QuTiP lecture: The Dicke model

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Latest version of this ipython notebook lecture is available at: http://github.com/jrjohansson/qutip-lectures

In [49]:
```
%pylab inline
```

```
Welcome to pylab, a matplotlib-based Python environment [backend: module://IPython.zmq.pylab.backend_inline].
For more information, type 'help(pylab)'.
```

In [50]:
```
from qutip import *
```

Introduction

The Dicke Hamiltonian consists of a cavity mode and N spin-1/2 coupled to the cavity:

$$H_D = \omega_0 \sum_{i=1}^{N} \sigma_z^{(i)} + \omega a^\dagger a + \sum_{i}^{N} \frac{\lambda}{\sqrt{N}} (a + a^\dagger)(\sigma_+^{(i)} + \sigma_-^{(i)})$$

$$H_D = \omega_0 J_z + \omega a^\dagger a + \frac{\lambda}{\sqrt{N}} (a + a^\dagger)(J_+ + J_-)$$

where J_z and $J_\pm$ are the collective angular momentum operators fpr a pseudospin of length $j = N/2$:

$$J_z = \sum_{i=1}^{N} \sigma_z^{(i)}$$

$$J_\pm = \sum_{i=1}^{N} \sigma_\pm^{(i)}$$

References

- R.H. Dicke, Phys. Rev. 93, 99–110 (1954)
- Lambert et al., Phys. Rev. B 80, 165308 (2009)
- Lambert et al., Phys. Rev. Lett. 92, 073602 (2004)

Setup problem in QuTiP

In [51]:
```
w  = 1.0
w0 = 1.0

g = 1.0
gc = sqrt(w * w0)/2 # critical coupling strength

kappa = 0.05
gamma = 0.15
```

In [52]:
```
M = 20
N = 8
j = N/2.0
n = 2*j + 1

a  = tensor(destroy(M), qeye(n))
Jp = tensor(qeye(M), jmat(j, '+'))
Jm = tensor(qeye(M), jmat(j, '-'))
Jz = tensor(qeye(M), jmat(j, 'z'))

H0 = w * a.dag() * a + w0 * Jz
H1 = 1.0 / sqrt(N) * (a + a.dag()) * (Jp + Jm)
H = H0 + g * H1

H
```

Out [52]: Quantum object: dims = [[20, 9], [20, 9]], shape = [180, 180], type = oper, isHerm = True

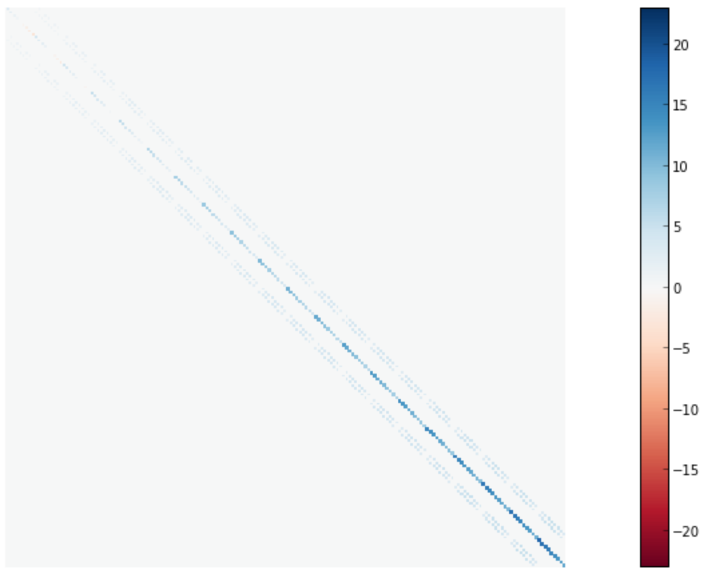
$$\begin{pmatrix}4.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 3.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 2.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 1.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 0.0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 19.0 & 0.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 18.0 & 0.0 & 0.0 & 0.0\\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 17.0 & 0.0 & 0.0\end{pmatrix}$$

$$\begin{pmatrix} 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 16.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 & 0.0 & 15.0 \end{pmatrix}$$

Structure of the Hamiltonian

In [53]:
```
fig, ax = subplots(1, 1, figsize=(10,10))
hinton(H, ax=ax)
```

Out [53]: <matplotlib.axes.AxesSubplot at 0x2be01b90>



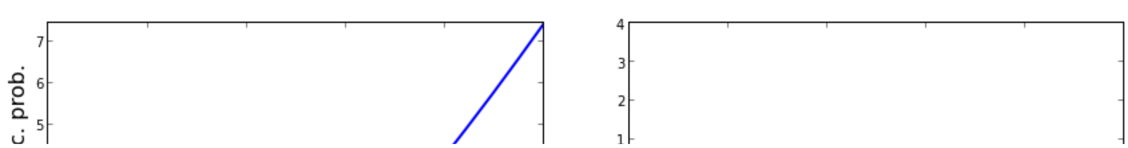
Find the ground state as a function of cavity-spin interaction strength

