Mathematics

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June 1, 2023

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3.1 Fluid Mechanics

3.1.1 Kinematics

Coordinates

Lagrangian $\underline{x}(\underline{a}, t)$: The *motion of individual particles* is studied; the position $\underline{x}$ of a particle at time t is related to its position at a reference point in time $\underline{a}$ (typically at $t = 0$).

Eulerian $(\underline{x}, t)$: The *'flow field' is considered as a whole* and the state of a fluid is described in terms of the values at a fixed location $\underline{x}$ and at a fixed time t

Velocity

In Cartesian coordinates the velocity of a fluid particle at position $\underline{x}(x, y, z)$ is given by:

$$\underline{u}(x, y, z) = u(x, y, z)\hat{\underline{i}} + v(x, y, z)\hat{\underline{j}} + w(x, y, z)\hat{\underline{k}}$$

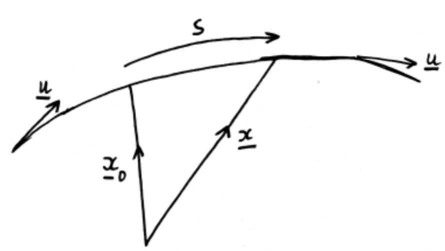
Stagnation Points

Stagnation points occur when the velocity vector $\underline{u}$ is equal to $\underline{0}$

$$u = 0$$
$$v = 0$$
$$w = 0$$

Streamlines

A streamline is a curve C drawn at one point in time such that the fluid velocity vector $\underline{u}$ is tangent to C at every point along C.



$$\frac{d\underline{x}}{ds} = \underline{u}$$

$$\frac{dx}{ds} = u, \frac{dy}{ds} = v, \frac{dz}{ds} = w$$

$$\boxed{\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} (= ds)}$$

Particle Paths

Particle path is obtained by solving the initial value problem:

$$\tfrac{d\underline{x}}{dt} = \underline{u}(\underline{x}, t)\ ,\ \underline{x} = x_0 \text{ at } t = 0$$

$$\boxed{\begin{array}{c} \tfrac{dx}{dt} = u\ ,\ x(0) = x_0 \\[2ex] \tfrac{dy}{dt} = v,\ y(0) = y_0 \\[2ex] \tfrac{dz}{dt} = w,\ z(0) = z_0 \end{array}}$$

Steady Flow

Steady Flow: The flow velocity vector $\underline{u}$ is independent of time t
Unsteady Flow: $\underline{u}$ depends on t; the pattern of streamlines changes with t

Convective Derivative

The convective derivative tells us how a property changes *as it moves with a flow.*
General

$$\boxed{\frac{D*}{Dt} = \frac{\partial *}{\partial t} + (\underline{u} \cdot \nabla)* = \frac{\partial *}{\partial t} + u\frac{\partial *}{\partial x} + v\frac{\partial *}{\partial y} + w\frac{\partial *}{\partial z}}$$

Scalar

$$\boxed{\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\underline{u} \cdot \nabla)\rho = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}}$$

Vector

$$\boxed{\frac{D\underline{\mathbf{u}}}{Dt} = \frac{\partial \underline{\mathbf{u}}}{\partial t} + (\underline{u} \cdot \nabla)\underline{\mathbf{u}} = \frac{\partial \underline{\mathbf{u}}}{\partial t} + u\frac{\partial \underline{\mathbf{u}}}{\partial x} + v\frac{\partial \underline{\mathbf{u}}}{\partial y} + w\frac{\partial \underline{\mathbf{u}}}{\partial z}}$$

Vorticity

Vorticity $\underline{\omega}$ is a measure of the local rotation of fluid particles in flow.

$$\boxed{\underline{\omega} = \nabla \times \underline{u}}$$

$$\nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Irrotational Flow:

$$\boxed{\underline{\omega} = \underline{0}}$$

Incompressible Flow

$$\boxed{\nabla \cdot \underline{u} = 0}$$

When this is true the convective derivative of the fluid density is zero:

$$\tfrac{D\rho}{Dt} = 0$$

Velocity Potential

For an irrotational flow the velocity can be described as the gradient of a scalar field known as the *Velocity Potential*.

