QuTiP lecture: Evolution and quantum statistics of a quantum parameter amplifier

Author: J. R. Johansson, robert@riken.jp

http://dml.riken.jp/~rob/

Latest version of this ipython notebook lecture is available at: http://github.com/jrjohansson/qutip-lectures

In [1]:
```
%pylab inline
```

```
Welcome to pylab, a matplotlib-based Python environment [backend:
module://IPython.zmq.pylab.backend_inline].
For more information, type 'help(pylab)'.
```

In [2]:
```
from qutip import *
```

Parameters

In [3]:
```
chi = 0.2
N1 = 75
N2 = 75
```

Operators and Hamiltonian

In [4]:
```
a  = tensor(destroy(N1), qeye(N2))
na = tensor(num(N1),     qeye(N2))
b  = tensor(qeye(N1),    destroy(N2))
nb = tensor(qeye(N1),    num(N2))
```

In [5]:
```
H = - chi * (a * b + a.dag() * b.dag())
```

In [6]:
```
# start in the ground (vacuum) state
psi0 = tensor(basis(N1,0), basis(N2,0))
```

Evolution

In [7]:
```
tlist = linspace(0, 10, 100)
```

In [8]:
```
c_ops = []
```

In [9]:
```
e_ops = []
```

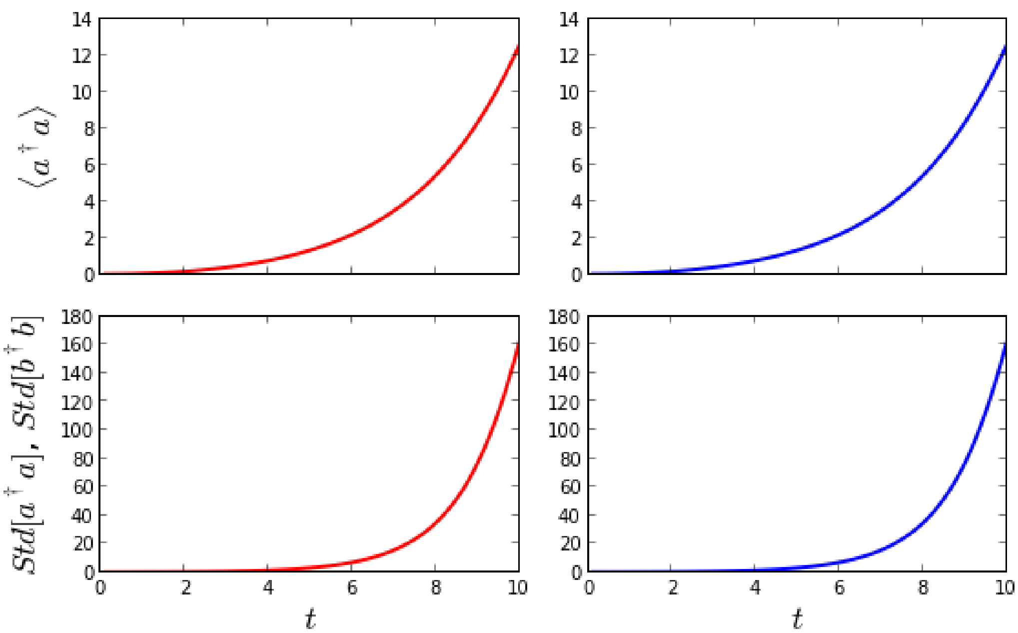
In [10]:
```
output = mesolve(H, psi0, tlist, c_ops, e_ops)
output
```

Out [10]:
```
Odedata object with mesolve data.
---------------------------------
states = True
num_collapse = 0
```

Expectation values and standard deviations

In [11]:
```
na_e = zeros(shape(tlist))
na_s = zeros(shape(tlist))
nb_e = zeros(shape(tlist))
nb_s = zeros(shape(tlist))

for idx, psi in enumerate(output.states):
    na_e[idx] = expect(na, psi)
    na_s[idx] = expect(na*na, psi)
    nb_e[idx] = expect(nb, psi)
    nb_s[idx] = expect(nb*nb, psi)

# substract the average squared to obtain variances
na_s = na_s - na_e ** 2
nb_s = nb_s - nb_e ** 2
```