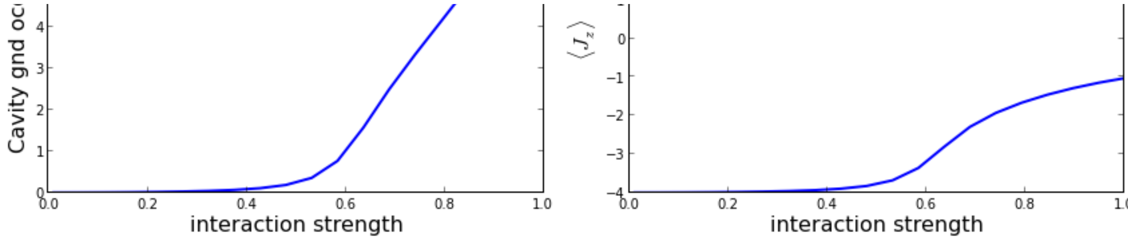
In [54]:
```
g_vec = linspace(0.01, 1.0, 20)

# Ground state and steady state for the Hamiltonian: H = H0 + g * H1
psi_gnd_list = [(H0 + g * H1).groundstate()[1] for g in g_vec]
```

Cavity ground state occupation probability

In [55]:
```
n_gnd_vec = expect(a.dag() * a, psi_gnd_list)
Jz_gnd_vec = expect(Jz, psi_gnd_list)
```

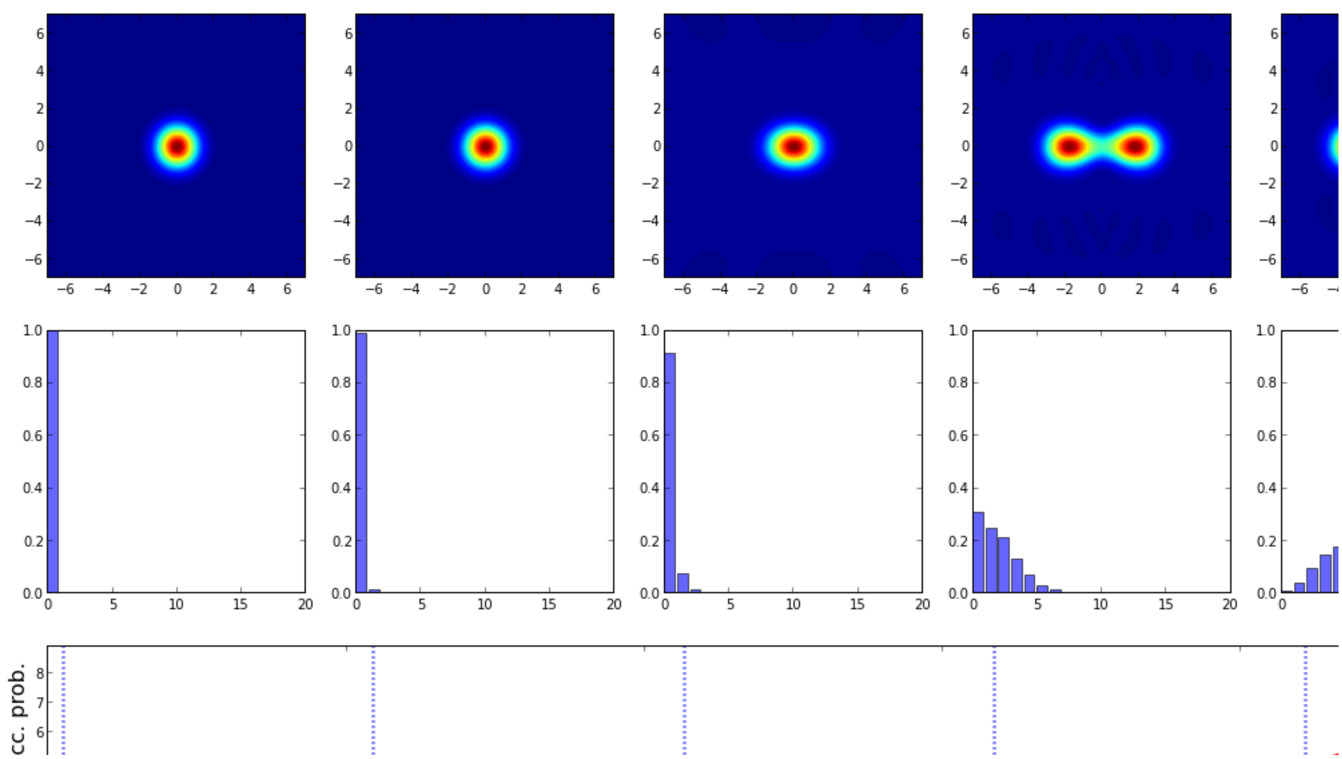
In [56]:
```
fig, axes = subplots(1, 2, sharex=True, figsize=(12,4))

axes[0].plot(g_vec, n_gnd_vec, 'b', linewidth=2, label="cavity occupation")
axes[0].set_ylim(0, max(n_gnd_vec))
axes[0].set_ylabel("Cavity gnd occ. prob.", fontsize=16)
axes[0].set_xlabel("interaction strength", fontsize=16)

axes[1].plot(g_vec, Jz_gnd_vec, 'b', linewidth=2, label="cavity occupation")
axes[1].set_ylim(-j, j)
axes[1].set_ylabel(r"$\langle J_z\rangle$", fontsize=16)
axes[1].set_xlabel("interaction strength", fontsize=16)

fig.tight_layout()
```

