Linear Systems TTK4115 - Helicopter lab

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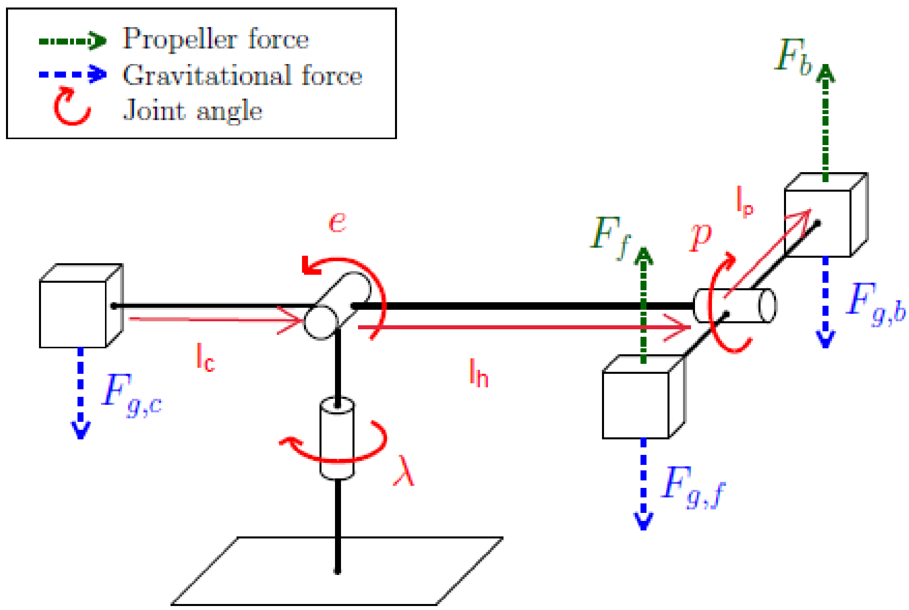
Contents

1 Part 1 - Mathematical modeling

1.1 Problem 1

We use the helicopter model fig. 1 as our starting point for deriving the equations of motion.

Figure 1: the helicopter model figure 7 from the assignment [2, p.12] with relevant distances drawn in.



The equation of motion for the pitch angle is found through the momentum around the pitch axis in the clockwise direction as shown in fig. 1. It becomes:

$$\begin{aligned} J_p\ddot{p} &= l_p(F_{g,b} - F_b - F_{g,f} + F_f) \\ &= l_p(m_p g - mp_g + K_f V_f - V_b) \\ &= l_p K_f (V_f - V_b) \end{aligned}$$

Since $V_d = V_f - V_b$, we can write this as:

$$J_p\ddot{p} = l_p K_f V d \tag{1}$$

Here, we can see that $L_1 = l_p K_f$.

The equation of motion for the elevation angle is found similarly through the momentum in the counter-clockwise direction around the elevation axis.

$$J_e\ddot{e} = arm_c F_{g,c} - arm_h(F_{g,f} + F_{g_b} - K_f cos(p)(V_f + V_b))$$

where arm_c is the moment arm between the counterweight point mass and the elevation axis, and arm_h is the moment arm between any of the two motor point masses and the elevation axis. As shown in fig. 2, $arm_c = l_c cos(e)$, and $arm_h = l_h cos(e)$. We can immediately substitute $V_s = V_f + V_b$, and simplify:

Husk at figuren må være som jeg (Daniel) har i notatene her, da er det 100 prosent tydelig hva arm_c og arm_h blir. Kan godt gjøre det enda litt mer detaljert.

$$
\begin{aligned}
J_e \ddot{e} &= l_c cos(e) m_c g - l_h cos(e)(2m_p g - K_f cos(p) V_s) \\
&= cos(e)(l_c m_c g - 2 l_h m_p g) + l_h K_f V_s cos(e) cos(p)
\end{aligned}
$$

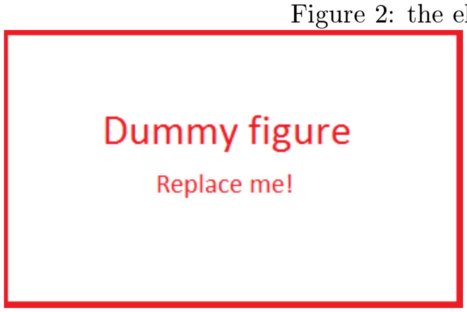
Here, we have (as the author of the exercise) counted the cos(e) factor of the V_s term as negligible, and set it to 1. The resulting equation has the form:

