Ollscoil na Gaillimhe
University of Galway

Semester 1 Examinations 2022-23

Course Instance Code(s)	4BCT1, 4BS2, 4BMS2, 4FM2, 3BA1
Examinations	4th Science
Module Codes	MA490
Module	Measure Theory
External Examiner(s)	Prof C. Roney-Dougal
Internal Examiner(s)	Prof G. Pfeiffer
	Dr James Cruickshank
Duration	2 hours
No. of Pages	3 pages (including this cover page)
Discipline	Mathematics
Instructions	**Attempt ALL questions.**
Requirements	Release in Exam Venue: Yes

Q1 (40%) Answer **five** of the following. **(8% each)**

(a) Let $X = \{a, b, c, d, e\}$. How many subsets of X are in the σ-algebra generated by $\{\{a, b, c\}\{c, d, e\}\}$.

(b) In Bernoulli space let E_n be the event that the nth term of the sequence is H. Write down a formula, in terms of the events E_n, for the event that HTH appears infinitely often.

(c) For each integer $n \geqslant 1$, let A_n be the interval $\left(\frac{(-1)^n}{n}, n\right)$. Calculate $\liminf A_n$ and $\limsup A_n$.

(d) Let A be the set of numbers in the interval $[0, 1]$ that have a finite decimal expansion. Compute the Lebesgue measure of A.

(e) Define $f : [0, 1] \to \mathbb{R}$ by setting $f(x) = 0$ if x is rational and $f(x) = x$ if x is irrational. Show that f is a measurable function with respect to the Borel σ-algebra on $\mathbb{R}$.

(f) Evaluate $\lim_{n\to\infty} \int_0^1 1 - \frac{e^{-x^2}}{(x+1)^n} dx$ and justify each step of the calculation.

(g) For $n = 1, 2, 3, \cdots$ let $f_n(x) = x^n$. Prove that the sequence (f_n) converges almost uniformly with respect to the Lebesgue measure on the the interval $[0, 1]$.

(h) Evaluate

$$\sum_{n=0}^{\infty} \sum_{k=2}^{\infty} \frac{k-1}{2^k k^{n+1}}.$$

Justify each step of your calculation.

(i) Let

$$f = \sum_{n=1}^{\infty} (-1)^n n^{-1} \chi_{[n,n+1)},$$

where χ_A denotes the characteristic function of a subset $A \subset X$. Prove that f is not integrable with respect to Lebesgue measure on $\mathbb{R}$.

(j) Give an example of a element of the Cantor set that is not the endpoint of one of the intervals that is removed in the standard construction of the Cantor set.

Q2 (20%) (a) Define the following terms: **(2% each)**

(i) a σ-*algebra*.

(ii) a *measure*.

(iii) a *probability space*.

(iv) the *Borel* σ-*algebra*.

(b) **(6%)** Suppose that $(X, \mathcal{A}, \mu)$ is a measure space and let (A_n) be a sequence of sets in $\mathcal{A}$. Prove that

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu(A_n).$$

(c) **(6%)** State the second Borel-Cantelli Lemma. In Bernoulli space let E be the event that infnitely many heads occur. Use the second Borel-Cantelli Lemma to show that $P(E) = 1$.

Q3 (20%) Let X be a set and let $\mathcal{A}$ be a σ-algebra of subsets of X.

(a) **(3%)** Define what it means for a function $f : X \to \mathbb{R}$ to be *measurable* with respect to $\mathcal{A}$.

(b) **(7%)** Suppose that $f : X \to \mathbb{R}$ is measurable. Prove that f^2 is also measurable. Give an example of a measurable space $(Y, \mathcal{B})$ and a function $f : Y \to \mathbb{R}$ such that f^2 is measurable, but f is not measurable.

(c) **(10%)** For each positive integer n, let $f_n : X \to \mathbb{R}$ be a measurable function. Suppose that the sequence (f_n) converges pointwise to the function $f : X \to \mathbb{R}$. Show that f is a measurable function.

Q4 (20%) Let $(X, \mathcal{A}, \mu)$ be a complete measure space.

(a) **(6%)** Define the *integral of a measurable simple function* on X. Define the *integral of a nonnegative measurable function* $f : X \to \mathbb{R}$. What does it mean to say that a measurable function $f : X \to \mathbb{R}$ is *integrable*?

(b) **(7%)** Let f be a nonnegative measurable function on X and let E be measurable set. Using the definitions from part (a), prove that $\int_E f d\mu = 0$ if and only if $f = 0$ almost everywhere.

(c) **(7%)** State the Lebesgue Monotone Convergence Theorem (no proof is required). Use this theorem to prove that if f and g are nonnegative measurable functions on X then

$$\int_X f + g \; d\mu = \int_X f \; d\mu + \int_X g \; d\mu.$$