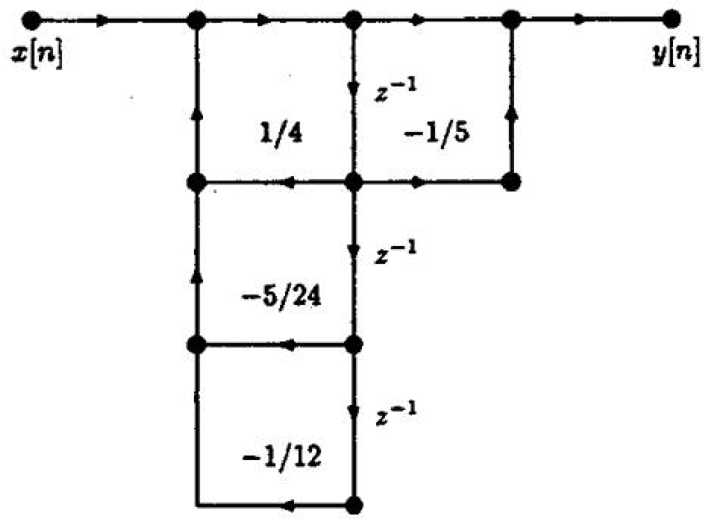
(a) (i) Direct form I.

$$H(z) = \frac{1 - \frac{1}{5}z^{-1}}{1 - \frac{1}{4}z^{-1} + \frac{5}{24}z^{-2} + \frac{1}{12}z(-3)}$$

so

$$b_0 = 1\ ,\ b_1 = -\frac{1}{5} \text{ and } a_1 = \frac{1}{4}\ ,\ a_2 = -\frac{5}{24}\ ,\ a_3 = -\frac{1}{12}.$$

(ii) Direct form II.

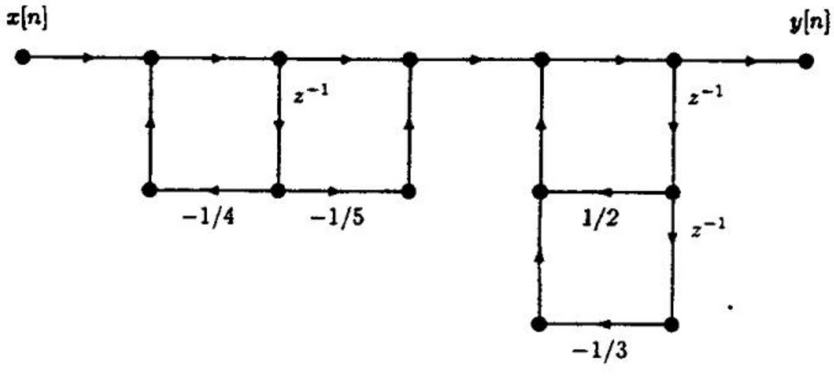


(iii) Cascade form using first and second order direct form II sections.

$$H(z) = \left(\frac{1 - \frac{1}{5}z^{-1}}{1 + \frac{1}{4}z^{-1}}\right)\left(\frac{1}{1 - \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2}}\right).$$

So

$$b_{01} = 1\,,\ b_{11} = -\tfrac{1}{5}\,,\ b_{21} = 0\,,$$
$$b_{02} = 1\,,\ b_{12} = 0\,,\ b_{22} = 0 \text{ and}$$
$$a_{11} = -\tfrac{1}{4}\,,\ a_{21} = 0\,,\ a_{12} = \tfrac{1}{2}\,,\ a_{22} = -\tfrac{1}{3}.$$

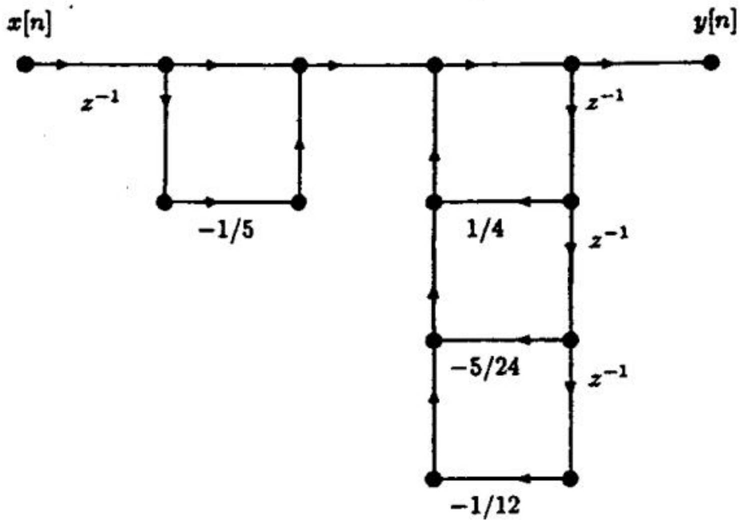


(iv) Parallel form using first and second order direct form II sections.
We can rewrite the transfer function as:

$$H(z) = \frac{\frac{27}{125}}{1+\frac{1}{4}z^{-1}} + \frac{\frac{98}{125} - \frac{36}{125}z^{-1}}{1-\frac{1}{2}z^{-1}-\frac{1}{3}z^{-2}}.$$

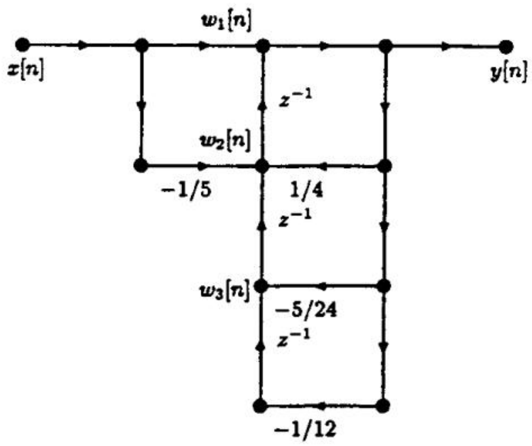
So

$$e_{01} = \tfrac{27}{125}\ ,\ e_{11} = 0\ ,$$
$$e_{02} = \tfrac{98}{125}\ ,\ e_{12} = -\tfrac{36}{125}\ ,\ \text{and}$$
$$a_{11} = -\tfrac{1}{4}\ ,\ a_{21} = 0\ ,\ a_{12} = \tfrac{1}{2}\ ,\ a_{22} = -\tfrac{1}{3}.$$



(v) Transposed direct form II

We take the direct form II derived in part (ii) and reverse the arrows as well as exchange the input and output. Then redrawing the flow graph, we get:



(b) To get the difference equation for the flow graph of part (v) in (a), we first define the intermediate variables: $w_1[n]$, $w_2[n]$ and $w_3[n]$. We have: