Mathematics

Oliver Brady

April 4, 2023

Contents

1 Calculus

1.1 Differentiation

2 Series

3 Multivariable Calculus

4 Linear Algebra

5 Differential Equations

6 Partial Differential Equations

7 Vector Calculus

7.1 Operators

7.1.1 Grad

7.1.2 Div

7.1.3 Curl

7.1.4 Laplacian

7.2 Integral Theorems

7.2.1 Divergence Theorem

7.2.2 Stokes's Theorem

8 Fluid Mechanics

8.1 Kinematics

8.1.1 Coordinates

Lagrangian $\underline{x}(\underline{a}, t)$: The *motion of individual particles* is studied; the position $\underline{x}$ of a particle at time t is related to its position at a reference point in time $\underline{\mathrm{a}}$ (typically at $t = 0$).

Eulerian $(\underline{x}, t)$: The *'flow field' is considered as a whole* and the state of a fluid is described in terms of the values at a fixed location $\underline{x}$ and at a fixed time t

8.1.2 Velocity

In Cartesian coordinates the velocity of a fluid particle at position $\underline{x}(x, y, z)$ is given by:

$$\underline{u}(x,y,z) = u(x,y,z)\underline{\hat{i}} + v(x,y,z)\underline{\hat{j}} + w(x,y,z)\underline{\hat{k}}$$

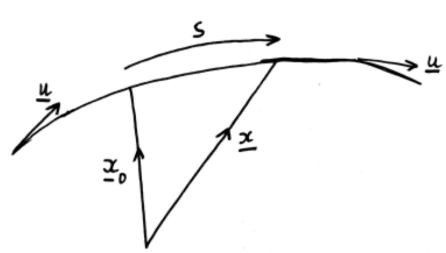
8.1.3 Stagnation Points

Stagnation points occur when the velocity vector $\underline{u}$ is equal to $\underline{0}$

$$\begin{aligned} u &= 0 \\ v &= 0 \\ w &= 0 \end{aligned}$$

8.1.4 Streamlines

A streamline is a curve C drawn at one point in time such that the fluid velocity vector $\underline{u}$ is tangent to C at every point along C.



$$\frac{d\underline{x}}{ds} = \underline{u}$$

$$\frac{dx}{ds} = u, \frac{dy}{ds} = v, \frac{dz}{ds} = w$$

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} (= ds)}$$

8.1.5 Particle Paths

Particle path is obtained by solving the initial value problem:

$$\tfrac{d\underline{x}}{dt} = \underline{u}(\underline{x}, t)\ ,\ \underline{x} = x_0 \text{ at } t = 0$$

$$\boxed{\begin{array}{c} \tfrac{dx}{dt} = u\ ,\ x(0) = x_0 \\ \tfrac{dy}{dt} = v,\ y(0) = y_0 \\ \tfrac{dz}{dt} = w,\ z(0) = z_0 \end{array}}$$

8.1.6 Steady Flow

Steady Flow: The flow velocity vector $\underline{u}$ is independent of time t
Unsteady Flow: $\underline{u}$ depends on t; the pattern of streamlines changes with t

8.1.7 Convective Derivative

The convective derivative tells us how a property changes *as it moves with a flow.*
General

$$\boxed{\frac{D*}{Dt} = \frac{\partial *}{\partial t} + (\underline{u} \cdot \nabla)* = \frac{\partial *}{\partial t} + u\frac{\partial *}{\partial x} + v\frac{\partial *}{\partial y} + w\frac{\partial *}{\partial z}}$$

Scalar

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\underline{u} \cdot \nabla)\rho = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}}$$

Vector

$$\boxed{\frac{D\underline{\mathbf{u}}}{Dt} = \frac{\partial \underline{\mathbf{u}}}{\partial t} + (\underline{u} \cdot \nabla)\underline{\mathbf{u}} = \frac{\partial \underline{\mathbf{u}}}{\partial t} + u\frac{\partial \underline{\mathbf{u}}}{\partial x} + v\frac{\partial \underline{\mathbf{u}}}{\partial y} + w\frac{\partial \underline{\mathbf{u}}}{\partial z}}$$

8.1.8 Vorticity

Vorticity $\underline{\omega}$ is a measure of the local rotation of fluid particles in flow.

