

# Linear Systems TTK4115 - Helicopter lab

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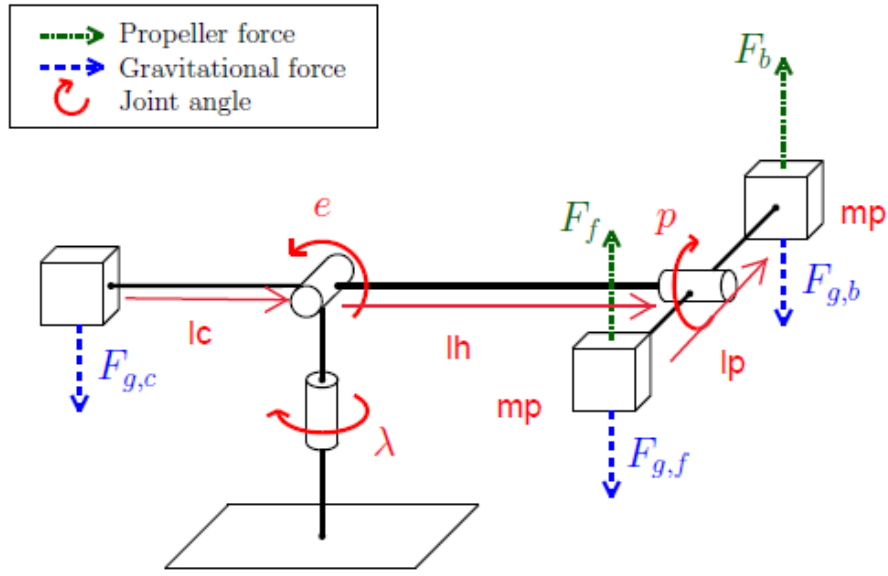
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# 1 Part 1 - Mathematical modeling

## 1.1 Problem 1

We use the helicopter model fig. 1 as our starting point for deriving the equations of motion.

Figure 1: the helicopter model figure 7 from the assignment [2, p.12] with relevant distances drawn in.



The equations of motion for the pitch is the momentum around the point p in the clockwise direction as shown in fig. 1:

$$\begin{aligned} J_p \ddot{p} &= l_p (F_{g,b} - F_b - F_{g,f} + F_f) \\ &= l_p (m_p g - m_p g + K_f V_f - V_b) \\ &= l_p K_f (V_f - V_b) \end{aligned}$$

Since  $V_d = V_f - V_b$ , we can write this as:

$$J_p \ddot{p} = l_p K_f V_d \quad (1)$$

Here, we can see that  $L_1 = l_p K_f$ .

Figure 2: the elevation model



## 1.2 Problem 2

To linearize the system about the point with all state variables equal to zero  $(p, e, \lambda)^T = (\dot{p}, \dot{e}, \dot{\lambda})^T = (0, 0, 0)$  the inputs in the equations of motion ( $V_s^*$  and  $V_d^*$ ) must be set to values that make this an equilibrium point.

At the linearization point, the equation of motion for pitch () reduces to  $0 = L_1 * V_d^*$  therefore:

$$V_d^* = 0 \quad (2)$$

At the linearization point, the equation of motion for elevation () reduces to  $0 = L_2 + L_3 * V_s^*$  therefore:

$$V_s^* = -L_2/L_3 \quad (3)$$

While the equation of motion for travel () at the linearization point simply reduces to  $0 = 0$ .

The following transformation is performed to simplify the analysis :

$$\begin{bmatrix} \tilde{p} \\ \tilde{e} \\ \tilde{\lambda} \end{bmatrix} = \begin{bmatrix} p \\ e \\ \lambda \end{bmatrix} - \begin{bmatrix} p^* \\ e^* \\ \lambda^* \end{bmatrix} \text{ and } \begin{bmatrix} \tilde{V}_s \\ \tilde{V}_d \end{bmatrix} = \begin{bmatrix} V_s \\ V_d \end{bmatrix} - \begin{bmatrix} V_s^* \\ V_d^* \end{bmatrix} \quad (4)$$

The equations of motion in the transformed system are therefore:

$$J_p \ddot{\tilde{p}} = L_1 \tilde{V}_d \quad (5a)$$

$$J_e \ddot{\tilde{e}} = L_2 \cos(\tilde{e}) + L_3(\tilde{V}_s + L_2/L_3) \cos(\tilde{p}) \quad (5b)$$

$$J_\lambda \ddot{\tilde{\lambda}} = L_4(\tilde{V}_s + L_2/L_3) \cos(\tilde{e}) \sin(\tilde{p}) \quad (5c)$$

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By choosing the state to be  $\mathbf{x} = (\tilde{p}, \tilde{e}, \tilde{\lambda}, \dot{p}, \dot{e}, \dot{\lambda})$  the nonlinear state equations become:

$$\dot{x}_1 = x_4 \quad (6a)$$

$$\dot{x}_2 = x_5 \quad (6b)$$

$$\dot{x}_3 = x_6 \quad (6c)$$

$$\dot{x}_4 = (L_1/J_p)V_d \quad (6d)$$

$$\dot{x}_5 = (L_2/J_e)\cos(x_2) + (L_3/J_e)(V_s + L_2/L_3)\cos(x_1) \quad (6e)$$

$$\dot{x}_6 = (L_4/J_\lambda)(V_s + L_2/L_3)\cos(x_2)\sin(x_1) \quad (6f)$$

If the above system is expressed as  $\dot{x} = h(x, u)$ , where  $\mathbf{x}$  is the state and  $u$  is the input, the system is linearized by finding the Jacobians of  $h$  with respect to the state and the input and then plugging in the equilibrium values.

