Semester I Examinations 2016/2017

Exam Codes	3BMS2, 4BMS2, 3BS9, 4BS2, 1OA1, 1HA1
Exam	Third and Fourth Science Examination
Module	Euclidean and Non-Euclidean Geometry 1
Module Code	MA3101/MA544
Paper No.	1
External Examiner	Dr Mark Lawson
Internal Examiners	Prof. Graham Ellis
	Dr John Burns
Instructions	**Answer all questions.**
Duration	2 hours
No. of Pages	3 pages including this page
School	Mathematics, Statistics & Applied Mathematics
Requirements	No special requirements
Release to Library:	Yes
Other Materials	Non-programmable calculators

Q1. Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius one.

(i) [9 marks] Prove that the shortest path joining any two points of S^2 is an arc of a great circle.

(ii) [8 marks] Prove than $SO(3)$ acts transitively on S^2.

(iii) [8 marks] A sailor circumnavigates Australia by a route consisting of a geodesic triangle. Prove that at least one interior angle of the triangle is $\geq \pi/3 + 10/169$ radians. (Take the Earth to be a sphere of radius 6,500 km and assume that the area of Australia is 7.5 million square km.)

Q2. Let $H \subset \mathbb{C}$ be the upper half plane with the hyperbolic metric (the Hyperbolic Plane).

(i) [6 marks] Prove than any matrix $M \in SL(2, \mathbb{R})$ representing a Möbius transformation is an isometry of H.

(ii) [6 marks] Prove that the positive $y-$axis is a geodesic of the Hyperbolic Plane.

(iii) [6 marks] Find a Möbius transformation that maps the semicircle C in H with polar equation $C = \{z \in \mathbb{C} : z = 6 \cos\theta e^{i\theta}; \theta \in (0, \pi/2)\}$ onto the positive y-axis.

(iv) [7 marks] Find the area of the hyperbolic (geodesic) triangle with vertices $(0, 0), (1, 0)$ and $(-1, 0)$.

Q3. Recall that for a surface patch $x(u, v)$ of an oriented surface M in $\mathbb{R}^3$, the matrix of the shape operator or Weingarten map with respect to the basis $\{x_u, x_v\}$ is $\Pi_1^{-1}\Pi_2$, where Π_1 and Π_2 are the matrices of the first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$, respectively.

(i) [7 marks] Prove that the mean curvature H of M is given by

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

(ii) [12 marks] Calculate the mean curvature H of the helicoid $x(u, v) = (v \cos u, v \sin u, 2u)$ and hence show that it has negative Gaussian curvature.

(iii) [6 marks] Prove that there is no surface homeomorphic to S^2 in $\mathbb{R}^3$ with everywhere non-positive Gaussian curvature.

Q4. (a) (i) [8 marks] Determine the rotation in the space of pure imaginary Quaternions $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ induced by conjugation by the unit quaternion $t = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$.

(ii) [7 marks] Prove that $SU(2)$ is a double cover of $SO(3)$ and that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.

(b) [10 marks] State Synge's theorem, explaining the assumptions of the theorem. Give an example of a surface with negative curvature for which the conclusion of the theorem does not hold.