BOUNDARY CONDITIONS

So far, we've only considered Electric Field $\vec{E}$ in a homogeneous medium.

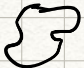
if field $\vec{E}$ exists in region with two different mediums,
the boundary condition is the condition needed to be satisfied at the interface

★ useful for determining the field on one side of boundary if $\vec{E}$ known on other side

1. Dielectric (ε_1) and dielectric (ε_2)
2. Conductor (σ) and dielectric (ε)
3. Conductor (σ) and free space (ε_0)

→ to determine boundary conditions, use Maxwell's Equations

$$\left\{ \begin{aligned} &\oint \vec{E} \cdot d\vec{l} = 0 \\ &\oint \vec{D} \cdot d\vec{A} = Q_{enc} \end{aligned} \right\}$$

→ given any arbitrary surface (), we can zoom closely on any one point until the surface looks like a straight line,

→ decompose electric field into $\perp$ and $//$ components

$$\vec{E} = \vec{E}_t + \vec{E}_n$$

→ decompose density into $\perp$ and $//$ components

$$\vec{D} = \vec{D}_t + \vec{D}_n$$

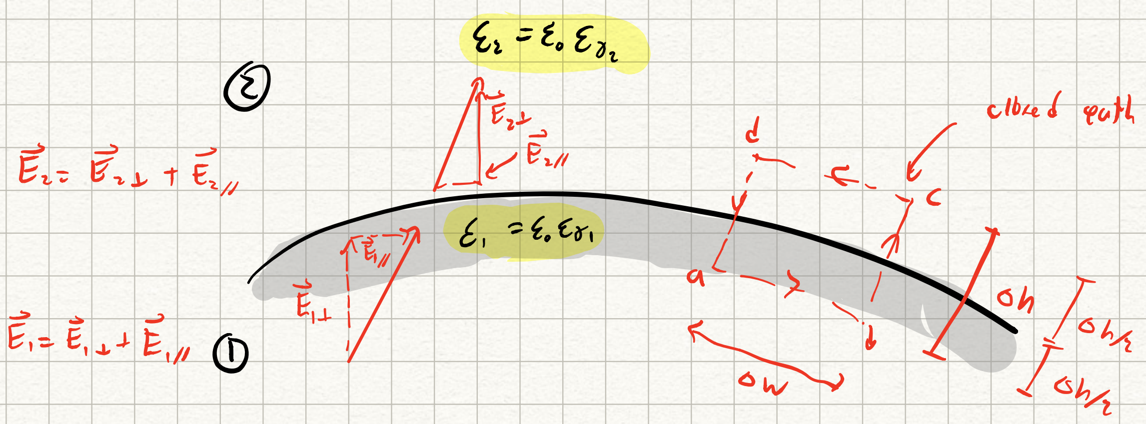
★ Electric flux density is the electric flux passing through unit area.
→ # of electric field lines through area.

/ $\vec{E}$ from pt. charge @ dist. R

$$D = \varepsilon E = \frac{\varepsilon Q}{4\pi\varepsilon R^2} = \frac{Q}{4\pi R}\hat{r}$$

permittivity of medium

→ similar to $\vec{E}$ but does not depend on permittivity
→ units $\frac{C}{m^2}$



Applying $\oint \vec{E}\cdot d\vec{l} = 0$ to the loop: [take magnitude of $\vec{E}_1$ & $\vec{E}_2$]

$$E_{1//}\Delta w + \cancel{E_{1\perp}\frac{\Delta h}{2}} + \cancel{E_{2\perp}\frac{\Delta h}{2}} - E_{2//}\Delta w - \cancel{E_{2\perp}\frac{\Delta h}{2}} - \cancel{E_{1\perp}\frac{\Delta h}{2}} = 0$$

$$(E_{1//} - E_{2//})\Delta w = 0$$

$$\boxed{E_{1//} = E_{2//}} \quad \text{as } \Delta h \to 0$$

→ tangential component of $\vec{E}$ is the same across the interface
→ continuous

