

# Linear Systems TTK4115 - Helicopter lab

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# 1 Part 1 - Mathematical modeling

## 1.1 Problem 1

We use the helicopter model shown in fig. 1 and fig. 2 as our starting point for deriving the equations of motion.

Figure 1: the helicopter model figure 7 from the assignment depicting forces and joint axes [2, p.12].

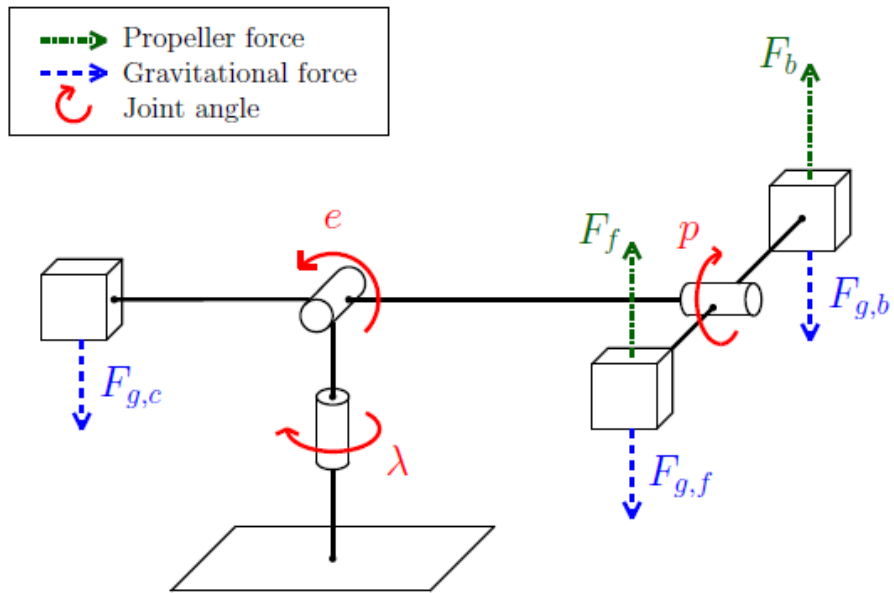
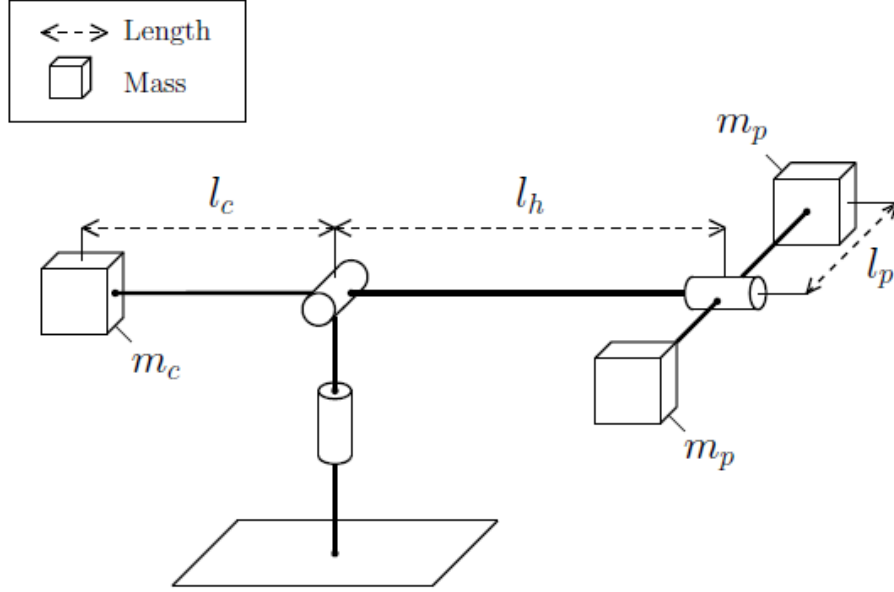


Figure 2: the helicopter model figure 8 from the assignment depicting masses and distances between the joint axes and the point masses [2, p.13].



