Linear Systems TTK4115 - Helicopter lab

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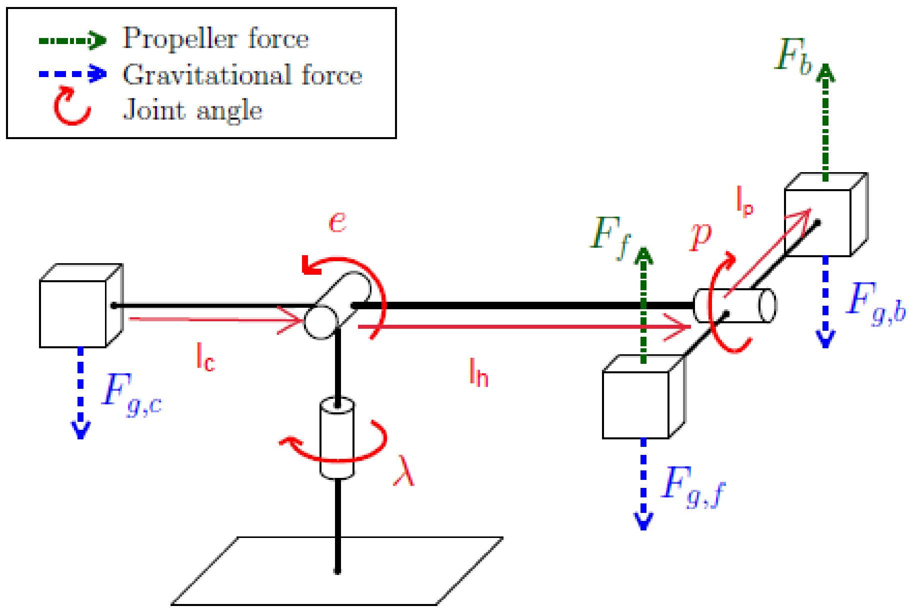
Contents

1 Part 1 - Mathematical modeling

1.1 Problem 1

The helicopter model in fig. 1 is used as a starting point for deriving the equations of motion.

Figure 1: the helicopter model figure 7 from the assignment [2, p.12] with relevant distances drawn in.



The equation of motion for the pitch is found through the momentum around the pitch axis in the clockwise direction as shown in fig. 1. It becomes:

$$\begin{aligned} J_p\ddot{p} &= l_p(F_{g,b} - F_b - F_{g,f} + F_f) \\ &= l_p(m_pg - mp_g + K_fV_f - V_b) \\ &= l_pK_f(V_f - V_b) \end{aligned}$$

Since $V_d = V_f - V_b$, this can be simplified to:

$$J_p\ddot{p} = l_pK_fVd \tag{1}$$

Therefore $L_1 = l_pK_f$.

The equation of motion for the elevation angle is found similarly through the momentum in the counter-clockwise direction around the elevation axis.

$$J_e\ddot{e} = arm_c F_{g,c} - arm_h(F_{g,f} + F_{g_b} - K_f cos(p)(V_f + V_b))$$

where arm_c is the moment arm between the counterweight point mass and the elevation axis, and arm_h is the moment arm between any of the two motor point masses and the elevation axis. As shown in fig. 2, $arm_c = l_c cos(e)$, and $arm_h = l_h cos(e)$. $V_s = V_f + V_b$ can be immediately substituted in and the equation simplified:

Husk at figuren må være som jeg (Daniel) har i notatene her, da er det 100 prosent tydelig hva arm_c og arm_h blir. Kan godt gjøre det enda litt mer detaljert.

$$\begin{aligned} J_e\ddot{e} &= l_c cos(e) m_c g - l_h cos(e)(2m_p g - K_f cos(p) V_s) \\ &= cos(e)(l_c m_c g - 2l_h m_p g) + l_h K_f V_s cos(e) cos(p) \end{aligned}$$

Here, we have (as the author of the exercise) counted the cos(e) factor of the V_s term as negligible, and set it to 1. The resulting equation has the form:

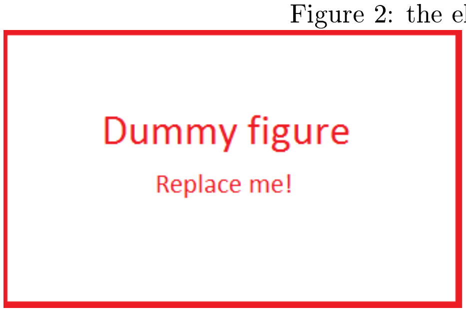
Should we cite this?

