Summer Examinations 2011

Exam Code(s)	4BS3, 1PAH1, 1HA1
Exam(s)	Bachelor of Science (Hons. Maths) Higher Diploma in Applied Science Postgraduate Diploma in Arts
Module(s)	FUNCTIONAL ANALYSIS ADVANCED ANALYSIS II
Module Code(s)	MA482, MA541
Paper No	1
Repeat Paper	
External Examiner(s)	Dr. C. Campbell
Internal Examiner(s)	Dr. G. Ellis and Professor D. O'Regan
Instructions:	**Full marks for THREE complete answers.**
Duration	2 Hours
No. of Pages	3 Pages
Requirements:	
MCQ	Release to Library: Yes
Handout	
Statistical Tables/ Log Tables	Yes
Cambridge Tables	
Graph paper	
Log Graph Paper	
Other Materials	

1. (a) Consider $C[0,1]$ (with the usual sup norm) and let $T : C[0,1] \to C[0,1]$ be defined by

$$T(f) = \int_0^1 t\, f(t)\, dt.$$

Show T is bounded with norm $\frac{1}{2}$.

(b) Let X, Y and Z be normed spaces and let $T : X \to Y$, $S : Y \to Z$ be bounded linear operators and $V : X \to Y$ a bounded invertible linear operator with inverse V^{-1}. Show $||S\,T|| \leq ||S||\,||T||$ and $||V^{-1}|| \geq \frac{1}{||V||}$.

(c) Let T be a bounded linear operator from a normed space X onto a normed space Y with $||T\,x|| \geq 3\,||x||$ for all $x \in X$. Show $T^{-1} : Y \to X$ exists and is bounded.

(d) Let X be a normed space and Y a Banach space. Show $B(X,Y)$ (the space of all bounded linear operators from X to Y) is a Banach space.

2. (a) State the Hahn-Banach Theorem. Let X and Y be normed spaces with $X \neq \{0\}$. If $B(X,Y)$ is complete, show Y is complete.

(b) Let $p > 1$ and $q = \frac{p}{p-1}$. Show that the dual space $(l^p)'$ is isometrically isomorphic to l^q.

(c) Let H be an inner product space. Prove the Parallelogram Law:

$$||x+y||^2 + ||x-y||^2 = 2\,||x||^2 + 2\,||y||^2 \quad \text{for every} \quad x, y \in H.$$

Show $C[0,1]$ (with the usual sup norm) is not an inner product space.

3. (a) Let K be a nonempty closed convex subset of a Hilbert space H. Show for every $x \in H$ there is a unique $y \in K$ with

$$\|x-y\| = \inf_{z \in K} \|x-z\|.$$

(b) Let Y be a closed subspace of a Hilbert space H. Define $Y^\perp$ and show it is a closed subspace of H. Also show $H = Y \oplus Y^\perp$.

(c) Let A and $A \subseteq B$ be nonempty subsets of an inner product space X. Show

$$A \subseteq A^{\perp\perp}, \ B^\perp \subseteq A^\perp \quad \text{and} \quad A^{\perp\perp\perp} = A^\perp.$$

4. (a) State the Riesz representation theorem. Let H_1 and H_2 be Hilbert spaces over the same field K ($\mathbf{R}$ or $\mathbf{C}$) and $h : H_1 \times H_2 \to K$ a bounded sesquilinear form. Show h has the representation $h(x, y) = \langle S(x), y\rangle$ where $S : H_1 \to H_2$ is a bounded linear operator and S is uniquely determined by h with norm $||S|| = ||h||$.

(b) Let $S : l^2 \to l^2$ be given by

$$S(x) = (0,\, x_1,\, x_2,\, ...) \quad \text{when} \quad x = (x_1, x_2, ...)$$

For $n \in \{1, 2, ...\}$ show $\langle\, e_j\,,\, S^*(e_{n+1}) - e_n\,\rangle = 0$ for each $j \in \{1, 2, ...\}$ and $S^*(e_{n+1}) = e_n$. Find $S^*(0, 2, 5, 0, -3, 0, 0, ...)$.