Linear Systems TTK4115 - Helicopter lab

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Contents

1 Part 1 - Mathematical modeling

1.1 Problem 1

We use the helicopter model shown in fig. 1 and fig. 2 as our starting point for deriving the equations of motion.

Figure 1: the helicopter model figure 7 from the assignment depicting forces and joint axes [2, p.12].

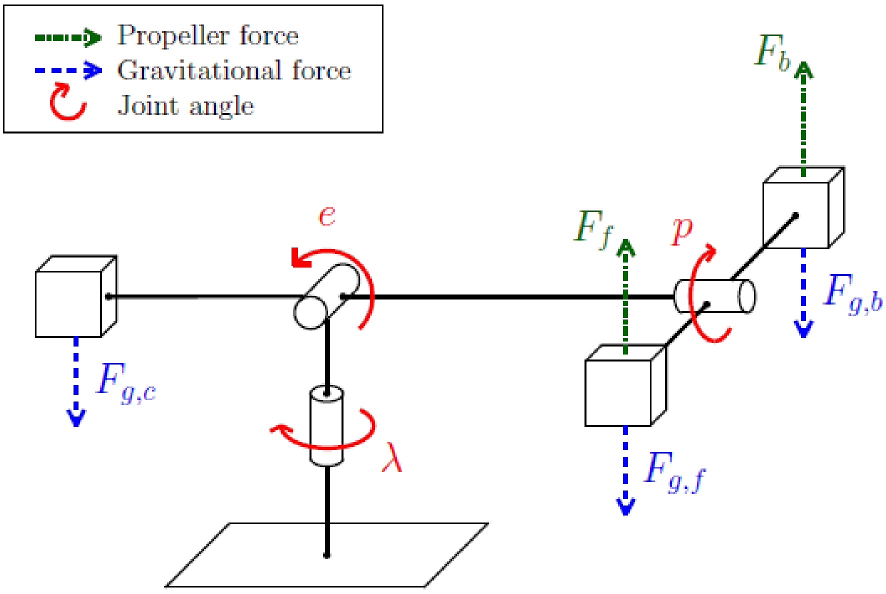
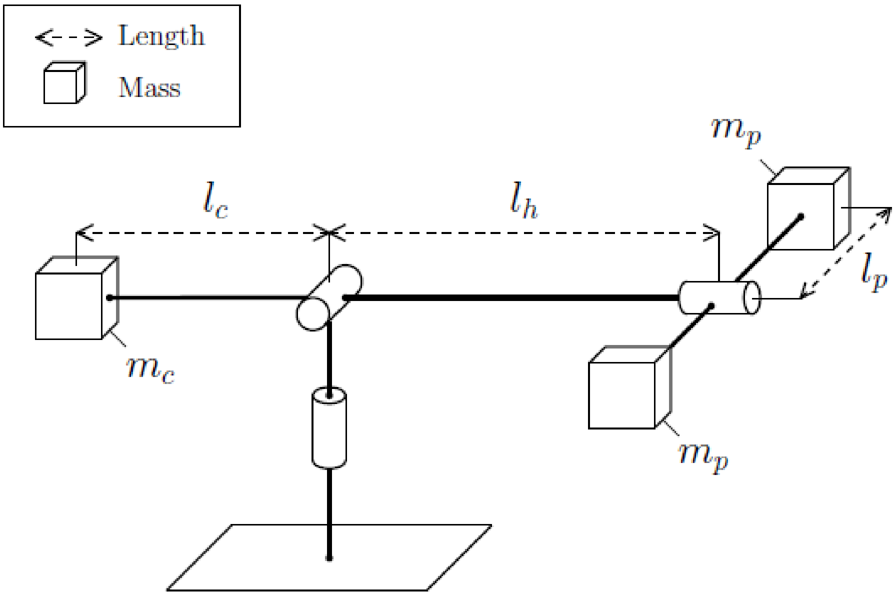


Figure 2: the helicopter model figure 8 from the assignment depicting masses and distances between the joint axes and the point masses [2, p.13].



Only forces perpendicular to a moment arm, with the moment arm again perpendicular to the axis of rotation in question produces moment, or torque. The equation of motion for the pitch is found through the momentum around the pitch axis in the clockwise direction as shown in fig. 1. It becomes:

$$\begin{aligned} J_p\ddot{p} &= l_p(F_{g,b} - F_b - F_{g,f} + F_f) \\ &= l_p(m_p g - mp_g + K_f V_f - V_b) \\ &= l_p K_f (V_f - V_b) \end{aligned}$$

Where the lengths including l_p is shown in fig. 2. Since $V_d = V_f - V_b$, we can write this as:

$$J_p\ddot{p} = l_p K_f V d \tag{1}$$

Therefore $L_1 = l_p K_f$.

