OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Semester 2 Examinations 2024

Exam Codes	3BA1, 4BDS1, 3BME1, 4BME1, 4BCT1, 1BMG1, 4BMS2, 4BS2, 4FM2, 1GDS1
Exam	4th Science
Module	Networks
Module Code	CS4423
External Examiner(s)	Prof. Colva Roney–Dougal
Internal Examiner(s)	Dr N. Madden Dr A. Carnevale*
Instructions	Answer all **four** questions.
Duration	2 hours
No. of Pages	3 pages (including this cover page)
Discipline	Mathematics

Requirements:

Release to Library	Yes
Release in Exam Venue	Yes
Statistical Tables/ Log Tables	Yes
Non-programmable calculators	Yes

Q1. [25 Marks]

(a) [10 Marks]

(i) What is a *graph*?

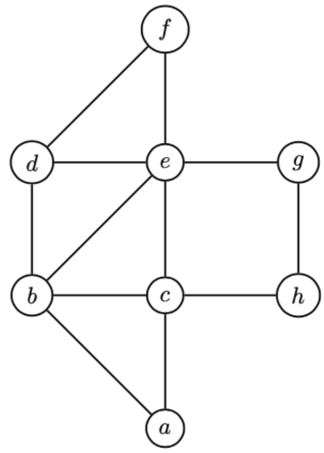
(ii) What are the *order* and the *size* of a graph?

(iii) What is the *adjacency matrix* of a graph?

(iv) Let A be the adjacency matrix of a graph G. Explain how one can compute the size of G as a function of the entries of A.

(v) Let A be the adjacency matrix of a graph G. Explain how one can compute the degree of the nodes of G as a function of the entries of A^2.

(b) [4 Marks] In the graph on the right, use the Breadth First Search algorithm to determine all the shortest distances between a and all the other nodes.



(c) [7 Marks] In the graph on the right, use the Breadth First Search algorithm for predecessors to determine all the predecessors in a shortest path from a to all the other nodes. Use this information to construct all the shortest paths between nodes a and f.

(d) [4 Marks] Determine a spanning tree of the graph on the right by using the Breadth First Search algorithm for spanning trees with inputs the graph itself and its node a.

Q2. [25 Marks]

(a) [5 Marks] Let G be the graph on the set of nodes $\{1, 2, 3, 4, 5, 6\}$ with edges $1-2$, $1-3$, $2-4$, $3-4$, $3-6$, $4-5$, $4-6$. Draw the graph G and determine the normalised degree centrality of all its nodes.

(b) [5 Marks] Define the concept of *normalised closeness centrality for a node in a connected graph G*. Compute the normalised closeness centrality of node 1 of G in (a).

(c) [5 Marks] Compute the normalised betweenness centrality of node 5 of G in (a). Justify your answer.

(d) [3 Marks] Is the graph G in (a) bipartite? Justify your answer.

(e) [7 Marks] Let H be a complete graph on 3 nodes. The eigenvalues of the adjacency matrix of A are 2 (with multiplicity 1), and -1 (with multiplicity 2). Use this information to compute the normalised eigenvector centrality of each node in H.

Q3. [25 Marks]

(a) [5 Marks] Describe how to generate a graph from the Erdős–Rényi models $G(n, m)$ and $G(n, p)$. In each model, what is the probability that a randomly chosen graph G has *exactly* m edges? Justify your answer.

(b) [5 Marks] For which value of p does sampling graphs from the $G(n, p)$ model yield an expected size m?

(c) [5 Marks] Suppose one constructed a graph G on 120 nodes by tossing a (fair, 6-sided) die once for each possible edge, adding the edge only if the die shows 3 or 6. Then pick a node at random in this graph. What is the probability that this node has degree 50? (You do not need to return a numerical value. It is enough to give an explicit formula in terms of the given data.)

(d) [5 Marks] Describe how to generate an (n, d, p)-WS graph in the *Watts–Strogatz small-world model* from an (n, d)-circle graph.

(e) [5 Marks] How do the clustering and characteristic path length of a random graph sampled from the WS model relate to those in a random graph sampled from the $G(n, p)$ model, or in an (n, d)-circle graph?

Q4. [25 Marks]

(a) [8 Marks] Compute the Prüfer code of the tree on the set of nodes $X = \{1, 2, 3, 4, 5, 6\}$ with edges 1–2, 1–3, 1–4, 4–5, 5–6. Explain how the degree sequence of this tree can be determined from its Prüfer code.

(b) [5 Marks] What is a *directed graph*? What is the in-degree and what is the out-degree of node a in the directed graph on the right?

(c) [7 Marks] For a directed graph G on a set of nodes X, define an equivalence relation on X that has the *strongly connected components* of G as its equivalence classes. Explain why the graph on the right has 5 strongly connected components, each consisting of a single node.

(d) [5 Marks] Many directed networks, like the World Wide Web, have a giant strongly connected component which covers a substantial part of the network. Name the other parts of the so-called *Bow-Tie diagram* of the components in a directed network, and describe each in terms of a suitable relation on the set of strongly connected components.