

# TAYLOR SERIES

Given a Taylor polynomial used to approximate a function, if we add on as terms, our approximation may become exact.

Suppose that  $f(x) = \underbrace{T_n(x)}_{\text{Taylor Approx.}} + \underbrace{R_n(x)}_{\text{Error}}$

Then:

IFF  $\lim_{n \rightarrow \infty} R_n(x) = 0$  for  $|x-a| < R$

$$\text{then } f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Ex Find Taylor Series for  $e^x$

$$f^{(n)} = e^x$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

Ex Find Taylor Series for  $e^{-x}$

a)  $f(x) = e^{-x}$

$$f'(x) = -e^{-x}$$

$$f''(x) = e^{-x}$$

⋮

$$f^{(n)}(x) = (-1)^n e^{-x}$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n$$

b) we know  $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

let  $x = -x$ .

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

Ex  $x^4 e^{-3x^2}$

$$x^4 \cdot e^{-m} = x^4 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{m^n}{n!}$$

let  $m = 3x^2$

$$x^4 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3x^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+4}}{n!}$$

ex.  $\sin(x)$

$f = \sin(x)$	0
$f' = \cos(x)$	1
$f'' = -\sin(x)$	0
$f''' = -\cos(x)$	-1
$f^{(4)} = \sin(x)$	0

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

When does this not work so well?

Consider

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n + R_n(x).$$

From the ratio test,  $|x| < 1$ , so  $R=1$ .

So this approximation only works on  $(-1, 1)$

## Constructing Taylor Series

If  $\sum c_n (x-x_0)^n$  has radius of convergence  $R$ , we can

- $\left. \begin{array}{l} \bullet \text{ Differentiate it} \\ \bullet \text{ Integrate it} \\ \bullet \text{ Multiply by constant} \\ \bullet \text{ Add it to another series w/ radius of convergence } \geq R \end{array} \right\} \text{ term-by-term}$

and the result will still have radius of convergence  $R$ .

⚠ This only guarantees  $R$  will be the same. We might lose/gain endpoints.

Ex

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \text{ with } |x| < 1, R=1$$

differentiating,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots, R \text{ is still } 1!$$

1st term disappears

We could instead integrate...

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \dots$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{4}x^4 - \dots$$

R is still 1. However, when  $x=-1$ ,  $-\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+1}$ , which converges.

↳ original:  $(-1, 1)$

We can then add these terms...

$$\begin{aligned} \frac{1}{(1-x)^2} + \ln(1-x) &= \sum_{n=1}^{\infty} n x^{n-1} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \\ &\stackrel{\text{re-index}}{=} \sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{needs to be the same.} \\ &= 1 + \sum_{n=1}^{\infty} (n+1)x^n - \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{need to have same index} \\ &= 1 + \sum_{n=1}^{\infty} \left[ n+1 - \frac{1}{n} \right] x^n \\ &= 1 + x + \frac{5}{2}x^2 + \frac{11}{3}x^3 + \frac{19}{4}x^4 + \dots \end{aligned}$$

## USING GEOMETRIC SERIES

↳ useful for constructing for  $\ln$ ,  $\tan^{-1}$ , and rational functions.

$$\text{Ex. } \frac{x}{3+2x} = x \left( \frac{1}{3+2x} \right)$$

$$= \frac{x}{3} \left( \frac{1}{1 + \frac{2}{3}x} \right)$$

$$= \frac{x}{3} \left( \frac{1}{1 - (-\frac{2}{3}x)} \right)$$

Consider as  
finite series of  
1 term.  
 $R = \infty$

$$\left| -\frac{2}{3}x \right| < 1$$

$$= \frac{x}{3} \sum_{n=0}^{\infty} \left( -\frac{2}{3}x \right)^n$$

$$|x| < \frac{3}{2} \text{ so } R = \frac{3}{2}.$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^n x^{n+1}}{3^{n+1}}$$

→  $R > \frac{3}{2}$  solve we take smaller R.

## USING BINOMIAL SERIES

$$(1+x)^m = \sum_{n=0}^m \binom{m}{n} x^n, \text{ where } \binom{m}{n} = \frac{m!}{(m-n)! n!} \quad \text{if } m \text{ is integer.}$$

$$= 1 + mx + m(m-1) \frac{x^2}{2!} + m(m-1)(m-2) \frac{x^3}{3!} + \dots$$

$$\dots + m(m-1)(m-2) \dots (m-n+1) \frac{x^n}{n!}$$

- converges for  $|x| < 1$

$$\text{ex. } \frac{1}{\sqrt{2-x}} = \frac{1}{\sqrt{2}} \left( \frac{1}{\sqrt{1-\frac{x}{2}}} \right)$$

$$= \frac{1}{\sqrt{2}} \left( 1 - \frac{x}{2} \right)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{2}} \left[ 1 + \left( -\frac{1}{2} \right) \left( -\frac{x}{2} \right) + \frac{1}{2!} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \left( -\frac{x}{2} \right)^2 + \frac{1}{3!} \left( -\frac{1}{2} \right) \left( -\frac{3}{2} \right) \left( -\frac{5}{2} \right) \left( -\frac{x}{2} \right)^3 + \dots \right]$$

$$= \frac{1}{\sqrt{2}} \left[ 1 + \frac{1}{4}x + \frac{3}{32}x^2 + \frac{5}{128}x^3 + \dots \right]$$

$(1+u)^m$  converges for  $|u| < 1$ . let  $u = -\frac{x}{2}$ ,

so this series converges for  $\left| -\frac{x}{2} \right| < 1$ , so for  $|x| < 2$ .

## BUILDING BLOCKS

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x$$