The equation of motion for the pitch is found through the momentum around the pitch axis in the clockwise direction as shown in fig. 1. It becomes:

$$\begin{aligned} J_p \ddot{p} &= l_p (F_{g,b} - F_b - F_{g,f} + F_f) \\ &= l_p (m_p g - m_p g + K_f V_f - V_b) \\ &= l_p K_f (V_f - V_b) \end{aligned}$$

Where the lengths including  $l_p$  is shown in fig. 2. Since  $V_d = V_f - V_b$ , we can write this as:

$$J_p \ddot{p} = l_p K_f V_d \quad (1)$$

Here, we can see that  $L_1 = l_p K_f$ .

The equation of motion for the elevation angle is found similarly through the momentum in the counter-clockwise direction around the elevation axis, again as shown in fig. 1.

$$J_e \ddot{e} = arm_c F_{g,c} - arm_h (F_{g,f} + F_{g,b} - K_f \cos(p) (V_f + V_b))$$

where  $arm_c$  is the moment arm between the counterweight point mass and the elevation axis, and  $arm_h$  is the moment arm between any of the two motor

point masses and the elevation axis. As shown in fig. 3,  $arm_c = l_c \cos(e)$ , and  $arm_h = l_h \cos(e)$ .

We can immediately substitute  $V_s = V_f + V_b$ , and simplify:

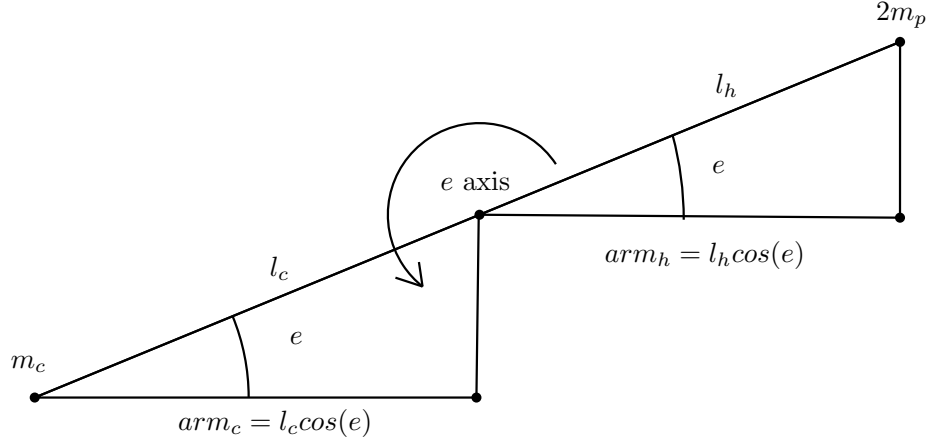
$$\begin{aligned} J_e \ddot{e} &= l_c \cos(e) m_c g - l_h \cos(e) (2m_p g - K_f \cos(p) V_s) \\ &= \cos(e) (l_c m_c g - 2l_h m_p g) + l_h K_f V_s \cos(e) \cos(p) \end{aligned}$$

Here, we have (as the author of the exercise) counted the  $\cos(e)$  factor of the  $V_s$  term as negligible, and set it to 1. The resulting equation has the form:

$$J_e \ddot{e} = g(l_c m_c - 2l_h m_p) \cos(e) + l_h K_f V_s \cos(p) \quad (2)$$

Note that  $L_2 = g(l_c m_c - 2l_h m_p)$ , and  $L_3 = l_h K_f$ .

Figure 3: moment arms around the elevation axis.



Forklar hvor  $\cos(p)$  kommer fra - og lag ny figur. Bruk denne til å forklare (2c)/(3) også.

Finally, the equation of motion for the travel angle is found through the momentum around the travel axis in the clockwise direction. As seen in fig. 3, the only forces with a moment arm perpendicular to the travel axis are the components of the motor forces in the horizontal direction, which have length  $arm_h = l_h \cos(e)$ . Furthermore, the horizontal components

Figure 4: the pitch model



$$J_{\lambda} \ddot{\lambda} = arm_h$$

The equations (1), (2), and (3) corresponds to (2a), (2b), and (2c) respectively of the assignment [2, p.13]

## 1.2 Problem 2

To linearize the system about the point with all state variables equal to zero  $(p, e, \lambda)^T = \dot{p}, (\dot{e}, \dot{\lambda})^T = (0, 0, 0)$  the inputs in the equations of motion ( $V_s^*$  and  $V_d^*$ ) must be set to values that make this an equilibrium point.

