OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2008

B.Sc. (Part II) EXAMINATION
HIGHER DIPLOMA IN MATHEMATICS EXAMINATION

MATHEMATICS — [MA490]

MEASURE THEORY

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Time allowed: **Two** hours.
Answer *three* questions.

1. Answer *four* of the following:

(a) In Bernoulli space, let E be the event that Heads is tossed exactly twice. Express E in terms of the simple events E_n (E_n is the event that the nth toss is Heads.)

(b) Let E and F be subsets of a set X. Show that the algebra generated by E and F has at most 16 elements.

(c) The function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = -2$ if $x \leq 1$ and $f(x) = 5$ is $x > 1$. What is the smallest σ-algebra for which f is a measurable function?

(d) For the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f = 2\chi_{[0,1]} - 3\chi_{(2,4)} + \chi_{\{5,6,7\}}$$

and the measure $\mu = 4\delta_0 + 3\delta_2 + \delta_6$, evaluate the integral $\displaystyle\int_{\mathbb{R}} f \, d\mu$

(e) If f is a nonnegative measurable function on a measure space $(X, \mathcal{A}, \mu)$ and $\displaystyle\int_X f \, d\mu = 0$, show that $f = 0$ almost everywhere.

(f) Explain why the following equation is valid:

$$\int_0^{1/2} \frac{dx}{1 - \sqrt{x}} = \sum_{n=0}^{\infty} \int_0^{1/2} x^{n/2} \, dx$$

and hence evaluate the integral on the left hand side.

p.t.o.

2. Let $(X, \mathcal{A}, \mu)$ be a *Probability Space.*

(a) If (E_n) is a *decreasing* sequence of sets in $\mathcal{A}$, show that

$$\mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)$$

(b) Let (E_n) be a sequence of sets in $\mathcal{A}$. Give the definitions of the sets $\liminf E_n$ and $\limsup E_n$. Show that $\liminf E_n \subset \limsup E_n$.

(c) Prove the First Borel-Cantelli Lemma: if the series $\sum_n \mu(E_n)$ converges, then $\mu(\limsup E_n) = 0$

3. Let $(X, \mathcal{A}, \mu)$ be a (complete) measure space.

(a) A function $f\colon X \to \mathbb{R}$ is aa *measurable function* if the set $\{x \in X : f(x) > \lambda\}$ is measurable for every $\lambda \in \mathbb{R}$. If f is a measurable function, show that the set

$$\{x \in X : f(x) < 0\}$$

is measurable

(b) Let (f_n) be a sequence of measurable functions on X. Show that the function $\liminf f_n$ is measurable.

(c) Let f and g be functions on X. If f is measurable and $f = g$ almost everywhere, show that g is measurable.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give an account (without proofs) of the definition of the integral $\int_X f \, d\mu$, starting with the integral of a simple measurable function. Explain how this integral can be interpreted in the following cases:
(i) $(X, \mathcal{A}, \mu)$ is a probability space.
(ii) $X = \mathbb{N}$ and μ is counting measure.

(b) Give an example of a sequence (f_n) of integrable functions that converges at every point to an integrable function f, with the property that

$$\int_X f \, d\mu \neq \lim_{n \to \infty} \int_X f_n \, d\mu \, .$$

(c) State, without proof, a theorem that allows term by term integration of a sequence of functions. Why does this result not apply to the example you have given in part (b)?

5. (a) Describe the construction of the *Cantor Set* C and show that
(i) C has Lebesgue measure zero.
(ii) C is uncountable

(b) Describe the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$ and show that there exists a set that does not belong to $\mathcal{M}$.