NUI Galway
OÉ Gaillimh

Semester 1 Examinations 2019/2020

Exam Codes	4BMS2, 4BS2, 4FM2, 4BCT1, 4BME1
Exams	B.Sc. B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics M.Sc. Mathematics/Mathematical Science
Module	Measure Theory
Module Code	MA490
External Examiner	Prof. T. Brady
Internal Examiner(s)	Prof. G. Ellis Dr M. Kerin
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	4 pages, including this one
School	Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes
Release in Exam Venue:	Yes

Q1 [40%] Answer **five** of the following, justifying your response in each case. **[8% each]**

(a) Let X be a set. How many σ-algebras of subsets of X contain exactly five elements?

(b) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be σ-algebras of subsets of a set X. Show that $\mathcal{A}_1 \cap \mathcal{A}_2$ is also a σ-algebra.

(c) Determine $\liminf A_n$ and $\limsup A_n$ for the sequence $(A_n)_{n\in\mathbb{N}}$ of sets given by

$$A_n = \begin{cases} \left(\frac{1}{n} - 2, 1\right), & n \text{ odd}, \\ \left(0, 3 + \frac{1}{n}\right), & n \text{ even}. \end{cases}$$

(d) Determine the Lebesgue measure of the subset of $[0, 1]$ whose elements have neither first nor second digit in their decimal expansion equal to "0".

(e) Let $(X, \mathcal{A}, \mathbb{P})$ be a probability space and $(A_n)_{n\in\mathbb{N}}$ a sequence of events in $\mathcal{A}$ such that $\mathbb{P}(A_n) = \frac{1}{7^n}$ for $n \in \mathbb{N}$. Determine the probability of the event $\limsup A_n$.

(f) Suppose $(X, \mathcal{A}, \mu) = (\Omega, \mathcal{E}, \mathbb{P})$ is the Bernoulli probability space associated to our share-price model, where $\Omega = \{(\omega_j)_{j\in\mathbb{N}} \mid \omega_j \in \{u, d\} \ \forall\, j \in \mathbb{N}\}$ and $\mathcal{E}$ is the σ-algebra generated by the simple events $E_n = \{(\omega_j)_{j\in\mathbb{N}} \in \Omega \mid \omega_n = u\}$, $n \in \mathbb{N}$. Determine whether the sequence $(A_n)_{n\in\mathbb{N}}$ of events in $\mathcal{E}$ defined by

$$A_n = E_n^c \cap E_{n+3}\,, \quad n \in \mathbb{N},$$

is an independent sequence of events.

(g) Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the Lebesgue measure space and $f : \mathbb{R} \to \mathbb{R}$ a function with $f(x) = 127\, e^x$ for all $x \in \mathbb{Q}^c$. Show that f is measurable with respect to $\mathcal{L}$.

(h) Show, with justification, that

$$\sum_{n=1}^{\infty} \int_2^4 \frac{1}{x^n}\, dx = \ln(3)\,.$$

(i) Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the Lebesgue measure space and $f : \mathbb{R} \to \mathbb{R}$ a non-negative, measurable function. If $f_n = f \cdot \chi_{[\frac{1}{n},1]} : \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, where $\chi_A : \mathbb{R} \to \mathbb{R}$ denotes the characteristic function of the subset $A \subset \mathbb{R}$, show that

$$\int_{(0,1]} f\, d\lambda = \lim_{n\to\infty} \int_{[\frac{1}{n},1]} f\, d\lambda\,.$$

(j) Which of the following numbers belong to the Cantor set?

$$\text{(i)}\ \frac{1}{27}\,, \qquad \text{(ii)}\ \frac{4}{27}\,, \qquad \text{(iii)}\ (0.21)_3\,, \qquad \text{(iv)}\ \sum_{k=1}^{\infty} \frac{2}{9^k}\,.$$

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Q2 [20%]

(a) Give the definitions of the following terms:

(i) an algebra; (ii) an additive set function; (iii) a σ-algebra; (iv) a measure. **[8%]**

(b) Let $(X, \mathcal{A})$ be a measurable space and let $Y \subset X$ be a proper, non-trivial subset of X. Show that

$$\mathcal{A}_Y = \{A \cap Y \mid A \in \mathcal{A}\}$$

is a σ-algebra of subsets of Y. **[6%]**

(c) Let $(X, \mathcal{A}, \mu)$ be a measure space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in $\mathcal{A}$. Show that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n).$$ **[6%]**

Q3 [20%] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) Define what it means for a function $f : X \to \mathbb{R}$ to be measurable with respect to $\mathcal{A}$. **[2%]**

(b) Suppose $g : X \to \mathbb{R}$ is measurable with respect to $\mathcal{A}$ and that $g(x) \neq 0$ for all $x \in X$. Show that the function $\frac{1}{g} : X \to \mathbb{R}$ is also measurable. **[6%]**

(c) Suppose $A \subset X$ is not an element of the σ-algebra $\mathcal{A}$. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions given by

$$f_n : X \to \mathbb{R}\,;\, x \mapsto 1 - \frac{1}{n}\,\chi_A(x)\,, \quad n \in \mathbb{N},$$

where $\chi_A : X \to \mathbb{R}$ denotes the characteristic function of A.

(i) Show that, for all $n \in \mathbb{N}$, the function f_n is not measurable with respect to $\mathcal{A}$. **[6%]**

(ii) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f : X \to \mathbb{R}$ which is measurable with respect to $\mathcal{A}$. **[6%]**

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Q4 [20%] Let $(X, \mathcal{A}, \mu)$ be a complete measure space.

(a) Define the μ-integral of a measurable, simple function, and use this to define μ-integrability for measurable, simple functions and for non-negative, measurable functions. **[6%]**

(b) If $f : X \to \mathbb{R}$ is a μ-integrable, measurable function, show that $|f| : X \to \mathbb{R}$ is μ-integrable. **[4%]**

(c) Suppose that $(X, \mathcal{A}, \mu) = (\mathbb{R}, \mathcal{L}, \lambda)$ is the Lebesgue measure space and that $A \subset [-1, 2]$ is not an element of the Lebesgue σ-algebra $\mathcal{L}$. Consider the function

$$f : \mathbb{R} \to \mathbb{R}\,;\; x \mapsto \begin{cases} 3, & x \in A, \\ -3, & x \in [-1, 2] \backslash A, \\ 0, & x \notin [-1, 2]. \end{cases}$$

(i) Show that f is not Lebesgue integrable. **[2%]**

(ii) Show that $|f|$ is Lebesgue integrable and compute $\int_{\mathbb{R}} |f|\, d\lambda$. **[4%]**

(d) Suppose $(X, \mathcal{A}, \mu) = (\Omega, \mathcal{E}, \mathbb{P})$ is the Bernoulli probability space associated to our share-price model, where $\Omega = \{(\omega_j)_{j\in\mathbb{N}} \mid \omega_j \in \{u, d\}\ \forall\ j \in \mathbb{N}\}$ and $\mathcal{E}$ is the (completion of the) σ-algebra generated by the simple events $E_n = \{(\omega_j)_{j\in\mathbb{N}} \in \Omega \mid \omega_n = u\}$, $n \in \mathbb{N}$. Compute the expected value $\mathbb{E}(Y)$ of the random variable $Y : \Omega \to \mathbb{R}$ defined by

$$Y = 3\,\chi_{E_1 \cap E_2} - 10\,\chi_{E_1^c \cap E_3^c \cap E_4^c} + 12\,\chi_{E_5}\,,$$

where $\chi_A : \Omega \to \mathbb{R}$ denotes the characteristic function of a subset $A \subset \Omega$. **[4%]**