Linear Systems TTK4115 - Helicopter lab

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October 2016

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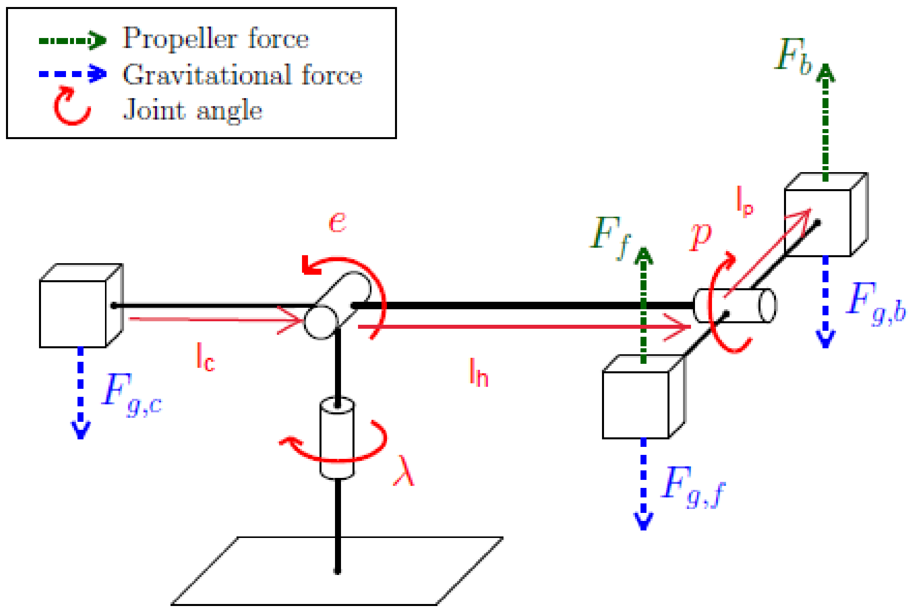
Contents

1 Part 1 - Mathematical modeling

1.1 Problem 1

We use the helicopter model fig. 1 as our starting point for deriving the equations of motion.

Figure 1: the helicopter model figure 7 from the assignment [2, p.12] with relevant distances drawn in.



The equation of motion for the pitch is found through the momentum around the pitch axis in the clockwise direction as shown in fig. 1. It becomes:

$$
\begin{aligned}
J_p\ddot{p} &= l_p(F_{g,b} - F_b - F_{g,f} + F_f) \\
&= l_p(m_p g - mp_g + K_f V_f - V_b) \\
&= l_p K_f (V_f - V_b)
\end{aligned}
$$

Since $V_d = V_f - V_b$, we can write this as:

$$J_p\ddot{p} = l_p K_f V d \tag{1}$$

Here, we can see that $L_1 = l_p K_f$.

The equation of motion for the elevation angle is found similarly through the momentum in the counter-clockwise direction around the elevation axis.

$$J_e\ddot{e} = arm_c F_{g,c} - arm_h(F_{g,f} + F_{g_b} - K_f cos(p)(V_f + V_b))$$

where arm_c is the moment arm between the counterweight point mass and the elevation axis, and arm_h is the moment arm between any of the two motor point masses and the elevation axis. As shown in fig. 2, $arm_c = l_c cos(e)$, and $arm_h = l_h cos(e)$. We can immediately substitute $V_s = V_f + V_b$, and simplify:

Husk at figuren må være som jeg (Daniel) har i notatene her, da er det 100 prosent tydelig hva arm_c og arm_h blir. Kan godt gjøre det enda litt mer detaljert.

$$\begin{aligned} J_e\ddot{e} &= l_c cos(e) m_c g - l_h cos(e)(2m_p g - K_f cos(p) V_s) \\ &= cos(e)(l_c m_c g - 2l_h m_p g) + l_h K_f V_s cos(e) cos(p) \end{aligned}$$

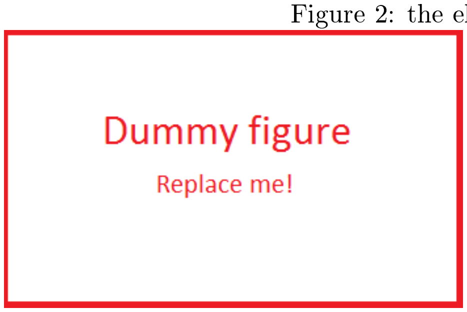
Here, we have (as the author of the exercise) counted the cos(e) factor of the V_s term as negligible, and set it to 1. The resulting equation has the form:

Forklar hvor cos(p) kommer fra - og lag ny figur. Bruk denne til å forklare (2c)/(3) også.

$$J_e\ddot{e} = g(l_c m_c - 2l_h m_p) cos(e) + l_h K_f V_s cos(p) \tag{2}$$

Note that $L_2 = g(l_c m_c - 2l_h m_p)$, and $L_3 = l_h K_f$.

