LAGRANGE MULTIPLIERS

Say we want to optimize $f(x,y,z)$ along path $g(x,y,z)=k$

method of lagrange multipliers

1. solve the system of eqns,
   $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ → i.e. when gradient of f & g pointing in same dir
   $g(x,y,z)=k$

2. plug in all solutions, (x,y,z) into $f(x,y,z)$ and find min/max, provided they exist & $\nabla g \neq \vec{0}$ at the pt.

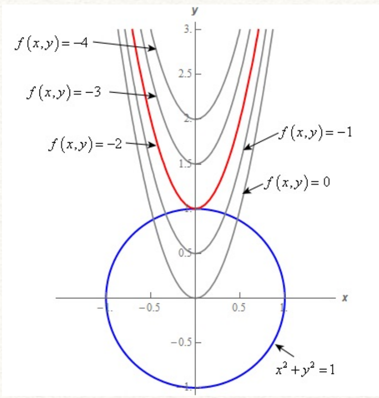
$$\langle f_x, f_y, f_z \rangle = \lambda \langle g_x, g_y, g_z \rangle = \langle \lambda g_x, \lambda g_y, \lambda g_z \rangle$$

$$\left.\begin{aligned} f_x &= \lambda g_x \\ f_y &= \lambda g_y \\ f_z &= \lambda g_z \end{aligned}\right\}$$ actually three equations w/ 4 unknowns $[x, y, z, \lambda]$.

- max/min won't always exist

<u>intuition</u>

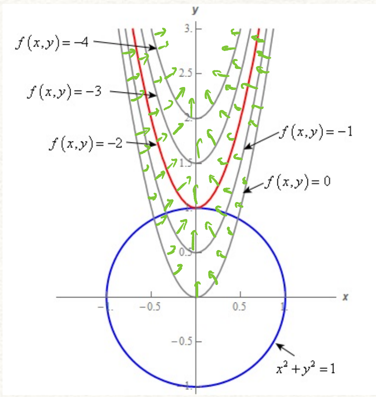
consider min/max values for $f(x,y) = 8x^2 - 2y$ w/ constraint $x^2+y^2=1$.



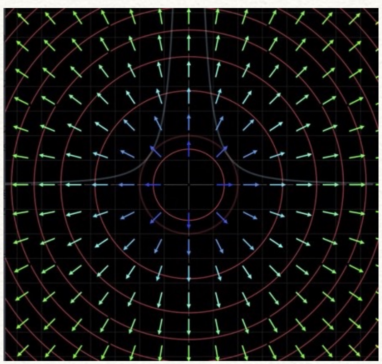
- the solutions must be somewhere on the graph of the constraint, $x^2+y^2=1$
- since we are looking for max/min of $f(x,y)$, intersection must take place at min/max of $f(x,y)=k$.
  ↳ i.e. when does a global min/max of $f(x,y)=k$ intersect with constraint?

- because $x^2+y^2=1$ is a constraint on x and y, z is constant so we can only consider the xy-plane.
  - ↳ think about sweeping thru vertical cross sections and finding min/max that still intersect.
- in the example above, we say: what is min value of k such that the constraint still holds?
- from the graph, min of $f(x,y) = -2$ just intersects/is tangent with constraint.

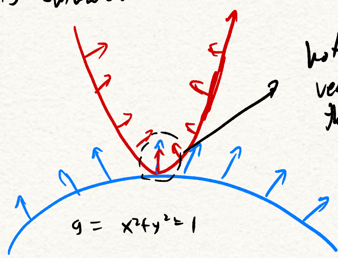
How do we actually find this mathematically?



- green represents the gradient field
  - ↳ always $\perp$ to contour lines
    - ↳ walking along contour lines is always const. elev. so makes sense to walk $\perp$ (no component is useless, max incr.)



- similarly contour lines for $x^2+y^2=1$ are also pointing outwards.



notice that gradient vectors are // at the point of tangency!!

