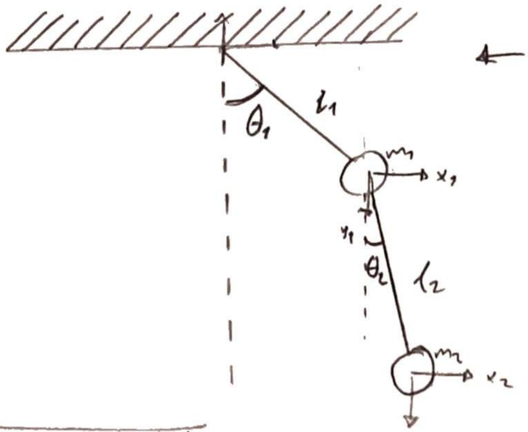
Descripción matematica del pendulo doble



← Dos grados de libertad

Posiciones

$x_1 = l_1 \operatorname{sen}\theta_1$ / *

$y_1 = -l_1 \cos\theta_1$

$x_2 = l_1 \operatorname{sen}\theta_1 + l_2 \operatorname{sen}\theta_2$ / *

$y_1 = l_1 \cos\theta_1 - l_2 \cos\theta_2$

Energia Cinetica

$$T = \tfrac{1}{2} m_1 (\dot{x}^2 + \dot{y}_1^2) + \tfrac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \tfrac{1}{2} m_1 [(l_1\dot\theta_1 \cos\theta_1)^2 + (-l_1\dot\theta_1 \operatorname{sen}\theta_2)^2] + \tfrac{1}{2} m_2 [(l_1\dot\theta_1 \cos\theta_1 + l_2\dot\theta_2 \cos\theta_2)^2 + (l_1\dot\theta_1 \operatorname{sen}\theta_1 + l_2\dot\theta_2 \operatorname{sen}\theta_2)^2]$$

$$= \tfrac{1}{2} m_1 [l_1^2 \dot\theta_1^2 \cos^2\theta_1 + l_1^2\dot\theta_1^2 \operatorname{sen}^2\theta_2]$$

$$+ \tfrac{1}{2} m_2 \Big[(l_1^2\dot\theta_1^2 \cos^2\theta_1 + l_2^2\dot\theta_2^2\cos^2\theta_2 + 2 l_1 l_2 \dot\theta_1\dot\theta_2 \frac{\cos(a+b)+\cos(a-b)}{2}$$

$$+ (l_1^2\dot\theta_1^2 \operatorname{sen}^2\theta_1 + l_2^2\dot\theta_2^2 \operatorname{sen}^2\theta_2 + 2 l_1 l_2 \dot\theta_1^2\dot\theta_2^2 \frac{\cos(a-b)-\cos(a+b)}{2}\Big]$$

$$= \tfrac{1}{2} m_1 [l_1^2\dot\theta_1^2] + \tfrac{1}{2} m_2 [l_1^2\dot\theta_1^2 + l_2^2\dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1\dot\theta_2 \cos(\theta_1 - \theta_2)]$$

Energia Potencial

$$U = m_1 g y_1 + m_2 g y_2$$

$$= m_1 g [l_1 \cos\theta_1] + m_2 g [-l_2 \cos\theta_2]$$

$$= -(m_1 + m_2) g l_1 \cos\theta_1 - m_2 l_2 g \cos\theta_2$$

Lagrangeano

$$L = T - V = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [l_1^2\dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$+ (m_1 + m_2) g l_1 \cos\theta_1 - m_2 l_2 g \cos\theta_2]$$

Ecuaciones de movimiento

① $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \,, \quad ② \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0$$

① $$\rightarrow \frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \cancel{\dot{\theta}_1 \dot{\theta}_2}\, \mathrm{sen}(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \,\mathrm{sen}\,\theta_1$$

$$* \quad \frac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$
$$= (m_1 + m_2) l_1^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 (\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - (\dot{\theta}_2 + \dot{\theta}_2)\,\mathrm{sen}(\theta_2 - \theta_1)\dot{\theta}_2]$$
$$= (m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_2^2\, \mathrm{sen}(\theta_2 - \theta_1)$$
$$+ m_2 l_1 l_2 \cancel{\dot{\theta}_1 \dot{\theta}_2}\, \mathrm{sen}(\theta_2 - \theta_1)$$

$$(m_1 + m_2) l_1^2 \ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 l_1 l_2 \dot{\theta}^2\, \mathrm{sen}(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \,\mathrm{sen}\,\theta_1 = 0$$

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② $$\frac{\partial L}{\partial \theta_2} = -m_2 l_1 l_2 \cancel{\dot{\theta}_1 \dot{\theta}_2}\, \mathrm{sen}(\theta_1 - \theta_2) - m_2 g l_2 \,\mathrm{sen}\,\theta_2$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 + m_2 l_1 l_2 \boxed{\dot{\theta}_1 \cos(\theta_1 - \theta_2)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 [\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - (\dot{\theta}_1 - \dot{\theta}_2)\,\mathrm{sen}(\theta_1 - \theta_2)\,\dot{\theta}_1]$$

$$= m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2\, \mathrm{sen}(\theta_1 - \theta_2)$$
$$+ m_2 l_1 l_2 \cancel{\dot{\theta}_1 \dot{\theta}_2}\, \mathrm{sen}(\theta_1 - \theta_2)$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 \dot{\theta}^2\, \mathrm{sen}(\theta_1 - \theta_2)$$
$$+ m_2 g l_2 \,\mathrm{sen}\,\theta_2 = 0$$

Sistema de 2 ecuaciones

$$\begin{cases} (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}^2 \operatorname{sen}(\theta_1-\theta_2) + (m_1+m_2) g l_1 \operatorname{sen}\theta_1 = 0 \\ m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}^2 \operatorname{sen}(\theta_1-\theta_2) + m_2 g l_2 \operatorname{sen}\theta_2 = 0 \end{cases}$$

solucion por wolfram

$$\ddot{\theta}_1 = \frac{m_2 g \operatorname{sen}\theta_2 \cos(\theta_1-\theta_2) - m_2 \operatorname{sen}(\theta_1-\theta_2)\left[l_1\dot{\theta}_1^2 \cos(\theta_1-\theta_2) + l_2\dot{\theta}_2^2\right] - (m_1+m_2) g \operatorname{sen}\theta_1}{l_1\left[m_1 + m_2 \operatorname{sen}^2(\theta_1-\theta_2)\right]}$$

$$\ddot{\theta}_2 = \frac{(m_2+m_1)\left[l_1\dot{\theta}_1^2 \operatorname{sen}(\theta_1-\theta_2) - g\operatorname{sen}\theta_2 + g\operatorname{sen}\theta_1 \cos(\theta_1-\theta_2)\right] + m_2 l_2 \dot{\theta}_2^2 \operatorname{sen}(\theta_1-\theta_2)\cos(\theta_1-\theta_2)}{l_2\left[m_1 + m_2 \operatorname{sen}^2(\theta_1-\theta_2)\right]}$$