## absolute vs. conditional convergence

- if $\sum a_n$ and $\sum |a_n|$ converge, absolute convergence

  if $\sum a_n$ converges to $s$, any rearrangement also has value $s$.

- if $\sum a_n$ converges and $\sum |a_n|$ diverges, conditional convergence

  if $\sum a_n$ is conditionally convergent,
  $\forall r \in \mathbb{R}$, $\exists$ an arrangement of $\sum a_n$ s.t. $\sum a_n = r$.

**ex.** consider $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots = \ln 2$ ①

- multiplying by $\frac{1}{2}$, $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} \dots = \frac{1}{2} \ln 2$
- adding zeroes, $0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} = \frac{1}{2} \ln 2$ ②
- adding ① and ②, $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} - \dots = \frac{3}{2} \ln 2$
- since rearranging gives different value, conditionally convergent

## divergence testing

$\boxed{\text{1. if } \lim_{n\to\infty} a_n \neq 0, \ \sum a_n \text{ will diverge}}$

→ however if $\lim_{n\to\infty} a_n = 0$, this tells you nothing!

**ex.**

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverge

$\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

most of the time, if $a_n \to 0$ faster than $\frac{1}{n}$, converges,
else, diverges.

**ex.**

$$\sum_{n=0}^{\infty} \frac{4n^2 - n^3}{10 + 2n^3}$$

$$\Rightarrow \lim_{n\to\infty} \frac{\cancel{n^3}\left(\frac{4}{n} - 1\right)}{\cancel{n^3}\left(\frac{10}{n^3} + 2\right)} = -\frac{1}{2}, \text{ this series diverges.}$$

## 2. geometric series converge if $|r|<1$, diverge otherwise.

if converge, $\sum_{n=0}^{\infty} a_1 r^n = \frac{a}{1-r}$

ex)

$$\sum_{n=1}^{\infty} 9^{-n+2} 4^{n+1}$$

let $n = i+1$

$$= \sum_{i=0}^{\infty} 9^{-i+1} 4^{i+2}$$

$$= 16 \cdot 9 \sum_{i=0}^{\infty} \left(\frac{4}{9}\right)^i$$

$$= 144 \left(\frac{1}{1-4/9}\right)$$

$$= \frac{1296}{5}$$

ex)

$$\sum_{n=0}^{\infty} 9^{-n+2} 4^{n+1}$$

$$= \sum_{n=0}^{1} 9^{-n+2} 4^{n+1} + \frac{1296}{5}$$

$$= (9^2 \cdot 4) + \frac{1296}{5}$$

$$= \frac{2916}{5}$$

## 3. Telescoping series usually converge, but not always

ex)

$$\sum_{n=1}^{\infty} \left(\frac{1}{n^2+4n+3}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2} \sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3}\right) \quad \text{(pfd)}$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\dots+\left(\frac{1}{n}-\frac{1}{n+2}\right)+\left(\frac{1}{n+1}-\frac{1}{n+3}\right)\right]$$

consider 1st and 2nd of each pair:

1st: $\frac{1}{2} + \frac{1}{3} + \cancel{\frac{1}{4}} + \cancel{\frac{1}{5}} + \dots + \cancel{\frac{1}{n}} + \cancel{\frac{1}{n+1}}$

2nd: $-\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}} - \cancel{\frac{1}{6}} - \cancel{\frac{1}{7}} - \dots - \frac{1}{n+2} - \frac{1}{n+3}$

$$= \lim_{n\to\infty} \frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}-\frac{1}{n+2}-\frac{1}{n+3}\right)$$

$= 5/12$ converges.

ex.

$$\sum_{n=1}^{\infty} n-(n+1)$$

$$= \lim_{n\to\infty} \left[ (1-\cancel{2}) + (\cancel{2}-\cancel{3}) + (\cancel{3}-\cancel{4}) + \ldots + ((\cancel{N-1})-\cancel{N}) + (\cancel{N}-(N+1)) \right]$$

$$= \lim_{n\to\infty} (1-(n+1))$$

$$= \lim_{n\to\infty} -n$$

$$= -\infty$$

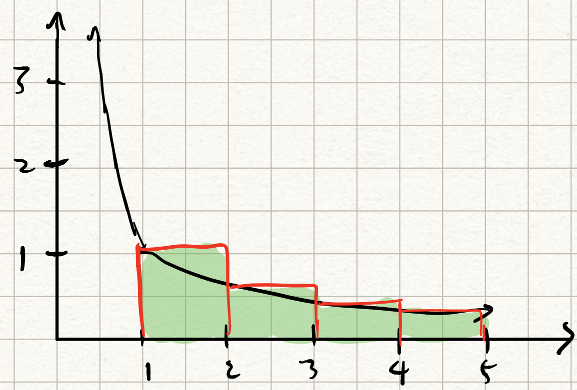
$\boxed{\text{4. The harmonic series, } \sum_{n=1}^{\infty} \frac{1}{n}, \text{ diverges}}$

