OÉ Gaillimh
NUI Galway

Winter Examinations 2018/2019

Exam Codes	4BCT1, 4BS2, 4BMS2, 1AL1, 1HA1
Exams	Bachelor of Science (Comp. Sci. & IT)
	Bachelor of Science (Hons)
	Bachelor of Science (Math. Sci.)
Module	Ring Theory
Module Code	MA416, MA538
External Examiner	Professor T. Brady
Internal Examiner	Professor G. Ellis
Internal Examiner	Professor D. Flannery*
Instructions	**Answer ALL questions**
Duration	2 hours
No. pages	3, including this one
School	Mathematics, Statistics and Applied Mathematics

Requirements

Release to Library	Yes
Release in Exam Venue	No

1. (a) State, with justification, whether each of the following is a ring.

(i) The set of all complex numbers $a + b\omega$ where a, $b \in \mathbb{Q}$ and ω is a primitive cube root of unity.

(ii) The set of 2×2 matrices $[a_{ij}]$ with entries a_{ij} in the real field $\mathbb{R}$ such that $a_{21} = 0$.

(iii) The set of all polynomials in $\mathbb{Z}[x]$ with even coefficients.

[**9 marks**]

(b) Let I be an ideal of a ring R. Define addition and multiplication on the set $R/I = \{r + I \mid r \in R\}$ of cosets of I in R that turn R/I into a ring (explain why each binary operation is well-defined).

[**6 marks**]

(c) Let $\phi : R \to S$ be a ring homomorphism. Prove that $\phi(R) = \{\phi(r) \mid r \in R\}$ is a subring of S isomorphic to $R/\ker\phi$. Deduce that the ring $\mathbb{Z}_n$ of integers modulo n is isomorphic to the quotient ring $\mathbb{Z}/n\mathbb{Z}$.

[**10 marks**]

2. (a) State, with justification, whether each of the following pairs of rings are isomorphic (demonstrate an isomorphism if one exists).

(i) $3\mathbb{Z}$ and $9\mathbb{Z}$.

(ii) $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4$.

(iii) $\mathbb{C}$ and the subring $\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$ of $\mathrm{Mat}(2, \mathbb{R})$.

[**10 marks**]

(b) Prove that a finite integral domain is a field.

[**8 marks**]

(c) Let R be a commutative ring with 1.

(i) Show that if I is a prime ideal of R then R/I is an integral domain.

(ii) Is a maximal ideal of R necessarily a prime ideal of R? Why?

[**7 marks**]

3. (a) Let R be an integral domain. Prove that $\ell \in R$ is a least common multiple of $a, b \in R \setminus \{0\}$ if and only if $aR \cap bR = \ell R$.

[**7 marks**]

(b) Let R be a UFD (unique factorization domain). Prove that the irreducible elements of R are precisely the prime elements of R.

[**8 marks**]

(c) Prove that $\mathbb{Z}[\sqrt{-7}]$ is an integral domain that is not a UFD.

[**7 marks**]

(d) Give an example of an integral domain in which not every irreducible element is prime.

[**3 marks**]

4. (a) Let R be a PID (principal ideal domain), S be any integral domain, and $\phi : R \to S$ be a ring epimorphism. Prove that either ϕ is an isomorphism or S is a field.

[**8 marks**]

(b) Prove that $\mathbb{Z}[x]$ is not a PID, by showing that the ideal generated by 2 and x in $\mathbb{Z}[x]$ is not principal.

[**7 marks**]

(c) (i) Define *Euclidean domain* (ED). Prove that every ED is a PID.

(ii) Calculate a greatest common divisor of $x^3 + 2x^2 + 2x + 1$ and $x^2 + x + 2$ in $\mathbb{Q}[x]$.

[**10 marks**]