Semester 1 Examinations 2021/2022

Exam Codes	3BA1, 4BCT1, 4BMS2, 4BS2, 4FM2
Exams	B.A. B.Sc. B.Sc. Mathematical Science B.Sc. Financial Mathematics & Economics M.Sc. Mathematics/Mathematical Science
Module	Measure Theory
Module Code	MA490
External Examiner	Prof. C. Roney-Dougal
Internal Examiners	Prof. G. Pfeiffer *Dr M. Kerin
Instructions	**Answer all four questions.**
Duration	2 Hours
No. of Pages	4 pages, including this one
School	Mathematical and Statistical Sciences
Requirements	No special requirements
Release to Library:	Yes
Release in Exam Venue:	Yes

Q1 [40%] Answer **five** of the following, justifying your response in each case. **[8% each]**

(a) Let X be a set and let $A, B \subseteq X$ be proper, non-empty subsets of X such that $A \subseteq B$ and $A \neq B$. Compute the σ-algebra $\mathcal{A}(S)$ of subsets of X generated by $S = \{A, B\}$.

(b) Give an example, with justification, of two σ-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ of subsets of a set X such that $\mathcal{A}_1 \cup \mathcal{A}_2$ is <u>not</u> a σ-algebra.

(c) Determine the Lebesgue measure of the subset of $[0, 1]$ whose elements have neither first nor second digit in their ternary expansion equal to '2'.

(d) Compute the ternary expansion of the number $\frac{8}{11}$ and use it to determine whether $\frac{8}{11}$ is an element of the Cantor set.

(e) Determine $\liminf A_n$ and $\limsup A_n$ for the sequence $(A_n)_{n \in \mathbb{N}}$ of subsets of $\mathbb{R}$ given by

$$A_n = \begin{cases} \left[\frac{1}{n^2} - 3\,, 1 - \frac{2}{n}\right], & n \text{ odd}, \\ \left(1 - \frac{1}{n}\,, 2 + \frac{1}{n+1}\right), & n \text{ even}. \end{cases}$$

(f) Let $(X, \mathcal{A}, \mathbb{P})$ be a probability space and $(A_n)_{n \in \mathbb{N}}$ an increasing, independent sequence of events in $\mathcal{A}$ with $\mathbb{P}(A_1) > 0$. Determine the probability of the event $\limsup A_n$.

(g) Suppose $(X, \mathcal{A}, \mu) = (\Omega, \mathcal{E}, \mathbb{P})$ is the Bernoulli probability space associated to our share-price model, where $\Omega = \{(\omega_j)_{j \in \mathbb{N}} \mid \omega_j \in \{u, d\} \ \forall\, j \in \mathbb{N}\}$ and $\mathcal{E}$ is the σ-algebra generated by the simple events $E_n = \{(\omega_j)_{j \in \mathbb{N}} \in \Omega \mid \omega_n = u\}$, $n \in \mathbb{N}$. Determine whether the sequence $(A_n)_{n \in \mathbb{N}}$ of events in $\mathcal{E}$ defined by

$$A_n = E_n^c \cap E_{n+3}^c \cap E_{2n+1}$$

is an independent sequence of events.

(h) Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the Lebesgue measure space and $f : \mathbb{R} \to \mathbb{R}$ a function with $f(x) = 3 - 5\,x$ for all $x \in \mathbb{Q}^c$. Show that f is measurable with respect to $\mathcal{L}$.

(i) Show, with justification, that

$$\sum_{n=0}^{\infty} \int_3^5 \left(\frac{x}{6}\right)^n \, dx = 6 \, \ln(3)\,.$$

(j) Let $(\mathbb{R}, \mathcal{L}, \lambda)$ be the Lebesgue measure space and, for each $n \in \mathbb{N}$, let $f_n : \mathbb{R} \to \mathbb{R}$ be the non-negative, measurable function defined by $f_n(x) = 3 - \frac{5x}{n^2}$. If $A \in \mathcal{L}$, show that

$$\lim_{n \to \infty} \int_A f_n \, d\lambda = 3\, \lambda(A)\,.$$

Continued on the next page.

Q2 [20%] Let $\mathcal{A}$ be a σ-algebra of subsets of a set X and let $\mu : \mathcal{A} \to [0, \infty]$ be a measure on $\mathcal{A}$.

