$$\int x\sec^2(2y) + 4xy \, dy$$

$$= \frac{x}{2}\tan(2y) + 2xy^2 + g(x)$$

constant is function of other variable

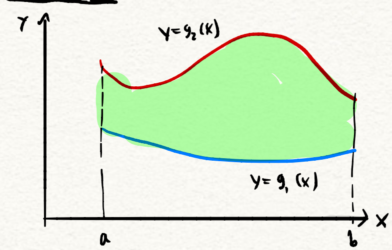
$$\int x^3 - e^{-\frac{x}{4}} \, dx$$

$$= \frac{1}{4}x^4 + y e^{-\frac{x}{4}} + h(y)$$

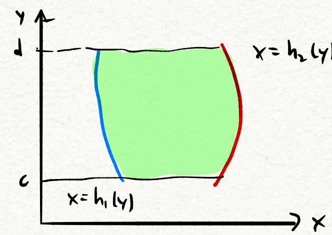
DOUBLE INTEGRALS: NON-RECTANGULAR REGIONS

$\iint_D f(x,y)\, dA \rightarrow$ what if D is any region?

<u>type 1 (x)</u>



<u>type 2 (y)</u>



<u>type 3</u>: where type 1 and type 2 both work

these regions are often described using set builder notation:

$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$$

$$\iint_D f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

$$D = \{(x,y) \mid c \le y \le d,\ h_1(y) \le x \le h_2(y)\}$$

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy$$

properties of double integrals

1. $$\iint_D f(x,y) + g(x,y)\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$$

2. $$\iint_D c f(x,y)\, dA = c \iint_D f(x,y)\, dA$$

3. if D can be split into two regions,

$$\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA$$

ex.

a) $$\iint_D e^{\frac{x}{y}}\, dA, \quad D = \{(x,y) \mid 1 \le y \le 2,\ y < x < y^3\}$$

$$= \int_1^2 \int_y^{y^3} e^{\frac{x}{y}}\, dx\, dy = \int_1^2 y e^{\frac{x}{y}} \Big|_y^{y^3} dy$$

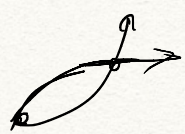
$$= \int_1^2 y e^{y^2} - y e\, dy$$

$$= \tfrac{1}{2} e^{y^2} - \tfrac{1}{2} e y^2 \Big|_1^2$$

$$= \tfrac{1}{2}(e^4 - 4e \cancel{- e} \cancel{+ e})$$

$$= \tfrac{1}{2} e^4 - 2e$$

b) $$\iint_D 4xy - y^3\, dA, \text{ bounded by } y = \sqrt{x} \text{ and } y = x^3$$



$\because$ intersection pts are 0 and 1.

$$\therefore \int_0^1 \int_{x^3}^{\sqrt{x}} 4xy - y^3\, dy\, dx$$

$$= \int_0^1 2xy^2 - \tfrac{1}{4} y^4 \Big|_{x^3}^{\sqrt{x}} dx$$

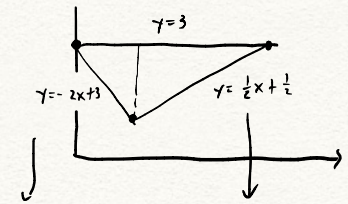
$$= \int_0^1 \frac{7}{4}x^2 - 2x^7 + \frac{1}{4}x^{12} \, dx$$

$$= \frac{7}{12}x^3 - \frac{1}{4}x^8 + \frac{1}{52}x^{13} \Big|_0^1$$

$$= \frac{7}{12} - \frac{1}{4} + \frac{1}{52}$$

$$= \frac{55}{156}$$

c) $\iint_D 6x^2 - 40y \, dA$, D is triangle w/ vertices $(0,3)$, $(1,1)$, $(5,3)$



