Semester 1 Examinations 2015/2016

Exam Codes	4BS3, 4BS9, 4FM2, 4BMS2, 3BA1, 4BA4, 1AL1
Exams	B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics B.Sc. M.Sc. Mathematics/Mathematical Science
Module	Measure Theory
Module Code	MA490 MA540
External Examiner	Dr Mark Lawson
Internal Examiner(s)	Prof. Graham Ellis Dr Ray Ryan
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	3 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

Q1 [**49%**] Answer *six* of the following:

(a) Let X be a set with n elements. How many algebras of subsets of X contain exactly five elements? Explain your answer.

(b) Let $A_n = [-n, 0]$ if n is odd and $[-1, 2 + 1/n)$ if n is even. Find $\liminf A_n$ and $\limsup A_n$.

(c) In Bernoulli Space Ω, let E_n be the event that the nth toss is heads. Write down a description in words of the following event:

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} (E_k \cap E_{k+1} \cap E_{k+2})$$

(d) Prove that in an infinite sequence of coin tosses, the probability that Heads will be tossed at least once is one.

(e) Find the Jordan content and the Lebesgue measure of the set of natural numbers in $\mathbb{R}$.

(f) Let $\mathcal{A}$ be the σ-algebra of subsets of $\mathbb{R}$ generated by the intervals $(0, 1)$ and $(3, 4)$. Give an example, with proof, of a function that is not measurable with respect to $\mathcal{A}$.

(g) Let $f_n(x) = \dfrac{nx + 1}{n + x^3}$ for $x \in \mathbb{R}$. Show that this sequence converges pointwise but not uniformly on $\mathbb{R}$.

(h) Let $f(x) = |x|$ and $g(x) = x$. Find a measure on the Borel σ-algebra for which it is true that $f(x) = g(x)$ almost everywhere.

(i) Give an example of a sequence of Lebesgue integrable functions on $\mathbb{R}$ that converges pointwise but for which term by term integration is not valid. Explain why the Lebesgue Monotone Convergence Theorem does not apply to your example.

(j) Find the value of the following sum:

$$\sum_{n=0}^{\infty} \int_0^{\pi/4} \sin^{2n} x \, dx \, .$$

Justify each step in your calculation.

(k) Give a description of the Cantor Set C. Which of the following numbers belong to C?

$$\text{(i)} \ \frac{6}{13}, \quad \text{(ii)} \ \frac{20}{27} \, .$$

You must answer every question.

Q2 [**17%**]

(a) Give the definitions of the following: an algebra, a σ-algebra, an additive set function and a measure. [**5%**]

(b) In Bernoulli space Ω, let $\mathcal{E}$ be the σ-algebra generated by $E_1, E_2, \ldots$ (E_n is the event that the nth toss is Heads.) Let A be the event that Heads are tossed no more than twice. Show that A belongs to the σ-algebra $\mathcal{E}$. [**6%**]

(c) Let μ be a measure on a σ-algebra $\mathcal{A}$ of subsets of a set X. Show that if (E_n) is an increasing sequence of measurable subsets of X, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)\,.$$

[**6%**]

Q3 [**17%**] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) Explain how Lebesgue's approach to the definition of the integral leads to the definition of measurability for a real-valued function on X. Give the definition of a measurable function. [**5%**]

(b) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = 4 - x^2\,.$$

(i) Working directly from the definition, show that f is measurable with respect to the Borel σ-algebra. [**3%**]

(ii) Give an example of a σ-algebra for which f is not measurable. [**3%**]

(c) Let f and g be measurable functions. Show that the function $f+g$ is measurable. [**6%**]

Q4 [**17%**] Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give the definition of the integral of a measurable simple function on X. If f, g are measurable simple functions and $f(x) = g(x)$ almost everywhere, show that the integrals of f and g are equal. [**5%**]

(b) Give the definition of the integral for a non-negative measurable function f and show that if and that

$$\int_X f\, d\mu = 0\,,$$

then $f(x) = 0$ almost everywhere. [**5%**]

(c) State, without proof, Lebesgue's Monotone Convergence Theorem. Explain how this theorem is used to prove Beppo Levi's Theorem and give an example of an application of Beppo Levi's Theorem. [**7%**]