(a) Give the definitions of the terms

(i) σ-algebra (ii) measure. **[4%]**

(b) Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets in the σ-algebra $\mathcal{A}$. Show that

$$\bigcap_{n=1}^{\infty} A_n \in \mathcal{A}\,.$$ **[6%]**

(c) Let $B \subseteq X$ be a subset of X.

(i) If $B \in \mathcal{A}$, show that the function

$$\mu_B : \mathcal{A} \to \mathbb{R} \cup \{\infty\}\,;\; A \mapsto \mu(A \cap B)$$

is also a measure on $\mathcal{A}$. **[8%]**

(ii) If $B \notin \mathcal{A}$, explain why the function μ_B above would not define a measure on $\mathcal{A}$. **[2%]**

Q3 [20%] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) Define what it means for a function $f : X \to \mathbb{R}$ to be measurable with respect to $\mathcal{A}$. **[2%]**

(b) Show that every constant function $f : X \to \mathbb{R}$ is measurable with respect to $\mathcal{A}$. **[4%]**

(c) Suppose that a function $g : X \to \mathbb{R}$ is both measurable with respect to $\mathcal{A}$ and positive, i.e. $g(x) > 0$ for all $x \in X$. Prove that

$$h : X \to \mathbb{R}\,;\; x \mapsto \ln(g(x) + 3)$$

is also a measurable function with respect to $\mathcal{A}$. **[4%]**

(d) Suppose $A \subseteq X$ is not an element of the σ-algebra $\mathcal{A}$. Let $(f_n)_{n\in\mathbb{N}}$ be the sequence of functions given by

$$f_n : X \to \mathbb{R}\,;\; x \mapsto 4 - \frac{7}{3n^2+1}\,\chi_A(x)\,, \quad n \in \mathbb{N},$$

where $\chi_A : X \to \mathbb{R}$ denotes the characteristic function of A.

(i) Show that the function f_n is not measurable with respect to $\mathcal{A}$ for any $n \in \mathbb{N}$. **[5%]**

(ii) Show that the sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f : X \to \mathbb{R}$ which is measurable with respect to $\mathcal{A}$. **[5%]**

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Q4 [20%] (a) Let $(X, \mathcal{A}, \mu)$ be a complete measure space and let $\chi_Y : X \to \mathbb{R}$ denote the characteristic function of a subset $Y \subseteq X$.

(i) If $A, B \subseteq X$, show that

$$\chi_A + \chi_B = \chi_{A\cup B} + \chi_{A\cap B}\,.$$

[3%]

(ii) If $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint subsets of X, show that the characteristic function of the set $\bigcup_{n=1}^{\infty} A_n \subseteq X$ satisfies the identity

$$\chi_{\bigcup_{n=1}^{\infty} A_n} = \sum_{n=1}^{\infty} \chi_{A_n}$$

and that the function

$$\sum_{n=1}^{\infty} \chi_{A_n} : X \to \mathbb{R}\,;\; x \mapsto \sum_{n=1}^{\infty} \chi_{A_n}(x)$$

converges for all $x \in \mathbb{R}$. **[5%]**

(iii) Let $f : X \to \mathbb{R}$ be a measurable, non-negative function and let $\eta : \mathcal{A} \to \mathbb{R} \cup \{\infty\}$ be the function defined by

$$\eta(A) = \int_A f\, d\mu\,, \quad A \in \mathcal{A}\,.$$

If $(A_n)_{n\in\mathbb{N}}$ is a sequence of pairwise disjoint sets in $\mathcal{A}$, show that η satisfies the identity

$$\eta\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \eta(A_n)$$

by applying an appropriate convergence theorem. **[8%]**

(b) Suppose $(\Omega, \mathcal{E}, \mathbb{P})$ is the Bernoulli probability space associated to our share-price model, where $\Omega = \{(\omega_j)_{j\in\mathbb{N}} \mid \omega_j \in \{u, d\}\ \forall\ j \in \mathbb{N}\}$ and $\mathcal{E}$ is the σ-algebra generated by the simple events $E_n = \{(\omega_j)_{j\in\mathbb{N}} \in \Omega \mid \omega_n = u\}$, $n \in \mathbb{N}$. Compute the expected value $\mathbb{E}(Y)$ of the random variable $Y : \Omega \to \mathbb{R}$ defined by

$$Y = 3\,\chi_{(E_1\cap E_3^c)\cup E_5}\,,$$

where $\chi_A : \Omega \to \mathbb{R}$ denotes the characteristic function of a subset $A \subseteq \Omega$. **[4%]**