NUI Galway
OÉ Gaillimh

Summer Examinations 2018/19

Exam Code(s)	4BS2, 4BMS2, 4FM2, 4BCT1, 4BME1, 1SPA1, 4SPE1
Exam(s)	4th Science
Module(s)	Networks
Module Code(s)	CS4423
External Examiner(s)	Prof. T. Brady
Internal Examiner(s)	Prof. G. Ellis
	*Prof. G. Pfeiffer
Instructions	Answer *all* questions.
Duration	2 hours
No. of Pages	3 pages (including this cover page)
Discipline	Mathematics

Requirements:

Release to Library	Yes	☑	No	☐
Release in Exam Venue	Yes	☑	No	☐
MCQ	Yes	☐	No	☑
Statistical/Log Tables	Yes	☐	No	☑

1.

(a) [**5** marks] What is a *graph*? What is a *directed graph*? What is a *tree*?

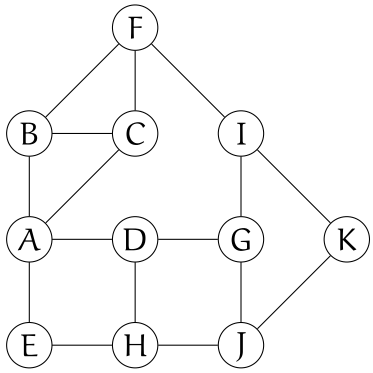
(b) [**5** marks] At a party with $n = 5$ people, some people know each other already while others don't. Each of the 5 guests is asked how many friends they have at this party. Two report that they have one friend each. Two other guests have two friends each, and the fifth guest has three friends at the party. Understanding friendship as a symmetric relation, is this network possible? Why, or why not?

(c) [**10** marks]

(i) Describe *Breadth First Search* as an algorithm for undirected graphs. What is its input, what is its output, and what sequence of steps is taken to produce the output from the input?

(ii) In the network on the side, apply Breadth First Search to determine a *spanning tree* with root G and the *shortest distances* from node G to each of the other nodes in the graph.

(iii) Use the Breadth First Search from part (ii) to also determine, for each node x, its possible *predecessors* on a shortest path from G to x. Use this information to list *all shortest paths* from G to B.



2.

(a) [**5** marks] Define the concepts of *degree centrality* and of *normalized degree centrality* for a graph G.

Let G be the graph on the vertex set $\{1, 2, 3, 4, 5, 6, 7\}$ with edges $1-2$, $1-3$, $2-4$, $2-5$, $3-4$, $4-6$, $5-6$ and $6-7$. For each node in this graph G, determine its normalized degree centrality. (A sketch of G might be useful.)

(b) [**5** marks] The fact that

$$\begin{pmatrix} 2 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

makes $\lambda = 4$ an *eigenvalue* with *eigenvector* $v = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ of the matrix $A = \left(\begin{smallmatrix} 2 & 2 \\ 3 & 1 \end{smallmatrix}\right)$. The other eigenvalue of A is -1. With this example in mind, formulate the Perron–Frobenius Theorem on square, irreducible, nonnegative matrices. Define the concept of *eigenvector centrality* for a connected graph G.

(c) [**5** marks] Define the concepts of *closeness centrality* and *normalized closeness centrality* for a connected graph G. Give an example of a connected graph on 5 vertices with a node of normalized closeness centrality 1.

(d) [**5** marks] Define the concepts of *betweenness centrality* and *normalized betweenness centrality* for a connected graph G.

3.

(a) [**10** marks]

(i) Define the two Erdős–Rényi models, $G(n,m)$ and $G(n,p)$ of random graphs.

(ii) For a fixed value of n, describe each of the two models as a probability distribution on the set G_n of *all* graphs on the n nodes $X = \{1, 2, \dots, n\}$.

(iii) For $n = 100$ and $p = \frac{1}{99}$, what is the probability that a graph G sampled from the $G(n,p)$ model has exactly 50 edges?

(b) [**10** marks]

(i) Define the degree distribution of a graph G.

(ii) What is the degree distribution in a random graph G in the $G(n,p)$ model? What is the expected average degree in such a graph G?

(iii) Construct a graph G on 100 nodes by throwing a (fair, 6-sided) dice once for each possible edge, adding the edge only if the dice shows a 6. Then pick a node at random in this graph. What is the probability that this node has degree 10?

4.

(a) [**10** marks]

(i) What is the *node clustering coefficient* of a vertex x in a graph G? What is the *graph clustering coefficient* C of G?

(ii) Determine the graph clustering coefficient C of a random graph in the $G(n,p)$ model. How does C behave in the limit $n \to \infty$, when the average node degree is kept constant? What practical consequence does this observation have?

(iii) Describe the *Watts–Strogatz small-world model* (WS model). What properties does a random graph sampled from the WS model have, that one wouldn't find in a random graph sampled from the $G(n,p)$ model, or in an (n,d)-circle graph?

(b) [**10** marks] Suppose that $G = (X, E)$ is a *directed* network with node set X and edge set E.

(i) Define what it means for G to be *weakly connected*, and what it means to be *strongly connected*.

(ii) In general, a directed network is partitioned into strongly connected components. Describe the equivalence relation that yields strongly connected components as its equivalence classes in terms of the edge set E, regarded as a relation on X.

(iii) Many directed networks, like the World Wide Web, have a giant strongly connected component which covers a substantial part of the network. Name the other parts of the so-called Bow-Tie diagram of the components in a directed network, and describe each in terms of a suitable relation on the set of strongly connected components.