REMAINDER THM FOR POWER RULE

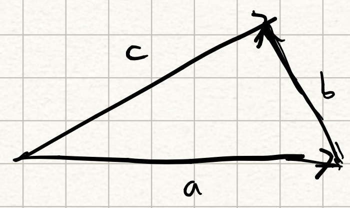
Mar. 03

→ how good is the approximation $f(x) \approx P_{n,x_0}(x)$?

→ can't calculate directly, but can find upper bound.

Recall the triangle inequality:

$$\|a+b\| \leq \|a\| + \|b\|$$



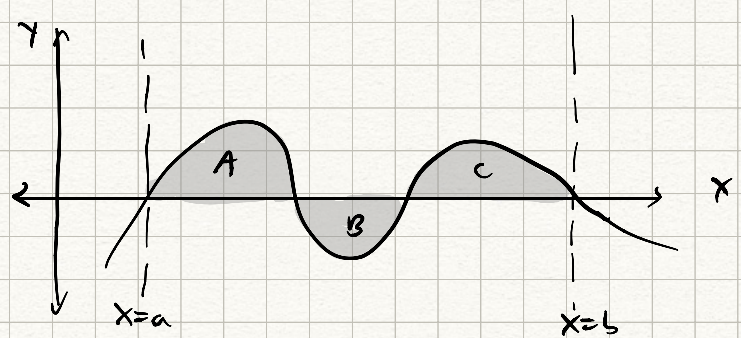
We can extend this to

$$|a_1 + a_2 + \dots a_n| \leq |a_1| + |a_2| + \dots + |a_n|$$

Applying to Riemann Sums,

$$\boxed{\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx, \text{ assuming } a<b.}$$

→ actual area $\leq$ area of $|f(x)|$



above,

$$\left| \int_a^b f(x)\,dx \right| = A - B + C \quad \text{and} \quad \int_a^b |f(x)| = A + B + C$$

Also recall FTC II:

If $F'(x) = f(x)$, then $\int_a^b f(x)\,dx = F(b) - F(a)$

We can write this as

$$\int_{x_0}^{x} f'(t)\,dt = f(x) - f(x_0)$$

$$\boxed{f(x) = f(x_0) + \int_{x_0}^{x} f'(t)\,dt}$$

↳ $P_{0,x_0}(x)$ with a remainder expressed as integral

let $u = f'(t)$ and $dv = dt$
$du = f''(t)dt$ $\quad v = t$

$$\int_{x_0}^{x} f'(t)\,dt = tf'(t)\Big|_{x_0}^{x} - \int_{x_0}^{x} tf''(t)\,dt$$

$$= xf'(x) - x_0 f'(x_0) - \int_{x_0}^{x} tf''(t)\,dt$$

$$f(x) = f(x_0) + xf'(x) - x_0 f'(x_0) - \int_{x_0}^{x} tf''(t)\,dt$$

$$= f(x_0) + xf'(x) \underbrace{- xf'(x_0) + xf'(x_0)}_{0} - x_0 f'(x_0) - \int_{x_0}^{x} tf''(t)\,dt$$

$$= f(x_0) + f'(x_0)(x - x_0) + \underbrace{x[f'(x) - f'(x_0)]}_{x\int_{x_0}^{x} f''(t)\,dt} - \int_{x_0}^{x} tf''(t)\,dt$$

This is $P_{1,x_0}(x)$!

$$= f(x_0) + f'(x_0)(x - x_0) + \underbrace{\int_{x_0}^{x} (x-t) f''(t)\,dt}_{\text{error}}$$

⟶ dependent on 2nd derivative from $[x_0, x]$

We can generalize to higher orders using the same procedure

$$u = f''(t) \qquad dv = (x-t)\,dt$$
$$du = f'''(t)\,dt \qquad v = -\tfrac{1}{2}(x-t)^2 \, dt$$

$$\int_{x_0}^{x} (x-t) f''(t)\,dt = -\frac{1}{2}(x-t)^2 f''(t)\Big|_{x_0}^{x} + \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t)\,dt$$

$$= -\frac{1}{2}(x-x_0)^2 f''(t) + \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t)\,dt$$

so $f(x) = f(x_0) + f'(x_0)(x-x_0) - \frac{1}{2} f''(t)(x-x_0)^2 + \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f'''(t)\,dt$

$\rightarrow$ $P_{2,x_0}(x)$!!

More generally, if f has $n+1$ derivatives at x_0,

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x), \text{ where} \quad R_n(x) = \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt}$$

$\rightarrow$ how do we evaluate $R_n(x)$?

if we can show that $\left|f^{(n+1)}(t)\right| \leq K$ for all values of t in $[x_0, x]$, where K is a constant, then,

$$|\text{error}| = \left|f(x) - P_{n,x_0}(x)\right|$$
$$= |R_n(x)|$$
$$= \left|\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt\right|$$

if $x_0 < x$, then we can apply the triangle inequality for integrals

$$|R_n(x)| \le \int_{x_0}^{x} \left| \frac{(x-t)^n}{n!} f^{(n+1)}(t) \right| dt$$

$$= \int_{x_0}^{x} \frac{(x-t)^n}{n!} \left| f^{(n+1)}(t) \right| dt$$

$$\le \int_{x_0}^{x} \frac{(x-t)^n}{n!} K \, dt$$

$$= K \int_{x_0}^{x} \frac{(x-t)^n}{n!} \, dt$$

$$= -K \left. \frac{(x-t)^{n+1}}{(n+1)!} \right|_{x_0}^{x}$$

$$= K \frac{(x-x_0)^{n+1}}{(n+1)!}$$

if $x_0 > x$, then,

$$|R_n(x)| \le \int_{x}^{x_0} \left| \frac{(x-t)^n}{n!} f^{(n+1)}(t) \right| dt$$

$$= \int_{x}^{x_0} \frac{(x-t)^n}{n!} \left| f^{(n+1)}(t) \right| dt$$

$$\le \int_{x}^{x_0} \frac{(x-t)^n}{n!} K \, dt$$

$$= K \int_{x}^{x_0} \frac{(x-t)^n}{n!} \, dt$$

$$= -K \left. \frac{(x-t)^{n+1}}{(n+1)!} \right|_{x}^{x_0}$$

$$= -K \frac{(x_0-x)^{n+1}}{(n+1)!}$$