Semester 1 Examinations 2013/2014

Exam Codes	4BS3, 4BS9, 4FM2, 4BMS2, 3BA1, 4BA4
Exams	B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics B.Sc. B.A. (Mathematics)
Module	Measure Theory
Module Code	MA490
External Examiner	Dr Colin Campbell
Internal Examiner(s)	Dr Goetz Pfeiffer Dr Ray Ryan
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	3 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

Q1 [**49%**] Answer *six* of the following:

(a) Let X be a set with five elements. How many algebras of subsets of X contain exactly four subsets?

(b) In Bernoulli Space Ω, let E_n be the event that the nth toss is heads. Write down a formula in terms of the E_n for the following event: "Every time two Heads appear in succession, the next two tosses are Tails".

(c) Find the Jordan content of the set $\{1, 1/2, 1/4, 1/8, \ldots\}$.

(d) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function $f(x) = 1 - x^2$. Prove directly from the definition of a measurable function that f is Borel measurable.

(e) Let $f_n(x) = x(1-x)^n$. Prove that f_n converges uniformly to zero on the interval $[0, 1]$.

(f) If f is a nonnegative measurable function, then there is an increasing sequence (f_n) of simple measurable functions that converges pointwise to f. Give the construction of the first two terms, f_1 and f_2, in such a sequence.

(g) Consider the following sequence of random variables on Bernoulli Space Ω:

$$X_n(\omega) = \begin{cases} 1 & \text{if } \omega \text{ has a run of } n \text{ Heads starting with the } n\text{th toss,} \\ 0 & \text{otherwise.} \end{cases}$$

Show that X_n converges to 0 in probability. Does this sequence converge pointwise? Explain your answer.

(h) Let μ be counting measure on the σ-algebra $\mathcal{P}(\mathbb{N})$. What is the integral $\displaystyle\int_{\mathbb{N}} s\,d\mu$ for a function $s\colon \mathbb{N} \to \mathbb{R}$? What does it mean for s to be integrable?

(i) Give an example of a sequence (f_n) of Lebesgue integrable functions on $\mathbb{R}$ that converges but for which term by term integration is not valid. State one of the convergence theorems and explain why it does not apply to your example.

(j) Find the value of the following sum:

$$\sum_{n=1}^{\infty} \int_0^{\pi/2} \left(1 - \sqrt{\sin x}\right)^n \cos x \, dx\,.$$

Justify each step in your calculation.

(k) Give the definition of the Cantor Set C. Which of the following numbers belong to C?

$$\text{(i) } \frac{5}{6}, \quad \text{(ii) } \frac{20}{81}, \quad \text{(iii) } (0.02021)_3\,.$$

You must answer every question.

Q2 [**17%**]

(a) Give the definition of the Jordan Content, $c(E)$, of a set E of real numbers. Explain why the set function c is not additive.

(b) Let μ be a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X. Show that if (E_n) is a sequence of measurable subsets of X, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n=1}^{\infty} \mu(E_n)\,.$$

(c) Give the definition of the lower limit and the upper limit of a sequence of sets. State one result that enables you to find the measure of a lower limit or an upper limit set.

Q3 [**17%**] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) Give the definition of a measurable function and use it to show if f is measurable then so is f^2.

(b) Let (f_n) be a sequence of measurable functions on X. Give the definition of each of the following:
(i) f_n converges to f almost everywhere.
(ii) f_n converges to f almost uniformly.
(iii) f_n converges to f uniformly.
(iv) f_n converges to f in measure.
Give an example of a sequence that satisfies (i) but not (ii) and a sequence that satisfies (ii) but not (iii).

(c) State Egoroff's Theorem for sequences of measurable functions on X when $\mu(X)$ is finite. Give an example to show that the finiteness condition is essential.

Q4 [**17%**]

(a) Give a brief account of the construction of the Riemann integral. Explain briefly how Lebesgue's construction took a different approach.

(b) Let $f(x) = 1$ if x is a rational number and $f(x) = 0$ if x is irrational. Explain why f is not Riemann integrable on the interval $[0, 1]$. What is the value of the Lebesgue integral of f on $[0, 1]$?

(c) Let f be a nonnegative measurable function on a measure space $(X, \mathcal{A}, \mu)$. Show that if $\displaystyle\int_X f\, d\mu = 0$ then $f(x) = 0$ almost everywhere.