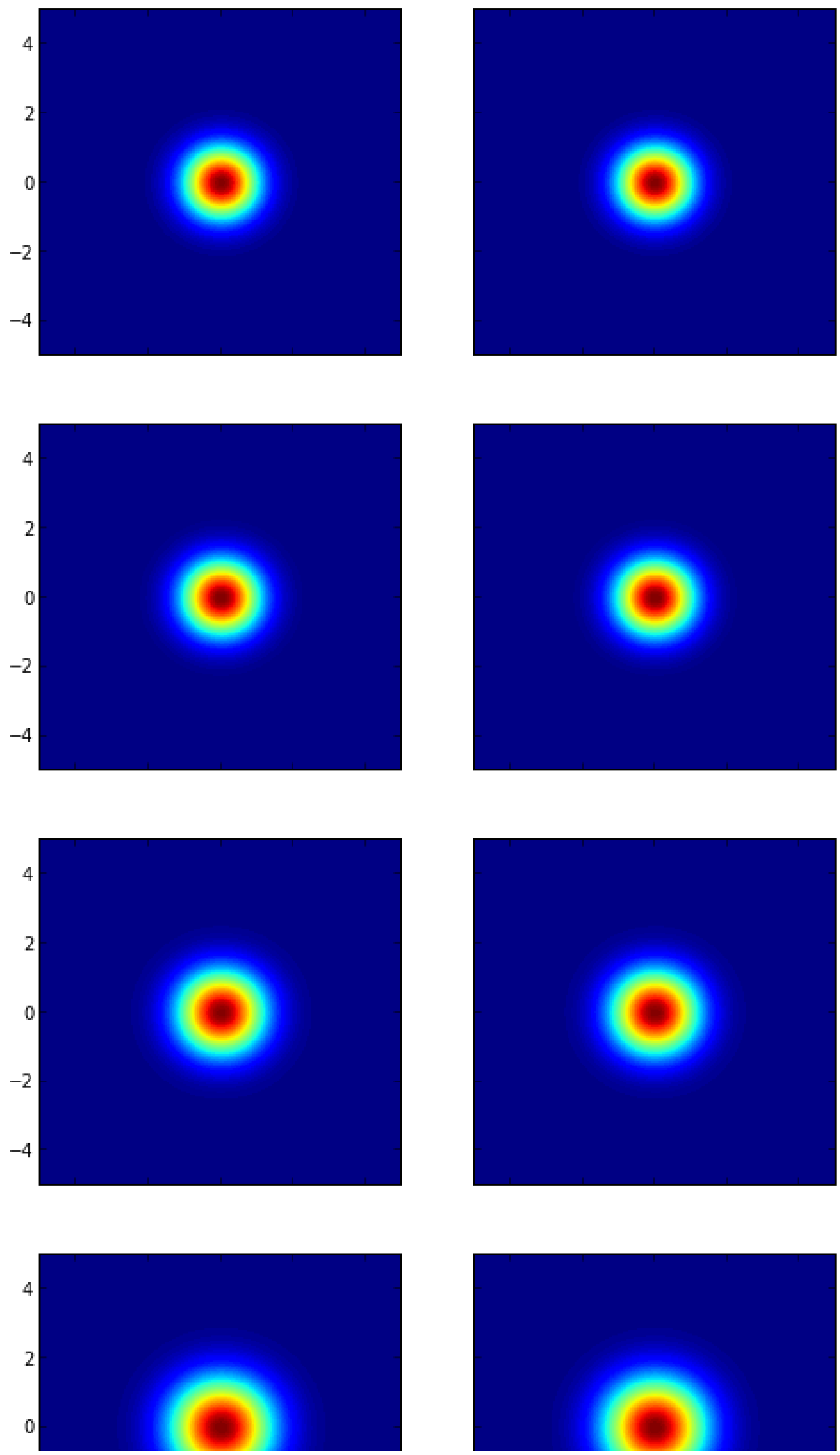
In [12]:
```
fig, axes = subplots(2, 2, sharex=True, figsize=(8,5))

line1 = axes[0,0].plot(tlist, na_e, 'r', linewidth=2)
axes[0,0].set_ylabel(r'$\langle a^\dagger a \rangle$', fontsize=18)

line2 = axes[0,1].plot(tlist, nb_e, 'b', linewidth=2)

line3 = axes[1,0].plot(tlist, na_s, 'r', linewidth=2)
axes[1,0].set_xlabel('$t$', fontsize=18)
axes[1,0].set_ylabel(r'$Std[a^\dagger a]$, $Std[b^\dagger b]$', fontsize=18)

line4 = axes[1,1].plot(tlist, nb_s, 'b', linewidth=2)
axes[1,1].set_xlabel('$t$', fontsize=18)

fig.tight_layout()
```