At the linearization point, the equation of motion for pitch () reduces to  $0 = L_1 * V_d^*$  therefore :

$$V_d^* = 0 \quad (3)$$

At the linearization point, the equation of motion for elevation () reduces to  $0 = L_2 + L_3 * V_s^*$  therefore:

$$V_s^* = -L_2/L_3 \quad (4)$$

While the equation of motion for travel () at the linearization point simply reduces to  $0 = 0$ .

The following transformation is performed to simplify the analysis :

$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} - \begin{bmatrix} p^* \\ e^* \\ \lambda^* \end{bmatrix} \text{ and } \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} = \begin{bmatrix} V_s \\ V_d \end{bmatrix} - \begin{bmatrix} V_s^* \\ V_d^* \end{bmatrix} \quad (5)$$

The equations of motion in the transformed system are therefore:

$$J_p \ddot{\tilde{p}} = L_1 \tilde{V}_d \quad (6a)$$

$$J_e \ddot{\tilde{e}} = L_2 \cos(\tilde{e}) + L_3 (\tilde{V}_s + L_2/L_3) \cos(\tilde{p}) \quad (6b)$$

$$J_{\lambda} \ddot{\tilde{\lambda}} = L_4 (\tilde{V}_s + L_2/L_3) \cos(\tilde{e}) \sin(\tilde{p}) \quad (6c)$$

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By choosing the state to be  $\mathbf{x} = (\tilde{p}, \tilde{e}, \tilde{\lambda}, \dot{p}, \dot{e}, \dot{\lambda})$  the nonlinear state equations become:

$$\dot{x}_1 = x_4 \quad (7a)$$

$$\dot{x}_2 = x_5 \quad (7b)$$

$$\dot{x}_3 = x_6 \quad (7c)$$

$$\dot{x}_4 = (L_1/J_p)V_d \quad (7d)$$

$$\dot{x}_5 = (L_2/J_e)\cos(x_2) + (L_3/J_e)(V_s + L_2/L_3)\cos(x_1) \quad (7e)$$

$$\dot{x}_6 = (L_4/J_\lambda)(V_s + L_2/L_3)\cos(x_2)\sin(x_1) \quad (7f)$$

If the above system is expressed as  $\dot{x} = h(x, u)$ , where  $\mathbf{x}$  is the state and  $u$  is the input, the system is linearized by finding the Jacobians of  $h$  with respect to the state and the input and then plugging in the equilibrium values.

$$\frac{\partial h}{\partial x} = A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{L_4 L_2}{J_\lambda L_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{\partial h}{\partial u} = B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_1/J_p \\ L_3/J_e & 0 \\ 0 & 0 \end{bmatrix} \quad (8)$$

where  $L_1, L_2, L_3$  and  $L_4$  were calculated in Section 1.1 and  $J_p, J_e$  and  $J_\lambda$  were given in the assignment description .

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The linearized equations of motion can therefore be written in the following form:

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$$\ddot{\tilde{p}} = K_1 \tilde{V}_d \quad K_1 = \frac{L_1}{J_p} \quad (9a)$$

$$\ddot{\tilde{e}} = K_2 \tilde{V}_s \quad K_2 = \frac{L_3}{J_e} \quad (9b)$$

$$\ddot{\tilde{\lambda}} = K_3 \tilde{p} \quad K_3 = \frac{L_4 L_2}{J_\lambda L_3} \quad (9c)$$

### 1.3 Problem 3

### 1.4 Problem 4

## 2 Part 2 – Monovariable control

### 2.1 Problem 1

We are given the controller shown in eq. (10).

$$\tilde{V}_d = K_{pp}(\tilde{p}_c - \tilde{p}) - K_{pd}\dot{\tilde{p}} \quad (10)$$

We take this controller and substitute it in the equation for pitch angle (??).

$$\ddot{\tilde{p}} = K_1 K_{pp}(\tilde{p}_c - \tilde{p}) - K_1 K_{pd}\dot{\tilde{p}} \quad (11)$$

