Semester 1 Examinations 2011/2012

Exam Codes	1AH1, 1AH2, 1HA1, 1PAH1, 1MA1, 4BS3, 4FM2
Exams	Postgraduate Diploma in Arts (Mathematics/Mathematical Science) Higher Diploma (Mathematical Science) M.A. (First Year) B.Sc. Financial Mathematics & Economics B.Sc. (Mathematics)
Module	Measure Theory (Advanced Analysis I)
Module Code	MA490/MA540
External Examiner	Dr Colin Campbell
Internal Examiner(s)	Prof. Graham Ellis Dr Ray Ryan
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	3 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

You must answer every question

Q1 [49%] Answer *seven* of the following:

(a) Let A, B, C, be three subsets of a set X and let $\mathcal{A}$ be the algebra generated by them. What is the biggest possible number of elements $\mathcal{A}$ can have? Explain your answer.

(b) U, V are two subsets of a set X and a sequence of subsets of X is defined as follows: $E_1 = \varnothing$, $E_2 = X$, $E_n = U$ if $n > 2$ is odd and $E_n = V$ if $n > 2$ is even. Find $\liminf E_n$ and $\limsup E_n$.

(c) Give a verbal description of the following events in Bernoulli space:

$$\text{(i)} \quad \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \cap E_{k+1} \cap E_{k+2} \qquad \text{(ii)} \quad \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k^c$$

(d) (i) If (A_n) is a sequence of measurable events in a probability space satisfying $P(A_n) = 3^{-n}$, find $P\big([A_n, \text{i.o.}]\big)$.
(ii) If (B_n) is an independent sequence of measurable events with the property that $P(B_n) = 1/n$, find $P\big([B_n^c, \text{ev.}]\big)$.

(e) Let D be the set of numbers in the interval $[0, 1]$ that have a decimal expansion in which the digit 2 does not appear. Show that D is a Borel set. State one other interesting property of this set.

(f) Show that, if f is a measurable function, then the function f^2 is also measurable. Show that the converse is false.

(g) Let $f_n(x) = n(1 - x)x^n$. You may take it that $f_n(x) \to 0$ as $n \to \infty$ for every $x \in (0, 1)$. Show that this sequence does not converge uniformly on the interval $(0, 1)$. What does Egorov's Theorem tell you about this sequence?

(h) Give an example of a series $\sum g_n$ of Lebesgue integrable functions on $\mathbb{R}$, that converges but for which term by term integration is not valid. Sketch the graphs of g_1 and g_2.

(i) Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be a nonnegative measurable function on X such that $\displaystyle\int_X f\, d\mu = 0$. Show that $f = 0$ almost everywhere.

(j) Determine which (if any) of the following numbers belong to the Cantor Set:

$$\frac{1}{9} + \frac{2}{27}\,, \qquad \frac{2}{\pi}\,, \qquad (0.02001)_3\,, \qquad \frac{8}{9}$$

(k) Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

$$f(x) = \begin{cases} x + 3 & \text{if } x \notin \mathbb{Q} \\ \sin e^x & \text{if } x \in \mathbb{Q} \end{cases}$$

Find the Lebesgue integral of f on the interval $[0, 1]$.

You must answer every question.

Q2 [17%]

(a) Give the definition of a *measure*. If (E_n) is an sequence of measurable sets, show that

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) \leq \sum_{n}^{\infty} \mu(E_n)$$

(b) Give the definition of $\liminf B_n$ and $\limsup B_n$, where (B_n) is a sequence of sets. Show that, if the sequence (B_n) is decreasing, then

$$\limsup B_n = \bigcap_{n=1}^{\infty} B_n$$

(c) State the two Borel-Cantelli Lemmas.

Q3 [17%] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) Suppose the function $f\colon X \to \mathbb{R}$ is *measurable*, so that $\{x \in X : f(x) > \lambda\}$ is a measurable set for every $\lambda \in \mathbb{R}$. Show that the sets

$$\text{(i)} \quad \{x \in X : f(x) < \lambda\} \quad \text{and} \quad \text{(ii)} \quad \{x \in X : f(x) = \lambda\}$$

are measurable for every $\lambda \in \mathbb{R}$.

(b) If f and g are measurable functions, show that the function $f - g$ is measurable.

Q4 [17%] Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give an account of the definition of the integral $\int_X f \, d\mu$ for nonnegative measurable functions f on X.

(b) State Lebesgue's Monotone Convergence Theorem (LMCT). Give an example of a sequence of functions for which term by term integration is not permissable. Explain why LMCT cannot be applied to your example.

(c) Describe the construction of the Cantor Set and prove that this set is uncountable.