The equation of motion for the elevation angle is found similarly through the momentum in the counter-clockwise direction around the elevation axis, again as shown in fig. 1.

$$J_e\ddot{e} = arm_c F_{g,c} - arm_h(F_{g,f} + F_{g,b}) + l_h(F_{f,p} + F_{b,p})$$

where arm_c is the moment arm between the counterweight point mass and the elevation axis, and arm_h is the moment arm between any of the two motor point masses and the elevation axis. $F_{f,p} = F_{f,perpendicular}$ is the perpendicular component of F_f, and $F_{b,p} = F_{b,perpendicular}$ is the perpendicular component of F_b. As shown in fig. 3, $arm_c = l_c cos(e)$, and $arm_h = l_h cos(e)$, and the moment arm for the motor forces is indeed l_h.

Figure 3: moment arms around the elevation axis, with relevant forces. Also, the positive direction of momentum is displayed as counter-clockwise around the e-axis.

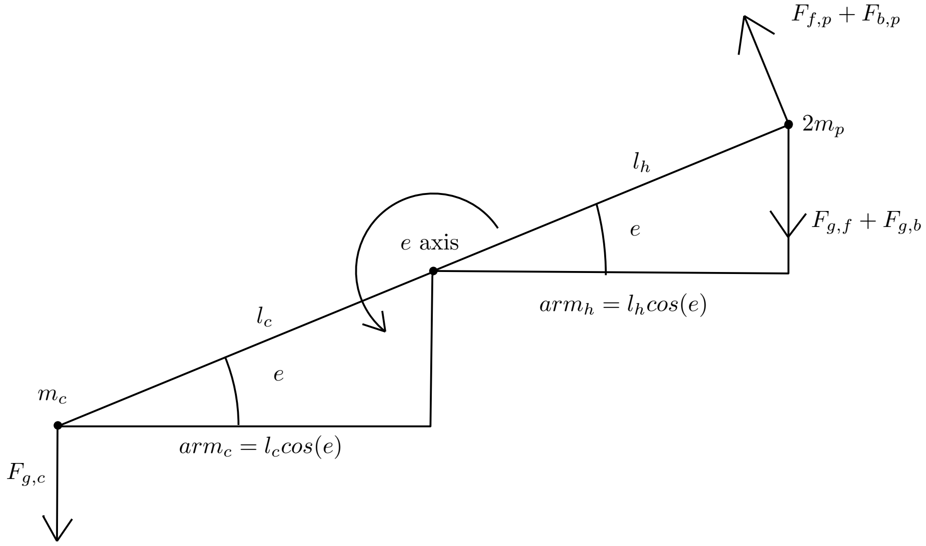
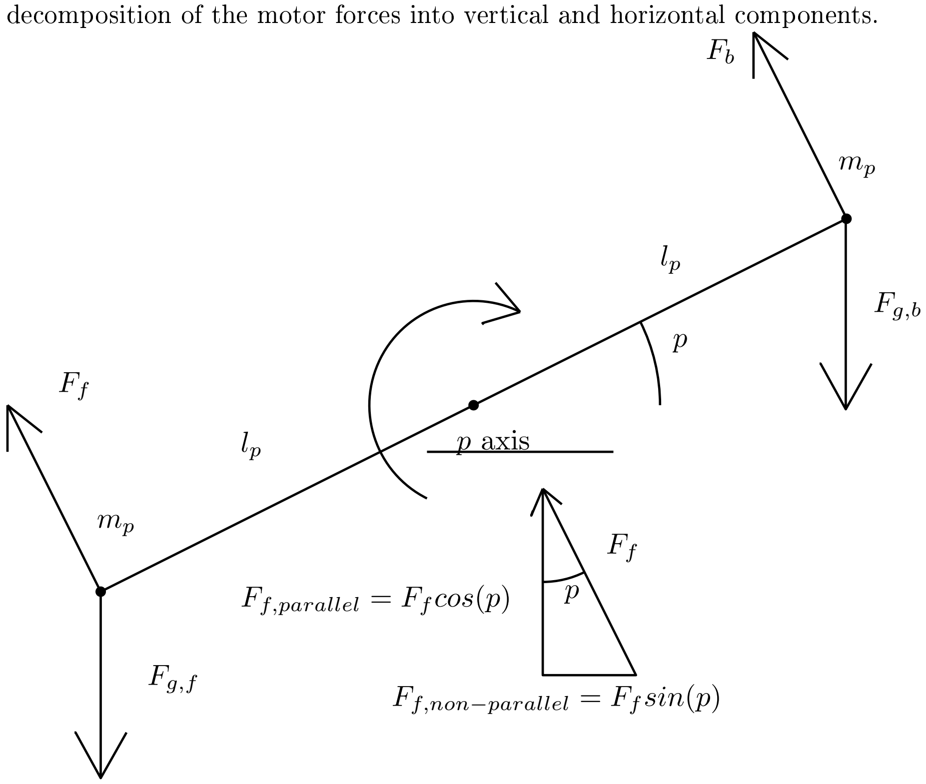


Figure 4: gravitational and motor forces on the helicopter head, as well as the decomposition of the motor forces into vertical and horizontal components.