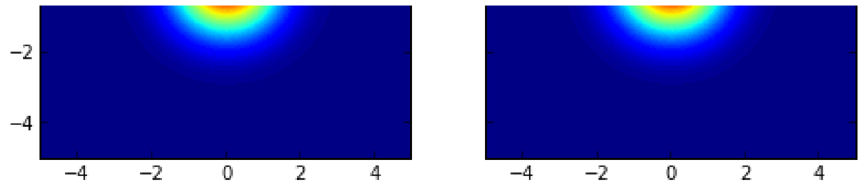


Wigner functions

In [13]:
```
# pick an arbitrary time and calculate the wigner functions for each mode
```

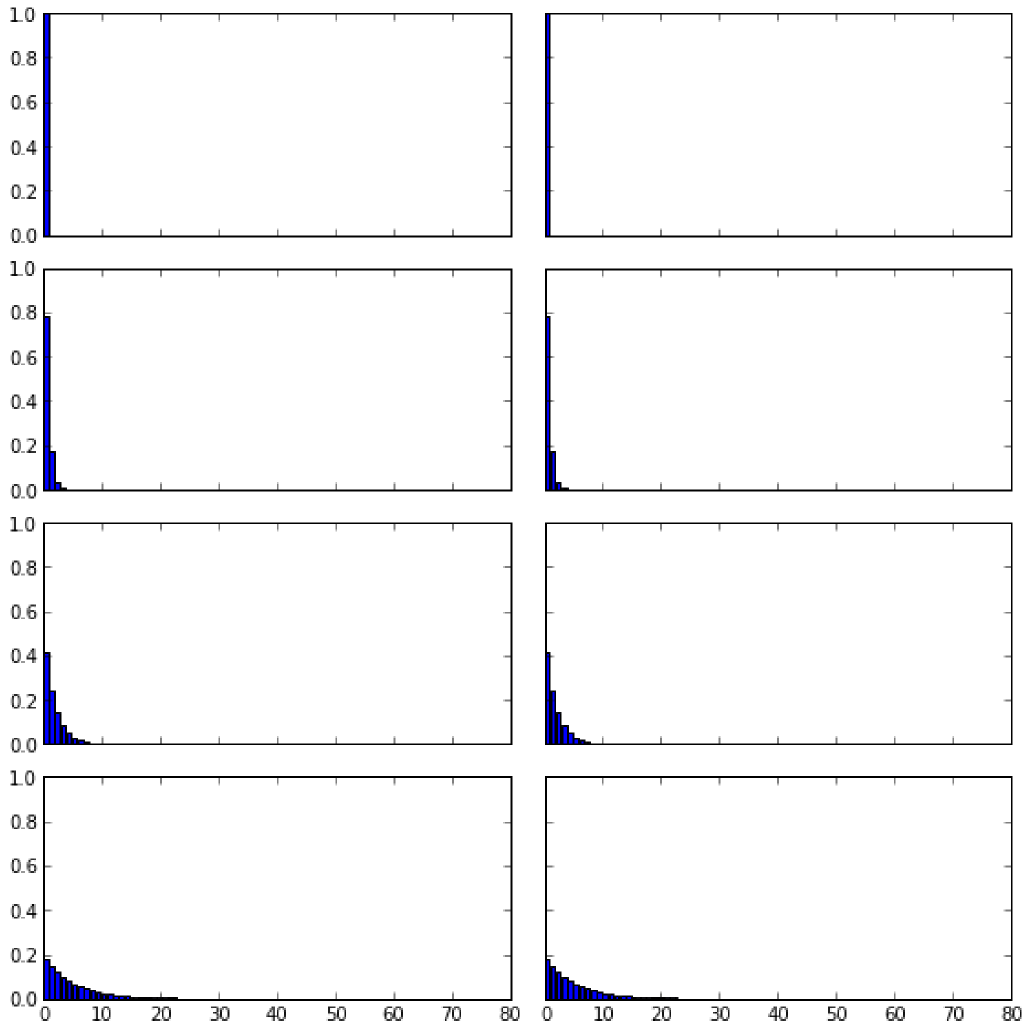
```
xvec = linspace(-5,5,200)
t_idx_vec = [0, 10, 20, 30]

fig, axes = subplots(len(t_idx_vec), 2, sharex=True, sharey=True,
figsize=(8,4*len(t_idx_vec)))

for idx, t_idx in enumerate(t_idx_vec):
    psi_a = ptrace(output.states[t_idx], 0)
    psi_b = ptrace(output.states[t_idx], 1)
    W_a = wigner(psi_a, xvec, xvec)
    W_b = wigner(psi_b, xvec, xvec)

    cont1 = axes[idx,0].contourf(xvec, xvec, W_a, 100)
    cont2 = axes[idx,1].contourf(xvec, xvec, W_b, 100)
```





