Proseminar on computer-assisted mathematics

Session 4 - Kernels, images and diagonalisation in Sagemath

```
# Example 1
A = matrix(QQ, [[2,0,4],[3,-4,12],[1,-2,5]])
f_A = A.charpoly("t")
show( f_A )
```

$t^3 - 3t^2 + 2t$

```
# We can factorise f_A
show( f_A.factor() )
```

$(t-2) \cdot (t-1) \cdot t$

```
# And its roots are indeed the eigenvalues of A
ev_A = A.eigenvalues()
show( ev_A )
```

$[2, 1, 0]$

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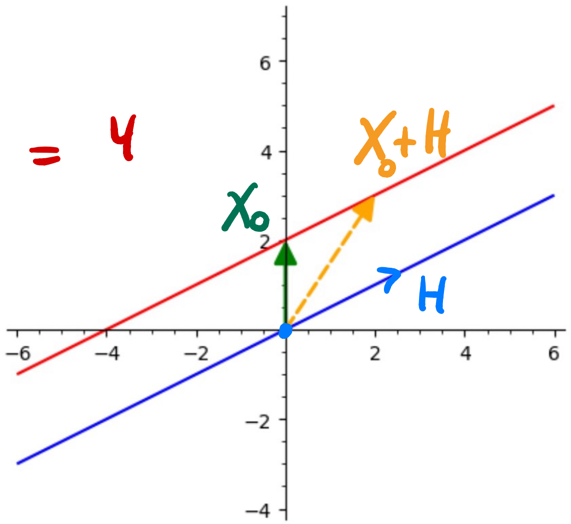
What we want to be able to do:

- Parameterise the set of solutions of a non-homogeneous linear system $AX = Y$ (which is an affine space).
- Extract, from a family of vectors, a basis of the, subspace that they generate.
- Complete a basis of a subspace to a basis of the ambient space.
- Determine whether a given matrix is diagonalisable and, if so, construct a basis of eigenvectors and the associated eigenvalues.

1. Kernels and images

Recall that the set of i u solutions of a linear system $AX = Y$ is an affine space of direction $\ker A$.

$-x + 2y = 4$



ker A

$A = (-1 \quad 2)$ $\qquad Y = 4$

$X_0 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $\qquad \ker A = \mathrm{span}_{\mathbb{R}}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$

Proof:

Assume $\exists X_0,\ AX_0 = Y$.

Then, for all X,

$AX = Y$ iff $A(X - X_0) = 0$

i.e. $H := (X - X_0) \in \ker A$.

So $AX = Y$

iff

$\exists H \in \ker A,\ X = X_0 + H$. □

$AX = Y$

iff

$\exists t \in \mathbb{R},\ X = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So, to solve $AX = Y$, we need to find one particular solution of that equation, as well as the general solution of the equation $AX = 0$.

Example:

$$\underbrace{\begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -1 & 3 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}_{Y}$$

Both can be obtained from the Gaussian reduction of the augmented matrix $(A|Y)$.

```
y = vector( QQ, [ 2, -1 ] )
M = A.augment( y, subdivide = True )
show( M )
```

$$\left(\begin{array}{rrrr|r} 1 & 1 & -1 & 5 & 2 \\ 0 & -1 & 3 & 0 & -1 \end{array}\right)$$

```
show( M.echelon_form() )
```

$$\left(\begin{array}{rrrr|r} 1 & 0 & 2 & 5 & 1 \\ 0 & 1 & -3 & 0 & 1 \end{array}\right)$$

$X_0 := \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ satisfies $AX_0 = Y$

$\ker A = \operatorname{span}\left(\begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$

The Gaussian reduction can also be used to:

- Find a basis of the column space of a matrix.
- Find linear dependence relations between the columns of a matrix.
- Complete a family of linearly independent vectors to a basis of the ambient space.

```
# Let us retake the previous matrix A
show(A)
```

$$\begin{pmatrix} 1 & 1 & -1 & 5 \\ 0 & -1 & 3 & 0 \end{pmatrix}$$

```
# The rank of A is equal to the number of the number of pivots in A1
A1 = A.echelon_form()
show(A1)
```

$$\begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -3 & 0 \end{pmatrix}$$

$(C_1(A), C_2(A))$ is a basis of the column space.

Moreover:

$C_3(A) = 2\,C_1(A) - 3\,C_2(A)$

$C_4(A) = 5\,C_1(A) + 0\,C_2(A)$

2. Diagonalisation

Definition. Let $\mathbb{k}$ be a field and let $n > 0$ be an integer. A matrix $A \in \mathrm{Mat}(n \times n; \mathbb{k})$ is called **diagonalisable over** $\mathbb{k}$ if there exists a pair of matrices (D, P) in $\mathrm{Mat}(n \times n; \mathbb{k})$ such that:

1. D is diagonal.
2. P is invertible.
3. $AP = PD$.

The last equality means that, for all $j \in \{1; \ldots; n\}$, the j-th column of P is an eigenvector for A, associated to the j-th diagonal coefficient d_j of D:

$$\forall\, j \in \{1; \ldots; n\}, AC_j(P) = d_j C_j(P)$$

where

$$D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

and $P = [C_1(P), \ldots, C_n(P)]$.

$$A \begin{pmatrix} \vert & & \vert \\ C_1(P) & \cdots & C_n(P) \\ \vert & & \vert \end{pmatrix} = \begin{pmatrix} \vert & & \vert \\ C_1(P) & \cdots & C_n(P) \\ \vert & & \vert \end{pmatrix} \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix}$$

$$= \begin{pmatrix} \vert & & \vert \\ d_1 C_1(P) & \cdots & d_n C_n(P) \\ \vert & & \vert \end{pmatrix}$$

Theorem A matrix $A \in \mathrm{Mat}(n \times n; \Bbbk)$ is diagonalisable over $\Bbbk$ if and only if its characteristic polynomial

$$f_A(t) := \det(tI_n - A)$$

splits into a product of linear factors

$$f_A(t) = (t - a_1)^{m_1} \dots (t - a_r)^{m_r},\ a_j \in \Bbbk$$

and

$$\forall\, j \in \{1; \dots ; r\},\ \ \dim\ker(A - a_j I_n) = m_j.$$

In other words, A is diagonalisable over $\Bbbk$ if and only if its characteristic polynomial $f_A(t)$ splits over $\Bbbk$ and *the geometric multiplicity of of a_j as an eigenvalue of A is equal to its algebraic multiplicity as a root of $f_A(t)$*.

We will now see how to apply this theorem using `Sage`. Note that sometimes the characteristic polynomial of A is defined as $\det(A - tI_n)$, which is equal to $(-1)^n \times f_A(t)$ with $f_A(t)$ as above. We have chosen to follow Sage's convention here.

```
# Example 2, with multiple eigenvalues
A = matrix(QQ, [[2,-3,1],[1,-2,1],[1,-3,2]])
f_A = A.charpoly("t")
show( f_A.factor() )
```

$t \cdot (t-1)^2$

```
# Sage can show us the eigenvalues of A, counted with their respective mutiplicities
show( A.eigenvalues() )
```

$[0, 1, 1]$

```
# Similarly, it can show us eigenvectors for A
show( A.eigenvectors_right() )
```

$[(0, [(1,\ 1,\ 1)]\,, 1)\,, (1, [(1,\ 0,\ -1)\,, (0,\ 1,\ 3)]\,, 2)]$

the eigenvalue

a basis of the corresponding eigenspace

the (algebraic) multiplicity

```
D, P  = A.eigenmatrix_right()
show( D, P )
```

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & 3 \end{pmatrix}$$

To check: $AP = PD$