As shown in fig. 4, the perpendicular components of the motor forces $F_{f,p}$ and $F_{b,p}$ are $F_f cos(p)$ and $F_b cos(p)$ respectively.

$$\begin{aligned} J_e\ddot{e} &= arm_c F_{g,c} - arm_h(F_{g,f} + F_{g,b}) + l_h cos(p)(F_f + F_b)) \\ &= arm_c m_c g - arm_h(m_p g + m_p g) + l_h K_f cos(p)(V_f + V_b)) \end{aligned}$$

We substitute in $V_s = V_f + V_b$ and the two moment arms:

$$J_e\ddot{e} = l_c cos(e) m_c g - l_h cos(e) 2 m_p g + l_h K_f cos(p) V_s)$$

The eqation of motion for the elevation angle then has the final form:

$$J_e\ddot{e} = g(l_c m_c - 2 l_h m_p) cos(e) + l_h K_f V_s cos(p) \tag{2}$$

We see that $L_2 = g(l_c m_c - 2 l_h m_p)$, and $L_3 = l_h K_f$.

Finally, the equation of motion for the travel angle is found through the momentum around the travel axis in the clockwise direction, as shown in fig. 1. As seen in fig. 4, the only forces with a moment arm perpendicular to the travel axis are the components of the motor forces in the horizontal direction, $F_{f,h} = F_f sin(p)$ and $F_{b,h} = F_b sin(p)$, has fig. 3 have the moment arm of length $arm_h = l_h cos(e)$:

$$\begin{aligned} J_\lambda\ddot{\lambda} &= arm_h(F_{f,h} + F_{b,h}) \\ &= l_h cos(e)(K_f sin(p)(V_f + V_b)) \end{aligned}$$

Again substituting $V_s = V_f + V_b$, and we get the final equation of motion for the travel angle:

$$J_\lambda\ddot{\lambda} = l_h K_f V_s cos(e) sin(p) \tag{3}$$

We see that $L_4 = l_h K_f$.

Equations (1) to (3) respectively correspond to equations (2a) to (2c) from the assignment [2, p.13].

1.2 Problem 2

To linearize the system about the point with all state variables equal to zero $((p, e, \lambda)^T = (\dot{p}, \dot{e}, \dot{\lambda})^T = (0,\ 0,\ 0))$ the inputs in the equations of motion (V_s^* and V_d^*) must be set to values that make this an equilibrium point.

At the linearization point, the equation of motion for pitch, eq. (1), reduces to $0 = L_1 * V_d^*$ therefore:

$$V_d^* = 0 \tag{4}$$

At the linearization point, the equation of motion for elevation, eq. (2), reduces to $0 = L_2 + L_3 * V_s^*$ therefore:

$$V_s^* = -L_2/L_3 \tag{5}$$

While the equation of motion for travel, eq. (3), at the linearization point simply reduces to $0 = 0$. The following transformation is performed to simplify the analysis [2, p.14]:

$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} - \begin{bmatrix} p^* \\ e^* \\ \lambda^* \end{bmatrix} \; and \; \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} = \begin{bmatrix} V_s \\ V_d \end{bmatrix} - \begin{bmatrix} V_s^* \\ V_d^* \end{bmatrix} \tag{6}$$

The equations of motion in the transformed system are therefore:

$$J_p\ddot{\tilde{p}} = L_1\tilde{V}_d \tag{7a}$$
$$J_e\ddot{\tilde{e}} = L_2cos(\tilde{e}) + L_3(\tilde{V}_s + L_2/L_3)cos(\tilde{p}) \tag{7b}$$
$$J_\lambda\ddot{\tilde{\lambda}} = L_4(\tilde{V}_s + L_2/L_3)cos(\tilde{e})sin(\tilde{p}) \tag{7c}$$

By choosing the state to be x $= (\tilde{p},\ \tilde{e},\ \tilde{\lambda},\ \dot{p},\ \dot{e},\dot{\lambda})$ the nonlinear state equations become:

$$\dot{x}_1 = x_4 \tag{8a}$$
$$\dot{x}_2 = x_5 \tag{8b}$$
$$\dot{x}_3 = x_6 \tag{8c}$$
$$\dot{x}_4 = (L_1/J_p)V_d \tag{8d}$$
$$\dot{x}_5 = (L_2/J_e)cos(x_2) + (L_3/J_e)(V_s + L_2/L_3)cos(x_1) \tag{8e}$$
$$\dot{x}_6 = (L_4/J_\lambda)(V_s + L_2/L_3)cos(x_2)sin(x_1) \tag{8f}$$

If the above system is expressed as $\dot{x} = h(x, u)$, where x is the state and u is the input, the system is linearized by finding the Jacobians of h with respect to the state and the input and then inserting the equilibrium values.

$$\frac{\partial h}{\partial x} = A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{L_4L_2}{J_\lambda L_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial h}{\partial u} = B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_1/J_p \\ L_3/J_e & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

where L_1, L_2, L_3 and L_4 were calculated in section 1.1 and J_p, J_e and J_λ were given in the assignment description [2, p.14]. The linearized equations of motion can therefore be written in the following form:

$$\ddot{\tilde{p}} = K_1\tilde{V}_d \qquad K_1 = \frac{L_1}{J_p} \tag{10a}$$
$$\ddot{\tilde{e}} = K_2\tilde{V}_s \qquad K_2 = \frac{L_3}{J_e} \tag{10b}$$
$$\ddot{\tilde{\lambda}} = K_3\tilde{p} \qquad K_3 = \frac{L_4L_2}{J_\lambda L_3} \tag{10c}$$

1.3 Problem 3

Check if we added gain to the output of joystick

The helicopter is difficult to control using only feed-forward. The physical behaviour of the helicopter differs from the (2a) - (2c) [2, p.13] model because it does not take into consideration drag, ground effects, etc.

