Figure 1. The geometric representation of G.

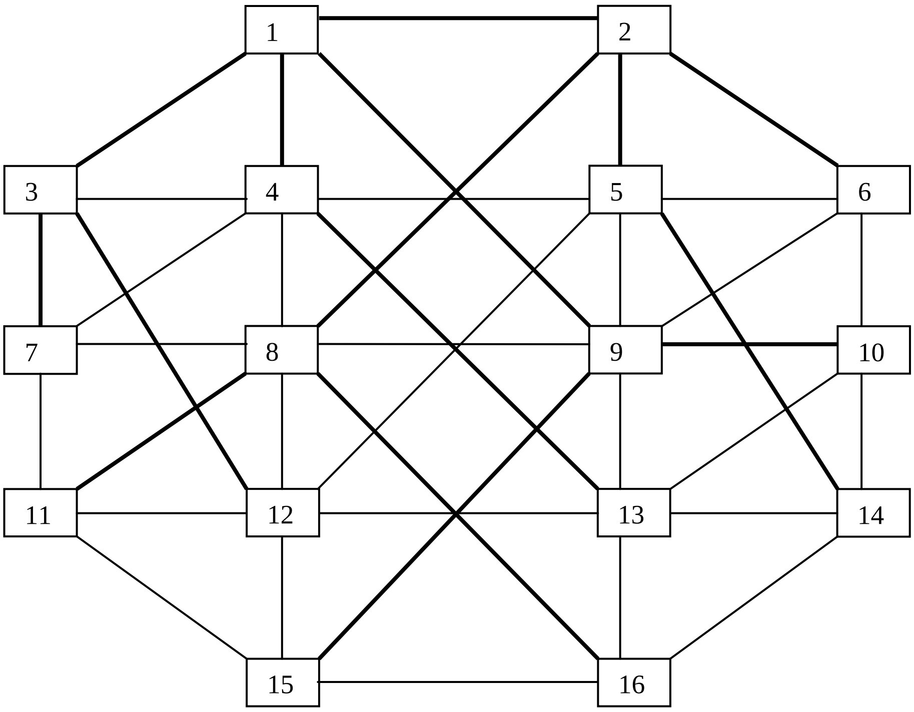

Table 1. The adjacency matrix of G.

										1	1	1	1	1	1	1
		1	2	3	4	5	6	7	8	9	0	1	2	3	4	5	6
	1	×	●	●	●					●							
	2	●	×			●	●		●								
	3	●		×	●			●					●				
	4	●		●	×	●		●	●					●			
	5		●		●	×	●			●			●		●		
	6		●			●	×			●	●						
	7			●	●			×	●			●					
	8		●		●			●	×	●		●	●				●
	9	●				●	●		●	×	●			●		●	
1	0						●			●	×			●	●		
1	1							●	●			×	●			●	
1	2			●		●			●			●	×	●		●	
1	3				●					●	●		●	×	●		●
1	4					●					●			●	×		●
1	5									●		●	●			×	●
1	6								●					●	●	●	×

