ITERATED INTEGRALS

fubini's theorem

if $f(x,y)$ is continuous on $R = [a,b] \times [c,d]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_c^d f(x,y)\, dy\, dx = \int_c^d \int_a^b f(x,y)\, dx\, dy$$

ex.

a) $\iint_R 6xy^2\, dA$, $R = [2,4] \times [1,2]$

$= \int_2^4 \int_1^2 6xy^2\, dy\, dx$

$= \int_2^4 2xy^3 \Big|_1^2\, dx$

$= \int_2^4 16x - 2x\, dx$

$= \int_2^4 14x\, dx$

$= 7x^2 \Big|_2^4$

$= 7(16-4)$

$= 84$

b) $\iint_R 2x - 4y^3\, dA$, $R = [-5,4] \times [0,3]$

$= \int_{-5}^4 \int_0^3 2x - 4y^3\, dy\, dx$

$= \int_{-5}^4 2xy - y^4 \Big|_0^3\, dx$

$= \int_{-5}^4 6x - 81\, dx$

$= 3x^2 - 81x \Big|_{-5}^4$

$= -276 - 480$

$= -756$

c) $\iint_R x^2y^2 + \cos(\pi x) + \sin(\pi y)\, dA$, $R = [-2,-1] \times [0,1]$

$$= \int_{-2}^{-1}\int_0^1 x^2y^2 + \cos(\pi x) + \sin(\pi y)\, dy\,dx$$

$$= \int_{-2}^{-1} \frac{1}{3}x^2y^3 + \cos(\pi x)\, y - \frac{1}{\pi}\cos(\pi y) \Big|_0^1 \, dx$$

$$= \int_{-2}^{-1} \frac{1}{3}x^2 + \cos(\pi x) + \frac{1}{\pi} + \frac{1}{\pi}\, dx$$

$$= \frac{1}{9}x^3 + \frac{1}{\pi}\sin(\pi x) + \frac{2}{\pi}x \Big|_{-2}^{-1}$$

$$= -\frac{1}{9} - \frac{2}{\pi} - \left(\frac{-8}{9} - \frac{4}{\pi}\right)$$

$$= \frac{7}{9} + \frac{2}{\pi}$$

$$= \int_0^1\int_{-2}^{-1} x^2y^2 + \cos(\pi x) + \sin(\pi y)\, dx\,dy$$

$$= \int_0^1 \frac{1}{3}x^3y^2 + \frac{1}{\pi}\sin(\pi x) + \sin(\pi y)\, x \Big|_{-2}^{-1} dy$$

$$= \int_0^1 -\frac{y^2}{3} - \sin(\pi y) - \left(\frac{-8}{3}y^2 - 2\sin(\pi y)\right) dy$$

$$= \int_0^1 \frac{7}{3}y^2 + \sin(\pi y)\, dy$$

$$= \frac{7}{9}y^3 - \frac{1}{\pi}\cos(\pi y) \Big|_0^1$$

$$= \frac{7}{9} + \frac{1}{\pi} - \left(-\frac{1}{\pi}\right)$$

$$= \frac{7}{9} + \frac{2}{\pi}$$

d) $\iint_R \frac{1}{(2x+3y)^2}\, dA$, $R = [0,1] \times [1,2]$

$$= \int_1^2\int_0^1 \frac{1}{(2x+3y)^2}\, dx\,dy$$

$$= \int_1^2\int_0^1 (2x+3y)^{-2}\, dx\,dy$$

$$= \int_1^2 -\frac{1}{2(2x+3y)} \Big|_0^1 dy$$

$$= -\frac{1}{2}\int_1^2 \frac{1}{2+3y} - \frac{1}{3y}\, dy$$

$$= -\frac{1}{2}\left(\frac{1}{3}\ln|2+3y| - \frac{1}{3}\ln|y|\right)\Big|_1^2$$

$$= -\frac{1}{6}(\ln 8 - \ln 2 - \ln 5)$$

$$= -\frac{1}{6}(\ln 4 - \ln 5)$$

e) $\iint_R x e^{xy}\, dA$, $R = [-1,2] \times [0,1]$

$$= \int_{-1}^2\int_0^1 x e^{xy}\, dy\,dx$$

* integrating by y significantly easier

$$= \int_{-1}^2 e^{xy}\Big|_0^1 dy$$

$$= \int_{-1}^2 e^y - 1\, dy$$

$$= e^y - y \Big|_{-1}^2$$

$$= e^2 - 2 - (e^{-1} + 1)$$

$$= e^2 - e^{-1} - 3$$

if we can break up the function, we can multiply the parts.

let $f(x) = g(x)\,h(y)$ and we integrate over $R = [a,b] \times [c,d]$.

$$\iint_R f(x,y)\,dA = \iint_R g(x)\,h(y)\,dA = \int_c^d \int_a^b g(x)\,h(y)\,dx\,dy$$

↳ if we integrate w/ respect to x first, we can factor out $h(y)$ (constant)

$$= \int_c^d h(y) \int_a^b g(x)\,dx\,dy$$

if we then integrate w/ respect to y, we can factor out the inner $\int$.

$$= \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy$$

$$\boxed{\text{Thus, if } f = g(x)\,h(y), \text{ then, } \iint_R f(x,y)\,dA = \iint_R g(x)\,h(y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right)}$$

ex.

$$\iint_R x\cos^2(y)\,dA, \quad R = [-2,3] \times [0, \tfrac{\pi}{2}]$$

$$= \int_{-2}^{3} x\,dx \cdot \int_0^{\pi/2} \cos^2(y)\,dy$$

$$= \frac{1}{2}x^2\Big|_{-2}^{3} \cdot \frac{1}{2}\int_0^{\pi/2} 1 + \cos 2y\,dy$$

$$= \frac{5}{2} \cdot \frac{1}{2}\left(y + \frac{1}{2}\sin 2y\right)\Big|_0^{\pi/2} = \frac{5}{4}\left(\frac{\pi}{2}\right) = \frac{5\pi}{8}$$