Forklar hvor cos(p) kommer fra - og lag ny figur. Bruk denne til å forklare (2c)/(3) også.

$$J_e\ddot{e} = g(l_c m_c - 2l_h m_p) cos(e) + l_h K_f V_s cos(p) \tag{2}$$

Note that $L_2 = g(l_c m_c - 2l_h m_p)$, and $L_3 = l_h K_f$.

Figure 2: the elevation model



Finally, the equation of motion for the travel angle is found through the momentum around the travel axis in the clockwise direction. As seen in fig. 2, the only forces with a moment arm perpendicular to the travel axis are the components of the motor forces in the horizontal direction, which have length $arm_h = l_h cos(e)$. Furthermore, the horizontal components

Figure 3: the pitch model

$$J_\lambda \ddot{\lambda} = arm_h$$

The equations eq. (1), 2, and (3) correspond to equations (2a), (2b), and (2c) respectively from the assignment [2, p.13]

1.2 Problem 2

To linearize the system about the point with all state variables equal to zero $((p, e, \lambda)^T = (\dot{p}, \dot{e}, \dot{\lambda})^T = (0,\ 0,\ 0))$ the inputs in the equations of motion (V_s^* and V_d^*) must be set to values that make this an equilibrium point.

At the linearization point, the equation of motion for pitch, eq. (1), reduces to $0 = L_1 * V_d^*$ therefore :

$$V_d^* = 0 \tag{3}$$

At the linearization point, the equation of motion for elevation, eq. (2), reduces to $0 = L_2 + L_3 * V_s^*$ therefore:

$$V_s^* = -L_2/L_3 \tag{4}$$

While the equation of motion for travel () at the linearization point simply reduces to $0 = 0$. The following transformation is performed to simplify the analysis [2, p.14]:

ADD EQUATION REFERENCE

$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} - \begin{bmatrix} p^* \\ e^* \\ \lambda^* \end{bmatrix} \ and \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} = \begin{bmatrix} V_s \\ V_d \end{bmatrix} - \begin{bmatrix} V_s^* \\ V_d^* \end{bmatrix} \tag{5}$$

The equations of motion in the transformed system are therefore:

$$J_p \ddot{\tilde{p}} = L_1 \tilde{V}_d \tag{6a}$$

$$J_e \ddot{\tilde{e}} = L_2 cos(\tilde{e}) + L_3(\tilde{V}_s + L_2/L_3) cos(\tilde{p}) \tag{6b}$$

$$J_\lambda \ddot{\tilde{\lambda}} = L_4(\tilde{V}_s + L_2/L_3) cos(\tilde{e}) sin(\tilde{p}) \tag{6c}$$

By choosing the state to be x = ($\tilde{p}$, $\tilde{e}$, $\tilde{\lambda}$, $\dot{p}$, $\dot{e}$,$\dot{\lambda}$) the nonlinear state equations become:

$$\dot{x}_1 = x_4 \tag{7a}$$
$$\dot{x}_2 = x_5 \tag{7b}$$
$$\dot{x}_3 = x_6 \tag{7c}$$
$$\dot{x}_4 = (L_1/J_p)V_d \tag{7d}$$
$$\dot{x}_5 = (L_2/J_e)cos(x_2) + (L_3/J_e)(V_s + L_2/L_3)cos(x_1) \tag{7e}$$
$$\dot{x}_6 = (L_4/J_\lambda)(V_s + L_2/L_3)cos(x_2)sin(x_1) \tag{7f}$$

If the above system is expressed as $\dot{x} = h(x, u)$, where x is the state and u is the input, the system is linearized by finding the Jacobians of h with respect to the state and the input and then plugging in the equilibrium values.

$$\frac{\partial h}{\partial x} = A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{L_4 L_2}{J_\lambda L_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial h}{\partial u} = B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_1/J_p \\ L_3/J_e & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

where L_1, L_2, L_3 and L_4 were calculated in Section 1.1 and J_p, J_e and J_λ were given in the assignment description [2, p.14].

