Mathematics

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5.1 Kinematics

5.1.1 Coordinates

Lagrangian $\underline{x}(\underline{a}, t)$: The *motion of individual particles* is studied; the position $\underline{x}$ of a particle at time t is related to its position at a reference point in time $\underline{\mathrm{a}}$ (typically at $t = 0$).

Eulerian $(\underline{x}, t)$: The *'flow field' is considered as a whole* and the state of a fluid is described in terms of the values at a fixed location $\underline{x}$ and at a fixed time t

5.1.2 Velocity

In Cartesian coordinates the velocity of a fluid particle at position $\underline{x}(x, y, z)$ is given by:

$$\underline{u}(x, y, z) = u(x, y, z)\underline{\hat{i}} + v(x, y, z)\underline{\hat{j}} + w(x, y, z)\underline{\hat{k}}$$

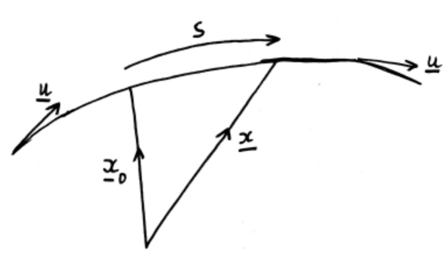
5.1.3 Stagnation Points

Stagnation points occur when the velocity vector $\underline{u}$ is equal to $\underline{0}$

$$\begin{aligned} u &= 0 \\ v &= 0 \\ w &= 0 \end{aligned}$$

5.1.4 Streamlines

A streamline is a curve C drawn at one point in time such that the fluid velocity vector $\underline{u}$ is tangent to C at every point along C.



$$\frac{d\underline{x}}{ds} = \underline{u}$$

$$\frac{dx}{ds} = u, \frac{dy}{ds} = v, \frac{dz}{ds} = w$$

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} (= ds)}$$

5.1.5 Particle Paths

Particle path is obtained by solving the initial value problem:

$$\tfrac{d\underline{x}}{dt} = \underline{u}(\underline{x}, t) \ , \ \underline{x} = x_0 \text{ at } t = 0$$

$$\boxed{\begin{array}{c} \frac{dx}{dt} = u \ , \ x(0) = x_0 \\[1em] \frac{dy}{dt} = v, \ y(0) = y_0 \\[1em] \frac{dz}{dt} = w, \ z(0) = z_0 \end{array}}$$

5.1.6 Steady Flow

Steady Flow: The flow velocity vector $\underline{u}$ is independent of time t
Unsteady Flow: $\underline{u}$ depends on t; the pattern of streamlines changes with t

5.1.7 Convective Derivative

The convective derivative tells us how a property changes *as it moves with a flow.*
General

$$\boxed{\frac{D*}{Dt} = \frac{\partial *}{\partial t} + (\underline{u} \cdot \nabla)* = \frac{\partial *}{\partial t} + u\frac{\partial *}{\partial x} + v\frac{\partial *}{\partial y} + w\frac{\partial *}{\partial z}}$$

Scalar

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\underline{u} \cdot \nabla)\rho = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}}$$

Vector

$$\boxed{\frac{D\underline{\mathbf{u}}}{Dt} = \frac{\partial \underline{\mathbf{u}}}{\partial t} + (\underline{u} \cdot \nabla)\underline{\mathbf{u}} = \frac{\partial \underline{\mathbf{u}}}{\partial t} + u\frac{\partial \underline{\mathbf{u}}}{\partial x} + v\frac{\partial \underline{\mathbf{u}}}{\partial y} + w\frac{\partial \underline{\mathbf{u}}}{\partial z}}$$

5.1.8 Vorticity

Vorticity $\underline{\omega}$ is a measure of the local rotation of fluid particles in flow.

$$\boxed{\underline{\omega} = \nabla \times \underline{u}}$$

$$\nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Irrotational Flow:

$$\boxed{\underline{\omega} = \underline{0}}$$

5.1.9 Incompressible Flow

$$\boxed{\nabla \cdot \underline{u} = 0}$$

When this is true the convective derivative of the fluid density is zero:

$$\frac{D\rho}{Dt} = 0$$

5.1.10 Velocity Potential

For an irrotational flow the velocity can be described as the gradient of a scalar field known as the *Velocity Potential*.

$$\nabla \times \underline{u} = 0$$

The curl of the gradient of a scalar field is zero:

$$\nabla \times (\nabla \phi) = 0$$

$$\underline{u} = \nabla \phi$$

If the flow is also incompressible

$$\nabla \cdot \underline{u} = 0$$

$$\nabla \cdot (\nabla \phi) = 0$$

Therefore the velocity potential of an irrotational, incompressible flow satisfies Laplace's Equation:

$$\boxed{\nabla^2 \phi = 0}$$

5.1.11 Equipotential Surfaces

Lines/surfaces of constant ϕ are *equipotentials*.
The velocity potential can be considered a surface:

$$\phi(x, y, z) = c$$

Let $\underline{a}$ be tangent to the surface, the derivative of ϕ in the direction of $\underline{a}$:

$$\underline{a} \cdot \nabla\phi = 0$$

Because the derivative of a constant c is zero: $\nabla\phi$ is normal to the surface.

$$\boxed{\hat{\underline{n}} = \frac{\nabla\phi}{|\nabla\phi|}}$$

5.1.12 The Stream Function

Considering incompressible two dimensional flow:

$$\nabla \cdot \underline{u} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

We can introduce the stream function $\psi(x, y, t)$ such that:

$$\boxed{u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}}$$

This satisfies the previous equations:

$$\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial^2\psi}{\partial y\partial x} = 0$$

Vorticity of an incompressible two dimensional flow:

$$\underline{\omega} = \nabla \times \underline{u}$$
$$\underline{\omega} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$
$$\underline{\omega} = -\frac{\partial^2\psi}{\partial^2 x} - \frac{\partial^2\psi}{\partial^2 y}$$
$$\boxed{\underline{\omega} = -\nabla^2\psi}$$

5.2 Pressure in a Fluid

5.3 Flow Dynamics

5.4 Tow-dimensional Flow

5.5 Vorticity Dynamics

5.6 Free Surface Waves