$\boxed{\text{5. Integral test}}$

- To prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, consider area under $\frac{1}{x}$:

$$\int_1^{\infty} \frac{1}{x}\,dx = \infty$$

notice that ...




$$A \approx \left(\tfrac{1}{1}\right)(1) + \left(\tfrac{1}{2}\right)(1) + \ldots$$

$$= \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \ldots$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}$$

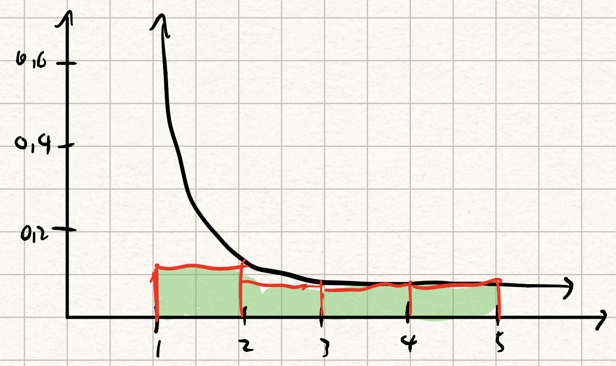
Since this approximation overestimates,

$$\sum_{n=1}^{\infty} \frac{1}{n} > \int_1^{\infty} \frac{1}{x}\,dx = \infty$$

thus, since $\int_1^\infty \frac{1}{x}\,dx = \infty$, $\sum_{n=1}^{\infty} \frac{1}{n} > \infty \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} = \infty \Rightarrow$ diverges.

- consider now $\sum_{n=1}^{\infty} \frac{1}{x^2}$

$$\int_1^\infty \frac{1}{x^2} = \left(-\frac{1}{x}\right)\Big|_1^\infty = 1$$




$$A \approx \left(\frac{1}{2^2}\right)(1) + \left(\frac{1}{3^2}\right)(1) + \dots$$

$$= \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots < 1 + \int_1^\infty \frac{1}{x^2}$$

$$< 2$$

→ this doesn't mean it's convergent since $-\infty < 2$.

since all terms are positive, this is an increasing series,

since increasing & bounded, it's convergent.

$$\boxed{\begin{array}{l}\text{if } f(x) \text{ is continuous, positive, and decreasing on } [k, \infty) \\ \bullet \text{ if } \int_k^\infty f(x)\,dx \text{ is convergent, so is } \sum_{n=k}^{\infty} a_n \\ \bullet \text{ if } \int_k^\infty f(x)\,dx \text{ is divergent, so is } \sum_{n=k}^{\infty} a_n.\end{array}}$$

ex.

a) $\sum_{n\geq 2}^{\infty} \frac{1}{n \ln n}$

$u = \ln n$
$du = \frac{1}{n} dn$

$\int_{\ln 2}^{\infty} \frac{1}{u} du = \infty$

so divergent.

b) $\sum_{n=0}^{\infty} n e^{-n^2} dn$

$u = -n^2$
$du = -2n\, dn$

$-\frac{1}{2}\int_0^{-\infty} e^u du = -\frac{1}{2}(0-1) = \frac{1}{2}$

so convergent.

# 6. Alternating Series Test

Alternating series: $a_n = (-1)^n b_n$ or $a_n = (-1)^{n+1} b_n$, where $b_n \geq 0$

ex.

$(-1)^{n-1} = (-1)^{n+1}(-1)^2 = (-1)^{n+2}$

$(-1)^{n-1} \frac{2}{n^{5/3}} = (-1)^{n+1} \frac{2}{n^{5/3}}$ — $b_n$

if we have an alternating series, if

1. $\lim_{n\to\infty} b_n = 0$ and
2. $\{b_n\}$ is decreasing, then

$\sum a_n$ is convergent.

