ABSOLUTE EXTREMA

- A region in $\mathbb{R}^2$ is closed if it includes boundary points
  ↳ open if doesn't include

ex.

open	closed
$-5<x<3$	$-5 \le x \le 3$
$1<y<6$	$1 \le y \le 6$

- bounded if can be contained in a disk

extreme value theorem

if $f(x,y)$ is continuous in a closed, bounded region, then $\exists (x_1, y_1)$ & (x_2, y_2) s.t. (x_1, y_1) is an absolute maximum and (x_2, y_2) is an absolute maximum

finding abs extrema!

1. find all critical points in region, & find function value at those pts.
2. find all extrema of function @ boundary, & evaluate
3. compare extrema

ex. find abs min/max of $f(x,y) = x^2 + 4y^2 - 2x^2y + 4$ on $-1 \le x \le 1$, $-1 \le y \le 1$

2

$$f_x = 2x - 4xy$$

$$f_y = 8y - 2x^2$$

3

$f_x = 0 \Rightarrow 2x(1-2y) = 0$
$\Rightarrow x = 0$ or $y = \frac{1}{2}$

$f_y = 0 \Rightarrow 8y - 2x^2 = 0$

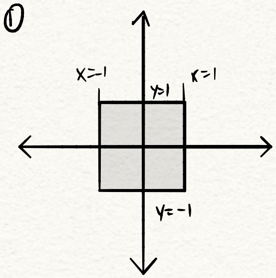
$x = 0$: $y = 0$

$y = \frac{1}{2}$: $4 = 2x^2$, $x = \pm\sqrt{2}$

so $(0,0)$, $(\sqrt{2}, \frac{1}{2})$, $(\sqrt{2}, -\frac{1}{2})$ are crit. pts.

notice $x = |\pm\sqrt{2}| > 1$, so outside boundary!

1



right: $x=1$, $-1 \le y \le 1$

left: $x=-1$, $-1 \le y \le 1$

top: $y=1$, $-1 \le x \le 1$

bottom: $y=-1$, $-1 \le x \le 1$

- $(0,0)$ is only critical point.

  $f(0,0) = 4$

- we need to walk along each side of the rectangle

right:

since $x=1$, let

$g(y) = 1 + 4y^2 - 2y + 4$

$g(y) = 4y^2 - 2y + 5$

$g'(y) = 8y - 2$

$g'(y) = 0 \Rightarrow 8y = 2$

$\Rightarrow y = \frac{1}{4}$

evaluating @ crit pt. & bounds,

should use g instead

$$\begin{cases} f(1,-1) = 1 + 4 + 2 + 4 = 11 \\ f(1,1) = 7 \\ f(1,\frac{1}{4}) = \frac{1}{4} - \frac{1}{2} + 5 = 4.75 \end{cases}$$

left:

$x=-1$, so

$g(y) = 1 + 4y^2 - 2y + 4$

$\vdots$

$y = \frac{1}{4}$

$= 1 - \cancel{4} - 2 + \cancel{5}4$

$= -1$

$f(-1,\frac{1}{4}) = 4.75$

$f(-1,-1) = 1 + 4 + 2 + 4 = 11$

$f(-1,1) = 7$

top:

Since $y=1$,

$g(x) = x^2 + 4 - 2x^2 + 4$

$= -x^2 + 8$

$g'(x) = -2x$

$g'(x) = 0 \Rightarrow x = 0$

$f(0,1) = 4+4 = 8$

$f(-1,1) = 7$

$f(1,1) = 7$

bottom:

Since $y=-1$,

$g(x) = x^2 + 4 + 2x^2 + 4$

$= 3x^2 + 8$

$g'(x) = 6x$

$g'(x) = 0 \Rightarrow x = 0.$

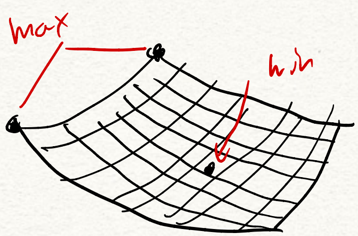
$f(0,-1) = 4 + 4 = 8$

$f(-1,-1) = 11$

$f(1,-1) = 11$

so max is $(1,-1)$ and $(-1,-1)$

min is $(0,0)$



ex. find abs max/min of $f(x,y) = 2x^2 - y^2 + 6y$, w/ disk of $r=4$: $x^2+y^2 \le 16$

$f_x = 4x$ $\qquad f_x = 0 \Rightarrow x = 0$

$f_y = -2y + 6$ $\qquad f_y = 0 \Rightarrow 2y = 6$

$y = 3$

$(0, 3)$ is the only critical pt.

$f(0,3) = -9 + 18 = 9$

- how do we "travel" along the circle?

$x^2 + y^2 \le 16$

$\Rightarrow x^2 \le 16 - y^2$

plugging into $f(x,y)$, $g(y) = 2(16-y^2) - y^2 + 6y = 32 - 3y^2 + 6y$

note that $-4 \le y \le 4$

$g'(y) = -6y + 6$

$g'(y) = 0 \Rightarrow y = 1$

$g(1) = 35$

$$\begin{array}{r} 16 \\ \times 3 \\ \hline 6 \end{array}$$

also...

$g(-4) = 32 - 48 - 24 = -40$

$g(4) = 32 - 48 + 24 = 8$

also $f(0,3) = 9$

$y = -4 \Rightarrow x^2 = 16 - y^2$

$x = 0$

$y = 1 \Rightarrow x^2 = 16 - (1)^2$

$x = \pm\sqrt{15}$

$y = 4 \Rightarrow x = 0$

so max: $(\sqrt{15}, 1)$, $(-\sqrt{15}, 1)$

min: $(0, -4)$