since $\vec{D} = \varepsilon\vec{E} = \vec{D}_t + \vec{D}_n$

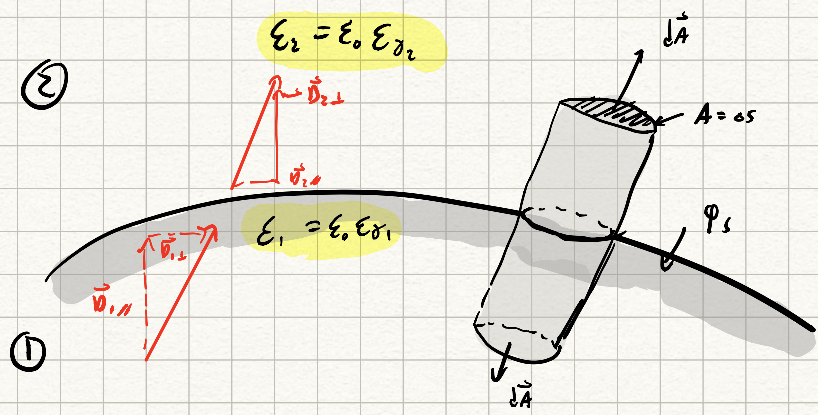
$\therefore E_{1//} = \frac{D_{1//}}{\varepsilon_1}$ and $E_{2//} = \frac{D_{2//}}{\varepsilon_2}$

since $E_{1//} = E_{2//}$,

$$\boxed{\frac{D_{1//}}{\varepsilon_1} = \frac{D_{2//}}{\varepsilon_2}}$$

→ tangential component of $\vec{D}$, $\vec{D}_{//}$, changes across the interface
→ discontinuous

Similarly, apply $\oint_s \vec{D} \cdot d\vec{A} = Q_{enc}$



$$\int_{top} \vec{D} \cdot d\vec{A} + \int_{bottom} \vec{D} \cdot d\vec{A} + \cancel{\int_{side} \vec{D} \cdot d\vec{A}}^{\,0} = Q_{enc}$$

notice that $\cos\theta$ between $\vec{D}_{1\parallel}$ and $d\vec{A} = -1$.

$\therefore \quad D_{2\perp} A - D_{1\perp} A = \rho_s A$

$$\boxed{D_{2\perp} - D_{1\perp} = \rho_s}$$

* if there is no free charges at the interface ($\rho_s = 0$), then:

$$\boxed{D_{2\perp} = D_{1\perp}}$$

→ normal component of $D_\perp$ continuous over interface

since $\vec{D} = \varepsilon \vec{E}$,

$$\boxed{\varepsilon_1 \vec{E}_{1\perp} = \varepsilon_2 \vec{E}_{2\perp}} \quad \text{or} \quad \boxed{\frac{\vec{E}_{1\perp}}{\varepsilon_2} = \frac{\vec{E}_{2\perp}}{\varepsilon_1}}$$

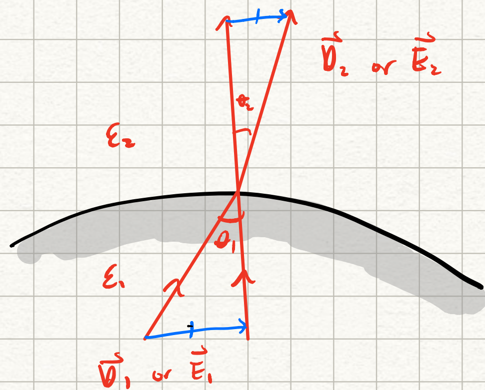
→ normal component of $\vec{E}$ is discontinuous at interface.

REFRACTION OVER A BOUNDARY

Mar 02

Boundary conditions can be used to determine "refraction" over an interface

$$\begin{cases} E_{1\parallel} = E_{2\parallel} \\ D_{1\perp} = D_{2\perp} \quad \text{(no charge)} \end{cases}$$



$E_{1\parallel} = E_{2\parallel}$

① $\boxed{E_1 \sin\theta_1 = E_2 \sin\theta_2}$

$D_{1\perp} = D_{2\perp}$

$D_1 \cos\theta_1 = D_2 \cos\theta_2$

② $\boxed{\varepsilon_1 E_1 \cos\theta_1 = \varepsilon_2 E_2 \cos\theta_2}$

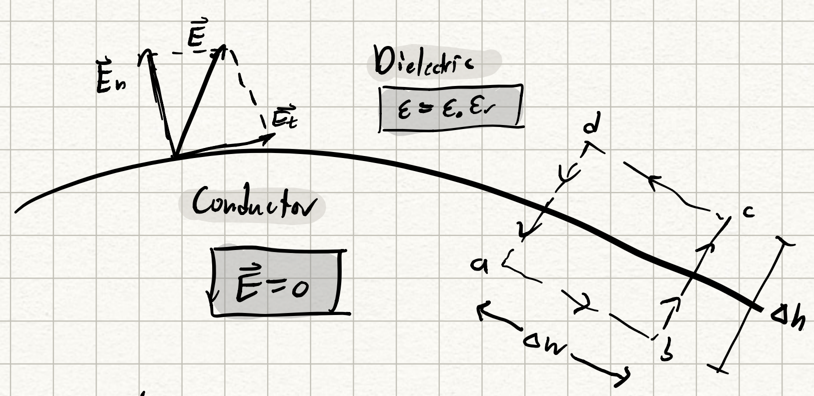
$$①/② \Rightarrow \frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