Forklar hvor cos(p) kommer fra - og lag ny figur. Bruk denne til å forklare (2c)/(3) også.

$$J_e \ddot{e} = g(l_c m_c - 2 l_h m_p) cos(e) + l_h K_f V_s cos(p) \tag{2}$$

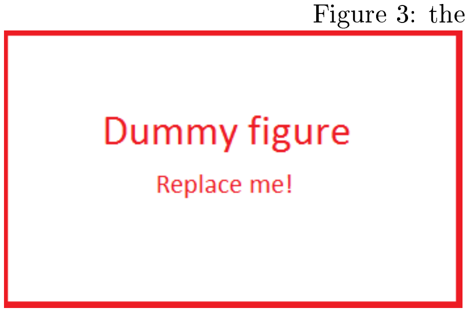
Note that $L_2 = g(l_c m_c - 2 l_h m_p)$, and $L_3 = l_h K_f$.

Figure 2: the elevation model



Finally, the equation of motion for the travel angle is found through the momentum around the travel axis in the clockwise direction. As seen in fig. 2, the only forces with a moment arm perpendicular to the travel axis are the components of the motor forces in the horizontal direction, which have length $arm_h = l_h cos(e)$. Furthermore, the horizontal components

Figure 3: the pitch model



$$J_\lambda \ddot{\lambda} = arm_h$$

The equations (1), (2), and (3) corresponds to (2a), (2b), and (2c) respectively of the assignment [2, p.13]

1.2 Problem 2

1.3 Problem 3

1.4 Problem 4

2 Part 2 – Monovariable control

2.1 Problem 1

We are given the controller shown in eq. (3).

$$\tilde{V_d} = K_{pp}(\tilde{p_c} - \tilde{p}) - K_{pd}\dot{\tilde{p}} \tag{3}$$

We take this controller and substitute it in the equation for pitch angle (**??**).

$$\ddot{\tilde{p}} = K_1 K_{pp}(\tilde{p_c} - \tilde{p}) - K_1 K_{pd}\dot{\tilde{p}} \tag{4}$$

Now we Laplace transform eq. (4) to find the transfer function $\frac{\tilde{p}(s)}{\tilde{p_c}(s)}$.

$$\ddot{\tilde{p}} + K_1 K_{pd}\dot{\tilde{p}} + K_1 K_{pp}\tilde{p} = K_1 K_{pp}\tilde{p_c}$$
$$\mathcal{L} \rightarrow$$
$$s^2\tilde{p}(s) + sK_1 K_{pd}\tilde{p}(s) + K_1 K_{pp}\tilde{p}(s) = K_1 K_{pp}\tilde{p_c}(s)$$

Which gives us our transfer function

$$\frac{\tilde{p}(s)}{\tilde{p_c}(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd} s + K_1 K_{pp}} \tag{5}$$

The linearized pitch dynamics can be regarded as a second-order linear system, which means that if we place eq. (5) on the form shown in eq. (6) we can determine K_{pp} and K_{pd} from ω and ζ.

$$h(s) = \frac{\omega^2}{s^2 + 2\zeta\omega^2 s + \omega^2} \tag{6}$$

Discuss how the tuning of the controller gains Kpp and Kpd influences the closed-loop eigenvalues and the pitch response.

2.2 Problem 2

3 Part 3 - Multivariable control

3.1 Problem 1

The matrices A and B are:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \end{bmatrix} \tag{7}$$

3.2 Problem 2

The systems controllability is examined through its controllability matrix $\boldsymbol{\mathcal{C}}$:

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K_1 \\ 0 & K_1 & 0 & 0 \\ K_2 & 0 & 0 & 0 \end{bmatrix} \tag{8}$$

which has full rank: $rank(\boldsymbol{\mathcal{C}}) = 3$, and is thus controllable.

Here, we are using LQR with reference feed-forward in order to control our system. Our $\boldsymbol{P}$ is defined such that as time goes to infinity, our states tend to the reference values. This happens when $\dot{\boldsymbol{x}} = 0$, as the system reaches a stable equilibrium around the reference values:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{Ax} - \boldsymbol{Bu} \\ &= \boldsymbol{Ax} - \boldsymbol{B}(\boldsymbol{Pr} - \boldsymbol{Kx}) = 0 \end{aligned}$$

3.3 Problem 3

3.4 Problem 4

References

[1] Chi-Tsong Chen, *Linear System Theory and Design*, Oxford University Press, 4th edition, 2014

[2] Kristoffer Gryte, *Helicopter lab assignment*, Department of Engineering Cybernetics, NTNU, Version 4.5, 2015