$$O(T(n))$$

$$O(T(n)\log n)$$

$$O(\log n)$$

$$O(T(n))$$

$$O(T(n))$$

$$n$$

$$m$$

$$O($$

$$) \sim O($$

$$)$$

$$O(m\log m) \sim O(\log n)$$

$$O(n)$$

$$O(n\log n) \sim O(1)$$

$$O(n\log n)$$

$$O(n) \sim O(\log n)$$

$$O(n)$$

$$S$$

$$n/S$$

$$n/S$$

$$S = \log n$$

$$O((n/\log n)\log n + (n/\log n) \times \log n \times \log\log n) = O(n\log\log n)$$

$$O(n\log\log n) \sim O(1)$$

$$O(n\log\log n)$$

$$O(n)$$

$$\sqrt{n}$$

$$O(n)$$

$$O(1)$$

$$O(\sqrt{n})$$

$$O(\frac{\sqrt{n}}{n})$$

$$O(n)$$

$$\sqrt{n}$$

$$O(n) \sim O(1)$$

$$O(n)$$

$$\pm 1$$

$$\log n$$

$$\sum_{i=1}^{\log n} 2^{i-1}$$

$$n$$

$$n$$

$$O(n)$$

$$O(n) \sim O(1)$$

$$O(n) \sim O(1)$$

$$O(n) \sim O(1)$$

$$O(n) \sim O(1)$$

$$O(n) \sim O(\log n)$$

$$n$$

$$\log_2 n \leq 64$$

$$A[1 \cdots n]$$

$$O(\log_2 n)$$

$$O(\frac{n}{\log_2 n})$$

$$1.5 \times \log_2 n$$

$$O(n)$$

$$Pre[1 \cdots n], Sub[1 \cdots n]$$

$$Pre[i]$$

$$A[i]$$

$$O(n)$$

$$l, r$$

$$bl, br$$

$$[bl+1, br-1]$$

$$Sub[l]$$

$$Pre[r]$$

$$l, r$$

$$A[1 \cdots r]$$

$$A[l, r]$$

$$p \geq l$$

$$A[p]$$

$$A[l, r]$$

$$A[p]$$

$$A[l \cdots p-1]$$

$$A[p+1 \cdots r]$$

$$A[p]$$

$$0/1$$

$$p$$

$$l$$

$$1$$

$$O(\log_2 n)$$

$$64$$

$$n$$

$$m$$

$$i$$

$$a_i$$

$$b_i$$

$$L(i, j)$$

$$i$$

$$j$$

$$L(i, j)$$

$$m$$

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} L(i, j)$$

$$i$$

a_i

b_i

i

$O(n\alpha(n))$

n

m

i

a_i

b_i

q

i

u_i

v_i

i

a_i

b_i

(a_i, b_i)

i

u

v

$O(n \log n)$

n

m

i

a_i

b_i

q

i

x_i

t_i

t_i

$O(q \log q + (n+q)\alpha(n))$

i

$$a_i$$
$$b_i$$
$$u$$
$$a_i$$
$$b_i$$
$$u$$
$$u$$
$$u$$
$$i$$
$$n$$
$$0$$
$$x_i$$
$$t_i$$
$$0$$
$$x_i$$
$$t_i$$
$$O(n \log n)$$
$$n$$
$$a_1, ..., a_n$$
$$0$$
$$m$$
$$a_x = 1$$
$$a_x, a_{x+1}, ..., a_n$$
$$0$$
$$f_i$$
$$a_i, a_{i+1}, ..., a_n$$
$$0$$
$$f_i = i$$
$$a_x = 1$$
$$a_x$$
$$1$$
$$f_x = f_{x+1}$$
$$O(n \log n)$$

$$O(n\alpha(n))$$

$$n$$

$$a$$

$$b$$

$$c$$

$$1 \le i \le j \le n$$

$$a_i \cdot b_j \cdot \min_{i \le k \le j} c_k$$

$$c_k$$

$$k$$

$$k-1$$

$$k+1$$

$$a$$

$$b$$

$$O(n \log n)$$

$$n$$

$$m$$

$$a_i$$

$$b_i$$

$$u_i$$

$$v_i$$

$$u_i$$

$$v_i$$

$$a_i$$

$$b_i$$

$$a_i$$

$$b_i$$

$$(a_i, b_i)$$

$$(a_j, b_j)$$

$$a_i$$

$$b_i$$

$$u_i$$

$$v_i$$

$$u_i$$

v_i

f_i

i

f

$O(n \log n)$

$O(n\alpha(n))$

G

n

m

i

a_i

b_i

G

(x, y)

(x, y)

p_i

i

i

a_i

b_i

a_i

b_i

1

$n - 1$

$O(n\alpha(n))$

a_i

b_i

a_i

b_i

a_i

b_i

a_i

$$b_i$$

$$a_i$$

$$b_i$$

$$1$$

$$a_i$$

$$b_i$$

$$O(n \log n)$$

$$O(n \log n + m \log n)$$

$$f_n = \sum_{i=0}^{n-1} f_i f_{n-i-1}$$

$$($$

$$)$$

$$\varepsilon$$

$$s$$

$$(s)$$

$$s, t$$

$$st$$

$$(())()$$

$$)()$$

$$[()]\{\}$$

$$s$$

$$i = 1, 2, ..., \mid s \mid$$

$$s_i$$

$$s_i$$

$$s_i$$

$$s$$

$$s$$

$$O(n)$$

$$2n$$

$$s$$

$$f_n$$

$$s_1$$

$$2i+2$$

$$f_n = \frac{1}{n+1}\binom{2n}{n}$$

$$k$$

$$f'_n = \frac{1}{n+1}\binom{2n}{n}k^n$$

$$s$$

$$\mid s \mid$$

$$s$$

$$i$$

$$s_i$$

$$s[i+1, \mid s \mid]$$

$$i$$

$$s[1, i-1]$$

$$s_i$$

$$s[1, i]$$

$$k$$

$$s$$

$$k$$

$$s[i+1, \mid s \mid - k]$$

$$((...(())...))$$

$$O(n)$$

$$s$$

$$s$$

$$p$$

$$p_i < s_i$$

$$\forall 1 \le j < i, p_j = s_i$$

$$p_i$$

$$s_i$$

$$i$$

$$s_i$$

$$p[1, i]$$

$$k$$

$$\mid s \mid - i$$

$$k$$

$$f(i,j)$$

$$i$$

$$j$$

$$f$$

$$f(i,j) = f(i-1,j-1) + f(i-1,j+1)$$

$$f(0,0) = 1$$

$$f$$

$$O(\mid s \mid^2)$$

$$s_i$$

$$s_i$$

$$f$$

$$k$$

$$1+2=3$$

$$\frac{a}{3}$$

$$\frac{y}{\frac{3}{x}+b}$$

$$\sqrt{y^2}$$

$$\sqrt[x]{y^2}$$

$$\sum_{x=1}^{5} y^z$$

$$\int_a^b f(x)$$

$$\to$$

$$\to$$

$$\to$$

$$\to$$

$$\to$$

$$\to$$

$$\to$$

$$\to$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

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$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

ê

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$1 + 2 = 3$$

$$n^2$$

$$2_a$$

$$b_{a-2}$$

$$\alpha$$

$$\beta$$

$$\delta, \Delta$$

$$\pi, \Pi$$

$$\sigma, \Sigma$$

$$\phi, \Phi, \varphi$$

$$\psi, \Psi$$

$$\omega, \Omega$$

$$\rightarrow$$

Dispersal in the contemporary united states

Dispersal in the contemporary United States

[1]

$$[Kop10]$$

$$[Koppe, 2010]$$

$$Koppe[2010]$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$\rightarrow$$

$$0$$

$$n$$

$$s$$

$$z[i]$$

$$s$$

$$s[i, n-1]$$

$$s[i]$$

$$z$$

$$s$$

$$z[0] = 0$$

$$O(n)$$

$$z(\texttt{aaaaa}) = [0, 4, 3, 2, 1]$$

$$z(\texttt{aaabaab}) = [0, 2, 1, 0, 2, 1, 0]$$

$$z(\texttt{abacaba}) = [0, 0, 1, 0, 3, 0, 1]$$

$$O(n^2)$$

$$1$$

$$n-1$$

$$z[i]$$

$$z[0]=0$$

$$z[i]$$

$$z[0],...,z[i-1]$$

$$i$$

$$[i,i+z[i]-1]$$

$$i$$

$$[l,r]$$

$$s[l,r]$$

$$s$$

$$z[i]$$

$$l\leq i$$

$$l=r=0$$

$$z[i]$$

$$i\leq r$$

$$[l,r]$$

$$s[i,r]=s[i-l,r-l]$$

$$z[i]\geq\min(z[i-l],r-i+1)$$

$$z[i-l]<r-i+1$$

$$z[i]=z[i-l]$$

$$z[i-l]\geq r-i+1$$

$$z[i]=r-i+1$$

$$z[i]$$

$$i>r$$

$$s[i]$$

$$z[i]$$

$$z[i]$$

$$i+z[i]-1>r$$

$$[l, r]$$

$$l = i, r = i + z[i] - 1$$

$$r$$

$$1$$

$$r < n - 1$$

$$n$$

$$O(n)$$

$$t$$

$$p$$

$$t$$

$$p$$

$$s = p + \diamond + t$$

$$p$$

$$t$$

$$\diamond$$

$$\diamond$$

$$p$$

$$t$$

$$s$$

$$[0, \mid t \mid - 1]$$

$$i$$

$$t[i]$$

$$s$$

$$k = z[i + \mid p \mid + 1]$$

$$k = \mid p \mid$$

$$p$$

$$t$$

$$i$$

$$p$$

$$t$$

$$i$$

$$O(\mid t \mid + \mid p \mid)$$

$$n$$

$$s$$

$$i + z[i] = n$$

$$s$$

$$s$$

$$s$$

$$k$$

$$s$$

$$c$$

$$s$$

$$c$$

$$c$$

$$t$$

$$s + c$$

$$t$$

$$t$$

$$t$$

$$z_{\max}$$

$$t$$

$$z_{\max}$$

$$s$$

$$c$$

$$c$$

$$s$$

$$\mid t \mid - z_{\max}$$

$$O(n^2)$$

$$O(n)$$

$$n$$

$$s$$

$$t$$

$$s$$

$$t$$

$$s$$

$$n$$

$$i$$

$$i + z[i] = n$$

$$1 \to 4 \to 8 \to 12$$

$$\delta(u, c)$$

$$u$$
$$c$$
$$u$$
$$c$$
$$c$$
$$0 \sim 26$$
$$n$$
$$m$$
$$1 \leq n \leq 10^4$$
$$1 \leq m \leq 10^5$$
$$50$$
$$\{0, 1\}$$
$$(u, v)$$
$$u$$
$$v$$
$$10^5$$
$$[0, 2^{31})$$
$$root$$
$$T(u, v)$$
$$u$$
$$v$$
$$T(u, v) = T(root, u) \oplus T(root, v)$$
$$T(root, u)$$
$$T(root, u)$$
$$T(root, v)$$
$$T(root, u)$$
$$1$$
$$0$$
$$\{0, 1\}$$
$$V_1, V_2, V_3 ... V_n$$
$$V_1 + 1, V_2 + 1, V_3 + 1 ... V_n + 1$$
$$n$$
$$m$$

$$x$$
$$1$$
$$+1$$
$$x$$
$$-v$$
$$x$$
$$1$$
$$100\%$$
$$1 \leq n \leq 5 \times 10^5$$
$$1 \leq m \leq 5 \times 10^5$$
$$0 \leq a_i \leq 10^5$$
$$1 \leq x \leq n$$
$$opt \in \{1, 2, 3\}$$
$$n$$
$$T$$
$$1$$
$$1$$
$$v_i$$
$$x$$
$$x$$
$$c_1, c_2, ..., c_k$$
$$x$$
$$val(x) = (v_{c_1} + d(c_1, x)) \oplus (v_{c_2} + d(c_2, x)) \oplus \cdots \oplus (v_{c_k} + d(c_k, x))$$
$$d(x, y)$$
$$x$$
$$y$$
$$d(x, x) = 0$$
$$\oplus$$
$$\sum_{i=1}^{n} val(i)$$
$$S$$
$$n$$
$$\Sigma$$
$$S[i]$$

S

i

$1 \leq i \leq n$

$S[l, r]$

S

l

r

S

$S[i, n]$

S

i

$S[1, i]$

S

i

S

S

S

$O(n^2)$

x

x

fa_x

fa_x

x

str_x

`cabab`

S

$2n$

S

`abbbc`

0

$[l, r]$

$S[l, r]$

m

0

a

0

$[1, \infty]$

∞

b

0

$[2, \infty\,\text{f()}]$

$[1, \infty]$

a

ab

b

b

b

k

$S[k, m]$

b

b

$k = 3$

bb

bb

k

$S[k, m]$

$l > k$

$S[l, m]$

$S[k, m]$

c

$S[k, m] + c$

S

$S[l, m] + c$

S

$S[l, m]$

c

$k = 3$

`bbc`

`bb`

$[5, \infty]$

$k \to k + 1$

$S[k, m]$

k

$k > m$

$O(n)$

$O(n^2)$

(now, rem)

$S[k, m]$

now

$S[m - rem + 1]$

rem

now

rem

$k \to k + 1$

(now, rem)

$now = 0$

$rem \to rem - 1$

-1

str_{now}

$S[l, r]$

now'

$S[l + 1, r]$

$now \to now'$

x

y

str_y

str_x

s

$$str_x$$

$$c_1, c_2$$

$$str_x + c1$$

$$str_x + c_2$$

$$S$$

$$s + c_1$$

$$s + c_2$$

$$S$$

$$s$$

$$y$$

$$str_y = s$$

$$\text{Link}(x) = y$$

$$now' = \text{Link}(now)$$

$$\text{Link}$$

$$S$$

$$0$$

$$S[1, m]$$

$$S[1, m]$$

$$S[k, m]$$

$$(now, rem)$$

$$S[m+1] = x$$

$$x$$

$$S[1, m]$$

$$x$$

$$\infty$$

$$S[k, m]$$

$$(now, rem)$$

$$x$$

$$S[k, m+1]$$

$$l > k$$

$$S[l, m+1]$$

$$rem \to rem + 1$$

$$(now, rem)$$

$$x$$

$$(now, rem)$$

$$x$$

$$x$$

$$l > k$$

$$S[l, m]$$

$$S[k+1, m]$$

$$S[k+1, m]$$

$$now$$

$$0$$

$$now = \mathrm{Link}(now)$$

$$rem \to rem - 1$$

$$k \to k + 1$$

$$O(n)$$

$$S$$

$$S$$

$$S$$

$$S$$

$$S$$

$$1$$

$$> 1$$

$$O(\mid S \mid \mid \Sigma \mid)$$

$$S$$

$$n$$

$$x_i$$

$$(now, rem)$$

$$now \to \mathrm{Link}(now)$$

$$now = 1$$

$$rem \to rem - 1$$

$$O(\mid S \mid \mid \Sigma \mid + \sum \mid x_i \mid)$$

$$T$$

T

n

T

X

S

X

$\mathsf{c}S$

S

S

c

$O(\log n)$

$O(\log n)$

$O(n)$

$O(n \log n)$

n

$O(n^2 \log n)$

$\mathsf{c}S$

S

c

S

X

$\mathsf{c}S$

A

$A, S \in X$

X

$O(\log n)$

$O(\log^2 n)$

n

$O(n \log^2 n)$

$O(1)$

$O(n \log n)$

val_i

$$i$$

$$tag_i$$

$$tag_i > tag_j \iff val_i > val_j$$

$$tag_i$$

$$O(1)$$

$$(0,1)$$

$$i$$

$$(l,r)$$

$$tag_i = \frac{l+r}{2}$$

$$(l, tag_i)$$

$$(tag_i, r)$$

$$tag_i$$

$$O(\log n)$$

$$(0, 10^{18})$$

$$n$$

$$\mathsf{c}S$$

$$S$$

$$\mathsf{c}S$$

$$\mathsf{c}S$$

$$\mathsf{c}$$

$$\mathsf{c}S$$

$$s$$

$$q$$

$$t$$

$$\leq 8 \times 10^5$$

$$q \leq 10^5$$

$$\leq 3 \times 10^6$$

$$s$$

$$s$$

$$O(\log n)$$

$$O(\log n)$$

$$N$$

$$O(N \log n)$$

$$t$$

$$t$$

$$t$$

$$t$$

$$t$$

$$t$$

$$t$$

$$t$$

$$t$$

$$O(\mid t \mid)$$

$$O(\log n)$$

$$O(\mid t \mid \log n)$$

$$L$$

$$O(L \log n)$$

Input. A string S
Output. The state transition of the sequence automaton of S
Method.
for $c \in \Sigma$
    $next[c] \leftarrow null$
for $i \leftarrow \mid S \mid$ **downto** 1
    $next[S[i]] \leftarrow i$
    for $c \in \Sigma$
        $\delta(i-1, c) \leftarrow next[c]$
return δ

$$s$$

$$s$$

$$n$$

$$n+1$$

$$t$$

$$s$$

$$\delta(start, t)$$

$$t$$

$$s$$

$$i$$

$$s[1..i]$$

$$s[1..i-1]$$

$$\delta(u,c)=\min\{i \mid i>u, s[i]=c\}$$

$$c$$

$$i>j$$

$$s[i..\mid s\mid]$$

$$s[j..\mid s\mid]$$

$$O(n\mid\Sigma\mid)$$

$$A$$

$$B$$

$$A$$

$$B$$

$$A$$

$$B$$

$$A$$

$$B$$

$$A$$

$$B$$

$$1\leq\mid A\mid,\mid B\mid\leq 2000$$

$$f(i,j)$$

$$i$$

$$j$$

$$f(i,null)=0$$

$$f(i,j)=\min_{\delta_A(i,c)\neq null} f(\delta_A(i,c),\delta_B(j,c))+1$$

$$\begin{cases}\operatorname{endpos}(w)\subseteq\operatorname{endpos}(u) & \text{if } u \text{ is a suffix of } w\\ \operatorname{endpos}(w)\cap\operatorname{endpos}(u)=\emptyset & \text{otherwise}\end{cases}$$

$$\operatorname{endpos}(v)\subsetneq\operatorname{endpos}(\operatorname{link}(v)),$$

$$\operatorname{minlen}(v)=\operatorname{len}(\operatorname{link}(v))+1.$$

$$d_v=1+\sum_{w:(v,w,c)\in DAWG} d_w$$

$$ans_v=\sum_{w:(v,w,c)\in DAWG} d_w+ans_w$$

$$cnt_{\operatorname{link}(v)}+=cnt_v$$

$$\operatorname{firstpos}(cur) = \operatorname{len}(cur) - 1$$

$$\operatorname{firstpos}(clone) = \operatorname{firstpos}(q)$$

$$d_v = 1 + \min_{w:(v,w,c) \in SAM} d_w$$

$$v = \operatorname{link}(v)$$

$$T = S_1 + D_1 + S_2 + D_2 + \cdots + S_k + D_k.$$

$$\Sigma$$

$$|\Sigma|$$

$$s$$

$$|s|$$

$$n$$

$$O(n)$$

$$O(n)$$

$$2n - 1$$

$$3n - 4$$

$$s$$

$$s$$

$$t_0$$

$$t_0$$

$$t_0$$

$$s$$

$$s$$

$$t_0$$

$$s$$

$$t_0$$

$$s$$

$$s$$

$$t_0$$

$$t_0$$

$$s = \emptyset$$

$$s = \texttt{a}$$

$$s = \texttt{aa}$$

$$s = \texttt{ab}$$

$$s = \texttt{abb}$$

$$s = \texttt{abbb}$$

$$s$$

$$t$$

$$\text{endpos}(t)$$

$$s$$

$$t$$

$$\texttt{abcbc}$$

$$\text{endpos}(\texttt{bc}) = 2, 4$$

$$t_1$$

$$t_2$$

$$\text{endpos}$$

$$\text{endpos}(t_1) = \text{endpos}(t_2)$$

$$s$$

$$\text{endpos}$$

$$\text{endpos}$$

$$\text{endpos}$$

$$+1$$

$$\text{endpos}$$

$$s$$

$$u$$

$$w$$

$$|u| \leq |w|$$

$$\text{endpos}$$

$$u$$

$$s$$

$$w$$

$$u$$

$$w$$

$$\text{endpos}$$

$$u$$

$$w$$

$$s$$

$$w$$

$$u$$

w

s

endpos

u

w

$|u| \leq |w|$

$\mathrm{endpos}(u) \cap \mathrm{endpos}(w) = \emptyset$

$\mathrm{endpos}(w) \subseteq \mathrm{endpos}(u)$

u

w

$\mathrm{endpos}(u)$

$\mathrm{endpos}(w)$

u

w

u

w

w

u

$\mathrm{endpos}(w) \subseteq \mathrm{endpos}(u)$

endpos

$[x, y]$

endpos

1

endpos

w

u

u

w

$[|u|, |w|]$

w

s

w

u

s

w

w

endpos

t_0

v

v

endpos

w

w

w

t

v

t

$\text{link}(v)$

w

endpos

t_0

$\text{endpos}(t_0) = \{-1, 0, ..., |S| - 1\}$

t_0

t_0

v

$\text{link}(v)$

t_0

endpos

$subset$

$subset$

link

endpos

t_0

v

$\text{link}(v)$

$\subsetneq$

$\subseteq$

$\text{endpos}(v) = \text{endpos}(\text{link}(v))$

v

$\mathrm{link}(v)$

endpos

`abcbc`

s

endpos

t_0

endpos

v

$\mathrm{longest}(v)$

$\mathrm{len}(v)$

$\mathrm{shortest}(v)$

$\mathrm{minlen}(v)$

$\mathrm{longest}(v)$

$[\mathrm{minlen}(v), \mathrm{len}(v)]$

t_0

v

$\mathrm{longest}(v)$

$\mathrm{minlen}(v) - 1$

t_0

endpos

t_0

v

$\mathrm{link}(v)$

$\mathrm{minlen}(v)$

v_0

t_0

$[\mathrm{minlen}(v_i), \mathrm{len}(v_i)]$

$[0, \mathrm{len}(v_0)]$

len

link

$$t_0$$

$$0$$

$$1, 2, \ldots$$

$$t_0$$

$$\text{len} = 0$$

$$\text{link} = -1$$

$$-1$$

$$c$$

$$last$$

$$c$$

$$last = 0$$

$$last$$

$$cur$$

$$\text{len}(cur)$$

$$\text{len}(last) + 1$$

$$\text{link}(cur)$$

$$last$$

$$c$$

$$cur$$

$$c$$

$$p$$

$$p$$

$$-1$$

$$\text{link}(cur)$$

$$0$$

$$p$$

$$c$$

$$q$$

$$\text{len}(p) + 1 = \text{len}(q)$$

$$\text{len}(p) + 1 = \text{len}(q)$$

$$\text{link}(cur)$$

$$q$$

$$q$$

$$clone$$
$$q$$
$$\mathrm{len}$$
$$\mathrm{len}(clone)$$
$$\mathrm{len}(p)+1$$
$$cur$$
$$clone$$
$$q$$
$$clone$$
$$p$$
$$p$$
$$q$$
$$clone$$
$$last$$
$$cur$$
$$s$$
$$last$$
$$s$$
$$s$$
$$(p,q)$$
$$\mathrm{len}(p)+1=\mathrm{len}(q)$$
$$\mathrm{len}(p)+1<\mathrm{len}(q)$$
$$c$$
$$s$$
$$cur$$
$$s+c$$
$$s$$
$$c$$
$$cur$$
$$c$$
$$-1$$
$$s$$
$$c$$
$$c$$
$$s$$

cur

0

(p, q)

$x + c$

x

s

$x + c$

s

s

cur

$x + c$

len

$\mathrm{len}(p) + 1$

$\mathrm{len}(q) > \mathrm{len}(p) + 1$

q

(p, q)

$\mathrm{len}(q) = \mathrm{len}(p) + 1$

cur

q

$\mathrm{len}(q) > \mathrm{len}(p) + 1$

q

$\mathrm{len}(p) + 1$

$s + c$

s

q

$\mathrm{len}(p) + 1$

q

$clone$

$\mathrm{len}(clone)$

$\mathrm{len}(p) + 1$

q

q

$clone$

$$clone$$
$$q$$
$$q$$
$$clone$$
$$cur$$
$$clone$$
$$q$$
$$clone$$
$$w + c$$
$$w$$
$$p$$
$$p$$
$$-1$$
$$q$$
$$\Sigma$$
$$|\Sigma|$$
$$O(n \log|\Sigma|)$$
$$O(n)$$
$$|\Sigma|$$
$$O(n)$$
$$O(n|\Sigma|)$$
$$O(1)$$
$$last$$
$$c$$
$$q$$
$$clone$$
$$q$$
$$clone$$
$$q$$
$$clone$$
$$v = \mathrm{longest}(p)$$
$$s$$
$$s$$

$$v$$
$$last$$
$$k$$
$$(k \geq 2)$$
$$v + c$$
$$cur$$
$$last$$
$$\text{longest}(\text{link}(\text{link}(last))$$
$$n$$
$$O(n)$$
$$O(n \log|\Sigma|)$$
$$O(n|\Sigma|)$$
$$|\Sigma|$$
$$O(n)$$
$$|\Sigma|$$
$$n$$
$$s$$
$$2n - 1$$
$$n \geq 2$$
$$n - 2$$
$$2$$
$$\text{endpos}$$
$$\text{endpos}$$
$$n$$
$$2n - 1$$
$$\texttt{abbb} \cdots \texttt{bbb}$$
$$2n - 1$$
$$n$$
$$s$$
$$3n - 4$$
$$n \geq 3$$
$$t_0$$
$$2n - 2$$

(p, q)

c

$u + c + w$

u

p

w

q

$u + c + w$

u

w

$u + c + w$

s

s

n

$u + c + w$

s

$n - 1$

$3n - 3$

$\texttt{abbb} \cdots \texttt{bbb}$

$3n - 3$

$3n - 4$

$\texttt{abbb} \cdots \texttt{bbbc}$

n

n

n

i

v_i

$S_{1 \ldots i}$

i

v

$\mathrm{link}(v)$

$S_{1 \ldots p}$

$S_{1 \ldots q}$

v_p

v_q

S

i

$\mathrm{len}(i) - \mathrm{len}(\mathrm{link}(i))$

0

$\mathrm{link}(i)$

i

i

$\mathrm{link}(i)$

k

T

P

P

T

$O(|T|)$

T

P

T

t_0

P

P

T

P

T

P

$O(|P|)$

P

S

S

S

t_0

d_v

v

d_v

$$v$$

$$d_{t_0} - 1$$

$$O(|S|)$$

$$\operatorname{len}(i) - \operatorname{len}(\operatorname{link}(i))$$

$$S$$

$$d_v$$

$$ans_v$$

$$d_v$$

$$ans_v$$

$$w$$

$$d_w$$

$$v$$

$$O(|S|)$$

$$\frac{\operatorname{len}(i) \times (\operatorname{len}(i) + 1)}{2}$$

$$\operatorname{link}$$

$$S$$

$$K_i$$

$$S$$

$$K_i$$

$$k$$

$$k$$

$$k$$

$$O(|S|)$$

$$O(|ans| \cdot |\Sigma|)$$

$$ans$$

$$|\Sigma|$$

$$S$$

$$S + S$$

$$S$$

$$S + S$$

$$|S|$$

$O(|S|)$

T

P

P

T

P

0

T

v

cnt_v

$\text{endpos}(v)$

v

T

endpos

endpos

cnt

t_0

cnt

len

cnt_v

$|T|$

i

i

cnt

1

cnt

0

v

$cnt_{\text{link}(v)} += cnt_v$

v

cnt_v

cnt_v

$O(|T|)$

cnt

cnt_t

t

0

$O(|P|)$

T

P

T

P

firstpos

v

$\text{firstpos}[v]$

endpos

endpos

firstpos

cur

q

$clone$

$\text{firstpos}(cur)$

$\text{firstpos}(t) - |P| + 1$

t

P

$O(|P|)$

T

T

firstpos

t

T

$\text{firstpos}(t)$

P

P

t

t

firstpos

$O(answer(P))$

endpos

$firstpos$

S

S

S

d_v

v

v

d_v

$d_v = 1$

d_{t_0}

d

S

T

S

T

X

S

T

S

T

S

T

l

v

l

$v = t_0$

$l = 0$

T_i

v

T_i

$$l$$
$$l$$
$$\mathrm{len}(v)$$
$$l$$
$$-1$$
$$T_i$$
$$S$$
$$v = l = 0$$
$$O(|T|)$$
$$l$$
$$l$$
$$k$$
$$S_i$$
$$X$$
$$T$$
$$D_i$$
$$T$$
$$S_i$$
$$S_j$$
$$D_j$$
$$D_1, ..., D_{j-1}, D_{j+1}, ..., D_k$$
$$D_i$$
$$v$$
$$\mathrm{longest}(v)$$
$$1$$
$$s$$
$$n$$
$$i$$
$$i$$
$$i$$
$$s$$
$$s[i...n]$$

$$sa$$

$$rk$$

$$sa[i]$$

$$i$$

$$sa$$

$$rk[i]$$

$$i$$

$$rk$$

$$sa[rk[i]] = rk[sa[i]] = i$$

$$O(n \log n)$$

$$O(n)$$

$$O(n^2 \log n)$$

$$s$$

$$1$$

$$sa_1$$

$$rk_1$$

$$1$$

$$rk_1[i]$$

$$rk_1[i+1]$$

$$s$$

$$2$$

$$\{s[i \dots \min(i+1, n)] \mid i \in [1, n]\}$$

$$sa_2$$

$$rk_2$$

$$2$$

$$rk_2[i]$$

$$rk_2[i+2]$$

$$s$$

$$4$$

$$\{s[i \dots \min(i+3, n)] \mid i \in [1, n]\}$$

$$sa_4$$

$$rk_4$$

$$w/2$$

$$rk_{w/2}[i]$$

$$rk_{w/2}[i + w/2]$$

$$s$$

$$w$$

$$s[i... \min(i + w - 1, n)]$$

$$sa_w$$

$$rk_w$$

$$i + w > n$$

$$rk_w[i + w]$$

$$rk_w[i]$$

$$s[i...i + w - 1]$$

$$w \geqslant n$$

$$sa_w$$

$$O(\log n)$$

$$O(n \log n)$$

$$2$$

$$O(n)$$

$$rk$$

$$O(n \log n)$$

$$O(n \log^2 n)$$

$$O(n \log^2 n)$$

$$O(n \log n)$$

$$O(n)$$

$$O(n \log n)$$

$$O(n)$$

$$O(n)$$

$$sa[i] + w > n$$

$$sa[i]$$

$$sa$$

$$rk$$
$$p$$
$$p$$
$$rk$$
$$rk$$
$$[1, n]$$
$$O(n \log n)$$
$$O(n)$$
$$S$$
$$SS$$
$$T$$
$$S$$
$$T$$
$$S$$
$$T$$
$$S$$
$$S$$
$$T$$
$$T$$
$$p$$
$$S$$
$$S$$
$$O(\mid S \mid)$$
$$O(\mid S \mid \log \mid T \mid)$$
$$T$$
$$p$$
$$O(n)$$
$$O(1)$$
$$S$$
$$T$$
$$x$$
$$x \leq \min(\mid S \mid, \mid T \mid)$$

$$S_i = T_i \ (\forall\ 1 \le i \le x)$$

$$lcp(i, j)$$

$$i$$

$$j$$

$$height[i] = lcp(sa[i], sa[i-1])$$

$$i$$

$$height[1]$$

$$0$$

$$height[rk[i]] \ge height[rk[i-1]] - 1$$

$$height[rk[i-1]] \le 1$$

$$0$$

$$height[rk[i-1]] > 1$$

$$height$$

$$lcp(sa[rk[i-1]], sa[rk[i-1]-1]) = height[rk[i-1]] > 1$$

$$i-1$$

$$sa[rk[i-1]-1]$$

$$height[rk[i-1]]$$

$$aA$$

$$a$$

$$A$$

$$height[rk[i-1]] - 1$$

$$i-1$$

$$aAD$$

$$sa[rk[i-1]-1]$$

$$aAB$$

$$B < D$$

$$B$$

$$D$$

$$i$$

$$AD$$

$$sa[rk[i-1]-1]+1$$

$$AB$$

$$sa[rk[i]-1]$$

$$sa[rk[i]]$$

$$i$$

$$AB < AD$$

$$AB \leqslant$$

$$sa[rk[i]-1] < AD$$

$$i$$

$$sa[rk[i]-1]$$

$$A$$

$$lcp(i, sa[rk[i]-1])$$

$$height[rk[i-1]]-1$$

$$height[rk[i]] \geq height[rk[i-1]]-1$$

$$k$$

$$n$$

$$n$$

$$2n$$

$$O(n)$$

$$lcp(sa[i], sa[j]) = \min\{height[i+1..j]\}$$

$$height$$

$$A = S[a..b]$$

$$B = S[c..d]$$

$$lcp(a, c) \geq \min(\mid A \mid, \mid B \mid)$$

$$A < B \Longleftrightarrow \mid A \mid < \mid B \mid$$

$$A < B \Longleftrightarrow rk[a] < rk[c]$$

$$n(n+1)/2$$

$$lcp(sa[i], sa[j]) = \min\{height[i+1..j]\}$$

$$\frac{n(n+1)}{2} - \sum_{i=2}^{n} height[i]$$

$$k$$

$$k$$

$$k-1$$

$$height$$

$$O(n)$$

$$\mid s \mid$$

$$h$$

$$\mid s \mid$$

$$\mid s \mid$$

$$\mid s \mid$$

$$\mid s \mid$$

$$h$$

$$h$$

$$n(n-1)(n+1)/2$$

$$n-1$$

$$n(n+1)/2$$

$$lcp(i,j)=k$$

$$\min\{height[i+1..j]\}=k$$

$$lcp(i,j)$$

$$\min\{x \mid i+1 \leq x \leq j, height[x]=lcp(i,j)\}$$

$$height$$

$$height$$

```
Index :   0123456789101112
  Pat :   2113311331210
 Type :   LSSLLSSLLSLLS
Bucket :  (0)(111111)(22)(3333)
```

```
Index :   0123456789101112
 Pat‘ :   7669966996710
```

```
Index :   0123456789101112
  Pat :   7669966996710
  LMS :   * ***
   SA :   (U)(E)(EEEEM)(EE)(EEEE)
```

Index :	0123456789101112
Pat :	7669966996710
SA :	(12)(E)(EEEEM)(EE)(EEEE)
SA :	(12)(E)(EE91M)(EE)(EEEE)
SA :	(12)(E)(E592M)(EE)(EEEE)
SA :	(12)(E)(1593M)(EE)(EEEE)
SA :	(12)(E)(EE159)(EE)(EEEE)

Index :	0123456789101112
SA :	(12)(11)(15926)(100)(4837)

Index :	0123456789101112
SA :	EEEEEEEEE12159

Index :	0123456789101112
SA :	1E1E2E0EE12159

Index :	0123456789101112
SA :	1120EEEEE12159

Index :	0123456789101112
SA :	1120EEEEE3012

Index :	0123456789101112
SA :	3012EEEEE15912

Index :	0123456789101112
SA :	12159EEEEE15912

Index :	0123456789101112
SA :	(12)(E)(EE159)(EE)(EEEE)

Index :	0123456789101112
Pat :	7669966996710
SA :	(12)(E)(EE159)(EE)(EEEE)

Index :	0123456789101112
Pat :	7669966996710
SA :	(12)(U)(EE159)(ME)(MEEE)

Index :	0123456789101112
SA :	($\overrightarrow{12}$)(11)(EE159)(ME)(MEEE)
SA :	(12)($\overrightarrow{11}$)(EE159)(10E)(MEEE)
SA :	(12)(11)(EE$\vec{1}$59)(100)(MEEE)
SA :	(12)(11)(EE1$\vec{5}$9)(100)(M14E)
SA :	(12)(11)(EE15$\vec{9}$)(100)(M248)
SA :	(12)(11)(EE159)(100)($\vec{4}$83E)
SA :	(12)(11)(EE159)(100)(4$\vec{8}$37)

$$\begin{array}{rl} \texttt{Index:} & \texttt{0123456789101112} \\ \texttt{SA:} & \texttt{(12)(11)(EE\underline{EEE})(100)(4837)} \end{array}$$

