Linear Systems TTK4115 - Helicopter lab

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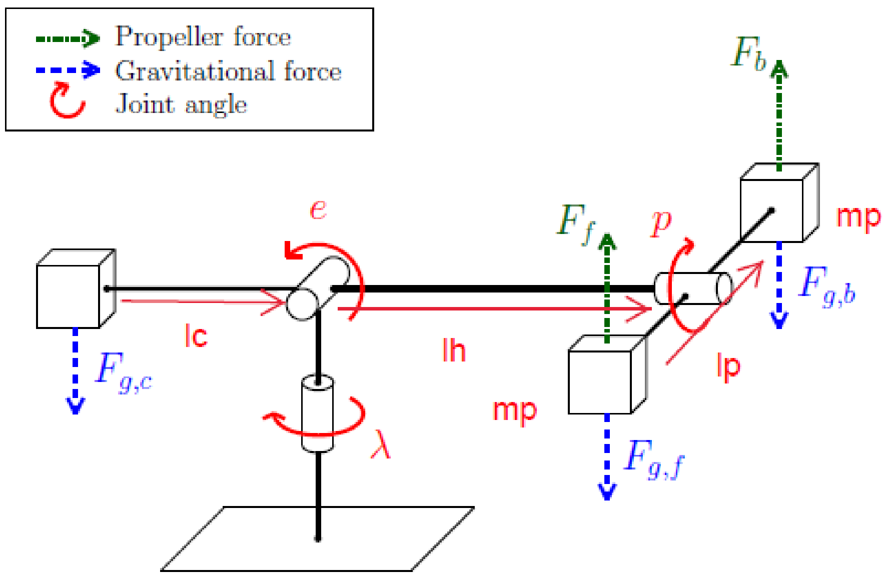
Contents

1 Part 1 - Mathematical modeling

1.1 Problem 1

We use the helicopter model **??** as our starting point for deriving the equations of motion.

Figure 1: the helicopter model figure 7 from the assignment [**?**, p.12] with relevant distances drawn in.



The equations of motion for the pitch is the momentum around the point p in the clockwise direction as shown in **??**:

$$\begin{aligned} J_p\ddot{p} &= l_p(F_{g,b} - F_b - F_{g,f} + F_f) \\ &= l_p(m_p g - mp_g + K_f V_f - V_b) \\ &= l_p K_f (V_f - V_b) \end{aligned}$$

Since $V_d = V_f - V_b$, we can write this as:

$$J_p\ddot{p} = l_p K_f V d \tag{1}$$

Here, we can see that $L_1 = l_p K_f$.

Figure 2: the elevation model

1.2 Problem 2

To linearize the system about the point with all state variables equal to zero $(p, e, \lambda)^{\mathrm{T}} = (\dot{p},\ \dot{e}, \dot{\lambda})^{\mathrm{T}} = (0,\ 0,\ 0))$ the inputs in the equations of motion ($\mathrm{V}_{\mathrm{s}}^{*}$ and $\mathrm{V}_{\mathrm{d}}^{*}$) must be set to values that make this an equilibrium point. At the linearization point, the equation of motion for pitch () reduces to $0 = L_1 * V_{\mathrm{d}}^{*}$ therefore:

ADD EQUATION REFERENCE

$$V_{\mathrm{d}}^{*} = 0 \tag{2}$$

At the linearization point, the equation of motion for elevation () reduces to $0 = L_2 + L_3 * V_{\mathrm{s}}^{*}$ therefore:

ADD EQUATION REFERENCE

$$V_{\mathrm{s}}^{*} = -L_2/L_3 \tag{3}$$

While the equation of motion for travel () at the linearization point simply reduces to $0 = 0$.
The following transformation is performed to simplify the analysis :

ADD EQUATION REFERENCE

Add cite of assignment here

$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} - \begin{bmatrix} p^{*} \\ e^{*} \\ \lambda^{*} \end{bmatrix} \mathit{and} \begin{bmatrix} \tilde{V}_{\mathrm{s}} \\ \tilde{V}_{\mathrm{d}} \end{bmatrix} = \begin{bmatrix} V_{\mathrm{s}} \\ V_{\mathrm{d}} \end{bmatrix} - \begin{bmatrix} V_{\mathrm{s}}^{*} \\ V_{\mathrm{d}}^{*} \end{bmatrix} \tag{4}$$

The equations of motion in the transformed system are therefore:

$$J_p\ddot{\tilde{p}} = L_1\tilde{V}_d \tag{5a}$$

$$J_e\ddot{\tilde{e}} = L_2 cos(\tilde{e}) + L_3(\tilde{V}_s + L_2/L_3)cos(\tilde{p}) \tag{5b}$$

$$J_\lambda\ddot{\tilde{\lambda}} = L_4(\tilde{V}_s + L_2/L_3)cos(\tilde{e})sin(\tilde{p}) \tag{5c}$$