Quasi coloring

Let G=(V,E) be a finite simple graph. Let $C_1,\dots,C_k$ be a partition of V and suppose that t_i is the number of all the edges connecting the nodes of C_i. Set $t=t_1+\dots+t_k$. If $t_i=0$, then C_i is an independent set in G. If t=0, then each C_i is an independent set in G and $C_1,\dots,C_k$ can be viewed as color classes of a legal k-coloring of G. We call the partition $C_1,\dots,C_k$ a (k,t)-quasi coloring of G. Intuitively the closer is t to 0 the closer is a (k,t)-quasi coloring to a k-coloring. The quantity t can be called the number of disturbing nodes.
Given a partition $C_1,\dots,C_k$ of V the d(v,j) degree of a node v is the number of the neighbors of v in C_j. We record the d(v,j) degrees in a D matrix. The rows of D are labeled by the nodes of G and the columns of D are labeled by the colors 1,…,k. The map f: V→{1,…,k} is used to express the fact that the node v receives the color f(v). Moving a node v from C_p to C_q creates a new quasi coloring $(C_1)',\dots,(C_k)'$. Here $(C_p)'=C_p-\{v\}$, $(C_q)'=C_q\cup\{v\}$ and $(C_i)'=C_i$ for each $i\neq p$, $i\neq q$. Suppose that d(v,1),…,d(v,k) are the degrees of v with respect to the quasi coloring $C_1,\dots,C_k$. Then the new degrees d'(v,1),…,d'(v,k) of v with respect to the quasi coloring $(C_1)',\dots,(C_k)'$ can be computed in the following way. If {v,u} is an edge of G, then d'(u,p)=d(u,p)–1, d'(u,q)=d(u,q)+1 and d'(u,i)=d(u,i) for $i\neq p$, $i\neq q$. If {v,w} is not an edge of G, then d'(w,1)=d(w,1),…,d'(w,k)=d(w,k).
One can observe that if d(v,p)>d(v,q) holds for a v in C_p, then moving v from C_p to C_q reduces the number of the disturbing edges. This move improves the quality of the quasi coloring and forms the base of a greedy quasi coloring algorithm.

Let us see an example. We use the graph in Figure 3. The initial quasi coloring is $C_1=\{1,2,3,4,5\}$, $C_2=\{6,7,8,9,10\}$, $C_3=\{11,12,13,14,15,16\}$. The degrees of the nodes of G with respect to this quasi 3-coloring can be seen in Table 19. The rows are labeled with the nodes and the names of the nodes are in the 1-st column. The columns are labeled with the colors and the names of the colors are in the 1-st row. The colors of the nodes are in the 5-th column. The bracketed degrees inside the table are used to compute the number of disturbing nodes. Which is this time t=(1/2)[12+10+16]=19.

Table 2. The initial degrees.

	C_1	C_2	C_3	
1	[3]	1	0	1
2	[2]	2	0	1
3	[2]	1	1	1
4	[3]	2	1	1
5	[2]	2	2	1
6	2	[2]	0	2
7	2	[1]	1	2
8	2	[2]	3	2
9	2	[3]	2	2
10	0	[2]	2	2
11	0	2	[2]	3
12	2	1	[3]	3
13	1	2	[3]	3
14	1	1	[2]	3
15	0	1	[3]	3
16	0	1	[3]	3

An inspection reveals that d(1,1)>d(1,3) and so it is advantageous to move node 1 from C_1 to C_3. We compute the new degrees in a new table. We will call the element 1 the pivot element of the computation. We mark the pivot element with an ← symbol and also we mark the neighbors of the pivot element with a + symbol. To hold these marking symbols we add a new column to the table. We will work with extended tables. The extended initial table and the extended new table are presented in Table 20.

Table 3. The initial degrees and the new degrees.

	C_1	C_2	C_3		
1	[3]	1	0	1	←
2	[2]	2	0	1	+
3	[2]	1	1	1	+
4	[3]	2	1	1	+
5	[2]	2	2	1	
6	2	[2]	0	2	
7	2	[1]	1	2	
8	2	[2]	3	2	
9	2	[3]	2	2	+
10	0	[2]	2	2	
11	0	2	[2]	3	
12	2	1	[3]	3	
13	1	2	[3]	3	
14	1	1	[2]	3	
15	0	1	[3]	3	
16	0	1	[3]	3	

	C_1	C_2	C_3		
1	3	1	[0]	3	
2	[1]	2	1	1	+
3	[1]	1	2	1	
4	[2]	2	2	1	
5	[2]	2	2	1	+
6	2	[2]	0	2	←
7	2	[1]	1	2	
8	2	[2]	3	2	
9	1	[3]	3	2	+
10	0	[2]	2	2	+
11	0	2	[2]	3	
12	2	1	[3]	3	
13	1	2	[3]	3	
14	1	1	[2]	3	
15	0	1	[3]	3	
16	0	1	[3]	3	