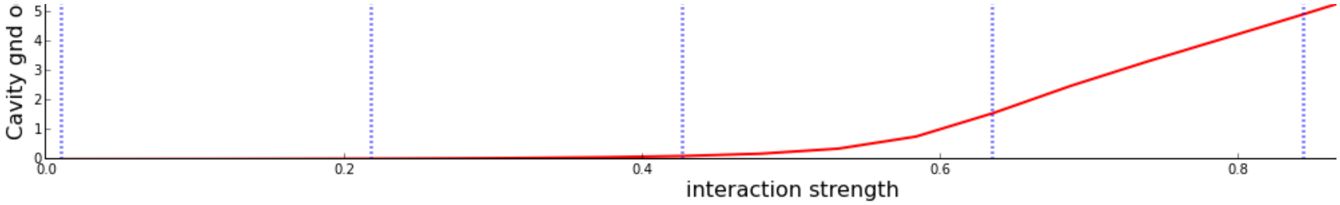




Cavity Wigner function and Fock distribution as a function of coupling strength

In [57]:
```
psi_gnd_sublist = psi_gnd_list[::4]

xvec = linspace(-7,7,200)

fig_grid = (3, len(psi_gnd_sublist))
fig = figure(figsize=(4*len(psi_gnd_sublist),12))

for idx, psi_gnd in enumerate(psi_gnd_sublist):

    # trace out the cavity density matrix
    rho_gnd_cavity = ptrace(psi_gnd, 0)

    # calculate its wigner function
    W = wigner(rho_gnd_cavity, xvec, xvec)

    # plot its wigner function
    ax = subplot2grid(fig_grid, (0, idx))
    ax.contourf(xvec, xvec, W, 100)

    # plot its fock-state distribution
    ax = subplot2grid(fig_grid, (1, idx))
    ax.bar(arange(0, M), real(rho_gnd_cavity.diag()), color="blue", alpha=0.6)
    ax.set_ylim(0, 1)
    ax.set_xlim(0, M)

# plot the cavity occupation probability in the ground state
ax = subplot2grid(fig_grid, (2, 0), colspan=fig_grid[1])
ax.plot(g_vec, n_gnd_vec, 'r', linewidth=2, label="cavity occupation")
ax.set_xlim(0, max(g_vec))
ax.set_ylim(0, max(n_gnd_vec)*1.2)
ax.set_ylabel("Cavity gnd occ. prob.", fontsize=16)
ax.set_xlabel("interaction strength", fontsize=16)

for g in g_vec[::4]:
    ax.plot([g,g],[0,max(n_gnd_vec)*1.2], 'b:', linewidth=2.5)
```

