OLLSCOIL NA hÉIREANN, GAILLIMH
NATIONAL UNIVERSITY OF IRELAND, GALWAY

AUTUMN EXAMINATIONS 2008

B.Sc. (Part II) EXAMINATION

MATHEMATICS — [MA490/482]

MEASURE THEORY AND FUNCTIONAL ANALYSIS

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Time allowed: **Three** hours.
Full marks for five questions.
Use separate answer books for each section.

SECTION A — *MEASURE THEORY*

A1. Answer *four* of the following:

(a) In Bernoulli space, let E be the event that the sequence HHH appears only a finite number of times. Express E in terms of the simple events E_n.

(b) Let $X = \{a, b, c, d\}$ be a set with four elements and let $\mathcal{A} = \{\varnothing, X, \{a\}, \{b, c, d\}\}$. Give an example, with proof, of function on X that is not measurable.

(c) Let $\mathcal{A}$ be the σ-algebra of all subsets of $\mathbb{N}$ and let μ be counting measure on $\mathcal{A}$. Give an example of a bounded function $f : \mathbb{N} \to \mathbb{R}$ that is *not* integrable with respect to μ.

(d) Let P be the probability measure on $\mathbb{R}$ given by

$$P = \frac{1}{4}\delta_{-1} + \frac{1}{2}\delta_0 + \frac{1}{4}\delta_1 \,.$$

Compute the expected value $E(X)$, where X is the random variable defined by $X(t) = t^2$.

(e) Show that the set of irrational numbers belongs to the σ-algebra generated by the intervals.

(f) Explain why the function

$$\sum_{n=0}^{\infty} 2^{-n} \chi_{[n,n+1)}$$

is Lebesgue integrable and evaluate its integral.

A2. Let $(X, \mathcal{A}, \mu)$ be a *Probability Space.*

(a) If (A_n) is an *decreasing* sequence of sets in $\mathcal{A}$, show that

$$\mu\Big(\bigcap_{n=1}^{\infty} A_n\Big) = \lim_{n\to\infty} \mu(A_n)$$

(b) Give the definitions of the sets $\liminf E_n$ and $\limsup E_n$. What are these sets if (E_n) is an increasing sequence?

(c) Give examples to show that each of the following inequalities can be *strict*:

$$\mu(\liminf E_n) \le \liminf \mu(E_n) \le \limsup \mu(E_n) \le \mu(\limsup E_n)\,.$$

A3. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give the definition of a *measurable function* on X. If f and g are measurable, show that the function $f + g$ is also measurable.

(b) Let (f_n) be a sequence of measurable functions. Show that the functions $\sup_n f_n$ and $\inf_n f_n$ are measurable.
Explain how these results are used to show that the limit of a pointwise convergent sequence of measurable functions is measurable.

(c) Let f and g be functions on a complete measure space. If f is measurable and $f = g$ almost everywhere, show that g is measurable.

A4. Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give an account (without proofs) of the definition of the integral $\displaystyle\int_X f\, d\mu$, starting with the integral of a simple measurable function.

(b) Give an example of a sequence (f_n) of integrable functions that converges at every point to an integrable function f, with the property that

$$\int_X f\, d\mu \ne \lim_n \int_X f_n\, d\mu\,.$$

State (without proof) a result that permits term-by-term integration of a sequence of integrable functions. Why does this result not apply to the example you have given?

(c) Give the definition of the Cantor Set, C. Show that C is uncountable and has Lebesgue measure zero.

p.t.o.

SECTION B — *FUNCTIONAL ANALYSIS*

B1. (a) What is meant by saying that a normed space is *complete*? Give, without proof, an example of a normed space that is complete and an example of a normed space that is not complete.

(b) State the Hölder and Minkowski inequalities in ℓ_p, where $1 \leq p < \infty$, and prove *one* of them.

(c) What is a convex set? Show that the closed unit ball of a normed space is convex. Show that the set

$$B = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x_1|^{\frac{1}{3}} + |x_2|^{\frac{1}{3}} \leq 1\}$$

is not convex and sketch this set.

B2. Let T be a linear operator between normed spaces X and Y.

(a) Let T be bounded. Define $\|T\|$, and prove that $\|T(x)\| \leq \|T\|\|x\|$ for all $x \in X$.

(b) Let X be $\mathbb{R}^2$ with the ℓ_1-norm, let Y be $\mathbb{R}^2$ with the ℓ_2-norm and let $T((x_1, x_2)) = (2x_1 + 3x_2, x_1 + 2x_2)$. Determine the norm of T.

(c) Prove that if X is finite dimensional, then T is bounded.

B3. Let X and Y be normed spaces and $L(X, Y)$ the set of all bounded linear operators from X into Y.

(a) Show that if Y is complete, then $L(X, Y)$ is also complete.

(b) Define the *dual space* X^* of X.
A linear functional ϕ is defined on ℓ_2 by $\phi(x) = x_1 + x_2 + x_3 + x_5 + x_7$, where $x = (x_1, x_2, \ldots) \in \ell_2$. Find the norm of ϕ.

(c) Let e^n be the sequence whose nth term is 1 and all other terms are 0. Show that (e^n) is a Schauder basis for c_0 but not for ℓ_∞.

B4. Let H be a Hilbert space.

(a) The *parallelogram law* $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|x\|^2$ for all $x, y \in H$ fails in $C[0,1]$ with the norm

$$\|f\| = \sup_{t\in[0,1]} \{|f(t)|\}.$$

What may we deduce from this failure?

(b) Given a closed subspace F of H, define the orthogonal complement, $F^{\perp}$, of F and show that it is a closed subspace of H.

(c) Show that for every $x \in H$ there exist unique $x_1 \in F$ and $x_2 \in F^{\perp}$ such that $x = x_1 + x_2$.