NUI Galway
OÉ Gaillimh

Semester I Examinations 2019/2020

Exam Codes	3BMS2, 4BMS2, 3BS9, 4BS2, 3BA1, 4BA4, 4BCW1, 4BCT1/1HA1
Exam	Third and Fourth Science Examination
Module	Euclidean and Non-Euclidean Geometry
Module Code	MA3101/MA544
Paper No.	1
External Examiner	Prof. Thomas Brady
Internal Examiners	Prof. Graham Ellis
	Dr. John Burns
Instructions	**Answer all questions.**
Duration	2 hours
No. of Pages	3 pages including this page
School	Mathematics, Statistics & Applied Mathematics
Requirements	No special requirements
Release to Library:	Yes
Other Materials	Non-programmable calculators

Q1. Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius one.

(i) [9 marks] Prove that the shortest path joining any two points of S^2 is an arc of a great circle.

(ii) [8 marks] Prove than $SO(3)$ acts transitively on S^2.

(iii) [8 marks] Calculate the shortest distance between Prague $((\theta, \phi) = (50°05', 14°25'))$ and Winnipeg $((\theta, \phi) = (49°55', -97°06'))$. (Take the radius of the earth to be 6377.5 km.)

Q2. Let $H \subset \mathbb{C}$ be the upper half plane with the hyperbolic metric (the Hyperbolic Plane).

(i) [6 marks] Prove than any matrix $M \in SL(2, \mathbb{R})$ representing a Möbius transformation is an isometry of H.

(ii) [6 marks] Prove than any pair of elements in H can be mapped to the positive y-axis by an element of $SL(2, \mathbb{R})$.

(iii) [6 marks] Let γ denote the positive y-axis and let $p = (1, \frac{1}{2})$. Find two geodesics of the Hyperbolic Plane containing p that do not intersect γ.

(iv) [7 marks] Find the area of the hyperbolic (geodesic) triangle with vertices $(3, 0), (4, 0)$ and $(5, 0)$.

Q3. Recall that for a surface patch $x(u, v)$ of an oriented surface M in $\mathbb{R}^3$, the matrix of the shape operator or Weingarten map with respect to the basis $\{x_u, x_v\}$ is $\Pi_1^{-1}\Pi_2$, where Π_1 and Π_2 are the matrices of the first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$, respectively.

(i) [7 marks] Prove that the mean curvature H of M is given by

$$H = \frac{LG - 2MF + NE}{2(EG - F^2)}.$$

(ii) [12 marks] Calculate the mean curvature H of the helicoid $x(u, v) = (v \cos u, v \sin u, 2u)$ and hence prove that it has negative Gaussian curvature. Describe a geodesic of the helicoid passing through the point $(0, 0, 2)$.

(iii) [6 marks] Prove that there is no surface homeomorphic to S^2 in $\mathbb{R}^3$ with everywhere non-positive Gaussian curvature.

Q4. (a) (i) [9 marks] Use Hamilton's quaternions to prove that the unit 3-sphere S^3 in $\mathbb{R}^4$ is a disjoint union of isometric circles, parametrized by S^2.

(ii) [6 marks] Determine the rotation in $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ induced by conjugation by the unit quaternion $t = \frac{\sqrt{3}}{2} + \frac{k}{2}$.

(b) [10 marks] State Synge's theorem, explaining the assumptions of the theorem. Give an example of a compact surface having negative curvature at some points for which the conclusion of the theorem does not hold.