

# Mathematics

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# 1 Calculus

## 1.1 Differentiation

- 2 Series**
- 3 Multivariable Calculus**
- 4 Linear Algebra**
- 5 Differential Equations**
- 6 Partial Differential Equations**

## 7 Vector Calculus

### 7.1 Operators

#### 7.1.1 Grad

#### 7.1.2 Div

#### 7.1.3 Curl

#### 7.1.4 Laplacian

### 7.2 Integral Theorems

#### 7.2.1 Divergence Theorem

#### 7.2.2 Stokes's Theorem

## 8 Fluid Mechanics

### 8.1 Kinematics

#### 8.1.1 Coordinates

**Lagrangian**  $\underline{x}(\underline{a}, t)$ : The *motion of individual particles* is studied; the position  $\underline{x}$  of a particle at time  $t$  is related to its position at a reference point in time  $\underline{a}$  (typically at  $t = 0$ ).

**Eulerian**  $(\underline{x}, t)$ : The *'flow field' is considered as a whole* and the state of a fluid is described in terms of the values at a fixed location  $\underline{x}$  and at a fixed time  $t$ .

#### 8.1.2 Velocity

In Cartesian coordinates the velocity of a fluid particle at position  $\underline{x}(x, y, z)$  is given by:

$$\underline{u}(x, y, z) = u(x, y, z)\hat{i} + v(x, y, z)\hat{j} + w(x, y, z)\hat{k}$$

#### 8.1.3 Stagnation Points

Stagnation points occur when the velocity vector  $\underline{u}$  is equal to  $\underline{0}$

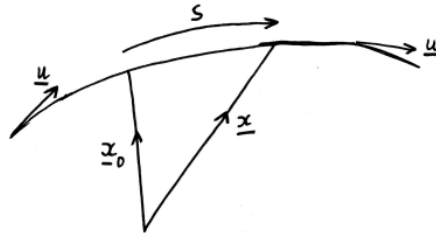
$$u = 0$$

$$v = 0$$

$$w = 0$$

#### 8.1.4 Streamlines

A streamline is a curve  $C$  drawn at one point in time such that the fluid velocity vector  $\underline{u}$  is tangent to  $C$  at every point along  $C$ .



$$\frac{d\underline{x}}{ds} = \underline{u}$$

$$\frac{dx}{ds} = u, \frac{dy}{ds} = v, \frac{dz}{ds} = w$$

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} (= ds)}$$

### 8.1.5 Particle Paths

Particle path is obtained by solving the initial value problem:

$$\frac{d\underline{x}}{dt} = \underline{u}(\underline{x}, t), \underline{x} = x_0 \text{ at } t = 0$$

$$\frac{dx}{dt} = u, x(0) = x_0$$

$$\frac{dy}{dt} = v, y(0) = y_0$$

$$\frac{dz}{dt} = w, z(0) = z_0$$

### 8.1.6 Steady Flow

**Steady Flow:** The flow velocity vector  $\underline{u}$  is independent of time  $t$

**Unsteady Flow:**  $\underline{u}$  depends on  $t$ ; the pattern of streamlines changes with  $t$

### 8.1.7 Convective Derivative

The convective derivative tells us how a property changes *as it moves with a flow*.

**General**

$$\boxed{\frac{D*}{Dt} = \frac{\partial*}{\partial t} + (\underline{u} \cdot \nabla)* = \frac{\partial*}{\partial t} + u \frac{\partial*}{\partial x} + v \frac{\partial*}{\partial y} + w \frac{\partial*}{\partial z}}$$

**Scalar**

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + (\underline{u} \cdot \nabla)\rho = \frac{\partial\rho}{\partial t} + u \frac{\partial\rho}{\partial x} + v \frac{\partial\rho}{\partial y} + w \frac{\partial\rho}{\partial z}}$$

**Vector**

$$\boxed{\frac{D\underline{u}}{Dt} = \frac{\partial\underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} = \frac{\partial\underline{u}}{\partial t} + u \frac{\partial\underline{u}}{\partial x} + v \frac{\partial\underline{u}}{\partial y} + w \frac{\partial\underline{u}}{\partial z}}$$

### 8.1.8 Vorticity

Vorticity  $\underline{\omega}$  is a measure of the local rotation of fluid particles in flow.

$$\underline{\omega} = \nabla \times \underline{u}$$

$$\nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

**Irrotational Flow:**

$$\underline{\omega} = \underline{0}$$

### 8.1.9 Incompressible Flow

$$\nabla \cdot \underline{u} = 0$$

When this is true the convective derivative of the fluid density is zero:

$$\frac{D\rho}{Dt} = 0$$

### 8.1.10 Velocity Potential

For an irrotational flow the velocity can be described as the gradient of a scalar field known as the *Velocity Potential*.

$$\nabla \times \underline{u} = 0$$

The curl of the gradient of a scalar field is zero:

$$\nabla \times (\nabla \phi) = 0$$

$$\underline{u} = \nabla \phi$$

If the flow is also incompressible

$$\nabla \cdot \underline{u} = 0$$

$$\nabla \cdot (\nabla \phi) = 0$$

Therefore the velocity potential of an irrotational, incompressible flow satisfies

Laplace's Equation:

$$\nabla^2 \phi = 0$$

### 8.1.11 Equipotential Surfaces

Lines/surfaces of constant  $\phi$  are *equipotentials*.