Fock-state distribution

In [14]:
```
# pick arbitrary times and plot the photon distributions at those times
#t_idx_vec = [0, 10, 20, 30]
t_idx_vec = range(0,len(tlist),25)

fig, axes = subplots(len(t_idx_vec), 2, sharex=True, sharey=True,
figsize=(8,2*len(t_idx_vec)))

for idx, t_idx in enumerate(t_idx_vec):
    psi_a = ptrace(output.states[t_idx], 0)
    psi_b = ptrace(output.states[t_idx], 1)

    cont1 = axes[idx,0].bar(range(0, N1), real(psi_a.diag()))
    cont2 = axes[idx,1].bar(range(0, N2), real(psi_b.diag()))

fig.tight_layout()
```



Nonclassical correlations

In [26]:
```
# second-order photon correlations
g2_1  = zeros(shape(tlist))
g2_2  = zeros(shape(tlist))
g2_12 = zeros(shape(tlist))

ad_ad_a_a = a.dag() * a.dag() * a * a
bd_bd_b_b = b.dag() * b.dag() * b * b
ad_a_bd_b = a.dag() * a * b.dag() * b

cs_rhs = zeros(shape(tlist))
cs_lhs = zeros(shape(tlist))

for idx, psi in enumerate(output.states):
    # g2 correlations
    g2_1[idx]  = expect(ad_ad_a_a, psi)
    g2_2[idx]  = expect(bd_bd_b_b, psi)
    g2_12[idx] = expect(ad_a_bd_b, psi)

    # cauchy-schwarz
    cs_lhs[idx] = expect(ad_a_bd_b, psi)
    cs_rhs[idx] = expect(ad_ad_a_a, psi)

# normalize the correlation functions
g2_1  = g2_1  / (na_e ** 2)
g2_2  = g2_2  / (nb_e ** 2)
g2_12 = g2_12 / (na_e * nb_e)
```

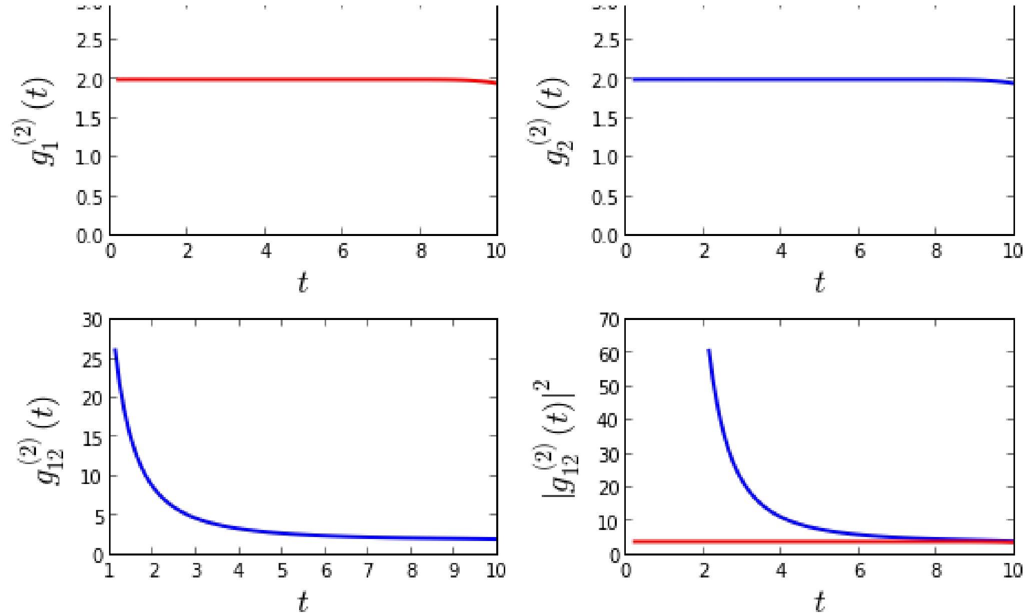
Second-order coherence functions: Cauchy-Schwarz inequality

Walls and Milburn, page 78: Classical states satisfy

$$[g_{12}^{(2)}]^2 \leq g_1^{(2)} g_2^{(2)}$$

(variant of the Cauchy-Schwarz inequality)

