Semester I Examinations 2011/2012

Exam Code(s)	1AH2, 1PAH1, 1HA1, 4BS3, 4CS2
Exam(s)	Bachelor of Arts (Mathematics) Bachelor of Science (Mathematics) Postgraduate Diploma in Arts (Mathematics) Higher Diploma in Applied Science (Mathematics)
Modules	Ring Theory, Advanced Algebra I
Module Codes	MA416, MA538
Paper	1
External Examiner(s)	Dr Colin Campbell
Internal Examiner(s)	Prof. Graham Ellis Dr Emil Sköldberg
<u>**Instructions:**</u>	**Students should attempt ALL questions.**
Duration	2 Hours
No. of Pages	3 (including this page)
Discipline	Mathematics

1. (a) In each of the following cases, determine if the object is a ring. If you believe that it is a ring, it is enough to say so; if not, you should give a reason.

   (i) The set of all symmetric 2×2 matrices with real entries under the usual addition and multiplication of matrices.

   (ii) The set $\mathbb{N}$ of natural numbers $0, 1, 2, \ldots$ under the usual addition and multiplication.

   (b) Find all units and zero-divisors in the following rings: (i) $\mathbb{Z}_{12}$ and (ii) $\mathbb{Z}[i]$.

   (c) Show that if r, s are elements in a ring R, then (i): $0 \cdot r = 0$, and (ii): $(-r)s = r(-s) = -(rs)$.

2. (a) Let R and S be rings. Give the definition of a *ring homomorphism* $\varphi : R \to S$.

   (b) Show that there is no ring homomorphism from $\mathbb{Q}$ to $\mathbb{Z}$.

   (c) Define the *field of fractions* of an integral domain. (You need to describe its elements, and the definition of the addition and multiplication.)

   (d) Show that if R is a field, then its field of fractions is isomorphic to R.

3. (a) Give the definition of a *primitive* polynomial in $\mathbb{Z}[x]$, and show that the product of two primitive polynomials is primitive.

   (b) For each of the following polynomials, determine if it is irreducible or not, and if it is reducible, factor it into irreducibles.

   (i) $x^3 + 1 \in \mathbb{Z}_2[x]$

   (ii) $x^4 + 2x^3 + 2x^2 + 2x \in \mathbb{Q}[x]$

4. (a) Let I and J be ideals in the ring R. Show that the sets $I \cap J$ and $I + J$ are ideals in R.

   (b) Give the definition of *maximal ideal* and *prime ideal* in a commutative ring. Show that every maximal ideal is prime.

   (c) Let F be a field. Show that the function $\psi : F[x] \to F$ defined by

$$\psi\left(\sum_{i=0}^{\infty} a_i x^i\right) = a_0$$

   is a ring homomorphism. Determine the kernel of ψ and hence, or otherwise, show that the ideal $(x) \subset F[x]$ is a maximal ideal.

5. (a) Give the definition of *Euclidean ring.*

   (b) Let F be a field. Show that $F[t, t^{-1}]$, the Laurent polynomial ring, is a Euclidean ring.

   (c) Given the two elements $2 + 3i$ and $1 - i$ in $\mathbb{Z}[i]$, find their greatest common divisor. Hence, or otherwise, find a generator for the ideal $I + J$, where $I = (2 + 3i)$ and $J = (1 - i)$.