`Pat`

`SA`

`Pat`

$O(n)$

`Pat`

`Pat`

`Pat`

`Pat‘`

`Pat‘`

`Pat`

`Pat`

`SA`

`Pat`

`SA`

`Pat`

$\texttt{Pat}[\texttt{i}]$

$\texttt{SA}[\texttt{Pat}[\texttt{i}]]$

$\texttt{SA}[\texttt{Pat}[\texttt{i}]]$

$\texttt{SA}[\texttt{Pat}[\texttt{i}]]$

`Pat`

$\texttt{Pat}[\texttt{i}]$

$\texttt{SA}[\texttt{Pat}[\texttt{i}]]$

$\texttt{SA}[\texttt{Pat}[\texttt{i}]] = \texttt{i}$

`SA`

$O(n)$

`SA`

$\texttt{SA}[\texttt{i}]$

$$\texttt{SA}\left[\left\lfloor \frac{\texttt{SA}[i]}{2} \right\rfloor\right]$$

`Pat`

`SA`

SA

SA

Pat1

SA

SA

SA1

SA

Pat

Pat

SA

SA

SA1

SA

Pat

SA

suf[SA[i]-1]

$$\frac{1}{3}$$

SA

$$dp[i] = 1 + \min_{s[j+1..i]\ \square\square\square\square} dp[j]$$

$$-1, 0$$

$$1$$

$$p - 1$$

$$p$$

$$s_p = s_{p-len-1}$$

$$p$$

$$-1$$

$$s_p = s_p$$

$$len_B$$

$$len_B$$

$$len_B$$

$$0$$

$$s$$

$$| s |$$

$$| s | = 1$$

$$s$$

$$| s | > 1$$

$$t = sc$$

$$t$$

$$s$$

$$c$$

$$s$$

$$c$$

$$l_1, l_2, ..., l_k$$

$$t[l_1.. | t |]$$

$$l_1 \leq p \leq | t |$$

$$t[p.. | t |] = t[l_1..l_1 + | t | - p]$$

$$1 < i \leq k$$

$$t[l_i.. | t |]$$

$$t[1.. | t | - 1]$$

$$1$$

$$O(| s |)$$

$$O(| s |)$$

$$s$$

$$n = | s |$$

$$O(n)$$

$$-1$$

$$0$$

$$-1$$

$$-1$$

$$+2$$

$$n$$

$$n$$

$$2n$$

$$s$$

$$O(| s |)$$

$$s$$

$$s$$

$$s$$

$$s(1 \leq |\ s\ | \leq 10^5)$$

$$k$$

$$s_1, s_2, ..., s_k$$

$$s_i(1 \leq i \leq k)$$

$$s_1, s_2, ..., s_k$$

$$s$$

$$dp[i]$$

$$s$$

$$i$$

$$i$$

$$O(n^2)$$

$$O(n^2)$$

$$s$$

$$i$$

$$pre(s, i)$$

$$i$$

$$suf(s, i)$$

$$0 < p \leq |\ s\ |$$

$$\forall 1 \leq i \leq |\ s\ | - p, s[i] = s[i + p]$$

$$p$$

$$s$$

$$0 \leq r < |\ s\ |$$

$$pre(s, r) = suf(s, r)$$

$$pre(s, r)$$

$$s$$

$$t$$

$$s$$

$$|\ s\ | - |\ t\ |$$

$$s$$

$$t$$

$$s$$

$$pre(s, \mid t \mid) = suf(s, \mid t \mid)$$

$$\forall 1 \leq i \leq \mid t \mid, s[i] = s[\mid s \mid - \mid t \mid + i]$$

$$\mid s \mid - \mid t \mid$$

$$s$$

$$\mid s \mid - \mid t \mid$$

$$s$$

$$\forall 1 \leq i \leq \mid s \mid - (\mid s \mid - \mid t \mid) = \mid t \mid, s[i] = s[\mid s \mid - \mid t \mid + i]$$

$$pre(s, \mid t \mid) = suf(s, \mid t \mid)$$

$$t$$

$$s$$

$$t$$

$$s$$

$$t$$

$$s$$

$$t$$

$$1 \leq i \leq \mid t \mid$$

$$s$$

$$t$$

$$s[i] = s[\mid s \mid - i + 1] = s[\mid s \mid - \mid t \mid + i]$$

$$t$$

$$s$$

$$1 \leq i \leq \mid t \mid$$

$$t$$

$$s$$

$$s[i] = s[\mid s \mid - \mid t \mid + i]$$

$$s$$

$$s[i] = s[\mid s \mid - i + 1]$$

$$s[\mid s \mid - i + 1] = s[\mid s \mid - \mid t \mid + i]$$

$$t$$

$$t$$

$$s$$

$$\mid s \mid \leq 2 \mid t \mid$$

$$s$$
$$t$$
$$s$$
$$1$$
$$t$$
$$t$$
$$t$$
$$s$$
$$\forall 1 \leq i \leq \mid t \mid, s[i] = s[\mid s \mid - \mid t \mid + i] = s[\mid s \mid - i + 1]$$
$$\mid s \mid \leq 2 \mid t \mid$$
$$s$$
$$t$$
$$s$$
$$\mid s \mid - \mid t \mid$$
$$s$$
$$\mid s \mid - \mid t \mid$$
$$s$$
$$t$$
$$s$$
$$x$$
$$y$$
$$x$$
$$z$$
$$y$$
$$u, v$$
$$x = uy, y = vz$$
$$\mid u \mid \geq \mid v \mid$$
$$\mid u \mid > \mid v \mid$$
$$\mid u \mid > \mid z \mid$$
$$\mid u \mid = \mid v \mid$$
$$u = v$$
$$3$$
$$\mid u \mid = \mid x \mid - \mid y \mid$$
$$x$$

$$| v | = | y | - | z |$$

y

$$| u | < | v |$$

y

x

u

x

y

$$| v |$$

y

$$| u | \geq | v |$$

y

x

v

x

w

$$x = vw$$

z

w

$$| u | \leq | z |$$

$$| zu | \leq 2 | z |$$

2

w

1

w

x

$$| u | > | v |$$

$$| w | > | y |$$

$$| u | > | z |$$

u, v

x

$$| u | = | v |$$

$$u = v$$

s

$$\log|\ s\ |$$

$$s$$

$$l_1, l_2, ..., l_k$$

$$2 \leq i \leq k-1$$

$$l_i - l_{i-1} = l_{i+1} - l_i$$

$$l_{i-1}, l_i, l_{i+1}$$

$$l_i - l_{i-1} \neq l_{i+1} - l_i$$

$$4$$

$$l_{i+1} - l_i > l_i - l_{i-1}$$

$$l_{i+1} - l_i > l_{i-1}$$

$$l_{i+1} > 2l_{i-1}$$

$$O(\log|\ s\ |)$$

$$s$$

$$\log|\ s\ |$$

$$s$$

$$x$$

$$x \in [2^0, 2^1), [2^1, 2^2), ..., [2^k, n)$$

$$\log|\ s\ |$$

$$dp$$

$$u$$

$$diff[u]$$

$$slink[u]$$

$$diff[u]$$

$$u$$

$$fail[u]$$

$$len[u] - len[fail[u]]$$

$$slink[u]$$

$$u$$

$$v$$

$$diff[v] \neq diff[u]$$

$$u$$

$$slink$$

$$O(\log|\ s\ |)$$

$$dp$$

$$\min$$

$$g[v]$$

$$v$$

$$dp$$

$$v$$

$$g[v] = \sum_{slink[x]=slink[v]} dp[i - len[x]]$$

$$i$$

$$g$$

$$dp$$

$$i$$

$$x$$

$$g[x]$$

$$dp$$

$$slink[x]$$

$$fail[x]$$

$$i - diff[x]$$

$$i - diff[x]$$

$$g[fail[x]]$$

$$dp$$

$$g[x]$$

$$g[fail[x]]$$

$$dp$$

$$i - (len[slink[x]] + diff[x])$$

$$g[x]$$

$$dp[i]$$

$$slink[x]$$

$$x$$

$$fail[x]$$

$$fail[x]$$

$$i - diff[x]$$

$$1$$

$$fail[x]$$

$$x$$

$$i - diff[x]$$

$$fail[x]$$

$$(i - diff[x], i)$$

$$j$$

$$x$$

$$fail[x]$$

$$2 \mid fail[x]| \geq x$$

$$fail[x]$$

$$i - diff[x]$$

$$fail[x]$$

$$w$$

$$u$$

$$uw = fail[x]$$

$$1$$

$$w$$

$$x$$

$$s[i - len[x] + 1..j] = uwu$$

$$fail[x]$$

$$x$$

$$s$$

$$s$$

$$t_1, t_2, ..., t_k$$

$$k$$

$$t_i = t_{k-i+1}$$

$$t = s[0]s[n-1]s[1]s[n-2]s[2]s[n-3]...s[n/2-1]s[n/2]$$

$$t$$

$$dp$$

$$O(n \log n)$$

$$O(n)$$

$$S[i \cdots n] + S[1 \cdots i-1] = T$$

$$S[i \cdots i+k-1] = S[j \cdots j+k-1]$$

$$S$$

$$i$$

$$S$$

$$T$$

$$S$$

$$S$$

$$i$$

$$j$$

$$k$$

$$\texttt{aaa} \cdots \texttt{aab}$$

$$O(n^2)$$

$$A, B$$

$$S$$

$$i, j$$

$$k$$

$$S[i+k] > S[j+k]$$

$$l$$

$$i \le l \le i+k$$

$$S_{i+p}$$

$$i+p$$

$$p \in [0, k]$$

$$S_{j+p}$$

$$l \in [i, i+k]$$

$$S_{i+k+1}$$

$$O(n)$$

$$i$$

$$0$$

j

1

k

0

k

i, j

S

T

S

T

T

S

T

T

S

S

T

n

m

$m \ll n$

m

$n - m$

n

$O(n)$

$n - m + 1$

m

$m(n - m + 1)$

$O(mn)$

$n - m + 1$

$O(n)$

$m(n - m + 1)$

$O(mn)$

$O(n)$

$$a\ \overbrace{b\ a\ \underset{s_3}{b}\ a\ b}^{d_1[3]=3}\ c$$

$$c\ \overbrace{b\ a\ \underset{s_3}{a}\ b}^{d_2[3]=2}\ d$$

$$\ldots\ \overbrace{s_l\ \ldots\ \underbrace{s_{j-d_1[j]+1}\ \ldots\ s_j\ \ldots\ s_{j+d_1[j]-1}}_{\text{palindrome}}\ \ldots\ \underbrace{s_{i-d_1[j]+1}\ \ldots\ s_i\ \ldots\ s_{i+d_1[j]-1}}_{\text{palindrome}}\ \ldots\ s_r}^{\text{palindrome}}\ \ldots$$

$$\ldots\ \overbrace{\underbrace{s_l\ \ldots\ s_j\ \ldots\ s_{j+(j-l)}}_{\text{palindrome}}\ \ldots\ \underbrace{s_{i-(r-i)}\ \ldots\ s_i\ \ldots\ s_r}_{\text{palindrome}}}^{\text{palindrome}}\ \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{try moving here}}$$

$$n$$

$$s$$

$$(i,j)$$

$$s[i \ldots j]$$

$$t = t_{\text{rev}}$$

$$t$$

$$t_{\text{rev}}$$

$$t$$

$$O(n^2)$$

$$i = 0 \ldots n-1$$

$$d_1[i]$$

$$d_2[i]$$

$$i$$

$$i$$

$$d_1[i]$$

$$d_2[i]$$

$$i$$

$$s = \texttt{abababc}$$

$$s[3] = b$$

$$3$$

$$d_1[3] = 3$$

$$s = \texttt{cbaabd}$$

$$s[3] = a$$

$$2$$
$$d_2[3] = 2$$
$$i$$
$$l$$
$$i$$
$$l - 2$$
$$l - 4$$
$$d_1[i]$$
$$d_2[i]$$
$$d_1[]$$
$$d_2[]$$
$$O(n \log n)$$
$$O(n)$$
$$i$$
$$1$$
$$O(n^2)$$
$$d_1[]$$
$$d_2[]$$
$$(l, r)$$
$$r$$
$$l$$
$$r$$
$$l = 0$$
$$r = -1$$
$$i$$
$$d_1[i]$$
$$d_1[]$$
$$i$$
$$i > r$$
$$d_1[i]$$
$$[i - d_1[i] ... i + d_1[i]]$$

$$d_1[i]$$
$$s$$
$$d_1[i]$$
$$(l, r)$$
$$i \le r$$
$$d_1[]$$
$$(l, r)$$
$$i$$
$$j = l + (r - i)$$
$$d_1[j]$$
$$j$$
$$i$$
$$d_1[i] = d_1[j]$$
$$j$$
$$i$$
$$j - d_1[j] + 1 \le l$$
$$i + d_1[j] - 1 \ge r$$
$$d_1[i] = d_1[j]$$
$$i$$
$$d_1[i] = r - i$$
$$d_1[i]$$
$$j$$
$$j$$
$$i$$
$$i$$
$$d_1[i]$$
$$(l, r)$$
$$d_2[]$$
$$d_1[]$$
$$r$$
$$1$$

$$r$$
$$O(n)$$
$$O(n)$$
$$d_1[]$$
$$d_2[]$$
$$d_1[]$$
$$d_2[]$$
$$d_1[]$$
$$n$$
$$s$$
$$n+1$$
$$\#$$
$$2n+1$$
$$s'$$
$$s = \texttt{abababc}$$
$$s' = \#\texttt{a}\#\texttt{b}\#\texttt{a}\#\texttt{b}\#\texttt{a}\#\texttt{b}\#\texttt{c}\#$$
$$\#$$
$$s$$
$$\#$$
$$s'$$
$$d_1[]$$
$$i$$
$$d_1[i]$$
$$\#$$
$$\#$$
$$s$$
$$m+1$$
$$s'$$
$$2m+3$$
$$s$$
$$m$$
$$s'$$

$$\#$$

$$2m+1$$

$$m$$

$$s'$$

$$d_1[i]$$

$$s$$

$$s'$$

$$d_1[]$$

$$s$$

$$d_1[]$$

$$d_2[]$$

$$s'$$

$$d_1[]$$

$$s'$$

$$d_1[]$$

$$O(N)$$

$$\begin{aligned} t_0 &= a, \\ t_1 &= b, \\ t_i &= t_{i-1} + t_{i-2}, \end{aligned}$$

$$l_1 + l_2 = l = \mid u \mid - cntr$$

$$\begin{aligned} l_1 &\leq k_1, \\ l_2 &\leq k_2. \end{aligned}$$

$$n$$

$$s$$

$$(i, j)$$

$$s[i...j]$$

$$s$$

$$s$$

$$\overline{s}$$

$$s$$

$$\overline{\texttt{abc}} = \texttt{cba}$$

$$\texttt{acababaee}$$

$$s[2...5] = \texttt{abab}$$

$$s[3...6] = \texttt{baba}$$

$$s[7...8] = \texttt{ee}$$

$$\texttt{abaaba}$$

$$s[0...5] = \texttt{abaaba}$$

$$s[2...3] = \texttt{aa}$$

$$n$$

$$O(n^2)$$

$$n$$

$$O(n \log n)$$

$$O(n \log n)$$

$$(i, p, r)$$

$$i$$

$$p$$

$$r$$

$$O(n \log n)$$

$$f_i$$

$$t_i$$

$$O(f_i \log f_i)$$

$$O(f_i \log f_i)$$

$$u$$

$$v$$

$$s = u + v$$

$$u, v$$

$$s$$

$$s[i...j]$$

$$s[(i + j + 1)/2]$$

$$u$$

$$2l$$

$$v$$

$$s[\mid u \mid]$$

$$u$$

$$u[cntr]$$

$$cntr$$

$$\texttt{c}\ \underset{\texttt{cntr}}{\texttt{a}}\ \texttt{c} \mid \texttt{a d a}$$

$$|$$

$$cntr = 1$$

$$\texttt{caca}$$

$$cntr$$

$$l$$

$$0$$

$$\mid u \mid -1$$

$$cntr$$

$$cntr$$

$$\texttt{abcabcac}$$

$$\overbrace{\texttt{a}}^{l_1}\ \overbrace{\underset{cntr}{\texttt{b}}\ \texttt{c}}^{l_2}\ \overbrace{\texttt{a}}^{l_1} \mid \overbrace{\texttt{b c}}^{l_2}$$

$$l_1$$

$$s[cntr-1]$$

$$l_2$$

$$s[cntr]$$

$$2l = 2(l_1 + l_2) = 2(\mid u \mid - cntr)$$

$$k_1$$

$$u[cntr - k_1 ... \ cntr - 1] = u[\mid u \mid - k_1 ... \mid u \mid - 1]$$

$$k_2$$

$$u[cntr \ ... \ cntr + k_2 - 1] = v[0 ... k_2 - 1]$$

$$l_1 \leq k_1$$

$$l_2 \leq k_2$$

$$(l_1, l_2)$$

$$cntr$$

$$2l = 2(\mid u \mid - cntr)$$

$$l_1$$

$$l_2$$

$$k_1$$

$$k_2$$

$$k_1$$

$$k_2$$

$$O(1)$$

$$k_1$$

$$\overline{u}$$

$$k_2$$

$$v + \# + u$$

$$\#$$

$$u$$

$$v$$

$$u$$

$$s[\mid u \mid - 1]$$

$$v$$

$$v$$

$$cntr$$

$$k_1$$

$$v[cntr - k_1 + 1 ... \ cntr] = u[\mid u \mid - k_1 ... \mid u \mid - 1]$$

$$k_2$$

$$v[cntr + 1 ... \ cntr + k_2] = v[0 ... k_2 - 1]$$

$$\overline{u} + \# + \overline{v}$$

$$v$$

$$k_1$$

$$k_2$$

$$cntr$$

$$(cntr, l, k_1, k_2)$$

$$O(n \log n)$$

$$O(n^2)$$

$$s$$

$$s$$

$$s$$

$$s$$

$$s$$

s

s

$s = w_1 w_2 \cdots w_k$

w_i

$w_1 \geq w_2 \geq \cdots \geq w_k$

$O(n)$

t

$t = ww \cdots \overline{w}$

w

$\overline{w}$

w

$\overline{w}$

t

s

$s = s_1 s_2 s_3$

s_1

s_2

s_3

s_3

s_2

s_2

s_2

s_1

i

s_2

i

1

n

j

s_3

k

s_2

j

s_2

$s[j]$

s_2

$s[j]$

$s[k]$

$s[j] = s[k]$

$s[j]$

s_2

j, k

$s[j] > s[k]$

$s_2 s[j]$

j

k

s_2

s_2

$s[j] < s[k]$

$s_2 s[j]$

s_2

$j - k$

s_2

j, k

s

n

i

$O(n)$

n

$O(n)$

$4n - 3$

$O(n)$

n

s

ss

t

$$n$$

$$n$$

$$t$$

$$s$$

$$t$$

$$n$$

$$s$$

$$\Theta(N)$$

$$\Theta(N)$$

$$\pi[i] = \max_{k=0...i}\{k : s[0...k-1] = s[i-(k-1)...i]\}$$

$$\underbrace{\overbrace{s_0\ s_1\ s_2}^{\pi[i]=3}\ s_3}_{\pi[i+1]=4}\ ...\ \underbrace{\overbrace{s_{i-2}\ s_{i-1}\ s_i}^{\pi[i]=3}\ s_{i+1}}_{\pi[i+1]=4}$$

$$\overbrace{\underbrace{s_0\ s_1}_{j}\ s_2\ s_3}^{\pi[i]}\ ...\ \overbrace{s_{i-3}\ s_{i-2}\ \underbrace{s_{i-1}\ s_i}_{j}}^{\pi[i]}\ s_{i+1}$$

$$s[0...j-1] = s[i-j+1...i] = s[\pi[i]-j...\pi[i]-1]$$

$$\overbrace{r_0\ r_1\ r_2\ r_3\ r_4\ r_5}^{p}\ \overbrace{r_0\ r_1\ r_2\ r_3\ r_4 r_5}^{p}$$

$$r_0\ r_1\ r_2\ r_3\ \underbrace{\overbrace{r_0\ r_1\ r_2\ r_3\ r_4\ r_5}^{p}\ r_0\ r_1}_{\pi[11]=8}$$

$$\underbrace{s_0\ s_1\ ...\ s_{n-1}\ \#}_{\text{need to store}}\ \underbrace{t_0\ t_1\ ...\ t_{m-1}}_{\text{do not need to store}}$$

$$g_1 = \texttt{a}$$
$$g_2 = \texttt{aba}$$
$$g_3 = \texttt{abacaba}$$
$$g_4 = \texttt{abacabadabacaba}$$

$$\text{mid} = \text{aut}[G[i-1][j]][i]$$
$$G[i][j] = G[i-1][\text{mid}]$$

$$K[i][j] = K[i-1][j] + [\text{mid} == \mid s \mid] + K[i-1][\text{mid}]$$

$$t_1 = \texttt{abdeca}$$
$$t_2 = \texttt{abc} + t_1^{30} + \texttt{abd}$$
$$t_3 = t_2^{50} + t_1^{100}$$
$$t_4 = t_2^{10} + t_3^{100}$$

$$n$$

$$s$$

$$n$$
$$\pi$$
$$\pi[i]$$
$$s[0...i]$$
$$s[0...k-1]$$
$$s[i-(k-1)...i]$$
$$\pi[i]$$
$$\pi[i]=k$$
$$\pi[i]$$
$$\pi[i]=0$$
$$\pi[i]$$
$$s[0...i]$$
$$\pi[0]=0$$
$$\pi[0]=0$$
$$\pi[1]=0$$
$$\pi[2]=0$$
$$\pi[3]=1$$
$$\pi[4]=2$$
$$\pi[5]=3$$
$$\pi[6]=0$$
$$[0,1,0,1,2,2,3]$$
$$i=1\rightarrow n-1$$
$$\pi[i]$$
$$\pi[0]$$
$$0$$
$$\pi[i]$$
$$j$$
$$i$$
$$\pi[i]$$
$$j=0$$

$$j = 0$$

$$\pi[i] = 0$$

$$i + 1$$

$$O(n^3)$$

$$1$$

$$\pi[i+1]$$

$$s[i+1]$$

$$s[i+1] = s[\pi[i]]$$

$$\pi[i+1] = \pi[i] + 1$$

$$\pi[i]$$

$$O(n)$$

$$O(n^2)$$

$$\pi[i+1]$$

$$s[i+1] = s[\pi[i]]$$

$$\pi[i+1] = \pi[i] + 1$$

$$s[i+1] \neq s[\pi[i]]$$

$$s[0...i]$$

$$\pi[i]$$

$$j$$

$$i$$

$$s[0...j-1] = s[i-j+1...i]$$

$$j$$

$$s[i+1]$$

$$s[j]$$

$$\pi[i+1] = j + 1$$

$$s[0...i]$$

$$j$$

$$j^{(2)}$$

$$j = 0$$

$$s[i+1] \neq s[0]$$

$$\pi[i+1]=0$$

$$s[0...\pi[i]-1]=s[i-\pi[i]+1...i]$$

$$s[0...i]$$

$$j$$

$$j$$

$$s[\pi[i]-1]$$

$$j=\pi[\pi[i]-1]$$

$$j$$

$$s[j-1]$$

$$j^{(2)}=\pi[j-1]$$

$$j$$

$$j^{(n)}=\pi[j^{(n-1)}-1],\ \ (j^{(n-1)}>0)$$

$$O(n)$$

$$M$$

$$M+1$$

$$t$$

$$s$$

$$s$$

$$t$$

$$n$$

$$s$$

$$m$$

$$t$$

$$s+\#+t$$

$$\#$$

$$s$$

$$t$$

$$n+1$$

$$s$$

$$\pi[i]$$

$$i$$

$$s$$

$$i$$

$$n$$

$$\pi[i] = n$$

$$s$$

$$i$$

$$s + \# + t$$

$$i$$

$$\pi[i] = n$$

$$s$$

$$t$$

$$i - (n-1) - (n+1) = i - 2n$$

$$s + \#$$

$$t$$

$$O(n+m)$$

$$O(n)$$

$$s$$

$$0 < p \le \mid s \mid$$

$$s[i] = s[i+p]$$

$$i \in [0, \mid s \mid - p - 1]$$

$$p$$

$$s$$

$$s$$

$$0 \le r < \mid s \mid$$

$$s$$

$$r$$

$$r$$

$$s$$

$$r$$

$$s$$

$$s$$

$$r$$

$$\mid s \mid - r$$

$$s$$

$$s$$

$$\pi[n-1], \pi[\pi[n-1]-1], ...$$

$$O(n)$$
$$s$$
$$\pi[n-1]$$
$$s$$
$$n-\pi[n-1]$$
$$s$$
$$n$$
$$s$$
$$s[0...i]$$
$$s[0...i]$$
$$t$$
$$i$$
$$\pi[i]$$
$$s$$
$$\pi[i]$$
$$i$$
$$i$$
$$j$$
$$i$$
$$k<j$$
$$i$$
$$i$$
$$\pi[i]$$
$$\pi[\pi[i]-1]$$
$$\pi[\pi[\pi[i]-1]-1]$$
$$0$$
$$\pi$$
$$i$$
$$\text{ans}[i]$$
$$1$$
$$s+\#+t$$
$$t$$
$$i \geq n+1$$

$$\pi[i]$$
$$n$$
$$s$$
$$s$$
$$k$$
$$s$$
$$c$$
$$s$$
$$c$$
$$t = s + c$$
$$t^{\sim}$$
$$t^{\sim}$$
$$t^{\sim}$$
$$t^{\sim}$$
$$\pi_{\max}$$
$$s$$
$$\pi_{\max}$$
$$\mid s \mid + 1 - \pi_{\max}$$
$$O(n)$$
$$O(n^2)$$
$$n$$
$$s$$
$$t$$
$$s$$
$$t$$
$$t$$
$$s$$
$$s$$
$$\pi[n-1]$$
$$k = n - \pi[n-1]$$
$$k$$
$$n$$
$$k$$
$$n$$
$$n$$

$$k$$
$$k$$
$$n-k$$
$$s$$
$$k$$
$$k$$
$$\pi[n-1]$$
$$n-k$$
$$k$$
$$n$$
$$k$$
$$n$$
$$r$$
$$p$$
$$p$$
$$n$$
$$s$$
$$n/p \geq 2$$
$$\pi[n-1]$$
$$n-p$$
$$n$$
$$k$$
$$r_0 r_1 ... r_{p-1}$$
$$r_{p-k} r_{p-k+1} ... r_{p-1} r_0 r_1 ... r_{p-k-1}$$
$$p$$
$$r_{(i+k) \bmod p} = r_{i \bmod p}$$
$$\cdot \bmod p$$
$$p$$
$$x$$
$$y$$
$$xk + yp = \gcd(k, p)$$
$$pk - kp = 0$$
$$x' > 0$$
$$y' < 0$$

$$x'k + y'p = \gcd(k, p)$$

$$r_{(i+\gcd(k,p)) \bmod p} = r_{i \bmod p}$$

$$\gcd(k, p)$$

$$p$$

$$\gcd(k, p)$$

$$r$$

$$\pi[n-1] > n - p$$

$$n - \pi[n-1] = k < p$$

$$\gcd(k, p)$$

$$p$$

$$r$$

$$s$$

$$\gcd(k, p) < p$$

$$p$$

$$k$$

$$k$$

$$s$$

$$t$$

$$s + \# + t$$

$$\#$$

$$\mid s \mid$$

$$s + \#$$

$$t$$

$$c$$

$$t$$

$$t$$

$$(\text{old } \pi, c) \rightarrow \text{new\_}\pi$$

$$O(\mid \Sigma \mid n^2)$$

$$j$$

$$\pi[j-1]$$

$$(j, c)$$

$$(\pi[j-1], c)$$

$$O(\mid \Sigma \mid n)$$

$$s + \# + t$$

$$s$$

$$t$$

$$s + \# + t$$

$$s + \#$$

$$s$$

$$t$$

$$k \leq 10^5$$

$$\leq 10^5$$

$$s$$

$$s$$

$$k$$

$$t$$

$$k$$

$$2^k - 1$$

$$G[i][j]$$

$$j$$

$$g_i$$

$$K[i][j]$$

$$j$$

$$g_i$$

$$s$$

$$g_i$$

$$K[i][j]$$

$$\mid s \mid$$

$$K[k][0]$$

$$G[0][j] = j$$

$$K[0][j] = 0$$

$$i$$

$$g_i$$

$$g_{i-1}$$

$$i$$

$$g_{i-1}$$

$$K[i][j]$$

$$[\cdot]$$

$$1$$

$$0$$

$$s$$

$$t_i$$

$$t_k^{\text{cnt}}$$

$$t_k$$

$$\text{cnt}$$

$$100^{100}$$

$$s$$

$$f$$

$$f$$

$$f$$

$$10^9$$

$$10^{18}$$

$$l$$

$$s$$

$$f(s) = \sum_{i=1}^{l} s[i] \times b^{l-i} \pmod M$$

$$xyz$$

$$xb^2 + yb + z$$

$$f(s) = \sum_{i=1}^{l} s[i] \times b^{i-1} \pmod M$$

$$xyz$$

$$x + yb + zb^2$$

$$b$$

$$f(s) = \sum_{i=1}^{l} s[i] \times b^{l-i} \pmod M$$

$$M$$

$$M$$

$$b$$

$$x$$

$$b$$

$$f(s)$$

$$\mathbb{Z}_M[x]$$

$$s, t$$

$$f(s) = f(t)$$

$$h(x) = f(s) - f(t) = \sum_{i=1}^{l} (s[i] - t[i]) x^{l-i} \pmod M$$

$$l = \max(|\ s\ |, |\ t\ |)$$

$$h(x)$$

$$l - 1$$

$$s$$

$$t$$

$$x = b$$

$$b$$

$$h(x)$$

$$h(x)$$

$$\mathbb{Z}_M$$

$$M$$

$$M$$

$$l - 1$$

$$b$$

$$[0, M)$$

$$f(s)$$

$$f(t)$$

$$\frac{l-1}{M}$$

$$1$$

$$\frac{1-1}{M} = 0$$

$$M$$

$$n$$

$$\prod_{i=0}^{n-1} \frac{M-i}{M}$$

$$i$$

$$\frac{M-i}{M}$$

$$M = 10^9 + 7$$

$$n = 10^6$$

$$O(n)$$

$$n$$

$$b$$

$$M$$

$$f_i(s)$$

$$f(s[1..i])$$

$$i$$

$$f_i(s) = s[1] \cdot b^{i-1} + s[2] \cdot b^{i-2} + ... + s[i-1] \cdot b + s[i]$$

$$f(s[l..r])$$

$$s[l..r]$$

$$f(s[l..r]) = s[l] \cdot b^{r-l} + s[l+1] \cdot b^{r-l-1} + ... + s[r-1] \cdot b + s[r]$$

$$f(s[l..r]) = f_r(s) - f_{l-1}(s) \times b^{r-l+1}$$

$$b^{r-l+1}$$

$$O(n)$$

$$O(1)$$

$$O(\log n)$$

$$k$$

$$n$$

$$s$$

$$m$$

$$p$$

$$s'$$

$$s$$

$$s'$$

$$s$$

$$k$$
$$1 \leq n, m \leq 10^6$$
$$0 \leq k \leq 5$$
$$s'$$
$$s'$$
$$p$$
$$s'$$
$$p$$
$$k$$
$$O(m + kn \log_2 m)$$
$$O(n \log n)$$
$$O(n)$$
$$O(n)$$
$$R_i$$
$$i$$
$$\max_{i=1}^{n} R_i$$
$$R_i \leq R_{i-1} + 2$$
$$R_{i-1} + 2$$
$$z$$
$$R_i$$
$$0$$
$$z$$
$$i$$
$$2$$
$$1$$
$$2n$$
$$O(n)$$
$$m$$
$$n$$
$$1 \leq m, n \leq 10^6$$
$$k$$
$$k - 1$$

$$k$$

$$k$$

$$n$$

$$O(m + n \log n)$$

$$n$$

$$l = 1, \cdots, n$$

$$l$$

$$b$$

$$h[i]$$

$$h[0] = 0$$

$$i$$

$$i$$

$$i$$

$$10^5$$

$$10^6$$

$$i$$

$$k$$

$$S_1, S_2, S_3 ... S_k$$

$$0$$

$$1$$

$$\sum_{i=1}^{k} | \ S_i |$$

$$i$$

$$len[i] - len[link[i]]$$

$$k$$

$$0$$

$$len$$

pat :	`EXAMPLE`
string :	`HERE IS A SIMPLE EXAMPLE` ...
	⇑

int *delta*1(**char** *char*)
    if char □□ pat □ || char □ pat □□□□□□□
        return *patlen*
    else
        return $patlastpos - i$ // i □□□□ pat □□□□□□□ char □□□□□□□ pat[i]=char

int $delta2($**int** $j)$ // j □□□□□□ pat □□□□□□□□

return $patlastpos - rpr(j)$

```
pat :         AT-THAT
string :  ... WHICH-FINALLY-HALTS.–AT-THAT-POINT ...
                    ⇑

pat :                AT-THAT
string :  ... WHICH-FINALLY-HALTS.–AT-THAT-POINT ...
                            ⇑

pat :                    AT-THAT
string :  ... WHICH-FINALLY-HALTS.–AT-THAT-POINT ...
                                ⇑

pat :                    AT-THAT
string :  ... WHICH-FINALLY-HALTS.–AT-THAT-POINT ...
                               ⇑

pat :                           AT-THAT
string :  ... WHICH-FINALLY-HALTS.–AT-THAT-POINT ...
                                      ⇑

pat :                           AT-THAT
string :  ... WHICH-FINALLY-HALTS.–AT-THAT-POINT ...
                                     ⇑

pat :                                AT-THAT
string :  ... WHICH-FINALLY-HALTS.–AT-THAT-POINT ...
                                            ⇑
```

$i \leftarrow patlastpos.$
$j \leftarrow patlastpos.$
loop
    if $j < 0$
        return $i + 1$

    if $string[i] = pat[j]$
        $j \leftarrow j - 1$
        $i \leftarrow i - 1$
        continue

    $i \leftarrow i + max(delta_1(string[i]), delta_2(j))$

    if $i > stringlastpos$
        return $false$
    $j \leftarrow patlastpos$

j :	0 1 2 3 4 5 6 7 8
pat :	A B C X X X A B C
$rpr(j)$:	5 4 3 2 1 0 2 1 8
sgn :	$\bullet$ - - - - - - - +

j :	0 1 2 3 4 5 6 7 8
pat :	A B Y X C D E Y X
$rpr(j)$:	8 7 6 5 4 3 2 1 8
sgn :	$\bullet$ - - - - - + - +

```
int delta0(char char)
    if char = pat[patlastpos]
        return large  // large □□□□□□□□□□ large>stringlastpos+patlen
    return delta1(char)

i ← patlastpos
loop
    if i > stringlastpos
        return false

    while i < stringlen
        i ← i + delta0(string(i))  // □□ string[i]□ pat □□□□□□□□□□□□□□□□ patlen
    if i ⩽ large                   //□□□□ string □□□□□□□□□ pat □□□□□□□
        return false

    i ← i − large
    j ← patlastpos.
    while j ⩾ 0 and string[i] = pat[j]
        j ← j − 1
        i ← i − 1

    if j < 0
        return i + 1
    i ← i + max(delta_1(string[i]), delta_2(j))
```

j :	0 1 2 3 4 5 6 7 8
pat :	A B C X X X A B C

j :	0 1 2 3 4 5 6 7 8 9
pat :	A B A A B A A B A A

$$\frac{\sum_{m=0}^{patlen-1} cost(m) \times prob(m)}{\sum_{m=0}^{patlen-1} prob(m) \times \sum_{k=1}^{patlen} skip(m,k) \times k}$$

$$cost(m) = m + 1$$

$$prob(m) = \frac{p^m(1-p)}{1-p^{patlen}}$$

$$probdelta1\_worthless(m) = 1 - (1-p)^m$$

$$probdelta1(m,k) = \begin{cases} lcl(1-p)^m \times \begin{cases} lcl1 & \text{for } m+1=patlen \\ p & \text{for } m+1 \neq patlen \end{cases} & \text{for } k=1 \\ (1-p)^{m+k-1} \times p & \text{for } 1 < k < patlen - m \\ (1-p)^{patlen-1} & \text{for } k = patlen - m \\ 0 & \text{for } patlenpatlen - m < k \leqslant patlen \end{cases}$$

$$probpr(m,k) = \begin{cases} lcl(1-p) \times p^m & \text{for } 1 \leqslant k < patlen - m \\ p^{patlen-k} & \text{for } patlen - m \leqslant k \leqslant patlen \end{cases}$$

$$probdelta2(m,k) = probpr(m,k)(1 - \sum_{n=1}^{k-1} probdelta2(m,n))$$

$$probdelta2'(m,k) = \begin{cases} 0 & \text{for } k=1 \\ probpr(m,k)\left(1 - \sum_{n=2}^{k-1} probdelta2'(m,n)\right) & \text{for } 1 \leqslant k \leqslant patlen \end{cases}$$

$$skip(m,k) = \begin{cases} probdelta1(m,1) \times probdelta2(m,1) & \text{for } k=1 \\ prodelta1\_\ worthless(m) \times probdelta2'(m,1) & \\ + probdelta1(m,k) \times \sum_{n=1}^{k-1} probdelta2(m,n) & \\ + probdelta2(m,k) \times \sum_{n=1}^{k-1} probdelta1(m,n) & \\ + probdelta1(m,k) \times probdelta2(m,k) & \text{for } 1 < k \leqslant patlen \end{cases}$$

$$pat$$

$$string$$

$$pat$$

$$patlen$$

$$patlastpos = patlen - 1$$

$$pat$$

$$string$$

$$stringlen$$

$$stringlastpos = stringlen - 1$$

$$string$$

$$patlen$$

$$char$$

$$pat$$

$$char$$

$$pat$$

$$pat$$

$$string$$
$$1$$
$$2$$
$$patlen$$
$$pat$$
$$patlen$$
$$pat$$
$$char$$
$$delta_1$$
$$pat$$
$$delta_1$$
$$delta_1$$
$$char$$
$$pat$$
$$char$$
$$pat$$
$$string$$
$$delta_1$$
$$pat$$
$$delta_1$$
$$delta_1$$
$$delta_1(char)$$
$$patlastpos - 1$$
$$char$$
$$pat$$
$$char$$
$$pat$$
$$pat$$
$$string$$
$$pat$$
$$pat$$
$$m$$

$$m + 1$$
$$pat$$
$$pat$$
$$m$$
$$m + 1$$
$$string$$
$$pat$$
$$pat$$
$$k$$
$$k + m$$
$$pat$$
$$string$$
$$k = delta_1 - m$$
$$string$$
$$delta_1 - m + m = delta_1$$
$$string$$
$$m$$
$$pat$$
$$m$$
$$subpat$$
$$string$$
$$char$$
$$subpat$$
$$pat$$
$$subpat$$
$$pat$$
$$char$$
$$pat$$
$$subpat$$
$$string$$
$$subpat$$
$$pat$$

$$subpat$$

$$pat$$

$$k$$

$$pat$$

$$subpat$$

$$string$$

$$k + m$$

$$delta_2(j)$$

$$rpr(j)$$

$$subpat = pat[j + 1...patlastpos]$$

$$pat[j]$$

$$rpr(j) < j$$

$$k = j - rpr(j), m = patlastpos - j$$

$$string$$

$$\max(delta_1, delta_2)$$

$$char$$

$$\texttt{F}$$

$$pat$$

$$pat$$

$$patlen$$

$$pat$$

$$\texttt{T}$$

$$string$$

$$\texttt{L}$$

$$\texttt{L}$$

$$pat$$

$$pat$$

$$k = delta_1 - m = 7 - 1 = 6$$

$$string$$

$$delta_1 = 7$$

$$char$$

$$pat$$

T

$$string$$

A

-

$$pat$$

$$k = 5$$

AT

$$string$$

$$delta_2 = k + patlastpos - j = 5 + 6 - 4 = 7$$

$$string$$

$$delta_1 = 7 - 1 - 2 = 4,\, delta_2 = 7, \max(delta_1, delta_2) = 7$$

$$pat$$

$$string$$

$$string$$

$$pat$$

$$string$$

$$patlen = 7$$

$$delta_1$$

$$delta_2$$

$$\textbf{return } false$$

$$pat$$

$$string$$

$$pat$$

$$string$$

$$delta_2$$

$$rpr(j)$$

$$rpr(j)$$

$$pat(j)$$

$$subpat = pat[j + 1...patlastpos]$$

$$pat[j]$$

$$k$$

$$pat[k...k+patlastpos-j-1]=pat[j+1...patlastpos]$$

$$k<0$$

$$pat$$

$$delta_2$$

$$k>0$$

$$pat[k-1]=pat[j]$$

$$pat[k...k+patlastpos-j-1]$$

$$subpat$$

$$pat[j]$$

$$pat$$

$$k$$

$$pat[k-1]$$

$$k<j$$

$$k=j$$

$$pat[k]=pat[j]$$

$$pat[j]$$

$$pat[k]$$

$$delta_2(patlastpos)=0$$

$$rpr(patlastpos)=patlastpos$$

$$rpr(j)$$

$$rpr(0)$$

$$subpat$$

`BCXXXABC`

$$pat[0]$$

[(`BCXXX`)`ABC`]`XXXABC`

$$rpr(j)=-5$$

$$rpr(1)$$

$$subpat$$

`CXXXABC`

$$pat[1]$$

$[(\texttt{CXXX})\texttt{ABC}]\texttt{XXXABC}$

$$rpr(j) = -4$$

$$rpr(2)$$

$$subpat$$

XXXABC

$$pat[2]$$

$[(\texttt{XXX})\texttt{ABC}]\texttt{XXXABC}$

$$rpr(j) = -3$$

$$rpr(3)$$

$$subpat$$

XXABC

$$pat[3]$$

$[(\texttt{XX})\texttt{ABC}]\texttt{XXXABC}$

$$rpr(j) = -2$$

$$rpr(4)$$

$$subpat$$

XABC

$$pat[4]$$

$[(\texttt{X})\texttt{ABC}]\texttt{XXXABC}$

$$rpr(j) = -1$$

$$rpr(5)$$

$$subpat$$

ABC

$$pat[5]$$

$[\texttt{ABC}]\texttt{XXXABC}$

$$rpr(j) = 0$$

$$rpr(6)$$

$$subpat$$

BC

$$string[0] = string[6]$$

$$string[0]$$

$$string[6]$$

$$string[0...2]$$

$$subpat$$

$[(\texttt{BC})]\texttt{ABCXXXABC}$

$$rpr(j) = -2$$

$$rpr(7)$$

$$subpat$$

C

$$string[7] = string[1]$$

$$string[1...2]$$

$$subpat$$

$[(\texttt{C})]\texttt{ABCXXXABC}$

$$rpr(j) = -1$$

$$rpr(8)$$

$$delta_2$$

$$rpr(patlastpos) = patlastpos$$

$$rpr(8) = 8$$

$$rpr(0)$$

$$subpat$$

BYXCDEYX

$$pat[0]$$

$[(\texttt{BYXCDEYX})]\texttt{ABYXCDEYX}$

$$rpr(j) = -8$$

$$rpr(1)$$

$$subpat$$

YXCDEYX

$$pat[1]$$

$[(\texttt{YXCDEYX})]\texttt{ABYXCDEYX}$

$$rpr(j) = -7$$

$$rpr(2)$$

$$subpat$$

`XCDEYX`

$$pat[2]$$

[(`XCDEYX`)]`ABYXCDEYX`

$$rpr(j) = -6$$

$$rpr(3)$$

$$subpat$$

`CDEYX`

$$pat[3]$$

[(`CDEYX`)]`ABYXCDEYX`

$$rpr(j) = -5$$

$$rpr(4)$$

$$subpat$$

`DEYX`

$$pat[4]$$

[(`DEYX`)]`ABYXCDEYX`

$$rpr(j) = -4$$

$$rpr(5)$$

$$subpat$$

`EYX`

$$pat[5]$$

[(`EYX`)]`ABYXCDEYX`

$$rpr(j) = -3$$

$$rpr(6)$$

$$subpat$$

`YX`

$$string[2...3] = string[7...8]$$

$$string[6] \neq string[1]$$

$pat[6]$

`AB[YX]CDEYX`

$rpr(j) = 2$

$rpr(7)$

$subpat$

`X`

$string[3] = string[8]$

$string[2] = string[7]$

$pat[7]$

`[X]ABYXCDEYX`

$rpr(j) = -1$

$rpr(8)$

$delta_2$

$rpr(patlastpos) = patlastpos$

$rpr(8) = 8$

$string[i]$

$pat[patlastpos]$

$patlen$

$delta0$

$delta0$

$delta_1$

$large$

$delta_2$

$delta_2$

$delta_2$

$delta_2$

$delta_1$

$delta_2$

$delta_2$

$delta_2$

$$delta_1$$

$$delta_2$$

$$delta_2$$

$$delta_2$$

$$delta_2$$

$$O(n^3)$$

$$delta_2$$

$$delta_2$$

$$pat$$

$$delta_2$$

$$subpat$$

$$delta_2(lastpos) = 0$$

$$O(n^3)$$

$$O(n)$$

$$delta_2$$

$$O(n)$$

$$O(n)$$

$$pat$$

$$delta_2$$

$$delta_2$$

$$delta_2$$

$$delta_2$$

$$delta_2$$

$$delta_2$$

$$subpat$$

$$subpat$$

$$pat$$

$$[(\texttt{EYX})]\texttt{ABYXCDEYX}$$

$$delta_2(j) = patlastpos \times 2 - j$$

$$subpat$$

$$pat$$

$$pat$$

$$[(\texttt{XX})\texttt{ABC}]\texttt{XXXABC}$$

$$patlastpos < delta_2(j) < patlastpos \times 2 - j$$

$$subpat$$

$$pat$$

$$[\texttt{ABC}]\texttt{XXXABC}$$

$$patlastpos = delta_2(j)$$

$$subpat$$

$$pat$$

$$\texttt{AB}[\texttt{YX}]\texttt{CDEYX}$$

$$delta_2(j) < patlastpos$$

$$delta_2$$

$$subpat$$

$$pat$$

$$pat$$

$$subpat$$

$$pat$$

$$delta_2(3)$$

$$[(\texttt{XX})\texttt{ABC}]\texttt{XXXABC}$$

$$subpat$$

$$\texttt{XXABC}$$

$$\texttt{ABC}$$

$$j$$

$$subpat$$

$$subpat$$

$$j \leqslant 5$$

$$j = 5$$

$$subpat$$

$$pat$$

$$delta_2(j)$$

$$prefixlen$$

$$subpatlen = patlastpos - j$$

$$pat$$

$$subpatlen - prefixlen$$

$$rpr(j) = -(subpatlen - prefixlen)$$

$$delta_2(j) = patlastpos - rpr(j) = patlastpos \times 2 - j - prefixlen$$

$$j \leq 2$$

`ABAABAA`

$$2 < j \leq 5$$

`ABAA`

$$5 < j \leq 8$$

`A`

$$subpat$$

$$pat$$

$$pat$$

$$j$$

$$delta_2(j)$$

$$pat$$

$$j^{(n)} = \pi[j^{(n-1)} - 1]$$

$$\pi[patlastpos]$$

$$delta_2$$

$$subpat$$

$$pat$$

$$pat$$

$$pat[0...patlastpos - 1]$$

$$subpat$$

$$patlen \leqslant 2$$

$$delta_2$$

$$subpat$$

$$pat$$

$$subpat$$

$$delta_2$$

$$string$$

$$pat$$

$$string$$

$$pat$$

$$string$$

$$patlen$$

$$O(mn)$$

$$m$$

$$patlen$$

$$n$$

$$stringlen$$

$$pat$$

`AAA`

$$string$$

`AAAAA...`

$$patlen$$

$$string$$

$$2^m$$

$$pat$$

$$\frac{1}{2}m^2 + m$$

$$O(qm)$$

$$q$$

$$qm$$

$$pat$$

$$U$$

$$UUUU...$$

$$U$$

$$pat$$

pat : `ABCABCAB`

`ABC`

`ABCABCABC`

ABC

3

pat

pat

$pat[i] = pat[i + 3]$

ABCABC...

pat

pat

k

$k \leq patlen$

pat

$string$

k

k

pat

pat

$prefixlen$

$pat[i] = pat[i + (patlen - prefixlen)]$

$patlen - prefixlen$

pat

pat

$\pi[patlastpos]$

$delta_2$

pat

$string$

pat

AAA

$string$

AAAAA...