In eq. (10) the model is linearized around a point with all angles equal to zero, where elevation is defined as zero when the elevation arm is parallel to the table, pitch is defined as zero when the pitch arm is parallel to the table and travel angle is defined as zero at the initial travel angle. Since the physical system is not linear, the linear assumption breaks down when the system is too far from this point.

1.4 Problem 4

Elevation is the only variable that needs to be changed, since travel and pitch are zeroed correctly upon initialization. To correct for the offset of the arm at startup, subtract 30 degrees from the elevation to get the correct point of zero elevation.

The value of V_s^* in order to stabilize the helicopter at the equilibrium point, was measured to be 7.5 V.

The motor force constant, K_f, which relates F_f and V_f is calculated to be $-(L_2/l_h) * V_s^*$, and has the value 0.1332.

2 Part 2 – Monovariable control

2.1 Problem 1

The following controller form is given in the assignment[2, p.15]:

$$\tilde{V_d} = K_{pp}(\tilde{p_c} - \tilde{p}) - K_{pd}\dot{\tilde{p}} \tag{11}$$

Substituting in the equation for pitch angle (eq. (1)) gives the following.

$$\ddot{\tilde{p}} = K_1 K_{pp}(\tilde{p_c} - \tilde{p}) - K_1 K_{pd}\dot{\tilde{p}} \tag{12}$$

To find the transfer function, $\frac{\tilde{p}(s)}{\tilde{p_c}(s)}$, the Laplace transform is taken.

$$\begin{gathered}
\ddot{\tilde{p}} + K_1 K_{pd}\dot{\tilde{p}} + K_1 K_{pp}\tilde{p} = K_1 K_{pp}\tilde{p_c} \\
\mathcal{L} \rightarrow \\
s^2\tilde{p}(s) + sK_1 K_{pd}\tilde{p}(s) + K_1 K_{pp}\tilde{p}(s) = K_1 K_{pp}\tilde{p_c}(s)
\end{gathered}$$

Which gives the transfer function

$$\frac{\tilde{p}(s)}{\tilde{p_c}(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd} s + K_1 K_{pp}} \tag{13}$$

The linearized pitch dynamics can be regarded as a second-order linear system, which means that by putting eq. (13) in the form shown in eq. (14), K_{pp} and K_{pd} can be determine from ω and ζ.

$$h(s) = \frac{\omega^2}{s^2 + 2\zeta\omega^2 s + \omega^2} \tag{14}$$

This gives the following relations:

$$\omega = \sqrt{K_1 K_{pp}} \tag{15}$$

$$2\zeta\omega^2 = K_1 K_{pd}$$

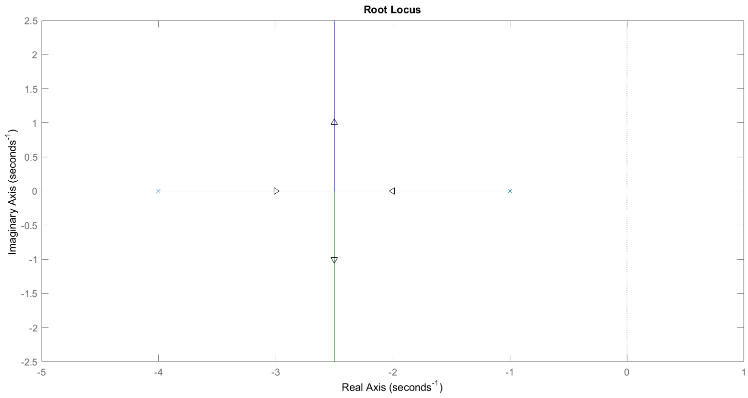
$$\zeta = \frac{K_1 K_{pd}}{2\omega^2} = \frac{K_{pd}}{2K_{pp}} \tag{16}$$

For a critically damped system $\zeta = 1$, which gives the following relationship

$$K_{pd} = 2K_{pp} \tag{17}$$

Beginning with a $K_{pp} = 3$, and then from the relation in eq. (17) a $K_{pd} = 6$, the response of the pitch angle to the input was slower than desired. Therefore, K_{pp} was increased to $K_{pp} = 12.5$ and K_{pd} was lowered to underdamp the system, until it was sufficiently responsive at $K_{pd} = 0.7K_{pp} = 8.75$. At these values the system responded faster with only minor oscillations. It was observed that larger values of K_{pp} gave rise to larger oscillations.