$2x = 3 - y$
$x = \frac{3}{2} - \frac{y}{2}$

$x = 2y - 1$

$$= \int_0^1 \int_{2x+3}^3 6x^2 - 40y \, dy dx + \int_1^5 \int_{\frac{1}{2}x+\frac{1}{2}}^3 6x^2 - 40y \, dy dx$$

$$= \int_0^1 6x^2y - 20y^2 \Big|_{-2x+3}^3 dx + \int_1^5 6x^2y - 20y^2 \Big|_{\frac{1}{2}x+3}^3 dx$$

$$= \int_0^1 18x^2 - 180 - 6x^2(-2x+3) - 20(3-2x)^2 + \int_1^5 18x^2 - 180 - 6x^2(\tfrac{1}{2}x+3) - 20(\tfrac{1}{2}x+3)^2 dx$$

$$= \int_0^1 -12x^3 - 180 - 20(3-2x)^2 dx + \int_1^5 -3x^3 - 180 - 20(\tfrac{1}{2}x+3)^2 dx$$

$u = 3-2x$, $du = -2\,dx$ $\qquad$ $u = \frac{1}{2}x+3$, $du = \frac{1}{2}dx$

$$= (-3x^4 - 180x)\Big|_0^1 + 10\int_0^1 u^2 \, du + \left(-\frac{3}{4}x^4 - 180x\right)\Big|_1^5 - 40\int_1^5 u^2 \, du$$

$$= -183 + \frac{10}{3}(3-2x)^3\Big|_0^1 + \left(-\frac{1875}{4} - 400 + \frac{3}{4} + 180\right) - \frac{40}{3}(\tfrac{1}{2}x+3)\Big|_1^5$$

$$= -183 - \frac{80}{3} - \frac{1875}{4} - 400 + \frac{3}{4} + 180 - \frac{40}{3}\left(\frac{4}{2}\right)$$

$$= -\frac{935}{3}$$

<u>easier method!</u> $\Rightarrow$ only one integral!

$$\iint_D 6x^2 - 40y \, dA = \int_1^3 \int_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} 6x^2 - 40y \, dx \, dy$$

$$= \int_1^3 2x^3 - 40xy \Big|_{-\frac{y}{2}+\frac{3}{2}}^{2y-1} dy$$

$$= \int_1^3 2(2y-1)^3 - 40y(2y-1) - 2\left(-\frac{y}{2}+\frac{3}{2}\right)^3 + 40y\left(-\frac{y}{2}+\frac{3}{2}\right) dy$$

$$= \int_1^3 -100y^2 + 100y + 2(2y-1)^3 - 2\left(-\frac{y}{2}+\frac{3}{2}\right)^3 dy$$

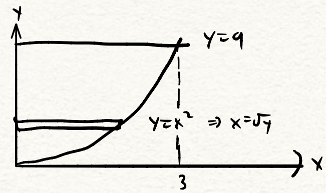
$$= \left[-\frac{100}{3}y^3 + 50y + \frac{1}{4}(2y-1)^4 + \left(-\frac{y}{2}+\frac{3}{2}\right)^4\right]\Big|_1^3$$

$= -\frac{935}{3}$

- can't multiply everything out if you can use a calc I substitution!

<u>ex.</u> $\int_0^3 \int_{x^2}^9 x^3 e^{y^3}\, dy\, dx$

- can't integrate w/ respect to y (need y^2)
- we can't just switch limits since x can't be in integral bounds.
  - ↳ if we ignored, ans wouldn't be const.



$0 \le y \le 9$
$0 \le x \le \sqrt{y}$

$$= \int_0^9 \int_0^{\sqrt{y}} x^3 e^{y^3}\, dx\, dy$$

$$= \int_0^9 \frac{1}{4} x^4 e^{y^3} \Big|_0^{\sqrt{y}}\, dy$$

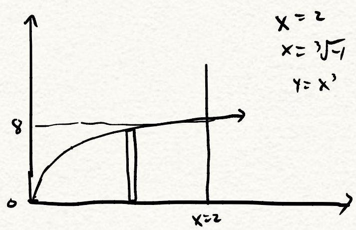
$$= \frac{1}{4} \int_0^9 y^2 e^{y^3}\, dy$$