$delta_1$

$delta_1$

$$O(1)$$

$$patlen$$

$$string$$

$$delta_2$$

$$delta_1$$

$$delta_1$$

$$pat$$

$$delta_1$$

$$pat$$

$$delta_1$$

$$patlen + 1$$

$$pat$$

$$patlastpos$$

$$0...patlastpos - 1$$

$$patlastpos$$

$$pat$$

$$patlen + 1$$

$$pat$$

$$patlastpos$$

$$delta_1$$

$$delta_1$$

$$pat$$

$$pat$$

$$delta_1(pat[patlastpos])$$

$$delta_1$$

$$delta_1$$

$$delta_1$$

$$pat$$

$$pat$$

$$string$$

$$string$$

$$string$$

$$pat$$

$$string$$

$$p$$

$$p$$

$$q$$

$$p = \frac{1}{q}$$

$$q = \frac{1}{256}$$

$$rate(patlen, p)$$

$$cost(m)$$

$$m$$

$$prob(m)$$

$$m$$

$$1 - p^{patlen}$$

$$pat$$

$$skip(m, k)$$

$$pat$$

$$k$$

$$k$$

$$k$$

$$pat$$

$$patlastpos$$

$$skip(m, k)$$

$$delta_1$$

$$delta_2$$

$$skip(m, k)$$

$$skip(m, k)$$

$$delta_1$$

$$delta_1$$

$$pat$$

$$m$$

$$probdelta\_1\_worthless$$

$$delta_1$$

$$k = 1$$

$$pat$$

$$1 < k < patlen - m$$

$$pat$$

$$k = patlen - m$$

$$pat$$

$$pat$$

$$k > patlen - m$$

$$delta_1$$

$$delta_1$$

$$delta_2$$

$$delta_2$$

$$subpat$$

$$delta_2(m, k)$$

$$pr(m, k)$$

$$k$$

$$delta_2$$

$$delta_1$$

$$delta_2$$

$$delta_2$$

$$k$$

$$delta_2$$

$$pat[-(m+1)] = pat[-m] = pat[-(m-1)]...pat[-1]$$

$$pat[-n]$$

$$pat$$

$$n$$

$$delta_1$$

$$delta_1$$

$$pat[-(m+1)]$$

$$pat[-m]...pat[-1]$$

$$m+1$$

$$delta_1$$

$$delta_2$$

$$delta_1$$

$$delta_2$$

$$skip$$

$$O(n)$$

$$\frac{1}{patlen}$$

$$O(\frac{n}{m})$$

$$delta_1$$

$$delta_2$$

$$delta_1$$

$$delta_1$$

$$delta_2$$

$$delta_2$$

$$delta_1$$

$$delta_2$$

$$delta_1$$

$$delta_1$$

$$delta_1$$

$$delta_2$$

$$O(m)$$

$$\Sigma$$

$$\Sigma$$

$$\alpha$$

$$\beta$$

$$\alpha < \beta$$

$$\beta < \alpha$$

$$\Sigma$$

$$S$$

$$n$$

$$n$$

$$S$$

$$\mid S \mid$$

$$1$$

$$S$$

$$i$$

$$S[i]$$

$$0$$

$$S$$

$$i$$

$$S[i-1]$$

$$S$$

$$S[i..j]\text{□}i \leq j$$

$$S$$

$$i$$

$$j$$

$$S[i], S[i+1], ..., S[j]$$

$$S[i..j]$$

$$i > j$$

$$S$$

$$S$$

$$S[p_1], S[p_2], ..., S[p_k]$$

$$1 \leq p_1 < p_2 < \cdots < p_k \leq \mid S \mid$$

$$i$$

$$S$$

$$i$$

$$Suffix(S, i)$$

$$Suffix(S, i) = S[i.. \mid S \mid -1]$$

$$S$$

$$S$$

$$i$$

$$S$$

$$i$$

$$Prefix(S, i)$$

$$Prefix(S, i) = S[0..i]$$

$$S$$

$$S$$

$$i$$

$$i$$

$$a < aa$$

$$\forall 1 \leq i \leq |\ s\ |, s[i] = s[|\ s\ | + 1 - i]$$

$$s$$

$$\delta(i, c) = \begin{cases} i + 1 & s[i+1] = c \\ 0 & s[1] \neq c \wedge i = 0 \\ \delta(\pi(i), c) s[i+1] \neq c \wedge i > 0 \end{cases}$$

$$\Sigma$$

$$Q$$

$$start$$

$$start \in Q$$

$$s$$

$$start$$

$$F$$

$$F \subseteq Q$$

$$\delta$$

$$\delta$$

$$A$$

$$S$$

$$A(S) = \text{True}$$

$$A(S) = \text{False}$$

$$v$$

$$c$$

$$\delta(v, c) = \text{null}$$

$$\text{null}$$

$$\text{null}$$

$$\text{null}$$

$$\text{null}$$

$$\text{null}$$

$$\delta$$

$$\delta(v, s) = \delta(\delta(v, s[1]), s[2.. \mid s \mid])$$

$$A(s) = [\delta(start, s) \in F]$$

$$s$$

$$s$$

$$\mid s \mid$$

$$\text{trans}_{i,c} = \begin{cases} i+1, & \text{if } b_i = c \\ [2ex]\text{trans}_{\text{next}_i,c}, & \text{otherwise} \end{cases}$$

$$s_1, s_2 ... s_n$$

$$Q$$

$$u$$

$$v$$

$$v \in Q$$

$$v$$

$$u$$

$$u$$

$$u$$

$$p$$

$$p$$

$$u$$

$$trie[p, \mathsf{c}] = u$$

$$u$$

$$\text{trie}[\text{fail}[p], \mathsf{c}]$$

$$\text{trie}[\text{fail}[p], \mathsf{c}]$$

$$p$$

$$\text{fail}[p]$$

$$u$$

$$fail[u]$$

$$\text{trie}[\text{fail}[p], \texttt{c}]$$

$$\text{trie}[\text{fail}[\text{fail}[p]], \texttt{c}]$$

$$\text{fail}[u]$$

$$u$$

$$\text{fail}[5] = 10$$

$$\text{fail}[10] = 0$$

$$\text{fail}[6] = 7$$

$$\text{trie}[u, c]$$

$$u$$

$$\text{trans}(u, c)$$

$$u$$

$$\text{fail}[u]$$

$$u$$

$$\text{trans}[u][\texttt{i}]$$

$$\text{trans}[u][\texttt{i}]$$

$$\text{trans}[\text{fail}[u]][\texttt{i}]$$

$$\text{trans}[u][\texttt{i}]$$

$$\text{trans}[\text{fail}[u]][\texttt{i}]$$

$$S$$

$$\text{trans}[S][\texttt{c}]$$

$$S$$

$$S'$$

$$S'$$

$$S'$$

$$\text{trans}[S][\texttt{c}]$$

$$\text{trans}[\text{fail}[S]][\texttt{c}]$$

$$\text{fail}[S]$$

$$S$$

$$\text{trans}[\text{fail}[S]][\texttt{c}]$$

$$S'$$

$$S$$

$$S'$$

$$S'$$

$$S$$

$$S'$$

$$S$$

$$S$$

$$S$$

$$S$$

$$u$$

$$\text{fail}[u]$$

$$\text{trans}[5][s] = 6$$

$$\text{fail}[5] = 10$$

$$\text{fail}[10] = 0$$

$$\text{trie}[0][s] = 7$$

$$\text{fail}[6] = 7$$

$$\text{fail}[5] = 10$$

$$\text{trans}[10][s] = 7$$

$$7$$

$$u$$

$$u$$

$$p$$

$$p$$

$$p$$

$$20$$

$$O(1)$$

$$20$$

$$\mid s_i|$$

$$\mid S \mid$$

$$\mid \Sigma \mid$$

$$O(\sum \mid s_i| + n \mid \Sigma \mid + \mid S \mid)$$

$$n$$

$$O(\sum| \ s_i|)$$

$$O(\sum| \ s_i| + | \ S \ |)$$

$$Q$$

$$\Sigma$$

$$\delta : Q \times \Sigma \to Q$$

$$\delta(q, \sigma) = q',\, q, q' \in Q, \sigma \in \Sigma$$

$$s \in Q$$

$$F \subseteq Q$$

$$(Q, \Sigma, \delta, s, F)$$

$$\text{trans}(u, c)$$

$$\text{trans}[u][c]$$

$$m$$

$$i$$

$$i$$

$$\text{trans}_{i,c}$$

$$i$$

$$c$$

$$\text{next}_i$$

$$\text{next}_0 = 0$$

$$\text{trans}_i$$

$$O(m \mid \Sigma \mid)$$

$$n$$

$$n$$

$$i$$

$$k$$

$$t_{i,k}$$

$$1 \leq i, k \leq n$$

$$n$$

$$n + 1$$

$$n$$

$$n + 1$$

$$n$$

$$n \times n$$

$$i$$

$$k$$

$$t_{i,k}$$

$$i$$

$$k$$

$$1 \leq n \leq 15$$

$$1 \leq t_{i,k} \leq 10^4$$

$$i$$

$$j$$

$$i$$

$$i$$

$$j$$

$$n$$

$$d$$

$$d$$

$$d$$

$$limit$$

$$\frac{1}{a}$$

$$a \in \mathbb{N}^*$$

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{6}$$

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3}$$

$$\frac{a}{b}$$

$$\frac{19}{45} = \frac{1}{5} + \frac{1}{6} + \frac{1}{18}$$

$$a, b$$

$$0 < a < b < 500$$

$$maxd$$

$$maxd$$

$$maxd$$

$$i$$

$$\frac{c}{d}$$

$$i$$

$$\frac{1}{e}$$

$$\frac{\frac{a}{b} - \frac{c}{d}}{\frac{1}{e}}$$

$$\frac{a}{b}$$

$$\frac{19}{45} = \frac{1}{5} + \frac{1}{100} + \cdots$$

$$\frac{1}{101}$$

$$\frac{\frac{19}{45} - \frac{1}{5}}{\frac{1}{101}} = 23$$

$$\frac{19}{45}$$

$$22$$

$$maxd$$

$$n$$

$$g(n)$$

$$h(n)$$

$$g(n) + h(n) > maxd$$

$$g(n)$$

$$h(n)$$

$$N$$

$$W$$

$$w_i$$

$$v_i$$

$$f$$

$$0$$

$$f$$

$S_1 = \{5, 9, 17\}$

$S_2 = \{1, 8, 119\}$

$S_3 = \{3, 5, 17\}$

$S_4 = \{1, 8\}$

$S_5 = \{3, 119\}$

$S_6 = \{8, 9, 119\}$

$X = \{1, 3, 5, 8, 9, 17, 119\}$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} \\ \color{blue}{1} & \color{red}{0} & \color{blue}{1} & \color{blue}{0} \\ \color{blue}{0} & \color{red}{1} & \color{blue}{0} & \color{blue}{1} \end{pmatrix}$$

$$\begin{pmatrix} \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} \\ \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{0} \\ \color{blue}{0} & \color{blue}{1} & \color{blue}{0} & \color{blue}{1} \end{pmatrix}$$

$$()$$

$$\begin{pmatrix} \color{red}{1} & 0 & \color{red}{1} & 1 \\ \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{0} \\ \color{red}{0} & 1 & \color{red}{0} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \color{blue}{1} & \color{red}{0} & \color{blue}{1} & \color{red}{1} \\ \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{0} \\ \color{blue}{0} & 1 & \color{blue}{0} & 1 \end{pmatrix}$$

$$\begin{pmatrix} \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{1} \\ \color{blue}{1} & \color{blue}{0} & \color{blue}{1} & \color{blue}{0} \\ \color{blue}{0} & 1 & \color{blue}{0} & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$()$$

$$S_i(1 \leq i \leq n)$$

$$X$$

$$(T_1, T_2, \cdots, T_m)$$

$$\forall i, j \in [1, m], T_i \bigcap T_j = \emptyset (i \neq j)$$

$$X = \bigcup_{i=1}^{m} T_i$$

$$\forall i \in [1, m], T_i \in \{S_1, S_2, \cdots, S_n\}$$

$$(S_1, S_4, S_5)$$

$$\bigcup_{i=1}^{n} S_i$$

$$i$$

$$S_i$$

$$[1 \in S_i], [3 \in S_i], [5 \in S_i], \cdots, [119 \in S_i]$$

$$O(2^n)$$

$$O(nm)$$

$$O(nm \cdot 2^n)$$

$$m$$

n

m

$2^m - 1$

$O(2^n)$

$O(n)$

$O(n \cdot 2^n)$

3

1

3

1

3

1

2

1

2

1

3

1

1

1

1

1

3

1

2

1

2

1

1

1

1

1

1

2

1

1

$1, 4, 5$

M

r

r

S

r

r

r_i

c_i

M'

M'

r

1

S

M'

r

1

r

r_i

c_i

M'

i

i

$c + 1$

0

c

c

c

c

c

c

j

j

j

j

j

c

$r \times c$

r

c

$c + 1$

i

$i - 1$

$i + 1$

i

i

0

c

c

0

r

c

r

r

c

$first(r)$

idx

idx

c

idx

c

idx

c

idx

idx

c

c

idx

idx

idx

$first(r)$

idx

$first(r)$

$first(r)$

idx

idx

$first(r)$

$first(r)$

idx

0

1

1

r, c

$O(c^n)$

c

1

n

1

i

P_i

(r, c, w)

(r, c)

$9 \times 9 \times 9 = 729$

(r, c, w)

(r, c)

b

r

w

$9 \times 9 = 81$

c

w

$$9 \times 9 = 81$$

$$b$$

$$w$$

$$9 \times 9 = 81$$

$$(r, c)$$

$$9 \times 9 = 81$$

$$81 \times 4 = 324$$

$$729 \times 4 = 2916$$

$$1$$

$$9 \times 9$$

$$729$$

$$324$$

$$2916$$

$$1$$

$$d$$

$$90^\circ$$

$$f$$

$$(v, d, f, i)$$

$$i$$

$$v$$

$$d$$

$$90^\circ$$

$$f = 1$$

$$f = -1$$

$$55 \times 4 \times 2 \times 12 = 5280$$

$$(1, 0, 1, 4)$$

$$2730$$

$$(v, d, f, i)$$

$$55$$

$$i$$

$$12$$

$$55 + 12 = 67$$

$$5280 \times (5 + 1) = 31680$$

1

5280

67

31680

1

n

3

$6 = 1 + 2 + 3$

m

n

m

n

k

$a_1, a_2, ..., a_k$

i

a_i

i

$$n - \sum_{j=1}^{i} a_j$$

a_{i-1}

i

m

a_i

i

$O(a^b)$

$O(a^{b/2})$

n

$1 \leq n \leq 35$

$O(2^n)$

$O(n2^{n/2})$

1

mid

6×6

$$\{2, 4, 6, 1, 3, 5\}$$

$$i$$

$$i$$

$$a_i$$

$$i$$

$$\{1, 2, 3, 4, 5, 6\}$$

$$a_i$$

$$\{2, 4, 6, 1, 3, 5\}$$

$$3$$

$$N \times M$$

$$T$$

$$s$$

$$t$$

$$g(x)$$

$$h(x)$$

$$h^*(x)$$

$$f(x) = g(x) + h(x)$$

$$f$$

$$h \leq h^*$$

$$h$$

$$h = 0$$

$$h = 0$$

$$1$$

$$3 \times 3$$

$$1$$

$$8$$

$$0$$

$$h$$

$$h$$

$$s$$

$$t$$

$$k$$

$$s$$

$$t$$
$$s$$
$$t$$
$$t$$
$$t$$
$$f(x) = g(x) + h(x)$$
$$k$$
$$t$$
$$s$$
$$t$$
$$k$$
$$x$$
$$v$$
$$k$$
$$k + 1$$
$$k$$
$$k$$
$$\alpha - \beta$$
$$\alpha - \beta$$
$$v$$
$$\alpha \leq v \leq \beta$$
$$\alpha$$
$$\beta$$
$$\alpha \geq \beta$$
$$\alpha = \beta$$
$$\alpha = -\infty, \beta = +\infty$$
$$-\infty \leq v \leq +\infty$$
$$\beta$$
$$\beta$$
$$\beta = +\infty$$
$$\beta$$
$$\beta$$
$$\beta = 3$$

$$\alpha$$
$$\alpha$$
$$\alpha = -\infty$$
$$\alpha$$
$$\beta$$
$$\beta$$
$$\alpha \geq \beta$$
$$\beta$$
$$\beta$$
$$\alpha$$
$$\beta$$
$$\beta$$
$$\beta$$
$$\alpha$$
$$\alpha \geq \beta$$
$$\alpha$$
$$\alpha$$
$$\alpha$$
$$\beta$$
$$\beta$$
$$\alpha \geq \beta$$
$$\alpha$$
$$\alpha$$
$$\alpha$$
$$\alpha$$
$$\beta$$
$$\beta = 3$$
$$\alpha \geq \beta$$
$$2k = n \times C + l + (k - l)$$
$$k = n \times C$$
$$n$$
$$nums$$
$$k$$
$$k$$

$$1 \leq n \leq 3 \times 10^4, 1 \leq nums[i] \leq 1000, 0 \leq k \leq 10^6$$

$$l, r$$

$$tmp$$

$$[l, r]$$

$$l, r$$

$$r$$

$$tmp \geq k$$

$$r$$

$$l$$

$$tmp < k$$

$$r$$

$$l$$

$$r$$

$$r - l + 1$$

$$n$$

$$n$$

$$t$$

$$u$$

$$u$$

$$u$$

$$t$$

$$1 \leq n \leq 2 \times 10^5, 1 \leq t \leq 10^{18}, 1 \leq p_i < i, 1 \leq w_i \leq 10^{12}$$

$$u$$

$$p$$

$$u$$

$$t$$

$$p$$

$$u$$

$$p$$

$$u$$

$$p$$

$$O(n^2)$$

$$s$$

$$dictionary$$

$$s$$
$$s$$
$$t$$
$$t$$
$$s$$
$$i$$
$$s$$
$$j$$
$$t$$
$$s[i] = t[j]$$
$$t$$
$$j$$
$$s$$
$$i$$
$$j$$
$$t$$
$$j$$
$$i$$
$$s$$
$$j$$
$$t$$
$$s$$
$$\log n$$
$$num[1] + num[n] > target$$
$$num[1]$$
$$n$$
$$num[1] + num[n] < target$$
$$num[n]$$
$$l, r$$
$$l < r$$
$$num[l] + num[r] > target$$
$$r$$
$$num[l] + num[r] < target$$

$$l$$

$$l$$

$$r$$

$$O(n+m)$$

$$O(n)$$

$$k$$

$$l$$

$$2k$$

$$C$$

$$n$$

$$k=C$$

$$P(\Delta E)=\begin{cases}1, & S' \text{ is better than } S,\\ \mathrm{e}^{\frac{-\Delta E}{T}}, & \text{otherwise.}\end{cases}$$

$$T$$

$$S'$$

$$S$$

$$\Delta E$$

$$\Delta E \geqslant 0$$

$$T_0$$

$$d$$

$$T_k$$

$$T_0$$

$$d$$

$$1$$

$$1$$

$$T_k$$

$$0$$

$$T=T_0$$

$$T=d\cdot T$$

$$T<T_k$$

$$n$$

$$O(n\sqrt{m})$$

$$n$$

$$A$$
$$m$$
$$(1 \leq n, m \leq 10^5)$$
$$[L, R]$$
$$i$$
$$i$$
$$i$$
$$B$$
$$B$$
$$1$$
$$B$$
$$B$$
$$1$$
$$b$$
$$O(b)$$
$$O(n)$$
$$b$$
$$\frac{n}{b}$$
$$O(mb + \frac{n^2}{b})$$
$$b = \frac{n}{\sqrt{m}}$$
$$O(n\sqrt{m})$$
$$s_i \equiv s_{i-1} \times A + C \bmod M$$
$$R_{i+1} = (A \times R_i + B) \bmod P$$
$$R_i \equiv R_{i-j} \star R_{i-k} \bmod P$$
$$2^{31} - 1$$
$$[0, 2^{15})$$
$$2^{15} - 1$$
$$2^{15}$$
$$2 \cdot 10^6$$
$$A, C, M$$

$$s$$

$$A$$

$$C$$

$$M$$

$$s$$

$$A$$

$$C$$

$$M$$

$$\{R_i\}$$

$$A, B, P$$

$$P$$

$$P$$

$$\{R_i\}$$

$$0 < j < k$$

$$P$$

$$2$$

$$2^{32}$$

$$2^{64}$$

$$\star$$

$$\begin{aligned} &\frac{(n-1)(m-1)}{nm} \cdot \frac{(n-2)(m-2)}{(n-1)(m-1)} \cdots \frac{(n-k)(m-k)}{(n-k+1)(m-k+1)} \\ &= \frac{(n-k)(m-k)}{nm} \\ &\geq \frac{1}{4} \end{aligned}$$

$$S(N)_{i,j} - {F'}_1 S(N-1)_{i,j} - \cdots - {F'}_l S(N-l)_{i,j} \not\equiv 0 \pmod{P}$$

$$T_{i,j} := \Big(x_{i,j} \cdot \big(S(N)_{i,j} - {F'}_1 S(N-1)_{i,j} - \cdots - {F'}_l S(N-l)_{i,j} \big) \bmod P \Big) = 0$$

$$\Pr[A \equiv B \pmod{Q}] = O\Big(\frac{\log N \log Q_{max}}{Q_{max}} \Big)$$

$$\Pr[A \equiv B \pmod{Q}] = O\Big(\frac{L \log(m_1 + m_2) \log Q_{max}}{Q_{max}} \Big)$$

$$
\begin{aligned}
h(A) &= 1 + \frac{h(L) + h(R)}{2} \\
&\le 1 + \frac{\log_2(\mid L \mid + 1) + \log_2(\mid R \mid + 1)}{2} \\
&= \log_2 2\sqrt{(\mid L \mid + 1)(\mid R \mid + 1)} \\
&\le \log_2 \frac{2\big((\mid L \mid + 1) + (\mid R \mid + 1)\big)}{2} \\
&= \log_2(\mid A \mid + 1)
\end{aligned}
$$

$$\Pr[A]$$

$$A$$

$$\mathrm{E}[X]$$

$$X$$

$$:=$$

$$Y := 1926$$

$$1926$$

$$Y$$

$$n$$

$$m$$

$$v$$

$$\{R, G, B\}$$

$$C_v$$

$$v$$

$$C_v$$

$$(\frac{2}{3})^n$$

$$(u, v)$$

$$u, v$$

$$C_u, C_v$$

$$X$$

$$u$$

$$X$$

$$v$$

$$\{R, G, B\} \setminus \{X, C_v\}$$

$$v$$

$$B_v$$

$$v$$

$$O(n+m)$$

$$(\frac{2}{3})^n$$

$$-(\frac{3}{2})^n \log \epsilon$$

$$1-\epsilon$$

$$\{R, G, B\}^n$$

$$\leq K$$

$$S$$

$$S$$

$$S$$

$$n$$

$$\frac{2}{2^n} = 2^{1-n}$$

$$a_1, \cdots, a_l$$

$$(a_1 - 1) + \cdots + (a_l - 1) \leq K$$

$$K$$

$$2^{1-a_1} \cdots 2^{1-a_l} \geq 2^{-K}$$

$$S$$

$$T$$

$$v$$

$$S$$

$$-A_v$$

$$\infty$$

$$v$$

$$T$$

$$-A_v$$

$$\infty$$

$$(u, v, B)$$

$$u$$

$$v$$

$$u$$

$$v$$

$$B$$

$$K$$

$$O\big(K^2(n+m)\big)$$

$$\leq K$$

$$O(K)$$

$$3K$$

$$K$$

$$L$$

$$\frac{L}{2}-2$$

$$6K+4$$

$$O\big(K^2(n+m)\big)$$

$$-2^K \log \epsilon$$

$$1-\epsilon$$

$$O\big(2^K K^2(n+m)\cdot -\log \epsilon\big)$$

$$C$$

$$C$$

$$C$$

$$C$$

$$S$$

$$S$$

$$e$$

$$e$$

$$e$$

$$O(1)$$

$$O(\mid S \mid)$$

$$e'$$

$$e' \in S$$

$$S'$$

$$e'$$

$$\mid S'\mid \leq \mid S \mid + 1$$

$$O(\mid S'\mid) = O(\mid S \mid)$$

$$\frac{| S |}{n}$$

$$n$$

$$-\frac{n}{| S |} \log \epsilon$$

$$1 - \epsilon$$

$$O\big(| S | \cdot -\frac{n}{| S |} \log \epsilon\big) = O(-n \log \epsilon)$$

$$t$$

$$s$$

$$t \to s$$

$$s$$

$$t$$

$$t$$

$$s$$

$$s$$

$$t$$

$$(s, t)$$

$$t \to s$$

$$(s', t')$$

$$s - t$$

$$(s, t)$$

$$n + m - 1$$

$$n$$

$$m$$

$$(s, t)$$

$$s, t$$

$$(s, t)$$

$$\geq \frac{(n-1)(m-1)}{nm}$$

$$\frac{\min(n, m)}{2}$$

$$(s, t)$$

$$t \to s$$

$$\geq \frac{1}{4}$$

$$k$$

$$(s,t)$$

$$n,m$$

$$k$$

$$k = \frac{\min(n,m)}{2}$$

$$\min(n,m)$$

$$O\big(\log(n+m)\big)$$

$$O\big((\mid V \mid + \mid E \mid)\log\mid V \mid\big)$$

$$(s,t)$$

$$(s,t)$$

$$n$$

$$\frac{2n}{3}$$

$$n \leq 50$$

$$V_L, V_R$$

$$\frac{n}{2}$$

$$f_{L,k}$$

$$L \subseteq V_L$$

$$\geq k$$

$$C_R$$

$$C_R$$

$$N_L$$

$$f_{N_L, \frac{2}{3}n - \mid C_R \mid} + value(C_R)$$

$$O(1)$$

$$f_{L,k}$$

$$d$$

$$L$$

$$d$$

$$f_{L\setminus\{d\},k}$$

$$d$$

$$f_{L\cap N(d),k} + value(d)$$

$$N(d)$$

$$d$$

$$f_{L,k+1}$$

$$f_{L,k}$$

$$C_{res}$$

$$|\ C_{res} \cap V_L| = |\ C_{res} \cap V_R|$$

$$|\ C_{res}|$$

$$C_{res}$$

$$L$$

$$C_R$$

$$\geq \frac{n}{3}$$

$$f$$

$$\geq \frac{n}{3}$$

$$L$$

$$C_R$$

$$1.8 \cdot 10^6$$

$$O\left(2^{|\ V_L|} + 2^{|\ V_R|}\right)$$

$$V_L, V_R$$

$$e$$

$$(s, t)$$

$$e$$

$$s, t$$

$$e$$

$$S_e$$

$$(a, b)$$

$$S_e$$

$$S_e$$

$$e$$

$$(a, b)$$

$$2^{64}$$

$$H_{(a,b)}$$

$$H$$

$$R := \{0, 1, \cdots, 2^{64} - 1\}$$

$$R$$

$$H_{(a,b)}$$

$$R$$

$$R$$

$$S(N)$$

$$N$$

$$O(K)$$

$$S(N)$$

$$P := 998244353$$

$$F$$

$$F$$

$$F'$$

$$F'$$

$$F$$

$$(i, j)$$

$$< P$$

$$x_{i,j}$$

$$P$$

$$F$$

$$F$$

$$l$$

$$F'$$

$$F'$$

$$(i, j)$$

$$S_{i,j}$$

$$N$$

$$F'$$

$$(i, j)$$

$$x_{i,j} = 0$$

$$P^{-1}$$

$$(i, j)$$

$$T_{i,j}$$

$$R := \{0, 1, \cdots, P-1\}$$

$$R$$

$$R$$

$$P^{-1}$$

$$G_0, G_1$$

$$f_{K,i,j}$$

$$G_K$$

$$i$$

$$j$$

$$j$$

$$i$$

$$= L$$

$$\{f_{0,*,L}\}$$

$$\{f_{1,*,L}\}$$

$$\{f_{0,*,L}\}$$

$$f_{0,i,L}$$

$$i$$

$$n_1 + n_2$$

$$a_1 a_2 \cdots a_k$$

$$\left(a_1 + P a_2 + P^2 a_3 + \cdots + P^{k-1} a_k\right) \bmod Q$$

$$Q$$

$$P, Q$$

$$(c, j)$$

$$c$$

$$j$$

$$c$$

$$x_{c,j}$$

$$a_1 a_2 \cdots a_k$$

$$x_{a_1,1} x_{a_2,2} \cdots x_{a_k,k} \bmod Q$$

$$Q$$

$$Q$$

$$f \in F[z_1, \cdots, z_k]$$

$$F$$

$$k$$

$$d$$

$$S$$

$$F$$

$$d \cdot \mid S \mid^{k-1}$$

$$(z_1, \cdots, z_k) \in S^k$$

$$f(z_1, \cdots, z_k) = 0$$

$$z_1, \cdots, z_k$$

$$S$$

$$\Pr[f(z_1, \cdots, z_k) = 0] \leq \frac{d}{\mid S \mid}$$

$$F$$

$$Q$$

$$L \leq n_1 + n_2$$

$$\sum_i f_{0,i,L}$$

$$\sum_i f_{1,i,L}$$

$$F$$

$$\{x_{*,*}\}$$

$$L$$

$$P_0, P_1$$

$$P_0 \equiv P_1 \pmod{Q}$$

$$P_0, P_1$$

$$Q$$

$$P_0 \not\equiv P_1 \pmod{Q}, P_0(x_{*,*}) \equiv P_1(x_{*,*}) \pmod{Q}$$

$$P_0, P_1$$

$$\{x_{*,*}\}$$

$$A \neq B; A, B \leq N$$

$$Q \leq Q_{\max}$$

$$A \equiv B$$

$$Q$$

$$Q \mid (A - B)$$

$$Q$$

$$\omega(A - B) \leq \log_2 N$$

$$Q_{\max}$$

$$\Theta\left(\frac{Q_{\max}}{\log Q_{\max}}\right)$$

$$A, B$$

$$A \neq B$$

$$P_0, P_1$$

$$G_0, G_1$$

$$A, B \leq (m_1 + m_2)^L$$

$$Q_{\max} \approx 10^{12}$$

$$F$$

$$Q$$

$$f(x_{*,*}) = P_0(x_{*,*}) - P_1(x_{*,*})$$

$$L$$

$$S = F$$

$$\leq \frac{L}{Q}$$

$$L$$

$$n \times m$$

$$q$$

$$\epsilon$$

$$\delta$$

$$(1-\delta)q$$

$$\epsilon$$

$$n \cdot m \leq 2 \cdot 10^5; q \leq 10^6; \epsilon = 0.5, \delta = 0.2$$

$$X_{1 \cdots k}$$

$$[0,1]$$

$$\mathrm{E}[\min_i X_i] = \frac{1}{k+1}$$

$$k+1$$

$$0, X_1, X_2, \cdots, X_k$$

$$\min_i X_i$$

$$k+1$$

$$\frac{1}{k+1}$$

$$k$$

$$\min_i X_i$$

$$k$$

$$[0,1]$$

$$\{X_1, \cdots, X_k\}$$

$$k$$

$$[0,1]$$

$$M$$

$$\frac{1}{M} - 1$$

$$\mathrm{E}\Big[\frac{1}{\min_i X_i} - 1\Big]$$

$$\frac{1}{\mathrm{E}[\min_i X_i]} - 1 = k$$

$$\infty$$

$$\min_i X_i$$

$$C$$

$$C$$
$$M$$
$$\overline{M}$$
$$(\overline{M})^{-1} - 1$$
$$C \approx 80$$
$$O(nm \log n \log m)$$
$$O(1)$$
$$A$$
$$h(A) \leq \log_2(\mid A \mid + 1)$$
$$A, B$$
$$O\big(h(A) + h(B)\big)$$
$$A$$
$$A$$
$$A$$
$$L, R$$
$$\left(1 - \frac{1}{n}\right)^n \leq \frac{1}{\mathrm{e}}, \forall n \geq 1$$
$$n \geq 1$$
$$+\infty$$
$$\frac{1}{\mathrm{e}}$$
$$n$$
$$1 - \frac{1}{n}$$
$$\frac{1}{\mathrm{e}}$$
$$n \in \mathbf{N}^*; p, q \in [0, 1]$$
$$q \leq p$$
$$G_1(n, p)$$
$$G_2(n, q)$$
$$G(n, \alpha)$$
$$n$$
$$G$$

$$\frac{n(n-1)}{2}$$

α

G_1

T_1

p

G_2

T_2

q

$$\frac{n(n-1)}{2}$$

T

q

$p-q$

$1-p$

T

$$\frac{n(n-1)}{2}$$

G_1

G_2

G_1

G_2

G_1

G_1

G_2

G_1

G_2

G_2

G_1

G_2

G_1

c_i

$$i$$
$$c_i$$
$$x$$
$$\frac{x}{2}$$
$$x$$
$$f_k$$
$$k$$
$$f_k$$
$$f_k$$
$$f_k = \frac{x}{2} \cdot (R-1) + x$$
$$R$$
$$p$$
$$\frac{1}{p}$$
$$\frac{1}{p} \cdot p = 1$$
$$n \to \infty$$
$$\frac{n-k}{n}$$
$$R = \frac{n}{n-k}$$
$$f_k$$
$$k$$
$$A$$
$$x$$
$$B$$
$$x$$
$$x$$
$$A$$
$$B$$
$$A$$
$$B$$
$$A$$
$$B$$