In [16]:
```
fig, axes = subplots(2, 2, figsize=(8,5))

line1 = axes[0,0].plot(tlist, g2_1, 'r', linewidth=2)
axes[0,0].set_xlabel("$t$", fontsize=18)
axes[0,0].set_ylabel(r'$g_1^{(2)}(t)$', fontsize=18)
axes[0,0].set_ylim(0,3)

line2 = axes[0,1].plot(tlist, g2_2, 'b', linewidth=2)
axes[0,1].set_xlabel("$t$", fontsize=18)
axes[0,1].set_ylabel(r'$g_2^{(2)}(t)$', fontsize=18)
axes[0,1].set_ylim(0,3)

line3 = axes[1,0].plot(tlist[10:], g2_12[10:], 'b', linewidth=2)
axes[1,0].set_xlabel("$t$", fontsize=18)
axes[1,0].set_ylabel(r'$g_{12}^{(2)}(t)$', fontsize=18)


line4 = axes[1,1].plot(tlist[20:], abs(g2_12[20:])**2, 'b', linewidth=2)
line5 = axes[1,1].plot(tlist, g2_1 * g2_2, 'r', linewidth=2)
axes[1,1].set_xlabel("$t$", fontsize=18)
axes[1,1].set_ylabel(r'$|g_{12}^{(2)}(t)|^2$', fontsize=18)

fig.tight_layout()
```



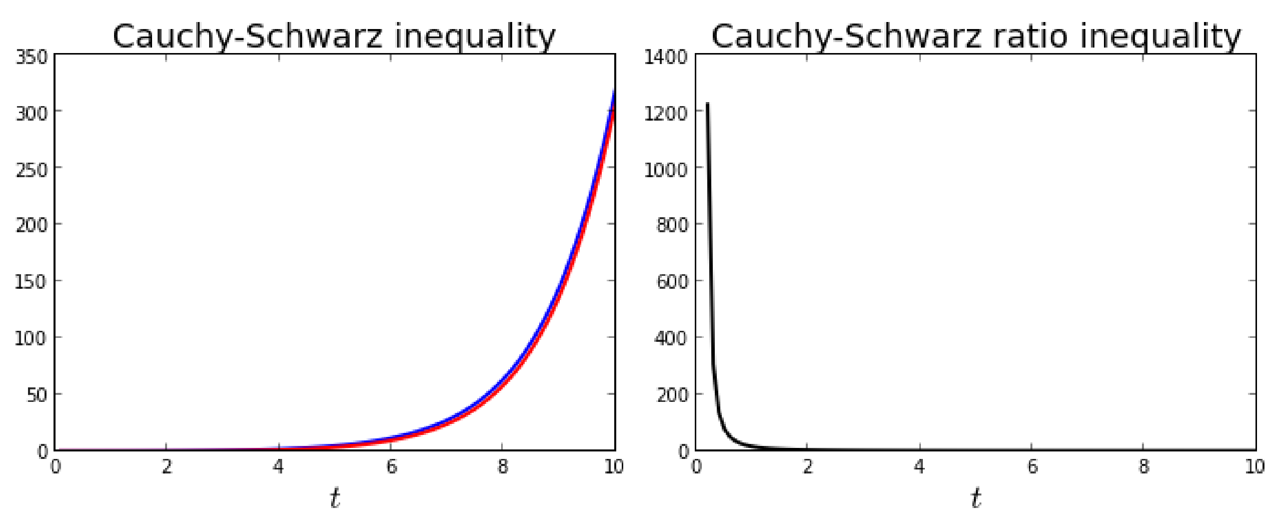
Clearly the two-mode squeezed state from the parameteric amplifier does not satisfy this inequality, and is therefore not classical.

Cauchy-Schwarz inequality

Walls and Milburn, page 78: the Cauchy-Schwarz inequality for symmetric modes

$\langle a^\dagger a b^\dagger b\rangle \leq \langle (a^\dagger)^2 a^2\rangle$

In [28]:
```
fig, axes = subplots(1, 2, figsize=(10,4))

line1 = axes[0].plot(tlist, cs_lhs, 'b', tlist, cs_rhs, 'r', linewidth=2)
axes[0].set_xlabel("$t$", fontsize=18)
axes[0].set_title(r'Cauchy-Schwarz inequality', fontsize=18)

line1 = axes[1].plot(tlist, cs_lhs / (cs_rhs), 'k', linewidth=2)
axes[1].set_xlabel("$t$", fontsize=18)
axes[1].set_title(r'Cauchy-Schwarz ratio inequality', fontsize=18)

fig.tight_layout()
```



