Semester I Examinations 2021/2022

Exam Codes	3BMS2, 4BMS2, 3BS9, 4BS2, 1OA1, 1HA1
Exam	Third and Fourth Science Examination
Module	Euclidean and Non-Euclidean Geometry 1
Module Code	MA3101/MA544
Paper No.	1
External Examiner	Prof. Colva Roney-Dougal
Internal Examiners	Prof. Goetz Pfeiffer
	Dr. John Burns
Instructions	**Answer all questions.**
Duration	2 hours
No. of Pages	3 pages including this page
School	Mathematical & Statistical Sciences
Requirements	No special requirements
Release to Library:	Yes
Other Materials	Non-programmable calculators

Q1. Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius one and centre at the origin.

(i) [9 marks] Prove that the shortest path joining any two points of S^2 is an arc of a great circle.

(ii) [8 marks] Consider the curve $\gamma(t) = (\cos t, 0, \sin t)$ on S^2. Calculate the geodesic curvature of $\gamma(t)$.

(iii) [8 marks] Use the spherical cosine rule in the geodesic triangle whose vertices are New York (41 degrees N, 74 degrees W), New Orleans (30 degrees N, 90 degrees W) and the north pole to find the distance from New Orleans to New York. Take the Earth to be a sphere of radius 6,500 km.
(Recall: The spherical cosine rule for a geodesic triangle with angles α, β, γ and corresponding opposite sides a, b, c says that
$\cos a = \cos \alpha \sin b \sin c + \cos b \cos c$.)

Q2. Let $H \subset \mathbb{C}$ be the upper half plane with the hyperbolic metric (the Hyperbolic Plane).

(i) [6 marks] Prove than any matrix $M \in SL(2, \mathbb{R})$ representing a Möbius transformation is an isometry of H.

(ii) [6 marks] Prove that the positive $y-$axis is a geodesic of the Hyperbolic Plane.

(iii) [6 marks] Prove that the semicircle C in H with polar equation $C = \{z \in \mathbb{C} : z = 6\cos\theta e^{i\theta};\ \theta \in (0, \pi/2)\}$ is a geodesic of H.

(iv) [7 marks] Find the area of the hyperbolic (geodesic) triangle with vertices $(2, 0), (2, 1)$ and $(3, 0)$.

Q3. Recall that for a surface patch $x(u, v)$ of an oriented surface M in $\mathbb{R}^3$, the matrix of the shape operator or Weingarten map with respect to the basis $\{x_u, x_v\}$ is $\Pi_1^{-1}\Pi_2$, where Π_1 and Π_2 are the matrices of the first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$, respectively.

(i) [7 marks] Prove that the mean curvature H of M is given by

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

(ii) [13 marks] Calculate the mean curvature H of the helicoid $x(u, v) = (v \cos u, v \sin u, 2u)$ and hence show that it has negative Gaussian curvature. Write down a parametrised geodesic of the helicoid passing through the point (0,0,2).

(iii) [5 marks] Prove that there is no surface homeomorphic to the torus T^2 in $\mathbb{R}^3$ with everywhere positive Gaussian curvature.

Q4. (a) (i) [5 marks] Determine the rotation in the space of pure imaginary quaternions $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ induced by conjugation by the unit quaternion $t = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$.

(ii) [10 marks] Prove that $SU(2)$ is a double cover of $SO(3)$. Use the fact that an arbitrary element of $SO(4)$ can be realised as $v \to q_1 v q_2^{-1}$, where q_1 and q_2 are unit quaternions to prove that $SO(4)$ is a double cover of $SO(3) \times SO(3)$.

(b) [10 marks] State Synge's theorem. Define the words compact, orientable and simply connected in the statement of the theorem. Give an example of a compact surface having negative curvature at some points for which the conclusion of the theorem does not hold.