A

$y \neq x$

A

y

B

y

B

A

A

B

A

B

A

$[l, r]$

$[L, R]$

$[L, R]$

$[l, r]$

mid

mid

mid

mid

$q1$

$q2$

$l = r$

$[l, r]$

$[l, r]$

k

k

$[l, r]$

l

r

$$m = \left\lfloor \frac{l+r}{2} \right\rfloor$$

$$m$$

$$O(\log n)$$

$$q$$

$$O(q \log n)$$

$$mid$$

$$mid$$

$$mid$$

$$mid$$

$$k$$

$$mid$$

$$t$$

$$k-t$$

$$l=r$$

$$[1, maxans]$$

$$O(\log maxans)$$

$$O(T)$$

$$O(T \log n)$$

$$k$$

$$m$$

$$m$$

$$t$$

$$k$$

$$k \le t$$

$$m$$

$$m$$

$$[l, r]$$

$$k$$

$$q, a$$

$$L, R$$

$$-1$$

$$k$$

$$k$$
$$O(n\log^2 n)$$
$$k$$
$$O(n\log n)$$
$$k$$
$$O(n\log^2 n)$$
$$mid$$
$$t$$
$$k-t$$
$$[L,R]$$
$$[l,r]$$
$$[L,l)$$
$$[l,r]$$
$$pos$$
$$mid$$
$$\le pos$$
$$1$$
$$0$$
$$pos$$
$$pos$$
$$n\log n$$
$$pos$$
$$k$$
$$k$$
$$O(n\log n)$$
$$k$$
$$k$$
$$k$$
$$k$$
$$x$$
$$x$$
$$k$$
$$x$$

$$x+1$$

$$k$$

$$+1$$

$$-1$$

$$[l,r]$$

$$[ql,qr]$$

$$mid=\left\lfloor\frac{ql+qr}{2}\right\rfloor$$

$$[mid+1,qr]$$

$$[ql,mid]$$

$$0$$

$$[l,r]$$

$$mid+1$$

$$[l,r]$$

$$i$$

$$[l,i]$$

$$mid$$

$$[i+1,r]$$

$$mid+1$$

$$[ql,mid]$$

$$mid$$

$$mid+1$$

$$O(n\log\log n)$$

$$O(n\log n)$$

$$x$$

$$[l,r]$$

$$[l,x)$$

$$[x,r]$$

$$[l,r]$$

$$[split(l),split(r+1))$$

$$1$$

$$\sum_{i=1}^{n/s}\sum_{j=i+1}^{n/s}(q_{i,j}\cdot s+t)$$

$$= ms + \left(\frac{n}{s}\right)^2 t$$

$$= ms + \frac{n^2 t}{s^2}$$

$$[l, r]$$

$$[l, r, \text{time}]$$

$$[l, r, \text{time}]$$

$$[l-1, r, \text{time}]$$

$$[l+1, r, \text{time}]$$

$$[l, r-1, \text{time}]$$

$$[l, r+1, \text{time}]$$

$$[l, r, \text{time}-1]$$

$$[l, r, \text{time}+1]$$

$$O(1)$$

$$O(n^{5/3})$$

$$n^{2/3}$$

$$n^{1/3}$$

$$n$$

$$m$$

$$t$$

$$n$$

$$t$$

$$s$$

$$\frac{n}{s}$$

$$i$$

$$j$$

$$q_{i,j}$$

$$i$$

$$j$$

$$(i, j)$$

$$O(s)$$

$$O(t)$$

$$f(s) = ms + \frac{n^2 t}{s^2}$$

$$f'(s) = m - \frac{2n^2 t}{s^3} = 0$$

$$s = \sqrt[3]{\frac{2n^2 t}{m}} = \frac{2^{1/3} n^{2/3} t^{1/3}}{m^{1/3}} = s_0$$

$$\frac{n^{2/3} t^{1/3}}{m^{1/3}}$$

$$O\left(n^{2/3} m^{2/3} t^{1/3}\right)$$

$$O\left(n^{5/3}\right)$$

$$n, m, t$$

$$n^{2/3}$$

$$[l, r]$$

$$0$$

$$+1$$

$$+1$$

$$0$$

$$-1$$

$$-1$$

$$i$$

$$j$$

$$i < j$$

$$i + 1$$

$$j$$

$$j < i$$

$$i$$

$$j + 1$$

$$pos$$

$$pos$$

$$a$$

$$b$$

$$[l,r]$$

$$pos$$

$$[l,r]$$

$$a$$

$$b$$

$$pos$$

$$b$$

$$[l,r]$$

$$pos$$

$$b$$

$$pos$$

$$b$$

$$a$$

$$\begin{aligned}
&O(\sqrt{n}(\max{}_1-1)+\sqrt{n}(\max{}_2-\max{}_1)+\sqrt{n}(\max{}_3-\max{}_2)+\cdots+\sqrt{n}(\max{}_{\lceil\sqrt{n}\rceil}-\max{}_{\lceil\sqrt{n}\rceil-1)})\\
&=O(\sqrt{n}\cdot(\max{}_1-1+\max{}_2-\max{}_1+\max{}_3-\max{}_2+\cdots+\max{}_{\lceil\sqrt{n}\rceil-1}-\max{}_{\lceil\sqrt{n}\rceil-2}+\max{}_{\lceil\sqrt{n}\rceil}-\max{}_{\lceil\sqrt{n}\rceil-1)})\\
&=O(\sqrt{n}\cdot(\max{}_{\lceil\sqrt{n}\rceil-1}))
\end{aligned}$$

$$n=m$$

$$[l,r]$$

$$O(1)$$

$$[l-1,r],[l+1,r],[l,r+1],[l,r-1]$$

$$[l,r]$$

$$O(n\sqrt{n})$$

$$[l,r]$$

$$l$$

$$r$$

$$n$$

$$m$$

$$O(\sqrt{n}\cdot\sqrt{n}\log\sqrt{n}+n\log n)=O(n\log n)$$

$$O(n\sqrt{n})$$

$$L$$

$$\max_1,\max_2,\max_3,\cdots,\max_{\lceil\sqrt{n}\rceil}$$

$$\max_1 \leq \max_2 \leq \cdots \leq \max_{\lceil\sqrt{n}\rceil}$$

$$O(n)$$

$$R$$

$$n$$

$$L$$

$$\max_{i-1}$$

$$\max_i$$

$$\max_i$$

$$\max_{i-1}$$

$$R$$

$$R$$

$$O(n)$$

$$O(n\sqrt{n})$$

$$L$$

$$O(\max_i - \max_{i-1})$$

$$O(\sqrt{n} \cdot (\max_i - \max_{i-1}))$$

$$L$$

$$L$$

$$\max_{\lceil\sqrt{n}\rceil}$$

$$n$$

$$L$$

$$O(n\sqrt{n})$$

$$O(n\sqrt{n})$$

$$m$$

$$m < n$$

$$S$$

$$n$$

$$\frac{n}{S}$$

$$\frac{n^2}{S}$$

$$mS$$

$$O\left(\frac{n^2}{S}+mS\right)$$

$$S$$

$$\frac{n}{\sqrt{m}}$$

$$O\left(\frac{n^2}{\frac{n}{\sqrt{m}}}+m\left(\frac{n}{\sqrt{m}}\right)\right)=O(n\sqrt{m})$$

$$m$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$O(n\sqrt{n})$$

$$O(n)$$

$$[l,r]$$

$$(l,r)$$

$$n,m$$

$$n$$

$$[a\sqrt{n},b\sqrt{n}](1\leq a,b\leq\sqrt{n})$$

$$n$$

$$\sqrt{n}$$

$$O(n\sqrt{n})$$

$$n$$

$$m$$

$$m$$

$$n$$

$$\{c_i\}$$

$$m$$

$$l,r$$

$$l$$

$$r$$

$$[l,r]$$

$$l$$

$$r$$

$$O(n)$$

$$[i, i]$$

$$col[i]$$

$$i$$

$$ans$$

$$k$$

$$ans$$

$$\binom{col[k]+1}{2} - \binom{col[k]}{2}$$

$$ans$$

$$\binom{col[k]}{2} - \binom{col[k]-1}{2}$$

$$\frac{ans}{\binom{r-l+1}{2}}$$

$$\binom{a}{2} = \frac{a(a-1)}{2}$$

$$\binom{a+1}{2} - \binom{a}{2} = \frac{(a+1)a}{2} - \frac{a(a-1)}{2} = \frac{a}{2} \cdot (a+1-a+1) = \frac{a}{2} \cdot 2 = a$$

$$\binom{col[k]+1}{2} - \binom{col[k]}{2} = col[k]$$

$$O(n\sqrt{n})$$

$$\sqrt{n}$$

$$[1, r]$$

$$[1, l-1]$$

$$l \leq r$$

$$[1, l-1]$$

$$[l, r]$$

$$[r+1, +\infty)$$

$$l = r+1$$

$$[1, r]$$

$$[r+1, +\infty)$$

$$l > r+1$$

$$[r+1, l-1]$$

$$l > r+1$$

$$4! = 24$$

$$12$$

$$12$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l < r < l' < r'$$

$$l \leq r+1$$

$$l \leq l' \leq r' \leq r$$

$$[l', r']$$

$$l \leq r+1$$

$$l = 2, r = 100$$

$$a < b$$

$$b < a$$

$$\times 3$$

$$10^5 \times 10^5$$

$$n$$

$$a$$

$$m$$

$$1 \leq n, m \leq 10^5$$

$$0 \leq a_i \leq 10^9$$

$$O(n\sqrt{n}\log n)$$

$$T = \sqrt{n \log n}$$

$$O(\log n)$$

$$O(1)$$

$$O(n\sqrt{n}+n\log n)$$
$$n$$
$$a$$
$$m$$
$$[l,r]$$
$$a_i$$
$$[l,r]$$
$$k$$
$$k\times a_i$$
$$1\le a_i\le 100000$$
$$1\le l\le r\le n$$
$$1\le n,m\le 500000$$
$$i$$
$$c_i$$
$$u_i$$
$$v_i$$
$$\sum_c val_c\sum_{i=1}^{cnt_c} w_i$$
$$val$$
$$cnt$$
$$w$$
$$i$$
$$O(1)$$
$$val_c\times w_{cnt_{c+1}}$$
$$vis$$
$$vis_x$$
$$vis_x=0$$
$$unit$$
$$unit$$
$$unit^2$$
$$unit\times dis_{\max}$$
$$n\times unit$$

$$unit^2 \times (\frac{n}{unit})$$

$$O(n)$$

$$\frac{n^2}{unit}$$

$$n \times (\frac{n}{unit})$$

$$n^{0.6}$$

$$q$$

$$(x_1, y_1)$$

$$(x_2, y_2)$$

$$B$$

$$x_1$$

$$\Theta(q \cdot B)$$

$$y_2$$

$$\Theta(n^4 \cdot B^{-3})$$

$$q \cdot B = n^4 \cdot B^{-3}$$

$$B = n \cdot q^{-\frac{1}{4}}$$

$$B$$

$$0$$

$$\Theta(n^2 \cdot q^{\frac{3}{4}})$$

$$\Theta(n^2 \cdot q^{\frac{3}{4}} + q \log q)$$

$$n \times m$$

$$q$$

$$x$$

$$p$$

$$p^2$$

$$1 \leq n, m \leq 200$$

$$0 \leq q \leq 10^5$$

$$|$$

$$| \leq 2 \times 10^9$$

$$n \times n$$

$$q$$

$$k$$

$$1 \leq n \leq 500$$

$$1 \leq q \leq 6 \times 10^4$$

$$0 \leq a_{i,j} \leq 10^9$$

$$n, q$$

$$11$$

$$n$$

$$a$$

$$\frac{n}{2}$$

$$+1$$

$$O(n+m)$$

$$\frac{n}{2}$$

$$O(n \log n)$$

$$O(n)$$

$$\frac{n}{2}$$

$$O(1)$$

$$c$$

$$\frac{1}{3}$$

$$a + b = b + a$$

$$(a+b) + c \neq a + (b+c)$$

$$S_{new} = S_{old} + a$$

$$a$$

$$S$$

$$a_{eff} = S_{new} - S_{old}$$

$$a_{eff}$$

$$a$$

$$a_{eff}$$

$$a$$

$$E_{roundoff} = a_{eff} - a$$

$$a - a_{eff}$$

$$E_{roundoff}$$

$$c$$

$$c_{new} = c_{old} + (a - a_{eff})$$

$$sum$$

$$c$$

$$sum$$

$$y$$

$$y$$

$$(t - sum)$$

$$y$$

$$y$$

$$y$$

$$c$$

$$y$$

$$n$$

$$i$$

$$a_i$$

$$\lambda$$

$$\lambda \times a_i$$

$$i$$

$$b_i$$

$$p$$

$$\lambda$$

$$\lambda \times p$$

$$0$$

$$0, 1, \cdots, (n-1)$$

$$p$$

$$q$$

$$Perm((Ord(p) + Ord(q)) \bmod n!)$$

$$Perm(x)$$

$$0, 1, \cdots, (n-1)$$

$$x$$

$$Ord(p)$$

$$p$$

$$Perm(0) = (0, 1, \cdots, n-2, n-1)$$

$$Perm(n!-1) = (n-1, n-2, \cdots, 1, 0))$$

$$p$$

$$q$$

$$J_{n,k} = (J_{n-1,k} + k) \bmod n$$

$$n\left(1 - \frac{1}{k}\right)^x = 1$$

$$x = -\frac{\ln n}{\ln\left(1 - \frac{1}{k}\right)}$$

$$\begin{aligned}
\lim_{k\to\infty} k \log\left(1 - \frac{1}{k}\right) &= \lim_{k\to\infty} \frac{\log\left(1 - \frac{1}{k}\right)}{1/k} \\
&= \lim_{k\to\infty} \frac{\frac{\mathrm{d}}{\mathrm{d}k}\log\left(1 - \frac{1}{k}\right)}{\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{k}\right)} \\
&= \lim_{k\to\infty} \frac{\frac{1}{k^2\left(1-\frac{1}{k}\right)}}{-\frac{1}{k^2}} \\
&= \lim_{k\to\infty} -\frac{k}{k-1} \\
&= -\lim_{k\to\infty} \frac{1}{1 - \frac{1}{k}} \\
&= -1
\end{aligned}$$

$$0, 1, \cdots, n-1$$

$$0$$

$$k$$

$$k = 2$$

$$n-1$$

$$\Theta(n^2)$$

$$0, 1, \cdots, n-1$$

$$k$$

$$J_{n,k}$$

$$n, k$$

$$0$$

$$k$$

$$k-1$$

$$n-1$$

$$n-1$$

$$k$$

$$\Theta(n)$$

$$k$$

$$n$$

$$\Theta(k\log n)$$

$$k$$

$$\left\lfloor\frac{n}{k}\right\rfloor$$

$$n-\left\lfloor\frac{n}{k}\right\rfloor$$

$$\left\lfloor\frac{n}{k}\right\rfloor\cdot k$$

$$n-n\bmod k$$

$$k$$

$$n-\left\lfloor\frac{n}{k}\right\rfloor$$

$$k$$

$$1$$

$$0$$

$$k$$

$$n$$

$$n$$

$$0$$

$$\frac{k}{k-1}$$

$$\Theta(k\log n)$$

$$x$$

$$n\left(1-\frac{1}{k}\right)$$

$$\Theta(k\log n)$$

$$\lim_{k\to\infty}k\log\left(1-\frac{1}{k}\right)$$

$$x \sim k \ln n, k \to \infty$$

$$-\frac{\ln n}{\ln\left(1-\frac{1}{k}\right)} = \Theta(k \log n)$$

$$F(p) = \sum_{i=1}^{n} f_{p_i}\left(\sum_{j=1}^{i-1} t_{p_j}\right)$$

$$\begin{aligned} F(p') - F(p) &= c_{p'_i} \sum_{j=1}^{i-1} t_{p'_j} + c_{p'_{i+1}} \sum_{j=1}^{i} t_{p'_j} - \left(c_{p_i} \sum_{j=1}^{i-1} t_{p_j} + c_{p_{i+1}} \sum_{j=1}^{i} t_{p_j}\right) \\ &= c_{p_i} t_{p_{i+1}} - c_{p_{i+1}} t_{p_i} \end{aligned}$$

$$n$$

$$i$$

$$t_i$$

$$i$$

$$t$$

$$f_i(t)$$

$$n$$

$$f_i$$

$$n$$

$$t_i$$

$$p$$

$$f_i(x) = c_i x + d_i$$

$$c_i$$

$$f_i(x) = c_i x$$

$$p$$

$$p'$$

$$p'$$

$$p$$

$$i$$

$$i+1$$

$$c_{p_i} t_{p_{i+1}} - c_{p_{i+1}} t_{p_i} > 0$$

$$\frac{c_{p_i}}{t_{p_i}} > \frac{c_{p_{i+1}}}{t_{p_{i+1}}}$$

$$\frac{c_i}{t_i}$$

$$f_i(x) = c_i \mathrm{e}^{ax}$$

$$c_i \geq 0, a > 0$$

$$i$$

$$i + 1$$

$$\frac{1 - \mathrm{e}^{at_i}}{c_i}$$

$$f_i(x)$$

$$t_i$$

$$f_i(t) = c_i t + d_i$$

$$c_i \geq 0$$

$$f_i(t) = c_i \mathrm{e}^{at} + d_i$$

$$c_i, a > 0$$

$$f_i(t) = \phi(t)$$

$$\phi(t)$$

$$O(n \log n)$$

$$n$$

$$1$$

$$i$$

$$h_i$$

$$O(n^2)$$

$$O(n)$$

$$l_i$$

$$i$$

$$l_i$$

$$i$$

$$l_i = 1$$

$$a_i > a_{l_i - 1}$$

$$a_i \leq a_{l_i - 1}$$

$$l_i - 1$$

$$i$$

$$l_i$$

$$l_{l_i-1}$$

$$l_i$$

$$l_j$$

$$O(n)$$

$$n$$

$$a_i$$

$$i$$

$$\min_{j=l}^{r} a_j = a_i$$

$$[l, r]$$

$$n \times m$$

$$\times 3$$

$$(x, y)$$

$$x$$

$$x'$$

$$x'$$

$$\Uparrow$$

$$\Downarrow$$

$$1$$

$$[0.985, 0.999]$$

$$n$$

$$n+1$$

$$n$$

$$n \leq 10$$

$$20000$$

$$cans$$

$$cans$$

$$n$$

$$000, 001, 011, 010, 110, 111, 101, 100$$

$$
\begin{array}{c} k=1 \\ 0 \\ 1 \end{array}
\rightarrow
\begin{array}{c} {\color{red}0} \\ {\color{red}1} \\ {\color{blue}1} \\ {\color{blue}0} \end{array}
\rightarrow
\begin{array}{c} k=2 \\ {\color{red}0}0 \\ {\color{red}0}1 \\ {\color{blue}1}1 \\ {\color{blue}1}0 \end{array}
\rightarrow
\begin{array}{c} {\color{red}00} \\ {\color{red}01} \\ {\color{red}11} \\ {\color{red}10} \\ {\color{blue}10} \\ {\color{blue}11} \\ {\color{blue}01} \\ {\color{blue}00} \end{array}
\rightarrow
\begin{array}{c} k=3 \\ {\color{red}0}00 \\ {\color{red}0}01 \\ {\color{red}0}11 \\ {\color{red}0}10 \\ {\color{blue}1}10 \\ {\color{blue}1}11 \\ {\color{blue}1}01 \\ {\color{blue}1}00 \end{array}
$$

$$G(n) = n \oplus \left\lfloor \frac{n}{2} \right\rfloor$$

$$(n)_2 = \cdots 0 \underbrace{11 \cdots 11}_{k\square}$$

$$(n+1)_2 = \cdots 1 \underbrace{00 \cdots 00}_{k\square}$$

$$
\begin{aligned}
n_k &= g_k \\
n_{k-1} &= g_{k-1} \oplus n_k &&= g_k \oplus g_{k-1} \\
n_{k-2} &= g_{k-2} \oplus n_{k-1} &&= g_k \oplus g_{k-1} \oplus g_{k-2} \\
n_{k-3} &= g_{k-3} \oplus n_{k-2} &&= g_k \oplus g_{k-1} \oplus g_{k-2} \oplus g_{k-3} \\
&\vdots \\
n_{k-i} &= \bigoplus_{j=0}^{i} g_{k-j}
\end{aligned}
$$

$$3$$

$$0$$

$$G(0) = 000, G(4) = 110$$

$$k$$

$$0$$

$$000 \rightarrow 001$$

$$1$$

$$001 \rightarrow 011$$

$$2^{k-1}$$

$$k$$

$$k$$

$$k-1$$

$$n$$

$$G(n)$$

$$G(n)$$

$$i$$

$$1$$

$$n$$

$$i$$

$$1$$

$$i+1$$

$$0$$

$$i$$

$$0$$

$$i+1$$

$$1$$

$$n$$

$$n+1$$

$$n$$

$$1$$

$$n$$

$$1$$

$$0$$

$$1$$

$$n$$

$$n+1$$

$$g(n)$$

$$g(n+1)$$

$$k$$

$$\underbrace{100\cdots00}_{k\square}$$

$$k+1$$

$$n$$

$$n+1$$

$$k+1$$

$$k+1$$

$$1$$

$$0$$

$$k+1$$

$$k+1$$

$$g$$

$$n$$
$$1$$
$$k$$
$$n$$
$$i$$
$$g$$
$$i$$
$$g_i$$
$$k$$
$$k$$
$$n$$
$$n$$
$$0$$
$$G(0)$$
$$G(i)$$
$$G(i+1)$$
$$i$$
$$i$$
$$i$$
$$i$$
$$n$$
$$f \to t \to r \to f \to t \to r \to \cdots$$
$$f$$
$$t$$
$$r$$
$$n$$
$$f \to r \to t \to f \to r \to t \to \cdots$$
$$n$$
$$n+1$$
$$w_0, w_1, ..., w_n$$
$$n$$
$$n+1$$
$$n+1$$
$$n$$

$$n$$

$$n$$

$$n+1$$

$$n+1$$

$$0$$

$$1$$

$$\infty$$

$$n$$

$$num$$

$$num[0] = num[n+1] = \infty$$

$$x$$

$$y$$

$$z$$

$$x \leq z$$

$$x$$

$$y$$

$$x$$

$$y$$

$$x+y$$

$$x$$

$$x+y$$

$$x$$

$$x$$

$$y$$

$$y$$

$$x+y$$

$$z$$

$$x \leq z$$

$$n$$

$$O(n \log n)$$

$$n$$

$$1 \leq n \leq 50000$$

$$n$$

$$1 \leq n \leq 50000$$

$$n$$

$$num[n]$$

$$num[0] = num[n+1] = \infty$$

$$i$$

$$num[i-1] \le num[i+1]$$

$$num[i-1], num[i]$$

$$temp$$

$$j$$

$$num[j] > temp$$

$$temp$$

$$j$$

$$1$$

$$num[j] \ge num[i-1] + num[i]$$

$$num[j+1]$$

$$num[i-2]$$

$$num[mid]$$

$$sum$$

$$N$$

$$i$$

$$a_i$$

$$x+y$$

$$x$$

$$y$$

$$x+y$$

$$\frac{\sum\limits_{i=1}^{n} a_i \times w_i}{\sum\limits_{i=1}^{n} b_i \times w_i}$$

$$\begin{aligned} & \frac{\sum a_i \times w_i}{\sum b_i \times w_i} > mid \\ \Longrightarrow & \sum a_i \times w_i - mid \times \sum b_i \cdot w_i > 0 \\ \Longrightarrow & \sum w_i \times (a_i - mid \times b_i) > 0 \end{aligned}$$

$$a_i$$

$$b_i$$

$$w_i \in \{0,1\}$$
$$a$$
$$b$$
$$\frac{\sum a}{\sum b}$$
$$W$$
$$mid$$
$$0$$
$$mid$$
$$L$$
$$\sum w_i \times (a_i - mid \times b_i)$$
$$n$$
$$a$$
$$b$$
$$w_i \in \{0,1\}$$
$$\frac{\sum a_i \times w_i}{\sum b_i \times w_i}$$
$$a_i - mid \times b_i$$
$$i$$
$$0$$
$$n$$
$$a$$
$$b$$
$$n-k$$
$$p_1, p_2, \cdots, p_{n-k}$$
$$\frac{\sum a_{p_i}}{\sum b_{p_i}}$$
$$100$$
$$i$$
$$a_i - mid \times b_i$$
$$n-k$$
$$n$$
$$a$$
$$b$$

$$w_i \in \{0, 1\}$$

$$\frac{\sum w_i \times a_i}{\sum w_i \times b_i}$$

$$\sum w_i \times b_i \geq W$$

$$W$$

$$b_i$$

$$i$$

$$a_i - mid \times b_i$$

$$i$$

$$dp[n][W]$$

$$\sum w_i \times b_i$$

$$W$$

$$W$$

$$a_i$$

$$b_i$$

$$T$$

$$\frac{\sum_{e \in T} a_e}{\sum_{e \in T} b_e}$$

$$a_i - mid \times b_i$$

$$w$$

$$C$$

$$\frac{\sum_{e \in C} w}{\mid C \mid}$$

$$a_i - mid$$

$$0$$

$$O(nm)$$

$$\sum w_i \times (a_i - mid \times b_i)$$

$$a + b{*}c{*}d + (e - f){*}(g{*}h + i)$$

$$abc{*}d{*} + ef - gh{*}i + {*} +$$

$$a + b{*}c$$

$$n$$

$$O(n)$$

$$3\ 2\ \ 1\ -$$

$$3 \times 2 = 6$$

$$6 - 1 = 5$$

$$O(n)$$

$$+$$

$$-$$

$$*$$

$$/$$

$$O(n)$$

$$a \wedge b \wedge c$$

$$a^{b^c}$$

$$(a^b)^c$$

$$+$$

$$-$$

$$*$$

$$/$$

$$+$$

$$-$$

$$O(n)$$

$$O(n \log n)$$

$$n$$

$$O(n \log n)$$

$$O(n \log n)$$

$$O(n \log n)$$

$$O(n)$$

$$(i, j)$$

$$(i, j)$$

$$mid$$

$$(i, j)$$

$$1 \leq i \leq mid, 1 \leq j \leq mid$$

$$1 \leq i \leq mid, mid + 1 \leq j \leq n$$

$$mid + 1 \leq i \leq n, mid + 1 \leq j \leq n$$

$$(1, n)$$

$$(1, mid)$$

$$(mid + 1, n)$$

$$l \leq i \leq r, l \leq j \leq r$$

$$a_i, b_i, c_i$$

$$(i, j)$$

$$a_j \leq a_i$$

$$b_j \leq b_i$$

$$c_j \leq c_i$$

$$j \neq i$$

$$a$$

$$l \leq i \leq mid$$

$$mid + 1 \leq j \leq r$$

$$(i, j)$$

$$a_i < a_j$$

$$b_i < b_j$$

$$c_i < c_j$$

$$a_i < a_j$$

$$a$$

$$a_i < a_j$$

$$i < j$$

$$i$$

$$mid$$

$$j$$

$$mid$$

$$i$$

$$j$$

$$b_i < b_j$$

$$c_i < c_j$$

$$j$$

$$i$$

$$(l, mid)$$

$$(mid + 1, r)$$

$$b$$

$$j$$

$$b_i < b_j$$

$$i$$

$$c$$

$$c_j$$

$$j$$

$$i$$

$$c$$

$$x$$

$$x$$

$$x$$

$$c$$

$$j$$

$$b_i < b_j$$

$$i$$

$$i$$

$$j$$

$$b$$

$$O(n^2)$$

$$O(n)$$

$$O(n \log n)$$

$$T(n) = T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lceil \frac{n}{2} \right\rceil) + O(n \log n) = O(n \log^2 n)$$

$$a$$

$$\{(i, j) |\ i < j \land a_i > a_j\}$$

$$1 \sim n$$

$$m$$

$$O(n)$$

$$O(n^2)$$

$$O(n \log^2 n)$$

$$a$$

$$b$$

$$dp_i = 1 + \max_{j=1}^{i-1} dp_j [a_j < a_i][b_j < b_i]$$

$$j < i, a_j < a_i$$

$$b_j < b_i$$

$$j$$

$$i$$

$$O(n^2)$$

$$dp_j$$

$$dp_i$$

$$(l, r)$$

$$l = r$$

$$dp_r$$

$$dp_r + +$$

$$l \le j \le mid$$

$$mid + 1 \le i \le r$$

$$l \le j \le mid$$

$$mid + 1 \le i \le r$$

$$j < i$$

$$i$$

$$j$$

$$a$$

$$j$$

$$dp_i$$

$$l \le j \le mid$$

$$mid + 1 \le i \le r$$

$$solve(l, mid)$$

$$solve(mid + 1, r)$$

$$dp_i$$

dp_j

dp_i

dp_j

dp_i

dp_j

$O(n^2)$

8

$(1,4)$

$(5,6)$

$(7,7)$

5

$(1,4)$

i

$\log$

$(1,i)$

$\log$

j

i

dp_1

dp_2

dp_2

$(1,2)$

dp

dp_3

dp_3

dp_4

$(1,2)$

$(3,3)$

dp_4

dp_4

$$(3,4)$$

$$(1,4)$$

$$(l,r)$$

$$O(n^2)$$

$$(l,r)$$

$$(l,mid)$$

$$(mid,r)$$

$$l \le i \le mid$$

$$mid+1 \le j \le r$$

$$i$$

$$j$$

$$l \le i \le mid$$

$$mid+1 \le j \le r$$

$$O(N^2)$$

$$mid$$

$$O(n \log n)$$

$$mid$$

$$O(\log n)$$

$$T(n) = T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lceil \frac{n}{2} \right\rceil) + O(n \log n) = O(n \log^2 n)$$

$$n$$

$$a_i$$

$$m$$

$$[l,r]$$

$$x$$

$$[l,r]$$

$$pre_i$$

$$pre$$

$$O(n+m)$$

$$O(1)$$

$$pre$$

$$pre$$

$$(l, r)$$

$$(l, r)$$

$$+\infty$$

$$-\infty$$

$$+\infty$$

$$(l, r)$$

$$(l, r)$$

$$-\infty$$

$$(l, r)$$

$$(l, r)$$

$$O(len)$$

$$+\infty$$

$$O(len)$$

$$O(len)$$

$$O(len)$$

$$-\infty$$

$$-\infty$$

$$len$$

$$O(len)$$

$$O(len)$$

$$O(len)$$

$$O(n \log n)$$

$$T(n) = T(\left\lfloor \frac{n}{2} \right\rfloor) + T(\left\lceil \frac{n}{2} \right\rceil) + O(n \log n) = O(n \log^2 n)$$

$$\mathsf{UC}(\alpha) = \begin{cases} 0 & M_\alpha(\alpha) = 1 \\ 1 & \text{otherwise} \end{cases}$$

$$\mathsf{P} = \bigcup_{k \in \mathbb{N}} \mathsf{DTIME}(n^k)$$

$$\mathsf{EXPTIME} = \bigcup_{k \in \mathbb{N}} \mathsf{DTIME}(2^{n^k})$$

$$\mathsf{NP} = \bigcup_{k \in \mathbb{N}} \mathsf{NTIME}(n^k)$$

$$\mathsf{NEXPTIME} = \bigcup_{k \in \mathbb{N}} \mathsf{NTIME}(2^{n^k})$$

$$\mathit{DTIME}\left(o\left(\frac{f(n)}{\log f(n)}\right)\right) \subsetneq \mathit{DTIME}(f(n))$$

$$\mathit{NSPACE}(f(n)) \subseteq \mathit{DSPACE}\left((f(n))^2\right)$$

$$\Sigma^*$$

$$\Sigma$$

$$\Sigma$$

$$\Sigma^*$$

$$\Sigma = \{0,1\}$$

$$f : \Sigma^* \to \{0,1\}, f(x) = 1 \iff x \in L$$

$$\Sigma$$

$$L$$

$$f$$

$$M = \langle Q, \Gamma, b, \Sigma, \delta, q_0, F\rangle$$

$$Q$$

$$\Gamma$$

$$b \in \Gamma$$

$$\Sigma \subseteq (\Gamma \setminus \{b\})$$

$$q_0 \in Q$$

$$F \subseteq Q$$

$$\delta : (Q \setminus F) \times \Gamma \not\to Q \times \Gamma \times \{L, R\}$$

$$\delta$$

$$x$$

$$y$$

$$\delta(x, y)$$

$$\delta(x, y) = (a, b, c)$$

$$a$$

$$b$$

$$c$$

$$L$$

$$R$$

$$M$$

$$x$$

$$M(x)$$
$$M(x) = 1$$
$$M$$
$$x$$
$$M(x) = 0$$
$$M$$
$$x$$
$$M$$
$$x$$
$$k$$
$$k - 1$$
$$k - 1$$
$$f : \mathbb{N} \to \mathbb{M}$$
$$\alpha$$
$$M_\alpha$$
$$\mathcal{U}$$
$$M_\alpha$$
$$x$$
$$\mathcal{U}(x, \alpha) = M_\alpha(x)$$
$$\mathcal{U}(x, \alpha)$$
$$x \in \{0, 1\}^*$$
$$M_\alpha$$
$$x$$
$$T(\mid x \mid)$$
$$x \in \{0, 1\}^*$$
$$\mathcal{U}(x, \alpha)$$
$$O(T(\mid x \mid) \log T(\mid x \mid))$$
$$\alpha$$
$$x$$
$$M_\alpha$$
$$x$$
$$\mathsf{UC} : \{0, 1\}^* \to \{0, 1\}$$

$$\mathsf{UC}$$

$$M_\beta$$

$$\mathsf{UC}$$

$$\mathsf{UC}$$

$$\mathsf{UC}(\beta) = 1 \iff M_\beta(\beta) \neq 1$$

$$M_\beta$$

$$\mathsf{UC}$$

$$M_\beta(\beta) = \mathsf{UC}(\beta)$$

$$\mathsf{UC}$$

$$M_{\mathsf{HALT}}$$

$$M_{\mathsf{HALT}}(x, \alpha)$$

$$M_\alpha$$

$$x$$

$$\mathsf{UC}$$

$$M_{\mathsf{UC}}$$

$$M_{\mathsf{UC}}$$

$$M_{\mathsf{HALT}}(\alpha, \alpha)$$

$$0$$

$$M_{\mathsf{UC}}(\alpha) = 1$$

$$M_{\mathsf{UC}}$$

$$\mathsf{UC}$$

$$M_{\mathsf{HALT}}$$

$$L$$

$$M$$

$$M$$

$$M(x) = 1 \iff x \in L$$

$$M$$

$$L$$

$$L$$

$$M$$

$$L$$

$$M$$

$$M(x) = 1 \iff x \in L$$
$$M$$
$$L$$
$$\mathit{R}$$
$$\mathit{RE}$$
$$\mathit{RE}$$
$$\mathit{R} \subseteq \mathit{RE}$$
$$x$$
$$O(f(\mid x \mid))$$
$$\mathit{DTIME}(f(n))$$
$$\mathit{P}$$
$$\mathit{P}$$
$$\mathit{EXPTIME}$$
$$k$$
$$k$$
$$\mathit{EXPTIME}$$
$$O(k)$$
$$k$$
$$O(\log k)$$
$$x$$
$$O(f(\mid x \mid))$$
$$\mathit{NTIME}(f(n))$$
$$\mathit{NP}$$
$$\mathit{P}$$
$$\mathit{NP}$$
$$\mathit{NP}$$
$$\mathit{NP}$$
$$H$$
$$H$$
$$H$$
$$\mathit{NP}$$
$$\mathit{NP}$$
$$k$$

$$NP$$

$$EXPTIME$$

$$co-NP$$

$$NP$$

$$\Sigma^*$$

$$n$$

$$k$$

$$n$$

$$k$$

$$co-NP$$

$$NP$$

$$P$$

$$P \neq NP$$

$$NEXPTIME$$

$$\#P$$

$$NP$$

$$NP$$

$$\#P$$

$$\#P$$

$$x$$

$$O(f(\mid x \mid))$$

$$DSPACE(f(n))$$

$$REG = DSPACE(O(1))$$

$$L = DSPACE(O(\log n))$$

$$PSPACE = \bigcup_{k \in \mathbb{N}} DSPACE(n^k)$$

$$EXPSPACE = \bigcup_{k \in \mathbb{N}} DSPACE(2^{n^k})$$

$$x$$

$$O(f(\mid x \mid))$$

$$NSPACE(f(n))$$

$$REG = DSPACE(O(1)) = NSPACE(O(1))$$

$$NL = NSPACE(O(\log n))$$

$$\mathsf{CSL} = \mathsf{NSPACE}(O(n))$$

$$\mathsf{PSPACE} = \mathsf{NPSPACE} = \bigcup_{k \in \mathbb{N}} \mathsf{NSPACE}(n^k)$$

$$\mathsf{EXPSPACE} = \mathsf{NEXPSPACE} = \bigcup_{k \in \mathbb{N}} \mathsf{NSPACE}(2^{n^k})$$

$$k$$

$$O(n^k)$$

$$\Theta(n^k)$$

$$n$$

$$\mathsf{P}$$

$$O(1)$$

$$O(\log a + \log b)$$

$$a$$

$$b$$

$$n$$

$$O(\log n)$$

$$\mathsf{P} \neq \mathsf{NP}$$

$$T(n)$$

$$T(n)$$

$$T(n)$$

$$M$$

$$1^n$$

$$n$$

$$M$$

$$O(f(n))$$

$$f(n)$$

$$f(n)$$

$$O(n)$$

$$o(n)$$

$$M$$

$$1^n$$

$$n$$

$$M$$

$$O(f(n))$$

$$f(n)$$

$$f(n)$$

$$f(n)$$

$$\mathsf{P} \subsetneq \mathsf{EXPTIME}$$

$$L = \{(x, y) \mid \mathcal{U}((x, y), x)□f(\mid x \mid + \mid y \mid) □□□□□□□□\}$$

$$f(n)$$

$$L$$

$$O(f(\mid x \mid + \mid y \mid))$$

$$L \in \mathsf{DTIME}(f(n))$$

$$L \in \mathsf{DTIME}(o\left(\frac{f(n)}{\log f(n)}\right))$$

$$M_z$$

$$o\left(\frac{f(n)}{\log f(n)}\right)$$

$$L$$

$$\mathcal{U}(x, z)$$

$$x$$

$$g(\mid x \mid)$$

$$g(n) = o(f(n))$$

$$y$$

$$g(\mid z \mid + \mid y \mid) < f(\mid z \mid + \mid y \mid)$$

$$y'$$

$$y$$

$$\mathcal{U}((z, y'), z)$$

$$f(\mid z \mid + \mid y' \mid)$$

$$M_z(z, y') \neq M_z(z, y')$$

$$g(n)$$

$$f(n + 1) = o(g(n))$$

$$\mathsf{NTIME}(f(n)) \subsetneq \mathsf{NTIME}(g(n))$$

$$NP \subsetneq NEXPTIME$$

$$f(n)$$

$$f(n) = \Omega(\log n)$$

$$SPACE(o(f(n))) \subsetneq SPACE(f(n))$$

$$SPACE$$

$$DSPACE$$

$$NSPACE$$

$$PSPACE \subsetneq EXPSPACE$$

$$f(n) = \Omega(\log n)$$

$$PSPACE = NPSPACE$$

$$EXPSPACE = NEXPSPACE$$

$$P$$

$$NP$$

$$P = NP$$

$$NP = co-NP$$

$$NP = co-NP$$

$$P = NP$$

$$NP$$

$$co-NP$$

$$co-NP$$

$$co-NP$$

$$NP = co-NP$$

$$co-NP$$

$$co-NP$$

$$co-NP$$

$$NP$$

$$P = NP$$

$$EXPTIME = NEXPTIME$$

$$P \neq NP$$

$$4 \times 4$$

$$n$$

$$8$$

$$3 \times 3$$

$$8$$

$$15$$

$$16$$

$$n \times m$$

$$4 \times 4$$

$$15$$

$$m$$

$$n$$

$$2$$

$$m \times n$$

$$m = n = 2$$

$$m$$

$$n$$

$$h(n)$$

$$A_{15}$$

$$2 \times k - 1$$

$$A_{2k-1}$$

$$\dim \pi_\lambda = \frac{n!}{\prod_{x \in Y(\lambda)} \mathrm{hook}(x)}.$$

$$\dim \pi_\lambda = \frac{10!}{7 \cdot 5 \cdot 4 \cdot 3 \cdot 1 \cdot 5 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 288.$$

