Semester I Examinations 2017/2018

Exam Codes	3BMS2, 4BMS2, 3BS9, 4BS2, 1OA1, 1HA1
Exam	Third and Fourth Science Examination
Module	Euclidean and Non-Euclidean Geometry 1
Module Code	MA3101
Paper No.	1
External Examiner	Prof. Thomas Brady
Internal Examiners	Prof. Graham Ellis
	Dr. John Burns
Instructions	**Answer all questions.**
Duration	2 hours
No. of Pages	3 pages including this page
School	Mathematics, Statistics & Applied Mathematics
Requirements	No special requirements
Release to Library:	Yes
Other Materials	Non-programmable calculators

Q1. Let $S^2 \subset \mathbb{R}^3$ be the sphere of radius one.

(i) [9 marks] Prove that the shortest path joining any two points of S^2 is an arc of a great circle.

(ii) [8 marks] Prove than $SO(3)$ acts transitively on S^2.

(iii) [8 marks] Use the spherical cosine rule in the geodesic triangle whose vertices are New York (41 degrees N, 74 degrees W), New Orleans (30 degrees N, 90 degrees W) and the north pole to find the distance from New Orleans to New York. Take the Earth to be a sphere of radius 6,500 km.
(Recall: The spherical cosine rule for a geodesic triangle with angles α, β, γ and corresponding opposite sides a, b, c says that
$\cos a = \cos\alpha \sin b \sin c + \cos b \cos c$.)

Q2. Let $H \subset \mathbb{C}$ be the upper half plane with the hyperbolic metric (the Hyperbolic Plane).

(i) [6 marks] Prove than any matrix $M \in SL(2, \mathbb{R})$ representing a Möbius transformation is an isometry of H.

(ii) [6 marks] Prove that the positive $y-$axis is a geodesic of the Hyperbolic Plane.

(iii) [6 marks] Let γ denote the positive y-axis and let $p = (1, \frac{1}{2})$. Find two geodesics of the Hyperbolic Plane containing p that do not intersect γ.

(iv) [7 marks] Find the area of the hyperbolic (geodesic) triangle with vertices $(0, 0), (1, 0)$ and $(0, 1)$.

Q3. Recall that for a surface patch $x(u, v)$ of an oriented surface M in $\mathbb{R}^3$, the matrix of the shape operator or Weingarten map with respect to the basis $\{x_u, x_v\}$ is $\Pi_1^{-1}\Pi_2$, where Π_1 and Π_2 are the matrices of the first and second

fundamental forms $Edu^2 + 2Fdudv + Gdv^2$ and $Ldu^2 + 2Mdudv + Ndv^2$, respectively.

(i) [7 marks] Prove that the Gaussian curvature K of M is given by

$$K = \frac{LN - M^2}{EG - F^2}.$$

(ii) [12 marks] Calculate the Gaussian curvature K of Enneper's surface $x(u, v) = (u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2)$.

(iii) [6 marks] Describe two different geodesics on the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ passing through the point $(1, 0, 0)$.

Q4. (a) (i) [8 marks] Use Hamilton's quaternions to prove that the unit 3-sphere S^3 in $\mathbb{R}^4$ is a disjoint union of isometric circles, parametrized by S^2.

(ii) [7 marks] Determine the rotation in $\mathbb{R}i + \mathbb{R}j + \mathbb{R}k$ induced by conjugation by the unit quaternion $t = \frac{1}{2} + \frac{i}{2} + \frac{j}{2} + \frac{k}{2}$.

(b) [10 marks] State the Cayley-Salmon theorem, explaining the assumptions of the theorem. Give an example of a smooth cubic surface containing twenty seven lines and describe all the lines.