- if this point is (x_m, y_m),

$$\boxed{\nabla f(x_m, y_m) = \underbrace{\lambda}_{\text{lagrange multiplier}} \nabla g(x_m, y_m)}$$

$$\nabla f = \nabla(8x^2-2y) = \begin{bmatrix} 16x \\ -2 \end{bmatrix}$$

$$\nabla g = \nabla(x^2+y^2) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

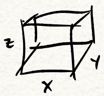
$$\Rightarrow \begin{cases} 16x = \lambda 2x \\ -2 = \lambda 2y \\ x^2+y^2=1 \end{cases} \Rightarrow \begin{array}{l} \lambda = 8 \text{ or } x=0 \\ -2 = 16y \Rightarrow \boxed{y=-\frac{1}{8}} \\ x^2+\frac{1}{64}=1 \Rightarrow \boxed{x = \pm\frac{3\sqrt{7}}{8}} \\ \text{if } \boxed{x=0,\ y=1.} \end{array}$$

we therefore have three "critical" points, which we plug into $f(x,y) = 8x^2 - 2y$

$$\left(\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) \Rightarrow f = 8.125$$
$$\left(-\frac{3\sqrt{7}}{8}, -\frac{1}{8}\right) \Rightarrow f = 8.125$$
max

$$(0, 1) \Rightarrow f = -2$$
min

<u>ex.</u> find dimensions of box w/ largest volume if total SA is 64 cm²



maximize $V = xyz$

constraint: $2xy + 2xz + 2yz = 64$

$\Rightarrow xy + xz + yz = 32$

$$\nabla V = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} \qquad \nabla A = \begin{bmatrix} y+z \\ x+z \\ x+y \end{bmatrix}$$

$\nabla V = \lambda \nabla A$

$$\Rightarrow \left.\begin{aligned} yz &= \lambda(y+z) \\ xz &= \lambda(x+z) \\ xy &= \lambda(x+y) \\ xy+xz+yz &= 32 \end{aligned}\right\}$$

$$\lambda = \frac{yz}{y+z} = \frac{xz}{x+z} = \frac{xy}{x+y}$$

$\Rightarrow y = x$

$\Rightarrow x = z$

$x^2 + x^2 + x^2 = 32$

$3x^2 = 32$

$x^2 = \sqrt{\frac{32}{3}}$

$x = y = z = 4\sqrt{\frac{2}{3}}$

<u>ex.</u> $f(x,y) = 5x - 3y \quad , \; x^2 + y^2 = 136$

$$\nabla f = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \qquad \nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$$\left.\begin{aligned} 5 &= \lambda 2x \\ -3 &= \lambda 2y \\ x^2+y^2 &= 136 \end{aligned}\right\}$$

$x = \frac{5}{2\lambda}$

$y = \frac{-3}{2\lambda}$

$\frac{25}{4\lambda^2} + \frac{9}{4\lambda^2} = 136$

$\frac{17}{2\lambda^2} = 136$

$17 = 272\lambda^2$

$\lambda = \pm\frac{1}{4}$

$\lambda = \frac{1}{4}$:
$10 = x$
$-6 = y$
$\Rightarrow (10, -6)$

$\lambda = -\frac{1}{4}$:
$-10 = x$
$6 = y$
$\Rightarrow (-10, 6)$

$f(10,-6) = 68 \rightarrow$ max

$f(-10,6) = -68 \rightarrow$ min

ex. $f(x,y,z) = xyz$, $x+y+z=1$. $x,y,z \geq 0$.

$$\nabla f = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix} \quad \nabla g = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \left.\begin{array}{l} yz = \lambda \\ xz = \lambda \\ xy = \lambda \\ x+y+z=1 \end{array}\right\}$$

consider $yz = xz$

$\Rightarrow z=0$ or $x=y$

$z=0$: $x=0 \Rightarrow y=1$, $(0,1,0)$; $y=0 \Rightarrow x=1$, $(1,0,0)$

$x=y$: $xz=xy$: $x=0 \Rightarrow z=1$, $(0,0,1)$; $z=y \Rightarrow x=y=z=\frac{1}{3}$, $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$

★ we used constraint to ensure closed boundary.

ex. $(-100, 100, 1) \Rightarrow f(-100,100,1) = -10000$

$(-50,-50,101) \Rightarrow f(-50,-50,101) = 252500$

$f(0,1,0) = f(1,0,0) = f(0,0,1) = 0$ (min)

$f(\frac{1}{3},\frac{1}{3},\frac{1}{3}) = \frac{1}{27}$ (max)

- we found abs. min but not all locations for abs. min.

consider, for $x,y,z \geq 0$,

$$\left.\begin{array}{l} (0,y,z), \ y+z=1 \\ (x,0,z), \ x+z=1 \\ (0,y,z), \ y+z=1 \end{array}\right\} \text{infinitely many!}$$

- lagrange method only finds values, not all possible locations.

ex. find max/min for $f(x,y) = 4x^2+10y^2$ on disk $x^2+y^2 \leq 4$

① get crit pts!