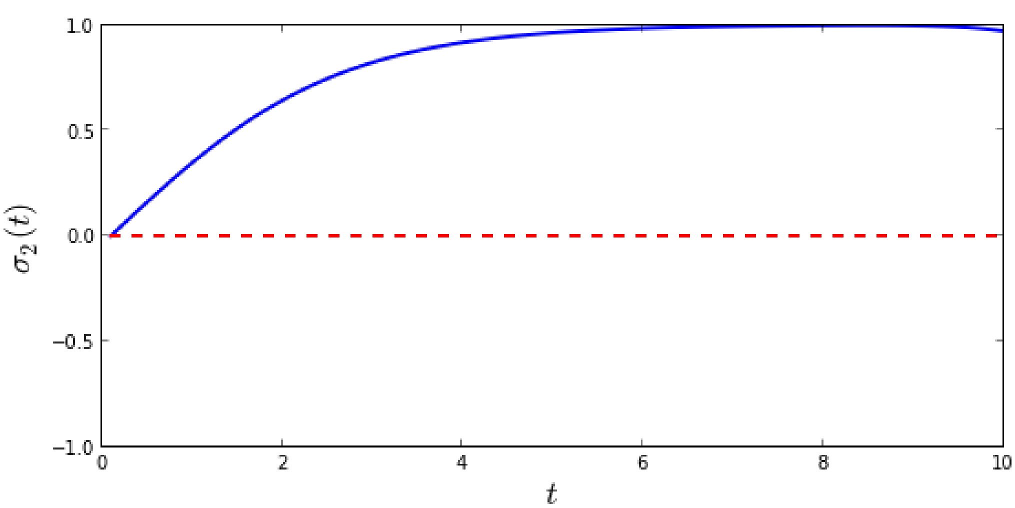
Two-mode squeezing correlations

The two-mode squeezing can be characterized by the parameter σ_2 defined as

$$\sigma_2 = \frac{2\sqrt{\omega_a\omega_b}\left[\langle ab\rangle e^{i2\theta_\Sigma} + \langle a^\dagger b^\dagger\rangle e^{-i2\theta_\Sigma}\right]}{\omega_a\langle a^\dagger a + aa^\dagger\rangle + \omega_b\langle b^\dagger b + bb^\dagger\rangle}$$

In [18]:
```
# pre-compute operators outside the loop
op_a_b = a * b
op_ad_bd = a.dag() * b.dag()
op_ad_a_p_a_ad  = a.dag() * a + a * a.dag()
op_bd_b_p_b_bd  = b.dag() * b + b * b.dag()

e_a_b   = zeros(shape(tlist), dtype=complex)
e_ad_bd = zeros(shape(tlist), dtype=complex)
e_ad_a_p_a_ad = zeros(shape(tlist), dtype=complex)
e_bd_b_p_b_bd = zeros(shape(tlist), dtype=complex)

for idx, psi in enumerate(output.states):

    e_a_b[idx]         = expect(op_a_b, psi)
    e_ad_bd[idx]       = expect(op_ad_bd, psi)
    e_ad_a_p_a_ad[idx] = expect(op_ad_a_p_a_ad, psi)
    e_bd_b_p_b_bd[idx] = expect(op_bd_b_p_b_bd, psi)

# calculate the sigma_2
theta = 3*pi/4
w_a = w_b = 1
sigma2 = 2 * sqrt(w_a * w_b) * (e_a_b * exp(2j * theta) + e_ad_bd * exp(-2j * theta))
/ (w_a * e_ad_a_p_a_ad + w_b * e_bd_b_p_b_bd)
```

In [19]:
```
fig, axes = subplots(1, 1, figsize=(8,4))

line1 = axes.plot(tlist, real(sigma2), 'b', tlist, imag(sigma2), 'r--', linewidth=2)
axes.set_xlabel("$t$", fontsize=18)
axes.set_ylabel(r'$\sigma_2(t)$', fontsize=18)
axes.set_ylim(-1, 1)
fig.tight_layout()
```



Quantum-classical indicator $\langle : f^\dagger f : \rangle$

Using $\langle : f^\dagger f : \rangle$ we can again show that the output of the parametric amplifier is nonclassical. If we choose

