Semester I Examinations 2017/2018

Exam Codes	4BS2, 4BMS2, 1AL1
Exams	Bachelor of Science (Hons), Mathematical Science (Hons), Master of Science (Mathematics)
Module	Ring Theory
Module Code	MA416, MA538
External Examiner	Prof. T. Brady
Internal Examiners	Prof. G. Ellis Prof. D. Flannery
Instructions	**Answer all four questions**
Duration	2 hours
Number of pages	3, including this one
School	Mathematics, Statistics and Applied Mathematics

Requirements

Release to Library	Yes
Release in Exam Venue	No

1. (a) State, with justification, whether each of the following is a ring.

(i) The real numbers, with addition defined as usual, and multiplication defined by $ab = ba = 0$ for all real numbers a, b.

(ii) $\{n/2^a \mid n, a \in \mathbb{Z}\}$, under ordinary addition and multiplication of rational numbers.

(iii) The set of 3×3 matrices with entries in a field $\mathbb{F}$ and zeros everywhere above the main diagonal, under ordinary addition and multiplication of matrices.

(iv) The set of all polynomials in $R[x]$ with zero constant term, where R is any ring.

[**11 marks**]

(b) Let I be an ideal of a ring R. Define addition and multiplication on the set $R/I = \{r + I \mid r \in R\}$ of cosets of I in R that turn R/I into a ring (explain why each binary operation is well-defined).

[**6 marks**]

(c) (i) Let $\phi : R \to S$ be a ring epimorphism. Prove that $R/\ker\phi \cong S$.

(ii) Let $n \geq 2$ be an integer. Apply (i) to obtain an isomorphism between $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_n$.

[**8 marks**]

2. (a) State, with justification, whether each of the following pairs of rings are isomorphic (demonstrate an isomorphism if one exists).

(i) $2\mathbb{Z}$ and $3\mathbb{Z}$.

(ii) $\mathbb{Z}_3 \oplus \mathbb{Z}_7$ and $\mathbb{Z}_{21}$.

(iii) $\mathbb{C}$ and the subring consisting of all matrices $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ in $\mathrm{Mat}(2, \mathbb{R})$.

[**9 marks**]

(b) (i) Prove that if p is a prime then $\mathbb{Z}_p$ is a field.

(ii) Does 5 have a multiplicative inverse in $\mathbb{Z}_{42}$? If so, calculate the inverse; if not, explain why not.

[**7 marks**]

(c) Let R be a commutative ring with 1.

(i) Let I be an ideal of R. Prove that $J = \{ar + s \mid r \in R, s \in I\}$ is an ideal of R for any $a \in R$. Deduce that if I is a maximal ideal of R then R/I is a field.

(ii) Is a maximal ideal of R necessarily a prime ideal? Is a prime ideal of R necessarily a maximal ideal? Why?

[**9 marks**]

3. (a) Let R be a UFD (unique factorization domain). Prove that the irreducible elements of R are precisely the prime elements of R.

[**9 marks**]

(b) Prove that $\mathbb{Z}[\sqrt{-6}]$ is an integral domain that is not a UFD.

[**10 marks**]

(c) Explain why 5 and $2+\sqrt{-6}$ have a greatest common divisor of 1 in $\mathbb{Z}[\sqrt{-6}]$. Do there exist $a,\, b \in \mathbb{Z}[\sqrt{-6}]$ such that $5a + (2+\sqrt{-6})b = 1$? Justify your answer.

[**6 marks**]

4. (a) Let R be a PID (principal ideal domain), S be any integral domain, and $\phi : R \to S$ be a ring epimorphism. Prove that either ϕ is an isomorphism or S is a field.

[**8 marks**]

(b) (i) Define *Euclidean domain* (ED). Prove that every ED is a PID.

(ii) Calculate a greatest common divisor of $x^3 + 2x^2 + 4x - 7$ and $x^2 + x - 2$ in $\mathbb{Q}[x]$.

[**11 marks**]

(c) Let R be a commutative ring with 1. Show that the polynomial ring $R[x]$ is a PID if and only if R is a field.

[**6 marks**]

END