NUI Galway
OÉ Gaillimh

Semester I Examinations 2013/2014

Exam Code(s)	4BCT1, 4BMS2, 4BS4, 1HA1
Exam(s)	Bachelor of Science (Applied Mathematics) Bachelor of Science (CS & IT) Bachelor of Science (Mathematical Science) Higher Diploma in Applied Science (Mathematics)
Modules	Ring Theory, Advanced Algebra I
Module Codes	MA416, MA538
Paper	1
External Examiner(s)	Dr Colin Campbell
Internal Examiner(s)	Dr Götz Pfeiffer Dr Emil Sköldberg
<u>Instructions:</u>	**Students should attempt ALL questions.** (All questions carry equal marks.)
Duration	2 Hours
No. of Pages	3 (including this page)
Discipline	Mathematics

1. (a) In each of the following cases, determine if the object is a ring. If you believe that it is a ring, it is enough to say so; if not, you should give a reason.

      (i) The set of all upper-triangular 2×2 matrices with integer entries under the usual addition and multiplication of matrices.

      (ii) The set of all polynomials $\sum_{i \geq 0} a_i x^i$ in $\mathbb{Q}[x]$ such that $a_i = 0$ whenever i is even.

   (b) Find all units and zerodivisors in the rings

      (i) $\mathbb{Z}[i]$,

      (ii) $\mathbb{Z}_2[x]/(x^2)$.

   (c) Show that if u is an element in a ring with $u^4 = 0$, then $1 - u$ is a unit.

   (d) Show that for r, s elements of a ring R, we have (i) $0 \cdot r = 0$ and (ii) $(-r)s = r(-s) = -(rs)$.

2. (a) Let R and S be rings. What is meant by a *ring homomorphism* from R to S?

   (b) If $\varphi : R \longrightarrow S$ is a ring homomorphism; define its *kernel*. Also explain what a *(two-sided) ideal* is, and show that the kernel of φ is an ideal.

   (c) Show that for every ring R, there is a unique ring homomorphism $\varphi : \mathbb{Z} \longrightarrow R$.

   (d) What is meant by a *principal ideal domain* (PID)? Let F be a field. Show that the polynomial ring $F[x]$ is a principal ideal domain.

3. (a) Give the definition of a *primitive* polynomial in $\mathbb{Z}[x]$, and show that the product of two primitive polynomials is primitive.

   (b) What is meant by an *irreducible polynomial* in $F[x]$, where F is a field? Give an example of a field F and a polynomial $f(x)$ in $F[x]$ such that $f(x)$ is reducible in $F[x]$, but $f(x)$ does not have a root in F.

   (c) For each of the following polynomials in $\mathbb{Q}[x]$, determine if it is irreducible or not, and if it is reducible, factor it into irreducibles.

   (i) $x^3 - 2x^2 + 3x - 1$.

   (ii) $x^4 - 4x^3 + 8x^2 - 10x + 6$.

   (d) Find all irreducible monic polynomials of degree 2 in $\mathbb{Z}_3[x]$.

4. (a) What is a *maximal ideal* in a commutative ring R? Show that the ideal $I \subseteq R$ is maximal if and only if R/I is a field.

   (b) Let I and J be ideals in the commutative ring R. Define the sets IJ and $I \cap J$ and show that they are ideals in R.

   (c) Let R be a PID, and let $I = (r)$, $J = (s)$ be two ideals in R. Show that the ideal $IJ = (rs)$.

   (d) Determine all maximal ideals in $R = \mathbb{Z}_2[x]/(x^4 + x)$, and for each maximal ideal I, determine the order of R/I.

5. (a) Give the definition of a *Euclidean ring*.

   (b) Show that the ring of integers $\mathbb{Z}$ is a Euclidean ring.

   (c) Show that in a Euclidean ring, $d(a) = d(1)$ if and only if a is a unit.

   (d) Given the elements $6+2i$ and $1-2i$ in $\mathbb{Z}[i]$, compute their greatest common divisor. Hence, or otherwise, find a generator for the ideal $I + J$, where $I = (6 + 2i)$ and $J = (1 - 2i)$.