Figure 2: the elevation model



Finally, the equation of motion for the travel angle is found through the momentum around the travel axis in the clockwise direction. As seen in fig. 2, the only forces with a moment arm perpendicular to the travel axis are the components of the motor forces in the horizontal direction, which have length $arm_h = l_h cos(e)$. Furthermore, the horizontal components

Figure 3: the pitch model

$$J_\lambda \ddot{\lambda} = arm_h$$

The equations (1), (2), and (3) corresponds to (2a), (2b), and (2c) respectively of the assignment [2, p.13]

1.2 Problem 2

To linearize the system about the point with all state variables equal to zero $(p, e, \lambda)^T = \dot{p}, (\dot{e}, \dot{\lambda} =^T = (0,\ 0,\ 0))$ the inputs in the equations of motion (V_s^*andV_d^*) must be set to values that make this an equilibrium point.

At the linearization point, the equation of motion for pitch () reduces to $0 = L_1 * V_d^*$ therefore :

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$$V_d^* = 0 \tag{3}$$

At the linearization point, the equation of motion for elevation () reduces to $0 = L_2 + L_3 * V_s^*$ therefore:

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$$V_s^* = -L_2/L_3 \tag{4}$$

While the equation of motion for travel () at the linearization point simply reduces to $0 = 0$.

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The following transformation is performed to simplify the analysis :

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$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} - \begin{bmatrix} p^* \\ e^* \\ \lambda^* \end{bmatrix} and \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} = \begin{bmatrix} V_s \\ V_d \end{bmatrix} - \begin{bmatrix} V_s^* \\ V_d^* \end{bmatrix} \tag{5}$$

The equations of motion in the transformed system are therefore:

$$J_p \ddot{\tilde{p}} = L_1 \tilde{V}_d \tag{6a}$$

$$J_e \ddot{\tilde{e}} = L_2 cos(\tilde{e}) + L_3(\tilde{V}_s + L_2/L_3) cos(\tilde{p}) \tag{6b}$$

$$J_\lambda \ddot{\tilde{\lambda}} = L_4(\tilde{V}_s + L_2/L_3) cos(\tilde{e}) sin(\tilde{p}) \tag{6c}$$

By choosing the state to be x = ($\tilde{p}$, $\tilde{e}$, $\tilde{\lambda}$, $\dot{p}$, $\dot{e}$,$\dot{\lambda}$) the nonlinear state equations become:

$$\dot{x}_1 = x_4 \tag{7a}$$
$$\dot{x}_2 = x_5 \tag{7b}$$
$$\dot{x}_3 = x_6 \tag{7c}$$
$$\dot{x}_4 = (L_1/J_p)V_d \tag{7d}$$
$$\dot{x}_5 = (L_2/J_e)cos(x_2) + (L_3/J_e)(V_s + L_2/L_3)cos(x_1) \tag{7e}$$
$$\dot{x}_6 = (L_4/J_\lambda)(V_s + L_2/L_3)cos(x_2)sin(x_1) \tag{7f}$$

If the above system is expressed as $\dot{x} = h(x, u)$, where x is the state and u is the input, the system is linearized by finding the Jacobians of h with respect to the state and the input and then plugging in the equilibrium values.

$$\frac{\partial h}{\partial x} = A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{L_4 L_2}{J_\lambda L_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \frac{\partial h}{\partial u} = B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_1/J_p \\ L_3/J_e & 0 \\ 0 & 0 \end{bmatrix} \tag{8}$$

where L_1, L_2, L_3 and L_4 were calculated in Section 1.1 and J_p, J_e and J_λ were given in the assignment description .

