Ollscoil na Gaillimhe
University of Galway

Semester 1 Examinations 2023/2024

Course Instance Codes	1OA2, 4BCT1, 4BMS2, 4BS2
Exams	4th Science, Study Abroad
Module Code	MA416
Module	Rings
External Examiner	Prof. C. Roney-Dougal
Internal Examiner	*Dr T. Rossmann
Instructions:	**Answer all four questions.** **Each question carries 25 marks.** **All workings must be shown.**
Duration	2 hours
No. of Pages	3 pages, including this one
Discipline	Mathematics
Requirements:	No special requirements
Release to Library	Yes
Release in Exam Venue	Yes
MCQ	No
Statistical / Log Tables	No
Non-programmable Calculator	Yes
Mathematical Tables	No
Graph paper	No

Q1. (a) Let $R = \{a + b\sqrt{17} : a, b \in \mathbf{Z}\}$.

(i) Show that R is a subring of $\mathbf{R}$.

(ii) Show that $4 + \sqrt{17}$ is a unit of R.

(iii) Using (ii) or otherwise, show that R has infinitely many units. [12 marks]

(b) (i) Give an example (with justification) of a ring R and a non-zero zero divisor $a \in R$.

(ii) Give an example (with justification) of a non-constant unit of $C(\mathbf{R})$, the ring of all continuous functions $\mathbf{R} \to \mathbf{R}$ with pointwise operations.

(iii) Let $\mathbf{R}[\varepsilon]$ denote the ring of dual numbers. Give an example (with justification) of a unit $f \in (\mathbf{R}[\varepsilon])[X]$ with $\deg(f) > 0$. [9 marks]

(c) Exhibit two different ring endomorphisms of the dual numbers $\mathbf{R}[\varepsilon]$. [4 marks]

Q2. (a) Let R be a ring. Let I and J be (two-sided) ideals of R.

(i) Show that the intersection $I \cap J = \{a \in R : a \in I \text{ and } a \in J\}$ is a (two-sided) ideal of R.

(ii) Give the definition of the product IJ. (You do not need to show that this is an ideal of R.)

(iii) Show by means of an explicit example (with justification) that the union $I \cup J = \{a \in R : a \in I \text{ or } a \in J\}$ need not be an ideal of R. [10 marks]

(b) Let R be a ring. Let I be a (two-sided) ideal of R. Give an example of a ring R' and ring homomorphism $f \colon R \to R'$ with $I = \mathrm{Ker}(f)$. [4 marks]

(c) Let R be a commutative ring and let I be an ideal of R. Show that I is maximal if and only if R/I is a field. [6 marks]

(d) Let $\mathrm{Fun}(\mathbf{R}, \mathbf{Z}/2\mathbf{Z})$ denote the ring of all functions $\mathbf{R} \to \mathbf{Z}/2\mathbf{Z}$ with pointwise addition and multiplication. Give an example (with justification) of a maximal ideal of $\mathrm{Fun}(\mathbf{R}, \mathbf{Z}/2\mathbf{Z})$. [5 marks]

Q3. (a) Let $f\colon R \to S$ and $g\colon S \to T$ be ring homomorphisms. Let $q\colon R \to R/\operatorname{Ker}(f)$ be the quotient map. Show that there exists a ring homomorphism $h\colon R/\operatorname{Ker}(f) \to T$ such that the diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle g} \\ R/\operatorname{Ker}(f) & \xrightarrow{h} & T \end{array}$$

commutes. [6 marks]

(b) Let R be an integral domain and let $a, b \in R$. Show that $aR = bR$ if and only if a and b are associates. [8 marks]

(c) State the definition of a *Euclidean domain.* [5 marks]

(d) Show that every Euclidean domain is a principal ideal domain. [6 marks]

Q4. Let R denote the integral domain $\mathbf{Z}[\sqrt{10}] = \{a + b\sqrt{10} : a, b \in \mathbf{Z}\}$. In the following, you may use without proof that every element $r \in R$ admits a *unique* representation $r = a+b\sqrt{10}$ with $a, b \in \mathbf{Z}$. For $r \in R$, $r = a+b\sqrt{10}$ with $a, b \in \mathbf{Z}$, define

$$N(r) = a^2 - 10b^2 = (a + b\sqrt{10})(a - b\sqrt{10}).$$

(a) Let $r, s \in R$. Show that $N(rs) = N(r)N(s)$. [6 marks]

(b) Let $r \in R$. Show that $r \in R^\times$ if and only if $N(r) = \pm 1$. [5 marks]

(c) Show that 2 is an irreducible element of R. [5 marks]

(d) Using $6 = 2 \cdot 3 = (4 + \sqrt{10})(4 - \sqrt{10})$ or otherwise, show that $2R$ is not a prime ideal of R. [6 marks]

(e) Show that R is not a unique factorisation domain. [3 marks]