Semester 1 Examinations 2014/2015

Exam Codes	4BS3, 4BS9, 4FM2, 4BMS2, 3BA1, 4BA4
Exams	B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics B.Sc. B.A. (Mathematics)
Module	Measure Theory
Module Code	MA490 MA540
External Examiner	Dr Mark Lawson
Internal Examiner(s)	Prof. Graham Ellis Dr Ray Ryan
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	3 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

Q1 [**49%**] Answer *six* of the following:

(a) Let X be a non empty set and let E, F be two subsets of X. Let $\mathcal{A}$ be the σ-algebra generated by E and F. What is the biggest possible number of sets that $\mathcal{A}$ can contain? What is the smallest possible number?

(b) In Bernoulli Space Ω, let E_n be the event that the nth toss is heads. Write down a formula in terms of the E_n for the following event: "The sequence $HTTH$ appears infinitely often".

(c) Prove that in an infinite sequence of coin tosses, the probability that Heads will never appear is zero.

(d) Find the Jordan content and the Lebesgue measure of the set of rational numbers in the interval $[0, 1]$.

(e) Let $\mathcal{A}$ be the σ-algebra of subsets of $\mathbb{R}$ consisting of $\varnothing$, $\mathbb{R}$, $(-\infty, 1]$ and $(1, \infty)$. Give an example, with proof, of a function that is not measurable with respect to $\mathcal{A}$.

(f) Let $f_n(x) = \sqrt{2n+1}\, x(1-x^2)^n$. The sequence (f_n) converges pointwise to zero on the interval $[0, 1]$. Is the convergence uniform? Explain your answer.

(g) A random variable is defined on Bernoulli space by the formula $X(\omega) = \sum 2^{-n}\omega_n$. Find the integral of X with respect to the Bernoulli probability measure P.

(h) Is the function $f = \sum_{n=1}^{\infty} (-1)^n n^{-1} \chi_{[n,n+1)}$ Lebesgue integrable on $\mathbb{R}$? Explain your answer.

(i) Give an example of a series $\sum g_n$ of Lebesgue integrable functions on $\mathbb{R}$ that converges pointwise but for which term by term integration is not valid.

(j) Find the value of the following sum:

$$\sum_{n=0}^{\infty} \int_0^{\pi/2} \left(1 - \sqrt{\sin x}\right)^n \cos x \, dx \, .$$

Justify each step in your calculation.

(k) Give the definition of the Cantor Set C. Which of the following numbers belong to C?

$$\text{(i) } \frac{1}{7}, \quad \text{(ii) } \frac{8}{81} \, .$$

You must answer every question.

Q2 [**17%**]

(a) Give the definition of an algebra and a σ-algebra and give an example of an algebra that is not a σ-algebra. **[5%]**

(b) Show that the set $\mathbb{Q}^c$ of irrational numbers belongs to the σ-algebra generated by the intervals in $\mathbb{R}$. **[6%]**

(c) Let μ be a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X. Show that if (E_n) is a sequence of measurable subsets of X, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)\,.$$

[6%]

Q3 [**17%**] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) Explain how Lebesgue's approach to the definition of the integral leads to the definition of measurability for a real-valued function on X. **[5%]**

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f = 2\chi_{[0,2]} + 3\chi_{[3,5]}\,.$$

(i) Working directly from the definition, show that f is measurable with respect to the Borel σ-algebra. **[3%]**

(ii) What is the smallest σ-algebra for which f is measurable? **[3%]**

(c) Let (f_n) be a sequence of measurable functions that converges pointwise to a function f. Show that f is measurable. **[6%]**

Q4 [**17%**] Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give a brief account of the construction of the integral $\int_X f\, d\mu$ for measurable functions on X. **[5%]**

(b) If f is an integrable function, show that $|f|$ is also integrable and that

$$\left|\int_X f\, d\mu\right| \leq \int_X |f|\, d\mu\,.$$

[5%]

(c) State, without proof, two of the convergence theorems that relate to term by term integration of a sequence or series of integrable functions. In each case, give an example of a sequence or series to which the theorem can be applied. **[7%]**