Figure 5: Change in pole position by increasing K_{pp} given a constant K_{pd}



At the critically damped point, the poles lie on the same point on the x-axis. When K_{pp} is decreased in relation to K_{pd} the system is over-damped the poles move away from each other along the x-axis. When K_{pp} is increased in relation to K_{pd}, the system is under damped and the poles move away from each other vertically from the critically damped point.

With the PD controller, it was significantly easier to control the helicopter than with just feed forward joystick control as no scaling of the joystick input was needed.

2.2 Problem 2

By plugging the P controller for travel

$$\tilde{p} = K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}}) \tag{18}$$

into the equation of motion for travel eq. (10c), the transfer function between $\dot{\tilde{\lambda}}$ and $\dot{\tilde{\lambda}}_c$ can be derived. By substituting the right side of the controller into the equation of motion for travel the equation becomes:

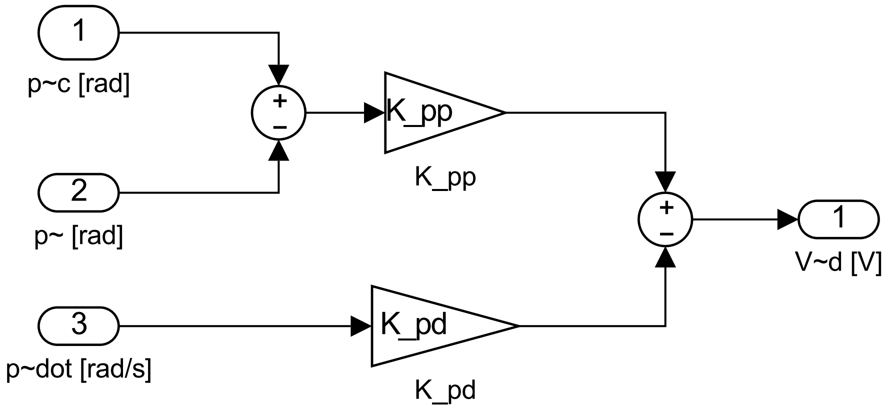
$$\ddot{\tilde{\lambda}} = K_3(K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}})) \tag{19}$$

After taking the laplace transform and rearranging terms, the transfer function is as follows:

$$\frac{\dot{\tilde{\lambda}}(s)}{\dot{\tilde{\lambda}}_c(s)} = \frac{K_3 K_{rp}}{s + K_3 K_{rp}} \tag{20}$$

This controller was quite quick and stable after correctly tuning K_{rp}. The value that was deemed best was K_{rp} = -40. Higher values, ie. more negative values, of K_{rp} led to large oscillations, while lower values, or less negative values, of K_{rp} led to a slower response to the joysticks input. It was necessary to add a gain to the x value of the joystick to limit the input range of the controller.

Figure 6: Simulink implementation of PD controller



3 Part 3 - Multivariable control

3.1 Problem 1

Here, we are to find the systems matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Trivially, they are:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \end{bmatrix} \tag{21}$$

3.2 Problem 2

First, we are to examine the controllability of the system. We do that by finding the controllability matrix $\boldsymbol{\mathcal{C}}$ of the system, and see whether it has full rank or not:

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K_1 \\ 0 & K_1 & 0 & 0 \\ K_2 & 0 & 0 & 0 \end{bmatrix} \tag{22}$$

which has full rank: $rank(\boldsymbol{\mathcal{C}}) = 3$, and is thus controllable.

Second, we are to implement a LQR controller with reference feed-forward, $u = \boldsymbol{Pr} - \boldsymbol{Kx}$. K is derived from the Matlab lqr function, which requires the A and B matrices, in addition to weighting matrices Q and R. When finding appropriate Q and R matrices, Byrson's rule guided the first iteration. The rule states:

$$Q_{ii} = \frac{1}{\text{maximum acceptable value of } x_i^2} \tag{23}$$

$$R_{jj} = \frac{1}{\text{maximum acceptable value of } u_j^2} \tag{24}$$

where x_i represents the i^{th} state and u_j represents the j^{th} input. All other entries to Q and R are 0. This resulted in:

$$Q = \begin{bmatrix} 1/(\pi/8)^2 & 0 & 0 \\ 0 & 1/(\pi/2)^2 & 0 \\ 0 & 0 & 1/(\pi/8)^2 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{25}$$

We chose these initial values to have a small range of motion in pitch, a large max speed in pitch to better control the pitch and a relatively slow maximum speed in elevation. Furthermore, both inputs max value is 1.

After tuning, the following Q and R matrices seemed to perform with the quickest response without large overshoots:

$$Q = \begin{bmatrix} 60 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 100 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{26}$$

A higher value for $q_{1,1}$ yields a more oscillatory pitch behavior, while a lower value yields a slower response. As for $q_{2,2}$ a higher value slowed the pitch response. With $q_{3,3} > 100$ the system becomes difficult to control.