Now we Laplace transform eq. (11) to find the transfer function  $\frac{\tilde{p}(s)}{\tilde{p}_c(s)}$ .

$$\begin{aligned} \ddot{\tilde{p}} + K_1 K_{pd}\dot{\tilde{p}} + K_1 K_{pp}\tilde{p} &= K_1 K_{pp}\tilde{p}_c \\ \mathcal{L} \rightarrow \\ s^2\tilde{p}(s) + sK_1 K_{pd}\tilde{p}(s) + K_1 K_{pp}\tilde{p}(s) &= K_1 K_{pp}\tilde{p}_c(s) \end{aligned}$$

Which gives us our transfer function

$$\frac{\tilde{p}(s)}{\tilde{p}_c(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd}s + K_1 K_{pp}} \quad (12)$$

The linearized pitch dynamics can be regarded as a second-order linear system, which means that if we place eq. (12) on the form shown in eq. (13) we can determine  $K_{pp}$  and  $K_{pd}$  from  $\omega$  and  $\zeta$ .

$$h(s) = \frac{\omega^2}{s^2 + 2\zeta\omega^2 s + \omega^2} \quad (13)$$

This gives us the following relations

$$\omega = \sqrt{K_1 K_{pp}} \quad (14)$$

$$\begin{aligned} 2\zeta\omega^2 &= K_1 K_{pd} \\ \zeta &= \frac{K_1 K_{pd}}{2\omega^2} = \frac{K_{pd}}{2K_{pp}} \end{aligned} \quad (15)$$

We know that for a critically damped system  $\zeta = 1$ , which gives us

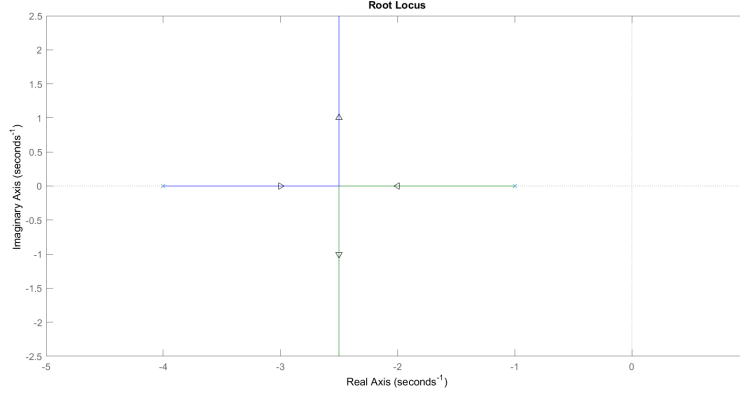
$$K_{pd} = 2K_{pp} \quad (16)$$

We chose a  $K_{pp} = 3$  and then from the relation in eq. (16) we get  $K_{pd} = 6$ . With these values the response of the pitch angle to the input was slower than desired. Therefore,  $K_{pp}$  was increased to  $K_{pp} = 12.5$  and  $K_{pd}$  was lowered to underdamp



the system, until it was sufficiently responsive at  $K_{pd} = 0.7K_{pp} = 8.75$ . At these values the system responded faster with only minor oscillations. It was observed that larger values of  $K_{pp}$  gave rise to larger oscillations.

Figure 5: Change in pole position by increasing  $K_{pp}$  given a constant  $K_{pd}$



At the critically damped point, the poles lie on the same point on the x-axis. When  $K_{pp}$  is decreased in relation to  $K_{pd}$  the system is over-damped the poles move away from each other along the x-axis. When  $K_{pp}$  is increased in relation to  $K_{pd}$ , the system is under damped and the poles move away from each other vertically from the critically damped point.

With the PD controller, it was significantly easier to control the helicopter than with just feed forward joystick control.

