TAYLOR POLYNOMIALS

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→ for smooth function $f(x)$, tangent line to graph at $(x_0, f(x_0))$ can be defined by considering secant line $(x_0, f(x_0))$ to $(x_1, f(x_1))$ and letting $x_1 \to x_0$.

→ this is the derivative, can use tangent line as approx. for x near x_0.

→ for any set of $n+1$ points with equidistant nodes, a polynomial of degree n can be found which passes through all of them.

if we have multiple points, what happens to the interpolating polynomial if all the points merge?

consider $f: x \to f(x)$ at

$$x_0 = x_0$$

$$x_1 = x_0 + \Delta x$$

$$x_2 = x_0 + 2\Delta x.$$

then,

$$y_0 = f(x_0)$$

$$y_1 = f(x_0 + \Delta x)$$

$$y_2 = f(x_0 + 2\Delta x)$$

using Newton's Forward Difference formula, the parabola joining these points is

$$y = y_0 + (x - x_0)\frac{\Delta y_0}{\Delta x} + \frac{1}{2}(x - x_0)(x - x_1)\frac{\Delta^2 y_0}{(\Delta x)^2}$$

what happens when $\Delta x \to 0$?

$$\frac{\Delta y_0}{\Delta x} = f'(x_0) \text{ as } \Delta x \to 0$$

∴ for the first two terms,

$$y = y_0 + f'(x_0)(x - x_0) + \ldots$$

$$\frac{\partial^2 y}{\partial x^2}$$

it's been proven that $\frac{\Delta^2 y_0}{(\Delta x)^2} = f''(x_0)$ as $\Delta x \to 0$.

thus,

$(x-x_0)(x-x_1) = (x-x_0)^2$ as $x_1 \to x_0$.

$$y = y_0 + f'(x_0)(x-x_0) + \frac{1}{2} f''(x_0)(x-x_0)^2$$

! ✱ for a good approximation, $P(x) = y_0$ when $x = x_0$.
→ why first term is y_0. (ex. first term for $P_{0,0}$ for $\sin x$ at $x=0$ is 0.

to improve approx, derivative should also be the same. or else approx. drifts too fast.

$y' = f'(x_0) + \ldots \to 0 + \cancel{\frac{1}{2} f''(x_0) x} \to 0$

so $P'(x_0) = f'(x_0)$ at x_0, and so on...

💡 this is similar to the tangent line for 2 points, but is instead a parabola for 3 points

→ quadratic approximation to $f(x)$!
→ more accurate approximation than linear (usually)

ex. consider $f(x) = \ln x$. To find linear approx. for $x \simeq 1$, we need f and f' at 1.

→ $f(1) = 0$
→ $f'(1) = \frac{1}{1} = 1$

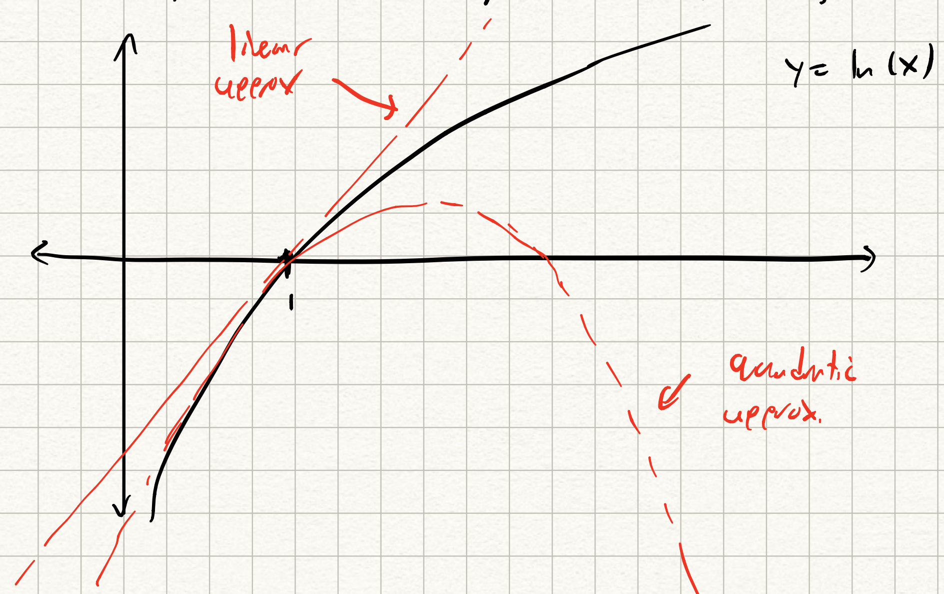
so linear approx is $\ln x \approx 0 + 1(x-1)$

to find quadratic approx, we need $f''(1) = -\frac{1}{x^2} = -1$

so quadratic approx. is

$$\ln x \approx 0 + 1(x-1) - \frac{1}{2}(x-1)^2$$

→ this takes concavity into account, leading to more accurate approx.