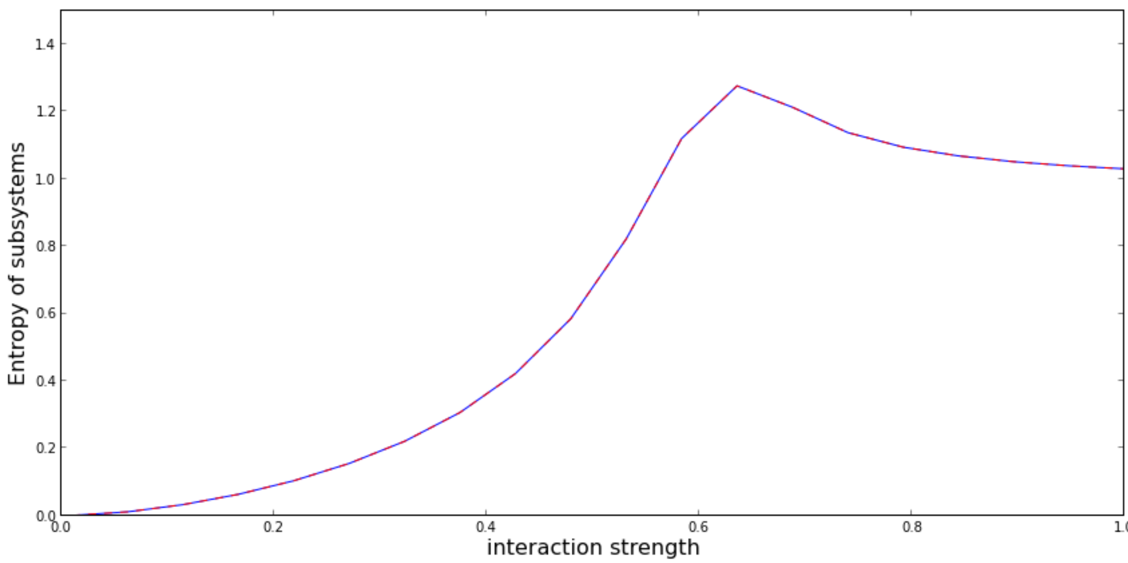




Entropy/Entanglement between spins and cavity

In [58]:
```
entropy_tot    = zeros(shape(g_vec))
entropy_cavity = zeros(shape(g_vec))
entropy_spin   = zeros(shape(g_vec))

for idx, psi_gnd in enumerate(psi_gnd_list):

    rho_gnd_cavity = ptrace(psi_gnd, 0)
    rho_gnd_spin   = ptrace(psi_gnd, 1)

    entropy_tot[idx]    = entropy_vn(psi_gnd, 2)
    entropy_cavity[idx] = entropy_vn(rho_gnd_cavity, 2)
    entropy_spin[idx]   = entropy_vn(rho_gnd_spin, 2)
```

In [59]:
```
fig, axes = subplots(1, 1, figsize=(12,6))
axes.plot(g_vec, entropy_tot, 'k', g_vec, entropy_cavity, 'b', g_vec, entropy_spin, 'r--')

axes.set_ylim(0, 1.5)
axes.set_ylabel("Entropy of subsystems", fontsize=16)
axes.set_xlabel("interaction strength", fontsize=16)

fig.tight_layout()
```



Entropy as a function interaction strength for increasing N

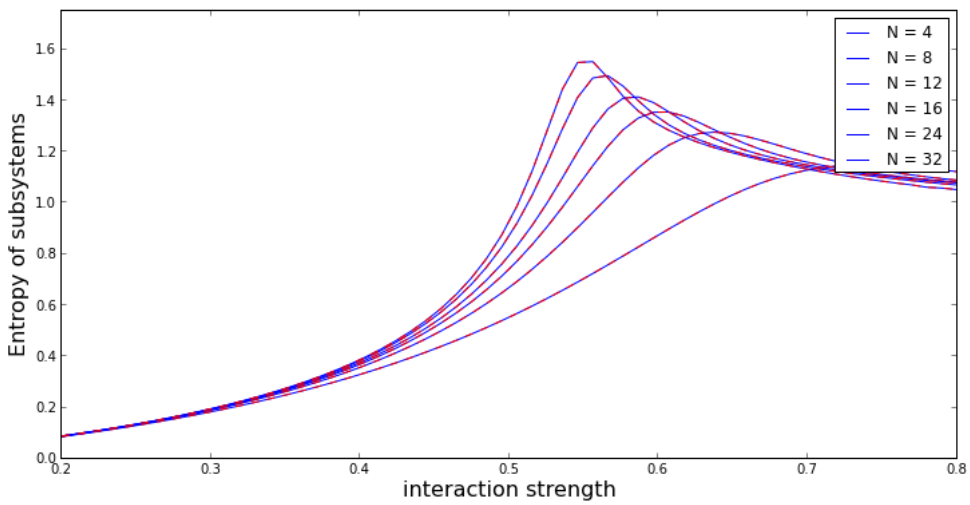
See Lambert et al., Phys. Rev. Lett. 92, 073602 (2004).