## 2.2 Problem 2

By plugging the P controller for travel:

$$\tilde{p} = K_{rp}(\dot{\lambda}_c - \dot{\lambda}) \quad (17)$$

into the equation of motion for travel, the transfer function between  $\dot{\lambda}$  and  $\dot{\lambda}_c$  can be derived. By substituting the right side of the controller into the equation of motion for travel the equation becomes:

$$\ddot{\lambda} = K_3(K_{rp}(\dot{\lambda}_c - \dot{\lambda})) \quad (18)$$

After taking the laplace transform and rearranging terms, the transfer function is as follows:

$$\frac{\ddot{\lambda}(s)}{\dot{\lambda}_c(s)} = \frac{K_3 K_{rp}}{s + K_3 K_{rp}} \quad (19)$$

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This controller was quite quick and stable after correctly tuning  $K_{rp}$ . The value that was deemed best was  $K_{rp} = .$  Higher values of  $K_{rp}$  led to , while lower values of  $K_{rp}$  led to . It was not nessecary to add a gain to the x value of the joystick to produce quick and accurate control.

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### 3 Part 3 - Multivariable control

#### 3.1 Problem 1

Here, we are to find the systems matrices  $\mathbf{A}$  and  $\mathbf{B}$ . Trivially, they are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \end{bmatrix} \quad (20)$$

#### 3.2 Problem 2

First, we are to examine the controllability of the system. We do that by finding the controllability matrix  $\mathbf{C}$  of the system, and see whether it has full rank or not:

$$\mathbf{C} = [\mathbf{B} \quad \mathbf{AB}] = \begin{bmatrix} 0 & 0 & 0 & K_1 \\ 0 & K_1 & 0 & 0 \\ K_2 & 0 & 0 & 0 \end{bmatrix} \quad (21)$$

which has full rank:  $\text{rank}(\mathbf{C}) = 3$ , and is thus controllable.

Second, we are to implement a LQR controller with reference feed-forward. Our  $\mathbf{P}$  is defined such that as time goes to infinity, our states  $\tilde{p}$  and  $\dot{\tilde{e}}$  tend to their reference values  $\tilde{p}_c$  and  $\dot{\tilde{e}}_c$ . This happens when  $\dot{\mathbf{x}} = 0$ , as the system reaches a stable equilibrium around the reference values:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{Ax} + \mathbf{Bu} \\ &= \mathbf{Ax} + \mathbf{B}(\mathbf{Pr} - \mathbf{Kx}) \\ &= (\mathbf{A} - \mathbf{BK})\mathbf{x} + \mathbf{BPr} = 0 \end{aligned}$$

When  $\dot{\mathbf{x}} = 0$ , our  $\mathbf{x}$  has reached its final value and we define  $\mathbf{x} = \mathbf{x}_\infty$ :

$$\begin{aligned} (\mathbf{BK} - \mathbf{A})\mathbf{x}_\infty &= \mathbf{BPr} \\ \Leftrightarrow \mathbf{x}_\infty &= (\mathbf{BK} - \mathbf{A})^{-1} \mathbf{BPr} \\ \Rightarrow \mathbf{y}_\infty &= \mathbf{Cx}_\infty = \mathbf{C}(\mathbf{BK} - \mathbf{A})^{-1} \mathbf{BPr} \end{aligned}$$

We see that our output  $\mathbf{y}_\infty$  is equal to our reference  $\mathbf{r}$  when:

$$\mathbf{P} = [\mathbf{C}(\mathbf{BK} - \mathbf{A})^{-1} \mathbf{B}]^{-1} \quad (22)$$

Therefore, we now have

$$\lim_{t \rightarrow \infty} \mathbf{y}(t) = \mathbf{y}_\infty = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{e}} \end{bmatrix} = \begin{bmatrix} \tilde{p}_c \\ \dot{\tilde{e}}_c \end{bmatrix} = \mathbf{r},$$

which is what we wanted.

### 3.3 Problem 3

Without integral effect, the system could track pitch without much of a problem. But the elevation rate had a noticeable deviation in the ranges of elevation not close to our linearization point  $\dot{\epsilon} = 0$ .

With integral effect, this deviation in elevation rate is asymptotically removed. This means that with a zero Y output from our joystick, we see a corresponding zero elevation rate, and the elevation angle stays asymptotically constant also for angles not close to its linearization point.

## References

- [1] Chi-Tsong Chen, *Linear System Theory and Design*, Oxford University Press, 4th edition, 2014
- [2] Kristoffer Gryte, *Helicopter lab assignment*, Department of Engineering Cybernetics, NTNU, Version 4.5, 2015