Explain why the system becomes difficult to control

$\boldsymbol{P}$ is defined such that as time goes to infinity, our states $\tilde{p}$ and $\dot{\tilde{e}}$ tend to their reference values $\tilde{p}_c$ and $\dot{\tilde{e}}_c$. This happens when $\dot{\boldsymbol{x}} = 0$, as the system reaches a stable equilibrium around the reference values:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}(\boldsymbol{P}\boldsymbol{r} - \boldsymbol{K}\boldsymbol{x}) \\ &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}\boldsymbol{P}\boldsymbol{r} = 0 \end{aligned}$$

When $\dot{\boldsymbol{x}} = 0$, our $\boldsymbol{x}$ has reached its final value and we define $\boldsymbol{x} = \boldsymbol{x}_\infty$:

$$\begin{aligned} (\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})\boldsymbol{x}_\infty &= \boldsymbol{B}\boldsymbol{P}\boldsymbol{r} \\ \Leftrightarrow \boldsymbol{x}_\infty &= (\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{P}\boldsymbol{r} \\ \Rightarrow \boldsymbol{y}_\infty = \boldsymbol{C}\boldsymbol{x}_\infty &= \boldsymbol{C}(\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})^{-1}\boldsymbol{B}\boldsymbol{P}\boldsymbol{r} \end{aligned}$$

We see that our output $\boldsymbol{y}_\infty$ is equal to our reference $\boldsymbol{r}$ when:

$$\boldsymbol{P} = [\boldsymbol{C}(\boldsymbol{B}\boldsymbol{K} - \boldsymbol{A})^{-1}\boldsymbol{B}]^{-1} \tag{27}$$

Therefore, we now have

$$\lim_{t\to\infty} \boldsymbol{y}(t) = \boldsymbol{y}_\infty = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{e}} \end{bmatrix} = \begin{bmatrix} \tilde{p}_c \\ \dot{\tilde{e}}_c \end{bmatrix} = \boldsymbol{r},$$

which is what we wanted.

3.3 Problem 3

With integral effect, the new state space matrices become:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{28}$$

The matrices used to calculate the controllers gains, Q and R, also needed to be updated:

$$Q = \begin{bmatrix} 60 & 0 & 0 & 0 & 0 \\ 0 & 0.01 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 30 \end{bmatrix} \qquad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{29}$$

Without integral effect, the system could track pitch without much of a problem. But the elevation rate had a noticeable deviation in the ranges of elevation not close to our linearization point $\dot{\tilde{e}} = 0$.

With integral effect, this deviation in elevation rate is asymptotically removed. This means that with a zero Y output from our joystick, we see a corresponding zero elevation rate, and the elevation angle stays asymptotically constant also for angles not close to its linearization point.