In [60]:
```
def calulcate_entropy(M, N, g_vec):

    j = N/2.0
    n = 2*j + 1

    # setup the hamiltonian for the requested hilbert space sizes
    a  = tensor(destroy(M), qeye(n))
    Jp = tensor(qeye(M), jmat(j, '+'))
    Jm = tensor(qeye(M), jmat(j, '-'))
    Jz = tensor(qeye(M), jmat(j, 'z'))

    H0 = w * a.dag() * a + w0 * Jz
    H1 = 1.0 / sqrt(N) * (a + a.dag()) * (Jp + Jm)

    # Ground state and steady state for the Hamiltonian: H = H0 + g * H1
    psi_gnd_list = [(H0 + g * H1).groundstate()[1]  for g in g_vec]
```

```
    entropy_cavity = zeros(shape(g_vec))
    entropy_spin   = zeros(shape(g_vec))

    for idx, psi_gnd in enumerate(psi_gnd_list):

        rho_gnd_cavity = ptrace(psi_gnd, 0)
        rho_gnd_spin   = ptrace(psi_gnd, 1)

        entropy_cavity[idx] = entropy_vn(rho_gnd_cavity, 2)
        entropy_spin[idx]   = entropy_vn(rho_gnd_spin, 2)

    return entropy_cavity, entropy_spin
```

In [61]:
```
g_vec = linspace(0.2, 0.8, 60)
N_vec = [4, 8, 12, 16, 24, 32]
MM = 25

fig, axes = subplots(1, 1, figsize=(12,6))

for NN in N_vec:

    entropy_cavity, entropy_spin = calulcate_entropy(MM, NN, g_vec)

    axes.plot(g_vec, entropy_cavity, 'b', label="N = %d" % NN)
    axes.plot(g_vec, entropy_spin, 'r--')

axes.set_ylim(0, 1.75)
axes.set_ylabel("Entropy of subsystems", fontsize=16)
axes.set_xlabel("interaction strength", fontsize=16)
axes.legend()
```

Out [61]: <matplotlib.legend.Legend at 0x4ec67790>



Dissipative cavity: steady state instead of the ground state

In [62]:
```
# average number thermal photons in the bath coupling to the resonator
n_th = 0.25

c_ops = [sqrt(kappa * (n_th + 1)) * a, sqrt(kappa * n_th) * a.dag()]
#c_ops = [sqrt(kappa) * a, sqrt(gamma) * Jm]
```

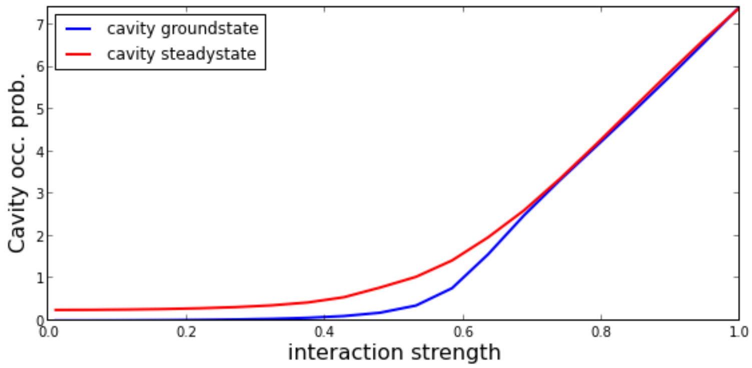
Find the ground state as a function of cavity-spin interaction strength

In [63]:
```
g_vec = linspace(0.01, 1.0, 20)

# Ground state for the Hamiltonian: H = H0 + g * H1
rho_ss_list = [steadystate(H0 + g * H1, c_ops) for g in g_vec]
```

Cavity ground state occupation probability

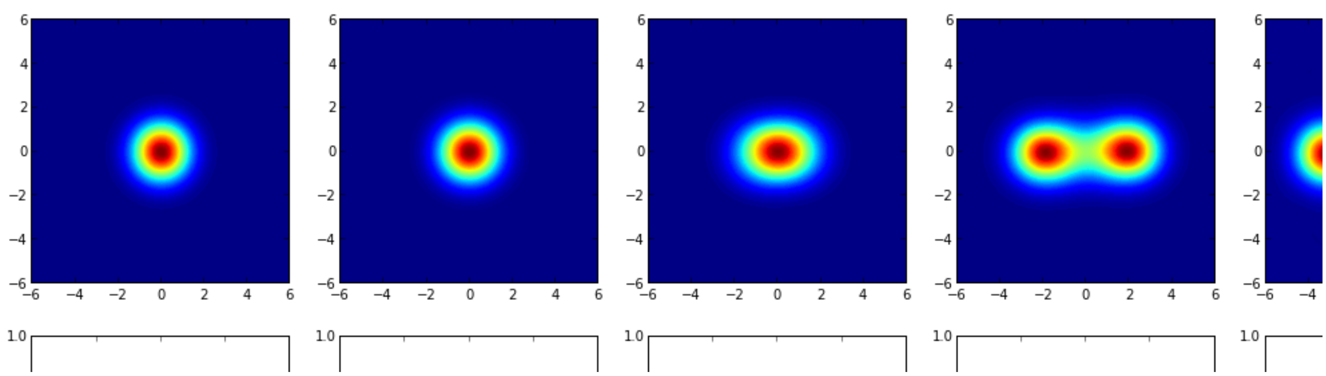
In [64]:
```
# calculate the expectation value of the number of photons in the cavity
n_ss_vec = expect(a.dag() * a, rho_ss_list)
```

