Semester 1 Examinations 2018/2019

Exam Codes	4BS3, 4BS9, 4FM2, 4BMS2, 3BA1, 4BA4, 1AL1
Exams	B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics M.Sc. Mathematics/Mathematical Science
Module	Measure Theory
Module Code	MA490
External Examiner	Prof. T. Brady
Internal Examiner(s)	Prof. Graham Ellis Dr Nina Snigireva
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	4 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

Q1 [**49%**] Answer **five** of the following:

(a) Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and let

$$\mathcal{A} = \{\varnothing, X, \{1, 3, 5\}, \{2, 4, 6\}, \{1, 2, 3, 4, 5, 6\}, \{1, 3, 5, 7\}, \{2, 4, 6, 8\}, \{7, 8\}\}.$$

Is $\mathcal{A}$ a σ-algebra on X? Justify your answer.

(b) In the share price movement example, we let $X = \{\omega = \omega_1, \omega_2 \ldots \mid \omega_n = u \text{ or } d\}$ and we let E_n be the event that the share price moves up at time $t = n$. Write down a formula in terms of these sets of the event that the share price never moves up three times in a row.

(c) Let $A_n = [0, 4 + 1/n)$ if n is odd and $[1, n + 2]$ if n is even. Find $\liminf A_n$ and $\limsup A_n$.

(d) In the share price movement example, let A be the event that the share price never moves up. Find, with a justification, the probability of A.

(e) Find the Lebesgue measure of the set of numbers in the interval $[0, 1]$ that do not have the digit "4" in the first or second place in their decimal expansion.

(f) (i) Show that the function $f(x) = 1 - x^2$ is measurable with respect to the Borel σ-algebra of subsets of $\mathbb{R}$.

(ii) Give an example of a σ-algebra for which this function is not measurable.

(g) Give an example, with a justification, of a sequence of functions that converges pointwise but not uniformly.

(h) Find the value of the following sum:

$$\sum_{n=0}^{\infty} \int_{\pi/4}^{\pi/3} (1 - \sqrt{\cos x})^n \sin x \, dx \, .$$

Justify each step in your calculations.

(i) Is the function $f = \sum_{n=0}^{\infty} 2^{-n} \chi_{[n,n+1)}$ Lebesgue integrable on $\mathbb{R}$? Justify your answer.

(j) Which of the following numbers belong to the Cantor set?

$$\text{(i) } \frac{8}{27}, \quad \text{(ii) } (0.2200221)_3, \quad \text{(iii) } \frac{1}{2}.$$

Explain your answer in each case.

continued on the next page

You must answer every question.

Q2 [**17%**]

(a) Give the definitions of the following: a σ-algebra and a measure. **[2%]**

(b) Let $(X, \mathcal{A}, \mu)$ be a finite measure space.

(i) If $(A_n)_n$ is a decreasing sequence of measurable sets, show that **[5%]**

$$\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)\,.$$

(You may use, without a proof, the fact that if $(B_n)_n$ is an increasing sequence of measurable sets then $\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} \mu(B_n)$.)

(ii) Give an example of a sequence $(E_n)_n$ and a measure μ so that **[4%]**

$$\mu(\liminf E_n) < \liminf \mu(E_n) < \limsup \mu(E_n) < \mu(\limsup E_n)\,.$$

(c) In the share price movement example, let A_n be the event that the share price moves up three times in a row from time $t = n$. Use, with a justification, the first or the second Borel-Cantelli lemma to find $P[A_n, i.o.]$. **[6%]**

Q3 [**17%**] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) What does it mean for a function $f : X \to \mathbb{R}$ to be measurable? **[2%]**

(b) If f and g are measurable and $\alpha, \beta \in \mathbb{R}$ show that the function $\alpha f + \beta g$ is also measurable. **[7%]**

(c) (i) Suppose that f is a measurable function. Is $|f|$ measurable? (Give a proof or a counterexample.) **[3%]**

(ii) Suppose that $|f|$ is a measurable function. Is f measurable? (Give a proof or a counterexample.) **[3%]**

(iii) Let $X = \mathbb{R}$ and let $f(x) = 3$ if x is rational and $f(x) = 1$ if x is not. What is the *smallest* σ-algebra of subsets of $\mathbb{R}$ with respect to which f is measurable? **[2%]**

continued on the next page

Q4 [17%] Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Write a short account of the definition of the integral for measurable functions on X, starting with the integral for simple, measurable functions. **[5%]**

(b) Let $X = \mathbb{R}$, $f = 4\chi_{[-2,2]} - \chi_{[0,4]} - 2\chi_{\{3\}}$.
Evaluate the integral $\int_X f \, d\mu$ (if possible) in each case:

(i) μ is Lebesgue measure, **[2%]**
(ii) μ is counting measure, **[2%]**
(iii) μ is the Dirac measure δ_3. **[2%]**

(c) State *three* convergence theorems that allow term by term integration for a sequence or series of integrable functions. In each case, give an example of a sequence or series to which the theorem can be applied. **[6%]**