$$\frac{\partial h}{\partial x} = A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{L_4 L_2}{J_\lambda L_3} & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \frac{\partial h}{\partial u} = B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & L_1/J_p \\ L_3/J_e & 0 \\ 0 & 0 \end{bmatrix} \quad (7)$$

where  $L_1, L_2, L_3$  and  $L_4$  were calculated in Section 1.1 and  $J_p, J_e$  and  $J_\lambda$  were given in the assignment description .

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The linearized equations of motion can therefore be written in the following form:

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$$\ddot{\tilde{p}} = K_1 \tilde{V}_d \quad K_1 = \frac{L_1}{J_p} \quad (8a)$$

$$\ddot{\tilde{e}} = K_2 \tilde{V}_s \quad K_2 = \frac{L_3}{J_e} \quad (8b)$$

$$\ddot{\tilde{\lambda}} = K_3 \tilde{p} \quad K_3 = \frac{L_4 L_2}{J_\lambda L_3} \quad (8c)$$

### 1.3 Problem 3

### 1.4 Problem 4

## 2 Part 2 – Monovvariable control

### 2.1 Problem 1

We are given the controller shown in eq. (9).

$$\tilde{V}_d = K_{pp}(\tilde{p}_c - \tilde{p}) - K_{pd}\dot{\tilde{p}} \quad (9)$$

We take this controller and substitute it in the equation for pitch angle (??).

$$\ddot{\tilde{p}} = K_1 K_{pp}(\tilde{p}_c - \tilde{p}) - K_1 K_{pd}\dot{\tilde{p}} \quad (10)$$

Now we Laplace transform eq. (10) to find the transfer function  $\frac{\tilde{p}(s)}{\tilde{p}_c(s)}$ .

$$\begin{aligned} \ddot{\tilde{p}} + K_1 K_{pd}\dot{\tilde{p}} + K_1 K_{pp}\tilde{p} &= K_1 K_{pp}\tilde{p}_c \\ \mathcal{L} \rightarrow \\ s^2\tilde{p}(s) + sK_1 K_{pd}\tilde{p}(s) + K_1 K_{pp}\tilde{p}(s) &= K_1 K_{pp}\tilde{p}_c(s) \end{aligned}$$

Which gives us our transfer function

$$\frac{\tilde{p}(s)}{\tilde{p}_c(s)} = \frac{K_1 K_{pp}}{s^2 + K_1 K_{pd}s + K_1 K_{pp}} \quad (11)$$

The linearized pitch dynamics can be regarded as a second-order linear system, which means that if we place eq. (11) on the form shown in eq. (12) we can determine  $K_{pp}$  and  $K_{pd}$  from  $\omega$  and  $\zeta$ .

$$h(s) = \frac{\omega^2}{s^2 + 2\zeta\omega^2 s + \omega^2} \quad (12)$$

This gives us the following relations

$$\omega = \sqrt{K_1 K_{pp}} \quad (13)$$

$$\begin{aligned} 2\zeta\omega^2 &= K_1 K_{pd} \\ \zeta &= \frac{K_1 K_{pd}}{2\omega^2} = \frac{K_{pd}}{2K_{pp}} \end{aligned} \quad (14)$$

We know that for a critically damped system  $\zeta = 1$ , which gives us

$$K_{pd} = 2K_{pp} \quad (15)$$

We chose a  $K_{pp} = 3$  and then from the relation in eq. (15) we get  $K_{pd} = 6$ . With these values the response of the pitch angle to the input was slower than desired. Therefore,  $K_{pp}$  was increased to  $K_{pp} = 12.5$  and  $K_{pd}$  was lowered to underdamp

the system, until it was sufficiently responsive at  $K_{pd} = 0.7K_{pp} = 8.75$ . At these values the system responded faster with only minor oscillations. It was observed that larger values of  $K_{pp}$  gave rise to larger oscillations.

Figure for that shows poles on Im/Re axis?

At the critically damped point, the poles lie on the same point on the x-axis. When  $K_{pd}$  is increased in relation to  $K_{pp}$  the system is over-damped the poles move away from each other along the x-axis. When  $K_{pd}$  decreased in relation to  $K_{pp}$ , the system is under damped and the poles move away from each other vertically from the critically damped point.

With the PD controller, it was significantly easier to control the helicopter than with just feed forward joystick control.

## 2.2 Problem 2

## References

- [1] Chi-Tsong Chen, *Linear System Theory and Design*, Oxford University Press, 4th edition, 2014
- [2] Kristoffer Gryte, *Helicopter lab assignment*, Department of Engineering Cybernetics, NTNU, Version 4.5, 2015