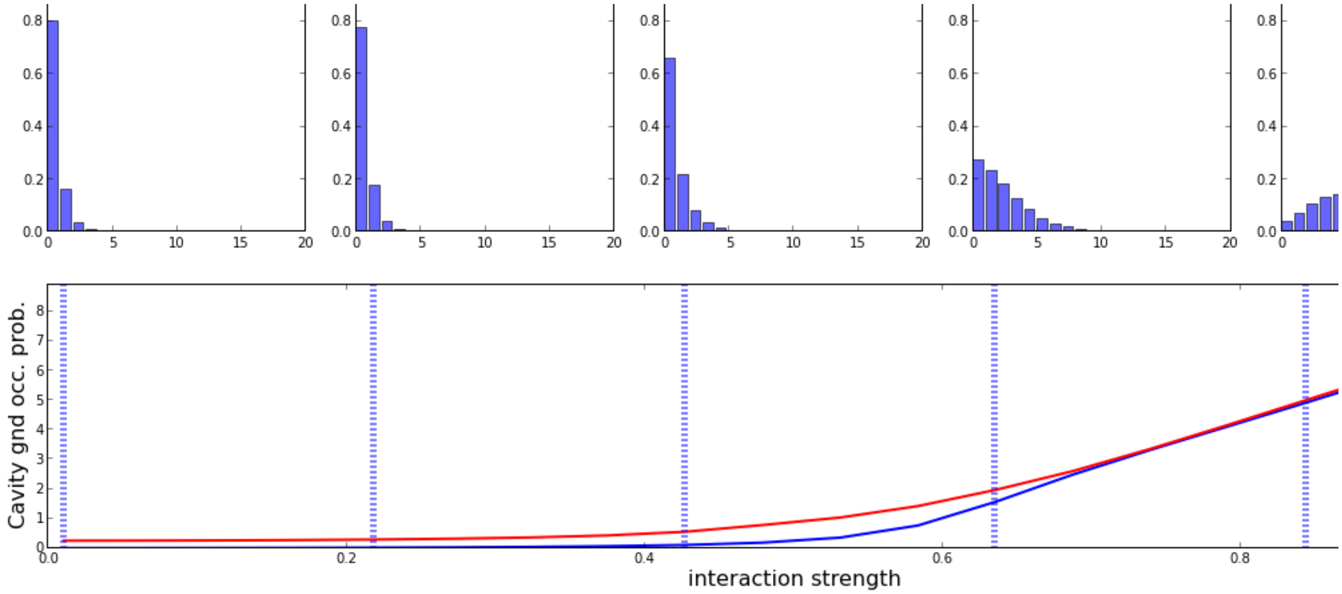
In [65]:
```
fig, axes = subplots(1, 1, sharex=True, figsize=(8,4))

axes.plot(g_vec, n_gnd_vec,'b', linewidth=2, label="cavity groundstate")
axes.plot(g_vec, n_ss_vec, 'r', linewidth=2, label="cavity steadystate")
axes.set_ylim(0, max(n_ss_vec))
axes.set_ylabel("Cavity occ. prob.", fontsize=16)
axes.set_xlabel("interaction strength", fontsize=16)
axes.legend(loc=0)

fig.tight_layout()
```