$$\nabla \times \underline{u} = 0$$

The curl of the gradient of a scalar field is zero:

$$\nabla \times (\nabla \phi) = 0$$

$$\underline{u} = \nabla \phi$$

If the flow is also incompressible

$$\nabla \cdot \underline{u} = 0$$

$$\nabla \cdot (\nabla \phi) = 0$$

Therefore the velocity potential of an irrotational, incompressible flow satisfies Laplace's Equation:

$$\boxed{\nabla^2 \phi = 0}$$

Expanded to:

$$\tfrac{\partial^2 \phi}{\partial x^2} + \tfrac{\partial^2 \phi}{\partial y^2} + \tfrac{\partial^2 \phi}{\partial z^2} = 0$$

Equipotential Surfaces

Lines/surfaces of constant ϕ are *equipotentials*.
The velocity potential can be considered a surface:

$$\phi(x, y, z) = c$$

Let $\underline{a}$ be tangent to the surface, the derivative of ϕ in the direction of $\underline{a}$:

$$\underline{a} \cdot \nabla\phi = 0$$

Because the derivative of a constant c is zero: $\nabla\phi$ is normal to the surface.

$$\boxed{\hat{\underline{n}} = \frac{\nabla\phi}{|\nabla\phi|}}$$

The Stream Function

Considering incompressible two dimensional flow:

$$\nabla \cdot \underline{u} = 0$$
$$\tfrac{\partial u}{\partial x} + \tfrac{\partial v}{\partial y} = 0$$

We can introduce the stream function $\psi(x, y, t)$ such that:

$$\boxed{u = \frac{\partial\psi}{\partial y}, v = -\frac{\partial\psi}{\partial x}}$$

This satisfies the previous equations:

$$\tfrac{\partial^2\psi}{\partial x\partial y} - \tfrac{\partial^2\psi}{\partial y\partial x} = 0$$

Vorticity of an incompressible two dimensional flow:

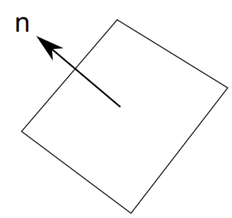
$$\underline{\omega} = \nabla \times \underline{u}$$
$$\underline{\omega} = \tfrac{\partial v}{\partial x} - \tfrac{\partial u}{\partial y}$$
$$\underline{\omega} = -\tfrac{\partial^2\psi}{\partial^2 x} - \tfrac{\partial^2\psi}{\partial^2 y}$$
$$\boxed{\underline{\omega} = -\nabla^2\psi}$$

3.1.2 Pressure in a Fluid

Pressure and Force

Let the normal vector $\underline{n}$ be pointing into the fluid.

$$\partial\underline{F} = -\partial F\underline{n}$$



∂F is the force exerted over a small area ∂S

$$p = \lim_{\partial S \to 0} (\tfrac{\partial F}{\partial S})$$
$$\boxed{\underline{F} = -\int\int_S p\underline{n}dS}$$

Equations of State

Ideal Gas Law

$$pV = nRT$$
$$\tfrac{pV}{T} = nR$$
$$V \propto 1/\rho$$
$$\boxed{\frac{p}{\rho T} = constant}$$

Isothermal Gas
For a gas at constant temperature T

Boyle's Law:
$$p \propto \rho$$
$$\boxed{p\rho^{-1} = p_0\rho_0^{-1}}$$

Adiabatic Gas
Quick fluctuations in pressure (such as in sound waves) are described by the adiabatic law.

$$\boxed{p\rho^{-\gamma} = c}$$

Where γ is the *Polytropic Index*

Hydrostatics

The force due to the pressure of the fluid
$$\underline{F} = -\int\int_S p\underline{n}dS$$
Applying the divergence theorem for a scalar:
$$\underline{F} = -\int\int\int_V \nabla p dV$$