By choosing the state to be x = ($\tilde{p}$, $\tilde{e}$, $\tilde{\lambda}$, $\dot{p}$, $\dot{e}$,$\dot{\lambda}$) the nonlinear state equations become:

$$\dot{x}_1 = x_4 \tag{6a}$$
$$\dot{x}_2 = x_5 \tag{6b}$$
$$\dot{x}_3 = x_6 \tag{6c}$$
$$\dot{x}_4 = (L_1/J_p)V_d \tag{6d}$$
$$\dot{x}_5 = (L_2/J_e)cos(x_2) + (L_3/J_e)(V_s + L_2/L_3)cos(x_1) \tag{6e}$$
$$\dot{x}_6 = (L_4/J_\lambda)(V_s + L_2/L_3)cos(x_2)sin(x_1) \tag{6f}$$

If the above system is expressed as $\dot{x} = h(x, u)$, where x is the state and u is the input, the system is linearized by finding the Jacobians of h with respect to the state and the input and then plugging in the equilibrium values.

$$\frac{\partial h}{\partial x} = A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{L_4 L_2}{J_\lambda L_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial h}{\partial u} = B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_1/J_p \\ L_3/J_e & 0 \\ 0 & 0 \end{bmatrix} \tag{7}$$

where L_1, L_2, L_3 and L_4 were calculated in Section 1.1 and J_p, J_e and J_λ were given in the assignment description . The linearized equations of motion can therefore be written in the following form:

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$$\ddot{\tilde{p}} = K_1 \tilde{V}_d \qquad K_1 = \frac{L_1}{J_p} \tag{8a}$$

$$\ddot{\tilde{e}} = K_2 \tilde{V}_s \qquad K_2 = \frac{L_3}{J_e} \tag{8b}$$

$$\ddot{\tilde{\lambda}} = K_3 \tilde{p} \qquad K_2 = \frac{L_4 L_2}{J_\lambda L_3} \tag{8c}$$

1.3 Problem 3

1.4 Problem 4

2 Part 2 – Monovariable control

2.1 Problem 1

We are given the controller shown in **??**.

$$\tilde{V_d} = K_{pp}(\tilde{p_c} - \tilde{p}) - K_{pd}\dot{\tilde{p}} \tag{9}$$

We take this controller and substitute it in the equation for pitch angle (**??**).

$$\ddot{\tilde{p}} = K_1 K_{pp}(\tilde{p_c} - \tilde{p}) - K_1 K_{pd}\dot{\tilde{p}} \tag{10}$$

Now we Laplace transform **??** to find the transfer function $\frac{\tilde{p}(s)}{\tilde{p_c}(s)}$.

$$\ddot{\tilde{p}} + K_1 K_{pd}\dot{\tilde{p}} + K_1 K_{pp}\tilde{p} = K_1 K_{pp}\tilde{p_c}$$
$$\mathcal{L} \rightarrow$$
$$s^2\tilde{p}(s) + sK_1 K_{pd}\tilde{p}(s) + K_1 K_{pp}\tilde{p}(s) = K_1 K_{pp}\tilde{p_c}(s)$$

Which gives us our transfer function

$$\frac{\tilde{p}(s)}{\tilde{p_c}(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd} s + K_1 K_{pp}} \tag{11}$$

The linearized pitch dynamics can be regarded as a second-order linear system, which means that if we place **??** on the form shown in **??** we can determine K_{pp} and K_{pd} from ω and ζ.

$$h(s) = \frac{\omega^2}{s^2 + 2\zeta\omega^2 s + \omega^2} \tag{12}$$

This gives us the following relations

$$\omega = \sqrt{K_1 K_{pp}} \tag{13}$$
$$2\zeta\omega^2 = K_1 K_{pd}$$
$$\zeta = \frac{K_1 K_{pd}}{2\omega^2} = \frac{K_{pd}}{2K_{pp}} \tag{14}$$

We know that for a critically damped system $\zeta = 1$, which gives us

$$K_{pd} = 2K_{pp} \tag{15}$$

We chose a $K_{pp} = 3$ and then from the relation in **??** we get $K_{pd} = 6$. With these values the response of the pitch angle to the input was slower than desired. Therefore, K_{pp} was increased to $K_{pp} = 12.5$ and K_{pd} was lowered to underdamp the system, until it was sufficiently responsive at $K_{pd} = 0.7K_{pp} = 8.75$. At these values the system responded faster with only minor oscillations. It was observed that larger values of K_{pp} gave rise to larger oscillations.

Figure for that shows poles on Im/Re axis?

At the critically damped point, the poles lie on the same point on the x-axis. When K_{pd} is increased in relation to K_{pp} the system is over-damped the poles move away from each other along the x-axis. When K_{pd} decreased in relation to K_{pp}, the system is under damped and the poles move away from each other vertically from the critically damped point.

With the PD controller, it was significantly easier to control the helicopter than with just feed forward joystick control.

2.2 Problem 2

References

[1] Chi-Tsong Chen, *Linear System Theory and Design*, Oxford University Press, 4th edition, 2014

[2] Kristoffer Gryte, *Helicopter lab assignment*, Department of Engineering Cybernetics, NTNU, Version 4.5, 2015