Semester I Examinations 2019/2020

Exam Codes	4BCT1, 4BS2, 4BMS2, 1AL1
Exams	Bachelor of Science (Comp. Sci. & IT)
	Bachelor of Science (Hons)
	Bachelor of Science (Math. Sci.)
Module	Ring Theory
Module Code	MA416, MA538
External Examiner	Professor T. Brady
Internal Examiner	Professor G. Ellis
Internal Examiner	Professor D. Flannery*
Instructions	**Answer ALL questions**
Duration	2 hours
No. pages	3, including this one
School	Mathematics, Statistics and Applied Mathematics

Requirements

Release to Library	Yes
Release in Exam Venue	Yes

1. (a) State, with justification, whether each of the following is a ring.

(i) The integers, with addition defined as usual, and multiplication defined by $ab = 0$ for all a, b.

(ii) $\{n/3^a \mid n, a \in \mathbb{Z}\}$, under ordinary addition and multiplication of rational numbers.

(iii) The set of all polynomials in $\mathbb{R}[x]$ with zero constant term.

[**10 marks**]

(b) Let R be a commutative ring with 1. Show that R is a field if and only if R has exactly two ideals.

[**7 marks**]

(c) Let $\phi : R \to S$ be a ring epimomorphism.

(i) Prove that $\ker \phi$ is an ideal of R.

(ii) Prove that $R/\ker \phi \cong S$.

[**8 marks**]

2. (a) State, with justification, whether each of the following pairs of rings are isomorphic (demonstrate an isomorphism if one exists).

(i) $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}$.

(ii) $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_{15}$.

(iii) $\mathbb{C}$ and the subring consisting of all matrices $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ in $\mathrm{Mat}(2, \mathbb{R})$.

[**9 marks**]

(b) (i) Prove that if p is a prime then $\mathbb{Z}_p$ is a field.

(ii) Does 5 have a multiplicative inverse in $\mathbb{Z}_{42}$? If so, calculate the inverse; if not, explain why not.

[**7 marks**]

(c) Let R be a commutative ring with 1.

(i) Let I be an ideal of R. Prove that $J = \{ar + s \mid r \in R, s \in I\}$ is an ideal of R for any $a \in R$. Deduce that if I is a maximal ideal of R then R/I is a field.

(ii) Is a maximal ideal of R necessarily a prime ideal? Is a prime ideal of R necessarily a maximal ideal? Why?

[**9 marks**]

3. (a) Let R be a UFD (unique factorization domain). Prove that the irreducible elements of R are precisely the prime elements of R.
[**9 marks**]

(b) Prove that $\mathbb{Z}[\sqrt{-6}]$ is an integral domain that is not a UFD.
[**10 marks**]

(c) Explain why 5 and $2+\sqrt{-6}$ have a greatest common divisor of 1 in $\mathbb{Z}[\sqrt{-6}]$. Do there exist α, $\beta \in \mathbb{Z}[\sqrt{-6}]$ such that $5\alpha + (2+\sqrt{-6})\beta = 1$? Justify your answer.
[**6 marks**]

4. (a) Let R be a PID (principal ideal domain), S be any integral domain, and $\phi : R \to S$ be a ring epimorphism. Prove that either ϕ is an isomorphism or S is a field.
[**8 marks**]

(b) (i) Define *Euclidean domain* (ED). Prove that every ED is a PID.
(ii) Calculate a greatest common divisor of $x^3 + 2x^2 + 4x - 7$ and $x^2 + x - 2$ in $\mathbb{Q}[x]$.
[**11 marks**]

(c) Let R be a commutative ring with 1. Show that the polynomial ring $R[x]$ is a PID if and only if R is a field (you will need to use part (a) of this question).
[**6 marks**]