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The linearized equations of motion can therefore be written in the following form:

$$\ddot{\tilde{p}} = K_1 \tilde{V}_d \qquad K_1 = \frac{L_1}{J_p} \tag{9a}$$
$$\ddot{\tilde{e}} = K_2 \tilde{V}_s \qquad K_2 = \frac{L_3}{J_e} \tag{9b}$$
$$\ddot{\tilde{\lambda}} = K_3 \tilde{p} \qquad K_2 = \frac{L_4 L_2}{J_\lambda L_3} \tag{9c}$$

1.3 Problem 3

1.4 Problem 4

2 Part 2 – Monovariable control

2.1 Problem 1

We are given the controller shown in eq. (10).

$$\tilde{V_d} = K_{pp}(\tilde{p_c} - \tilde{p}) - K_{pd}\dot{\tilde{p}} \tag{10}$$

We take this controller and substitute it in the equation for pitch angle (**??**).

$$\ddot{\tilde{p}} = K_1 K_{pp}(\tilde{p_c} - \tilde{p}) - K_1 K_{pd}\dot{\tilde{p}} \tag{11}$$

Now we Laplace transform eq. (11) to find the transfer function $\frac{\tilde{p}(s)}{\tilde{p_c}(s)}$.

$$\begin{gathered}\ddot{\tilde{p}} + K_1 K_{pd}\dot{\tilde{p}} + K_1 K_{pp}\tilde{p} = K_1 K_{pp}\tilde{p_c} \\ \mathcal{L} \rightarrow \\ s^2\tilde{p}(s) + sK_1 K_{pd}\tilde{p}(s) + K_1 K_{pp}\tilde{p}(s) = K_1 K_{pp}\tilde{p_c}(s)\end{gathered}$$

Which gives us our transfer function

$$\frac{\tilde{p}(s)}{\tilde{p_c}(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd} s + K_1 K_{pp}} \tag{12}$$

The linearized pitch dynamics can be regarded as a second-order linear system, which means that if we place eq. (12) on the form shown in eq. (13) we can determine K_{pp} and K_{pd} from ω and ζ.

$$h(s) = \frac{\omega^2}{s^2 + 2\zeta\omega^2 s + \omega^2} \tag{13}$$

This gives us the following relations

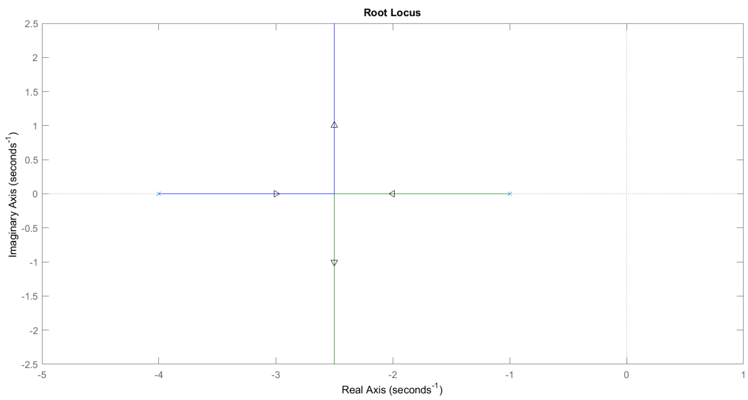
$$\begin{aligned}\omega &= \sqrt{K_1 K_{pp}} && (14) \\ 2\zeta\omega^2 &= K_1 K_{pd} \\ \zeta &= \frac{K_1 K_{pd}}{2\omega^2} = \frac{K_{pd}}{2K_{pp}} && (15)\end{aligned}$$

We know that for a critically damped system $\zeta = 1$, which gives us

$$K_{pd} = 2K_{pp} \tag{16}$$

We chose a $K_{pp} = 3$ and then from the relation in eq. (16) we get $K_{pd} = 6$. With these values the response of the pitch angle to the input was slower than desired. Therefore, K_{pp} was increased to $K_{pp} = 12.5$ and K_{pd} was lowered to underdamp the system, until it was sufficiently responsive at $K_{pd} = 0.7K_{pp} = 8.75$. At these values the system responded faster with only minor oscillations. It was observed that larger values of K_{pp} gave rise to larger oscillations.

Figure 4: Change in pole position by increasing K_{pp} given a constant K_{pd}



At the critically damped point, the poles lie on the same point on the x-axis. When K_{pp} is decreased in relation to K_{pd} the system is over-damped the poles move away from each other along the x-axis. When K_{pp} is increased in relation to K_{pd}, the system is under damped and the poles move away from each other vertically from the critically damped point.