$$n$$

$$\lambda$$

$$\lambda$$

$$\lambda$$

$$(5, 4, 1)$$

$$x_1$$

$$x_2$$

$$x_3$$

$$...$$

$$n$$

$$1$$

$$n$$

$$n$$

$$n$$

$$n$$

$$n$$

$$n$$

$$1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, ...$$

$$(1, 1, ..., 1)$$

$$1$$

$$n$$

$$S$$

$$S \leftarrow x$$

$$x$$

$$x$$

$$y$$

$$x$$

$$y$$

$$x \leftarrow y$$

$$x$$

$$x$$

$$s$$

$$t$$

$$(s, t)$$

$$(s, t)$$

$$(s + 1, t)$$

$$(s, t + 1)$$

$$3$$

$$(2, 5, 9)(6, 7)(8)$$

$$\lambda = (\lambda 1, \lambda 2...)$$

$$\mu = (\mu 1, \mu 2, ...)$$

$$\lambda$$

$$\mu$$

$$\mu i \leq \mu i$$

$$i$$

$$\lambda / \mu$$

$$\lambda$$

μ

λ

μ

$(5,4,1)$

1

n

λ

μ

λ/μ

λ

μ

λ/μ

λ

μ

μ

0

λ/μ

λ

n

π_λ

1

n

n

dim_{π_λ}

v

$\mathrm{hook}(v)$

dim_λ

$n!$

$10=5+4+1$

P

1

n

$X=x_1,...,x_n$

$$P_X$$
$$X$$
$$P$$
$$X$$
$$P_X$$
$$X^R$$
$$X$$
$$X^R$$
$$P_{X^R}$$
$$P_X$$
$$X = 1, 5, 7, 2, 8, 6, 3, 4$$
$$X^R = 4, 3, 6, 8, 2, 7, 5, 1$$
$$P_X$$
$$P_X$$
$$X$$
$$k$$
$$LIS/LDS$$
$$k - LIS$$
$$k - LDS$$
$$1 - LIS$$
$$1 - LIS$$
$$LDS$$
$$k$$
$$k - LIS$$
$$X$$
$$m$$
$$P$$
$$X^*$$
$$(P_{m,1}..., P_{m,\lambda_m}, P_{m-1,1}..., P_{1,1}...P_{1,\lambda_1})$$
$$X$$
$$X^*$$
$$k - LIS$$

$$F(k) = \sum_{i=1}^{m} \min(k, \lambda_i)$$

$$k$$

$$n$$

$$b$$

$$B_m = (b_1, b_2, ..., b_m)$$

$$C$$

$$B_m$$

$$C$$

$$k$$

$$C$$

$$k$$

$$O(n^2 \log n)$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$W \times H$$

$$K \le W$$

$$K > W$$

$$W$$

$$H$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$-A_i$$

$$O(n\sqrt{n} \log n)$$

$$n$$

$$n$$

$$a_1, a_2, ..., a_n \, (a_1 \ge a_2 \ge ... \ge a_n \ge 1)$$

$$a = [3, 2, 2, 2, 1]$$

$$1 \times 2$$

$$2 \times 1$$

$$\max z = x_1 + x_2$$

$$s.t\begin{cases}2x_1+x_2\le 12\\x_1+2x_2\le 9\\x_1,x_2\ge 0\end{cases}$$

$$\max z=\sum_{j=1}^{n}c_jx_j$$

$$s.t\begin{cases}\displaystyle\sum_{j=1}^{n}a_{ij}x_j=b_i,i=1,2,...,m\\x_j\ge 0,j=1,2,...,n\end{cases}$$

$$\max z=CX$$

$$AX=b$$

$$X\ge 0$$

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&\ddots&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix}$$

$$x_1+2x_2\le 9\Longrightarrow\begin{cases}x_1+2x_2+x_3=9\\x_3\ge 0\end{cases}$$

$$-\infty\le x_k\le+\infty\Longrightarrow\begin{cases}x_k=x_m-x_n\\x_m,x_n\ge 0\end{cases}$$

$$\max z=x_1+x_2$$

$$s.t\begin{cases}2x_1+x_2+x_3=12\\x_1+2x_2+x_4=9\\x_1,x_2,x_3,x_4\ge 0\end{cases}$$

$$x_i'=C_i+\sum_{j=m+1}^{n}m_{ij}x_j'(i=1,2,...,m)$$

$$x_i'=C_i$$

$$\max z=x_1+x_2$$

$$s.t\begin{cases}2x_1+x_2+x_3=12\\x_1+2x_2+x_4=9\\x_1,x_2,x_3,x_4\ge 0\end{cases}$$

$$\begin{bmatrix}\mathrm{X}&x_1&x_2&x_3&x_4&b\\&2&1&1&0&12\\&1&2&0&1&9\\\mathrm{C}&1&1&0&0&z\end{bmatrix}\rightarrow\begin{bmatrix}\mathrm{X}&x_1&x_2&x_3&x_4&b\\&\frac{3}{2}&0&1&-\frac{1}{2}&\frac{15}{2}\\&\frac{1}{2}&1&0&\frac{1}{2}&\frac{9}{2}\\\mathrm{C}&\frac{1}{2}&0&0&-\frac{1}{2}&z-\frac{9}{2}\end{bmatrix}$$

$$x_i' = C_i + \sum_{j=m+1}^{n} m_{ij} x_j' (i = 1, 2, ..., m)$$

$$z = z_0 + \sum_{j=m+1}^{n} \sigma_j x_j'$$

$$\begin{bmatrix} \mathrm{X} & x_1 & x_2 & x_3 & x_4 & b \\ & 2 & 1 & 1 & 0 & 12 \\ & 1 & 2 & 0 & 1 & 9 \\ \mathrm{C} & 1 & 1 & 0 & 0 & z \end{bmatrix} \to \begin{bmatrix} \mathrm{X} & x_1 & x_2 & x_3 & x_4 & b \\ & \frac{3}{2} & 0 & 1 & -\frac{1}{2} & \frac{15}{2} \\ & \frac{1}{2} & 1 & 0 & \frac{1}{2} & \frac{9}{2} \\ \mathrm{C} & \frac{1}{2} & 0 & 0 & -\frac{1}{2} & z - \frac{9}{2} \end{bmatrix}$$

$$\max x_1 + 14x_2 + 6x_3$$

$$s.t \begin{cases} x_1 + x_2 + x_3 \leq 4 \\ x_1 \leq 2 \\ x_3 \leq 3 \\ 3x_2 + x_3 \leq 6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

$$\max x_1 + 14x_2 + 6x_3$$

$$s.t \begin{cases} x_1 + x_2 + x_3 + x_4 = 4 \\ x_1 + x_5 = 2 \\ x_3 + x_6 = 3 \\ 3x_2 + x_3 + x_7 = 6 \\ x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{cases}$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, i = 1, 2, ..., m$$

$$x_j \geq 0, j = 1, 2, ..., n$$

$$Ax \leq b$$

$$x \geq 0$$

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \to x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, x_{n+i} \geq 0$$

$$\max \sum_{i=1}^{n} b_i y_i$$

$$s.t \sum_{l_i \leq j \leq r_i} y_j \leq c_i, y_i \geq 0$$

$$\max c^T x : Ax \leq b, x \geq 0$$

$$\min b^T y : A^T y \geq c, y \geq 0$$

$$\max \sum_{(u,v)\in E} c_{uv} d_{uv}$$

$$s.t\begin{cases} \sum_{(v)\in Y} d_{uv} \leq 1, u \in X \\ \sum_{(u)\in X} d_{uv} \leq 1, v \in Y \\ d_{u,v} \in \{0,1\} \end{cases}$$

$$\min \sum_{u\in x} p_u + \sum_{v\in Y} p_v$$

$$s.t\begin{cases} p_u + p_v \geq c_{uv} \\ u \in X, v \in Y \\ p_u, p_v \geq 0 \end{cases}$$

$$z$$

$$\max$$

$$m$$

$$n$$

$$n \geq m$$

$$m$$

$$x_j'(j = 1, 2, ..., m)$$

$$0$$

$$C_i$$

$$0$$

$$x_2$$

$$x_3$$

$$x_1$$

$$x_4$$

$$0$$

$$x_2$$

$$x_3$$

$$x_2 = \frac{9}{2}$$

$$x_3 = \frac{15}{2}$$

$$X_1 = 0$$

$$y$$

$$X_4 = 0$$

$$x_1 + 2x_2 = 9$$

$$z$$

$$\sigma_j$$

$$0$$

$$j$$

$$\sigma_j > 0$$

$$z_0$$

$$0$$

$$x_j' = 0$$

$$x_j'$$

$$x_j'$$

$$\sigma_j > 0$$

$$\sigma_j > 0$$

$$\sigma_j > 0$$

$$x_j'$$

$$x_s'$$

$$x_j'$$

$$0$$

$$x_j'$$

$$x_s'$$

$$x_s'$$

$$x_1$$

$$\frac{1}{2} > 0$$

$$x_2$$

$$x_3$$

$$x_1$$

$$x_2$$

$$x_3$$

$$x_1$$

$$x_3$$

$$x_3 = 0$$

$$x_1 = \frac{15/2}{3/2} = 5$$

$$x_1$$

$$x_2$$

$$x_2 = 0$$

$$x_1 = \frac{9/2}{1/2} = 9$$

$$x_2$$

$$x_1$$

$$x_3$$

$$x_3$$

$$x_2$$

$$x_4$$

$$x_2 = 0$$

$$x_1 + 2x_2 = 9$$

$$x_3$$

$$0$$

$$0$$

$$0$$

$$z$$

$$b$$

$$x_1$$

$$x_2$$

$$x_3$$

$$x_4$$

$$x_5$$

$$x_6$$

$$x_7$$

$$b$$

$$c_1 = 1$$

$$c_2 = 14$$

$$c_3 = 6$$

$c_4 = 0$

$c_5 = 0$

$c_6 = 0$

$c_7 = 0$

$-z = 0$

1

1

1

1

0

0

0

4

1

0

0

0

1

0

0

2

0

0

1

0

0

1

0

3

0

3

1

0

0

0

1

6

$*$

$*$

$*$

$*$

x

x

c

c_2

x

x_2

x_2

x

x_2

x

0

x

1

x_2

4

$\frac{4}{1}$

4

x_2

2

$\frac{6}{3}$

x_2

x

4

x

c_2

x_2

x_7

x_1

x_2

x_3

x_4

x_5

x_6

x_7

b

$c_1 = 1$

$c_2 = 0$

$c_3 = 1.33$

$c_4 = 0$

$c_5 = 0$

$c_6 = 0$

$c_7 = -4.67$

$-z = -28$

1

0

0.67

1

0

0

−0.33

2

1

0

0

0

1

0

0

2

0

0

1

0

0

1

0

3

0

1

0.33

0

0

0

0.33

2

$*$

$*$

$*$

$*$

x_1

x_2

x_3

x_4

x_5

x_6

x_7

b

$c_1 = -1$

$c_2 = 0$

$c_3 = 0$

$c_4 = -2$

$c_5 = 0$

$c_6 = 0$

$c_7 = -4$

$-z = -32$

1.5

0

1

1.5

0

0

-0.5

3

1

0

0

0

1

0

0

2

-1.5

0

0

-1.5

0

1

0.5

0

-0.5

1

0

-0.5

0

0

0.5

1

$$*$$

$$*$$

$$*$$

$$*$$

$$x$$

$$x$$

$$32$$

$$m+n$$

$$n$$

$$x$$

$$A$$

$$m \times n$$

$$c$$

$$n$$

$$b$$

$$m$$

$$cx$$

$$Ax \leq b, x > 0$$

$$\sum_{j=1}^{n} c_j x_j$$

$$n$$

$$m+n$$

$$m \times n$$

$$A$$

$$m$$

$$b$$

$$n$$

$$c$$

$$C^T x$$

$$f(x_1, x_2, ..., x_n) = b$$

$$f(x_1, x_2, ..., x_n) \leq b, -f(x_1, x_2, ..., x_n) \leq -b$$

$$f(x_1, x_2, ..., x_n) \geq b$$

$$-f(x_1, x_2, ..., x_n) \leq -b$$

$$x$$

$$x_0, x_1$$

$$x = x_0 - x_1$$

$$B$$

$$\mid B \mid = m$$

$$N$$

$$\mid N \mid = n$$

$$x_n + i = b_i - \sum a_{ij} x_j \geq 0$$

$$x_{n+i}$$

$$x_e$$

$$x_l$$

$$0$$

$$b_i \geq 0$$

$$B$$

$$N$$

$$n$$

$$x_e$$

$$x_l$$

$$b_i < 0$$

$$x_l$$

$$a_{le} < 0$$

$$x_e$$

$$pivot(l, e)$$

$$b_i$$

$$n$$

$$n$$

$$l$$

$$x_e$$

$$x_e$$

$$x_e$$

$$b$$

$$a$$

$$c$$

$$v$$

$$l$$

e

a

$a_{l,\square}$

$a_{e,\square}$

a

$= | B |$

$= | N |$

m

B

N

m

n

a

$a_{i,e}$

$c_e > 0$

e

l

l, e

ϵ

$c_e > \epsilon$

l

$a_{i,e} > \epsilon$

n

i

b_i

m

$[l_i, r_i] + 1$

a_i

m

n

$l_j \leq i \leq r_j$

$a_{ij} = 1$

n

m

$$d_{uv}$$

$$u, v$$

$$p_u, p_v$$

$$-1, 0, 1$$

$$A$$

$$\in -1, 0, 1$$

$$0$$

$$G_\alpha = \langle t_i s t_{i^s}^{-1} \mid i \in \alpha^G, s \in S \rangle$$

$$G \leq S_n$$

$$t$$

$$O(n^2 \log^3 | G | + tn \log | G |)$$

$$O(n^2 \log | G | + tn)$$

$$O(n^3 \log^3 | G | + tn^2 \log | G |)$$

$$O(n \log^2 | G | + tn)$$

$$O(n \log n \log^4 | G | + tn \log | G |)$$

$$O(n \log | G | + tn)$$

$$G$$

$$S$$

$$G = \langle S \rangle$$

$$i$$

$$G$$

$$s_1, \cdots, s_r$$

$$s_i \in S$$

$$s_i^{-1} \in S$$

$$S$$

$$\alpha$$

$$\alpha$$

$$\alpha$$

$$\alpha^G$$

$$\alpha$$

$$S$$

$$i, j$$

$$j$$

$$i$$

$$s \in S$$

$$i$$

$$j$$

$$\alpha$$

$$s \in S$$

$$\alpha^s$$

$$O(rn)$$

$$| \alpha^G|$$

$$\alpha$$

$$G_\alpha$$

$$G = \langle S \rangle$$

$$\alpha$$

$$G_\alpha$$

$$t_i$$

$$\alpha$$

$$i$$

$$G$$

$$i$$

$$G$$

$$B$$

$$B = (b_1, \cdots, b_k) \subseteq \Omega$$

$$G_{b_1, \cdots, b_k}$$

$$G$$

$$B$$

$$i$$

$$\langle S \cap G^{(i)} \rangle = G(i)$$

$$S \subseteq G$$

$$G^{(i)} := G_{b1, \cdots, bi}$$

$$G^{(0)} := G$$

$$G$$

$$G$$
$$G$$
$$b \in \Omega$$
$$b$$
$$\mid b^G \mid$$
$$G_b$$
$$G_b$$
$$\mid G_b \mid$$
$$(b_1, \cdots, b_m)$$
$$G$$
$$G^{(i)}$$
$$G$$
$$B = [b_1, \cdots, b_k]$$
$$S$$
$$B \in \Omega$$
$$S \subseteq G$$
$$S$$
$$B$$
$$T_{i+1}$$
$$G^{(i)}$$
$$G^{(i+1)}$$
$$S =$$
$$B, S$$
$$B = (b_1, \cdots, b_k]$$
$$S$$
$$C := [b_2, \cdots, b_k]$$
$$T := S \cap G_{b1}$$
$$C, T$$
$$H = \langle T \rangle$$
$$B := B \cup C$$
$$S := S \cap T$$

$$H \leqslant G_{b_1}$$

$$G_{b_1}$$

$$s$$

$$s \in H$$

$$H$$

$$H = G_{b_1}$$

$$B \square S$$

$$s \in G_{b_1}$$

$$s \notin H$$

$$S := S \cup s$$

$$s$$

$$B$$

$$s$$

$$\Omega$$

$$B$$

$$B$$

$$S$$

$$O(n^2 \log n)$$

$$O(n^8 \log^3 n)$$

$$n$$

$$t$$

$$O(n^2 + nt)$$

$$t > n$$

$$O(nt)$$

$$O(n^2 \log n)$$

$$t$$

$$nO(n(n^2 \log n)) = O(n^4 \log n)$$

$$R$$

$$N$$

$$1 \leqslant N \leqslant 15$$

$$1 \leqslant R \leqslant 1000$$

$$N$$

$$a_1, a_2, \cdots a_n$$

$$a_i$$

$$i$$

$$a_i = i$$

$$f = \begin{pmatrix} a_1, a_2, ..., a_n \\ a_{p_1}, a_{p_2}, ..., a_{p_n} \end{pmatrix}$$

$$g \circ f = \begin{pmatrix} a_1, a_2, ..., a_n \\ a_{q_1}, a_{q_2}, ..., a_{q_n} \end{pmatrix}$$

$$\sigma(1) = i_1 \quad \sigma(2) = i_2 \quad \cdots \quad \sigma(n) = i_n$$

$$S$$

$$S$$

$$S = \{a_1, a_2, ..., a_n\}$$

$$a_i$$

$$a_{p_i}$$

$$p_1, p_2, ..., p_n$$

$$1, 2, ..., n$$

$$S$$

$$n!$$

$$f = \begin{pmatrix} a_1, a_2, ..., a_n \\ a_{p_1}, a_{p_2}, ..., a_{p_n} \end{pmatrix}$$

$$g = \begin{pmatrix} a_{p_1}, a_{p_2}, ..., a_{p_n} \\ a_{q_1}, a_{q_2}, ..., a_{q_n} \end{pmatrix}$$

$$f$$

$$g$$

$$g \circ f$$

$$f$$

$$g$$

$$I = \{1, 2, \cdots, n\}$$

$$n$$

$$\sigma$$

$$I$$

$$i_1, i_2, \cdots, i_n$$

$$1, 2, \cdots, n$$

$$n$$

$$1, 2, \cdots, n$$

$$1, 2, \cdots, n$$

$$i_1, i_2, \cdots, i_n$$

$$1, 2, \cdots, n$$

$$n$$

$$1, 2, \cdots, n$$

$$n!$$

$$1, 2, \cdots, n$$

$$i$$

$$j$$

$$(i, j)$$

$$i_1 i_2 \cdots i_n$$

$$j_1 j_2 \cdots j_n$$

$$n$$

$$i_1 i_2 \cdots i_n$$

$$j_1 j_2 \cdots j_n$$

$$n$$

$$2$$

$$n$$

$$\frac{n!}{2}$$

$$O(n \log n)$$

$$(a_1, a_2, ..., a_m) = \binom{a_1, a_2, ..., a_{m-1}, a_m}{a_2, a_3, ..., a_m, a_1}$$

$$\binom{a_1, a_2, a_3, a_4, a_5}{a_3, a_1, a_2, a_5, a_4} = (a_1, a_3, a_2) \circ (a_4, a_5)$$

$$\mid X/G \mid = \frac{1}{\mid G \mid} \sum_{g \in G} \mid X^g \mid$$

$$X^g = \{x \mid x \in X, g(x) = x\}$$

$$\frac{1}{1+6+3+6+8}(3^6 + 6 \times 3^3 + 3 \times 3^4 + 6 \times 3^3 + 8 \times 3^2) = 57$$

$$\mid G \mid = \mid G^x \mid \mid G(x) \mid$$

$$\mid G \mid = \mid G^x \mid [G : G^x]$$

$$\sum_{g \in G} \mid X^g \mid = \mid \{(g,x) \mid (g,x) \in G \times X, g(x) = x\} \mid$$

$$= \sum_{x \in X} \mid G^x \mid$$

$$= \sum_{x \in X} \frac{\mid G \mid}{\mid G(x) \mid} \qquad \text{□□□□□□□□□}$$

$$= \mid G \mid \sum_{x \in X} \frac{1}{\mid G(x) \mid}$$

$$= \mid G \mid \sum_{Y \in X/G} \sum_{x \in Y} \frac{1}{\mid G(x) \mid}$$

$$= \mid G \mid \sum_{Y \in X/G} \sum_{x \in Y} \frac{1}{\mid Y \mid}$$

$$= \mid G \mid \sum_{Y \in X/G} 1$$

$$= \mid G \mid \mid X/G \mid$$

$$\mid X/G \mid = \frac{1}{\mid G \mid} \sum_{g \in G} \mid X^g \mid$$

$$\mid X/G \mid = \frac{1}{\mid G \mid} \sum_{g \in G} \mid B \mid^{c(g)}$$

$$\begin{pmatrix} 1,3,2,4,5,6 \\ 3,1,4,2,6,5 \end{pmatrix} = (1,3) \circ (2,4) \circ (5,6)$$

S

A

B

X

A

B

G

A

X

G

X/G

G

X

X

$$G$$
$$| S |$$
$$S$$
$$A$$
$$B$$
$$X$$
$$3^6$$
$$G$$
$$X/G$$
$$X^g$$
$$g$$
$$g$$
$$G$$
$$| X^g| = 3^6$$
$$90^\circ$$
$$| X^g| = 3^3$$
$$180^\circ$$
$$| X^g| = 3^4$$
$$180^\circ$$
$$| X^g| = 3^3$$
$$120^\circ$$
$$| X^g| = 3^2$$
$$G$$
$$X$$
$$\forall x \in X, G^x = \{g \mid g(x) = x, g \in G\}, G(x) = \{g(x)| \; g \in G\}$$
$$G^x$$
$$x$$
$$G(x)$$
$$x$$
$$G^x$$
$$G$$
$$f, g \in G$$

$$f \circ g(x) = f(g(x)) = f(x) = x$$

$$f \circ g \in G^x$$

$$I(x) = x$$

$$I \in G^x$$

$$I$$

$$g \in G^x$$

$$g^{-1}(x) = g^{-1}(g(x)) = g^{-1} \circ g(x) = I(x) = x$$

$$g^{-1} \in G^x$$

$$[G : G^x]$$

$$G^x$$

$$|\ G(x)| = [G : G^x]$$

$$G(x)$$

$$G^x$$

$$\varphi(g(x)) = gG^x$$

$$\varphi$$

$$g(x) = f(x)$$

$$f^{-1}$$

$$f^{-1} \circ g(x) = I(x) = x$$

$$f^{-1} \circ g \in G^x$$

$$(f^{-1} \circ g)G^x = G^x$$

$$gG^x = fG^x$$

$$gG^x = fG^x$$

$$g(x) = f(x)$$

$$g(x)$$

$$\varphi$$

$$G(x)$$

$$\varphi$$

$$\varphi$$

$$X$$

$$A$$

$$B$$

$$c(g)$$
$$g$$
$$180^\circ$$
$$c(g) = 3, \mid B \mid^{c(g)} = 3^3$$
$$g(x) = x$$
$$x$$
$$g$$
$$B$$
$$\mid X^g \mid = \mid B \mid^{c(g)}$$
$$(x, y)$$
$$x$$
$$y$$
$$X$$
$$Y$$
$$X \times Y$$
$$x \in X$$
$$y \in Y$$
$$(x, y)$$
$$A$$
$$B$$
$$B$$
$$A$$
$$\leq$$
$$P$$
$$\leq$$
$$P$$
$$\leq$$
$$a, b$$
$$c$$
$$P$$
$$a \leq a$$
$$a \leq b$$
$$b \leq a$$

$$a = b$$
$$a \leq b$$
$$b \leq c$$
$$a \leq c$$
$$P$$
$$a$$
$$b$$
$$a \leq b$$
$$b \leq a$$
$$X$$
$$\leq$$
$$|$$
$$m$$
$$n$$
$$m$$
$$n$$
$$n \mid m$$
$$1$$
$$m$$
$$a$$
$$m \leq a$$
$$0$$
$$0$$
$$1$$
$$0$$
$$|$$
$$\{2, 3, 4, 5, 6\}$$
$$2$$
$$3$$
$$5$$
$$4$$
$$5$$
$$6$$
$$m$$

$$a \leq m$$
$$a$$
$$a = m$$
$$\leq$$
$$\geq$$
$$5$$
$$\{2, 3, 4, 5, 6\}$$
$$P$$
$$S$$
$$S$$
$$P$$
$$b$$
$$S$$
$$S$$
$$s$$
$$s \leq b$$
$$-5$$
$$S$$
$$\sup(S)$$
$$\bigvee S$$
$$\inf(S)$$
$$\bigwedge S$$
$$x$$
$$y$$
$$\sup(\{x, y\})$$
$$\inf(\{x, y\})$$
$$1$$
$$|$$
$$(a, x) \leq (b, y)$$
$$a \leq b$$
$$x \leq y$$
$$\leq$$

$$a \leq b$$
$$b \leq a$$
$$a < b$$
$$\leq$$
$$<$$
$$a < b$$
$$a = b$$
$$a \leq b$$
$$T_0$$
$$a \leq b$$
$$a$$
$$b$$
$$f$$
$$f(x, y)$$
$$x < y$$
$$f(x, x)$$
$$f(x, y)$$
$$f(y, x)$$
$$f(x, y)$$
$$f(y, z)$$
$$f(x, z)$$
$$f(x, y)$$
$$f(y, x)$$
$$f(y, z)$$
$$f(z, y)$$
$$f(x, z)$$
$$f(z, x)$$
$$<$$
$$x > y$$
$$y < x$$
$$x \leq y$$

$$y \nless x$$

$$x \geq y$$

$$x \nless y$$

$$x = y$$

$$x \nless y$$

$$y \nless x$$

$$x \nless y$$

$$y \nless x$$

$$x$$

$$y$$

$$x \mid y$$

$$x$$

$$y$$

$$x$$

$$y$$

$$x \bmod y$$

$$x$$

$$y$$

$$x \perp y$$

$$x$$

$$y$$

$$\gcd(x, y)$$

$$(x, y)$$

$$\operatorname{lcm}(x, y)$$

$$[x, y]$$

$$\sum$$

$$\sum_{i=1}^{n} i$$

$$1 + 2 + \cdots + n$$

$$i$$

$$i$$

$$i$$

$$1$$
$$n$$
$$i$$
$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$
$$\sum_{S \subseteq T} \mid S \mid$$
$$T$$
$$\sum_{p \leq n, p \perp n} 1$$
$$n$$
$$n$$
$$\varphi(n)$$
$$\varphi$$
$$\prod$$
$$\prod_{i=1}^{n} i$$
$$n$$
$$n!$$
$$\prod_{i=1}^{n} a_i$$
$$a_1 \times a_2 \times a_3 \times \cdots \times a_n$$
$$\prod_{x \mid d} x$$
$$d$$
$$!$$
$$n!$$
$$1 \times 2 \times 3 \times \cdots \times n$$
$$0! = 1$$
$$\lfloor x \rfloor$$
$$x$$
$$\left\lfloor \frac{x}{y} \right\rfloor$$
$$\lceil x \rceil$$

$$x$$

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{Bmatrix} x \\ y \end{Bmatrix}$$

$$x_1$$

$$x_2$$

$$x_3$$

$$x_4$$

$$-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$$

$$5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$$

$$3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 + x_2 + x_3 + x_4$$

$$-2x_1 + 8x_2 + 0x_3 + 10x_4 \geq 50$$

$$5x_1 + 2x_2 + 0x_3 + 0x_4 \geq 100$$

$$3x_1 - 5x_2 + 10x_3 - 2x_4 \geq 25$$

$$x_1, x_2, x_3, x_4 \geq 0$$

$$[a_1..a_n]$$

$$[x_1..x_n]$$

$$f(x_1..x_n) = \sum_{i=1}^{n} a_i x_i$$

$$f(x_1..x_n) = b$$

$$f(x_1..x_n) \leq b$$

$$f(x_1..x_n) \geq b$$

$$x + y$$

$$3x + y \geq 9$$

$$x \leq 4$$

$$y \geq 3$$

$$x, y \in N$$

$$x \leq 4$$

$$y \geq 3$$

$$3x + y \geq 9$$

$$x + y$$

$$(2, 3)$$

$$x$$

$$y$$

$$x + y$$

$$\min(x + y) = (2 + 3) = 5$$

$$\varphi(a \cdot b) = \varphi(a) * \varphi(b)$$

$$gH = \{g \cdot h \mid h \in H\}$$
$$Hg = \{h \cdot g \mid h \in H\}$$

$$G/N = \{gN \mid g \in G\}$$

$$G = \langle S \mid R \rangle$$

$$G$$

$$G$$

$$\cdot$$

$$a$$

$$b$$

$$G$$

$$a \cdot b$$

$$G \neq \emptyset$$

$$G$$

$$\cdot$$

$$(G, \cdot)$$

$$G$$

$$a, b$$

$$a \cdot b$$

$$G$$

$$a, b, c$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$G$$

$$e$$

$$G$$

$$a$$
$$e \cdot a = a \cdot e = a$$
$$G$$
$$a$$
$$G$$
$$b$$
$$a \cdot b = b \cdot a = e$$
$$e$$
$$b$$
$$a$$
$$a^{-1}$$
$$(G, \cdot)$$
$$(\mathbb{Z}, +)$$
$$(G, \cdot)$$
$$(G, \cdot)$$
$$(G, \cdot)$$
$$(G, \cdot)$$
$$(G, \cdot)$$
$$G$$
$$a, b$$
$$a \cdot b = b \cdot a$$
$$(G, \cdot)$$
$$R$$
$$R$$
$$+$$
$$\cdot$$
$$(R, +, \cdot)$$
$$(R, +)$$
$$0$$
$$R$$
$$a$$
$$-a$$
$$(R, \cdot)$$

$$R$$
$$a, b, c$$
$$a \cdot (b + c) = a \cdot b + a \cdot c$$
$$(a + b) \cdot c = a \cdot c + b \cdot c$$
$$R$$
$$R$$
$$R$$
$$1$$
$$R$$
$$R$$
$$0$$
$$a$$
$$a^{-1}$$
$$R$$
$$X = x, y, z$$
$$x, y, z$$
$$X$$
$$x \in X$$
$$x$$
$$X$$
$$f : X \to Y$$
$$f$$
$$X$$
$$Y$$
$$(G, \cdot)$$
$$(H, *)$$
$$\varphi : G \to H$$
$$G$$
$$a$$
$$b$$
$$(G, \cdot), (H, \cdot)$$
$$H \subseteq G$$

$$(H, \cdot)$$

$$(G, \cdot)$$

$$G$$

$$H$$

$$G$$

$$G$$

$$H$$

$$h_1$$

$$h_2$$

$$H$$

$$h_1 \cdot h_2$$

$$h_1^{-1}$$

$$H$$

$$H$$

$$G$$

$$H$$

$$(G, \cdot)$$

$$H$$

$$G$$

$$(H, \cdot)$$

$$(H, \cdot)$$

$$(G, \cdot)$$

$$G$$

$$H$$

$$g, h \in H$$

$$g^{-1} \cdot h \in H$$

$$H$$

$$H$$

$$g$$

$$g$$

$$H$$

$$[G : H]$$

$$G$$

$$H$$

$$g$$

$$b = g^{-1}ag$$

$$a$$

$$b$$

$$g \in G, h \in H$$

$$g^{-1}hg \in H$$

$$H$$

$$G$$

$$H \triangleleft G$$

$$S \subset G$$

$$\langle S \rangle$$

$$G$$

$$S$$

$$G$$

$$S$$

$$S$$

$$G = \langle S \rangle$$

$$S$$

$$G$$

$$S$$

$$S$$

$$x$$

$$\langle S \rangle$$

$$\langle x \rangle$$

$$\langle x \rangle$$

$$x$$

$$\langle a \rangle = \{a^k, k \geq 1\}$$

$$x$$

$$N$$

$$G$$

$$\mathrm{ord}(G)$$

$$\mid G \mid$$

$$G$$

$$a$$

$$a^m = e$$

$$m$$

$$\mathrm{ord}(a)$$

$$\mid a \mid$$

$$\mathrm{ord}(\langle a \rangle)$$

$$a$$

$$Z_n^\times = \{a \in \{0, 1, \cdots, n-1\} \mid \gcd(a, n) = 1\}$$

$$\varphi(n)$$

$$x$$

$$x^r \equiv 1 \pmod n$$

$$r$$

$$x$$

$$n$$

$$H$$

$$G$$

$$\mid G \mid = [G : H] \mid H \mid$$

$$G$$

$$G$$

$$H$$

$$G$$

$$a$$

$$b$$

$$m$$

$$n$$

$$a^s b^t = e$$

$$a^s = e$$

$$b^t = e$$

$$a^s b^t = e$$

$$a^s = e$$

$$b^t = e$$

$$a^s b^t = e$$

$$a^s$$

$$b^t$$

$$e = a^s m = b^{-tm}$$

$$\gcd(-m, n) = 1$$

$$-m$$

$$e = b^t$$

$$G$$

$$\operatorname{lcm}$$

$$D_4$$

$$(\{0, 1, 2, 3\}, \oplus)$$

$$\oplus$$

$$4$$

$$e$$

$$1$$

$$2$$

$$G$$

$$x_1$$

$$x_2$$

$$d_1$$

$$d_2$$

$$G$$

$$d = \operatorname{lcm}(d_1, d_2)$$

$$S_3$$

$$X = \{1, 2, 3\}$$

$$2$$

$$3$$

$$6$$

$$X$$

$$X$$

$$G$$

$$G$$

X

C_n

G

a

$a = g^k$

k

g

G

g

G

G

e

G

g

i

j

$g^i = g^j$

g

0

n

$g^n = e$

$g^0 = e$

g

d

G

a

g

d

m

g

d

e

g

d

m

d

g

i

j

$g^i = g^j$

$g^n = e$

n

0

d

d

$d = m$

m

G

m

Z_m

f

$Z_m \to G$

$f(n) = g^n$

f

i

j

$f(i + j) = g^i g^j$

G

n

K

n

K^n

X

X

X

H

G

$$G/H$$

$$G$$

$$X$$

$$G/H$$

$$X$$

$$\max(A_1(x_1, ..., x_{r-1}), ..., A_{n_A}(x_1, ..., x_{r-1})) \leq x_r \leq \min(B_1(x_1, ..., x_{r-1}), ..., B_{n_B}(x_1, ..., x_{r-1})) \wedge \phi$$

$$\max(A_1(x_1, ..., x_{r-1}), ..., A_{n_A}(x_1, ..., x_{r-1})) \leq \min(B_1(x_1, ..., x_{r-1}), ..., B_{n_B}(x_1, ..., x_{r-1})) \wedge \phi$$

$$\max(A_1(x_1, ..., x_{r-1}), ..., A_{n_A}(x_1, ..., x_{r-1})) \leq \min(B_1(x_1, ..., x_{r-1}), ..., B_{n_B}(x_1, ..., x_{r-1}))$$

$$n$$

$$S$$

$$x_1$$

$$x_r$$

$$r$$

$$x_r$$

$$x_r$$

$$S$$

$$x_r \geq b_i - \sum_{k=1}^{r-1} a_{ik} x_k$$

$$1$$

$$n_A$$

$$n_A$$

$$j$$

$$x_r \geq A_j(x_1, ..., x_{r-1})$$

$$x_r \leq b_i - \sum_{k=1}^{r-1} a_{ik} x_k$$

$$1$$

$$n_B$$

$$n_B$$

$$j$$

$$x_r \leq B_j(x_1, ..., x_{r-1})$$

$$x_r$$

$$\phi$$

$$\exists x_r \ S$$

$$1 \leq i \leq n_A$$

$$1 \leq j \leq n_B$$

$$n_A n_B$$

$$A_i(x_1, ..., x_{r-1}) \leq B_j(x_1, ..., x_{r-1})$$

$$S$$

$$x_r$$

$$(n - n_A - n_B) + n_A n_B$$

$$n_A = n_B = n/2$$

$$n^2/4$$

$$2x - 5y + 4z \leq 10$$

$$3x - 6y + 3z \leq 9$$

$$-x + 5y - 2z \leq -7$$

$$-3x + 2y + 6z \leq 12$$

$$x$$

$$x$$

$$x \leq (10 + 5y - 4z)/2$$

$$x \leq (9 + 6y - 3z)/3$$

$$x \geq 7 + 5y - 2z$$

$$x \geq (-12 + 2y + 6z)/3$$

$$\leq$$

$$\geq$$

$$\leq$$

$$\geq$$

$$2 \times 2$$

$$7 + 5y - 2z \leq (10 + 5y - 4z)/2$$

$$7 + 5y - 2z \leq (9 + 6y - 3z)/3$$

$$(-12 + 2y + 6z)/3 \leq (10 + 5y - 4z)/2$$

$$(-12 + 2y + 6z)/3 \leq (9 + 6y - 3z)/3$$

$$n$$

$$n^2/4$$

$$d$$

$$4(n/4)^{2^d}$$

$$\mid \alpha \mid = \frac{l}{r}$$

$$k\,\mathrm{rad} = \frac{\pi}{180^\circ} n^\circ$$

$$x = \rho \cos\varphi$$
$$y = \rho \sin\varphi$$

$$\rho^2 = x^2 + y^2$$

$$\tan\varphi = \frac{y}{x} \quad (x \neq 0)$$

$$\mathrm{atan2}(y, x) = \begin{cases} \arctan(\frac{y}{x}) & \text{if } x > 0 \\ \arctan(\frac{y}{x}) + \pi & \text{if } y \geq 0, x < 0 \\ \arctan(\frac{y}{x}) - \pi & \text{if } y < 0, x < 0 \\ \pi/2 & \text{if } y > 0, x = 0 \\ -\pi/2 & \text{if } y < 0, x = 0 \\ \text{any} & \text{if } y = 0, x = 0 \end{cases}$$

$$r = \sqrt{\rho^2 + z^2}$$
$$\vartheta = \begin{cases} \arctan\left(\frac{\rho}{z}\right) & \text{if } z > 0 \\ \pi/2 & \text{if } z = 0, \rho \neq 0 \\ \arctan\left(\frac{\rho}{z}\right) + \pi & \text{if } z < 0 \end{cases}$$

$$\rho = r \sin\vartheta$$
$$z = r \cos\vartheta$$

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\vartheta = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$
$$\varphi = \mathrm{atan2}(y, x)$$

$$x = r \sin\vartheta \cos\varphi$$
$$y = r \sin\vartheta \sin\varphi$$
$$z = r \cos\vartheta$$

$$[0, 360^\circ]$$

$$720^\circ$$

$$0^\circ$$

$$360^\circ$$

$$1$$

rad

0

r

α

l

360°

2π

α

$\{\varphi \mid \varphi = \alpha + 2k\pi, k \in \mathbf{Z}\}$

π

τ

2π

τ

2π

2π

2π

π

π

π

π

3.14159265358979310000

π

3.14159265358979360000

π

3.1415926535897932

x

y

x

y

O

O

xOy

x

y

x

y

C

C

x

y

x

y

a, b

C

(a, b)

C

B

A

30°

100

A

$B(50, 50\sqrt{3})$

O

Ox

1

A

O

A

$\mid OA \mid$

ρ

OA

$\angle xOA$

φ

(ρ, φ)

A

(ρ, φ)

$(\rho, \varphi + 2k\pi)\ (k \in \mathbf{Z})$

$(0, \varphi)\ (\varphi \in \mathbf{R})$

$$\rho \geq 0, 0 \leq \varphi < 2\pi$$

$$(\rho, \varphi)$$

$$(\rho, \varphi)$$

$$A(\rho, \varphi)$$

$$(x, y)$$

$$\rho = \sqrt{x^2 + y^2}$$

$$\frac{y}{x}$$

$$\tan \varphi$$

$$\varphi$$

$$x, y$$

$$\varphi = \operatorname{atan2}(y, x)$$

$$(-\pi, \pi]$$

$$O$$

$$O$$

$$\overrightarrow{Ox}, \overrightarrow{Oy}, \overrightarrow{Oz}$$

$$x$$

$$y$$

$$z$$

$$z$$

$$x$$

$$y$$

$$z$$

$$1$$

$$O - xyz$$

$$O$$

$$x$$

$$y$$

$$xOy$$

$$yOz$$

$$zOx$$

$$O - xyz$$

$$M$$

M

x

y

z

x

y

z

P, Q, R

P, Q, R

M

x

y

z

P, Q, R

x

y

z

x, y, z

M

(x, y, z)

(x, y, z)

x

x

P

y

y

Q

z

z

R

P, Q, R

x

y

z

M

M

(x, y, z)

M

(x, y, z)

(x, y, z)

M

$M(x, y, z)$

x

y

z

O

z

(ρ, φ, z)

ρ

φ

z

z

z

(x, y)

(ρ, φ)

φ

ϑ

φ

ϑ

r

(r, ϑ, φ)

