Semester 1 Examinations 2016/2017

Exam Codes	4BS3, 4BS9, 4FM2, 4BMS2, 3BA1, 4BA4, 1AL1
Exams	B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics B.Sc. M.Sc. Mathematics/Mathematical Science
Module	Measure Theory
Module Code	MA490
External Examiner	Prof. Mark Lawson
Internal Examiner(s)	Prof. Graham Ellis Dr Ray Ryan
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	3 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

Q1 **[49%]** Answer **six** of the following:

(a) Let X be a set with n elements. How many algebras of subsets of X contain exactly four elements? Explain your answer.

(b) In Bernoulli Space Ω, let E_n be the event that the nth toss is heads. Write down a formula in terms of these sets of the event that the sequence $HTHT$ appears only a finite number of times.

(c) Let $A_n = [-1 + 1/n, 2 - 1/n]$ if n is odd and $[0, 2]$ if n is even. Find $\liminf A_n$ and $\limsup A_n$.

(d) In Bernoulli space, let A be the event that a run of 100 Heads in a row appears infinitely often. Find the probability of A.

(e) Find the Jordan content and the Lebesgue measure of the subset $\{1/n : n \in \mathbb{N}\}$ of $\mathbb{R}$.

(f) Let $\mathcal{A}$ be the σ-algebra $\{\varnothing, \mathbb{R}, (-\infty, 1), [1, \infty)\}$. Give an example of

(i) A function that is not measurable with respect to $\mathcal{A}$.

(ii) A function that is measurable with respect to $\mathcal{A}$.

(g) Find the pointwise limit of the sequence of functions on $\mathbb{R}$ given by $f_n(x) = 3n/(n + x^4)$. Does this sequence converge uniformly? Explain your answer.

(h) Give an example of a sequence of Lebesgue integrable functions on $\mathbb{R}$ that converges pointwise but for which term by term integration is not valid. Explain why the Lebesgue Monotone Convergence Theorem does not apply to your example.

(i) Find the value of the following sum:

$$\sum_{n=0}^{\infty} \int_0^{\pi/2} \cos^{2n} x \, dx \, .$$

Justify each step in your calculation.

(j) Which of the following numbers belong to the Cantor set?

$$\text{(i) } \frac{2}{81}\,, \quad \text{(ii) } (0.220221)_3\,, \quad \text{(iii) } \frac{e}{6}\,.$$

Explain your answer in each case.

(k) Describe the construction of a subset of $\mathbb{R}$ that is not Lebesgue measurable. (You are not required to prove that the set is not measurable.)

You must answer every question.

Q2 [17%]

(a) Give the definitions of the following: an algebra, a σ-algebra, an additive set function and a measure. **[4%]**

(b) Let (A_n) be a sequence of sets. Give the definitions of the sets $\liminf A_n$ and $\limsup A_n$. **[2%]**

(i) Give an example of a sequence for which $\liminf A_n = \limsup A_n$. **[2%]**

(ii) Give an example of a sequence for which $\liminf A_n \neq \limsup A_n$. **[3%]**

(c) Prove the First Borel-Cantelli Lemma:
if (A_n) is a sequence of measurable sets in a probability space for which the series $\sum_{n=1}^{\infty} P(A_n)$ converges, then $P(\limsup A_n) = 0$.

(Note: you may assume that if (B_n) is a decreasing sequence of measurable sets, then $P\big(\bigcap_n B_n\big) = \lim_{n\to\infty} P(B_n)$.) **[6%]**

Q3 [17%] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) A function $f\colon X \to \mathbb{R}$ is measurable if the set $\{x \in X : f(x) > \lambda\}$ belongs to $\mathcal{A}$ for every real number λ. Show that this holds if and only if the set $\{x \in X : f(x) \geq \lambda\}$ belongs to $\mathcal{A}$ for every $\lambda \in \mathbb{R}$. **[8%]**

(b) Let $f\colon X \to \mathbb{R}$ be a function.

(i) Show that if f is measurable then the function f^2 is measurable. **[3%]**

(ii) Give an example to show that the converse of (i) is false. **[3%]**

(iii) Explain how (i) is used to prove that the product of two measurable functions is measurable. **[3%]**

Q4 [17%] Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Write a short account of the definition of the integral for measurable functions on X, starting with the integral for simple, measurable functions. **[5%]**

(b) Let $X = \mathbb{N}$, with $\mathcal{A} = \mathcal{P}(\mathbb{N})$ and $\mu = \sigma$, counting measure. What is the integral of a function in this case? Explain your answer. What are the integrable functions? **[5%]**

(c) State *three* convergence theorems that allow term by term integration for a sequence or series of integrable functions. In each case, give an example of a sequence or series to which the theorem can be applied. **[7%]**