Autumn Examinations 2015/2016

Exam Code(s)	4BS2, 4BMS2
Exam(s)	Bachelor of Science (Mathematical Science) Bachelor of Science
Modules	Ring Theory
Module Codes	MA416
External Examiner(s)	Prof. Mark Lawson
Internal Examiner(s)	Prof. Graham Ellis Dr Emil Sköldberg
Instructions:	**Students should attempt ALL questions.**
Duration	2 Hours
No. of Pages	3 (including this page)
Discipline	Mathematics

Release to Library: No
Release in Exam Venue: No

Q1. (a) In each of the following cases, determine if the object is a ring. If you believe that it is a ring, it is enough to say so; if not, you should give a reason.

(i) The set of *all* 2×2 matrices with entries from $\mathbb{N}$ under the usual addition and multiplication of matrices. [**3 marks**]

(ii) The set of all 2×2 matrices with real entries and nonzero determinant under the usual addition and multiplication of matrices. [**3 marks**]

(b) Find all units and zero-divisors in the following rings: (i) $\mathbb{Z}_{13}$ and (ii) $\mathbb{Z}_{14}$. [**6 marks**]

(c) Show that the units of a ring R form a group under multiplication. [**8 marks**]

Q2. (a) Let R and S be rings. Give the definition of a *ring homomorphism* $\varphi : R \to S$. [**2 marks**]

(b) Show that, for an arbitrary ring R, there is a unique ring homomorphism $\phi : \mathbb{Z} \to R$. [**6 marks**]

(c) Define the *field of fractions* of an integral domain. (You need to describe its elements, and the definition of the addition and multiplication.) [**6 marks**]

(d) Show that the field of fractions of $\mathbb{Z}$ is isomorphic to $\mathbb{Q}$. [**6 marks**]

Q3. (a) State and prove Eisenstein's irreducibility criterion. [**10 marks**]

(b) For each of the following polynomials, determine if it is irreducible or not, and if it is reducible, factor it into irreducibles.

(i) $x^3 + x \in \mathbb{Z}_5[x]$

(ii) $x^3 - 3x + 1 \in \mathbb{Q}[x]$

[**10 marks**]

Q4. (a) Let I and J be ideals in the ring R. Show that the sets $I \cap J$ and $I + J$ are ideals in R. [**4 marks**]

(b) Give the definition of *maximal ideal* and *prime ideal* in a commutative ring. Show that every maximal ideal is prime. [**6 marks**]

(c) Consider the ring homomorphism $\psi : \mathbb{R}[x] \to \mathbb{C}$ defined by

$$\psi\left(\sum_{k=0}^{\infty} a_k x^k\right) = \sum_{k=0}^{\infty} a_k i^k, \qquad \text{where } i^2 = -1.$$

Determine the kernel of ψ and hence, or otherwise, show that the ideal $(x^2 + 1) \subset \mathbb{R}[x]$ is a maximal ideal. [**10 marks**]

Q5. (a) Give the definition of *Euclidean ring.* [**3 marks**]

(b) Show that the ring of integers is a Euclidean ring. [**8 marks**]

(c) Given the two elements $x^3 + 3x^2 + 3x + 1$ and $x^2 - 1$ in $\mathbb{Q}[x]$, find their greatest common divisor. Hence, or otherwise, find a generator for the ideal $I + J$, where $I = (x^3 + 3x^2 + 3x + 1)$ and $J = (x^2 - 1)$. [**9 marks**]