ϑ

φ

ϕ

θ

φ

$(0, 0, 0)$

ϑ

$$\operatorname{atan2}$$

$$(0,0,0)$$

$$\vartheta$$

$$\varphi$$

$$z_1 + z_2 = (a+c) + (b+d)\mathrm{i}$$

$$\begin{gathered} z_1 + z_2 = z_2 + z_1 \\ (z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \end{gathered}$$

$$z_1 - z_2 = (a-c) + (b-d)\mathrm{i}$$

$$\begin{aligned} z_1 z_2 &= (a + b\mathrm{i})(c + d\mathrm{i}) \\ &= ac + bc\mathrm{i} + ad\mathrm{i} + bd\mathrm{i}^2 \\ &= (ac - bd) + (bc + ad)\mathrm{i} \end{aligned}$$

$$\begin{aligned} \frac{a + b\mathrm{i}}{c + d\mathrm{i}} &= \frac{(a + b\mathrm{i})(c - d\mathrm{i})}{(c + d\mathrm{i})(c - d\mathrm{i})} \\ &= \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}\mathrm{i} (c + d\mathrm{i} \neq 0) \end{aligned}$$

$$\tan\theta = \frac{y}{x}$$

$$\theta = \operatorname{Arg} z$$

$$-\pi < \arg z \leq \pi$$

$$\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i}\sin x$$

$$\exp z = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)$$

$$\cos z = \frac{\exp(\mathrm{i}z) + \exp(-\mathrm{i}z)}{2}$$

$$\sin z = \frac{\exp(\mathrm{i}z) - \exp(-\mathrm{i}z)}{2\mathrm{i}}$$

$$\cos z = \operatorname{Re}(\mathrm{e}^{\mathrm{i}z})$$

$$\sin z = \operatorname{Im}(\mathrm{e}^{\mathrm{i}z})$$

$$z = x + y\mathrm{i}$$

$$z = r(\cos\theta + \mathrm{i}\sin\theta) = r\exp(\mathrm{i}\theta)$$

$$\begin{aligned} \omega_n^n &= 1 \\ \omega_n^k &= \omega_{2n}^{2k} \\ \omega_{2n}^{k+n} &= -\omega_{2n}^k \end{aligned}$$

$$\{\omega_n^k \mid 0 \leq k < n, \gcd(n,k) = 1\}$$

$$x^2+a=0(a>0)$$

$$x^2+1=0$$

$$x^2-2=0$$

$$x^2+1=0$$

$$\mathrm{i}$$

$$x=\mathrm{i}$$

$$x^2+1=0$$

$$\mathrm{i}^2=-1$$

$$\mathrm{i}$$

$$\mathrm{i}$$

$$b$$

$$\mathrm{i}$$

$$b\mathrm{i}$$

$$a$$

$$b\mathrm{i}$$

$$a+b\mathrm{i}$$

$$\mathrm{i}$$

$$a+b\mathrm{i}(a,b\in\mathbf{R})$$

$$a+b\mathrm{i}$$

$$a,b\in\mathbf{R}$$

$$\mathrm{i}$$

$$\mathbf{C}$$

$$z$$

$$z=a+b\mathrm{i}$$

$$a$$

$$z$$

$$\mathrm{Re}(z)$$

$$b$$

$$z$$

$$\mathrm{Im}(z)$$

$$a,b\in\mathbf{R}$$

$$z$$

$$b=0$$

$$b \neq 0$$

$$a = 0$$

$$b \neq 0$$

$$a + b\mathrm{i}$$

$$z_1 = a + b\mathrm{i}, z_2 = c + d\mathrm{i}$$

$$a = c$$

$$b = d$$

$$(a, b)$$

$$z = a + b\mathrm{i}$$

$$x$$

$$y$$

$$(a, b)$$

$$z = a + b\mathrm{i}$$

$$Z(a, b)$$

$$\overrightarrow{OZ} = (a, b)$$

$$0$$

$$z = a + b\mathrm{i}$$

$$|\ z\ | = \sqrt{a^2 + b^2}$$

$$z = a + b\mathrm{i}$$

$$Z$$

$$\overrightarrow{OZ}$$

$$z_1 = a + b\mathrm{i}, z_2 = c + d\mathrm{i}$$

$$z_1 = a + b\mathrm{i}, z_2 = c + d\mathrm{i}$$

$$\mathrm{i}^2$$

$$-1$$

$$z_1 z_2 = z_2 z_1$$

$$(z_1 z_2) z_3 = z_1 (z_2 z_3)$$

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

$$c - d\mathrm{i}$$

$$z = a + b\mathrm{i}$$

$$a - b\mathrm{i}$$

$$z$$

$$\bar{z}$$

$$z, w$$

$$z \cdot \bar{z} = \mid z \mid^2$$

$$\bar{\bar{z}} = z$$

$$\mathrm{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\mathrm{Im}(z) = \frac{z - \bar{z}}{2}$$

$$\overline{z \pm w} = \bar{z} \pm \bar{w}$$

$$\overline{zw} = \bar{z}\bar{w}$$

$$\overline{z/w} = \bar{z}/\bar{w}$$

$$1$$

$$\mathrm{i}$$

$$z$$

$$(r, \theta)$$

$$r$$

$$z$$

$$z = x + \mathrm{i}y$$

$$\theta$$

$$z$$

$$z$$

$$\operatorname{Arg} z$$

$$\arg z$$

$$\arg z$$

$$2k\pi$$

$$\operatorname{Arg} z = \{\arg z + 2k\pi \mid k \in \mathbf{Z}\}$$

$$1$$

$$1$$

$$x$$

$$z = x + \mathrm{i}y$$

$$f(z) = \mathrm{e}^x(\cos y + \mathrm{i}\sin y)$$

$$f(z_1 + z_2) = f(z_1)f(z_2)$$

$$\mid \exp z \mid = \exp x > 0$$

$$\arg \exp z = y$$

$$\exp(z_1 + z_2) = \exp(z_1) \exp(z_2)$$

$$\exp z$$

$$2\pi\mathrm{i}$$

$$f(z)$$

$$z \in \mathbf{R}$$

$$1$$

$$2\pi$$

$$1$$

$$x^n = 1$$

$$n$$

$$n$$

$$n$$

$$n$$

$$n$$

$$n$$

$$n$$

$$\omega_n = \exp \frac{2\pi\mathrm{i}}{n}$$

$$2\frac{\pi}{n}$$

$$x^n = 1$$

$$\{\omega_n^k \mid k = 0, 1 \cdots, n-1\}$$

$$n$$

$$1$$

$$\omega_n$$

$$\omega_n$$

$$n$$

$$k$$

$$n$$

$$n$$

$$n$$

$$\omega_n$$

$$\omega_n$$

$$\omega_n$$

$$\omega$$

$$\omega_n$$

$$0 < k < n$$

$$\omega$$

$$k$$

$$1$$

$$n$$

$$\varphi(n)$$

$$\mathrm{e}$$

$$\mathrm{e}^{\mathrm{i}\pi}$$

$$5 = (101)_2$$
$$6 = (110)_2$$
$$5 \,\&\, 6 = (100)_2 = 4$$
$$5 \,|\, 6 = (111)_2 = 7$$
$$5 \oplus 6 = (011)_2 = 3$$

$$5 = (00000101)_2$$
$$\text{space.nobreak } 5 = (11111010)_2 = -6$$
$$-5 \text{ □□□} = (11111011)_2$$
$$\text{space.nobreak } (-5) = (00000100)_2 = 4$$

$$11 = (00001011)_2$$
$$11 << 3 = (01011000)_2 = 88$$
$$11 >> 2 = (00000010)_2 = 2$$

$$6$$

$$\&$$

$$\text{and}$$

$$1$$

$$1$$

$$|$$

$$\text{or}$$

$$1$$

$$1$$

$$\oplus$$

$$\text{xor}$$

$$1$$

$$\wedge$$

$$\vee$$

$$a \oplus b \oplus b = a$$

$$num$$

$$num$$

$$0$$

$$1$$

$$0$$

$$1$$

$$1$$

$$0$$

$$0$$

$$num$$

$$i$$

$$num$$

$$i$$

$$0$$

$$0$$

$$1$$

$$0$$

$$0$$

$$0$$

$$0$$

$$-1$$

$$0$$

$$1$$

$$32$$

$$32$$

$$1$$

$$1$$

$$1$$

$$0$$

$$O(n)$$

$$n$$

$$2^n - 1$$

$$2^n - 1$$

$$O(n)$$

$$0 \sim n$$

$$x$$

$$(10110)_2$$

$$(10110)_2$$

$$1$$

$$1$$

$$(11010)_2$$

$$(11010)_2$$

$$1$$

$$1$$

$$1$$

$$010$$

$$001$$

$$(11001)_2$$

$$x$$

$$x$$

$$1$$

$$1$$

$$(10110)_2$$

$$(11000)_2$$

$$1$$

$$t$$

$$x$$

$$1$$

$$1$$

$$x$$

$$x$$

$$1$$

$$(10110)_2$$

$$t = (11000)_2$$

$$\operatorname{lowbit}(t) = (01000)_2$$

$$\operatorname{lowbit}(x) = (00010)_2$$

$$1$$

$$1$$

$$r$$

$$0$$

$$1$$

$$l$$

$$t$$

$$x$$

$$1$$

$$r-l$$

$$1$$

$$1$$

$$\frac{\text{lowbit(t)/2}}{\text{lowbit(x)}} = \frac{(00100)_2}{(00010)_2} = (00010)_2$$

$$1$$

$$0 \sim n$$

$$0$$

$$0$$

$$x$$

$$1$$

$$1$$

$$1$$

$$x$$

$$0$$

$$0$$

$$x$$

$$0$$

$$x$$

$$0$$

$$x$$

$$0$$

$$x$$

$$0$$

$$x$$

$$0$$

$$x$$
$$0$$
$$x$$
$$1$$
$$x$$
$$1$$
$$x$$
$$1$$
$$1$$
$$(1010)_2$$
$$(0010)_2$$
$$\sum_{k=0}^{n}\binom{n}{k}2^n$$
$$0$$
$$1$$
$$\{1,3,4,8\}$$
$$(100011010)_2$$
$$a \cap b$$
$$a \cup b$$
$$\bar{a}$$
$$a \setminus b$$
$$a \bigtriangleup b$$
$$2$$
$$mod-1$$
$$2$$
$$0$$
$$n$$
$$2$$
$$n$$
$$n-1$$
$$0$$
$$n$$
$$n-1$$
$$n$$

1

1

n

$n-1$

n

1

2

n

1

n

2

1

0

0

1

0

m

m

s

0

0

m

s

s

1

s

1

$s-1$

m

1

$(s-1)\&m$

$s-1$

s

$s=0$

$s-1$

-1

1

$(s-1)\&m$

s

m

$s=0$

$\mathrm{popcount}(m)$

m

1

$O(2^{\mathrm{popcount}(m)})$

m

n

$O(3^n)$

n

i

i

m

0

s

0

m

1

s

0

m

s

1

n

3^n

m

k

1

2^k

k

$$\binom{n}{k}$$

$$m$$

$$(1+2)^n$$

$$3^n$$

$$3^{13} = 3^{(1101)_2} = 3^8 \cdot 3^4 \cdot 3^1$$

$$\begin{aligned}
3^1 &= 3 \\
3^2 &= \left(3^1\right)^2 = 3^2 = 9 \\
3^4 &= \left(3^2\right)^2 = 9^2 = 81 \\
3^8 &= \left(3^4\right)^2 = 81^2 = 6561
\end{aligned}$$

$$3^{13} = 6561 \cdot 81 \cdot 3 = 1594323$$

$$n = n_t 2^t + n_{t-1} 2^{t-1} + n_{t-2} 2^{t-2} + \cdots + n_1 2^1 + n_0 2^0$$

$$\begin{aligned}
a^n &= \left(a^{n_t 2^t + \cdots + n_0 2^0}\right) \\
&= a^{n_0 2^0} \times a^{n_1 2^1} \times \cdots \times a^{n_t 2^t}
\end{aligned}$$

$$\begin{aligned}
n &= n_{t \ldots 0} \\
&= 2 \times n_{t \ldots 1} + n_0 \\
&= 2 \times (2 \times n_{t \ldots 2} + n_1) + n_0 \\
&\cdots
\end{aligned}$$

$$\begin{aligned}
a^n &= a^{n_{t \ldots 0}} \\
&= a^{2 \times n_{t \ldots 1} + n_0} = \left(a^{n_{t \ldots 1}}\right)^2 a^{n_0} \\
&= \left(a^{2 \times n_{t \ldots 2} + n_1}\right)^2 a^{n_0} = \left(\left(a^{n_{t \ldots 2}}\right)^2 a^{n_1}\right)^2 a^{n_0} \\
&\cdots
\end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$a \cdot b = \begin{cases} 0 & \text{if } a = 0 \\ 2 \cdot \frac{a}{2} \cdot b & \text{if } a > 0 \text{ and } a \text{ even} \\ 2 \cdot \frac{a-1}{2} \cdot b + b & \text{if } a > 0 \text{ and } a \text{ odd} \end{cases}$$

$$a \times b \bmod m = a \times b - \left\lfloor \frac{ab}{m} \right\rfloor \times m$$

$$a \times b \bmod m = a \times b - \left\lfloor \frac{ab}{m} \right\rfloor \times m = \left(a \times b - \left\lfloor \frac{ab}{m} \right\rfloor \times m \right) \bmod 2^{64}$$

$$\Theta(\log n)$$

$$a^n$$

$$\Theta(n)$$

$$a$$

$$n$$

$$n$$

$$a$$

$$a^n = \underbrace{a \times a \cdots \times a}_{n \,\square\, \mathrm{a}}$$

$$a, n$$

$$a^{b+c} = a^b \cdot a^c,\ a^{2b} = a^b \cdot a^b = (a^b)^2$$

$$n$$

$$n$$

$$\lfloor \log_2 n \rfloor + 1$$

$$a^1, a^2, a^4, a^8, ..., a^{2^{\lfloor \log_2 n \rfloor}}$$

$$\Theta(\log n)$$

$$a^n$$

$$2^k$$

$$3^{13}$$

$$n$$

$$(n_t n_{t-1} \cdots n_1 n_0)_2$$

$$n_i \in \{0, 1\}$$

$$2^i$$

$$2^{i+1}$$

$$\Theta(\log n)$$

$$\Theta(\log n)$$

$$2^k$$

$$\Theta(\log n)$$

$$2^{i+1}$$

$$2^i$$

$$a^{2^i}$$

$$n_{t\ldots i} = (n_t n_{t-1} \cdots n_i)_2$$

$$n_{t\ldots i}$$

$$n$$

$$t - i + 1$$

$$a^{n_{t\ldots i}} = (a^{n_{t\ldots(i+1)}})^2 a^{n_i}$$

$$\Theta(\log n)$$

$$x^n \bmod m$$

$$m$$

$$x^{n \bmod (m-1)}$$

$$n$$

$$F_n$$

$$F_n = F_{n-1} + F_{n-2}$$

$$2 \times 2$$

$$F_i, F_{i+1}$$

$$F_{i+1}, F_{i+2}$$

$$n$$

$$\Theta(\log n)$$

$$n$$

$$k$$

$$k$$

$$n$$

$$O(n \log k)$$

$$k$$

$$k$$

$$O(n)$$
$$n$$
$$p_i$$
$$m$$
$$k$$
$$O(n \cdot length)$$
$$n$$
$$length$$
$$4 \times 4$$
$$x$$
$$5$$
$$y$$
$$7$$
$$z$$
$$9$$
$$x$$
$$y, z$$
$$x$$
$$\theta$$
$$O(m \log k)$$
$$n$$
$$O(n + m \log k)$$
$$u, v$$
$$u$$
$$v$$
$$k$$
$$M_{i,j}$$
$$i$$
$$j$$
$$k$$
$$O(n^3 \log k)$$
$$a \times b \bmod m,\ a, b \leq m \leq 10^{18}$$
$$O(\log_2 m)$$

$$O(\log_2 m)$$

$$O(1)$$

$$\left\lfloor \frac{ab}{m} \right\rfloor$$

$$\left\lfloor \frac{ab}{m} \right\rfloor$$

$$\frac{a}{m}$$

$$b$$

$$80$$

$$1$$

$$15$$

$$64$$

$$64$$

$$\frac{a}{m}$$

$$65$$

$$(-2^{-64}, 2^{64})$$

$$b$$

$$64$$

$$(-0.5, 0.5)$$

$$0.5$$

$$(0, 1)$$

$$\{0, 1\}$$

$$-m$$

$$\{0, -m\}$$

$$m$$

$$r$$

$$[0, m)$$

$$r$$

$$r + m$$

$$(r + m) \bmod m$$

$$P$$

$$1000 < P < 3100000$$

$$2^P - 1$$

$$O(\sqrt{n})$$

$$O(1)$$

$$s$$

$$a^0$$

$$a^s$$

$$a^{0 \cdot s}$$

$$a^{\lceil \frac{p}{s} \rceil \cdot s}$$

$$a^b \bmod p$$

$$b$$

$$\left\lfloor \frac{b}{s} \right\rfloor \cdot s + b \bmod s$$

$$a^b = a^{\lfloor \frac{b}{s} \rfloor \cdot s} \times a^{b \bmod s}$$

$$O(1)$$

$$s$$

$$\sqrt{p}$$

$$2$$

$$\sqrt{p}$$

$$2$$

$$\begin{aligned} x &= x_1 \cdot 10^m + x_0, \\ y &= y_1 \cdot 10^m + y_0, \\ x \cdot y &= z_2 \cdot 10^{2m} + z_1 \cdot 10^m + z_0, \end{aligned}$$

$$\begin{aligned} z_2 &= x_1 \cdot y_1, \\ z_1 &= x_1 \cdot y_0 + x_0 \cdot y_1, \\ z_0 &= x_0 \cdot y_0. \end{aligned}$$

$$z_1 = (x_1 + x_0) \cdot (y_1 + y_0) - z_2 - z_0,$$

$$1000$$

$$0$$

$$10$$

$$1$$

$$10$$

$$1$$

$$a$$

$$b$$

$$a < b$$

$$a - b = -(b - a)$$

$$b - a$$

$$b > a$$

$$a - b$$

$$a$$

$$b$$

$$9$$

$$9b$$

$$-10$$

$$10$$

$$1337$$

$$42$$

$$b$$

$$10^9$$

$$a \times b_i \times 10^i$$

$$1337 \times 42$$

$$1337 \times 2 \times 10^0 + 1337 \times 4 \times 10^1$$

$$b$$

$$a$$

$$45$$

$$12$$

$$12$$

$$3$$

$$l_a$$

$$l_b$$

$$l_a - l_b$$

$$8192 \times 42$$

$$(8000 + 100 + 90 + 2) \times (40 + 2)$$

$$(8100 + 92) \times 42$$

$$100$$

$$a_4, a_3, a_2, a_1$$

$$b_3, b_2, b_1$$

$$base$$

$$\frac{a_4 base + a_3}{b_3 + b_2 base^{-1} + (b_1 + 1) base^{-2}}$$

$$q - 1 \leq q' \leq q$$

$$base^3$$

$$base^3 < 2^{53}$$

$$395081/9876$$

$$3950/988 = 3$$

$$395081 - (9876 \times 3 \times 10^1) = 98801$$

$$9880/988 = 10$$

$$98801 - (9876 \times 10 \times 10^0) = 41$$

$$3 \times 10^1 + 10 \times 10^0 = 40$$

$$n$$

$$O(n^2)$$

$$x$$

$$y$$

$$n$$

$$0 < m < n$$

$$x_0, y_0, z_0, z_1 < 10^m$$

$$z_1$$

$$(x_1 + x_0) \cdot (y_1 + y_0)$$

$$z_0$$

$$z_2$$

$$n$$

$$3$$

$$m = \left\lceil \frac{n}{2} \right\rceil$$

$$n$$

$$T(n)$$

$$T(n) = 3 \cdot T\left(\left\lceil \frac{n}{2} \right\rceil\right) + O(n)$$

$$T(n) = \Theta(n^{\log_2 3}) \approx \Theta(n^{1.585})$$

$$b$$

$$n \cdot b^2$$

$$x_1 + x_0$$

$$y_1 + y_0$$

$$2 \cdot b^m$$

$$x_1 - x_0$$

$$10^{10^5}$$

$$n$$

$$a$$

$$10$$

$$A = a_0 10^0 + a_1 10^1 + \cdots + a_{n-1} 10^{n-1}$$

$$O(n^2)$$

$$O(n \log n)$$

$$n$$

$$m$$

$$O(nm)$$

$$m = O(n)$$

$$O(n^2)$$

$$\{a_0 \dots a_{n-1}\}$$

$$\{r_0 \dots r_m\}$$

$$\sum_{j=0}^{m} r_j a_{i-j} = 0, \forall i \geq m$$

$$r_0 = 1$$

$$m$$

$$\{a_i\}$$

$$\{f_0 \dots f_{m-1}\}$$

$$a_i = \sum_{j=0}^{m-1} f_j a_{i-j-1}, \forall i \geq m$$

$$f_i = -r_{i+1}$$

$$m$$

$$\{a_i\}$$

$$\{f_i\}$$

$$i$$

$$F_i = \{f_{i,j}\}$$

$$F_0 = \{\}$$

$$F_{i-1}$$

$$\{a_i\}$$

$$i-1$$

$$i$$

$$a_i$$

$$F_i = F_{i-1}$$

$$a_i$$

$$F_{i-1}$$

$$F_i$$

$$\Delta_i = a_i - \sum_{j=0}^{m} f_{i-1,j} a_{i-j-1}$$

$$a_i$$

$$F_{i-1}$$

$$a_i$$

$$F_i$$

$$i$$

$$0$$

$$\{a_i\}$$

$$k$$

$$G = \{g_0 ... g_{m\prime-1}\}$$

$$\sum_{j=0}^{m\prime-1} g_j a_{i\prime-j-1} = 0, \forall i' \in [m', i)$$

$$\sum_{j=0}^{m\prime-1} g_j a_{i-j-1} = \Delta_i$$

$$F_k$$

$$G$$

$$F_i$$

$$G$$

$$G = \{0, 0, ..., 0, \frac{\Delta_i}{\Delta_k}, -\frac{\Delta_i}{\Delta_k} F_k\}$$

$$i - k - 1$$

$$0$$

$$-\frac{\Delta_i}{\Delta_k} F_k$$

$$F_k$$

$$-\frac{\Delta_i}{\Delta_k}$$

$$\sum_{j=0}^{m\prime-1} g_j a_{i-j-1} = \Delta_k \frac{\Delta_i}{\Delta_k} = \Delta_i$$

$$G$$

$$F_i$$

$$F_k$$

$$G$$

$$\{r_i\}$$

$$\{f_j\}$$

$$r_0 = 1$$

$$m$$

$$O(nm)$$

$$m = O(n)$$

$$O(n^2)$$

$$F_k$$

$$O(n)$$

$$2m$$

$$p$$

$$\boldsymbol{v}_i$$

$$n$$

$$n$$

$$\mathbf{u}^T$$

$$\{\boldsymbol{u}^T \boldsymbol{v}_i\}$$

$$1 - \frac{n}{p}$$

$$\{A_i\}$$

$$n \times m$$

$$1 \times n$$

$$\mathbf{u}^T$$

$$m \times 1$$

$$\boldsymbol{v}$$

$$\{\boldsymbol{u}^T A_i \boldsymbol{v}\}$$

$$1 - \frac{n+m}{p}$$

$$\boldsymbol{f}_i$$

$$n$$

$$\boldsymbol{f}_i = A\boldsymbol{f}_{i-1}$$

$$\{\boldsymbol{f}_i\}$$

$$n$$

$$\boldsymbol{f}_0 \cdots \boldsymbol{f}_{2n-1}$$

$$\{\boldsymbol{f}_i\}$$

$$\boldsymbol{f}_m$$

$$O(n^3 + n \log n \log m)$$

$$A$$

$$k$$

$$O(nk + n \log n \log m)$$

$$O(nk)$$

$$O(n^2 \log m)$$

$$A$$

$$f(A) = 0$$

$$f$$

$$\{A^i\}$$

$$A$$

$$n$$

$$n$$

$$A^i$$

$$O(n^4)$$

$$\{\boldsymbol{u}^T A^i \boldsymbol{v}\}$$

$$A^i \boldsymbol{v}$$

$$A$$

$$k$$

$$O(kn + n^2)$$

$$A$$

$$(-1)^n$$

$$A$$

$$B$$

$$AB$$

$$1 - \frac{2n^2 - n}{p}$$

$$\det B$$

$$A$$

$$n$$

$$k$$

$$O(kn + n^2)$$

$$A$$

$$n \times m$$

$$n \times n$$

$$P$$

$$m \times m$$

$$Q$$

$$QAPA^T Q$$

$$QAPA^T Q$$

$$x$$

$$A$$

$$k$$

$$n \leq m$$

$$O(kn + n^2)$$

$$A\mathbf{x} = \mathbf{b}$$

$$A$$

$$n \times n$$

$$\mathbf{b}$$

$$\mathbf{x}$$

$$1 \times n$$

$$A, \mathbf{b}$$

$$n^\omega$$

$$x$$

$$\mathbf{x} = A^{-1}\mathbf{b}$$

$$\{A^i\mathbf{b}\}$$

$$i \geq 0$$

$$\{r_0 ... r_{m-1}\}$$

$$m \leq n$$

$$A^{-1}\mathbf{b} = -\frac{1}{r_{m-1}} \sum_{i=0}^{m-2} A^i \mathbf{b} r_{m-2-i}$$

$$A$$

$$\mathbf{b} ... A^{2n-1}\mathbf{b}$$

$$A$$

$$k$$

$$O(kn + n^2)$$

$$0, 1$$

$$0, 1, 2, 3, 4, 5, 6, 7$$

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A(10), B(11), C(12), D(13), E(14), F(15)$$

$$33.25 = (100001.01)_2$$

$$2^i$$

$$i$$

$$(11010.01)_2 = (26.25)_{10}$$

$$2^3 = 8$$

$$2^4 = 16$$

$$64_{10} = 02101_3$$

$$\mathtt{1Z101} = 81 \times 1 + 27 \times (-1) + 9 \times 1 + 3 \times 0 + 1 \times 1 = 64_{10}$$

$$237_{10} = 22210_3$$

$$\mathtt{100Z10} = 243 \cdot 1 + 81 \cdot 0 + 27 \cdot 0 + 9 \cdot (-1) + 3 \cdot 1 + 1 \cdot 0 = 237_{10}$$

$$X_3$$
$$x_i$$
$$3^i$$
$$Y_{10}$$
$$Y_{10}$$
$$Y_{10}$$
$$A_3, B_3$$
$$Y_{10}$$
$$A_3 = B_3$$
$$Y_{10} = 0$$
$$A_3 = B_3 = 0_3$$
$$Y_{10} > 0$$
$$A_3$$
$$B_3$$
$$a_i$$
$$A_3$$
$$i$$
$$b_i$$
$$B$$
$$i$$
$$A_3, B_3$$
$$i$$
$$a_i \neq b_i$$
$$i-1, i-2, ..., 0$$
$$A_3, B_3$$
$$i$$
$$A_{(3)'}, B_{(3)'}$$
$$A_{(3)'} = B_{(3)'}$$
$$A_{(3)'}, B_{(3)'}$$
$$0$$

$$a_0 \neq b_0$$

$$b_0 > a_0$$

$$a_0 > b_0$$

$$b_0 - a_0 \in \{1, 2\}$$

$$A_{(3)'}$$

$$i = 1, 2, 3, ...$$

$$A_{(3)'}$$

$$S_1 = a_1 \times 3^1 + a_2 \times 3^2 + ...$$

$$B_{(3)'}$$

$$i = 1, 2, 3, ...$$

$$B_{(3)'}$$

$$S_2 = b_1 \times 3^1 + b_2 \times 3^2 + ...$$

$$A_{(3)'} = B_{(3)'}$$

$$S_1 - S_2 = b_0 - a_0$$

$$S_1, S_2$$

$$3$$

$$b_0 - a_0$$

$$3$$

$$A_{(3)'} \neq B_{(3)'}$$

$$Y_{10} < 0$$

$$Y_{10} > 0$$

$$Y_{10}$$

$$X_3$$

$$1$$

$$1$$

$$8$$

$$16$$

$$32$$

$$16$$

$$32$$

64

8

8

≥ 16

16

≥ 16

16

≥ 16

32

16

≥ 16

32

16

≥ 32

32

64

≥ 32

32

64

≥ 64

64

≥ 64

64

x

$-2^{x-1} \sim 2^{x-1} - 1$

$0 \sim 2^x - 1$

8

$-2^7 \sim 2^7 - 1$

$0 \sim 2^8 - 1$

16

$-2^{15} \sim 2^{15} - 1$

$0 \sim 2^{16} - 1$

32

$$-2^{31} \sim 2^{31}-1$$

$$0 \sim 2^{32}-1$$

$$64$$

$$-2^{63} \sim 2^{63}-1$$

$$0 \sim 2^{64}-1$$

$$8$$

$$-128 \sim 127$$

$$0 \sim 255$$

$$32$$

$$3.4 \times 10^{38}$$

$$6 \sim 9$$

$$64$$

$$1.8 \times 10^{308}$$

$$15 \sim 17$$

$$\geq 80$$

$$\geq 1.2 \times 10^{4932}$$

$$\geq 18 \sim 21$$

$$128$$

$$1.2 \times 10^{4932}$$

$$33 \sim 36$$

$$N$$

$$N$$

$$N$$

$$x$$

$$\bmod 2^x$$

$$x$$

$$\bmod 2^x$$

$$0$$

$$g$$

$$10$$

$$n(1 \leq n \leq 10^5)$$

$$m(1 \leq m \leq 2 \times 10^5)$$

$$s$$

$$s$$
$$a^b$$
$$a^b$$
$$O(1)$$
$$x$$
$$y$$
$$n$$
$$\lambda$$
$$this^e$$
$$\frac{3}{2}$$
$$O(1)$$
$$O(1)$$
$$O(n^2)$$
$$O(n \log n)$$
$$n$$
$$O(\log_n)$$
$$\Delta < 0$$
$$\Delta = 0$$
$$n$$
$$n$$
$$n \leq 1000$$
$$2^{31} - 1$$
$$idx$$
$$0 \leq idx < size$$
$$n$$
$$m$$
$$n, m \leq 1000$$
$$n$$
$$m$$
$$p \wedge q$$
$$p$$
$$q$$
$$p$$

$$q$$

$$p \vee q$$

$$p$$

$$q$$

$$p$$

$$q$$

$$p$$

$$q$$

$$p \vee q$$

$$\neg p$$

$$p$$

$$p$$

$$p \implies q$$

$$p$$

$$q$$

$$p$$

$$q$$

$$q \impliedby p$$

$$p \implies q$$

$$p \iff q$$

$$p$$

$$q$$

$$(p \implies q) \wedge (q \implies p)$$

$$p \iff q$$

$$(\forall\ x \in A) \quad p(x)$$

$$A$$

$$x$$

$$p(x)$$

$$A$$

$$(\forall\ x) \quad p(x)$$

$$\forall$$

$$x \in A$$

$$(\exists\ x \in A) \quad p(x)$$

A

x

$p(x)$

A

$(\exists\ x)\ \ p(x)$

$\exists$

$x \in A$

$(\exists!\ x)\ \ p(x)$

x

$p(x)$

$\exists!$

$\exists^1$

$x \in A$

x

A

x

A

$A \ni x$

$x \in A$

$y \notin A$

y

A

y

A

$\{x_1, x_2, ..., x_n\}$

$x_1, x_2, ..., x_n$

$\{x_i \ \ | \ \ i \in I\}$

I

$\{x \in A \ \ | \ \ p(x)\}$

A

$p(x)$

$\{x \in \boldsymbol{R} \ \ | \ \ x \geq 5\}$

$$A$$

$$\{x \ \mid\ p(x)\}$$

$$\{x \ \mid\ x \geq 5\}$$

$$\mid$$

$$\{x \in A : p(x)\}$$

$$\operatorname{card} A$$

$$\mid A \mid$$

$$A$$

$$A$$

$$\emptyset$$

$$\emptyset$$

$$B \subseteq A$$

$$B$$

$$A$$

$$B$$

$$A$$

$$B$$

$$A$$

$$\subset$$

$$A \supseteq B$$

$$B \subseteq A$$

$$B \subset A$$

$$B$$

$$A$$

$$B$$

$$A$$

$$B$$

$$A$$

$$A$$

$$B$$

$$\subset$$

$$\subsetneq$$

$$A \supset B$$

$$B \subset A$$

$$A \cup B$$

$$A$$

$$B$$

$$A \cup B := \{x \ \mid \ x \in A \vee x \in B\}$$

$$:=$$

$$A \cap B$$

$$A$$

$$B$$

$$A \cap B := \{x \ \mid \ x \in A \wedge x \in B\}$$

$$:=$$

$$\bigcup_{i=1}^{n} A_i$$

$$A_1, A_2, ..., A_n$$

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup ... \cup A_n$$

$$\textstyle\bigcup_{i=1}^{n}$$

$$\bigcup_{i \in I}$$

$$\textstyle\bigcup_{i \in I}$$

$$I$$

$$\bigcap_{i=1}^{n} A_i$$

$$A_1, A_2, ..., A_n$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap ... \cap A_n$$

$$\textstyle\bigcap_{i=1}^{n}$$

$$\bigcap_{i \in I}$$

$$\textstyle\bigcap_{i \in I}$$

$$I$$

$$A \setminus B$$

$$A$$

$$B$$

$$A \setminus B = \{x \ \mid \ x \in A \land x \notin B\}$$

$$A - B$$

$$B$$

$$A$$

$$\complement_A B$$

$$A$$

$$A$$

$$\overline{B}$$

$$B$$

$$(a, b)$$

$$a$$

$$b$$

$$a$$

$$b$$

$$(a, b) = (c, d)$$

$$a = c$$

$$b = d$$

$$(a_1, a_2, ..., a_n)$$

$$n$$

$$A \times B$$

$$A$$

$$B$$

$$A \times B = \{(x, y) \ \mid \ x \in A \land y \in B\}$$

$$\prod_{i=1}^{n} A_i$$

$$A_1, A_2, ..., A_n$$

$$\prod_{i=1}^{n} A_i = \{(x_1, x_2, ..., x_n) \ \mid \ x_1 \in A_1, x_2 \in A_2, ..., x_n \in A_n\}$$

$$A \times A \times ... \times A$$

$$A^n$$

$$n$$

$$\mathrm{id}_A$$

$$A \times A$$

$$\mathrm{id}_A = \{(x, x) \;\mid\; x \in A\}$$

$$A$$

$$A$$

$$\mathbf{N}$$

$$\mathbf{N} = \{0, 1, 2, 3, ...\}$$

$$\mathbf{N}^* = \mathbf{N}_+ = \{1, 2, 3, ...\}$$

$$\mathbf{N}_{>5} = \{n \in \mathbf{N} \;\mid\; n > 5\}$$

$$\mathbb{N}$$

$$\mathbf{Z}$$

$$\mathbf{Z}^* = \mathbf{Z}_+ = \{n \in \mathbf{Z} \;\mid\; n \neq 0\}$$

$$\mathbf{Z}_{>-3} = \{n \in \mathbf{Z} \;\mid\; n > -3\}$$

$$\mathbb{Z}$$

$$\mathbf{Q}$$

$$\mathbf{Q}^* = \mathbf{Q}_+ = \{r \in \mathbf{Q} \;\mid\; r \neq 0\}$$

$$\mathbf{Q}_{<0} = \{r \in \mathbf{Q} \;\mid\; r < 0\}$$

$$\mathbb{Q}$$

$$\mathbf{R}$$

$$\mathbf{R}^* = \mathbf{R}_+ = \{x \in \mathbf{R} \;\mid\; x \neq 0\}$$

$$\mathbf{R}_{>0} = \{x \in \mathbf{R} \;\mid\; x > 0\}$$

$$\mathbb{R}$$

$$\mathbf{C}$$

$$\mathbf{C}^* = \mathbf{C}_+ = \{z \in \mathbf{C} \;\mid\; z \neq 0\}$$

$$\mathbb{C}$$

$$\mathbf{P}$$

$$\mathbf{P} = \{2, 3, 5, 7, 11, 13, 17, ...\}$$

$$\mathbb{P}$$

$$[a, b]$$

$$a$$

$$b$$

$$[a,b] = \{x \in \mathbf{R} \ \mid \ a \leq x \leq b\}$$

$$(a,b]$$

$$a$$

$$b$$

$$(a,b] = \{x \in \mathbf{R} \ \mid \ a < x \leq b\}$$

$$(-\infty,b] = \{x \in \mathbf{R} \ \mid \ x \leq b\}$$

$$[a,b)$$

$$a$$

$$b$$

$$[a,b) = \{x \in \mathbf{R} \ \mid \ a \leq x < b\}$$

$$[a,+\infty) = \{x \in \mathbf{R} \ \mid \ a \leq x\}$$

$$(a,b)$$

$$a$$

$$b$$

$$(a,b) = \{x \in \mathbf{R} \ \mid \ a < x < b\}$$

$$(-\infty,b) = \{x \in \mathbf{R} \ \mid \ x < b\}$$

$$(a,+\infty) = \{x \in \mathbf{R} \ \mid \ a < x\}$$

$$a = b$$

$$a$$

$$b$$

$$\equiv$$

$$a \neq b$$

$$a$$

$$b$$

$$a := b$$

$$a$$

$$b$$

$$a \approx b$$

$$a$$

$$b$$

$$a \simeq b$$

$$a$$

$$b$$

$$x \to a$$

$$\frac{1}{\sin(x-a)} \simeq \frac{1}{x-a}$$

$$x \to a$$

$$a \propto b$$

$$a$$

$$b$$

$$a \sim b$$

$$\sim$$

$$M \cong N$$

$$M$$

$$N$$

$$M$$

$$N$$

$$a < b$$

$$a$$

$$b$$

$$b > a$$

$$b$$

$$a$$

$$a \leq b$$

$$a$$

$$b$$

$$b \geq a$$

$$b$$

$$a$$

$$a \ll b$$

$$a$$

$$b$$

$$b \gg a$$

$$b$$

$$a$$

$$\infty$$

$$+\infty$$

$$-\infty$$

$$x \to a$$

$$x$$

$$a$$

$$a$$

$$\infty$$

$$+\infty$$

$$-\infty$$

$$m \mid n$$

$$m$$

$$n$$

$$m$$

$$n$$

$$(\exists\, k \in \mathbf{Z}) \quad m \cdot k = n$$

$$m \perp n$$

$$m$$

$$n$$

$$m$$

$$n$$

$$(\nexists\, k \in \mathbf{Z}_{>1}) \quad (k \mid m) \wedge (k \mid n)$$

$$n \equiv k \pmod{m}$$

$$n$$

$$m$$

$$k$$

$$n$$

$$k$$

$$m$$

$$m \mid (n - k)$$

$$\|$$

$$\perp$$

$$\angle$$

$$\overline{\mathrm{AB}}$$

$$\mathrm{AB}$$

$$\overrightarrow{\mathrm{AB}}$$

$$\mathrm{AB}$$

$$d(\mathrm{A}, \mathrm{B})$$

$$\mathrm{A}$$

$$\mathrm{B}$$

$$\overline{\mathrm{AB}}$$

$$a + b$$

$$a$$

$$b$$

$$a - b$$

$$a$$

$$b$$

$$a \pm b$$

$$a$$

$$b$$

$$a \mp b$$

$$a$$

$$b$$

$$-(a \pm b) = -a \mp b$$

$$a \cdot b$$

$$a \times b$$

$$ab$$

$$a$$

$$b$$

$$\times$$

$$\frac{a}{b}$$

$$a/b$$

$$a : b$$

$$a$$

$$b$$

$$\frac{a}{b} = a \cdot b^{-1}$$

$$:$$

$$\div$$

$$\sum_{i=1}^{n} a_i$$

$$a_1 + a_2 + ... + a_n$$

$$\textstyle\sum_{i=1}^{n} a_i$$

$$\sum_{i} a_i$$

$$\textstyle\sum_{i} a_i$$

$$\sum a_i$$

$$\prod_{i=1}^{n} a_i$$

$$a_1 \cdot a_2 \cdot ... \cdot a_n$$

$$\textstyle\prod_{i=1}^{n} a_i$$

$$\prod_{i} a_i$$

$$\textstyle\prod_{i} a_i$$

$$\prod a_i$$

$$a^p$$

$$a$$

$$p$$

$$a^{1/2}$$

$$\sqrt{a}$$

$$a$$

$$1/2$$

$$a$$

$$\surd a$$

$$a^{1/n}$$

$$\sqrt[n]{a}$$

$$a$$

$$1/n$$

$$a$$

$$n$$

$$\sqrt[n]{\vphantom{}}a$$

$$\bar{x}$$

$$\bar{x}_a$$

$$x$$

$$\bar{x}_h$$

$$\bar{x}_g$$

$$\bar{x}_q$$

$$\bar{x}_{rms}$$

$$\bar{x}$$

$$x$$

$$\operatorname{sgn} a$$

$$a$$

$$a$$

$$\operatorname{sgn} a = 1 \quad (a > 0)$$

$$\operatorname{sgn} a = -1 \quad (a < 0)$$

$$\operatorname{sgn} 0 = 0$$

$$\inf M$$

$$M$$

$$M$$

$$\sup M$$

$$M$$

$$M$$

$$\mid a \mid$$

$$a$$

$$\operatorname{abs} a$$

$$\lfloor a \rfloor$$

$$a$$

$$\lfloor 2.4 \rfloor = 2$$

$$\lfloor -2.4 \rfloor = -3$$

$$\lceil a \rceil$$

$$a$$

$$\lceil 2.4 \rceil = 3$$

$$\lceil -2.4 \rceil = -2$$

$$\min(a, b)$$

$$\min\{a,b\}$$

$$a$$

$$b$$

$$\inf$$

$$\max(a,b)$$

$$\max\{a,b\}$$

$$a$$

$$b$$

$$\sup$$

$$n \bmod m$$

$$n$$

$$m$$

$$n$$

$$m$$

$$(\exists\; q \in \mathbf{N}, r \in [0,m)) \quad n = qm + r$$

$$r = n \bmod m$$

$$\gcd(a,b)$$

$$\gcd\{a,b\}$$

$$a$$

$$b$$

$$(a,b)$$

$$\mathrm{lcm}(a,b)$$

$$\mathrm{lcm}\{a,b\}$$

$$a$$

$$b$$

$$[a,b]$$

$$(a,b)[a,b] = \mid ab \mid$$

$$n$$

$$k$$

$$a$$

$$k \leq n$$

$$n!$$

$$n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot 3 \cdot ... \cdot n \quad (n > 0)$$