Cavity Wigner function and Fock distribution as a function of coupling strength

In [66]:
```
rho_ss_sublist = rho_ss_list[::4]

xvec = linspace(-6,6,200)

fig_grid = (3, len(rho_ss_sublist))
fig = figure(figsize=(4*len(rho_ss_sublist),12))

for idx, rho_ss in enumerate(rho_ss_sublist):

    # trace out the cavity density matrix
    rho_ss_cavity = ptrace(rho_ss, 0)

    # calculate its wigner function
    W = wigner(rho_ss_cavity, xvec, xvec)

    # plot its wigner function
    ax = subplot2grid(fig_grid, (0, idx))
    ax.contourf(xvec, xvec, W, 100)

    # plot its fock-state distribution
    ax = subplot2grid(fig_grid, (1, idx))
    ax.bar(arange(0, M), real(rho_ss_cavity.diag()), color="blue", alpha=0.6)
    ax.set_ylim(0, 1)

# plot the cavity occupation probability in the ground state
ax = subplot2grid(fig_grid, (2, 0), colspan=fig_grid[1])
ax.plot(g_vec, n_gnd_vec,'b', linewidth=2, label="cavity groundstate")
ax.plot(g_vec, n_ss_vec, 'r', linewidth=2, label="cavity steadystate")
ax.set_xlim(0, max(g_vec))
ax.set_ylim(0, max(n_ss_vec)*1.2)
ax.set_ylabel("Cavity gnd occ. prob.", fontsize=16)
ax.set_xlabel("interaction strength", fontsize=16)

for g in g_vec[::4]:
    ax.plot([g,g],[0,max(n_ss_vec)*1.2], 'b:', linewidth=5)
```

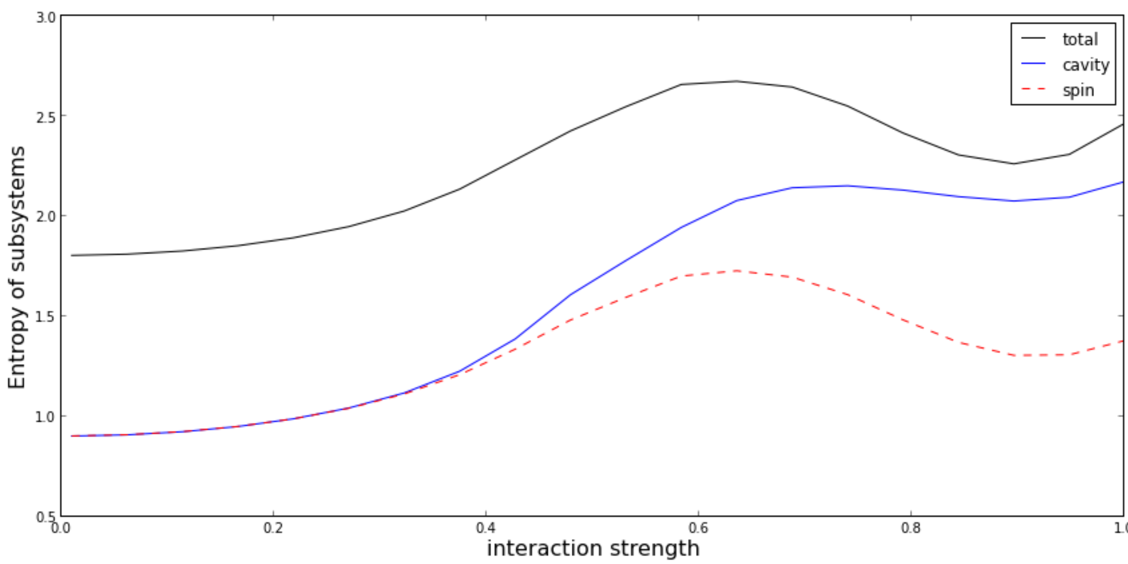




Entropy

In [67]:
```
entropy_tot    = zeros(shape(g_vec))
entropy_cavity = zeros(shape(g_vec))
entropy_spin   = zeros(shape(g_vec))

for idx, rho_ss in enumerate(rho_ss_list):

    rho_gnd_cavity = ptrace(rho_ss, 0)
    rho_gnd_spin   = ptrace(rho_ss, 1)

    entropy_tot[idx]    = entropy_vn(rho_ss, 2)
    entropy_cavity[idx] = entropy_vn(rho_gnd_cavity, 2)
    entropy_spin[idx]   = entropy_vn(rho_gnd_spin, 2)
```

In [68]:
```
fig, axes = subplots(1, 1, figsize=(12,6))
axes.plot(g_vec, entropy_tot, 'k', label="total")
axes.plot(g_vec, entropy_cavity, 'b', label="cavity")
axes.plot(g_vec, entropy_spin, 'r--', label="spin")

#axes.set_ylim(0, 1.5)
axes.set_ylabel("Entropy of subsystems", fontsize=16)
axes.set_xlabel("interaction strength", fontsize=16)
axes.legend(loc=0)
fig.tight_layout()
```