We can estimate the value using

$|s - s_n| \leq |s_{n+1} - s_n| = b_{n+1}$

$|R_n| = |s - S_n| \le b_{n+1}$

**ex.** Using $n = 15$, estimate $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

$$S_{15} = \sum_{n=1}^{15} \frac{(-1)^n}{n^2} = -0.8245417574$$

$$|R_{15}| = |s - S_{15}| \le b_{16} = \frac{1}{16^2} = 0.00390625$$

So our estimation will be $-0.8245417574 \pm 0.0390625$

**ex.** estimate $\sum_{n=1}^{\infty} \frac{(-1)^{(n+3)}}{n!}$ to 4 decimal places.

$$|R_n| = |s - S_n| \le b_{n+1} = \frac{1}{(n+1)!}$$

to guarantee accuracy to 4 decimal places,

$$\frac{1}{(n+1)!} < 5 \times 10^{-5} \Rightarrow (n+1)! > 20000$$

$7! = 5040$, $8! = 40230$, so let $n = 7$.

$$s \approx S_7 = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} + \frac{1}{7!} = 0.6321$$

## 7. p-series test

if $k > 0$ then $\sum_{n=k}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

**ex.**

$\sum_{n=4}^{\infty} \frac{1}{n^7}$ converges

$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges

## 8. comparison test

consider $\sum_{n=0}^{\infty} \frac{1}{3^n+n}$

↳ we can't use integral test since we need to eval. $\int_0^\infty \frac{1}{3^x+x}\,dx.$

notice that $\frac{1}{3^n+n} < \frac{1}{3^n}$ for all n,

$$\sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1-1/3} = \frac{3}{2}$$

thus, $\sum_{n=0}^{\infty} \frac{1}{3^n+n}$ is convergent.

if we have two series, $\sum a_n$, $\sum b_n$ and $\forall n$, $a_n, b_n \geq 0$ and $a_n < b_n$,

1. if $\sum b_n$ is convergent, so is $\sum a_n$.
2. if $\sum a_n$ is divergent, so is $\sum b_n$

ex.

a) $\sum_{n=1}^{\infty} \frac{n}{n^2-\cos^2(n)} \geq \frac{1}{n}$

So diverges, since $1/n$ diverges.

b) $\sum_{n=1}^{\infty} \frac{e^{-n}}{n+\cos^2(n)} \leq \frac{e^{-n}}{n} \leq \frac{e^{-n}}{1}$

So this series converges.

c) $\sum_{n=1}^{\infty} \frac{n^2+2}{n^4+5} = \sum \frac{n^2}{n^4+5} + \sum \frac{2}{n^4+5}$

$< \frac{n^2}{n^4} = \frac{1}{n^2}$ $\quad < \frac{2}{n^4} = 2 \cdot \frac{1}{n^4}$

converges $\quad$ converges.

thus converges.

## 9. Limit Comparison Test

given two series, $a_n, b_n \geq 0$ for all $n$,

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

if $L > 0$ and finite, then either both converge or both diverge.

↳ whether $a_n$ or $b_n$ is on top doesn't matter. Reciprocal also $> 0$ and finite.

<u>ex.</u> $\sum_{n=0}^{\infty} \frac{1}{3^n - n}$

let $b_n = \frac{1}{3^n}$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{3^n}\right)}{\left(\frac{1}{3^n - n}\right)} = \lim_{n\to\infty} \frac{3^n - n}{3^n} = 1 + \frac{n}{3^n} = 1$$

so the series converges.

## 10. Ratio Comparison Test

For $\sum a_n$, define

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

- if $L < 1$ absolutely convergent
- if $L > 1$ divergent
- if $L = 1$ divergent, absolutely convergent, or conditionally convergent.

<u>ex.</u> $\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$

$$\lim_{n\to\infty} \frac{\left(\frac{10^{n+1}}{4^{2n+3}(n+2)}\right)}{\left(\frac{10^n}{4^{2n+1}(n+1)}\right)} = \lim_{n\to\infty} 10 \frac{\cancel{4^{2n}} \cdot \cancel{4} (n+1)}{\cancel{4^{2n}} \cdot 4^{\cancel{3}\,2} (n+2)} = \frac{5}{8} \left(\lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n+2}} + \frac{1}{\cancel{n+2}}\right)$$

so absolutely convergent