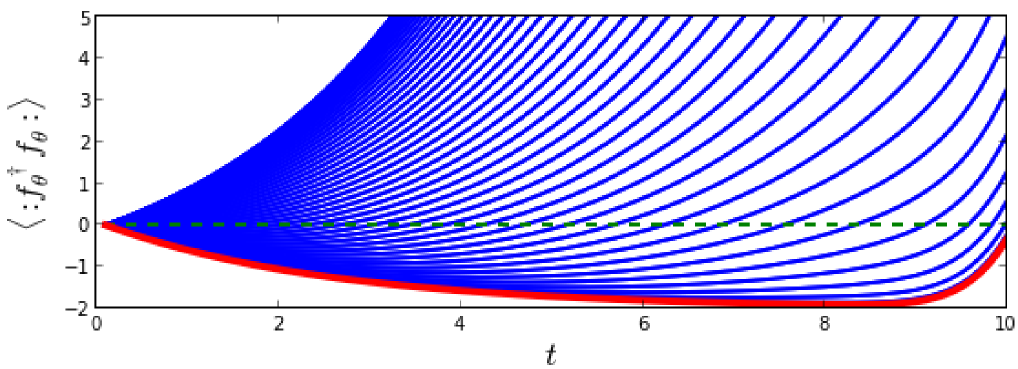
$$f_\theta = e^{i\theta} b_- + e^{-i\theta} b_-^\dagger + ie^{i\theta} b_+ - ie^{-i\theta} b_+^\dagger$$

we get

$F_\theta = \langle: f_\theta^\dagger f_\theta :\rangle = 2\langle a^\dagger a\rangle + 2\langle b^\dagger b\rangle + 2i\left(e^{2i\theta}\langle ab\rangle - e^{-2i\theta}\langle a^\dagger b^\dagger\rangle\right)$

In [20]:
```
# pre-compute operators outside the loop
op_ad_a  = a.dag() * a
op_bd_b  = b.dag() * b
op_a_b   = a * b
op_ad_bd = a.dag() * b.dag()

e_ad_a  = zeros(shape(tlist), dtype=complex)
e_bd_b  = zeros(shape(tlist), dtype=complex)
e_a_b   = zeros(shape(tlist), dtype=complex)
e_ad_bd = zeros(shape(tlist), dtype=complex)

for idx, psi in enumerate(output.states):

    e_ad_a[idx]  = expect(op_ad_a, psi)
    e_bd_b[idx]  = expect(op_bd_b, psi)
    e_a_b[idx]   = expect(op_a_b, psi)
    e_ad_bd[idx] = expect(op_ad_bd, psi)

# calculate the sigma_2, function of the angle parameter theta
def F_theta(theta):
    return 2 * e_ad_a + 2 * e_bd_b + 2j * (exp(2j * theta) * e_a_b - exp(-2j * theta)
* e_ad_bd)
```

In [21]:
```
fig, axes = subplots(1, 1, figsize=(8,3))

for theta in linspace(0.0, 2*pi, 100):
    line1 = axes.plot(tlist, real(F_theta(theta)), 'b', tlist, imag(F_theta(theta)),
'g--', linewidth=2)

line = axes.plot(tlist, real(F_theta(0)), 'r', linewidth=4)

axes.set_xlabel("$t$", fontsize=18)
axes.set_ylabel(r'$\langle:f_\theta^\dagger f_\theta:\rangle$', fontsize=18)
axes.set_ylim(-2, 5)
fig.tight_layout()
```



Entanglement: logarithmic negativity

Wigner covariance matrix:
In order to evaluate the logarithmic negativity we first need to construct the Wigner covariance matrix:

$V_{ij} = \frac{1}{2}\langle R_i R_j + R_j R_i\rangle$

where

$R^T = (q_1, p_1, q_2, p_2) = (q_a, p_a, q_b, p_b)$

is a vector with the quadratures for the two modes a and b:

$$q_a = e^{i\theta_a} a + e^{-i\theta_a} a^\dagger$$

$$p_a = -i(e^{i\theta_a} a - e^{-i\theta_a} a^\dagger)$$

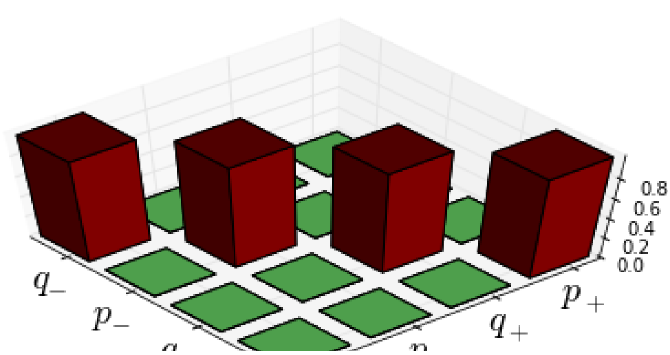
and likewise for mode b.

