Semester 1 Examinations 2012/2013

Exam Codes	1AH1, 1AH2, 1HA1, 1PAH1, 1MA1, 4BS3, 4FM2
Exams	Postgraduate Diploma in Arts (Mathematics/Mathematical Science) Higher Diploma (Mathematical Science) M.A. (First Year) B.Sc. Financial Mathematics & Economics B.Sc. (Mathematics)
Module	Measure Theory (Advanced Analysis I)
Module Code	MA490/MA540
External Examiner	Dr Colin Campbell
Internal Examiner(s)	Prof. Graham Ellis Dr Ray Ryan
Instructions:	**Answer every question.**
Duration	2 Hours
No. of Pages	3 pages, including this one
Department	School of Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes

You must answer every question

Q1 [49%] Answer *seven* of the following:

(a) Give an example, with proof, of an algebra that is not a σ-algebra.

(b) Let $E_n = [-2 + (-1)^n/n, -1/n]$ if n is odd and $E_n = (0, n)$ if n is even. Find the lower and upper limits, $\liminf E_n$ and $\limsup E_n$ of this sequence.

(c) Give a verbal description of the following events in Bernoulli space:

$$\text{(i)} \quad \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \cap E_{k+1}^c \cap E_{k+2} \qquad \text{(ii)} \quad \bigcup_{n=1}^{\infty} \left(E_{n-1} \cap E_n \cap \bigcap_{k>n} E_k^c \right)$$

(d) Suppose that μ is an additive set function on a σ-algebra $\mathcal{A}$ and has the following property: For every increasing sequence (E_n) of sets in $\mathcal{A}$,

$$\mu\Big(\bigcup_n E_n\Big) = \lim_{n\to\infty} \mu(E_n)\,.$$

Show that μ is a measure.

(e) Let $\mathcal{A}$ be the σ-algebra of subsets of $\mathbb{R}$ generated by the intervals $(-1, 0)$, $(1, 2)$ and $(4, 6)$.
(i) How many sets are there in $\mathcal{A}$?
(ii) Give an example of a function on $\mathbb{R}$ that is not measurable with respect to $\mathcal{A}$.

(f) (i) Give an example of a function f for which f^2 is measurable, but f is not.
(ii) What happens is we replace f^2 by f^3? Explain your answer.

(g) Let $f_n(x) = 2x^2/(x^2 + n)$. Show that the sequence (f_n) converges pointwise, but not uniformly, on $\mathbb{R}$.

(h) Give an example of a series $\sum g_n$ of Lebesgue integrable functions on $\mathbb{R}$ that converges but for which term by term integration is not valid.

(i) Let $(X, \mathcal{A}, \mu)$ be a measure space and let f be a nonnegative measurable function on X such that $\displaystyle\int_X f\, d\mu = 0$. Show that $f = 0$ almost everywhere.

(j) Evaluate the following limit using term by term integration and explain how Lebesgue's Dominated Convergence Theorem applies:

$$\lim_{n\to\infty} \int_0^1 (1 - e^{-x^2/n}) x^{-1/2}\, dx$$

(k) Does the Cantor set contain any irrational numbers? Explain your answer.

You must answer every question.

Q2 [**17%**]

(a) Give the definition of $\liminf B_n$ and $\limsup B_n$, where (B_n) is a sequence of sets. Show that, if the sequence (B_n) is increasing, then

$$\liminf B_n = \limsup B_n$$

(b) State and prove the first Borel-Cantelli Lemma.

Q3 [**17%**] Let X be a set and $\mathcal{A}$ a σ-algebra of subsets of X.

(a) Give the definition of a measurable function and use it to show that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 - 1$ is Borel measurable.

(b) Let u, v be real functions on a complete measure space. If $u(x) = v(x)$ almost everywhere and if u is measurable, show that v is measurable.

(c) If f and g are measurable functions, show that the function $f + g$ is measurable.

Q4 [**17%**] Let $(X, \mathcal{A}, \mu)$ be a measure space.

(a) Give the definition of the integral $\displaystyle\int_X f \, d\mu$ where f is a simple measurable function on X. If f, g are integrable simple functions, show that

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu \, .$$

Answer **either** part (b) **or** part (c):

(b) State both Lebesgue's Monotone Convergence Theorem (LMCT) and Fatou's Lemma. Explain how Lebesgue's Monotone Convergence Theorem is used to prove Fatou's Lemma.
Give an example to show that strict inequality can occur in Fatou's Lemma.

(c) Prove Lebesgue's Monotone Convergence Theorem.