NUI Galway
OÉ Gaillimh

Semester 1 Examinations 2016/ 2017

Exam Code(s)	4BS2, 4BMS2
Exam(s)	Bachelor of Science (Hons), Mathematical Science (Hons)
Module Code(s)	MA416/MA538
Module(s)	Rings
Paper No.	1
Repeat Paper	No
External Examiner(s)	Prof. Mark Lawson
Internal Examiner(s)	Prof. Dane Flannery.
Instructions:	**Students should attempt ALL questions.** (All questions carry equal marks.)
Duration	2 hours
No. of Pages	3 (including title page).
Discipline(s)	Mathematics
Course Co-ordinator(s)	

Requirements:

Release in Exam Venue Yes [√] No [ ]

PTO

1. (a) Justify that each of the following is a ring. Also state whether the ring is commutative, and whether it contains a 1.

(i) $n\mathbb{Z} = \{na \mid a \in \mathbb{Z}\}$ where n is any integer, under ordinary addition and multiplication of integers.

(ii) The set of 3×3 matrices with entries in a field $\mathbb{F}$ and zeros everywhere below the main diagonal, under ordinary addition and multiplication of matrices.

(iii) The set of rational numbers $\frac{m}{n}$, where $m \in \mathbb{Z}$ and n is a power of a fixed prime $p \in \mathbb{Z}$, under ordinary addition and multiplication of rational numbers.

[**8 marks**]

(b) Let I be an ideal of a ring R. Define addition and multiplication on the set $R/I = \{r + I \mid r \in R\}$ of cosets of I in R that turn R/I into a ring (explain why each binary operation is well-defined).

[**5 marks**]

(c) Let $\phi : R \to S$ be a ring homomorphism. Prove that

(i) $\ker \phi$ is an ideal of R

(ii) $\phi(R) = \{\phi(r) \mid r \in R\}$ is a subring of S isomorphic to $R/\ker \phi$.

[**7 marks**]

2. (a) State, with justification, whether each of the following pairs of rings are isomorphic (demonstrate an isomorphism if one exists).

(i) $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}$.

(ii) $\mathbb{Z}_2 \oplus \mathbb{Z}_3$ and $\mathbb{Z}_6$.

(iii) $\mathbb{C}$ and the subring $\left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ of $\mathrm{Mat}(2, \mathbb{R})$.

[**7 marks**]

(b) Let $n \in \mathbb{Z}$, $n \geq 2$. Prove that the following are equivalent: $\mathbb{Z}_n$ is an integral domain; $\mathbb{Z}_n$ is a field; n is prime.

[**7 marks**]

(c) Let R be an integral domain.

(i) Show that if I is a maximal ideal of R then R/I is a field.

(ii) Is a maximal ideal of R necessarily a prime ideal of R? Why?

[**6 marks**]

P.T.O.

3. (a) Let R be a UFD. Prove that the irreducible elements of R are precisely the prime elements of R.
[**7 marks**]

(b) Prove that $\mathbb{Z}[\sqrt{-5}]$ is an integral domain that is not a UFD (unique factorization domain).
[**8 marks**]

(c) Do 6 and $2 + 2\sqrt{-5}$ have a greatest common divisor in $\mathbb{Z}[\sqrt{-5}]$? Why?
[**5 marks**]

4. (a) Let R be an integral domain, and $a_1, \ldots, a_n$ be non-zero elements of R. Prove that $\ell \in R$ is a least common multiple of the a_is if and only if $a_1 R \cap \cdots \cap a_n R = \ell R$.
[**6 marks**]

(b) Let R be a PID (principal ideal domain), S be any integral domain, and $\phi : R \to S$ be a ring epimorphism. Show that either ϕ is an isomorphism or S is a field.
[**8 marks**]

(c) Prove that $\mathbb{Z}[x]$ is not a PID, by showing that the ideal generated by 2 and x in $\mathbb{Z}[x]$ is not principal.
[**6 marks**]

5. (a) What is the ACC (ascending chain condition) in a commutative ring? Let R be a PID. Prove that each non-zero element of R that is not a unit can be expressed as a product of finitely many irreducibles (assume that R satisfies the ACC).
[**10 marks**]

(b) (i) Define *Euclidean domain* (ED). Explain why $\mathbb{Z}$ and $\mathbb{F}[x]$ for any field $\mathbb{F}$ are EDs.
(ii) Prove that every ED is a PID.
(iii) Give an example of a PID that is not an ED.
[**10 marks**]

END