$$\boxed{\underline{\omega} = \nabla \times \underline{u}}$$

$$\nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Irrotational Flow:

$$\boxed{\underline{\omega} = \underline{0}}$$

8.1.9 Incompressible Flow

$$\boxed{\nabla \cdot \underline{u} = 0}$$

When this is true the convective derivative of the fluid density is zero:

$$\frac{D\rho}{Dt} = 0$$

8.1.10 Velocity Potential

For an irrotational flow the velocity can be described as the gradient of a scalar field known as the *Velocity Potential*.

$$\nabla \times \underline{u} = 0$$

The curl of the gradient of a scalar field is zero:

$$\nabla \times (\nabla \phi) = 0$$

$$\underline{u} = \nabla \phi$$

If the flow is also incompressible

$$\nabla \cdot \underline{u} = 0$$

$$\nabla \cdot (\nabla \phi) = 0$$

Therefore the velocity potential of an irrotational, incompressible flow satisfies Laplace's Equation:

$$\boxed{\nabla^2 \phi = 0}$$

8.1.11 Equipotential Surfaces

Lines/surfaces of constant ϕ are *equipotentials*.
The velocity potential can be considered a surface:

$$\phi(x, y, z) = c$$

Let $\underline{a}$ be tangent to the surface, the derivative of ϕ in the direction of $\underline{a}$:

$$\underline{a} \cdot \nabla \phi = 0$$

Because the derivative of a constant c is zero: $\nabla \phi$ is normal to the surface.

$$\boxed{\hat{\underline{n}} = \frac{\nabla \phi}{|\nabla \phi|}}$$

8.1.12 The Stream Function

Considering incompressible two dimensional flow:

$$\nabla \cdot \underline{u} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

We can introduce the stream function $\psi(x, y, t)$ such that:

$$\boxed{u = \frac{\partial \psi}{\partial y}, v = -\frac{\partial \psi}{\partial x}}$$

This satisfies the previous equations:

$$\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$$

Vorticity of an incompressible two dimensional flow:

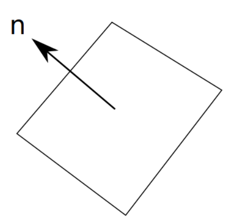
$$\underline{\omega} = \nabla \times \underline{u}$$
$$\underline{\omega} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$
$$\underline{\omega} = -\frac{\partial^2 \psi}{\partial^2 x} - \frac{\partial^2 \psi}{\partial^2 y}$$
$$\boxed{\underline{\omega} = -\nabla^2 \psi}$$

8.2 Pressure in a Fluid

8.2.1 Pressure and Force

Let the normal vector $\underline{n}$ be pointing into the fluid.

$$\partial \underline{F} = -\partial F \underline{n}$$



∂F is the force exerted over a small area ∂S

$$p = \lim_{\partial S \to 0} (\tfrac{\partial F}{\partial S})$$
$$\boxed{\underline{F} = -\int \int_S p\underline{n}dS}$$

8.2.2 Equations of State

Ideal Gas Law

$$pV = nRT$$
$$\tfrac{pV}{T} = nR$$
$$V \propto 1/\rho$$
$$\boxed{\frac{p}{\rho T} = constant}$$

Isothermal Gas
For a gas at constant temperature T

Boyle's Law:
$$p \propto \rho$$
$$\boxed{p\rho^{-1} = p_0\rho_0^{-1}}$$

Adiabatic Gas
Quick fluctuations in pressure (such as in sound waves) are described by the adiabatic law.

$$\boxed{p\rho^{-\gamma} = c}$$

Where γ is the *Polytropic Index*

8.2.3 Hydrostatics

The force due to the pressure of the fluid
$$\underline{F} = -\int \int_S p\underline{n}dS$$
Applying the divergence theorem for a scalar:
$$\underline{F} = -\int \int \int_V \nabla p dV$$
Calculating the weight of the fluid region:
$$\underline{W} = \int \int \int_V \rho \underline{g} dV$$
Since the fluid is stationary, the total force on the fluid is zero
$$\underline{W} + \underline{F} = 0$$
$$\int \int \int_V \rho \underline{g} dV - \int \int \int_V \nabla p dV = 0$$
$$\int \int \int_V (\rho \underline{g} - \nabla p) dV = 0$$
Therefore:
$$\boxed{\rho \underline{g} = \nabla p}$$

