Semester I Examinations 2015/2016

Exam Codes	3BMS2, 4BMS2, 3BS9, 4BS2, 1OA1, 1HA1
Exam	Third and Fourth Science Examination & Higher Diploma in Applied Science (Maths)
Module	Euclidean and Non-Euclidean Geometry 1
Module Code	MA3101/MA544
Paper No.	1
External Examiner	Dr Mark Lawson
Internal Examiners	Prof. Graham Ellis
	Dr John Burns

Instructions **Answer all questions.**

Duration	2 hours
No. of Pages	3 pages including this page
School	Mathematics, Statistics & Applied Mathematics

Requirements No special requirements

Release to Library:	Yes
Other Materials	Non-programmable calculators

Q1. Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius one.

(i) [9 marks] Prove that the shortest path joining any two points of S^2 is an arc of a great circle.

(ii) [8 marks] Prove than $SO(3)$ acts transitively on S^2.

(iii) [8 marks] Calculate the shortest distance between Prague $((\theta, \phi) = (50°05', 14°25'))$ and Winnipeg $((\theta, \phi) = (49°55', -97°06'))$. (Take the radius of the earth to be 6377.5 km.)

Q2. Let $H \subset \mathbb{C}$ be the upper half plane with the hyperbolic metric (the Hyperbolic Plane).

(i) [6 marks] Prove than any matrix $M \in SL(2, \mathbb{R})$ representing a Möbius transformation is an isometry of H.

(ii) [6 marks] Prove than any pair of elements in H can be mapped to the positive y-axis by an element of $SL(2, \mathbb{R})$. (Recall the Iwasawa decomposition $SL(2, \mathbb{R}) = KAN$.)

(iii) [6 marks] Find a Möbius transformation that maps the semicircle C in H with polar equation $C = \{z \in \mathbb{C} : z = 2p \cos\theta e^{i\theta}; \theta \in (0, \pi/2)\}$ onto the positive y-axis.

(iv) [7 marks] Find the area of the hyperbolic (geodesic) triangle with vertices $(0, 0), (0, 1)$ and $(1, 0)$.

Q3. Recall that for a surface patch $x(u, v)$ of an oriented surface M in $\mathbb{R}^3$, the matrix of the shape operator or Weingarten map with respect to the basis $\{x_u, x_v\}$ is $\Pi_1^{-1}\Pi_2$, where Π_1 and Π_2 are the matrices of the first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$, respectively.

(i) [7 marks] Prove that the Gaussian curvature K of M is given by

$$K = \frac{LN - M^2}{EG - F^2}.$$

(ii) [12 marks] Calculate the Gaussian curvature K of the helicoid $x(u, v) = (v \cos u, v \sin u, 2u)$ and give an example of a geodesic on this surface.

(iii) [6 marks] Prove that there is no surface homeomorphic to S^2 in $\mathbb{R}^3$ with everywhere non-positive Gaussian curvature.

Q4. (a) (i) [8 marks] Use Hamilton's quaternions to prove that the unit 3-sphere S^3 in $\mathbb{R}^4$ is a disjoint union of isometric circles, parametrized by S^2.

(ii) [7 marks] Determine the rotation in $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ induced by conjugation by the unit quaternion $t = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$.

(b) Answer (i) OR (ii) below, but not both: [10 marks]

(i) State the Cayley-Salmon theorem, explaining the assumptions of the theorem. Give an example of a smooth cubic surface and describe all its lines.

(ii) State Synge's theorem, explaining the assumptions of the theorem. Give an example of a surface with negative curvature for which the conclusion of the theorem does not hold.