The linearized equations of motion can therefore be written in the following form:

$$\ddot{\tilde{p}} = K_1 \tilde{V}_d \qquad K_1 = \frac{L_1}{J_p} \tag{9a}$$
$$\ddot{\tilde{e}} = K_2 \tilde{V}_s \qquad K_2 = \frac{L_3}{J_e} \tag{9b}$$
$$\ddot{\tilde{\lambda}} = K_3 \tilde{p} \qquad K_3 = \frac{L_4 L_2}{J_\lambda L_3} \tag{9c}$$

1.3 Problem 3

The helicopter is difficult to control using only feed-forward. The physical behaviour of the helicopter differs from the (2a) - (2c) [2, p.13] model because it does not take into consideration air resistance drag, ground effects nor the changes in length of the moment arm around the elevation axis. The equation

(9) model is linearized around a point with all angles equal to zero, where elevation is defined as zero when the elevation arm is parallel to the table, pitch is defined as zero when the pitch arm is parallel to the table and travel angle is defined as zero at the initial travel angle. Since the physical system is not linear, the linear assumption breaks down when the system is too far from this point.

1.4 Problem 4

Elevation is the only variable that needs to be changed, since travel and pitch are zeroed correctly upon initialization. To correct for the offset of the arm at startup, subtract 30 degrees from the elevation to get the correct point of zero elevation.

The motor force constant, K_f, which relates F_f and V_f is calculated to be $-(L_2/l_h) * V_s^*$, and has the value 0.1332.

2 Part 2 – Monovariable control

2.1 Problem 1

The following controller form is given in the lab report:

$$\tilde{V_d} = K_{pp}(\tilde{p_c} - \tilde{p}) - K_{pd}\dot{\tilde{p}} \tag{10}$$

Substituting in the equation for pitch angle (**??**) gives the following.

$$\ddot{\tilde{p}} = K_1 K_{pp}(\tilde{p_c} - \tilde{p}) - K_1 K_{pd}\dot{\tilde{p}} \tag{11}$$

To find the transfer function, $\frac{\tilde{p}(s)}{\tilde{p_c}(s)}$, the Laplace transform is taken.

$$\begin{gathered}
\ddot{\tilde{p}} + K_1 K_{pd}\dot{\tilde{p}} + K_1 K_{pp}\tilde{p} = K_1 K_{pp}\tilde{p_c} \\
\mathcal{L} \rightarrow \\
s^2\tilde{p}(s) + sK_1 K_{pd}\tilde{p}(s) + K_1 K_{pp}\tilde{p}(s) = K_1 K_{pp}\tilde{p_c}(s)
\end{gathered}$$

Which gives the transfer function

$$\frac{\tilde{p}(s)}{\tilde{p_c}(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd} s + K_1 K_{pp}} \tag{12}$$

The linearized pitch dynamics can be regarded as a second-order linear system, which means that by putting eq. (12) in the form shown in eq. (13), K_{pp} and K_{pd} can be determine from ω and ζ.