we can get even better approximations by assuming that

$\frac{\Delta^k y}{\ldots} = f^{(k)}(x_0)$ as $\delta x \to 0$. (which it does)

$(0x)^n$

an n-th order Taylor Polynomial is defined as:

$$P_{n,x_0}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2!}f''(x_0)(x-x_0)^2 + \dots + \frac{1}{n!}f^{(n)}(x_0)(x-x_0)^n$$

or, in sigma notation,

$$\boxed{P_{n,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = P_{n,x_0}(x_0)}$$

problem if $x = x_0$ and $k=0$.

In this context, we say $0^0 = 1$.

ex. find 3rd-order Taylor Polynomial for $f(x) = \ln x$, centered at 1

$$f'''(x) = \left(-\frac{1}{x^2}\right)' = \left(\frac{2}{x^3}\right) = \frac{2}{1} = 2, \text{ from previously,}$$

$$P_{3,1}(x) = 0 + 1(x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3$$

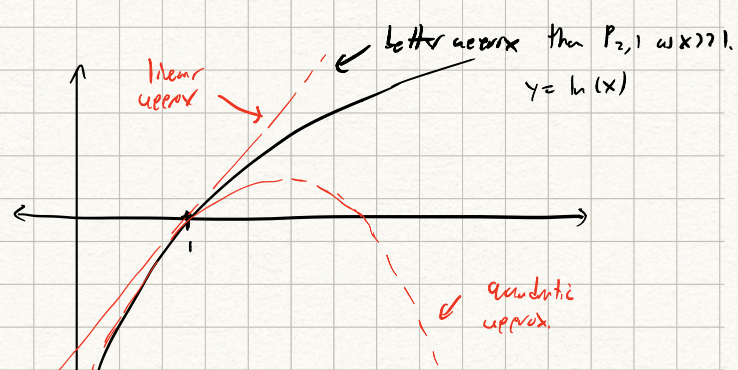
$$= 0 + 1(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

let's see how good these approximations are...

x	$\ln x$	$P_{1,1}(x)$	$P_{2,1}(x)$	$P_{3,1}(x)$
1	0	0	0	0
1.1	0.09531...	0.1	0.095	$0.095\overline{33}$
1.2	0.18233...	0.2	0.18	$0.18\overline{266}$
1.3	0.26236...	0.3	0.255	0.264
⋮	⋮	⋮	⋮	⋮
2	0.69314...	1	0.5	$0.8\overline{33}$
3	1.09861...	2	0	$2.\overline{66}$

- near $x=1$, approximations get better as more terms are added
- approximations get worse further from $x=1$ w/ more terms!!
  → for

→ how do we determine when it gets worse?



bad approx w/ $x >> 1$

what if $x < 1$?

x	$\ln x$	$P_{1,1}(x)$	$P_{2,1}(x)$	$P_{3,1}(x)$
1	0	0	0	0
0,9	$-0{,}10536\ldots$	$-0{,}1$	$-0{,}105$	$-0{,}115\overline{33}$
0,8	$-0{,}22314$	$-0{,}2$	$-0{,}22$	$-0{,}222\overline{66}$
0,7	$-0{,}35667$	$-0{,}3$	$-0{,}345$	$-0{,}354$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
0	$-\infty$	-1	$-1{,}8$	$-1{,}8\overline{33}$

$\rightarrow$ the polynomials in the above example "work" only for values in interval $[0, 2]$

$\rightarrow$ why?

$\leftrightarrow$ the interval is symmetrical about the center.

$\rightarrow$ radius of convergence

Ex. let's approximate $f(x) = \sin x$ centered at $x = \pi$ and $x = \frac{\pi}{4}$

approx is periodic, like the function.