	C_1	C_2	C_3		
1	3	1	[0]	3	+
2	[1]	1	2	1	
3	[1]	1	2	1	
4	[2]	2	2	1	
5	[2]	1	3	1	+
6	2	2	[0]	3	+
7	2	[1]	1	2	
8	2	[2]	3	2	+
9	1	[2]	4	2	←
10	0	[1]	3	2	+
11	0	2	[2]	3	
12	2	1	[3]	3	
13	1	2	[3]	3	+
14	1	1	[2]	3	
15	0	1	[3]	3	+
16	0	1	[3]	3	

The number of the disturbing nodes in the new quasi coloring is t=(1/2)[6+10+16]=16. As d(6,2)>d(6,3) we choose node 6 to be the pivot element. We mark the pivot and its neighbors in the last column. The computation continues in Table 21.

Table 4. The computation continues.

	C_1	C_2	C_3		
1	4	0	[0]	3	
2	[1]	1	2	1	+
3	[1]	1	2	1	
4	[2]	2	2	1	+
5	[3]	0	3	1	←
6	3	1	[0]	3	+
7	2	[1]	1	2	
8	3	[1]	3	2	
9	[1]	2	4	1	+
10	1	[0]	3	2	
11	0	2	[2]	3	
12	2	1	[3]	3	+
13	2	1	[3]	3	
14	1	1	[2]	3	+
15	1	0	[3]	3	
16	0	1	[3]	3	

	C_1	C_2	C_3		
1	4	0	[0]	3	
2	[0]	2	2	1	
3	[1]	1	2	1	
4	[1]	3	2	1	
5	3	[0]	3	2	
6	2	2	[0]	3	
7	2	[1]	1	2	+
8	3	[1]	3	2	+
9	[0]	3	4	1	
10	1	[0]	3	2	
11	0	2	[2]	3	←
12	1	2	[3]	3	+
13	2	1	[3]	3	
14	0	2	[2]	3	
15	1	0	[3]	3	+
16	0	1	[3]	3	

	C_1	C_2	C_3		
1	4	0	[0]	3	
2	[0]	2	2	1	
3	[1]	1	2	1	
4	[1]	3	2	1	
5	3	[0]	3	2	
6	2	2	[0]	3	
7	3	[1]	0	2	
8	4	[1]	2	2	+
9	[0]	3	4	1	
10	1	[0]	3	2	
11	[0]	2	2	1	
12	2	2	[2]	3	
13	2	1	[3]	3	+
14	0	2	[2]	3	+
15	2	0	[2]	3	+
16	0	1	[3]	3	←

	C_1	C_2	C_3		
1	4	0	[0]	3	
2	[0]	2	2	1	
3	[1]	1	2	1	
4	[1]	3	2	1	+
5	3	[0]	3	2	
6	2	2	[0]	3	
7	3	[1]	0	2	
8	5	[1]	1	2	
9	[0]	3	4	1	+
10	1	[0]	3	2	+
11	[0]	2	2	1	
12	2	2	[2]	3	+
13	3	1	[2]	3	←
14	1	2	[1]	3	+
15	3	0	[1]	3	

	C_1	C_2	C_3		
1	4	0	[0]	3	
2	[0]	2	2	1	
3	[1]	1	2	1	
4	[1]	4	1	1	
5	3	[0]	3	2	
6	2	2	[0]	3	
7	3	[1]	0	2	
8	5	[1]	1	2	
9	[0]	4	3	1	+
10	1	[1]	2	2	
11	[0]	2	2	1	+
12	2	3	[1]	3	+
13	3	[1]	2	2	
14	1	3	[0]	3	
15	3	0	[1]	3	←

	C_1	C_2	C_3		
1	4	0	[0]	3	
2	[0]	2	2	1	
3	[1]	1	2	1	+
4	[1]	4	1	1	+
5	3	[0]	3	2	
6	2	2	[0]	3	
7	3	[1]	0	2	←
8	5	[1]	1	2	+
9	[0]	5	2	1	
10	1	[1]	2	2	
11	[0]	3	1	1	+
12	2	4	[0]	3	
13	3	[1]	2	2	
14	1	3	[0]	3	
15	3	[0]	1	2	

