OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Semester 1 Examinations 2022/2023

Course Instance Codes	4BCT1, 4BMS2, 4BS2, 2SPS1
Exams	4th Science, PhD Science
Module Code	MA416
Module	Rings
External Examiner	Prof. C. Roney-Dougal
Internal Examiners	Prof. G. Pfeiffer *Dr R. Quinlan *Dr T. Rossmann
Instructions:	**Answer all four questions.** **Each question carries 25 marks.** **All workings must be shown.**
Duration	2 hours
No. of Pages	3 pages, including this one
Discipline	Mathematics
Requirements:	No special requirements
Release to Library	Yes
Release in Exam Venue	Yes
MCQ	No
Statistical / Log Tables	No
Non-programmable Calculator	Yes
Mathematical Tables	No
Graph paper	No

Q1. (a) Let $R = \{a + b\sqrt{3} : a, b \in \mathbf{Z}\}$.

1. Show that R is a subring of $\mathbf{R}$.
2. Show that $2 + \sqrt{3}$ is a unit of R.
3. Is R an integral domain? Justify your answer. [12 marks]

(b) True or false? Justify your answers.

1. The sum of two units of a ring is itself always a unit.
2. The product of two units of a ring is itself always a unit.
3. For nonzero polynomials $f, g \in R[X]$ with coefficients in a commutative ring R, we always have $\deg(fg) = \deg(f) + \deg(g)$.
4. The ring $\mathbf{R}[\varepsilon]$ of dual numbers is isomorphic to the polynomial ring $\mathbf{R}[X]$. [8 marks]

(c) Let R be a ring.

1. Let $f\colon R \to R'$ be a ring homomorphism. Show that $\mathrm{Ker}(f)$ is a (two-sided) ideal of R.
2. Is every (two-sided) ideal of R of the form $\mathrm{Ker}(f)$ for some ring R' and ring homomorphism $f\colon R \to R'$? Justify your answer. [5 marks]

Q2. (a) Let R be a ring. Let I and J be (two-sided) ideals of R.

1. Suppose that I contains a unit of R. Show that $I = R$.
2. Suppose that R is an integral domain and that $I \cap J = \{0\}$.
Show that $I = \{0\}$ or $J = \{0\}$. [7 marks]

(b)
1. Let $f\colon D \to R$ be a ring homomorphism. Suppose that D is a division ring and that $R \neq \{0\}$. Show that f is injective.
2. Show that there is no ring homomorphism $\mathbf{R} \to \mathbf{Q}$. [8 marks]

(c) Given an example (with justification) of a prime ideal of $C(\mathbf{R})$, the ring of continuous functions $\mathbf{R} \to \mathbf{R}$ with pointwise operations. [5 marks]

(d) Let $f\colon R \to S$ and $h\colon R \to T$ be surjective ring homomorphisms. Suppose that $\mathrm{Ker}(f) \subseteq \mathrm{Ker}(h)$. Show that there exists a unique ring homomorphism $g\colon S \to T$ such that the diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[dr, "h"'] & S \arrow[d, "g"] \\ & T \end{tikzcd}$$

commutes. [5 marks]

Q3. (a) Let R denote the integral domain $\mathbf{Z}[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in \mathbf{Z}\}$. For $z \in R$, $z = a + b\sqrt{-3}$, define $N(z) = a^2 + 3b^2$. By using the function N or otherwise, show that the only units in R are 1 and -1. [6 marks]

(b) Give an example, with explanation, of a non-zero non-unit element of R that is irreducible in R. [6 marks]

(c) Show that every non-zero non-unit element of R is a product of irreducible elements in R. [6 marks]

(d) Show that R is not a unique factorization domain (for example by exhibiting two different factorizations of 4 as a product of irreducible elements in R). Give an example (with explanation) of an element of R that is irreducible but not prime. [7 marks]

Q4. (a) What is meant by a *noetherian* ring? Show that every principal ideal domain is noetherian. [6 marks]

(b) Answer TRUE or FALSE to each of the following. [5 marks]

i. Every noetherian integral domain is a unique factorization domain.
ii. Every unique factorization domain is a principal ideal domain.
iii. Every unique factorization domain is a Euclidean domain.
iv. Every irreducible element in a noetherian integral domain is prime.
v. Every subring of a noetherian ring is noetherian.

(c) Give an example, with explanation, of an integral domain that is not noetherian. [7 marks]

(d) Explain how to construct the *field of fractions* of an integral domain. [7 marks]