$f(x) = \sin x$
$f'(x) = \cos x$
$f''(x) = -\sin x$
$f^{(3)}(x) = -\cos x$

$P_{4,x_0} = \sin x_0 + \cos x_0 (x - x_0) - \frac{1}{2} \sin x_0 (x - x_0)^2 - \frac{1}{6} \cos x_0 (x - x_0)^3$

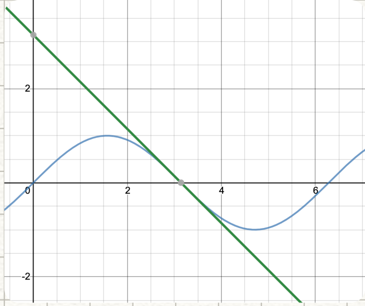
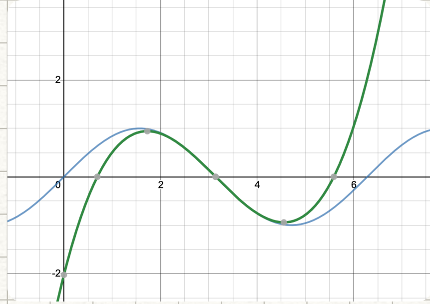
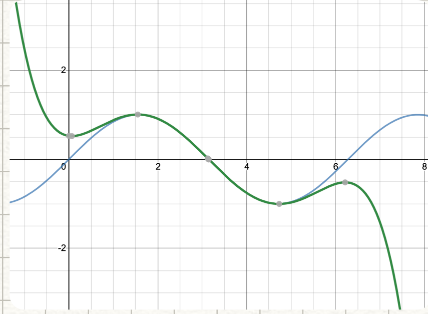
$\rightarrow$ 4th degree Taylor polynomial for any x_0

$\rightarrow$ since $\sin x$ is even, only even powers; similarly, for all odd fns, only odd powers

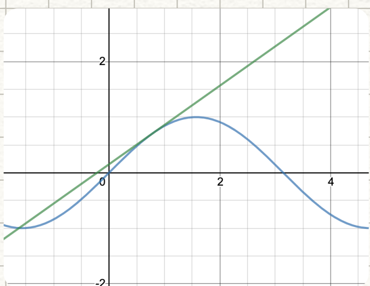
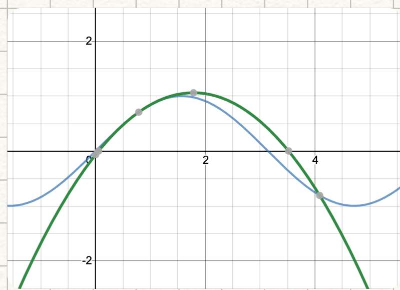
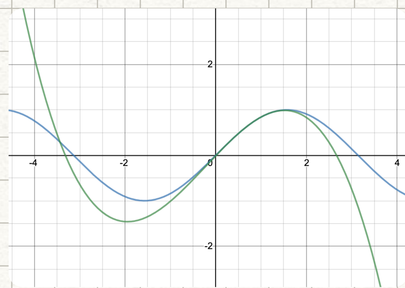
$\rightarrow$ notice that for $x_0 = \pi$, $\sin \pi = 0$

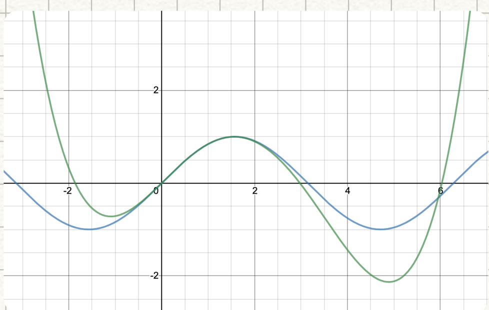
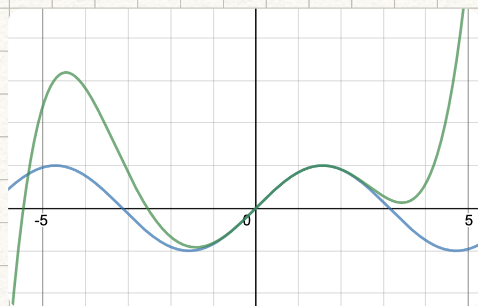
$\rightarrow$ all even terms are zero

$\Rightarrow$ no parabolic approx only odd functions!

→ for $x_0 = \pi/4$, we get better approx. for each term:

→ notice how parabola doesn't "sit" inside curve
  → sin / cos aren't parabolic!
  ↳ cos looks parabolic but isn't

→ technically, we can approximate the entire sin/ cos graph w/ infinite terms!!

★ where does $\sin(\theta) \approx \theta$ for small angles come from?