Figure 7: LQR Controller with no Integral Effect

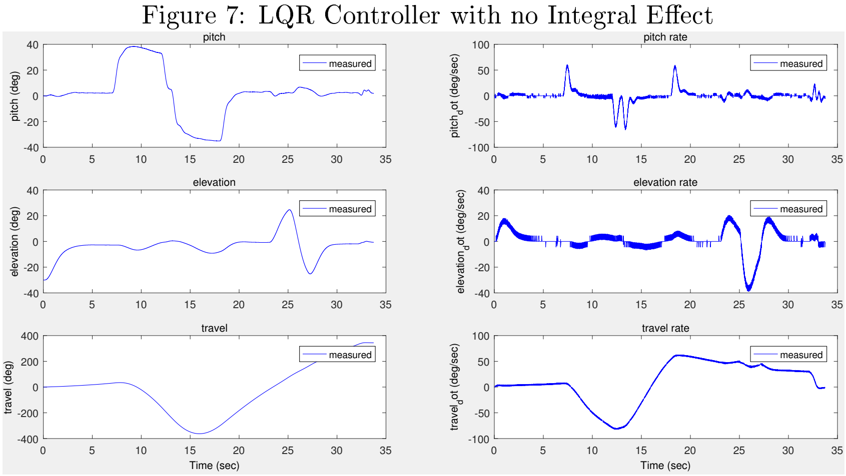


Figure 8: LQR Controller with Integral Effect

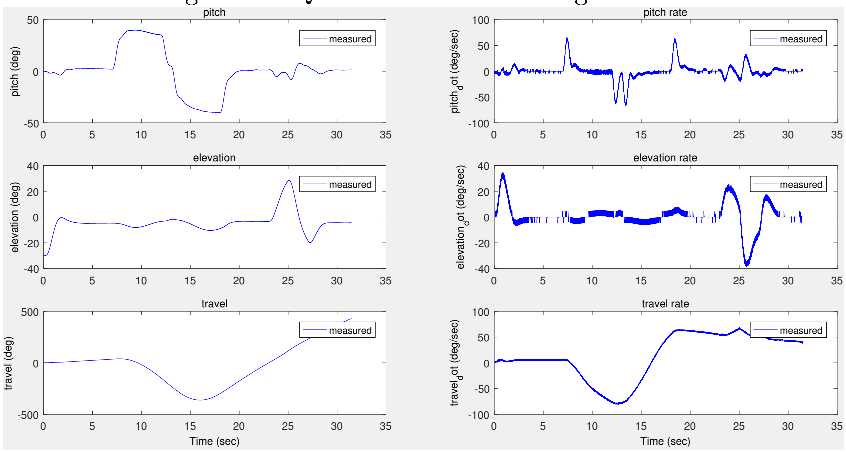


Figure 9: Input for no Integral Effect

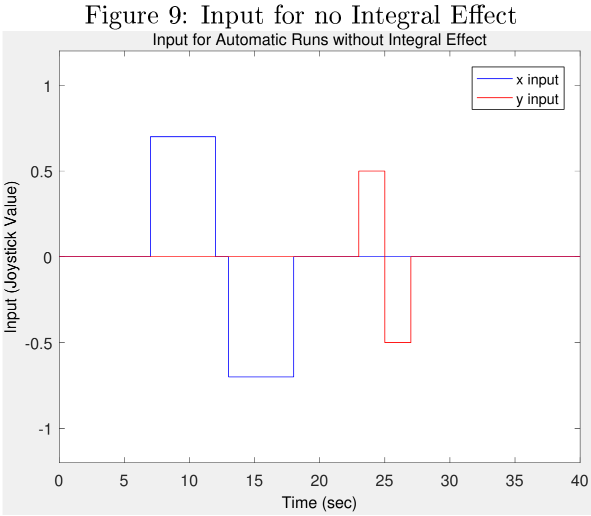
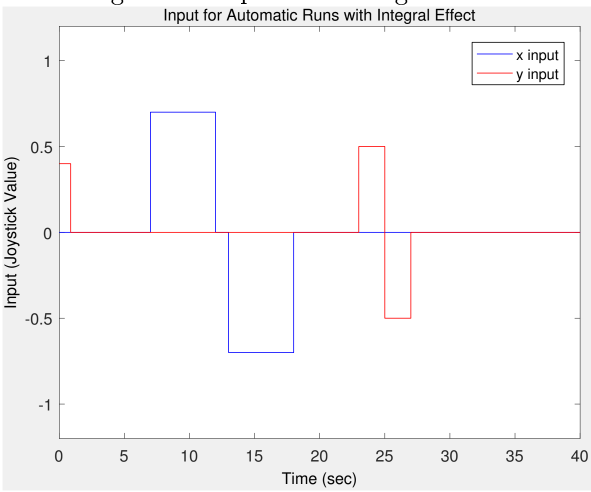


Figure 10: Input with Integral Effect



4 Part 4 – State estimation

This section consists of the development of an observer to estimate the nonmeasured angular velocities.

4.1 Problem 1

By describing the system in eq. (10) in the following state-space form

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned} \tag{30}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices. The state -, input - and output vector are given by

$$\boldsymbol{x} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{p}} \\ \tilde{e} \\ \dot{\tilde{e}} \\ \tilde{\lambda} \\ \dot{\tilde{\lambda}} \end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} \tag{31}$$

This gives the following $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ K_3 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ 0 & 0 \\ K_2 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{32}$$

Where K_1, K_2 and K_3 are given by eq. (10).

4.2 Problem 2

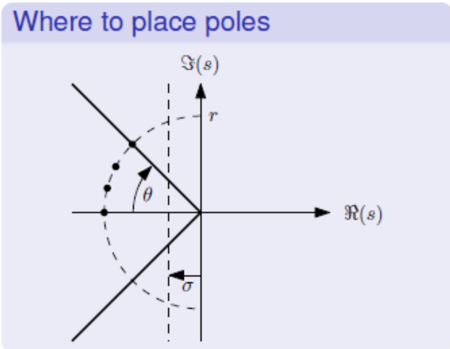
The observer matrix can be used. For a 6 state system, it is define by:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \boldsymbol{C} * \boldsymbol{A}^2 \\ \boldsymbol{C} * \boldsymbol{A}^3 \\ \boldsymbol{C} * \boldsymbol{A}^4 \\ \boldsymbol{C} * \boldsymbol{A}^5 \end{bmatrix} \tag{33}$$

This can be calculated using MATLAB's $obsv(\boldsymbol{A}, \boldsymbol{C})$ function. The resulting 18x6 matrix has rank 6, thereby full rank. It has full rank, and the system is therefore fully observable.

The observer gain matrix $\boldsymbol{L}$ is to be set in such a way that the poles of the observer is faster than the system, in order to drive the error to zero.

Figure 11: illustrating how to place poles during state, or estimated state feedback, on a semi-circle with the same radius, within the region shown.



Finn navnene på theta og sigma i forhold til regtek

The poles of the observer must be placed as in fig. 11. The real value of our poles must be larger than σ, which for the observer is the value of the largest real value of the controlled systems poles. This way, all of the linear observers poles are faster than the controlled system. θ is the biggest angle of our observer poles. If this is too large, we will get too much of an underdamped system and too much of an overshoot. However, if the radius is too large, undesired high frequency noise from the measurements becomes amplified to unwanted levels.

We chose $\theta = 15$, and r as 50 times the maximum length of the controlled systems poles.