$$0! = 1$$

$$a^{\underline{k}}$$

$$a^{\underline{k}} = a \cdot (a-1) \cdot ... \cdot (a-k+1) \quad (k > 0)$$

$$a^{\underline{0}} = 1$$

$$n^{\underline{k}} = \frac{n!}{(n-k)!}$$

$$a^{\overline{k}}$$

$$a^{\overline{k}} = a \cdot (a+1) \cdot ... \cdot (a+k-1) \quad (k > 0)$$

$$a^{\overline{0}} = 1$$

$$n^{\overline{k}} = \frac{(n+k-1)!}{(n-1)!}$$

$$\binom{n}{k}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = n \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

$$x^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix}$$

$$\begin{Bmatrix} n \\ k \end{Bmatrix} = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (k-i)^n$$

$$\sum_{k=0}^{n} \begin{Bmatrix} n \\ k \end{Bmatrix} x^{\underline{k}} = x^n$$

$$f$$

$$f(x)$$

$$f(x_1, ..., x_n)$$

$$f$$

$$x$$

$$f$$

$$(x_1, ..., x_n)$$

$$\operatorname{dom} f$$

$$f$$

$$\mathrm{D}(f)$$

$$\operatorname{ran} f$$

$$f$$

$$\mathrm{R}(f)$$

$$f : A \to B$$

$$f$$

$$A$$

$$B$$

$$\operatorname{dom} f = A$$

$$(\forall\ x \in \operatorname{dom} f) \quad f(x) \in B$$

$$x \mapsto T(x), x \in A$$

$$x \in A$$

$$T(x)$$

$$T(x)$$

$$x \in A$$

$$f$$

$$x \in A$$

$$f(x) = T(x)$$

$$T(x)$$

$$f$$

$$x \mapsto 3x^2y, x \in [0, 2]$$

$$3x^2y$$

$$x$$

$$3x^2y$$

$$f^{-1}$$

$$f$$

$$f$$

$$f^{-1}$$

$$f$$

$$f$$

$$\operatorname{dom}(f^{-1}) = \operatorname{ran} f$$

$$\operatorname{ran}(f^{-1}) = \operatorname{dom} f$$

$$(\forall\ x \in \operatorname{dom} f) \quad f^{-1}(f(x)) = x$$

$$f(x)^{-1}$$

$$g \circ f$$

$$f$$

$$g$$

$$(g \circ f)(x) = g(f(x))$$

$$f : x \mapsto y$$

$$f(x) = y$$

$$f$$

$$x$$

$$y$$

$$f\ |_a^b$$

$$f(..., u, ...)|_{u=a}^{u=b}$$

$$f(b) - f(a)$$

$$f(..., b, ...) - f(..., a, ...)$$

$$\lim_{x \to a} f(x)$$

$$\lim\nolimits_{x \to a} f(x)$$

$$x$$

$$a$$

$$f(x)$$

$$\lim\nolimits_{x \to a} f(x) = b$$

$$f(x) \to b \quad (x \to a)$$

$$\lim\nolimits_{x \to a+} f(x)$$

$$\lim\nolimits_{x \to a-} f(x)$$

$$f(x) = O(g(x))$$

$$|\ f(x)/g(x)|$$

$$f(x)$$

$$g(x)$$

$$f/g$$

$$g/f$$

$$f$$

$$g$$

$$=$$

$$\sin x = O(x) \quad (x \to 0)$$

$$f(x) = o(g(x))$$

$$f(x)/g(x) \to 0$$

$$f(x)$$

$$g(x)$$

$$=$$

$$\cos x = 1 + o(x) \quad (x \to 0)$$

$$\Delta f$$

$$f$$

$$\Delta x = x_2 - x_1$$

$$\Delta f(x) = f(x_2) - f(x_1)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x}$$

$$f'$$

$$f$$

$$x$$

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

$$f'(x)$$

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)_{x=a}$$

$$f'(a)$$

$$f$$

$$a$$

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n}$$

$$f^{(n)}$$

$$f$$

$$x$$

$$n$$

$$\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n}$$

$$f^{(n)}(x)$$

$$f''$$

$$f'''$$

$$f^{(2)}$$

$$f^{(3)}$$

$$\frac{\partial f}{\partial x}$$

$$f_x$$

$$f$$

$$x$$

$$\frac{\partial f(x, y, ...)}{\partial x}$$

$$f_x(x, y, ...)$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial(f_1, ..., f_m)}{\partial(x_1, ..., x_n)}$$

$$\mathrm{d}f$$

$$f$$

$$\mathrm{d}f(x, y, ...) = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y + ...$$

$$\delta f$$

$$f$$

$$\int f(x)\mathrm{d}x$$

$$f$$

$$\int\limits_a^b f(x)\mathrm{d}x$$

$$f$$

$$a$$

$$b$$

$$\int_a^b f(x)\mathrm{d}x$$

$$\int\limits_C$$

$$\int\limits_S$$

$$\int\limits_V$$

$$\oint$$

$$C$$

$$S$$

$$V$$

$$\iint$$

$$\iiint$$

$$f*g$$

$$f$$

$$g$$

$$(f*g)(x) = \int\limits_{-\infty}^{\infty} f(y)g(x-y)\mathrm{d}y$$

$$\frac{\partial(f_1,...,f_m)}{\partial(x_1,...,x_n)} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

$$x$$

$$\mathrm{e}$$

$$\mathrm{e} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.718\ 281\ 8...$$

$$e$$

$$a^x$$

$$x$$

$$a$$

$$\mathrm{e}^x$$

$$\exp x$$

$$x$$

$$\mathrm{e}$$

$$\log_a x$$

$$x$$

$$a$$

$$\log x$$

$$\log x$$

$$\ln x$$

$$\lg x$$

$$\mathrm{lb}\, x$$

$$\ln x$$

$$x$$

$$\ln x = \log_{\mathrm{e}} x$$

$$\lg x$$

$$x$$

$$\lg x = \log_{10} x$$

$$\mathrm{lb}\, x$$

$$x$$

$$2$$

$$\mathrm{lb}\, x = \log_2 x$$

$$\pi$$

$$\pi = 3.141\ 592\ 6...$$

$$\sin x$$

$$x$$

$$\sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}$$

$$(\sin x)^n$$

$$(\cos x)^n$$

$$n \geq 2$$

$$\sin^n x$$

$$\cos^n x$$

$$\cos x$$

$$x$$

$$\cos x = \sin(x + \pi/2)$$

$$\tan x$$

$$x$$

$$\tan x = \sin x / \cos x$$

$$\operatorname{tg} x$$

$$\cot x$$

$$x$$

$$\cot x = 1/\tan x$$

$$\operatorname{ctg} x$$

$$\sec x$$

$$x$$

$$\sec x = 1/\cos x$$

$$\csc x$$

$$x$$

$$\csc x = 1/\sin x$$

$$\operatorname{cosec} x$$

$$\arcsin x$$

$$x$$

$$y = \arcsin x \Longleftrightarrow x = \sin y \quad (-\pi/2 \leq y \leq \pi/2)$$

$$\arccos x$$

$$x$$

$$y = \arccos x \iff x = \cos y \quad (0 \le y \le \pi)$$

$$\arctan x$$

$$x$$

$$y = \arctan x \iff x = \tan y \quad (-\pi/2 \le y \le \pi/2)$$

$$\operatorname{arctg} x$$

$$\operatorname{arccot} x$$

$$x$$

$$y = \operatorname{arccot} x \iff x = \cot y \quad (0 \le y \le \pi)$$

$$\operatorname{arcctg} x$$

$$\operatorname{arcsec} x$$

$$x$$

$$y = \operatorname{arcsec} x \iff x = \sec y \quad (0 \le y \le \pi, y \ne \pi/2)$$

$$\operatorname{arccsc} x$$

$$x$$

$$y = \operatorname{arccsc} x \iff x = \csc y \quad (-\pi/2 \le y \le \pi/2, y \ne 0)$$

$$\operatorname{arccosec} x$$

$$\sinh x$$

$$x$$

$$\sinh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{2}$$

$$\operatorname{sh} x$$

$$\cosh x$$

$$x$$

$$\cosh^2 x = \sinh^2 x + 1$$

$$\operatorname{ch} x$$

$$\tanh x$$

$$x$$

$$\tanh x = \sinh x / \cosh x$$

$$\operatorname{th} x$$

$$\coth x$$

$$x$$

$$\coth x = 1/\tanh x$$

$$\operatorname{sech} x$$

$$x$$

$$\operatorname{sech} x = 1/\cosh x$$

$$\operatorname{csch} x$$

$$x$$

$$\operatorname{csch} x = 1/\sinh x$$

$$\operatorname{cosech} x$$

$$\operatorname{arsinh} x$$

$$x$$

$$y = \operatorname{arsinh} x \iff x = \sinh y$$

$$\operatorname{arsh} x$$

$$\operatorname{arcosh} x$$

$$x$$

$$y = \operatorname{arcosh} x \iff x = \cosh y \quad (y \geq 0)$$

$$\operatorname{arch} x$$

$$\operatorname{artanh} x$$

$$x$$

$$y = \operatorname{artanh} x \iff x = \tanh y$$

$$\operatorname{arth} x$$

$$\operatorname{arcoth} x$$

$$x$$

$$y = \operatorname{arcoth} x \iff x = \coth y \quad (y \neq 0)$$

$$\operatorname{arsech} x$$

$$x$$

$$y = \operatorname{arsech} x \iff x = \operatorname{sech} y \quad (y \geq 0)$$

$$\operatorname{arcsch} x$$

$$x$$

$$y = \operatorname{arcsch} x \iff x = \operatorname{csch} y \quad (y \geq 0)$$

$$\operatorname{arcosech} x$$

$$\mathrm{i}$$

$$\mathrm{i}^2 = -1$$

$$i$$

$$\operatorname{Re} z$$

$$z$$

$$\operatorname{Im} z$$

$$z$$

$$z = x + \mathrm{i}y \quad (x, y \in \mathbf{R})$$

$$x = \operatorname{Re} z$$

$$y = \operatorname{Im} z$$

$$|\ z\ |$$

$$z$$

$$|\ z\ | = \sqrt{(\operatorname{Re} z)^2 + (\operatorname{Im} z)^2}$$

$$\arg z$$

$$z$$

$$z = r\mathrm{e}^{\mathrm{i}\varphi}$$

$$r = |\ z\ |$$

$$-\pi < \varphi \leq \pi$$

$$\varphi = \arg z$$

$$\operatorname{Re} z = r \cos \varphi$$

$$\operatorname{Im} z = r \sin \varphi$$

$$\bar{z}$$

$$z^*$$

$$z$$

$$\bar{z} = \operatorname{Re} z - \mathrm{i} \operatorname{Im} z$$

$$\operatorname{sgn} z$$

$$z$$

$$\operatorname{sgn} z = z / \ |\ z\ | = \exp(\mathrm{i} \arg z) \quad (z \neq 0)$$

$$\operatorname{sgn} 0 = 0$$

$$A$$

$$m \times n$$

$$A$$

$$a_{ij} = (A)_{ij}$$

$$A = (a_{ij})$$

$$m$$

$$n$$

$$m = n$$

$$A + B$$

$$A$$

$$B$$

$$(A+B)_{ij} = (A)_{ij} + (B)_{ij}$$

$$A$$

$$B$$

$$xA$$

$$x$$

$$A$$

$$(xA)_{ij} = x(A)_{ij}$$

$$AB$$

$$A$$

$$B$$

$$(AB)_{ik} = \sum_j (A)_{ij}(B)_{jk}$$

$$A$$

$$B$$

$$I$$

$$E$$

$$(I)_{ik} = \delta_{ik}$$

$$\delta_{ik}$$

$$A^{-1}$$

$$A$$

$$AA^{-1} = A^{-1}A = I \quad (\det A \neq 0)$$

$$\det A$$

$$A^{\mathrm{T}}$$

$$A'$$

$$A$$

$$(A^{\mathrm{T}})_{ik} = (A)_{ki}$$

$$\overline{A}$$

$$A^*$$

$$A$$

$$\left(\overline{A}\right)_{ik} = \overline{(A)_{ik}}$$

$$A^{\mathrm{H}}$$

$$A^{\dagger}$$

$$A$$

$$A^{\mathrm{H}} = \left(\overline{A}\right)^{\mathrm{T}}$$

$$\det A$$

$$A$$

$$\mid A \mid$$

$$\operatorname{rank} A$$

$$A$$

$$\operatorname{tr} A$$

$$A$$

$$\operatorname{tr} A = \sum_i (A)_{ii}$$

$$\parallel A \parallel$$

$$A$$

$$A + B = C$$

$$\parallel A \parallel + \parallel B \parallel \geq \parallel C \parallel$$

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$

$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

$$\mathrm{O}$$

$$\mathrm{P}$$

$$\mathrm{O}$$

$$\mathrm{P}$$

$$x$$

$$y$$

$$z$$

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$$

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}x\ \boldsymbol{e}_x + \mathrm{d}y\ \boldsymbol{e}_y + \mathrm{d}z\ \boldsymbol{e}_z$$

$$\boldsymbol{e}_x$$

$$\boldsymbol{e}_y$$

$$\boldsymbol{e}_z$$

$$\boldsymbol{e}_1$$

$$\boldsymbol{e}_2$$

$$\boldsymbol{e}_3$$

$$\boldsymbol{i}$$

$$\boldsymbol{j}$$

$$\boldsymbol{k}$$

$$x_1$$

$$x_2$$

$$x_3$$

$$i$$

$$j$$

$$k$$

$$\rho$$

$$\varphi$$

$$z$$

$$\boldsymbol{r} = \rho\ \boldsymbol{e}_\rho + z\ \boldsymbol{e}_z$$

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}\rho\ \boldsymbol{e}_\rho + \rho\ \mathrm{d}\varphi\ \boldsymbol{e}_\varphi + \mathrm{d}z\ \boldsymbol{e}_z$$

$$\boldsymbol{e}_\rho(\varphi)$$

$$\boldsymbol{e}_\varphi(\varphi)$$

$$\boldsymbol{e}_z$$

$$z = 0$$

$$\rho$$

$$\varphi$$

$$r$$

$$\vartheta$$

$$\varphi$$

$$\boldsymbol{r} = r\boldsymbol{e}_r$$

$$\mathrm{d}\boldsymbol{r} = \mathrm{d}r\ \boldsymbol{e}_r + r\ \mathrm{d}\vartheta\ \boldsymbol{e}_\vartheta + r\ \sin\vartheta\ \mathrm{d}\varphi\ \boldsymbol{e}_\varphi$$

$$\boldsymbol{e}_r(\vartheta, \varphi)$$

$$\boldsymbol{e}_\vartheta(\vartheta, \varphi)$$

$$\boldsymbol{e}_\varphi(\varphi)$$

$$\boldsymbol{e}_1$$

$$\boldsymbol{e}_2$$

$\boldsymbol{e}_3$

n

$\boldsymbol{e}_1$

$\boldsymbol{e}_2$

$\boldsymbol{e}_3$

$\boldsymbol{a}$

$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3$

a_1

a_2

a_3

$a_1\boldsymbol{e}_1$

$a_2\boldsymbol{e}_2$

$a_3\boldsymbol{e}_3$

x

y

z

a_1

a_2

a_3

x_1

x_2

x_3

i

j

k

1

3

$\boldsymbol{a}$

$\vec{a}$

$\boldsymbol{a}$

$\boldsymbol{a} + \boldsymbol{b}$

$\boldsymbol{a}$

$\boldsymbol{b}$

$(\boldsymbol{a} + \boldsymbol{b})_i = a_i + b_i$

$$x\boldsymbol{a}$$

$$x$$

$$\boldsymbol{a}$$

$$(x\boldsymbol{a})_i = xa_i$$

$$|\ \boldsymbol{a}|$$

$$\boldsymbol{a}$$

$$\boldsymbol{a}$$

$$|\ \boldsymbol{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$$

$$\|\ a\ \|$$

$$\mathbf{0}$$

$$\vec{0}$$

$$0$$

$$\boldsymbol{e}_{\boldsymbol{a}}$$

$$\boldsymbol{a}$$

$$\boldsymbol{e}_{\boldsymbol{a}} = \boldsymbol{a}/|\ \boldsymbol{a}| \quad (\boldsymbol{a} \neq \mathbf{0})$$

$$\boldsymbol{e}_x$$

$$\boldsymbol{e}_y$$

$$\boldsymbol{e}_z$$

$$\boldsymbol{e}_1$$

$$\boldsymbol{e}_2$$

$$\boldsymbol{e}_3$$

$$\boldsymbol{i}$$

$$\boldsymbol{j}$$

$$\boldsymbol{k}$$

$$a_x$$

$$a_y$$

$$a_z$$

$$a_i$$

$$\boldsymbol{a}$$

$$\boldsymbol{a} = a_x\boldsymbol{e}_x + a_y\boldsymbol{e}_y + a_z\boldsymbol{e}_z$$

$$\boldsymbol{a} = (a_x, a_y, a_z)$$

$$a_x = \boldsymbol{a} \cdot \boldsymbol{e}_x$$

$$a_y = \boldsymbol{a} \cdot \boldsymbol{e}_y$$

$$a_z = \boldsymbol{a} \cdot \boldsymbol{e}_z$$

$$\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y + z\boldsymbol{e}_z$$

$$x$$

$$y$$

$$z$$

$$\delta_{ik}$$

$$\delta_{ik} = 1 \quad (i = k)$$

$$\delta_{ik} = 0 \quad (i \neq k)$$

$$\varepsilon_{ijk}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

$$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$$

$$\varepsilon_{ijk}$$

$$0$$

$$\boldsymbol{a} \cdot \boldsymbol{b}$$

$$\boldsymbol{a}$$

$$\boldsymbol{b}$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \sum_i a_i b_i$$

$$\boldsymbol{a} \times \boldsymbol{b}$$

$$\boldsymbol{a}$$

$$\boldsymbol{b}$$

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \sum_j \sum_k \varepsilon_{ijk} a_j b_k$$

$$\varepsilon_{ijk}$$

$$\boldsymbol{\nabla}$$

$$\boldsymbol{\nabla} = \boldsymbol{e}_x \frac{\partial}{\partial x} + \boldsymbol{e}_y \frac{\partial}{\partial y} + \boldsymbol{e}_z \frac{\partial}{\partial z} = \sum_i \boldsymbol{e}_i \frac{\partial}{\partial x_i}$$

$$\boldsymbol{\nabla}\varphi$$

$$\mathbf{grad}\,\varphi$$

$$\varphi$$

$$\boldsymbol{\nabla}\varphi = \sum_i \boldsymbol{e}_i \frac{\partial \varphi}{\partial x_i}$$

$$\mathbf{grad}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{a}$$

$$\mathbf{div}\, \boldsymbol{a}$$

$$\boldsymbol{a}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{a} = \sum_i \frac{\partial a_i}{\partial x_i}$$

$$\mathbf{div}$$

$$\boldsymbol{\nabla} \times \boldsymbol{a}$$

$$\mathbf{rot}\, \boldsymbol{a}$$

$$\boldsymbol{a}$$

$$(\boldsymbol{\nabla} \times \boldsymbol{a})_i = \sum_j \sum_k \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j}$$

$$\mathbf{rot}$$

$$\mathbf{curl}$$

$$\varepsilon_{ijk}$$

$$\boldsymbol{\nabla}^2$$

$$\Delta$$

$$\boldsymbol{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$z$$

$$w$$

$$k$$

$$n$$

$$k \leq n$$

$$\gamma$$

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln n \right) = 0.577\ 215\ 6...$$

$$\Gamma(z)$$

$$\Gamma(z) = \int_0^\infty t^{z-1} \mathrm{e}^{-t} \mathrm{d}t \quad (\mathrm{Re}\, z > 0)$$

$$\Gamma(n+1) = n!$$

$$\zeta(z)$$

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (\operatorname{Re} z > 1)$$

$$\mathrm{B}(z, w)$$

$$\mathrm{B}(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} \mathrm{d}t \quad (\operatorname{Re} z > 0$$

$$\operatorname{Re} w > 0)$$

$$\mathrm{B}(z, w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}$$

$$\frac{1}{(n+1)\,\mathrm{B}(k+1, n-k+1)} = \binom{n}{k}$$

$$\begin{aligned} \mathrm{e}^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= 1 + x + \frac{x^2}{2} \\ &\quad + \frac{x^3}{6} + \frac{x^4}{24} + \dots \end{aligned}$$

Input. The edges of the graph e, where each element in e is (u, v, w) denoting that there is an edge between u and v weighted w.
Output. The edges of the MST of the input graph.
Method.
result $\leftarrow \emptyset$
sort e into nondecreasing order by weight w
for each (u, v, w) in the sorted e
    if u and v are not connected in the union-find set
        connect u and v in the union-find set
        result $\leftarrow$ *result* $\bigcup \{(u, v, w)\}$
return *result*

LATEX

$$0$$

$$\pi$$

$$\delta x$$

$$\delta$$

$$\boldsymbol{a}$$

$$=$$

$$+$$

$$-$$

$$\pm$$

$$\mp$$

$$\times$$

$$\cdot$$

$$/$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$

$$\binom{n}{m}$$

$$\binom{n}{m}$$

$$\sum$$

$$\prod$$

$$\int$$

$$a \times b$$

$$a \cdot b$$

$$a * b$$

$$a_1, a_2, \cdots a_n$$

$$a_1, a_2, ... a_n$$

$$a = b$$

$$a == b$$

$$a \bmod b$$

$$a\%b$$

$$a_{i,j,k}$$

$$a[i][j][k]$$

$$f(i, j, k)$$

$$O$$

$$\Longleftrightarrow$$

$$\Leftrightarrow$$

$$log, ln, lg$$

$$\log$$

$$\ln$$

$$\lg$$

$$sin, cos, tan$$

$$\sin$$

$$\cos$$

$$\tan$$

$$gcd, lcm$$

$$\gcd$$

$$\operatorname{lcm}$$

$$e$$

$$e$$

$$\mathrm{e}$$

$$i$$

$$i$$

$$\mathrm{i}$$

□□a□□□

$$a$$

$$\cdots$$

$$\cdots$$

$$\cdots$$

$$\vdots$$

$$\ddots$$

$$a*b$$

$$a \times b$$

$$a \cdot b$$

$$SPFA$$

$$a == b$$

$$a = b$$

$$f[i][j][k]$$

$$f_{i,j,k}$$

$$f(i, j, k)$$

$$R, N^*$$

$$\mathbf{R}$$

$$\mathbf{N}^*$$

$$\emptyset$$

$$\emptyset$$

$$size$$

$$size$$

LATEX

$$2^{2^{2^{\cdots}}}$$

$$n$$

$$n-1$$

$$1$$

$$k$$

$$1$$

$$m$$

$$n$$

$$n-1$$

$$u, v, w$$

$$u$$

$$v$$

$$c$$

$$1 \leq u, v \leq n$$

$$1 \leq c \leq 10^5$$

$$n+1$$

$$m$$

$$m$$

$$k_i$$

$$i$$

$$k_i$$

$$k$$

$$h_1, h_2, \cdots, h_k$$

$$m$$

$$100\%$$

$$2 \leq n \leq 2.5 \times 10^5, 1 \leq m \leq 5 \times 10^5, \sum k_i \leq 5 \times 10^5, 1 \leq k_i \leq n-1$$

$$Dp(i)$$

$$i$$

$$w(a, b)$$

$$a$$

$$b$$

$$i$$

$$v$$

$$v$$

$$Dp(i) = Dp(i) + \min\{Dp(v), w(i, v)\}$$

$$v$$

$$Dp(i) = Dp(i) + w(i, v)$$

$$O(nq)$$

$$1$$

$$1$$

$$n$$

$$a$$

$$b$$

$$a$$

$$b$$

$$O(k^2)$$

$$1$$

$$lca$$

$$A$$

$$A$$

$$A$$

$$dfn$$

$$A$$

$$x, y$$

$$lca$$

$$lca, y$$

$$lca$$

$$y$$

$$x$$

$$y$$

$$x$$

$$y$$

$$x$$

$$y$$

$$x$$

y

x

y

$lca(x, y)$

y

$lca(x, y)$

y

lca

y

$m - 1$

$O(m \log n)$

m

n

1

$6, 4, 7$

$[4, 6, 7]$

1

4

1

1

4

$LCA(1, 4) = 1$

$LCA(1, 4) =$

4

$4, 1$

6

4

6

4

$LCA(6, 4) = 1$

$LCA(6, 4) \neq$

4

$LCA(6, 4)$

$$1$$

$$1 \to 4$$

$$4$$

$$6$$

$$6, 1$$

$$7$$

$$6$$

$$7$$

$$6$$

$$LCA(7,6) = 3$$

$$LCA(7,6) \neq$$

$$6$$

$$LCA(7,6)$$

$$1$$

$$3 \to 6$$

$$6$$

$$LCA(6,7)$$

$$7$$

$$1, 3, 7$$

$$1, 3, 7$$

$$1 \to 3$$

$$3 \to 7$$

$$i$$

$$v$$

$$v$$

$$Dp(i) = Dp(i) + \min\{Dp(v), w(i,v)\}$$

$$v$$

$$Dp(i) = Dp(i) + w(i,v)$$

$$f(u) = \frac{w(u, p_u) + \sum_{v \in son_u} (w(u,v) + f(v) + f(u))}{d(u)}$$

$$
\begin{aligned}
f(u) &= \frac{w(u,p_u) + \sum\limits_{v \in son_u} (w(u,v) + f(v) + f(u))}{d(u)} \\
&= \frac{w(u,p_u) + \sum\limits_{v \in son_u} (w(u,v) + f(v)) + (d(u)-1) f(u)}{d(u)} \\
&= w(u,p_u) + \sum_{v \in son_u} (w(u,v) + f(v)) \\
&= \sum_{(u,t) \in E} w(u,t) + \sum_{v \in son_u} f(v)
\end{aligned}
$$

$$
f(u) = d(u) + \sum_{v \in son_u} f(v)
$$

$$
g(u) = \frac{w(p_u,u) + \Big(w(p_u,p_{p_u}) + g(p_u) + g(u)\Big) + \sum\limits_{s \in sibling_u} (w(p_u,s) + f(s) + g(u))}{d(p_u)}
$$

$$
\begin{aligned}
g(u) &= \frac{w(p_u,u) + \Big(w(p_u,p_{p_u}) + g(p_u) + g(u)\Big) + \sum\limits_{s \in sibling_u} (w(p_u,s) + f(s) + g(u))}{d(p_u)} \\
&= \frac{w(p_u,u) + w(p_u,p_{p_u}) + g(p_u) + \sum\limits_{s \in sibling_u} (w(p_u,s) + f(s)) + (d(p_u)-1) g(u)}{d(p_u)} \\
&= w(p_u,u) + w(p_u,p_{p_u}) + g(p_u) + \sum_{s \in sibling_u} (w(p_u,s) + f(s)) \\
&= \sum_{(p_u,t) \in E} w(p_u,t) + g(p_u) + \sum_{s \in sibling_u} f(s) \\
&= \sum_{(p_u,t) \in E} w(p_u,t) + g(p_u) + \left(f(p_u) - \sum_{(p_u,t) \in E} w(p_u,t) - f(u) \right) \\
&= g(p_u) + f(p_u) - f(u)
\end{aligned}
$$

$$T = (V, E)$$

$$d(u)$$

$$u$$

$$w(u,v)$$

$$u$$

$$v$$

$$p_u$$

$$u$$

$$root$$

$$son_u$$

$$u$$

$$sibling_u$$

$$u$$

$$f(u)$$

$$u$$

$$p_u$$

$$d(u)$$

$$u$$

$$l$$

$$f(l) = w(p_l, l)$$

$$1$$

$$u$$

$$u$$

$$-1$$

$$u$$

$$p_u$$

$$1$$

$$2$$

$$g(u)$$

$$p_u$$

$$u$$

$$u$$

$$u$$

$$u$$

$$u$$

$$d(p_u)$$

$$p_u$$

$$g(\text{root}) = 0$$

$$h_x = f(\{h_i \mid i \in son(x)\})$$

$$f(S) = \left(c + \sum_{x \in S} g(x)\right) \bmod m$$

$$h_x$$

$$x$$

$$f$$

$$c$$

$$1$$

$$m$$

$$2^{32}$$

$$2^{64}$$

$$g$$

$$1$$

$$u$$

$$f(u)$$

$$u$$

$$|\ son(u)|\ !$$

$$v$$

$$v$$

$$f(v)$$

$$f(u) = |\ son(u)|\ ! \cdot \prod_{v \in son(u)} f(v)$$

$$f(u)$$

$$n$$

$$m$$

$$k$$

$$k$$

$$n \leq 10000, m \leq 100, k \leq 10000000$$

$$rt$$

$$rt$$

$$rt$$

$$ch$$

$$ch$$

$$ch$$

$$rt$$

$$i$$

$$rt$$

$$dist_i$$

$$tf_d$$

$$v$$

$$dist_v = d$$

$$k$$

$$tf_{k-dist_i} = true$$

$$k$$

$$ch$$

$$tf$$

$$tf$$

$$tf$$

$$h$$

$$O(hn)$$

$$O(n \log n)$$

$$n$$

$$k$$

$$k$$

$$n \leq 40000, k \leq 20000, w_i \leq 1000$$

$$[0, k]$$

$$s(i, j)$$

$$i$$

$$j$$

$$sum_i = \sum_{j=1}^{n} s(i, j)$$

$$1 \leq i \leq n$$

$$sum_i$$

$$1 \leq n, c_i \leq 10^5$$

$$sum_i$$

$$i$$

$$sum_i$$

$$j$$

$$i$$

$$j$$

$$cnt_j$$

$$sum_i$$

$$\sum cnt_j$$

$$sum_i$$

$$cnt_j$$

$$cnt_{col_u} += size_u$$

$$size_u$$

$$u$$

$$\log n$$

$$sum_i$$

$$u$$

$$d$$

$$d$$

$$v$$

$$v$$

$$(u, v)$$

$$num$$

$$u$$

$$d$$

$$siz1$$

$$v$$

$$num \times siz1$$

$$(u, v)$$

$$j$$

$$u$$

$$d$$

$$cnt_j$$

$$\sum_{j \notin (u,v)} cnt_j$$

$$size$$

$$size$$

$$1$$

$$0$$

$$O(n)$$

$$O(n \log n)$$

$$\log n$$

$$\log n$$

$$\log n$$

$$tot$$

$$O(n)$$

$$n$$

$$1 \leq n \leq 10^4$$

$$y$$

$$z$$

$$z$$

$$z$$

$$z'$$

$$\delta(z, z')$$

$$z$$

$$z'$$

$$z$$

$$y$$

$$z$$

$$y$$

$$\delta(s, t)$$

$$y$$

$$z$$

$$t$$

$$s$$

$$y$$

$$\delta(s, t)$$

$$\delta(y, z) > \delta(y, t) \Longrightarrow \delta(x, z) > \delta(x, t) \Longrightarrow \delta(s, z) > \delta(s, t)$$

$$\delta(s, t)$$

$$y$$

$$\delta(s, t)$$

$$\delta(y, z)$$

$$\delta(s, t)$$

$$\delta(y, z) > \delta(y, t) \Longrightarrow \delta(x, z) > \delta(x, t) \Longrightarrow \delta(s, z) > \delta(s, t)$$

$$\delta(s, t)$$

$$y$$

$$\delta(s,t)$$

$$\delta(y,z)$$

$$\delta(s,t)$$

$$\delta(y,z) > \delta(y,t) \Longrightarrow \delta(x',z) > \delta(x',t) \Longrightarrow \delta(x,z) > \delta(x,t) \Longrightarrow \delta(s,z) > \delta(s,t)$$

$$\delta(s,t)$$

$$1$$

$$d_1$$

$$d_2$$

$$d_1 + d_2$$

$$d$$

$$u$$

$$d = d_1[u] + d_2[u]$$

$$u$$

$$1$$

$$\delta(s,t)$$

$$\delta(s',t')$$

$$x$$

$$x'$$

$$\delta(s,x) = \delta(x,t) = \delta(s',x') = \delta(x',t')$$

$$\delta(s,t') = \delta(s,x) + \delta(x,x') + \delta(x',t') > \delta(s,x) + \delta(x,t) = \delta(s,t)$$

$$\delta(s,t)$$

$$n$$

$$n-1$$

$$n-1$$

$$1$$

$$1$$

$$n=1$$

$$u$$

$$v$$

$$v$$

$$u$$

$$u$$

$$v$$

$$v$$

$$u$$

$$2$$

$$u$$

$$u$$

$$u$$

$$O(\log n)$$

$$O(n)$$

$$\forall u, v \in V_1, (u, v) \in E_1 \iff (\varphi(u), \varphi(v)) \in E_2$$

$$\forall u, v \in V_1, (u, v) \in E_1 \iff (\varphi(u), \varphi(v)) \in E_2$$

Input. A rooted tree T
Output. The name of rooted tree T
ASSIGN-NAME(u)
    if u is a leaf
        NAME(u) = (0)
    else
        for all child v of u
            ASSIGN-NAME(v)
    sort the names of the children of u
    concatenate the names of all children u to temp
    NAME(u) = (temp)

