# TAYLOR'S INEQUALITY

The error in a $n^{th}$-order Taylor polynomial $P_{n,x_0}(x)$ as approx to $f(x)$

$$|R_n(x)| \le K \frac{|x-x_0|^{n+1}}{(n+1)!}$$

where $|f^{(n+1)}(z)| < K$ for all value in $[x_0, x]$

ex, estimate $\cos(\frac{1}{2})$ using $4^{th}$-order Maclaurin Polynomial

Give upper bound on magnitude of error.

$f(x) = \cos x, \quad f(0) = 1$

$f'(x) = -\sin x, \quad f'(0) = 0$

$f''(x) = -\cos x \quad f''(0) = -1$

$f^{(3)}(x) = \sin x \quad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x \quad f^{(4)}(0) = 1$

$$\cos x = P_{4,0}(x)$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$\cos(\tfrac{1}{2}) \approx 1 - \frac{(1/2)^2}{2} + \frac{(1/2)^4}{24}$$

$$\approx \frac{337}{384}$$

$$\approx 0.8776$$

$f^{(5)}(x) = -\sin(x)$

how large is $|\sin(x)|$ on $(0, \frac{1}{2})$?

$$K = \sin(\tfrac{1}{2}) \le 1$$

$$|\text{error}| \le \frac{|x-0|^5}{5!} = \frac{x^5}{5!} = \frac{1}{3840}$$

So $\cos(\frac{1}{2}) = \frac{337}{384} \pm \frac{1}{3840}$

$\star$ $P_{4,0}(x) = P_{5,0}(x)$ and $f^{(6)}x = -\cos x$, and $|-\cos x| \le 1$, so

$$|\text{error}| \le \frac{x^6}{6!} = \frac{1}{46{,}080} \le 10^{-4}, \text{ so}$$

$\cos(\frac{1}{2}) = 0.8776$ to 4 d.p.

ex. estimate value of $\sqrt{5}$, using $P_{3,4}(x)$ for $f(x) = \sqrt{x}$.

$f(x) = \sqrt{x}$, $\quad f(4) = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$

$\quad f'(4) = 1/4$

$f''(x) = -\frac{1}{4(x^{3/2})} \quad f''(4) = -\frac{1}{32}$

$f'''(x) = \frac{3}{8x^{5/2}} \quad f'''(4) = \frac{3}{256}$

$$P_{3,4}(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

$$P_{3,4}(5) = 2 + \frac{1}{4} - \frac{1}{64} + \frac{1}{512} = 2.236328125 \approx \sqrt{5}$$

accuracy:

$f^{(4)}(x) = \frac{-15}{16x^{7/2}}$ how large on $[4,5]$?

$$|f^{(4)}(x)| = \frac{15}{16x^{7/2}} \le \frac{15}{16(4)^{7/2}} \text{ on } [4,5]$$

$$= \frac{15}{2048}$$

$$|R_3(x)| = \frac{15}{2048}\frac{|x-4|^4}{4!} \quad \text{on } [4,5]$$

$$\Rightarrow R_4(T) = \frac{15}{2048}\cdot\frac{1}{4!} = \frac{5}{16384}$$

$$\sqrt{5} = 2.236328125 \pm 0.001$$

ex. What if we want $\sqrt{3}$ instead?

$$P_{3,4}(x) = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \frac{1}{512}(x-4)^3$$

$$P_{3,4}(3) = 2 - \frac{1}{4} - \frac{1}{64} - \frac{1}{512} = 1.7324187$$

$$f^{(4)} = -\frac{15}{16x^{7/2}} \rightarrow \text{max val. on } [3,4]?$$

$$\text{with } f^{(4)}(3) \leq \frac{15}{16(3)^{7/2}} \leq \frac{5}{144\sqrt{3}} < \frac{5}{144}$$

$$|\text{error}| \leq \frac{5}{144}\frac{|x-4|^4}{4!} = \frac{3}{3456} < 0.001$$

ex. Estimate $\tan^{-1}(1/10)$. Use 4th order poly.