$f_x = 8x \Rightarrow 8x=0 \Rightarrow x=0$

$f_y = 20y \Rightarrow 20y=0 \Rightarrow y=0$

$(0,0)$ satisfies the inequality

★ we need crit pts because we're not checking for intersection; we want min values when lagrange multiplier doesn't work.

$$\nabla f = \begin{bmatrix} 8x \\ 20y \end{bmatrix} \quad \nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$$

$8x = \lambda 2x \Rightarrow x=0$ or $\lambda = 4$

$20y = \lambda 2y$

$x^2+y^2=4$ ← treat as equality

if $x=0$: $y = \pm 2$

if $\lambda=4$: $20y = 8y \Rightarrow y=0$, $x=\pm 2$

$(0,2)$, $(0,-2)$, $(2,0)$, $(-2,0)$, $(0,0)$

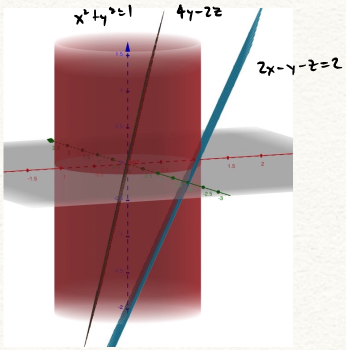
⇓

$f=40$ (max), $f=40$, $f=16$, $f=16$, $f=0$ (min) what if more than 1 constraint?

- optimize $f(x,y,z)$ with constraints $g(x,y,z)=c$ and $h(x,y,z)=k$,

$$\boxed{\begin{aligned} &\text{Solve} \quad \nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ &\qquad g(x,y,z) = c \\ &\qquad h(x,y,z) = k \end{aligned}}$$

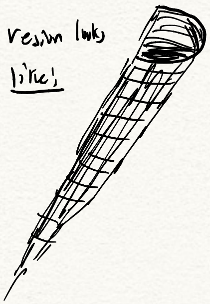
ex. find min/max of $f(x,y,z) = 4y-2z$ w/ constraint $2x-y-z=2$, $x^2+y^2=1$



$$\nabla f = \begin{bmatrix} 0 \\ 4 \\ -2 \end{bmatrix} \quad \nabla g = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix} \quad \nabla h = \begin{bmatrix} 2x \\ 2y \\ 0 \end{bmatrix}$$

$$\begin{cases} 0 = 2\lambda + 2\mu x & x = -\frac{2}{\mu} \quad ① \\ 4 = -\lambda + 2\mu y & y = \frac{3}{\mu} \\ -2 = -\lambda & \Rightarrow \lambda = 2 \\ 2x-y-z=2 \\ x^2+y^2=1 \end{cases}$$

region looks like:



① $\frac{4}{\mu^2} + \frac{9}{\mu^2} = 1$

$\mu = \pm\sqrt{13}$

$\mu = \sqrt{13}$:	$\mu = -\sqrt{13}$:
$x = -\frac{2}{\sqrt{13}}$	$x = \frac{2}{\sqrt{13}}$
$y = \frac{3}{\sqrt{13}}$	$y = -\frac{3}{\sqrt{13}}$
$z = -2 - \frac{7}{\sqrt{13}}$	$z = -2 + \frac{7}{\sqrt{13}}$

$f\left(-\frac{2}{\sqrt{13}}, \frac{3}{\sqrt{13}}, -2-\frac{7}{\sqrt{13}}\right) = 11.2111 \Rightarrow$ max

$f\left(\frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}, -2+\frac{7}{\sqrt{13}}\right) = -3.2111 \Rightarrow$ min