16	[0]	1	3	1	+
	C_1	C_2	C_3		
1	4	0	[0]	3	+
2	[0]	2	2	1	
3	[1]	0	3	1	←
4	[1]	3	2	1	+
5	3	[0]	3	2	
6	2	2	[0]	3	
7	3	1	[0]	3	+
8	5	[0]	2	2	
9	[0]	5	2	1	
10	1	[1]	2	2	
11	[0]	2	2	1	
12	2	4	[0]	3	+
13	3	[1]	2	2	
14	1	3	[0]	3	
15	3	[0]	1	2	
16	[0]	3	1	1	

16	[0]	2	2	1	+
	C_1	C_2	C_3		
1	3	1	[0]	3	
2	[0]	2	2	1	
3	1	[0]	3	2	
4	[0]	4	2	1	
5	3	[0]	3	2	
6	2	2	[0]	3	
7	2	2	[0]	3	
8	5	[0]	2	2	
9	[0]	5	2	1	
10	1	[1]	2	2	
11	[0]	2	2	1	
12	1	3	[0]	3	
13	3	[1]	2	2	
14	1	3	[0]	3	
15	3	[0]	1	2	
16	[0]	3	1	1	

16	[0]	3	1	1	

Table 5. The rearranged adjacency matrix of G.

					1	1				1	1	1				1	1
		2	4	9	1	6	3	5	8	0	3	5	1	6	7	2	4
	2	×						●	●				●	●			
	4		×				●	●	●		●		●		●		
	9			×				●	●	●	●	●	●	●			
1	1				×				●			●			●	●	
1	6					×			●		●	●					●
	3		●				×						●		●	●	
	5	●	●	●				×						●		●	●
	8	●	●	●	●	●			×						●	●	
1	0			●						×	●			●			●
1	3		●	●		●				●	×					●	●
1	5			●	●	●						×				●	
	1	●	●	●			●						×				
	6	●		●				●		●				×			
	7		●		●		●		●						×		
1	2				●		●	●	●		●	●				×	
1	4					●		●		●	●						×

Table 23 contains the rearranged adjacency matrix of the graph G. It is clearly visible that the sets C_1={2,4,9,11,16}, C_2={3,5,8,10,13,15}, C_3={1,6,7,12,14} form a (3,1)-quasi coloring of G. The only disturbing edge is {10,13}. After deleting the edge {10,13} the remaining graph has a legal 3-coloring and so it cannot have a 4-clique. If G has a 4-clique, then it must be in the graph spanned by the common neighbors of the nodes 10 and 13. The common neighbors of 10 and 13 are 9,14. If G has a 4-clique, then the graph spanned by {9,14}must contain a 2-clique. (But {9,14} is not an edge of G.)
In general, if $G_1,\ldots,G_t$ are the graphs spanned by the common neighbors of the disturbing nodes of a (k–1,t)-quasi coloring, then one has to look for (k–2)-cliques in $G_1,\ldots,G_t$ in order to find k-cliques in G. This is the quasi coloring based branching rule.

Let us work out a further example. We use the graph in Figure 1. A greedy sequential coloring and clique partition give the results in Table 24.

Table 6. A greedy coloring and a greedy clique partition.

color	1	2	3	4	5	6
	1	2	3	4	8	11
	5	6	9	10		
	7					
	12					

clique	1	2	3	4
	1	5	7	11
	2	6	8	12
	3	9		
	4	10		

Thus the clique size of G is between 5 and 6. The upper bound suggests constructing a quasi 5-coloring of G and test if the upper bound 6 can be reduced.

The initial quasi coloring is $C_1=\{1,2\}$, $C_2=\{3,4\}$, $C_3=\{5,6\}$, $C_4=\{7,8,9\}$, $C_5=\{10,11,12\}$. The degrees of the nodes of G with respect to this quasi 5-coloring can be seen in Table 25. The bracketed degrees inside the table are used to compute the number of disturbing nodes. Which is this time $t=(1/2)[2+2+2+4+6]=8$.