11-积分变换法

积分变换

D'Alembert 公式/弦振动方程

$$\begin{cases} u_{tt} - a^2 u_{xx} = f(t,x), -\infty < x < \infty, t > 0 \\ u|_{t=0} = \phi(x), u_t|_{t=0} = \psi(x) \end{cases}$$

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结论

$$u(x,t) = \frac{1}{2}[\phi(x-at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi + \frac{1}{2a}\int_0^t \int_{x-a(t-\tau)}^{x+a(t-\tau)} f(\xi,\tau)d\xi d\tau$$

两端 $\mathcal{F}$

$$\widehat{U}(t,\omega) = \mathcal{F}_x[u(t,x)] = \int_{-\infty}^{+\infty} u(t,x)e^{-i\omega x}dx$$

$$\mathcal{F}_x[u_{tt}] = \partial_{tt}\mathcal{F}_x[u] = \partial_{tt}\widehat{U}$$

$$\mathcal{F}_x[u_{xx}] = (i\omega)^2\widehat{U}$$

$$\partial_{tt}\widehat{U} + a^2\omega^2\widehat{U} = \widehat{f}$$

两端 $\mathcal{L}$

两端 $\mathcal{L}^{-1}$

两端 $\mathcal{F}^{-1}$

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, -\infty < x < \infty, t > 0 \\ u|_{t=0} = \phi(x), u_t|_{t=0} = \psi(x) \end{cases}$$

$$u(x,t) = \frac{1}{2}[\phi(x-at) + \phi(x+at)] + \frac{1}{2a}\int_{x-at}^{x+at} \psi(\xi)d\xi$$

Possion 方程/热传导方程

$$\begin{cases} u_t - a^2 u_{xx} = f(t,x), -\infty < x < \infty, t > 0 \\ u|_{t=0} = \phi(x) \end{cases}$$

$$\begin{cases} u_t - a^2 u_{xx} = 0, -\infty < x < \infty, t > 0 \\ u|_{t=0} = \phi(x) \end{cases}$$

结论

$$u(t,x)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{+\infty}\phi(x-y)e^{-\frac{y^2}{4a^2t}}dy$$

两端 $\mathcal{F}$

两端 $\mathcal{L}$

两端 $\mathcal{L}^{-1}$

两端 $\mathcal{F}^{-1}$

$$I(b)=\int_0^\infty e^{-\eta^2}cos(2b\eta)d\eta$$

求导

$$\begin{aligned}
I'(b)&=\int_0^\infty e^{-\eta^2}sin(2b\eta)(-2\eta)d\eta\\
&=\int_0^\infty sin(2b\eta)d(e^{-\eta^2})\\
&=\sin(2b\eta)e^{-\eta^2}|_0^\infty-\int_0^\infty e^{-\eta^2}(2b)\cos(2b\eta)d\eta\\
&=-2bI(b)
\end{aligned}$$

解得

$$I(b)=I(0)e^{-b^2}$$

$$I(0)=\int_0^\infty e^{-\eta^2}d\eta=\frac{\sqrt{\pi}}{2}$$