Leibniz Integral Rule

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In calculus, the Leibniz integral rule for differentiation under the integral sign states that for an integral of the form

$$\int_{a(x)}^{b(x)} f(x,t)dt \tag{1}$$

where $|a(x), b(x)| > \infty$ and the integrands are functions dependent on x, the derivative of this integral is expressible as

$$\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt. \tag{2}$$

If $a(x) = a$ is constant and $b(x) = x$, which is another common situation (for example, in the proof of Cauchy's repeated integration formula), the Leibniz integral rule becomes:

$$\frac{d}{dx}\left(\int_a^x f(x,t)\,dt\right) = f(x,x) + \int_a^x \frac{\partial}{\partial x} f(x,t)\,dt \tag{3}$$

General form: differentiation under the integral sign

Theorem — Let $f(x,t)$ be a function such that both $f(x,t)$ and its partial derivative $f_x(x,t)$ are continuous in t and x in some region of the xt-plane, including $a(x) \le t \le b(x)$, $x_o \le x \le x_1$. .Also suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_o \le x \le x_1$. Then for $x_o \le x \le x_1$,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x, b(x))\,b'(x) - f(x, a(x))\,a'(x) + \int_{a(x)}^{b(x)} f_x(x,t)\,dt. \tag{4}$$

[0] https://en.wikipedia.org/wiki/Leibniz_integral_rule