$$\Rightarrow \frac{\tan\theta_1}{\varepsilon_0 \varepsilon_{r1}} = \frac{\tan\theta_2}{\varepsilon_0 \varepsilon_{r2}}$$

$$\Rightarrow \boxed{\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_{r1}}{\varepsilon_{r2}}}$$

Law of Refraction for $\vec{E}$ at boundary w/o charge

CONDUCTOR-DIELECTRIC BOUNDARY COND.



① Apply $\oint \vec{E} \cdot d\vec{l} = 0$ along the loop,

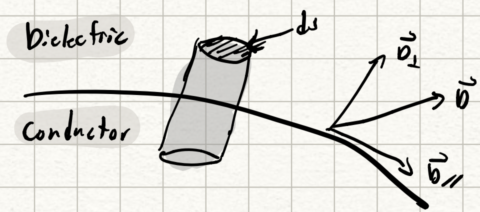
$$\cancelto{0}{\vec{E}_{c\parallel}}\,\cancel{\Delta w} + \frac{1}{2}\cancelto{0}{\vec{E}_{c\perp}}\,\cancel{\Delta h} + \cancel{\frac{1}{2}\vec{E}_{\perp}\Delta h} - \vec{E}_t \Delta w - \cancel{\frac{1}{2}\vec{E}_{\perp}\Delta h} - \frac{1}{2}\cancelto{0}{\vec{E}_{c\perp}}\,\cancel{\Delta h} = 0$$

$-\vec{E}_t \Delta w = 0$

$\Rightarrow \vec{E}_t = 0$ since Δw is non-zero.

$\longrightarrow$ this still satisfies $\vec{E}_{1\parallel} = \vec{E}_{2\parallel}$

② Apply $\int \vec{D} \cdot d\vec{s} = Q_{enc}$



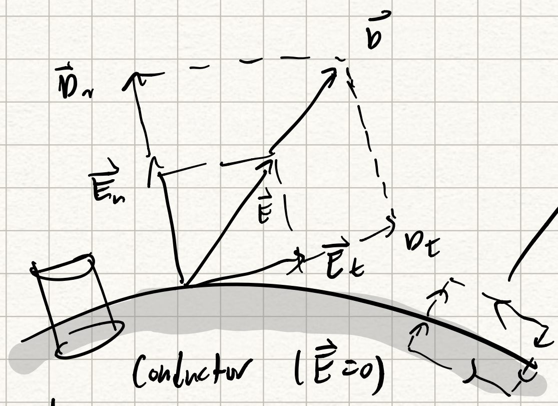
$$\int_{top} \vec{D} \cdot d\vec{s} + \cancel{\int_{bot} \vec{D} \cdot d\vec{s}} + \cancel{\int_{sides} \vec{D} \cdot d\vec{s}} = \rho_s A$$

$D_\perp A = \rho_s A$

$$\boxed{D_\perp = \rho_s}$$

$$\Rightarrow \boxed{D_\perp = 0 \quad \text{when } \rho_s = 0}$$

CONDUCTOR - FREE SPACE BOUNDARY



For free space, $\varepsilon_r = 1$.

$$\boxed{\vec{E}_t = 0}$$

→ this is why $\vec{E}$ always leaves ⊥ to surface

$$\Rightarrow \boxed{D_t = \varepsilon_0 \vec{E}_t = 0}$$

$$\int_{top} \vec{D}\cdot d\vec{s} + \cancel{\int_{bot} \vec{D}\cdot d\vec{s}} + \cancel{\int_{side} \vec{D}\cdot d\vec{s}} = Q_{enc}$$

$$D_n A = \rho_s A$$

$$\boxed{\begin{aligned} D_n &= \rho_s = \varepsilon_0 \vec{E}_n \\ \vec{E}_n &= \frac{\rho_s}{\varepsilon_0} \end{aligned}}$$

→ formula for electric field of a plane!!