Ollscoil na Gaillimhe
University of Galway

Semester I Examinations 2022/2023

Exam Codes	3BMS2, 4BMS2, 3BS9, 4BS2, 1OA1, 1HA1
Exam	Third and Fourth Science Examination
Module	Euclidean and Non-Euclidean Geometry 1
Module Code	MA3101/MA544
Paper No.	1
External Examiner	Prof. Colva Roney-Dougal
Internal Examiners	Prof. Graham Ellis
	Dr. John Burns

Instructions **Answer all questions.**

Duration	2 hours
No. of Pages	3 pages including this page
School	Mathematics, Statistics & Applied Mathematics

Requirements:

Release in Exam venue	Yes ☑	No ☐
Release to Library	Yes ☑	No ☐
MCQ	Yes ☐	No ☑
Statistical / Log Tables	Yes ☑	No ☐

Q1. Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius one centred at the origin.

(i) [9 marks] Prove that the shortest path joining any two distinct points of S^2 is an arc of a great circle.

(ii) [8 marks] Calculate the geodesic curvature of the curve $\gamma(t) = (\cos t, 0, \sin t)$ on S^2 and the equation of the tangent plane at the point $\gamma(t)$.

(iii) [8 marks] A sailor circumnavigates Australia by a route consisting of a geodesic triangle. Prove that at least one interior angle of the triangle is $\geq \pi/3 + 10/169$ radians. (Take the Earth to be a sphere of radius 6,500 km and assume that the area of Australia is 7.5 million square km.)

Q2. Let $H \subset \mathbb{C}$ be the upper half plane with the hyperbolic metric (the Hyperbolic Plane).

(i) [6 marks] Prove than every matrix $M \in SL(2, \mathbb{R})$ representing a Möbius transformation is an isometry of H.

(ii) [6 marks] Prove that $SL(2, \mathbb{R})$ (representing Möbius transformations of H) acts transitively on H.

(iii) [6 marks] Prove that the positive $y-$axis is a geodesic of the Hyperbolic Plane.

(iv) [7 marks] Find the area of the hyperbolic (geodesic) triangle with vertices $(0, 0), (1, 0)$ and $(-1, 0)$.

Q3. Recall that for a surface patch $x(u, v)$ of an oriented surface M in $\mathbb{R}^3$ the matrix of the shape operator or Weingarten map with respect to the basis $\{x_u, x_v\}$ is $\Pi_1^{-1}\Pi_2$, where Π_1 and Π_2 are the matrices of the first and second fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$, respectively.

(i) [7 marks] Prove that the Gaussian curvature K of M is given by

$$K = \frac{LN - M^2}{EG - F^2}.$$

(ii) [12 marks] Calculate the Gaussian curvature K of the torus T parametrised by:

$$x(u, v) = ((2 + \cos v)\cos u, (2 + \cos v)\sin u, \sin v).$$

Write down a parametrised geodesic on T.

(iii) [6 marks] For the torus T in part (ii) calculate $\int_T K dA$, stating clearly any theorem you use.

Q4. (a) (i) [9 marks] Use Hamilton's quaternions to prove that the unit 3-sphere S^3 in $\mathbb{R}^4$ is a disjoint union of isometric circles, parametrized by S^2.

(ii) [8 marks] Determine the rotation in $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ induced by conjugation by the unit quaternion $t = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$.

(b) Answer (i) OR (ii) below, but NOT both: [8 marks]

(i) State the Cayley-Salmon theorem. Give an example of a smooth cubic surface and describe all its lines.

(ii) State Synge's theorem. Define the words *compact*, *orientable* and *simply connected* in the statement of the theorem. Give an example of a compact, orientable surface with negative curvature at some point for which the conclusion of the theorem does not hold.