OLLSCOIL NA GAILLIMHE
UNIVERSITY OF GALWAY

Semester I Examinations 2023-24

Course Instance Code(s)	4BS2, 4BMS2, 4FM2, 1OA2
Examinations	4th Science
Module Codes	MA490
Module	Measure Theory
External Examiner(s)	Prof C. Roney-Dougal
Internal Examiner(s)	Dr T. Rossmann
	*Dr James Cruickshank
Duration	2 hours
No. of Pages	3 pages (including this cover page)
Subject	Mathematics
Instructions	**Attempt ALL questions**.

Requirements

Release in exam venue		Yes
MCQ answer aheet	No	
Handout	No	
Formula and Tables*		Yes
Cambridge Tables 2nd Edition	No	
Other materials	No	
Graphic material in colour	No	

Q1 (40%) Answer **five** of the following. **(8% each)**

(a) Let $X = \{a, b, c, d\}$. Write down all the subsets of X that are in the algebra generated by $\{\{a\}, \{a, b\}\}$.

(b) In Bernoulli space let E_n be the event that the nth term of the sequence is H. Write down a formula, in terms of the events E_n, for the event that HTH appears infinitely often.

(c) For each integer $n \geqslant 1$, let A_n be the interval $\left(\frac{(-1)^n}{n}, 2 + \frac{1}{n}\right)$. Calculate $\liminf A_n$ and $\limsup A_n$.

(d) Let $A = (\mathbb{Q} \cap [0, 1]) \cup ([2, 3] \cap \mathbb{Q}^c)$. Compute the Lebesgue measure of A. Give a brief justification of each step of you computation.

(e) Define $f : [0, 1] \to \mathbb{R}$ by setting $f(x) = 1$ if x is irrational and $f(x) = 2x + 1$ if x is rational. Show that f is a measurable function with respect to the Borel σ-algebra on $\mathbb{R}$.

(f) Evaluate

$$\sum_{n=0}^{\infty} \int_0^{1/2} x^n dx$$

and justify each step of the calculation.

(g) Let

$$f_n(x) = \frac{1}{1 + (x - n)^2}.$$

Show that this sequence of functions converges pointwise but not uniformly on $\mathbb{R}$.

(h) For $n = 1, 2, 3, \cdots$ let $f_n(x) = x^n$. Prove that the sequence (f_n) converges almost uniformly with respect to the Lebesgue measure on the interval $[0, 1]$.

(i) Let

$$f = \sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}} \chi_{[n,n+1)},$$

where χ_A denotes the characteristic function of a subset $A \subset X$. Prove that f is not integrable with respect to Lebesgue measure on $\mathbb{R}$.

(j) Does $(0.12)_3$ belong to the Cantor set? Does $(0.21)_3$ belong to the Cantor set? Briefly explain your answers with reference to an appropriate definition of the Cantor set.

Q2 (20%) (a) Define the following terms: **(2% each)**

(i) a σ-*algebra*.

(ii) an *additive function*.

(iii) a *measure*.

(iv) the *Borel* σ-*algebra*.

(b) **(6%)** Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and let (A_n) be an increasing sequence of sets in $\mathcal{A}$. Prove that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n).$$

(c) **(6%)** In Bernoulli space let $E_n = \{\omega : \omega_n = H\}$ and let $A_n = E_n^c \cap E_{n+1} \cap E_{n+2}$. Calculate $P(\limsup A_n)$. State clearly any theorem or lemma that you use without proof in your calculation.

Q3 (20%) Let X be a set and let $\mathcal{A}$ be a σ-algebra of subsets of X.

(a) **(3%)** Define what it means for a function $f : X \to \mathbb{R}$ to be *measurable* with respect to $\mathcal{A}$.

(b) **(9%)** Suppose that f is a measurable functions on X. Prove that f^2 is also measurable. Hence, or otherwise, prove that if f, g are both measurable functions on X then fg is a measurable function on X.

You can assume without proof that the sum of two measurable functions is measurable and that any constant multiple of a measurable function is measurable.

(c) **(8%)** Give an example, with proof, of a sequence of functions f_n on the interval $[0, 1]$ such that f_n is continuous for all n, $f_n \to 0$ almost everywhere with respect to the Lebesgue measure, but f_n does not converge pointwise to the zero function.

Q4 (20%) Let $(X, \mathcal{A}, \mu)$ be a complete measure space.

(a) **(6%)** Define the *integral of a measurable simple function* on X. Define the *integral of a nonnegative measurable function* $f : X \to \mathbb{R}$. What does it mean to say that a measurable function $f : X \to \mathbb{R}$ is *integrable*?

(b) **(7%)** Let f be a nonnegative measurable function on X. Using the definitions from part (a), prove that $\displaystyle\int f d\mu = 0$ if and only if $f = 0$ almost everywhere.

(c) **(7%)** State two different convergence theorems that allow term by term integration for a sequence or a sequence of integrable functions. Give an example of a sequence or series to which at least one of these theorems can be applied.