The velocity potential can be considered a surface:

$$\phi(x, y, z) = c$$



Let  $\underline{a}$  be tangent to the surface, the derivative of  $\phi$  in the direction of  $\underline{a}$ :

$$\underline{a} \cdot \nabla \phi = 0$$

Because the derivative of a constant  $c$  is zero:  $\nabla \phi$  is normal to the surface.

$$\underline{\hat{n}} = \frac{\nabla \phi}{|\nabla \phi|}$$

### 8.1.12 The Stream Function

Considering incompressible two dimensional flow:

$$\begin{aligned} \nabla \cdot \underline{u} &= 0 \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}$$

We can introduce the stream function  $\psi(x, y, t)$  such that:

$$u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}$$

This satisfies the previous equations:

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

**Vorticity of an incompressible two dimensional flow:**

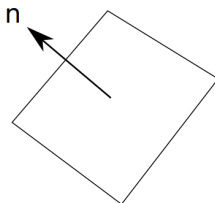
$$\begin{aligned} \underline{\omega} &= \nabla \times \underline{u} \\ \underline{\omega} &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\ \underline{\omega} &= -\frac{\partial^2 \psi}{\partial^2 x} - \frac{\partial^2 \psi}{\partial^2 y} \\ \underline{\omega} &= -\nabla^2 \psi \end{aligned}$$

## 8.2 Pressure in a Fluid

### 8.2.1 Pressure and Force

Let the normal vector  $\underline{n}$  be pointing into the fluid.

$$\partial \underline{F} = -\partial F \underline{n}$$



$\partial F$  is the force exerted over a small area  $\partial S$

$$p = \lim_{\partial S \rightarrow 0} \left( \frac{\partial F}{\partial S} \right)$$

$$\underline{F} = - \int \int_S p \underline{n} dS$$

### 8.2.2 Equations of State

#### Ideal Gas Law

$$pV = nRT$$

$$\frac{pV}{T} = nR$$

$$V \propto 1/\rho$$

$$\frac{p}{\rho T} = \text{constant}$$

#### Isothermal Gas

For a gas at constant temperature  $T$

Boyle's Law:

$$p \propto \rho$$

$$p\rho^{-1} = p_0\rho_0^{-1}$$

#### Adiabatic Gas

Quick fluctuations in pressure (such as in sound waves) are described by the adiabatic law.

$$p\rho^{-\gamma} = c$$

Where  $\gamma$  is the *Polytropic Index*

### 8.2.3 Hydrostatics

The force due to the pressure of the fluid

$$\underline{F} = - \int \int_S p \underline{n} dS$$

Applying the divergence theorem for a scalar:

$$\underline{F} = - \int \int_V \nabla p dV$$

Calculating the weight of the fluid region:

$$\underline{W} = \int \int_V \rho \underline{g} dV$$

Since the fluid is stationary, the total force on the fluid is zero

$$\underline{W} + \underline{F} = 0$$

$$\int \int \int_V \rho \underline{g} dV - \int \int \int_V \nabla p dV = 0$$

$$\int \int \int_V (\rho \underline{g} - \nabla p) dV = 0$$

Therefore:

$$\rho \underline{g} = \nabla p$$

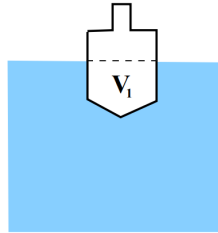
$$\begin{aligned}
 \underline{g} &= -g\hat{k} \\
 \frac{\partial p}{\partial z} &= -\rho g \\
 \text{Integrating gives:} \\
 \boxed{p &= p_a - \rho g z}
 \end{aligned}$$

### 8.2.4 Buoyancy

Buoyancy is the force exerted on a submerged body due to the pressure of the surrounding static fluid.

$$\begin{aligned}
 \underline{B} &= - \int_S \underline{p} \underline{n} dS \\
 \text{Applying the divergence theorem for a scalar:} \\
 \underline{B} &= - \int \int_V \nabla p dV \\
 \nabla p &= \rho \underline{g}, \underline{g} = -g\hat{k} \\
 \underline{B} &= \hat{k} g \int \int_V \rho dV \\
 \boxed{\underline{B} &= \rho V g \hat{k} = m_w g \hat{k}}
 \end{aligned}$$

**Archimedes' Principle** – the buoyancy force on a body is equal in magnitude to the weight of fluid that is displaced by the body.



$$\begin{aligned}
 V_1 &= \text{Volume of displaced water,} \\
 \rho_w &= \text{Density of water,} \\
 m_B &= \text{Mass of the body} \\
 \text{At equilibrium:} \\
 \underline{B} + \underline{W} &= 0 \\
 \rho_w V_1 g \hat{k} - m_B g \hat{k} &= 0 \\
 \boxed{m_B &= \rho_w V_1}
 \end{aligned}$$

**Center of Mass (Gravity):** A theoretical point in the body where the body's total mass (weight) is thought to be concentrated.

**Center of Buoyancy:** The centre of mass of the displaced water

A floating body is stable if its centre of gravity is below its centre of buoyancy.

### 8.2.5 Oscillation of a Floating Body

When a floating body is submerged below its equilibrium depth  $d$  it begins to oscillate.

Consider a cylindrical body with mass  $m$ , circular area  $A$ , equilibrium depth  $d$ :

$$\begin{aligned} mz''(t) &= \rho_w g(d - z(t))A - mg \\ &= -\rho_w g z(t)A \\ &= mg \frac{z(t)}{d} \end{aligned}$$

Therefore you can form the differential equation:

$$\begin{aligned} z''(t) &= \frac{g}{d} z(t) \\ z(0) &= -z_0 \end{aligned}$$

The solution:

$$z(t) = -z_0 \cos(\sqrt{\frac{g}{d}}t)$$

### 8.3 Flow Dynamics

### 8.4 Two-dimensional Flow

### 8.5 Vorticity Dynamics

### 8.6 Free Surface Waves

## 9 Numerical Methods