Calculating the weight of the fluid region:

$$\underline{W} = \int\int\int_V \rho \underline{g} dV$$

Since the fluid is stationary, the total force on the fluid is zero

$$\underline{W} + \underline{F} = 0$$

$$\int\int\int_V \rho \underline{g} dV - \int\int\int_V \nabla p dV = 0$$

$$\int\int\int_V (\rho \underline{g} - \nabla p) dV = 0$$

Therefore:

$$\boxed{\rho \underline{g} = \nabla p}$$

$$\underline{g} = -g\underline{\hat{k}}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

Integrating gives:

$$\boxed{p = p_a - \rho g z}$$

Buoyancy

Buoyancy is the force exerted on a submerged body due to the pressure of the surrounding static fluid.

$$\underline{B} = -\int\int_S p \underline{n} dS$$

Applying the divergence theorem for a scalar:

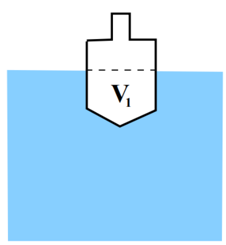
$$\underline{B} = -\int\int\int_V \nabla p dV$$

$$\nabla p = \rho \underline{g},\ \underline{g} = -g\underline{\hat{k}}$$

$$\underline{B} = \underline{\hat{k}} g \int\int\int_V \rho dV$$

$$\boxed{\underline{B} = \rho V g \underline{\hat{k}} = m_w g \underline{\hat{k}}}$$

Archimedes' Principle – the buoyancy force on a body is equal in magnitude to the weight of fluid that is displaced by the body.



V_1 = Volume od displaced water,

ρ_w = Density of water,

m_B = Mass of the body

At equilibrium:

$$\underline{B} + \underline{W} = 0$$

$$\rho_w V_1 g\underline{\hat{k}} - m_B g \underline{\hat{k}} = \underline{0}$$

$$\boxed{m_B = \rho_w V_1}$$

Center of Mass (Gravity): A theoretical point in the body where the body's total mass (weight) is thought to be concentrated.

Center of Buoyancy: The centre of mass of the displaced water

A floating body is stable if its centre of gravity is below its centre of buoyancy.

Oscillation of a Floating Body

When a floating body is submerged below its equilibrium depth d it begins to oscillate.

Consider a cylindrical body with mass m, circular area A, equilibrium depth d:

$$\begin{aligned} mz''(t) &= \rho_w g(d - z(t))A - mg \\ &= -\rho_w g z(t) A \\ &= mg\tfrac{z(t)}{d} \end{aligned}$$

Therefore you can form the differential equation:

$$z''(t) = \tfrac{g}{d} z(t)$$
$$z(0) = -z_0$$

The solution:

$$z(t) = -z_0 \cos(\sqrt{\tfrac{g}{d}}t)$$

3.1.3 Flow Dynamics

Reynolds Transport Theorem

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\Omega(t+\Delta t)} F(\underline{x},t+\Delta t)dV - \int_{\Omega(t)} F(\underline{x},t)dV]$$

Using the Taylor series expansion:

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\Omega(t+\Delta t)} F(\underline{x},t) + \tfrac{\partial F}{\partial t}(\underline{x},t)\Delta t + O(\Delta t)^2 dV - \int_{\Omega(t)} F(\underline{x},t)dV]$$

Let $\Delta\Omega$ be the difference between the regions $\Omega(t)$ and $\Omega(t+\Delta t)$:

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\Delta\Omega} F(\underline{x},t)dV + \int_{\Omega(t+\Delta t)} \tfrac{\partial F}{\partial t}(\underline{x},t)\Delta t + O(\Delta t)^2 dV]$$

