BIG-~~O~~ TOE

<u>motivation:</u>

if we have a 4th-order Maclaurin polynomial:

$$1 - x^3 + \frac{1}{10}x^4$$

and find that the 5th-derivative is zero when x is zero,

we used to express this as

$$f(x) = 1 - x^3 + \frac{1}{10}x^4 + R_5(x)$$

Given two functions f and g,

- if $\exists A \in \mathbb{Z}$, s.t. $|f(x)| \leq A|g(x)|$ on some interval near x_0,
- $f(x) = O(g(x))$ as $x \to x_0$

→ technically, we should use $\in$ instead of $=$

$g^{(x-x_0)}$ if Taylor

g^{x} if Maclaurin

☆ in CS, we let $n \to \infty$ instead of $x \to x_0$

<u>ex.</u>

$$|x^3| \leq |x^2| \text{ for all } x \in [-1, 1]$$

$$\Rightarrow x^3 = O(x^2) \text{ as } x \to 0$$

also, since $|x^3| \leq |x|$,

$$\Rightarrow x^3 = O(x) \text{ as } x \to 0$$

however,

$$x^3 \neq O(x^4) \text{ as } x \to 0, \text{ but}$$

$$x^3 = O(x^4) \text{ as } x \to \infty$$

ex.

$|\sin x| \leq |x|$, so

$$\sin x = O(x) \text{ as } x \to 0$$

alternatively,

$$\left|\frac{\sin x}{x}\right| \leq 1 \quad \forall x \in \mathbb{R}, x \neq 0.$$

$$\frac{\sin x}{x} = O(1) \text{ as } x \to 0$$

ex.

$\frac{x^2}{8+x^3}$ on $[-1, 1]$

$$\left|\frac{x^2}{8+x^3}\right| = \frac{1}{|8+x^3|}|x^2| \leq \frac{1}{7}|x^2|$$

$$\left|\frac{x^2}{8+x^3}\right| = O(x^2) \text{ as } x \to 0$$

BIG-O w/ TAYLOR'S INEQUALITY

for $|R_n(x)| \leq \frac{K|x-x_0|^{n+1}}{(n+1)!}$, when $|f^{(n+1)}(t)| < K \;\; \forall t \in [x_0, x]$,

we can now say:

$$R_n(x) = O\left((x-x_0)^{n+1}\right) \text{ as } x \to x_0$$

$$\Rightarrow f(x) - P_{n,x_0}(x) = O\left((x-x_0)^{n+1}\right) \text{ as } x \to x_0$$

$$\Rightarrow \boxed{f(x) = P_{n,x_0}(x) + O((x-x_0)^{n+1}) \text{ as } x \to x_0}$$

ex1

$$\because \sqrt{1+x} = 1 + \tfrac{1}{2}x + R_2(x)$$

$$\Rightarrow \quad = 1 + \tfrac{1}{2}x + o(x^2) \quad \text{as} \quad x \to 0$$

$$\because \sin x = 0 + x + 0 + R_3(x)$$

$$\Rightarrow \quad = x + o(x^3) \quad \text{as} \quad x \to 0$$

$$\therefore \sqrt{1+x} + \sin x = 1 + \tfrac{3}{2}x + o(x^2) + o(x^3), \qquad x \to 0$$

$$= 1 + \tfrac{3}{2}x + o(x^2), \quad x \to 0$$

ex1

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$= \frac{1}{2}\Big[\Big(\cancel{1} + x + \cancel{\tfrac{x^2}{2!}} + \tfrac{x^3}{3!} + \cancel{\tfrac{x^4}{4!}} + o(x^5)\Big)$$
$$- \Big(\cancel{1} - x + \cancel{\tfrac{x^2}{2!}} - \tfrac{x^3}{3!} + \cancel{\tfrac{x^4}{4!}} + o(x^5)\Big)\Big]$$

$$= x + \frac{x^3}{3!} + o(x^5), \quad x \to 0$$

* we originally defined sinh as <u>odd</u> component of e^x and cosh as <u>even</u> component of e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

$\square$ = cosh(x)
$\square$ = sin h(x)

$$e^x = \sinh(x) + \cosh(x)$$

BIG-O PROPERTIES

properties

1. $K\,O(x^n) = O(x^n), \ \forall K \in \mathbb{R}$
2. $O(x^m) + O(x^n) = O(x^q), \quad q = \min(m, n)$
3. $O(x^m) \cdot O(x^n) = O(x^{m+n})$
4. $[O(x^n)]^m = O(x^{mn})$
5. $\dfrac{O(x^m)}{x^n} = O(x^{m-n})$

↳ there is no rule for $\dfrac{O(x^m)}{O(x^n)}$ since $O(x^n)$ could be from $n \to \infty$

substitution

$$\sqrt{1+x} = 1 + \frac{x}{2} + O(x^2), \ x \to 0$$

letting $x = u^4$,

$$\sqrt{1+u^4} = 1 + \frac{u^4}{2} + O(u^8), \ u \to 0$$

multiplication

$$e^x \sin x = \left[1 + x + \frac{x^2}{2} + O(x^3)\right]\left[x + O(x^3)\right]$$

$$= x + x^2 + \underbrace{\frac{x^3}{2}}_{O(x^3)} + \underbrace{xO(x^3)}_{O(x^3)} + O(x^3) + \underbrace{x\,O(x^3)}_{O(x^3)} + \underbrace{\frac{x^2}{2}O(x^3)}_{O(x^3)} + \underbrace{[O(x^3)]^2}_{O(x^3)}$$

$$= x + x^2 + O(x^3), \ x \to 0$$

division

$$\tan(x) = \frac{\sin x}{\cos x} = \frac{x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)}{1 - \frac{x^2}{2} + \frac{x^4}{24} + O(x^6)}$$

$$\begin{array}{r|l} & x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7) \\ \hline 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + O(x^6) & x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + O(x^7) \end{array}$$

O(x^7) propogates, think of as collection of garbage terms

$$-\left(x - \frac{1}{2}x^3 + \frac{1}{24}x^5 + O(x^7)\right)$$

$$\frac{1}{3}x^3 - \frac{1}{30}x^5 + O(x^7)$$

$$-\left(\frac{1}{3}x^3 - \frac{1}{6}x^5 + O(x^7)\right)$$

$$\frac{2}{15}x^5 + O(x^7)$$

$$-\left(\frac{2}{15}x^5 + O(x^7)\right)$$

$$O(x^7)$$

$$\tan x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7)$$

USING BIG-O TO EVALUATE LIMITS

Ex1

$$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x + O(x^3)}{x}$$

$$= \lim_{x\to 0} 1 + O(x^2)$$

$$= 1$$

Ex2 $$\lim_{x\to 0} \frac{x\cos x - \sin x}{x^3} = \lim_{x\to 0} \frac{x\left[1 - \frac{x^2}{2!} + O(x^4)\right] - \left[x - \frac{x^3}{3!} + O(x^5)\right]}{x^3}$$

$$= \lim_{x\to 0} \frac{\cancel{x} - \frac{x^3}{2!} + O(x^5) \cancel{- x} + \frac{x^3}{3!} + O(x^4)}{x^3}$$

$$= \lim_{x\to 0} \frac{-\frac{1}{3}x^3 + O(x^5)}{x^3}$$

$$= \lim_{x\to 0} -\frac{1}{3} + O(x^2)$$

$$= -\frac{1}{3}$$