Semester I Examinations 2015/2016

Exam Code(s)	4BMS2, 4BS42, 1AL1, 1HA1
Exam(s)	Bachelor of Science (Hons.) Bachelor of Science (Mathematical Science) Honours Higher Diploma in Applied Science (Mathematics) Master of Science (Mathematics)
Modules	Ring Theory, Advanced Algebra I
Module Codes	MA416, MA538
Paper	1
External Examiner(s)	Dr Mark Lawson
Internal Examiner(s)	Prof. Graham Ellis Dr Emil Sköldberg
Instructions:	**Students should attempt ALL questions.** (All questions carry equal marks.)
Duration	2 Hours
No. of Pages	4 (including this page)
Discipline	Mathematics

1. (a) In each of the following cases, determine if the object is a ring. If you believe that it is a ring, it is enough to say so; if not, you should give a reason.

(i) The set of all complex numbers of the form $a+ai$ or $a-ai$ where $a \in \mathbb{R}$ with the usual addition and multiplication of complex numbers. [**3 marks**]

(ii) The set of all rational numbers of the form a/b where b is an *odd* number under the usual addition and multiplication of complex numbers. [**3 marks**]

(b) What is a *unit* in a ring? What is a *zero-divisor* in a ring? Show that the units of a ring form a group. [**6 marks**]

(c) Find all units and zero-divisors in the rings

(i) $\mathbb{Z}_{24}$,

(ii) $\mathbb{Z}_2[x]/(x^3+1)$.

[**4 + 4 marks**]

2. (a) Let R and S be rings. What is meant by a *ring homomorphism* from R to S? [**3 marks**]

(b) Show that the map $\theta : \mathbb{Z}_p \to \mathbb{Z}_p$ defined by $\theta(r) = r^p$ is a ring homomorphism. [**9 marks**]

(c) Define the *field of fractions* of an integral domain. (You need to describe its elements and the definition of the addition and multiplication). Show that the addition and the multiplication are well-defined. [**8 marks**]

P.T.O.

3. (a) State and prove Gauss' Lemma. [**6 marks**]

(b) For each of the following polynomials, determine, with explanation, if it is irreducible in the indicated ring. In the case the polynomial is reducible, factor it into irreducibles.

(i) $x^4 + x^3 + x^2 + x$ in $\mathbb{Z}_5[x]$

(ii) $x^4 - 5x^3 + 10x$ in $\mathbb{Q}[x]$

(iii) $x^6 - 1$ in $\mathbb{R}[x]$

(iv) $x^2 + x + 1$ in $\mathbb{Q}[x]$

[**8 marks**]

(c) Find all irreducible monic polynomials of degree 2 in $\mathbb{Z}_3[x]$. [**6 marks**]

4. (a) Explain what a (two-sided) ideal in a ring R is. Describe the quotient ring R/I (i.e. describe its elements and the addition and multiplication – you do *not* have to show that it satisfies the ring axioms). [**4 marks**]

(b) What is meant by a *principal ideal domain* (PID)? Show that the set I of all polynomials $f(x)$ such that either $\deg(f) \geq 1$ or $\deg(f) = 0$ and $f(x) = 2n$ for some $n \in \mathbb{Z}$, is an ideal in $\mathbb{Z}[x]$. Show that $\mathbb{Z}[x]$ is not a PID by showing that I is not a principal ideal. [**9 marks**]

(c) (i) What is a *prime ideal* in a commmutative ring?

(ii) What is a *maximal ideal* in a commutative ring?

(iii) Show that every maximal ideal is prime.

(iv) Give an example (with motivation) of each of the following: A *maximal ideal* in $\mathbb{Z}_2[x]$. A *non-maximal prime ideal* in $\mathbb{Z}_2[x]$. A *non-prime ideal* in $\mathbb{Z}_2[x]$.

[**7 marks**]

P.T.O

5. (a) Give the definition of a *Euclidean ring*. In particular state the conditions that the valuation function d must satisfy. [**3 marks**]

(b) Show that the ring of Gaussian integers $\mathbb{Z}[i]$ is a Euclidean ring. [**6 marks**]

(c) Show that in a Euclidean ring R, $d(a) = d(1)$ if and only if a is a unit, where d is the valuation function in R. Hence, or otherwise, characterise all the units in the ring of Gaussian integers. [**6 marks**]

(d) Given the elements $6 - 2i$ and $2 + i$ in $\mathbb{Z}[i]$, compute their greatest common divisor. Hence, or otherwise, find a generator for the ideal $I + J$, where $I = (6 - 2i)$ and $J = (2 + i)$. [**5 marks**]