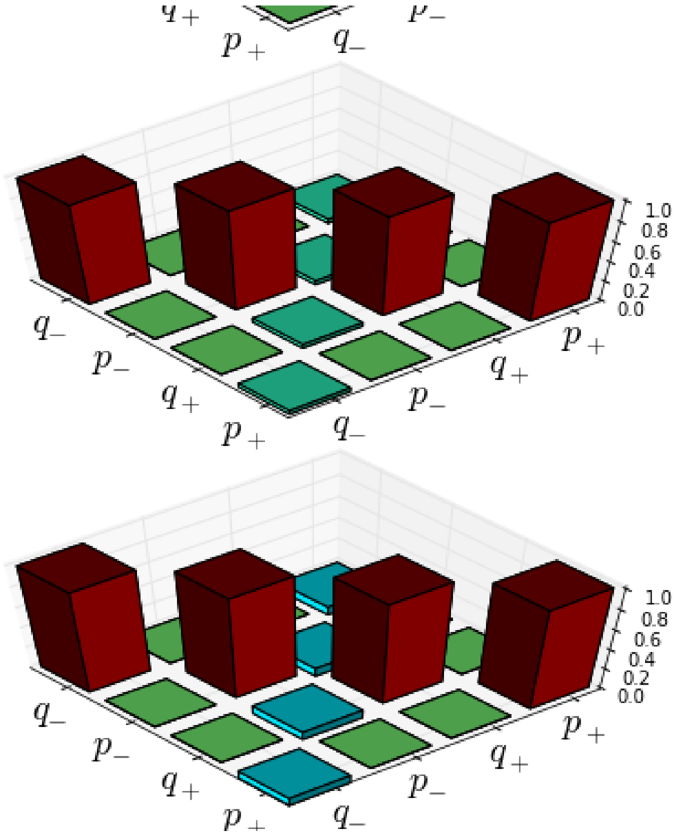
In [22]:
```
from continous_variables import *

R_op = correlation_matrix_quadrature(a, b)
```

In [23]:
```
def plot_covariance_matrix(V, ax):
    """
    Plot a matrix-histogram representation of the supplied Wigner covariance matrix.
    """
    num_elem = 16
    xpos,ypos = meshgrid(range(4),range(4))
    xpos = xpos.T.flatten()-0.5
    ypos = ypos.T.flatten()-0.5
    zpos = zeros(num_elem)
    dx = 0.75*ones(num_elem)
    dy = dx.copy()
    dz = V.flatten()

    nrm=mpl.colors.Normalize(-0.5,0.5)
    colors=cm.jet(nrm((sign(dz)*abs(dz)**0.75)))

    ax.view_init(azim=-40,elev=60)
    ax.bar3d(xpos, ypos, zpos, dx, dy, dz, color=colors)
    ax.axes.w_xaxis.set_major_locator(IndexLocator(1,-0.5))
    ax.axes.w_yaxis.set_major_locator(IndexLocator(1,-0.5))
    ax.axes.w_xaxis.set_ticklabels(("$q_-$", "$p_-$", "$q_+$", "$p_+$"), fontsize=20)
    ax.axes.w_yaxis.set_ticklabels(("$q_-$", "$p_-$", "$q_+$", "$p_+$"), fontsize=20)
```

In [24]:
```
# pick arbitrary times and plot the photon distributions at those times
t_idx_vec = [0, 20, 40]

fig, axes = subplots(len(t_idx_vec), 1, subplot_kw={'projection':'3d'},
figsize=(6,3*len(t_idx_vec)))

for idx, t_idx in enumerate(t_idx_vec):

    # calculate the wigner covariance matrix
    V = wigner_covariance_matrix(R_op, output.states[idx])

    plot_covariance_matrix(V, axes[idx])

fig.tight_layout()
```

```
/usr/lib/pymodules/python2.7/mpl_toolkits/mplot3d/axes3d.py:1481: RuntimeWarning: invali
encountered in divide
  for n in normals])
```





In [29]:
```python
"""
Calculate the wigner covariance matrix logarithmic negativity for each time step
"""
logneg = zeros(shape(tlist))

for idx, t_idx in enumerate(tlist):

    V = wigner_covariance_matrix(R_op, output.states[idx])

    logneg[idx] = wigner_logarithm_negativity(V)

fig, axes = subplots(1, 1, figsize=(8,4))
axes.plot(tlist, logneg, 'r')
axes.set_xlabel("$t$", fontsize=18)
axes.set_ylabel("Logarithmic negativity", fontsize=18)
fig.tight_layout()
```