Semester I Examinations 2021/2022

Exam Codes	3BA1, 4BCT1, 4BMU1, 4BS2, 4BMS2, 1AL1
Exams	Bachelor of Science, Maths Science, Computer Science & IT, Arts, Master of Science
Module	Ring Theory
Module Code	MA416, MA538
External Examiner	Prof. C. Roney-Dougal
Internal Examiner	Prof. G. Pfeiffer
Internal Examiner	Prof. D. Flannery*
Instructions	**Answer ALL questions**
Duration	2 hours
No. pages	3, including this one
School	Mathematical and Statistical Sciences

Requirements

Release to Library	Yes
Release in Exam Venue	No

1. (a) State, with justification, whether each of the following is a ring.

(i) $\frac{1}{2}\mathbb{Z} = \{n/2^a \mid n, a \in \mathbb{Z}\}$, under ordinary addition and multiplication of rational numbers.

(ii) The set of 3×3 matrices with entries in a field $\mathbb{F}$ and zeros everywhere above the main diagonal, under ordinary addition and multiplication of matrices.

(iii) The set of all polynomials in $R[x]$ with zero constant term, where R is any ring.

[**8 marks**]

(b) Let I be an ideal of a ring R. Define addition and multiplication on the set $R/I = \{r + I \mid r \in R\}$ of cosets of I in R that turn R/I into a ring (explain why each binary operation is well-defined).

[**5 marks**]

(c) Let $\phi : R \to S$ be a ring homomorphism. Prove that

(i) $\ker \phi$ is an ideal of R

(ii) $\phi(R) = \{\phi(r) \mid r \in R\}$ is a subring of S isomorphic to $R/\ker\phi$.

[**7 marks**]

2. (a) State, with justification, whether each of the following pairs of rings are isomorphic (demonstrate an isomorphism if one exists).

(i) $\mathbb{Z} \oplus \mathbb{Z}$ and $\mathbb{Z}$.

(ii) $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ and $\mathbb{Z}_{15}$.

(iii) $\mathbb{C}$ and the subring $\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$ of $\mathrm{Mat}(2, \mathbb{R})$.

[**7 marks**]

(b) Prove that a finite integral domain is a field.

[**7 marks**]

(c) Let R be an integral domain.

(i) Show that if I is a maximal ideal of R then R/I is a field.

(ii) Is a maximal ideal of R necessarily a prime ideal of R? Why?

[**6 marks**]

3. (a) Let R be a UFD. Prove that the irreducible elements of R are precisely the prime elements of R.
[**7 marks**]

(b) Prove that $\mathbb{Z}[\sqrt{-5}]$ is an integral domain that is not a UFD (unique factorization domain).
[**8 marks**]

(c) Do 6 and $2 + 2\sqrt{-5}$ have a greatest common divisor in $\mathbb{Z}[\sqrt{-5}]$? Why?
[**5 marks**]

4. (a) Let R be an integral domain, and $a_1, \ldots, a_n$ be non-zero elements of R. Prove that $\ell \in R$ is a least common multiple of the a_is if and only if $a_1 R \cap \cdots \cap a_n R = \ell R$.
[**6 marks**]

(b) Let R be a PID (principal ideal domain), S be any integral domain, and $\phi : R \to S$ be a ring epimorphism. Show that either ϕ is an isomorphism or S is a field.
[**8 marks**]

(c) Prove that $\mathbb{Z}[x]$ is not a PID, by showing that the ideal generated by 2 and x in $\mathbb{Z}[x]$ is not principal.
[**6 marks**]

5. (a) What is the ACC (ascending chain condition) in a commutative ring? Let R be a PID. Prove that each non-zero element of R that is not a unit can be expressed as a product of finitely many irreducibles (assume that R satisfies the ACC).
[**10 marks**]

(b) (i) Define *Euclidean domain* (ED). Explain why $\mathbb{Z}$ and $\mathbb{F}[x]$ for any field $\mathbb{F}$ are EDs.
(ii) Prove that every ED is a PID.
(iii) Give an example of a PID that is not an ED.
[**10 marks**]