$$h(s) = \frac{\omega^2}{s^2 + 2\zeta\omega^2 s + \omega^2} \tag{13}$$

This gives the following relations:

$$\omega = \sqrt{K_1 K_{pp}} \tag{14}$$

$$2\zeta\omega^2 = K_1 K_{pd}$$

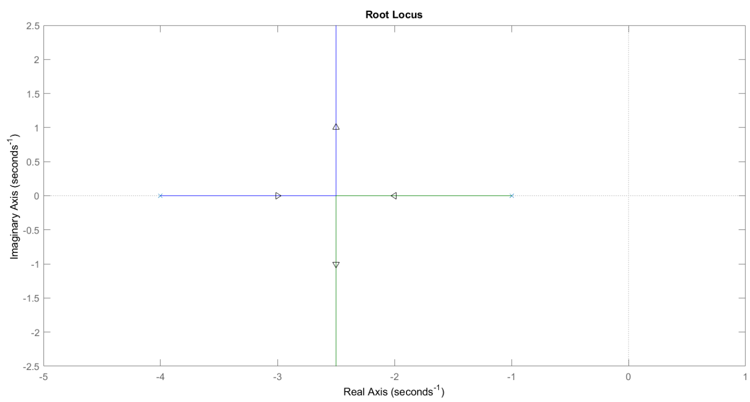
$$\zeta = \frac{K_1 K_{pd}}{2\omega^2} = \frac{K_{pd}}{2K_{pp}} \tag{15}$$

For a critically damped system $\zeta = 1$, which gives the following relationship

$$K_{pd} = 2K_{pp} \tag{16}$$

Beginning with a $K_{pp} = 3$, and then from the relation in eq. (16) a $K_{pd} = 6$, the response of the pitch angle to the input was slower than desired. Therefore, K_{pp} was increased to $K_{pp} = 12.5$ and K_{pd} was lowered to underdamp the system, until it was sufficiently responsive at $K_{pd} = 0.7K_{pp} = 8.75$. At these values the system responded faster with only minor oscillations. It was observed that larger values of K_{pp} gave rise to larger oscillations.

Figure 4: Change in pole position by increasing K_{pp} given a constant K_{pd}



At the critically damped point, the poles lie on the same point on the x-axis. When K_{pp} is decreased in relation to K_{pd} the system is over-damped the poles move away from each other along the x-axis. When K_{pp} is increased in relation to K_{pd}, the system is under damped and the poles move away from each other vertically from the critically damped point.

With the PD controller, it was significantly easier to control the helicopter than with just feed forward joystick control.

2.2 Problem 2

By plugging the P controller for travel

$$\tilde{p} = K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}}) \tag{17}$$

into the equation of motion for travel eq. (9c), the transfer function between $\dot{\tilde{\lambda}}$ and $\dot{\tilde{\lambda}}_c$ can be derived. By substituting the right side of the controller into the equation of motion for travel the equation becomes:

$$\ddot{\tilde{\lambda}} = K_3(K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}})) \tag{18}$$

After taking the laplace transform and rearranging terms, the transfer function is as follows:

$$\frac{\dot{\tilde{\lambda}}(s)}{\dot{\tilde{\lambda}}_c(s)} = \frac{K_3 K_{rp}}{s + K_3 K_{rp}} \qquad (19)$$

This controller was quite quick and stable after correctly tuning K_{rp}. The value that was deemed best was $K_{rp} =$. Higher values of K_{rp} led to , while lower values of K_{rp} led to . It was not nessecary to add a gain to the x value of the joystick to produce quick and accurate control.

fill in this value

Describe performance

describe performance

3 Part 3 - Multivariable control

3.1 Problem 1

Here, we are to find the systems matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Trivially, they are:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \end{bmatrix} \tag{20}$$

3.2 Problem 2

First, we are to examine the controllability of the system. We do that by finding the controllability matrix $\boldsymbol{\mathcal{C}}$ of the system, and see whether it has full rank or not:

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K_1 \\ 0 & K_1 & 0 & 0 \\ K_2 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

which has full rank: $rank(\boldsymbol{\mathcal{C}}) = 3$, and is thus controllable.