$$f(x) = \tan^{-1}(x) \Rightarrow f(0) = 0$$

$$f'(x) = \frac{1}{1+x^2} \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{-2x}{(1+x^2)^2} \Rightarrow f''(0) = 0$$

$$f'''(x) = \frac{6x^2-2}{(1+x^2)^3} \Rightarrow f'''(0) = -2$$

$$f^{(4)}(x) = \frac{24x(1-x^2)}{(1+x^2)^4} \Rightarrow f''''(0) = 0$$

$$P_{4,0}(x) = x - \frac{x^3}{3}$$

$$\tan^{-1}(1/10) = 0.1 - \frac{0.001}{3} = 0.09\overline{66}$$

$$f^{(5)}(x) = \frac{24 - 240x^2 + 120x^4}{(1+x^2)^5}$$

→ how tf do you find an error bound for this?
→ taking another deriv & looking for crit. pts. is cancer.

$$\text{on } [0, \tfrac{1}{10}],\ |f^{(5)}(x)| = \frac{|24 - 240x^2 + 120x^4|}{(1+x^2)^5}$$

$$\leq |24 - 240x^2 + 120x^4|$$

$$= 24|1 - 10x^2 + 5x^4|$$

$\leq 24(1+10x^2+5x^4)$ ($\Delta$ inequality)

$\leq 24(1+10(0.1)^2+5(0.1)^2)$

$< 30$

$|\text{error}| \leq \frac{30|x|^5}{5!} = \frac{|x|^5}{4}$

so $\tan^{-1}(\frac{1}{10}) = \frac{1}{10} - \frac{1}{3000} \pm \frac{1}{4\times 10^5}$

(4) consider function $\int_0^3 \sin(t^3)\,dt$

→ can't find anti derivative but can estimate w/ taylor poly!!!

Let's use 3 non-zero terms

$$\sin u \approx u - \frac{u^3}{3!} + \frac{u^5}{5!}$$

$$\Rightarrow \sin(t^3) = t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!}$$

→ $15^{th}$ order Mclaurin polynomial

$$\int_0^x \sin(t^3)\,dt \approx \int_0^x t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!}\,dt$$

$$= \frac{t^4}{4} - \frac{t^{10}}{60} + \frac{t^{16}}{1920}\Bigg|_0^x$$

$$= \frac{x^4}{4} - \frac{x^{10}}{60} + \frac{x^{16}}{1920}$$

## Error Band?

For $\sin u$, we can let $K = 1$, for any interval and any n.

$$|R_6(u)| \leq \frac{|u|^7}{7!}$$

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} \pm \frac{u^7}{7!}$$

→ error band looks like next term in series.

$$\Rightarrow \sin t^3 = t^3 - \frac{t^9}{3!} + \frac{t^{15}}{5!} \pm \frac{t^{21}}{7!}$$

$$\int_0^x \sin(t^3)\,dt = \frac{x^4}{4} - \frac{x^{10}}{60} + \frac{x^{16}}{1920} \pm \frac{x^{22}}{110880}$$

Ex. estimating $\tan^{-1}(1/10)$

Start w/ $f(u) = \frac{1}{1-u}$, then let $u = -t^2$, and integrate.

$$f(u) = \frac{1}{1-u} \qquad f(0) = 1$$

$$f'(u) = \frac{1}{(1-u)^2} \qquad f'(0) = 1$$

$$f''(u) = \frac{2}{(1-u)^3}$$

we want $x \in [0, \frac{1}{10}]$

$$\Rightarrow t \in [0, \tfrac{1}{10}]$$

$$\frac{1}{1-u} \approx 1 + u$$

$$\frac{1}{1+t^2} = 1 - t^2$$

$$\tan^{-1}(x) = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x (1-t^2)\,dt$$

$$= x = \frac{x^3}{3}$$