$u = y^3$
$du = 3y^2\, dy$

$$= \frac{1}{12} \int_0^{729} e^u\, du$$

$$= \frac{1}{12}\left(e^{729} - 1\right)$$

b) $\int_0^8 \int_{\sqrt[3]{y}}^2 \sqrt{x^4+1}\, dx\, dy$



$x = 2$
$x = \sqrt[3]{y}$
$y = x^3$

$0 \le x \le 2$
$0 \le y \le x^3$

$$= \int_0^2 \int_0^{x^3} \sqrt{x^4+1}\, dy\, dx$$

$$= \int_0^2 y\sqrt{x^4+1} \Big|_0^{x^3} dx$$

$$= \int_0^2 x^3\sqrt{x^4+1}\, dx$$

$u = x^4+1$

$du = 4x^3\, dx$

$$= \frac{1}{4} \int_0^2 \sqrt{u}\, du$$

$$= \frac{1}{6} (x^4+1)^{\frac{3}{2}} \Big|_0^2$$

$$= \frac{1}{6} (17^{\frac{3}{2}} - 1)$$

other interpretations

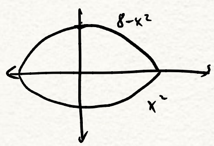
$V = \iint_D f(x,y)\, dA$ is volume below surface of $z=f(x,y)$ and above region D.

ex. volume enclosed by solid below $z=16xy+200$, above xy-plane bounded by $y=x^2$, $y=8-x^2$

$x^2 = 8-x^2$

$2x^2 - 8 = 0$

$2(x+2)(x-2) = 0$



$$\int_{-2}^{2} \int_{x^2}^{8-x^2} 16xy + 200\, dy\, dx$$

$$= \int_{-2}^{2} 8xy^2 + 200y \Big|_{x^2}^{8-x^2} dx$$

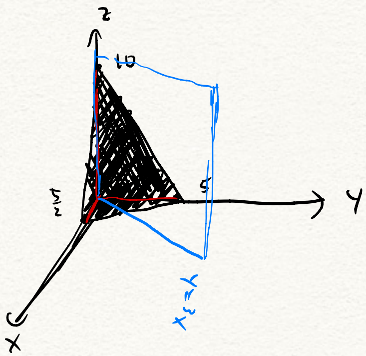
$$= \int_{-2}^{2} \underbrace{8x(8-x^2)^2}_{\text{odd}} + 200(8-x^2) - 8x^5 - 200x^2\, dx$$

$$= \int_{-2}^{2} 1600 - 400x^2 - 8x^5\, dx$$

$$= 1600x - \frac{400}{3}x^3 - \frac{4}{3}x^6 \Big|_{-2}^{2}$$

$$= \frac{12800}{3}$$

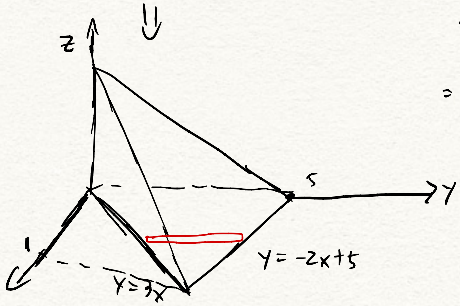
ex, find volume enclosed by planes $4x+2y+z=10$, $y=3x$, $z=0$, $x=0$



$$\frac{10}{4}=\frac{5}{2}$$

$$z=10-4x-2y$$

$\Downarrow$



$$3x=-2x+5$$
$$5x=5$$
$$x=1$$

$$=\int_0^1\int_{3x}^{-2x+5} 10-4x-2y \, dy \, dx$$

$$=\int_0^1 10y-4xy-y^2 \Big|_{3x}^{-2x+5} dx$$

$(4x^2-20x+25)$

$$=\int_0^1 -20x+50-(-8x^2+20x)-(-2x+5)^2-30x+12x^2+9x^2 \, dx$$

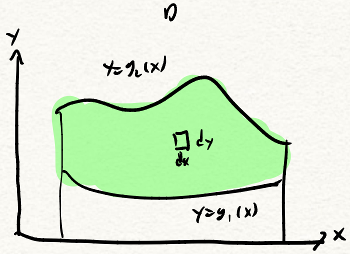
$$=\int_0^1 25x^2-50x+25 \, dx$$

$$=\frac{25}{3}x^3-25x^2+25x \Big|_0^1$$

$$=\frac{25}{3}$$

- regions below xy-plane give negative volume; above is positive.

$$\text{Area of } D=\iint_D dA$$



$$A=\int_a^b g_2(x)-g_1(x)\, dx$$

equal!!

$$A=\iint_D dA$$
$$=\int_a^b\int_{g_1(x)}^{g_2(x)} dy\, dx$$
$$=\int_a^b y\Big|_{g_1(x)}^{g_2(x)} dy$$
$$=\int_a^b g_2(x)-g_1(x)\, dx$$