Second, we are to implement a LQR controller with reference feed-forward. Our $\boldsymbol{P}$ is defined such that as time goes to infinity, our states $\tilde{p}$ and $\tilde{\dot{e}}$ tend to their reference values $\tilde{p}_c$ and $\tilde{\dot{e}}_c$. This happens when $\dot{\boldsymbol{x}} = 0$, as the system reaches a stable equilibrium around the reference values:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{Ax} + \boldsymbol{Bu} \\ &= \boldsymbol{Ax} + \boldsymbol{B}(\boldsymbol{Pr} - \boldsymbol{Kx}) \\ &= (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} + \boldsymbol{BPr} = 0 \end{aligned}$$

When $\dot{\boldsymbol{x}} = 0$, our $\boldsymbol{x}$ has reached its final value and we define $\boldsymbol{x} = \boldsymbol{x}_\infty$:

$$\begin{aligned} (\boldsymbol{BK} - \boldsymbol{A})\boldsymbol{x}_\infty &= \boldsymbol{BPr} \\ \Leftrightarrow \boldsymbol{x}_\infty &= (\boldsymbol{BK} - \boldsymbol{A})^{-1}\boldsymbol{BPr} \\ \Rightarrow \boldsymbol{y}_\infty = \boldsymbol{Cx}_\infty &= \boldsymbol{C}(\boldsymbol{BK} - \boldsymbol{A})^{-1}\boldsymbol{BPr} \end{aligned}$$

We see that our output $\boldsymbol{y}_\infty$ is equal to our reference $\boldsymbol{r}$ when:

$$\boldsymbol{P} = [\boldsymbol{C}(\boldsymbol{BK} - \boldsymbol{A})^{-1}\boldsymbol{B}]^{-1} \tag{22}$$

Therefore, we now have

$$\lim_{t \to \infty} \boldsymbol{y}(t) = \boldsymbol{y_\infty} = \begin{bmatrix} \tilde{p} \\ \dot{\tilde{e}} \end{bmatrix} = \begin{bmatrix} \tilde{p}_c \\ \dot{\tilde{e}}_c \end{bmatrix} = \boldsymbol{r},$$

which is what we wanted.

3.3 Problem 3

By describing the system in eq. (9) in state space form, with a state $x = (\tilde{p}, \dot{\tilde{p}}, \tilde{e}, \dot{\tilde{e}}, \tilde{\lambda}, \dot{\tilde{\lambda}})$, an input $u = (\tilde{V}_s, \tilde{V}_d)$ and an output $y = (\tilde{p}, \tilde{e}, \tilde{\lambda})$, the following matrices are found:

4 Part 4 – State estimation

This section consists of the development of an observer to estimate the nonmeasured angular velocities.

4.1 Problem 1

By describing the system in eq. (9) in the following state-space form

$$\begin{aligned}\dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}\\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x}\end{aligned} \tag{23}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ are matrices. The state -, input - and output vector are given by

$$\boldsymbol{x} = \begin{bmatrix}\tilde{p}\\ \dot{\tilde{p}}\\ \tilde{e}\\ \dot{\tilde{e}}\\ \tilde{\lambda}\\ \dot{\tilde{\lambda}}\end{bmatrix}, \quad \boldsymbol{u} = \begin{bmatrix}\tilde{V}_s\\ \tilde{V}_d\end{bmatrix} \quad \text{and} \quad \boldsymbol{y} = \begin{bmatrix}\tilde{p}\\ \tilde{e}\\ \tilde{\lambda}\end{bmatrix} \tag{24}$$

This gives the following $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ matrices

$$\boldsymbol{A} = \begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\\ K_3 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix}0 & 0\\ 0 & K_1\\ 0 & 0\\ K_2 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{bmatrix}1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\end{bmatrix} \tag{25}$$

Where K_1, K_2 and K_3 are given by eq. (9).

4.2 Problem 2

4.3 Problem 3

References

[1] Chi-Tsong Chen, *Linear System Theory and Design*, Oxford University Press, 4th edition, 2014

[2] Kristoffer Gryte, *Helicopter lab assignment*, Department of Engineering Cybernetics, NTNU, Version 4.5, 2015