With the PD controller, it was significantly easier to control the helicopter than with just feed forward joystick control.

2.2 Problem 2

By plugging the P controller for travel:

$$\tilde{p} = K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}}) \tag{17}$$

into the equation of motion for travel , the transfer function between $\dot{\tilde{\lambda}}$ and $\dot{\tilde{\lambda}}_c$ can be derived. By substituting the right side of the controller into the equation of motion for travel the equation becomes:

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$$\ddot{\tilde{\lambda}} = K_3(K_{rp}(\dot{\tilde{\lambda}}_c - \dot{\tilde{\lambda}})) \tag{18}$$

After taking the laplace transform and rearranging terms, the transfer function is as follows:

$$\frac{\ddot{\tilde{\lambda}}(s)}{\dot{\tilde{\lambda}}_c(s)} = \frac{K_3K_{rp}}{s + K_3K_{rp}} \tag{19}$$

This controller was quite quick and stable after correctly tuning K_{rp}. The value that was deemed best was $K_{rp} =$. Higher values of K_{rp} led to , while lower values of K_{rp} led to . It was not nessecary to add a gain to the x value of the joystick to produce quick and accurate control.

fill in this value

Describe performance

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3 Part 3 - Multivariable control

3.1 Problem 1

Here, we are to find the systems matrices $\boldsymbol{A}$ and $\boldsymbol{B}$. Trivially, they are:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{B} = \begin{bmatrix} 0 & 0 \\ 0 & K_1 \\ K_2 & 0 \end{bmatrix} \tag{20}$$

3.2 Problem 2

First, we are to examine the controllability of the system. We do that by finding the controllability matrix $\boldsymbol{\mathcal{C}}$ of the system, and see whether it has full rank or not:

$$\boldsymbol{\mathcal{C}} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{AB} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & K_1 \\ 0 & K_1 & 0 & 0 \\ K_2 & 0 & 0 & 0 \end{bmatrix} \tag{21}$$

which has full rank: $rank(\boldsymbol{\mathcal{C}}) = 3$, and is thus controllable.

Here, we are using LQR with reference feed-forward in order to control our system. Our $\boldsymbol{P}$ is defined such that as time goes to infinity, our states tend to the reference values. This happens when $\dot{\boldsymbol{x}} = 0$, as the system reaches a stable equilibrium around the reference values:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{Ax} + \boldsymbol{Bu} \\ &= \boldsymbol{Ax} + \boldsymbol{B}(\boldsymbol{Pr} - \boldsymbol{Kx}) \\ &= (\boldsymbol{A} - \boldsymbol{BK})\boldsymbol{x} + \boldsymbol{BPr} = 0 \end{aligned}$$

When $\dot{\boldsymbol{x}} = 0$, our $\boldsymbol{x}$ has reached its final value and we define $\boldsymbol{x} = \boldsymbol{x_\infty}$:

$$\begin{aligned} (\boldsymbol{BK} - \boldsymbol{A})\boldsymbol{x_\infty} &= \boldsymbol{BPr} \\ \Leftrightarrow \boldsymbol{x_\infty} &= (\boldsymbol{BK} - \boldsymbol{A})^{-1}\boldsymbol{BPr} \\ \Rightarrow \boldsymbol{y_\infty} = \boldsymbol{Cx_\infty} &= \boldsymbol{C}(\boldsymbol{BK} - \boldsymbol{A})^{-1}\boldsymbol{BPr} \end{aligned}$$

We see that our output $\boldsymbol{y_\infty}$ is equal to our reference $\boldsymbol{r}$ when:

$$\boldsymbol{P} = [\boldsymbol{C}(\boldsymbol{BK} - \boldsymbol{A})^{-1}\boldsymbol{B}]^{-1} \tag{22}$$

which is the $\boldsymbol{P}$ that makes we have used in our matlab code.

3.3 Problem 3

3.4 Problem 4

References

[1] Chi-Tsong Chen, *Linear System Theory and Design*, Oxford University Press, 4th edition, 2014

[2] Kristoffer Gryte, *Helicopter lab assignment*, Department of Engineering Cybernetics, NTNU, Version 4.5, 2015