PROSEMINAR ON COMPUTER-ASSISTED MATHEMATICS

HEIDELBERG UNIVERSITY, SUMMER SEMESTER 2023

Target audience: Bachelor students.
Instructor: Florent Schaffhauser.
Language of instruction: English.

Objectives

Through a learning-by-doing approach, the purpose of this proseminar is to give an introduction to computer-assisted mathematics. It will be split into the following two topics:

(1) Linear algebra (in *Sage*, *Python* or *R*).
(2) Proof assistants (in *Lean*, *Coq* or *Isabelle*).

In the first part, we will for instance write a short program that asks the user to input the matrix of a linear transformation T in the canonical basis of $\mathbb{R}^2$ or $\mathbb{R}^3$, then input a family of vectors, check that it is a basis and, if so, compute the matrix of T in that new basis.

In the second part, the goal is to learn what it means to formalise a proof in order to ask a computer to check it or even assist us with it. An example of an ambitious goal would be: to formalise a proof that the field of complex numbers is algebraically closed.

Contents and methodology

This proseminar is mathematically-oriented and intended for beginners: there are no pre-requisites beyond a first course in linear algebra (essentially the Gaussian elimination algorithm and its applications) and basic familiarity with polynomials with coefficients in $\mathbb{R}$ or $\mathbb{C}$ (Euclidean division, relation between roots and coefficients). No previous knowledge of computer algebra systems or proof assistants will be required.

The proseminar will be project-based and students will work in groups. Those already familiar with the material or computer software can act as tutors. We will devote approximately seven weeks to each of the main topics (first linear algebra, then proof assistants). Active participation in the group meetings is expected from all students looking to obtain credits for the seminar. The schedule for each part will be organised as follows:

- Weeks 1-2: Presentation of the material and choice of projects.
- Weeks 3-6: Group meetings to discuss the understanding of the mathematical content and the progress of computer implementation.
- Weeks 7-8: Project presentations.

Linear algebra. The list of linear algebra topics goes as follows:

(1) Computer algebra systems. Representations of vectors and matrices.
(2) Row operations. Gaussian elimination. Row-reduced echelon form of a matrix.
(3) Invertible matrices. Elementary matrices. Determinant.
(4) Linear independence. Bases for the kernel and the image of a linear transformation.
(5) Rank-nullity theorem and the row space of matrix. Basis for the row space.
(6) Base change. Coordinates of a vector, matrix of a linear transformation.
(7) Eigenvalues and the characteristic polynomial. Diagonalisation.
(8) The Gram-Schmidt process. Least-square approximation.

Optional additional topics include:

- Gaussian reduction of quadratic forms.
- The Smith normal form (over $\mathbb{Z}$ or $\mathbb{K}[x]$, for $\mathbb{K}$ a field).
- The Jordan normal form.

Proof assistants. The list of topics for the introduction to proof assistants goes as follows:

(1) Installation of a proof assistant. Familiarisation with the interface.
(2) The natural number game (Peano axioms and the induction principle).
(3) Equality and computations (tactics to prove algebraic identities).
(4) Implications and equivalences (propositional logic).
(5) Predicates and quantifiers (first-order logic).
(6) Contraposition and proof by contradiction (proof tactics).
(7) The complex number game (towards the fundamental theorem of algebra).

References

For the linear algebra part, a script will be provided by the instructor. For the proof assistant part, we will partly rely on the following online resources:

- *Mathematics in Lean*, an online textbook available at `https://leanprover-community.github.io/mathematics_in_lean/index.html`.
- *Formalising mathematics*, a GitHub repository for Kevin Buzzard's 2021 course on the subject, available at `https://github.com/ImperialCollegeLondon/formalising-mathematics`.