OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

SUMMER EXAMINATIONS 2009

B.Sc. EXAMINATION

MATHEMATICS — [MA490]

MEASURE THEORY

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Time allowed: **Two** hours.
Answer *three* questions.

1. Answer *four* of the following:

(a) Let $\mathcal{A}$ be an algebra of subsets of a set X and suppose that $\mathcal{A}$ is closed under *increasing* unions (i.e., if (E_n) is an increasing sequence of sets that belong to $\mathcal{A}$, then $\bigcup E_n$ also belongs to $\mathcal{A}$.) Show that $\mathcal{A}$ is a σ-algebra.

(b) Let
$$F_n = \begin{cases} [-1/n, 2+1/n] & \text{if } n \text{ is odd,} \\ [1, n^2] & \text{if } n \text{ is even.} \end{cases}$$
Find $\liminf F_n$ and $\limsup F_n$.

(c) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x$. Determine the smallest σ-algebra with respect to which f is measurable.

(d) If g is a measurable function, show that the function g^+ is also measurable ($g^+(x) = \max\{g(x), 0\}$.)

(e) Evaluate the integral of the function
$$f = \sum_{n=1}^{\infty} 3^{-n} \chi_{[n,n+1)}$$
using (i) Lebesgue measure, and (ii) the Dirac measure δ_0.

(f) Determine which, if any, of the following numbers belong to the Cantor Set:
$$\text{(i)}\ \frac{7}{8}, \qquad \text{(ii)}\ (0.0202021)_3, \qquad \text{(iii)}\ \frac{\pi}{9}.$$

p.t.o.

2. (a) Give the definitions of the terms *σ-algebra* and *measure*. If (E_n) is an increasing sequence of measurable sets and μ is a measure, show that

$$\mu\Big(\bigcup_{n=1}^{\infty} E_n\Big) = \lim_{n\to\infty} \mu(E_n)\,.$$

State, without proof, the corresponding result for decreasing sequences.

(b) Let (F_n) be a sequence of measurable sets. Give the definition of the set $\limsup F_n$ and show that, if μ is a finite measure, then

$$\limsup \mu(F_n) \le \mu\big(\limsup F_n\big)\,.$$

(c) Explain how the Borel-Cantelli Lemmas are used to find the probability of the event $\limsup F_n$ in a probability space.

3. (a) Give the definition of a *measurable function*. Use your definition to show that the function $f(x) = 3x^2 + 1$ is measurable with respect to the Borel σ-algebra $\mathcal{B}$. ($\mathcal{B}$ is the σ-algebra of subsets of $\mathbb{R}$ generated by the intervals.)

(b) Show that every measurable function is the limit of a sequence of measurable simple functions. Use the graph of the function in part (a) to illustrate your answer.

(c) In the case of the function f in part (a), does the sequence of simple functions in part (b) converge *uniformly* to f? Explain your answer.

4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give an account of the definition of the integral $\int_X f\, d\mu$ for non-negative measurable functions f on X.

(b) Give an example of a sequence of functions for which term by term integration is not valid. State Lebesgue's Dominated Convergence Theorem and explain why it cannot be applied to the example you have given.

(c) State and prove Lebesgue's Monotone Convergence Theorem.

5. (a) Give a brief description of the Lebesgue measure space $(\mathbb{R}, \mathcal{M}, m)$. The function $f\colon \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ e^{-x^2} & \text{if } x \notin \mathbb{Q} \end{cases}$$

Explain why f is Lebesgue integrable.

(b) Give the definition of the Cantor Set C and show that C is a Lebesgue measurable set.

(c) Show that there exist subsets of $\mathbb{R}$ that are not Lebesgue measurable.