SECOND DERIVATIVE TEST

if (a,b) is a critical point, define

$$D = D(a,b) = f_{xx}(a,b)\, f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

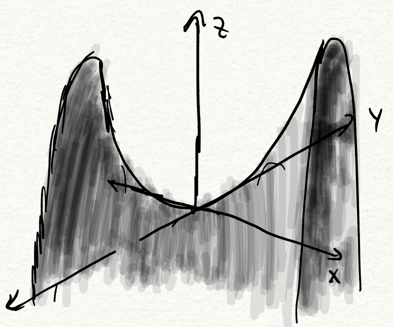
- if $D>0$ and $f_{xx} > 0$, (a,b) is relative minimum
- if $D>0$ and $f_{xx} < 0$, (a,b) is relative maximum
- if $D<0$ then (a,b) is saddle point
- if $D=0$ then (a,b) can be relative min, max, or saddle pt.

<u>intuition:</u>

$$D = \underbrace{f_{xx}(x_0,y_0)}_{x\text{-concavity}}\ \underbrace{f_{yy}(x_0,y_0)}_{y\text{-concavity}} - \underbrace{f_{xy}(x_0,y_0)^2}_{\text{always positive } (^2)} \longrightarrow \det\left(\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}\right)$$

Hessian Matrix

- so if x-concavity & y-concavity disagree, saddle point! $(+)(-) - (+)$
- if x-concavity & y-concavity agree, we have two cases!
  - $(+)(+) - (+)$
  - $(-)(-) - (+)$
- in both cases, it's a battle btw. first two terms & mixed partial derivative
  - ↳ mixed partial derivative wants to pull things toward a saddle pt.



Consider $f = xy$

$f_x = y \qquad f_{xx} = 0$

$f_y = x \qquad f_{yy} = 0$

walking along x & y axis yields a straight line.
however, $f_{xy} = 1$

the $f_{xy}(x_0,y_0)^2$ tells us how much a function behaves like $f = xy$

since f_{xy} always evaluates to some constant (multiple of f_{xy} for $f=xy$)

ex. find & classify critical points of $f(x,y) = 4 + x^3 + y^3 - 3xy$

$f_x = 3x^2 - 3y$

$f_y = 3y^2 - 3x$

$3x^2 - 3y = 0 \Rightarrow 3x^2 = 3y \Rightarrow y = x^2$

$--- \Rightarrow 3y^2 = 3x$

$\cancel{3}x^4 = \cancel{3}x$

$x^4 - x = 0$

$x(x^3 - 1) = 0$

$x = 0$ or $x^3 - 1 = 0$.

& $y = 0$ $\Rightarrow x = 1, y = 1$

$f_{xx} = 6x$

$f_{yy} = 6y$

$f_{xy} = -3$

$D(x,y) = 36xy - 9$

$D(0,0) = 36(0)(0) - 9 = -9 < 0$ (saddle pt.)

$D(1,1) = 36 - 9 = 27 > 0$ (local min.)

↳ $f_{xx}(1,1) > 0$, so positive concavity

↳ $f_{yy}(1,1)$ should >0 if $D > 0$.

ex. $f(x,y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$

$f_x(x,y) = 6xy - 6x$

$f_y(x,y) = 3x^2 + 3y^2 - 6y$

$f_{xx}(x,y) = 6y - 6$

$f_{yy}(x,y) = 6y - 6$

$f_{xy}(x,y) = 6x$

$D(x,y) = (6y-6)^2 - (6x)^2$

$6xy = 6x$

$y = 1$ or $x = 0$

$3x^2 = 6y - 3y^2$

$\underline{y=1:}$

$\Rightarrow 3x^2 = 6 - 3$

$x^2 = 1$

$x = \pm 1$

$\underline{x=0:}$

$3y^2 = 6y$

$3y^2 - 6y = 0$

$y(3y - 6) = 0$

$y = 0, y = 2$

$\underline{(0,0):}$

$D(0,0) = 36 > 0$

$f_{yy}(0,0) = -6$ (local max.)

$\underline{(0,2)}$

$D(0,2) > 0$

$f_{yy}(0,2) = 6$ (local min.)

$\underline{(1,1)}$

$D(1,1) < 0$ (saddle pt.)

$\underline{(-1,1)}$

$D(-1,1) < 0$ (saddle pt.)

ex. find point on plane $4x-2y+z=1$ closest to point $(-2,-1,5)$

let (x,y,z) be a point on plane.

$$D(x,y,z) = \sqrt{(x+2)^2+(y+1)^2+(z-5)^2}$$

$$4x-2y+z=1 \Rightarrow z = 1+2y-4x$$

$$r(x,y) = \sqrt{x^2+4x+4+y^2+2y+1+(2y-4x-4)^2}$$

$$= \sqrt{x^2+4x+4+y^2+2y+1+4y^2-8xy-8y-8xy+16x^2+16x-8y+16x+16}$$

$$= \sqrt{17x^2+36x-16xy+5y^2-14y+21}$$

* observe that since the inner expression is always defined, $r(x,y)$ will have the exact same critical pts. as r

Let $G = r^2$

$G_x = (34x+36-16y)$

$G_y = (10y-14-16x)$

$G_{xx} = 34$

$G_{yy} = 10$

$G_{xy} = -16$

$34x+36 = 16y$ $\qquad 10y = 16x+14$

$17x+18 = 8y$ $\qquad 5y = 8x+7$

$85x+90 = 40y$ $\qquad 40y = 64x+56$

$21x+34 = 0$

$x = -\frac{34}{21}$ $\qquad y = -\frac{25}{21}$

$z = 1+2\left(-\frac{25}{21}\right)-4\left(-\frac{34}{21}\right) = \frac{107}{21}$

