TAYLOR SERIES

Given a Taylor polynomial used to approximate a function, if we add on ∞ terms, our approximation may become exact.

Suppose that $f(x) = \underbrace{T_n(x)}_{\text{Taylor Approx.}} + \underbrace{R_n(x)}_{\text{Error}}$

Then:

$$\boxed{\begin{array}{l} \underline{\text{IFF}} \quad \lim_{n\to\infty} R_n(x) = 0 \quad \text{for} \quad |x-a| < R \\ \text{then} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)} a}{n!} (x-a)^n \end{array}}$$

ex. Find Taylor Series for e^x

$$f^{(n)} = e^x$$

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

ex. Find Taylor Series for e^{-x}

a)
$$f(x) = e^{-x}$$
$$f'(x) = -e^{-x}$$
$$f''(x) = e^{-x}$$
$$\vdots$$
$$f^{(n)}(x) = (-1)^n e^{-x}$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{n!} x^n$$

b) we know $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$

let $x = -x$.

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

$$e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$$

ex. $x^4 e^{-3x^2}$

$$x^4 \cdot e^{-m} = x^4 \sum_{n=0}^{\infty} (-1)^n \cdot \frac{m^n}{n!}$$

let $m = 3x^2$

$$x^4 \cdot \sum_{n=0}^{\infty} (-1)^n \cdot \frac{3x^{2n}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-3)^n x^{2n+4}}{n!}$$

ex. $\sin(x)$

	$f = \sin(x)$	0
0!	$f' = \cos(x)$	1
	$f'' = -\sin(x)$	0
1!	$f''' = -\cos(x)$	-1
	$f^{(4)} = \sin(x)$	0

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)}}{(2n+1)!}$$

When does this not work so well?

Consider

$$\frac{1}{1+x} = \sum_{n=0}^{n} (-1)^k x^k + R_n(x).$$

From the ratio test, $|x| < 1$, so $R = 1$.

So this approximation only works on $(-1, 1)$

Constructing Taylor Series

If $\sum c_k (x - x_0)^n$ has radius of convergence R, we can

term-by-term
- differentiate it
- integrate it
- multiply by constant
- add it to another series w/ radius of convergence $\geq R$

and the result will still have radius of convergence R.

! This only guarantees R will be the same. We might lose/gain endpoints.

Ex.

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{with } |x|<1,\ R=1$$

differentiating,

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + \dots, \quad R \text{ is still } 1!$$

1st term disappears

we could instead integrate...

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \dots$$

$$\ln(1-x) = -\sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 - \dots$$

R is still 1. However, when $x=-1$, $-\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{n+1}$, which converges.

↳ original: $(-1, 1)$

we can then add these terms...

$$\frac{1}{(1-x)^2} + \ln(1-x) = \sum_{n=1}^{\infty} n x^{n-1} - \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1}$$

re-indexing to add, power of n needs to be the same.

re-index

$$= \sum_{n=0}^{\infty} (n+1)x^n - \sum_{n=1}^{\infty} \frac{x^n}{n}$$

need to have same index to add...

$$= 1 + \sum_{n=1}^{\infty} (n+1)x^n - \sum_{n=1}^{\infty} \frac{x^n}{n}$$

$$= 1 + \sum_{n=1}^{\infty} \left[n+1-\tfrac{1}{n}\right] x^n$$

$$= 1 + x + \tfrac{5}{2}x^2 + \tfrac{11}{3}x^3 + \tfrac{19}{4}x^4 + \dots$$

USING GEOMETRIC SERIES

↳ useful for constructing for log, $\tan^{-1}$, and rational functions.

Ex.

$$\frac{x}{3+2x} = x\left(\frac{1}{3+2x}\right)$$

$$= \frac{x}{3}\left(\frac{1}{1+\frac{2}{3}x}\right)$$

$$= \frac{x}{3}\left(\frac{1}{1-\left(-\frac{2}{3}x\right)}\right)$$

consider as finite series of 1 term. $R = \infty$

$$= \frac{x}{3}\sum_{n=0}^{\infty}\left(-\frac{2}{3}x\right)^n$$

$\left|-\frac{2}{3}x\right| < 1$

$|x| < \frac{3}{2}$ so $R = \frac{3}{2}$.

$$= \sum_{n=0}^{\infty} (-1)^n \frac{2^n x^{n+1}}{3^{n+1}}$$

→ $R > \frac{3}{2}$ since we take smaller R.

Ex. $\arctan(x)$

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n, \quad |x^2| < 1 \Rightarrow |x| < 1$$

$$\arctan(x) = \int \sum_{n=0}^{\infty} (-x^2)^n \, dx = 1 + \ldots$$

$$= \sum_{n=0}^{\infty} \int (-1)^n \cdot x^{2n} \, dx$$

$$= \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1} + C$$

$\arctan(0) = 0$, so $C = 0$.

$$\therefore \arctan(x) = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{x^{2n+1}}{2n+1}$$

↳ R still $|x| < 1$.

USING BINOMIAL SERIES

$$(1+x)^m = \sum_{n=0}^{m} \binom{m}{n} x^n, \text{ where } \binom{m}{n} = \frac{m!}{(m-n)!\,n!} \quad \text{if } m \text{ is integer.}$$

$$= 1 + mx + m(m-1)\frac{x^2}{2!} + m(m-1)(m-2)\frac{x^3}{3!} + \ldots$$

$$\ldots + m(m-1)(m-2)\ldots(m-n+1)\frac{x^n}{n!}$$

- converges for $|x| < 1$

Ex.
$$\frac{1}{\sqrt{2-x}} = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{1-x/2}}\right)$$

$$= \frac{1}{\sqrt{2}}\left(1-\frac{x}{2}\right)^{-1/2}$$

$$= \frac{1}{\sqrt{2}}\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{2}\right) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{x}{2}\right)^2 + \frac{1}{3!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{x}{2}\right)^3 + \ldots\right]$$

$$= \frac{1}{\sqrt{2}}\left[1 + \frac{1}{4}x + \frac{3}{32}x^2 + \frac{5}{128}x^3 + \ldots\right]$$

$(1+u)^m$ converges for $|u| < 1$. let $u = -\frac{x}{2}$,

So this series converges for $\left|-\frac{x}{2}\right| < 1$, so for $|x| < 2$.

BUILDING BLOCKS

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{for } |x| < 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{for all } x$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \text{for all } x$$