RADIUS OF CONVERGENCE

A power series is essentially a Taylor Series:

$$\sum_{k=0}^{\infty} c_k (x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots$$

how do we determine values for convergence?

Apply ratio test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{c_{k+1}(x-x_0)^{k+1}}{c_k(x-x_0)^k}\right|$$

$$= \lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|\cdot|x-x_0|$$

$$= |x-x_0|\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$$

if $|x-x_0|\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right| < 1$, the series absolutely converges.

$$|x-x_0| < \frac{1}{\lim_{k\to\infty}\left|\frac{c_{k+1}}{c_k}\right|} = \lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$$

defining R as $\lim_{k\to\infty}\left|\frac{c_k}{c_{k+1}}\right|$, $\sum_{k=0}^{\infty} c_k(x-x_0)^k$ converges absolutely if $|x-x_0|<R$

radius of convergence

- if $R=0$, only converges at $x=x_0$
- if $R=\infty$, converges for all x.
- if $0<R<\infty$, converges for $x\in[x_0-R, x_0+R]$
  - ↳ at $x=x_0\pm R$, can't tell convergence/divergence.

ex.

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{x^{k+1}}{x^k}\cdot\frac{k!}{(k+1)!}\right|$$

$$= |x|\lim_{k\to\infty}\frac{1}{k+1}$$

thus converges for all x.

ex. $\sum_{n=2}^{\infty} \frac{(x-3)^n}{\ln(2n)}$

$= |x-3| \lim_{n\to\infty} \frac{\ln(2n)}{\ln(2n+2)}$

$\overset{H}{=} |x-3| \lim_{n\to\infty} \frac{2(2n+2)}{2 \cdot 2n}$

$= |x-3| \lim_{n\to\infty} \frac{n+1}{n} \to 1$

So radius of convergence is 1.

$\Rightarrow |x-3| = 1$

- if $x-3 = 1$
  $\Rightarrow x = 4$,

  $\sum_{n=2}^{\infty} \frac{1}{\ln(2n)}$, which is divergent

  $\hookrightarrow \ln(2n) < 2n$

  $\frac{1}{\ln(2n)} > \frac{1}{2n}$

  since $\frac{1}{2n}$ diverges, $\frac{1}{\ln(2n)}$ diverges.

- if $x-3 = -1$
  $\Rightarrow x = 2$

  $\sum_{n=2}^{\infty} (-1)^n \cdot \frac{1}{\ln(2n)}$

  according to alternating series test, this is convergent.

ex. $\sum_{n=0}^{\infty} \frac{x^{2n}}{10^n}$

$= \lim_{n\to\infty} \frac{x^{2n+2}}{x^{2n}} \cdot \frac{10^n}{10^{n+1}}$

$= \frac{x^2}{10}$

Converge when $x^2 < 10$

$|x| < \sqrt{10}$