$D = 340-16^2 = 84$ (any critical pt. is local. min)

↳ there should only be 1 crit pt. logically, and that pt. is abs. max.

so the closest pt. is $\left(-\frac{34}{21}, -\frac{25}{21}, \frac{107}{21}\right)$

* second derivative test fails when $D=0$

ex. $z = x^4+y^4$

$z_x = 4x^3$

$z_{xx} = 12x^2$

$z_{yy} = 12y^2$

↳ zero @ $(0,0)$

entire surface lies above tangent plane at $(0,0)$, so obviously local min. (by inspection) ex. $f(x,y) = xy + \ln x + y^2 - 10$ (for $x>0$)

$$\left.\begin{array}{l} f_x = y + \frac{1}{x} \\ \\ f_y = x + 2y \end{array}\right\}$$

$y + \frac{1}{x} = 0$

$y = -\frac{1}{x}$

$0 = x + 2y$

$0 = x - \frac{2}{x}$

$x = \frac{2}{x}$

$x^2 = 2$

$x = \sqrt{2}$

$\Rightarrow y = -\frac{1}{\sqrt{2}}$

$f_{xx} = -\frac{1}{x^2}$ $\qquad f_{xy} = 1$

$f_{yy} = 2$

$D(x,y) = -\frac{2}{x^2} - 1$

$(\sqrt{2}, -\frac{1}{\sqrt{2}})$:

$D(\sqrt{2}, -\frac{1}{\sqrt{2}}) = -\frac{2}{2} - 1 = -2 < 0$

so saddle point

ex. what $f(x,y) = x^3 - 3xy^2$

$f_x = 3x^2 - 3y^2$

$f_y = -6xy$

$f_y = 0 \Rightarrow x = 0$ or $y = 0$

$f_x = 0 \Rightarrow y = \pm x$

so $(0,0)$ crit. pt.

$$\left.\begin{array}{l} f_{xx} = 6x \\ f_{yy} = -6y \\ f_{yx} = -6y \end{array}\right\} D(x,y) = -36x^2 - 36y^2$$

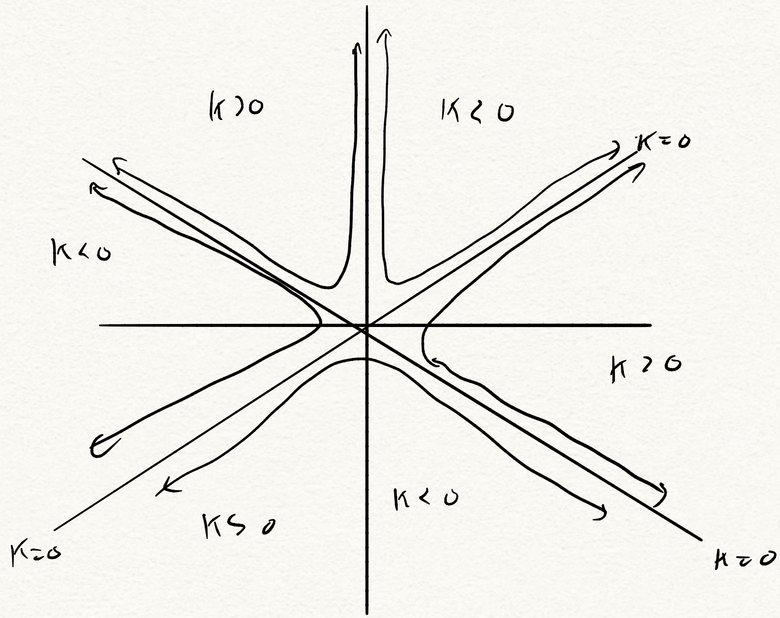
$D(0,0) = 0$

at $(0,0)$, consider $f=0$

$0 = x^3 - 3xy^2 = x(x^2 - 3y^2)$

$\Rightarrow x = 0$ or $x^2 = 3y^2$

$x = \sqrt{3}y$



<u>ex:</u> $f(x,y) = x^3 - 2y^2 - 2y^4 + 3x^2 y$

$f_x = 3x^2 + 6xy$
$f_y = -4y - 8y^3 + 3x^2$

$f_{xx} = 6x + 6y$ $\quad f_{xy} = 6x$
$f_{yy} = -4 - 24y^2$

$D(x,y) = (6x+6y)(-4-24y^2) - 36x^2$
$D(0,0) = 0 \quad ⇒$
$D(-1, \tfrac{1}{2}) = -6 \Rightarrow$ saddle
$D(-2,1) = 24 \Rightarrow$ local max

At $(0,0)$ observe $y=0$
$\Rightarrow z = x^3$, so not extremum

$3x = -6y$

$3x^2 + 6xy = 0$
$x(3x + 6y) = 0$
$\Rightarrow x = 0$ or $x = -2y$

$f_y = 0 \Rightarrow 8y^3 + 4y = 3x^2$

<u>$x=0$:</u>
$y(8y^2 + 4) = 0$
$\begin{cases} y = 0 \\ x = 0 \end{cases}$ ~~$, y = \sqrt{-\frac{1}{2}}$~~

<u>$x = -2y$</u>
$8y^3 + 4y = 12y^2$
$8y^3 - 12y^2 + 4y = 0$
$4y(2y^2 - 3y + 1) = 0$
$4y(2y-1)(y-1) = 0$
$y = 0,\ y = \tfrac{1}{2},\ y = 1$

so $(0,0), (-1, \tfrac{1}{2}), (-2, 1)$