Semester I Examinations 2014/2015

Exam Code(s)	4BMS2, 4BS42, 1OA1, 1HA1
Exam(s)	Bachelor of Science (Hons.) Bachelor of Science (Mathematical Science) Honours Higher Diploma in Applied Science (Mathematics)
Modules	Ring Theory, Advanced Algebra I
Module Codes	MA416, MA538
Paper	1
External Examiner(s)	Dr Mark Lawson
Internal Examiner(s)	Prof. Graham Ellis Dr Emil Sköldberg
Instructions:	**Students should attempt ALL questions.** (All questions carry equal marks.)
Duration	2 Hours
No. of Pages	4 (including this page)
Discipline	Mathematics

1. (a) In each of the following cases, determine if the object is a ring. If you believe that it is a ring, it is enough to say so; if not, you should give a reason.

(i) The set of *all* 2×2 matrices with entries from $\mathbb{N}$ under the usual addition and multiplication of matrices.

(ii) The set of all 2×2 matrices with real entries and nonzero determinant under the usual addition and multiplication of matrices.

(iii) The set of all rational numbers of the form $a/5$ where a is an integer, with the usual addition and multiplication of rational numbers.

(b) What is a *unit* in a ring? What is a *zero-divisor* in a ring? Show that the units of a ring form a group.

(c) Find all units and zero-divisors in the rings

(i) $\mathbb{Z}[i]$,

(ii) $\mathbb{Z}_2[x]/(x^2 + 1)$.

2. (a) Let R and S be rings. What is meant by a *ring homomorphism* from R to S?

(b) Let R be the set of all matrices in $M_2(\mathbb{Z})$ of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

(i) Show that R is a subring of $M_2(\mathbb{Z})$.

(ii) Show that the function $\varphi : \mathbb{Z}[i] \to R$ defined by

$$\varphi(a + bi) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is a ring homomorphism.

(iii) Show that φ is injective and surjective, and hence an isomorphism.

(c) Show that if $\varphi : \mathbb{Z}[i] \to \mathbb{Z}[i]$ is a ring homomorphism, then φ is completely determined by $\varphi(i)$. Hence, or otherwise, show that there are exactly two ring homomorphisms $\mathbb{Z}[i] \to \mathbb{Z}[i]$, namely $\varphi_1(z) = z$ and $\varphi_2(z) = \bar{z}$.

P.T.O.

3. (a) State and prove Eisenstein's irreducibility criterion.

(b) What is meant by an *irreducible polynomial* in $F[x]$, where F is a field? Give an example of a polynomial $f(x)$ in $\mathbb{Z}_2[x]$ such that $f(x)$ is reducible in $\mathbb{Z}_2[x]$, but $f(x)$ does not have a root in $\mathbb{Z}_2$.

(c) For each of the following polynomials, determine, with explanation, if it is irreducible in the indicated ring. In the case the polynomial is reducible, factor it into irreducibles.

(i) $5x^4 + 9x^3 - 12x^2 + 3x - 6$, in $\mathbb{Q}[x]$.

(ii) $x^4 - 4x^3 + 6x^2 - 5x - 12$, in $\mathbb{Z}[x]$.

(iii) $3x^3 + 2x^2 + 1$, in $\mathbb{Z}_5[x]$.

(iv) $x^4 - 1$, in $\mathbb{R}[x]$.

4. (a) Explain what a (two-sided) ideal in a ring R is. Describe the quotient ring R/I (i.e. describe its elements and the addition and multiplication – you do *not* have to show that it satisfies the ring axioms).

(b) What is meant by a *principal ideal domain* (PID)? Show that $F[x]$ is a PID when F is field.

(c) Determine all ideals in the ring $R = \mathbb{Z}_2[x]/(x^2 + 1)$, and for each such ideal I, describe the elements of R/I and the addition and multiplication in R/I.

(d) (i) What is a *prime ideal* in a commmutative ring?

(ii) What is a *maximal ideal* in a commutative ring?

(iii) Show that every maximal ideal is prime.

(iv) Give an example (with motivation) of each of the following: A *maximal ideal* in $\mathbb{Z}$. A *non-maximal prime ideal* in $\mathbb{Z}$. A *non-prime ideal* in $\mathbb{Z}$.

P.T.O

5. (a) Give the definition of a *Euclidean ring.* In particular state the conditions that the valuation function d must satisfy.

(b) Show that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean ring.

(c) Show that in a Euclidean ring R, $d(a) = d(1)$ if and only if a is a unit, where d is the valuation function in R.

(d) Given the elements 10 and $15 + 5i$ in $\mathbb{Z}[i]$, compute their greatest common divisor. Hence, or otherwise, find a generator for the ideal $I + J$, where $I = (10)$ and $J = (15 + 5i)$.