$f(0) = \sin(0)$
$f'(\theta) = \cos(\theta) \Rightarrow f'(0) = \cos(0) = 1$

using a 2nd-degree Taylor series,

$P_{2,0}(\theta) = 0 + 1(\theta - 0) = \theta$

so near $\theta = 0$, $\sin(\theta) \approx \theta$!!

★ for $P_{n,x_0}(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$,

think of 1st term = fixing value at x_0
2nd term = fixing derivative at x_0
3rd term = fixing concavity at x_0

ex. Taylor Polynomials for e^x

$\frac{d}{dx}e^x = e^x$

$$P(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \frac{x^n}{n!} = e^x \text{ as } n \to \infty!!$$

$\frac{d^2}{dx^2}e^x = e^x$

notice $\cosh(x) \approx 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!}$

$\sinh(x) \approx 1 + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!}$

$\vdots$

$\frac{d^n}{dx^n}e^x = e^x$ so $\cosh(x) + \sinh(x) \approx e^x$

ex. Taylor Polynomial for $\frac{1}{1-x}$ $\longrightarrow$ can get info from log and arctan

$f(x) = \frac{1}{1-x}$	$f(0) = 1$
$f'(x) = \frac{1}{(1-x)^2}$	$f'(0) = 1$
$f''(x) = \frac{2}{(1-x)^3}$	$f''(0) = 2$
$f'''(x) = \frac{6}{(1-x)^3}$	$f'''(0) = 6$
$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$	$f^{(n)}(0) = n!$

$$P_{n,0}(x) = 1 + x + x^2 + x^3 + \dots + x^n$$

ex. $f(x) = (1+x)^m$, $m \in \mathbb{R}$, m not positive integer

$f(x) = (1+x)^m$	$f(0) =$
$f'(x) = m(1+x)^{m-1}$	$f'(0) = m$
$f''(x) = m(m-1)(1+x)^{m-2}$	$f''(0) = m(m-1)$
$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$	$f'''(0) = m(m-1)(m-2)$

nth term

$\vee$

$$(1+x)^m = 1 + mx + \frac{m(m-1)x^2}{2!} + \frac{m(m-1)(m-2)x^3}{3!} + \dots \frac{m(m-1)\dots(m-n+1)x^n}{n!}$$

$$= \sum_{k=0}^{n} \binom{m}{k} x^k$$ → not standard choose notation

↳ ex. $\frac{1}{\sqrt{1+5x}} = (1-5x)^{-1/2}$

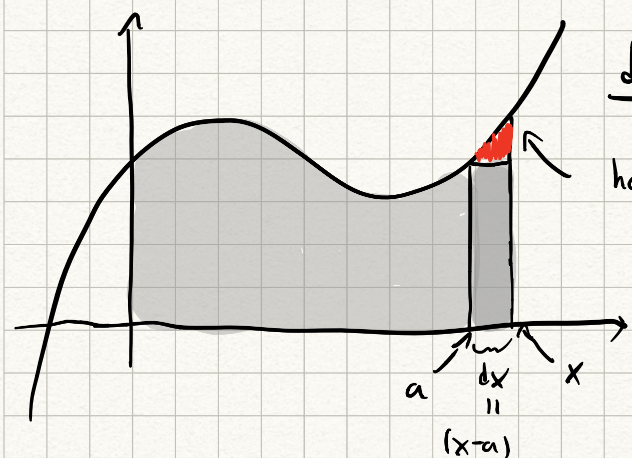
$$\approx 1 + \left(-\tfrac{1}{2}\right)(5x) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^2}{2} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) x^3}{6}$$

$$= 1 - \frac{5}{2}x + \frac{75}{8}x^2 - \frac{625}{16}x^3$$

geometric view

consider $p(x) = f(a) + \frac{df}{dx}(a)(x-a) + \frac{d^2f}{dx^2}(a)(x-a)^2$

how do we approximate area?



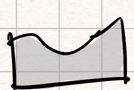
$\frac{df_{area}}{dx}(x)$

height = slope $(x-a)$ ← $\frac{\text{rise}}{\text{run}}$ = slope

$= \frac{d^2 f_{area}}{dx^2}(a)(x-a)$

so  $= \frac{1}{2}bh$

$= \frac{1}{2}\frac{d^2f}{dx^2}(a)(x-a)^2$

$f_{area}(x) \approx$  +  + 

$$= f(a) + \frac{df}{dx}(a)(x-a) + \frac{1}{2}\frac{d^2f}{dx^2}(a)(x-a)^2$$

A Taylor series is when we add infinitely many terms to a Taylor polynomial