NUI Galway
OÉ Gaillimh

Semester I Examinations 2012/2013

Exam Code(s)	4BS3, 4CS2
Exam(s)	Bachelor of Science (Mathematics)
Modules	Ring Theory
Module Codes	MA416, MA538
Paper	1
External Examiner(s)	Dr Colin Campbell
Internal Examiner(s)	Prof. Graham Ellis
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<u>Instructions:</u>	**Students should attempt ALL questions.**
Duration	2 Hours
No. of Pages	3 (including this page)
Discipline	Mathematics

1. (a) In each of the following cases, determine if the object is a ring. If you believe that it is a ring, it is enough to say so; if not, you should give a reason.

   (i) The set of all real 2×2 matrices with trace 0 under the usual addition and multiplication of matrices. (Recall that the trace of a matrix is the sum of its diagonal entries.)

   (ii) The set of all polynomials $\sum_{i \geq 0} a_i x^i$ in $\mathbb{Q}[x]$ such that $a_i = 0$ whenever i is odd.

   (b) Find all units and zerodivisors in the rings

   (i) $\mathbb{Z}_{15}$,

   (ii) $\mathbb{Z}_2[x]/(x^2 + 1)$.

   (c) Give an example of a ring R and two units $u, v \in R$ such that $u + v$ is a unit.

   (d) Show that the units in a ring R form a group under multiplication.

2. (a) Let R and S be rings. What is meant by a *ring homomorphism* from R to S?

   (b) If $\varphi : R \longrightarrow S$ is a ring homomorphism; define its *kernel*. Also explain what a *(two-sided) ideal* is, and show that the kernel of φ is an ideal.

   (c) What is meant by a *principal ideal domain (PID)*? Show that the integers $\mathbb{Z}$ form a PID.

   (d) Define the *field of fractions* of an integral domain. (You need to describe its elements and the definition of addition and multiplication). Show that addition and multiplication are well-defined operations.

3. (a) What is meant by an *irreducible polynomial* in $F[x]$, where F is a field? Give an example of a field F and a polynomial $f(x)$ in $F[x]$ such that $f(x)$ is reducible in $F[x]$, but $f(x)$ does not have a root in F.

   (b) State and prove Eisenstein's irreducibility criterion.

   (c) For each of the following polynomials, determine if it is irreducible or not, and if it is reducible, factor it into irreducibles.

      (i) $x^5 - 9x^2 + 3x - 15$ in $\mathbb{Q}[x]$.

      (ii) $x^3 + x^2 + x + 1$ in $\mathbb{Z}_2[x]$.

   (d) Find all irreducible monic polynomials of degree 3 in $\mathbb{Z}_2[x]$.

4. (a) What are *maximal ideals* and *prime ideals* in a commutative ring? Show that every maximal ideal is prime.

   (b) Let I and J be ideals in the commutative ring R. Define the sets IJ and $I + J$ and show that they are ideals in R.

   (c) Let R be a PID, and let $I = (r)$, $J = (s)$ be two ideals in R. Show that the ideal $IJ = (rs)$.

   (d) Determine all maximal ideals in $R = \mathbb{Z}_3[x]/(x^3 + x + 1)$, and for each maximal ideal I, determine the order of R/I.

5. (a) Give the definition of a *Euclidean ring.*

   (b) Let F be a field. Show that the polynomial ring $F[x]$ is a Euclidean ring.

   (c) Show that in a Euclidean ring, $d(a) = d(1)$ if and only if a is a unit.

   (d) Given the elements $4+i$ and $1-2i$ in $\mathbb{Z}[x]$, compute their greatest common divisor. Hence, or otherwise, find a generator for the ideal $I + J$, where $I = (4 + i)$ and $J = (1 - 2i)$.