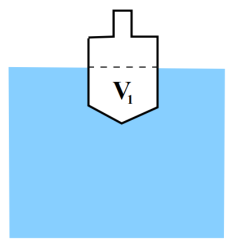
$$\underline{g} = -g\underline{\hat{k}}$$
$$\frac{\partial p}{\partial z} = -\rho g$$
Integrating gives:
$$\boxed{p = p_a - \rho g z}$$

8.2.4 Buoyancy

Buoyancy is the force exerted on a submerged body due to the pressure of the surrounding static fluid.

$$\underline{B} = -\int\int_S p\underline{n}dS$$
Applying the divergence theorem for a scalar:
$$\underline{B} = -\int\int\int_V \nabla p dV$$
$$\nabla p = \rho \underline{g},\ \underline{g} = -g\underline{\hat{k}}$$
$$\underline{B} = \underline{\hat{k}}g\int\int\int_V \rho dV$$
$$\boxed{\underline{B} = \rho V g\underline{\hat{k}} = m_w g\underline{\hat{k}}}$$

Archimedes' Principle – the buoyancy force on a body is equal in magnitude to the weight of fluid that is displaced by the body.



V_1 = Volume od displaced water,
ρ_w = Density of water,
m_B = Mass of the body
At equilibrium:
$$\underline{B} + \underline{W} = 0$$
$$\rho_w V_1 g\underline{\hat{k}} - m_B g\underline{\hat{k}} = \underline{0}$$
$$\boxed{m_B = \rho_w V_1}$$

Center of Mass (Gravity): A theoretical point in the body where the body's total mass (weight) is thought to be concentrated.
Center of Buoyancy: The centre of mass of the displaced water

A floating body is stable if its centre of gravity is below its centre of buoyancy.

8.2.5 Oscillation of a Floating Body

When a floating body is submerged below its equilibrium depth d it begins to oscillate.

Consider a cylindrical body with mass m, circular area A, equilibrium depth d:

$$mz''(t) = \rho_w g(d - z(t))A - mg$$
$$= -\rho_w g z(t) A$$
$$= mg\tfrac{z(t)}{d}$$

Therefore you can form the differential equation:

$$z''(t) = \tfrac{g}{d} z(t)$$
$$z(0) = -z_0$$

The solution:

$$z(t) = -z_0 \cos(\sqrt{\tfrac{g}{d}} t)$$

8.3 Flow Dynamics

8.3.1 Reynolds Transport Theorem

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\Omega(t+\Delta t)} F(\underline{x},t+\Delta t)dV - \int_{\Omega(t)} F(\underline{x},t)dV]$$

Using the Taylor series expansion:

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\Omega(t+\Delta t)} F(\underline{x},t) + \tfrac{\partial F}{\partial t}(\underline{x},t)\Delta t + O(\Delta t)^2 dV - \int_{\Omega(t)} F(\underline{x},t)dV]$$

Let $\Delta\Omega$ be the difference between the regions $\Omega(t)$ and $\Omega(t+\Delta t)$:

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\Delta\Omega} F(\underline{x},t)dV + \int_{\Omega(t+\Delta t)} \tfrac{\partial F}{\partial t}(\underline{x},t)\Delta t + O(\Delta t)^2 dV]$$

Letting U_n be the normal velocity to the boundary $\partial\Omega$

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\partial\Omega} F(\underline{x},t)U_n(\underline{x},t)\Delta t dS] + \lim_{\Delta t\to \infty}[\int_{\Omega(t+\Delta t)} \tfrac{\partial F}{\partial t}(\underline{x},t)\Delta t dV]$$

$$\boxed{\frac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \int_{\partial\Omega} F(\underline{x},t)U_n(\underline{x},t)dS + \int_{\Omega(t)} \frac{\partial F}{\partial t}(\underline{x},t)dV}$$

Leibniz's Integral Rule: The rule can be derived from considering a one dimensional case of *Reynold's Transport Theorem.*

$$\boxed{\frac{d}{dt}\int_{a(t)}^{b(t)} F(\underline{x},t)dx = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}(\underline{x},t)dx + F(b(t),t)\frac{db}{dt} - F(a(t),t)\frac{da}{dt}}$$

8.3.2 Conservation of Mass

8.4 Tow-dimensional Flow

8.5 Vorticity Dynamics

8.6 Free Surface Waves

9 Numerical Methods