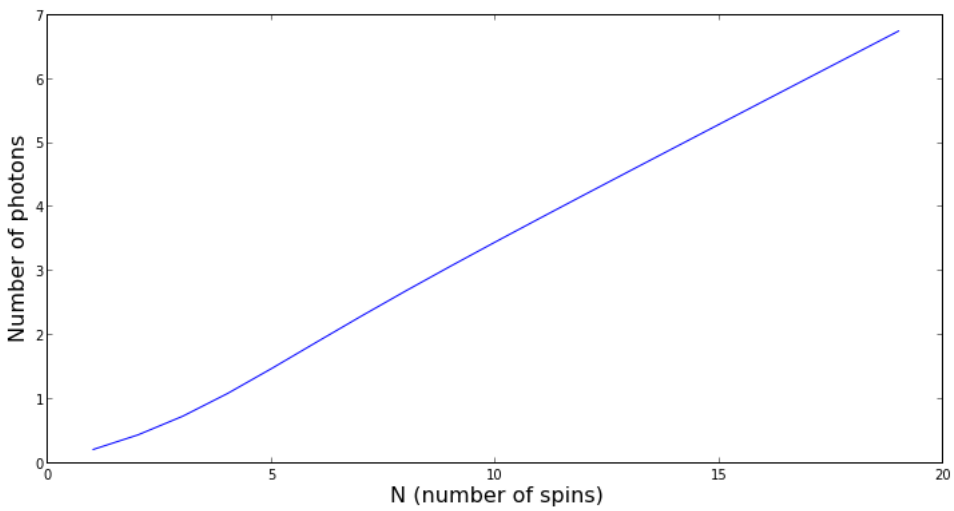


Superradiance ?

start with all atoms in the excited state

In [69]:
```
def calulcate_intensity(M, N, g):

    j = N/2.0
    n = 2*j + 1

    # setup the hamiltonian for the requested hilbert space sizes
    a  = tensor(destroy(M), qeye(n))
    Jp = tensor(qeye(M), jmat(j, '+'))
    Jm = tensor(qeye(M), jmat(j, '-'))
    Jz = tensor(qeye(M), jmat(j, 'z'))

    H0 = w * a.dag() * a + w0 * Jz
    H1 = 1.0 / sqrt(N) * (a + a.dag()) * (Jp + Jm)

    # Ground state and steady state for the Hamiltonian: H = H0 + g * H1
    psi_gnd = (H0 + g * H1).groundstate()[1]

    n_cavity = expect(a.dag() * a, psi_gnd)

    return n_cavity
```

In [71]:
```
N_vec = arange(1, 20, 1)
MM = 25
a

fig, axes = subplots(1, 1, figsize=(12,6))

n_cavity = [calulcate_intensity(MM, NN, 0.7) for NN in N_vec]

axes.plot(N_vec, n_cavity, 'b')

axes.set_ylabel("Number of photons", fontsize=16)
axes.set_xlabel("N (number of spins)", fontsize=16)
axes.legend()
```



In [72]:
```
def evolve_system(M, N, g, tlist):

    j = N/2.0
    n = 2*j + 1

    # setup the hamiltonian for the requested hilbert space sizes
    a  = tensor(destroy(M), qeye(n))
    Jp = tensor(qeye(M), jmat(j, '+'))
    Jm = tensor(qeye(M), jmat(j, '-'))
    Jz = tensor(qeye(M), jmat(j, 'z'))

    H0 = w * a.dag() * a + w0 * Jz
    H1 = 1.0 / sqrt(N) * (a + a.dag()) * (Jp + Jm)

    # Ground state and steady state for the Hamiltonian: H = H0 + g * H1
    N = H0 + g * H1

    psi0 = tensor(basis(M, 0), basis(int(n),int(n-1)))

    res = mesolve(H0 + g * H1, psi0, tlist, [], [a.dag() * a])
```

```
    return res.expect[0]
```

In [73]:
```
N_vec = arange(1, 20, 1)
MM = 45

g = 0.8
tlist = linspace(0.0, 10.0, 100)

fig, axes = subplots(1, 2, figsize=(16,6))

n_cavity_max = []

for NN in N_vec:
    n_cavity = evolve_system(MM, NN, g, tlist)
    n_cavity_max.append(max(n_cavity))
    axes[0].plot(tlist, n_cavity, linewidth=2, label="N = %d" % NN)

axes[0].set_ylabel("Photons in the cavity", fontsize=16)
axes[0].set_xlabel("time", fontsize=16)

axes[1].plot(N_vec, n_cavity_max, linewidth=2)
```

Out [73]:
```
[<matplotlib.lines.Line2D at 0x53937ad0>]
```