Semester 1 Examinations 2017/2018

Exam Codes	4BS3, 4BS9, 4FM2, 4BMS2, 3BA1, 4BA4, 1AL1
Exams	B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics B.Sc. M.Sc. Mathematics/Mathematical Science
Module	Measure Theory
Module Code	MA490
External Examiner	Prof. Mark Lawson
Internal Examiner(s)	Prof. Graham Ellis Dr Ray Ryan
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	3 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

Q1 [**49%**] Answer **five** of the following:

(a) Let X be a set with n elements. How many algebras of subsets of X contain exactly five elements? Explain your answer.

(b) In Bernoulli Space Ω, let E_n be the event that the nth toss is heads. Write down a formula in terms of these sets of the event that the sequence $HTHT$ appears infinitely often.

(c) Let $A_n = (-1 + 1/n, 2 - 1/n)$ if n is odd and $[0, n]$ if n is even. Find $\liminf A_n$ and $\limsup A_n$.

(d) In Bernoulli space, let A be the event that Heads is tossed exactly twice. Find the probability of A.

(e) Let A be the set of irrational numbers in the interval $[0, 1]$. Find the Jordan content and the Lebesgue measure of A.

(f) (i) Show that the function $f(x) = x^2 + 1$ is measurable with respect to the σ-algebra of Borel subsets of $\mathbb{R}$.

(ii) Give an example of a σ-algebra for which this function is not measurable.

(g) Let

$$f_n(x) = \frac{3nx}{x^2 + 4n} .$$

Show that this sequence converges pointwise but not uniformly on $\mathbb{R}$.

(h) Use term by term integration to evaluate

$$\lim_{n\to\infty} \int_{\pi/4}^{\pi/3} (1 - \sin^n x)\, dx .$$

Which of the convergence theorems can be used to justify your calculations?

(i) Give an example of a sequence of integrable functions on $\mathbb{R}$ for which term by term integration is not valid. Explain why Lebesgue's Dominated Convergence Theorem does not apply to your example.

(j) Which of the following numbers belong to the Cantor set?

$$\text{(i) } \frac{79}{81}, \quad \text{(ii) } (0.220212)_3 , \quad \text{(iii) } \sqrt{56} .$$

Explain your answer in each case.

You must answer every question.

Q2 [17%]

(a) Give the definitions of the following: an algebra, a σ-algebra, an additive set function and a measure. **[4%]**

(b) Let $(X, \mathcal{A}, \mu)$ be a measure space.

(i) If (A_n) is a sequence of measurable sets, show that **[5%]**

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu(A_n)\,.$$

(ii) If (A_n) is an increasing sequence of measurable sets, show that **[5%]**

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)\,.$$

(iii) *State* the result similar to part (ii) for a decreasing sequence of measurable sets. **[3%]**

Q3 [17%] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) What does it mean for a function $f\colon X \to \mathbb{R}$ to be measurable? **[2%]**
If f and g are measurable, show that the function $f - g$ is also measurable. **[6%]**

(b) Let (f_n) be a sequence of measurable functions.

(i) What does it mean to say that (f_n) converges pointwise to a function f? **[2%]**

(ii) If (f_n) converges pointwise to f, show that f is a measurable function. **[7%]**

Q4 [17%] Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Write a short account of the definition of the integral for measurable functions on X, starting with the integral for simple, measurable functions. **[5%]**

(b) Let f be a non-negative measurable function. If $\int_X f\, d\mu = 0$, show that $f(x) = 0$ almost everywhere. **[6%]**

(c) State *two* convergence theorems that allow term by term integration for a sequence or series of integrable functions. Give an example of a sequence or series to which one of these theorems can be applied. **[6%]**