The observer itself has the state space formulation:

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}})$$

$$
\begin{aligned}
\dot{\boldsymbol{e}} &= \dot{\tilde{\boldsymbol{x}}} - \dot{\hat{\boldsymbol{x}}} \\
&= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} - \boldsymbol{A}\hat{\boldsymbol{x}} - \boldsymbol{B}\boldsymbol{u} - \boldsymbol{L}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}) \\
&= \boldsymbol{A}(\boldsymbol{x} - \hat{\boldsymbol{x}}) - \boldsymbol{L}(\boldsymbol{C}\boldsymbol{x} - \boldsymbol{C}\hat{\boldsymbol{x}}) \\
&= \boldsymbol{A}\boldsymbol{e} - \boldsymbol{L}\boldsymbol{C}\boldsymbol{e}
\end{aligned}
$$

Meaning our error has the state space formulation:

$$\dot{\boldsymbol{e}} = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{e} \tag{34}$$

As previously stated, we need this error to converge to zero by placing the poles of this state space formulation such that they are faster than the poles of the system itself. The poles of this state space system can be placed arbitrarily because $\{\boldsymbol{A}, \boldsymbol{C}\}$ is observable:

$$det(\lambda \boldsymbol{I} - \boldsymbol{A} + \boldsymbol{L}\boldsymbol{C}) = 0$$

For this equation, λ are the values that solves the equation, and also the poles of the observer. By choosing values for λ, an $\boldsymbol{L}$ emerges in order to make the determinant equal to zero. The matlab function *place* places the poles as desired for us: $\boldsymbol{L} = place(\boldsymbol{A}^T, \boldsymbol{C}^T, \boldsymbol{\lambda})^T$, where $\boldsymbol{\lambda}$ is the vector of the observer poles in this instance.

Figure 12: LQR Controller Estimating p, e and λ

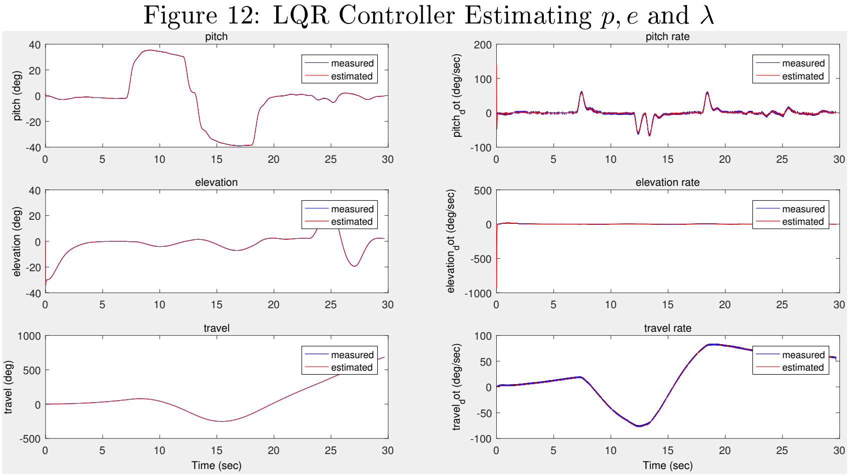
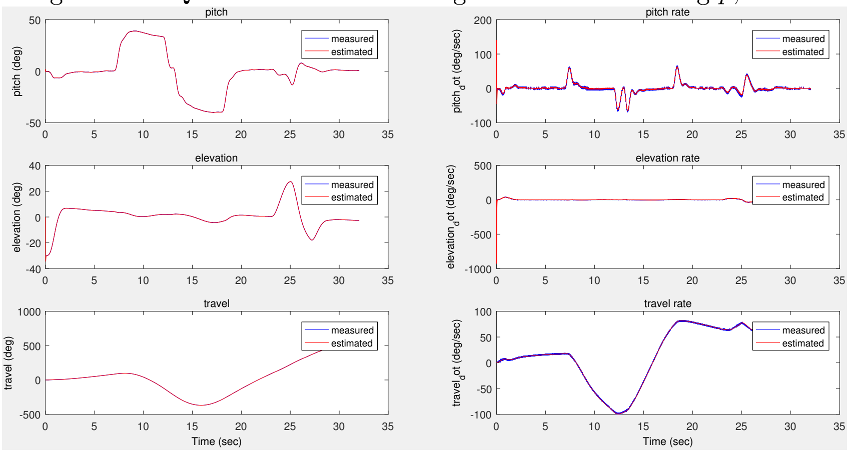


Figure 13: LQR Controller with Integral Effect Estimating p, e and λ



4.3 Problem 3

When only $\tilde{e}$ and $\tilde{\lambda}$ are measured, the output matrix $\boldsymbol{C}$ becomes: $\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. The observer matrix, found through $obsv(\boldsymbol{A}, \boldsymbol{C})$ is a 12x6 matrix with rank 6. Therefore, it is observable.

However, when only $\tilde{p}$ and $\tilde{e}$ are measured, the output matrix $\boldsymbol{C}$ becomes: $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$. The observer matrix, found through $obsv(\boldsymbol{A}, \boldsymbol{C})$ is a 12x6 matrix with rank 4. Therefore, it is not observable.

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