Letting U_n be the normal velocity to the boundary $\partial\Omega$

$$\tfrac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \lim_{\Delta t\to 0}[\int_{\partial\Omega} F(\underline{x},t)U_n(\underline{x},t)\Delta t dS] + \lim_{\Delta t\to \infty}[\int_{\Omega(t+\Delta t)} \tfrac{\partial F}{\partial t}(\underline{x},t)\Delta t dV]$$

$$\boxed{\frac{d}{dt}\int_{\Omega(t)} F(\underline{x},t)dV = \int_{\partial\Omega} F(\underline{x},t)U_n(\underline{x},t)dS + \int_{\Omega(t)} \frac{\partial F}{\partial t}(\underline{x},t)dV}$$

Leibniz's Integral Rule: The rule can be derived from considering a one dimensional case of *Reynold's Transport Theorem*.

$$\boxed{\frac{d}{dt}\int_{a(t)}^{b(t)} F(\underline{x},t)dx = \int_{a(t)}^{b(t)} \frac{\partial F}{\partial t}(\underline{x},t)dx + F(b(t),t)\frac{db}{dt} - F(a(t),t)\frac{da}{dt}}$$

Conservation of Mass

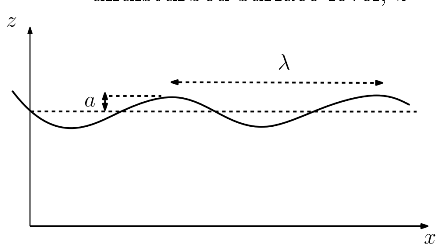
3.1.4 Tow-dimensional Flow

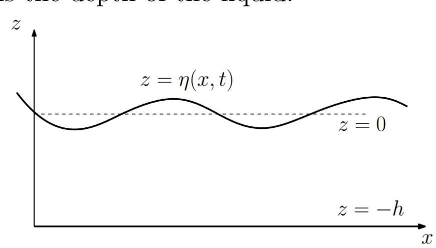
3.1.5 Vorticity Dynamics

3.1.6 Free Surface Waves

Periodic Gravity Waves

The free surface of the liquid is described by $z = \eta(x,t)$, $z = 0$ is the undisturbed surface level, $z = -h$ is the depth of the liquid.





η is assumed to take the form:

$$\boxed{\eta(x,t) = a\sin(kx - \omega t)}$$

k is the wave number, a is the amplitude, ω is the angular frequency, λ is the wavelength.

$$\boxed{k = \frac{2\pi}{\lambda}}$$

We also assume the liquid is *incompressible* and *irrotational*

$\nabla \times \underline{u} = 0$ and $\nabla \cdot \underline{u} = 0$

Therefore

$\underline{u} = \nabla\phi$ and $\nabla^2\phi = 0$

Boundary Conditions

No Normal Flow along the bottom at $z = 0$

$$\begin{aligned} &\underline{u}\cdot\underline{n} \\ &= \underline{u}\cdot\underline{\hat{k}} \\ &= \nabla\phi\cdot\underline{\hat{k}} \\ &= 0 \end{aligned}$$

Hence:

$$\boxed{\frac{\partial\phi}{\partial z} = 0 \text{ at } z = 0}$$

Dynamic Boundary Condition at $z = \eta(x,t)$

Fluid pressure at the free surface = Atmospheric pressure p_0

Bernoulli's Equation relates p_0 and ϕ:

$$p_0 + \rho(\tfrac{\partial\phi}{\partial t} + \tfrac{1}{2}\underline{u}^2 + gz) = f(t)$$

$$\tfrac{\partial\phi}{\partial t} + \tfrac{1}{2}\underline{u}^2 + gz = \tfrac{f(t)-p_0}{\rho} = 0$$

$$\boxed{\frac{\partial\phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta(x,t) = 0 \text{ at } z = \eta(x,t)}$$

Kinematic Boundary Condition at $z = \eta(x,t)$

Fluid particles can not leave the free surface thus the surface must move with the fluid.

3.2 Numerical Methods