Input. Two rooted trees $T_1(V_1, E_1, r_1)$ and $T_2(V_2, E_2, r_2)$
Output. Whether these two trees are isomorphic
AHU($T_1(V_1, E_1, r_1), T_2(V_2, E_2, r_2)$)
    ASSIGN-NAME(r_1)
    ASSIGN-NAME(r_2)
    if NAME(r_1) = NAME(r_2)
        return true
    else
        return false

$$T_1(V_1, E_1, r_1)$$

$$T_2(V_2, E_2, r_2)$$

$$\varphi : V_1 \to V_2$$

$$\varphi(r_1) = r_2$$

$$T_1(V_1, E_1, r_1)$$

$$T_2(V_2, E_2, r_2)$$

$$T_1(V_1, E_1)$$

$$T_2(V_2, E_2)$$

$\varphi : V_1 \to V_2$

$T_1(V_1, E_1)$

$T_2(V_2, E_2)$

T_1

T_1

T_2

$T_1(V_1, E_1)$

$T_2(V_2, E_2)$

1

c_1

c_2

$T_1(V_1, E_1, c_1)$

$T_2(V_2, E_2, c_2)$

$T_1(V_1, E_1)$

$T_2(V_2, E_2)$

2

c_1, c'_1

c_2, c'_2

$T_1(V_1, E_1, c_1)$

$T_2(V_2, E_2, c_2)$

$T_1(V_1, E_1, c'_1)$

$T_2(V_2, E_2, c_2)$

$T_1(V_1, E_1)$

$T_2(V_2, E_2)$

$O(|V|)$

$O(|V|)$

T_1

T_2

T_2

T_3

T_1

T_3

$NAME$

r

$NAME$

r

$NAME$

$NAME(r)$

$T_1(V_1, E_1, r_1)$

$T_2(V_2, E_2, r_2)$

$NAME(r_1) = NAME(r_2)$

T_1

T_2

n

n

$1 + 2 + \cdots + n$

$\Theta(n^2)$

$O(n^2)$

$NAME$

i

i

i

$NAME$

$i + 1$

$NAME$

$NAME$

u

i

u

T_1

T_2

i

$NAME$

$$NAME(u)$$

$$NAME$$

$$NAME$$

$$i$$

$$NAME$$

$$i$$

$$i+1$$

$$L_i$$

$$NAME$$

$$L$$

$$m$$

$$O(L+\mid \Sigma \mid)$$

$$\mid \Sigma \mid$$

$$O(L\log m)$$

$$O(\log m)$$

$$\ell_1$$

$$\ell_2$$

$$O(\min\{\ell_1, \ell_2\})$$

$$i$$

$$i+1$$

$$L_i$$

$$\sum_i L_i = O(n)$$

$$T(n) = O(n)$$

$$T(n) = O(n\log n)$$

$$ve(j) = \max\{ve(k) + dut(\langle k, j\rangle)\}, \quad \langle k, j\rangle \in T, 2 \le j \le n$$

$$vl(j) = \min\{vl(k) - dut(\langle j, k\rangle)\}, \quad \langle j, k\rangle \in S, 1 \le j \le n-1$$

$$u$$

$$(u, v)$$

$$u$$

$$v$$

$$(u, v)$$

$$u$$
$$v$$
$$i$$
$$j$$
$$j$$
$$i$$
$$i$$
$$j$$
$$i$$
$$j$$
$$j$$
$$i$$
$$a_j$$
$$e(j)$$
$$a_j$$
$$l(j)$$
$$v_j$$
$$ve(j)$$
$$ve(j) = e(j)$$
$$v_j$$
$$vl(j)$$
$$l(j) = vl(j) - dul(aj)$$
$$e$$
$$(j, k)$$
$$v_0$$
$$ve[0] = 0$$
$$ve[i],\ (i \leq i \leq n-1)$$
$$n$$
$$v_n$$
$$vl[n-1] = ve[n-1]$$
$$vl[i],\ (n-2 \geq i \geq 2)$$
$$ve$$

$$vl$$

$$s$$

$$e(s)$$

$$l(s)$$

$$e(s) = l(s)$$

$$S$$

$$0$$

$$L$$

$$S$$

$$u$$

$$L$$

$$u$$

$$(u, v_1), (u, v_2), (u, v_3) \cdots$$

$$(u, v)$$

$$v$$

$$0$$

$$v$$

$$S$$

$$S$$

$$L$$

$$L$$

$$G = (V, E)$$

$$0$$

$$S$$

$$O(E + V)$$

$$O(E + V)$$

$$O(E + V)$$

$$O(E + V)$$

$$O(V)$$

$$0$$

$$O(E + V \log V)$$

$$G = (V, E)$$

$$C$$

$$G$$

$$G$$

$$C$$

$$G$$

$$C$$

$$G$$

$$O(\mid V \mid \mid E \mid + \mid V \mid^2 \log \mid V \mid)$$

$$O(\mid V \mid^3)$$

$$G$$

$$S, T$$

$$G$$

$$C$$

$$S, T$$

$$C$$

$$S - T$$

$$G$$

$$s, t$$

$$G$$

$$S - T$$

$$s, t$$

$$G$$

$$\mid V \mid$$

$$1$$

$$s, t$$

$$s, t$$

$$(s, t)$$

$$G/\{s, t\}$$

$$k$$

$$(t, k)$$

$$d(t, k)$$

$$d(s, k)$$

$$s, t$$

$$G/\{s, t\}$$

$$k$$

$$(k, s) \in C_{\min}$$

$$(k, t) \in C_{\min}$$

$$s, t$$

$$k, t$$

$$s, k$$

$$C = C_{\min}/s$$

$$C_{\min}$$

$$s, t$$

$$s, t$$

$$\mid V \mid$$

$$1$$

$$\mid V \mid -1$$

$$G' = (V', E')$$

$$A$$

$$A = \emptyset$$

$$V'$$

$$i \notin A$$

$$w(A, i)$$

$$A$$

$$\mid A \mid = \mid V' \mid$$

$$w(A, i) = \sum_{j \in A} d(i, j)$$

$$(i, j) \notin E'$$

$$d(i, j) = 0$$

$$A$$

$$\operatorname{ord}(i)$$

$$i$$

$$A$$

$$t = \text{ord}(|\ V'|)$$

$$\text{pos}(v)$$

$$v$$

$$A$$

$$|\ A\ |$$

$$v$$

$$s$$

$$s$$

$$t$$

$$w(t)$$

$$v$$

$$v$$

$$A$$

$$A$$

$$u$$

$$v$$

$$G'' = (V', E'/C)$$

$$u$$

$$v$$

$$A$$

$$A_v = \{u \mid \text{pos}(u) < \text{pos}(v)\}$$

$$v$$

$$A$$

$$E_v$$

$$E'$$

$$A_v \cup \{v\}$$

$$v$$

$$C_v$$

$$C \cap E_v$$

$$w(C_v) = \sum_{(i,j) \in C_v} d(i,j)$$

$$v$$

$$w(A_v, v) \leq w(C_v)$$

$$v_0$$

$$w(A_{v_0}, v_0) = w(C_{v_0})$$

$$u, v$$

$$\mathrm{pos}(v) < \mathrm{pos}(u)$$

$$w(A_u, u) = w(A_v, u) + w(A_u - A_v, u)$$

$$w(A_v, u) \leq w(A_v, v)$$

$$w(A_v, v) \leq w(C_v)$$

$$w(A_u, u) \leq w(C_v) + w(A_u - A_v, u)$$

$$w(A_u - A_v, u)$$

$$w(C_u)$$

$$w(C_v)$$

$$w(A_u, u) \leq w(C_u)$$

$$\mathrm{pos}(s) < \mathrm{pos}(t)$$

$$s, t$$

$$t$$

$$w(A_t, t) \leq w(C_t) = w(C)$$

$$O(\mid E \mid + \mid V \mid \log \mid V \mid)$$

$$O(\mid V \mid)$$

$$O(\mid E \mid \mid V \mid + \mid V \mid^2 \log \mid V \mid)$$

$$\mid V \mid$$

$$\mid E \mid$$

$$O(\log \mid V \mid)$$

$$O(1)$$

$$O(\mid E \mid + \mid V \mid \log \mid V \mid)$$

$$A$$

$$B$$

$$C$$

$$P$$

$$a + b + c$$

$$a$$

$$b$$

$$c$$

P

A

B

C

ABC

120°

P

AB

BC

AC

120°

ABC

C

120°

P

C

A, B, C

n

$A_1, A_2, ..., A_n$

P

$a_1 + a_2 + ... + a_n$

a_i

PA_i

n

P

$a_1 \cdot w_1 + a_2 \cdot w_2 + ... + a_n \cdot w_n$

w_i

P

n

$A_1, A_2, \cdots, A_n$

n

n

$n-2$

120°

$$120^\circ$$

$$s_1, s_2$$

$$120^\circ$$

$$\{1, 2, 3, 4\}$$

$$\{1, 2, 3, 4\}$$

$$G$$

$$n$$

$$k$$

$$k$$

$$k$$

$$k$$

$$n - k$$

$$f(i, S)$$

$$i$$

$$S$$

$$f(i, S) \leftarrow \min(f(i, S), f(i, T) + f(i, S - T))$$

$$f(i, S) \leftarrow \min(f(i, S), f(j, S) + w(j, i))$$

$$i, j$$

$$f(i, S)$$

$$i$$

$$S$$

$$a_i$$

$$f(i, S) \leftarrow \min(f(i, S), f(i, T) + f(i, S - T) - a_i)$$

$$a_i$$

$$f(i, S) \leftarrow \min(f(i, S), f(j, S) + w(j, i))$$

$$i$$

$$s$$

$$S$$

$$n$$

$$n - 1$$

$$n$$

$$m$$

$$s$$

$$D(u)$$
$$s$$
$$u$$
$$dis(u)$$
$$s$$
$$u$$
$$dis(u) \geq D(u)$$
$$dis(u) = D(u)$$
$$w(u,v)$$
$$(u,v)$$
$$1$$
$$k$$
$$V' = 1, 2, ..., k$$
$$x$$
$$y$$
$$x$$
$$y$$
$$x$$
$$y$$
$$V' = 1, 2, ..., n$$
$$V$$
$$x$$
$$y$$
$$0$$
$$+\infty$$
$$+\infty$$
$$x$$
$$y$$
$$x = y$$
$$x$$
$$y$$
$$+\infty$$
$$k$$
$$k$$

$$O(N^3)$$

$$k$$

$$1$$

$$n$$

$$O(N^3)$$

$$O(N^2)$$

$$u$$

$$(u, x)$$

$$(u, y)$$

$$u$$

$$O(n^3)$$

$$1/0$$

$$\min$$

$$O(\frac{n^3}{w})$$

$$(u, v)$$

$$dis(v) = \min(dis(v), dis(u) + w(u, v))$$

$$S \to u \to v$$

$$S \to u$$

$$v$$

$$O(m)$$

$$+1$$

$$n - 1$$

$$n - 1$$

$$O(nm)$$

$$S$$

$$n - 1$$

$$n$$

$$S$$

$$S$$

$$S$$

$$s$$

$$n$$

$$s$$

$$O(nm)$$

$$S$$

$$T$$

$$T$$

$$dis(s) = 0$$

$$dis$$

$$+\infty$$

$$T$$

$$S$$

$$S$$

$$T$$

$$T$$

$$O(m)$$

$$O(n^2)$$

$$O(n^2 + m) = O(n^2)$$

$$(u, v)$$

$$v$$

$$v$$

$$O(m)$$

$$O(n)$$

$$O(\log n)$$

$$O((n + m) \log n) = O(m \log n)$$

$$O(m)$$

$$O(m \log m)$$

$$O(1)$$

$$O(n \log n + m)$$

$$O(m \log n)$$

$$m = O(n)$$

$$m = O(n^2)$$

$$u$$

$$D(u) = dis(u)$$

$$S = \emptyset$$

$$u$$

$$S$$

$$D(u) = dis(u)$$

$$s$$

$$D(u) = dis(u) = 0$$

$$S$$

$$u$$

$$S$$

$$S \neq \emptyset$$

$$s$$

$$u$$

$$D(u) = dis(u) = +\infty$$

$$s \to x \to y \to u$$

$$y$$

$$s \to u$$

$$T$$

$$x$$

$$y$$

$$x \in S$$

$$s = x$$

$$y = u$$

$$s \to x$$

$$y \to u$$

$$u$$

$$D(u) = dis(u)$$

$$x$$

$$S$$

$$D(x) = dis(x)$$

$$(x, y)$$

$$u$$

$$S$$

$$D(y) = dis(y)$$

$$D(u) = dis(u)$$

$$s \to x \to y \to u$$

$$D(y) \leq D(u)$$

$$dis(y) \leq D(y) \leq D(u) \leq dis(u)$$

$$u$$

$$T$$

$$y$$

$$T$$

$$dis(u) \leq dis(y)$$

$$dis(y) = D(y) = D(u) = dis(u)$$

$$D(u) \neq dis(u)$$

$$D(y) \leq D(u)$$

$$O(n^2)$$

$$O(m \log m)$$

$$n$$

$$O(n^2 m)$$

$$O(n^3)$$

$$n$$

$$O(nm \log m)$$

$$n$$

$$x$$

$$k$$

$$kx$$

$$1 \to 2$$

$$1 \to 5 \to 3 \to 2$$

$$-2$$

$$5$$

$$1 \to 2$$

$$1 \to 4 \to 2$$

$$0$$

$$0$$

$$0$$

$$h_i$$

$$u$$

$$v$$

$$w$$

$$w + h_u - h_v$$

$$n$$

$$O(nm \log m)$$

$$s$$

$$t$$

$$s \to p_1 \to p_2 \to \ldots \to p_k \to t$$

$$(w(s, p_1) + h_s - h_{p_1}) + (w(p_1, p_2) + h_{p_1} - h_{p_2}) + \ldots + (w(p_k, t) + h_{p_k} - h_t)$$

$$w(s, p_1) + w(p_1, p_2) + \ldots + w(p_k, t) + h_s - h_t$$

$$s$$

$$t$$

$$h_s - h_t$$

$$h_i$$

$$i$$

$$s \to t$$

$$s \to t$$

$$s \to t$$

$$s \to t$$

$$(u, v)$$

$$h_v \leq h_u + w(u, v)$$

$$w'(u, v) = w(u, v) + h_u - h_v \geq 0$$

$$O(N^3)$$

$$O(NM)$$

$$O(M \log M)$$

$$O(NM \log M)$$

$$7 \to 1$$

$$9 \to 7$$

$$3 \to 6$$

$$u$$

$$u$$

u

v

u

u

v

u

u

dfn_u

u

low_u

u

u

$Subtree_u$

low_u

dfn

$Subtree_u$

$Subtree_u$

u

v

v

u

v

v

low_v

low_u

u

v

v

u

v

dfn_v

low_u

v

v

u

$$dfn_u = low_u$$

$$dfn_u = low_u$$

$$u$$

$$O(n+m)$$

$$O(n+m)$$

$$w$$

$$w$$

$$w$$

$$w$$

$$n$$

$$m$$

$$d^+(u)$$

$$u$$

$$u$$

$$O(m)$$

$$O(m)$$

$$O(nm)$$

$$O(m)$$

$$u$$

$$v$$

$$u$$

$$v$$

$$O(1)$$

$$O(n)$$

$$O(n^2)$$

$$O(n^2)$$

$$O(1)$$

$$u$$

$$u$$

$$v$$

$$O(d^+(u))$$

$$O(\log(d^+(u)))$$

$$u$$

$$O(d^+(u))$$

$$O(n+m)$$

$$O(m)$$

$$u$$

$$v$$

$$O(d^+(u))$$

$$u$$

$$O(d^+(u))$$

$$O(n+m)$$

$$O(m)$$

$$\sum_{y\in son_x}\sum_{z\in son_x}[d(y)-1]\cdot[d(z)-1].$$

$$1\le n\le 19$$

$$f(s,i)$$

$$s$$

$$i$$

$$s$$

$$f(s,i)$$

$$u$$

$$u$$

$$s$$

$$A$$

$$f(s,i)$$

$$u$$

$$s$$

$$f(s,i)$$

$$f(s\cup\{u\},u)$$

$$\frac{A-m}{2}$$

$$m$$

$$O(2^n m)$$

$$G$$

$$(u,v,w)$$

$$(u, v)$$
$$(v, w)$$
$$(w, u)$$
$$u$$
$$u$$
$$v$$
$$v$$
$$w$$
$$u$$
$$w$$
$$O(m\sqrt{m})$$
$$O(n+m)$$
$$(v, w)$$
$$u$$
$$v$$
$$d^-(v)$$
$$d^-(v) \leq \sqrt{m}$$
$$w$$
$$n$$
$$O(n\sqrt{m})$$
$$d^-(v) > \sqrt{m}$$
$$v$$
$$w$$
$$d(v) \leq d(w)$$
$$d(w) > \sqrt{m}$$
$$m$$
$$w$$
$$\sqrt{m}$$
$$O(m\sqrt{m})$$
$$O(n+m+n\sqrt{m}+m\sqrt{m}) = O(m\sqrt{m})$$
$$u, w$$
$$n$$
$$m$$

$$2 \leq n \leq 10^5$$

$$1 \leq m \leq \min\left\{2 \times 10^5,\ \frac{n(n-1)}{2}\right\}$$

$$x$$

$$\binom{x}{2}$$

$$O(m\sqrt{m})$$

$$a,\ b,\ c,\ d$$

$$(a,\ b)$$

$$(b,\ c)$$

$$(c,\ d)$$

$$(d,\ a)$$

$$a$$

$$a$$

$$c$$

$$a$$

$$b$$

$$(a,\ b)$$

$$(b,\ c)$$

$$b$$

$$b$$

$$b$$

$$c$$

$$O(m\sqrt{m})$$

$$n,\ m$$

$$(a,\ b,\ c,\ d)$$

$$(a,\ c,\ b,\ d)$$

$$a \neq c$$

$$n$$

$$m$$

$$4 \leq n \leq 10^5$$

$$4 \leq m \leq 2 \times 10^5$$

$$(u,\ v,\ w)$$

$$[d(u)-2]+[d(v)-2]+[d(w)-2]$$

$$2$$

$$x$$

$$y$$

$$3$$

$$[d(x)-1]\cdot\binom{d(y)-1}{2}$$

$$y$$

$$x$$

$$3$$

$$x$$

$$y$$

$$s$$

$$z$$

$$t$$

$$y$$

$$t$$

$$s$$

$$z$$

$$s$$

$$z$$

$$y$$

$$t$$

$$y$$

$$t$$

$$s$$

$$z$$

$$s$$

$$t$$

$$2$$

$$x$$

$$x$$

$$3$$

$$4$$

$$O(n+m\sqrt{m})$$

$$\binom{k-2}{d_1-1, d_2-1, \cdots, d_k-1} = \frac{(k-2)!}{(d_1-1)!(d_2-1)!\cdots(d_k-1)!}$$

$$\binom{k-2}{d_1-1, d_2-1, \cdots, d_k-1} \cdot \prod_{i=1}^{k} s_i^{d_i}$$

$$\sum_{d_i \ge 1, \sum_{i=1}^{k} d_i = 2k-2} \binom{k-2}{d_1-1, d_2-1, \cdots, d_k-1} \cdot \prod_{i=1}^{k} s_i^{d_i}$$

$$(x_1 + \ldots + x_m)^p = \sum_{c_i \ge 0,\ \sum_{i=1}^{m} c_i = p} \binom{p}{c_1, c_2, \cdots, c_m} \cdot \prod_{i=1}^{m} x_i^{c_i}$$

$$\sum_{e_i \ge 0, \sum_{i=1}^{k} e_i = k-2} \binom{k-2}{e_1, e_2, \cdots, e_k} \cdot \prod_{i=1}^{k} s_i^{e_i+1}$$

$$(s_1 + s_2 + \cdots + s_k)^{k-2} \cdot \prod_{i=1}^{k} s_i$$

$$n^{k-2} \cdot \prod_{i=1}^{k} s_i$$

$$0$$

$$1$$

$$1$$

$$n$$

$$[1, n]$$

$$n-2$$

$$n-2$$

$$O(n \log n)$$

$$2, 2, 3, 3, 2$$

$$p$$

$$p$$

$$p$$

$$x$$

$$p, x$$

$$x > p$$

$$x$$

$$x$$

$$2$$

$$> p$$

p

$n-2$

p

p

x

$x > p$

p

$x < p$

p

x

p

x

x

$O(n)$

n

1

1

1

n

1

$O(n \log n)$

p

K_n

n^{n-2}

$n-2$

$[1, n]$

n^{n-2}

n

m

k

$k-1$

s_i

k

d_i

$$i$$

$$\sum_{i=1}^{k} d_i = 2k - 2$$

$$d$$

$$i$$

$$s_i^{d_i}$$

$$d$$

$$d$$

$$e_i = d_i - 1$$

$$\sum_{i=1}^{k} e_i = k - 2$$

$$G$$

$$S$$

$$G$$

$$S$$

$$G$$

$$G$$

$$K_{3,3}$$

$$K_5$$

$$G$$

$$G$$

$$G$$

$$G$$

$$m$$

$$G$$

$$G$$

$$G$$

$$n(n \geq 3)$$

$$G$$

$$G$$

$$G$$

$$n - m + r = 2$$

$$n, m, r$$

$$G$$

$$p(p \geq 2)$$

$$G$$

$$n - m + r = p + 1$$

$$G$$

$$G$$

$$l(l \geq 3)$$

$$m \leq \frac{l}{l-2}(n-2)$$

$$p(p \geq 2)$$

$$G$$

$$m \leq \frac{l}{l-2}(n-p-1)$$

$$G$$

$$n \geq 3$$

$$m$$

$$m \leq 3n - 6$$

$$G_1$$

$$G_2$$

$$G$$

$$G$$

$$K_5$$

$$K_{3,3}$$

$$G$$

$$G$$

$$K_5$$

$$K_{3,3}$$

$$G$$

$$G^*$$

$$G$$

$$R_i$$

$$G^*$$

$$v_i^*$$

$$e$$

$$G$$

$$e$$

$$G$$

$$R_i$$

$$R_j$$

$$G^*$$

$$e^*$$

$$e$$

$$e^*$$

$$G^*$$

$$v_i^*, v_j^*$$

$$e^* = (v_i^*, v_j^*)$$

$$e^*$$

$$e$$

$$G$$

$$R_i$$

$$e^*$$

$$R_i$$

$$v_i^*$$

$$e^* = (v_i^*, v_j^*)$$

$$G^*$$

$$G$$

$$G^*$$

$$G$$

$$G^*$$

$$G$$

$$G^*$$

$$G^*$$

$$G$$

$$R_i, R_j$$

$$v_i^*, v_j^*$$

$$G^*$$

$$G^{**}$$

$$G$$

$$G$$

$$G$$

$$G$$

$$\tilde{G}$$

$$G$$

$$\tilde{G}$$

$$G$$

$$G$$

$$G$$

$$u, v$$

$$G' = G \cup (u, v)$$

$$G'$$

$$G$$

$$n(n \geq 3)$$

$$G$$

$$n$$

$$n(n \geq 3)$$

$$n - 2$$

$$G$$

$$n(n \geq 3)$$

$$m = 2n - 3$$

$$G$$

$$G$$

$$G$$

$$\kappa$$

$$G$$

$$G$$

$$K_4$$

$$K_{2,3}$$

$$u, v$$

$$\langle u, v\rangle$$

$$u$$

$$v$$

$$\langle u, v\rangle$$

$$k$$

$$\mathrm{dis}_{i,j}$$

$$i$$

$$j$$

$$\mathrm{dis}$$

$$\mathrm{dis}_{i,j} = \min\{\min\{\mathrm{dis}_{from,j-1}\}, \min\{\mathrm{dis}_{from,j} + w\}\}$$

$$from$$

$$i$$

$$w$$

$$j - 1 \geq k$$

$$\mathrm{dis}_{from,j}$$

$$\infty$$

$$k + 1$$

$$u_i$$

$$i$$

$$u$$

$$n$$

$$m$$

$$k$$

$$s$$

$$t$$

Input. The edges of the graph e, where each element in e is (u, v, w) denoting that there is an edge between u and v weighted w.
Output. The edges of the MST of the input graph.
Method.
$result \leftarrow \emptyset$
sort e into nondecreasing order by weight w
for each (u, v, w) in the sorted e
    if u and v are not connected in the union-find set
        connect u and v in the union-find set
        $result \leftarrow result \bigcup \{(u, v, w)\}$
return $result$

Input. The nodes of the graph V; the function $g(u, v)$ which means the weight of the edge (u, v); the function $adj(v)$ which means the nodes adjacent to v.
Output. The sum of weights of the MST of the input graph.
Method.
$result \leftarrow 0$
choose an arbitrary node in V to be the $root$
$dis(root) \leftarrow 0$
for each node $v \in (V - \{root\})$
    $dis(v) \leftarrow \infty$
$rest \leftarrow V$
while $rest \neq \emptyset$
    $cur \leftarrow$ the node with the minimum dis in $rest$
    $result \leftarrow result + dis(cur)$
    $rest \leftarrow rest - \{cur\}$
    for each node $v \in adj(cur)$
        $dis(v) \leftarrow \min(dis(v), g(cur, v))$
return $result$

Input. A graph G whose edges have distinct weights.
Output. The minimum spanning forest of G.
Method.
Initialize a forest F to be a set of one-vertex trees
while True
    Find the components of F and label each vertex of G by its component
    Initialize the cheapest edge for each component to "None"
    for each edge (u, v) of G
        if u and v have different component labels
            if (u, v) is cheaper than the cheapest edge for the component of u
                Set (u, v) as the cheapest edge for the component of u
            if (u, v) is cheaper than the cheapest edge for the component of v
                Set (u, v) as the cheapest edge for the component of v
    if all components'cheapest edges are "None"
        return F
    for each component whose cheapest edge is not "None"
        Add its cheapest edge to F

$$O(m \log m)$$

$$O(m\alpha(m, n))$$

$$O(m \log n)$$

$$O(m\log m)$$

$$n-1$$

$$F$$

$$T$$

$$e$$

$$e$$

$$T$$

$$T+e$$

$$F$$

$$f$$

$$f$$

$$e$$

$$f$$

$$e$$

$$f$$

$$e$$

$$T+e-f$$

$$T$$

$$T+e-f$$

$$F$$

$$O(1)$$

$$O(n^2+m)$$

$$O((n+m)\log n)$$

$$O(n\log n+m)$$

$$dis$$

$$n-1$$

$$F$$

$$T$$

$$e$$

$$e$$

$$T$$

$$T+e$$

$$f$$

$$f$$
$$e$$
$$f$$
$$e$$
$$f$$
$$e$$
$$T + e - f$$
$$e$$
$$f$$
$$T + e - f$$
$$F$$
$$E'$$
$$E'$$
$$V' \subseteq V$$
$$u$$
$$v$$
$$E'$$
$$E' = \emptyset$$
$$(u, v)$$
$$u$$
$$v$$
$$u$$
$$v$$
$$E'$$
$$E'$$
$$O(\log V)$$
$$O(E \log V)$$
$$O(\alpha(m))$$
$$T$$
$$M$$
$$e = (u, v, w)$$
$$T$$
$$u$$

$$v$$
$$e' = (s, t, w')$$
$$T$$
$$e$$
$$e'$$
$$M' = M + w - w'$$
$$T'$$
$$M'$$
$$u, v$$
$$2^i$$
$$2^i$$
$$u$$
$$v$$
$$u$$
$$v$$
$$2^i$$
$$O(m \log m)$$
$$G$$
$$G$$
$$w$$
$$w$$
$$G$$
$$n$$
$$0$$
$$n - 1$$
$$n$$
$$x$$
$$\leq val$$
$$y$$
$$x$$
$$\leq val$$
$$\geq p$$
$$O((n + m + Q) \log n)$$

$$\sum_{i=0}^{x-1}\left(\frac{h-d_i}{x}+1\right)$$

$$n$$

$$n$$

$$n$$

$$n$$

$$n$$

$$K$$

$$p$$

$$f(i+y)=f(i)+y$$

$$f(v)=f(u)+edge(u,v)$$

$$x□y□z□h$$

$$k\in[1,h]$$

$$k$$

$$ax+by+cz=k$$

$$0\leq a,b,c$$

$$1\leq x,y,z\leq 10^5$$

$$h\leq 2^{63}-1$$

$$x<y<z$$

$$d_i$$

$$p \bmod x=i$$

$$p$$

$$x$$

$$i$$

$$i\xrightarrow{y}(i+y)\bmod x$$

$$i\xrightarrow{z}(i+z)\bmod x$$

$$a_i$$

$$x$$

$$d_0,d_1,d_2,...,d_{x-1}$$

$$d_i$$

$$\{a_1,a_2,\cdots,a_n\}$$

$$\{a_1+d,a_2+d,\cdots,a_n+d\}$$

$i = 1$

$dis_1 = 1$

d_i

n

n

$1 \leq n \leq 10^5$

$O(n \log^2 n)$

1

10

1

1

$0 \leq k \leq n - 1$

k

$10k$

0

k

$k + 1$

1

n

n

1

0

1

0

10

1

$O(n)$

n

$(n \geq 3)$

u

v

w

$dis(u, v)$

u

$$v$$
$$u$$
$$v$$
$$dis(u,v)+w$$
$$dis(v,u)+w$$
$$O(n^2m)$$
$$O(m(n+m)\log n)$$
$$u,v$$
$$val(u,v)$$
$$k$$
$$k$$
$$dis$$
$$dis_{u,v}$$
$$u$$
$$v$$
$$[1,k)$$
$$w$$
$$w$$
$$u,v$$
$$k=w$$
$$dis_{u,v}+val(v,w)+val(w,u)$$
$$k$$
$$i<k,j<k$$
$$(i,j)$$
$$O(n^3)$$
$$n$$
$$q$$
$$x$$
$$50\%$$
$$n,q\leq 100$$
$$x$$
$$x$$
$$x$$

$$O(qn^3)$$

$$100\%$$

$$n, q \leq 400$$

$$x$$

$$x$$

$$x+1$$

$$O(qn^2 \log q)$$

$$x$$

$$x$$

$$x$$

$$+$$

$$(u, v)$$

$$x$$

$$u$$

$$x$$

$$u$$

$$x$$

$$v$$

$$x \to u \to v \to x$$

$$O(n^2)$$

$$O(qn^2)$$

$$d(i, j)$$

$$i, j$$

$$rk(i, j)$$

$$i$$

$$j$$

$$w = (u, v)$$

$$c$$

$$u$$

$$x$$

$$x \leq w$$

$$v$$

$$w - x$$

$$i$$

$$c$$

$$i$$

$$d(c,i) = \min(d(u,i) + x, d(v,i) + (w - x))$$

$$i$$

$$c$$

$$c$$

$$d(c,i)$$

$$f = \max\{d(c,i)\}, i \in [1,n]$$

$$ans \leftarrow \min(ans, d(i, rk(i,n)) \times 2)$$

$$d$$

$$rk(i,j)$$

$$ans \leftarrow \min(ans, d(i, rk(i,n)) \times 2)$$

$$w(u,v)$$

$$u$$

$$d(v, rk(u,i)) > \max_{j=i+1}^{n} d(v, rk(u,j))$$

$$ans \leftarrow \min(ans, d(u, rk(u,i)) + \max_{j=i+1}^{n} d(v, rk(u,j)) + w(u,v))$$

$$v$$

$$v$$

$$v$$

$$R$$

$$P$$

$$X$$

$$R, X$$

$$P$$

$$P$$

$$v$$

$$R$$

$$X$$

$$P$$

$$v$$

$$v$$

$$R$$

$$R, P, X$$

$$P$$

$$v$$

$$v$$

$$X$$

$$P, X$$

$$R$$

$$n$$

$$m$$

$$v$$

$$R$$

$$P$$

$$X$$

$$P \cup X$$

$$u$$

$$R$$

$$P \cap N(u)$$

$$N(u)$$

$$u$$

$$u$$

$$u$$

$$v$$

$$u$$

$$v$$

$$u$$

$$D_{ii}(G) = \deg(i), D_{ij} = 0, i \neq j$$

$$A_{ij}(G) = A_{ji}(G) = \#e(i,j), i \neq j$$

$$L(G) = D(G) - A(G)$$

$$D_{ii}^{out}(G) = \deg^{\text{out}}(i), D_{ij}^{out} = 0, i \neq j$$

$$A_{ij}(G) = \#e(i,j), i \neq j$$

$$L^{out}(G) = D^{out}(G) - A(G)$$

$$L^{in}(G) = D^{in}(G) - A(G)$$

$$t(G) = \det L(G) \begin{pmatrix} 1, 2, \cdots, i-1, i+1, \cdots, n \\ 1, 2, \cdots, i-1, i+1, \cdots, n \end{pmatrix}$$

$$t^{root}(G,k)=\det L^{out}(G)\begin{pmatrix}1,2,\cdots,k-1,k+1,\cdots,n\\1,2,\cdots,k-1,k+1,\cdots,n\end{pmatrix}$$

$$t^{leaf}(G,k)=\det L^{in}(G)\begin{pmatrix}1,2,\cdots,k-1,k+1,\cdots,n\\1,2,\cdots,k-1,k+1,\cdots,n\end{pmatrix}$$

$$ec(G)=t^{root}(G,k)\prod_{v\in V}(\deg(v)-1)!$$

$$\mid AB\mid=\sum_{\mid S\mid=n,S\subseteq[m]}\mid A_{[n],[S]}\mid\mid B_{[S],[n]}\mid$$

$$A_{i,j}=\begin{cases}\sqrt{\omega(e_j)} & u_i\in e_j\wedge u_i<\zeta(e_j,u_i)\\-\sqrt{\omega(e_j)} u_i\in e_j\wedge u_i>\zeta(e_j,u_i)\\0 & \text{otherwise}\end{cases}$$

$$M_{1,1}=\sum_{\mid S\mid=n-1,S\subseteq[\mid E\mid]}\mid B_{[n-1],[S]}\mid\mid(B^T)_{[S],[n-1]}\mid=\sum_{\mid S\mid=n-1,S\subseteq[\mid E\mid]}\mid B_{[n-1],[S]}\mid^2$$

$$F_k=\sum_{T\in\mathcal{T}_k}\omega(T)Q(T)=(-1)^{\mid V\mid-k}[x^k]P(x)$$

$$A_{i,j}=\begin{cases}\sqrt{\omega(e_j)} & u_i\text{ is }e_j\text{'s head}\\-\sqrt{\omega(e_j)} u_i\text{ is }e_j\text{'s tail}\\0 & \text{otherwise}\end{cases}$$

$$B_{i,j}=\begin{cases}\sqrt{\omega(e_j)} u_i\text{ is }e_j\text{'s head}\\0 & \text{otherwise}\end{cases}$$

$$\begin{pmatrix}2&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&2\end{pmatrix}-\begin{pmatrix}0&1&0&1\\1&0&1&0\\0&1&0&1\\1&0&1&0\end{pmatrix}=\begin{pmatrix}2&-1&0&-1\\-1&2&-1&0\\0&-1&2&-1\\-1&0&-1&2\end{pmatrix}$$

$$\begin{vmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{vmatrix}=4$$

$$L_n=\begin{bmatrix}n&-1&-1&-1&\cdots&-1&-1\\-1&3&-1&0&\cdots&0&-1\\-1&-1&3&-1&\cdots&0&0\\-1&0&-1&3&\cdots&0&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\-1&0&0&0&\cdots&3&-1\\-1&-1&0&0&\cdots&-1&3\end{bmatrix}_{n+1}$$

$$M_n = \begin{bmatrix} 3 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 3 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 3 \end{bmatrix}_n$$

$$\det M_n = 3 \det \begin{bmatrix} 3 & -1 & \cdots & 0 & 0 \\ -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & \cdots & -1 & 3 \end{bmatrix}_{n-1} + \det \begin{bmatrix} -1 & 0 & \cdots & 0 & -1 \\ -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 3 & -1 \\ 0 & 0 & \cdots & -1 & 3 \end{bmatrix}_{n-1} + (-1)^n \det \begin{bmatrix} -1 & 0 & \cdots & 0 & -1 \\ 3 & -1 & \cdots & 0 & 0 \\ -1 & 3 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 3 & -1 \end{bmatrix}_{n-1}$$

$$\det M_n = 3d_{n-1} - 2d_{n-2} - 2$$

$$d_n = 3d_{n-1} - d_{n-2}$$

$$\det M_n = 3 \det M_{n-1} - \det M_{n-2} + 2$$

$$G$$

$$n$$

$$D(G)$$

$$\#e(i, j)$$

$$i$$

$$j$$

$$A$$

$$L$$

$$G$$

$$t(G)$$

$$G$$

$$n$$

$$D^{out}(G)$$

$$D^{in}(G)$$

$$\#e(i, j)$$

$$i$$

$$j$$

$$A$$

$$L^{out}$$

$$L^{in}$$

$$G$$

$$r$$

$$t^{root}(G, r)$$

$$G$$

$$r$$

$$t^{leaf}(G, r)$$

$$i$$

$$L(G)\binom{1, 2, \cdots, i-1, i+1, \cdots, n}{1, 2, \cdots, i-1, i+1, \cdots, n}$$

$$L(G)$$

$$1, \cdots, i-1, i+1, \cdots, n$$

$$1, \cdots, i-1, i+1, \cdots, n$$

$$n-1$$

$$\lambda_1, \lambda_2, \cdots, \lambda_{n-1}$$

$$L(G)$$

$$n-1$$

$$t(G) = \frac{1}{n}\lambda_1\lambda_2\cdots\lambda_{n-1}$$

$$k$$

$$k$$

$$t^{root}(G, k)$$

$$k$$

$$k$$

$$t^{leaf}(G, k)$$

$$G$$

$$G$$

$$ec(G)$$

$$G$$

$$k, k'$$

$$t^{root}(G, k) = t^{root}(G, k')$$

$$G$$

$$[n] = \{1, 2, \cdots, n\}$$

$$A$$

$$A_{[S],[T]}$$

$$A_{i,j} \quad (i \in S, j \in T)$$

$$e$$

$$\omega(e)$$

$$n \times m$$

$$A$$

$$m \times n$$

$$B$$

$$G(V, E)$$

$$V = L \cup R \cup D$$

$$L = \{l_1, l_2, \cdots, l_n\}$$

$$R = \{r_1, r_2, \cdots, r_n\}$$

$$D = \{d_1, d_2, \cdots, d_m\}$$

$$E = E_L \cup E_R$$

$$E_L = \{(l_i, d_j) \mid i \in [1, n], j \in [1, m]\}$$

$$E_R = \{(d_i, r_j) \mid i \in [1, m], j \in [1, n]\}$$

$$\omega(l_i, d_j) = a_{i,j}$$

$$\omega(d_i, r_j) = b_{i,j}$$

$$L$$

$$R$$

$$S$$

$$D$$

$$L$$

$$C_{i,j}$$

$$i$$

$$\boldsymbol{r}_1, \boldsymbol{r}_2, \cdots, \boldsymbol{r}_n$$

$$\sum \boldsymbol{r}_i = \mathbf{0}$$

$$M_{i,j}$$

$$\boldsymbol{r}_k$$

$$\boldsymbol{r}_i$$

$$\boldsymbol{r}_k$$

$$A=[\boldsymbol{r}_1,\cdots,\boldsymbol{r}_{j-1},\boldsymbol{r}_{j+1},\cdots,\boldsymbol{r}_{k-1},-\boldsymbol{r}_j,\boldsymbol{r}_{k+1},\cdots,\boldsymbol{r}_n]$$

$$-\boldsymbol{r}_j$$

$$\boldsymbol{r}_{j+1}$$

$$\mid A\mid=M_{i,k}=(-1)^{1+(k-1)-(r+1)+1}M_{i,j}$$

$$C_{i,j}=C_{i,k}$$

$$i$$

$$\mathbf{0}$$

$$C_{j,i}=C_{k,i}$$

$$G(V,E)$$

$$T(V,E_T)$$

$$G$$

$$\omega(T)=\prod_{e\in E_T}\omega(e)$$

$$\mathcal{T}$$

$$G$$

$$G$$

$$\sum_{T\in\mathcal{T}}\omega(T)$$

$$C_{1,1}=\sum_{T\in\mathcal{T}}\omega(T)$$

$$e=(u,v)$$

$$\zeta(e,u)=v$$

$$\zeta(e,v)=u$$

$$u_i<u_j\Longleftrightarrow i<j$$

$$E=\{e_1,e_2,\cdots,e_{\mid E\mid}\}$$

$$L=AA^T$$

$$A$$

$$B$$

$$M_{1,1}=BB^T$$

$$\mid B_{[n-1],[S]}\mid$$

$$u_2,\cdots,u_n$$

$$S$$

$$u_i$$

$$e_j$$

$$(u_i, \zeta(e_j, u_i))$$

$$S$$

$$u_2, \cdots, u_n$$

$$1$$

$$f(f(x)) = x$$

$$f$$

$$1$$

$$\mid B_{[n-1],[S]} \mid^2$$

$$\sum_{T \in \mathcal{T}} \omega(T)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{\mid V \mid}$$

$$\lambda_{\mid V \mid} = 0$$

$$L = AA^T$$

$$L$$

$$\mid M \mid = 0$$

$$\lambda = 0$$

$$k$$

$$(V, E)$$

$$k$$

$$\mathcal{T}_k$$

$$G$$

$$k$$

$$Q(T)$$

$$T$$

$$L$$

$$P(x)$$

$$P(x) = \det(xI - L)$$

$$[x^k]$$

$$k$$

$$k$$

$$(-1)^{|V|-k}$$

$$F_1 = \prod_{i=1}^{|V|-1} \lambda_i$$

$$n$$

$$A, B$$

$$B$$

$$O(n^3)$$

$$L$$

$$i$$

$$i$$

$$k$$

$$\mathbb{Z}_k$$

$$n$$

$$n+1$$

$$n$$

$$n \leq 100000$$

$$L_n$$

$$M_n$$

$$d_{n-1}, a_{n-1}, b_{n-1}$$

$$d_n$$

$$d_n$$

$$d_n = 3d_{n-1} - d_{n-2}$$

$$a_{n-1} = -d_{n-2} - 1$$

$$(-1)^n b_{n-1} = -d_{n-2} - 1$$

$$\det M_n$$

$$(\det M_n + 2) = 3(\det M_{n-1} + 2) - (\det M_{n-2} + 2)$$

$$w_i x + 1$$

$$\times$$

$$O(n^3)$$

$$k$$

$$S - T$$

$$S$$

$$T$$

$$O(a + b + c + d)$$

$$M = \begin{bmatrix} e(A_1, B_1) & e(A_1, B_2) & \cdots & e(A_1, B_n) \\ e(A_2, B_1) & e(A_2, B_2) & \cdots & e(A_2, B_n) \\ \vdots & \vdots & \ddots & \vdots \\ e(A_n, B_1) & e(A_n, B_2) & \cdots & e(A_n, B_n) \end{bmatrix}$$

$$\det(M) = \sum_{S:A \to B} (-1)^{t(\sigma(S))} \prod_{i=1}^{n} \omega(S_i)$$

$$\begin{aligned} \det(M) &= \sum_{\sigma} (-1)^{t(\sigma)} \prod_{i=1}^{n} e(a_i, b_{\sigma(i)}) \\ &= \sum_{\sigma} (-1)^{t(\sigma)} \prod_{i=1}^{n} \sum_{P:a_i \to b_{\sigma(i)}} \omega(P) \end{aligned}$$

$$\begin{aligned} &\sum_{\sigma} (-1)^{t(\sigma)} \prod_{i=1}^{n} \sum_{P:a_i \to b_{\sigma(i)}} \omega(P) \\ =& \sum_{\sigma} (-1)^{t(\sigma)} \sum_{P=\sigma} \omega(P) \\ =& \sum_{P:A \to B} (-1)^{t(\sigma)} \prod_{i=1}^{n} \omega(P_i) \end{aligned}$$

$$\begin{aligned} &\sum_{P:A \to B} (-1)^{t(\sigma)} \prod_{i=1}^{n} \omega(P_i) \\ =& \sum_{U:A \to B} (-1)^{t(U)} \prod_{i=1}^{n} \omega(U_i) + \sum_{V:A \to B} (-1)^{t(V)} \prod_{i=1}^{n} \omega(V_i) \end{aligned}$$

$$\begin{vmatrix} f(a_1, b_1) & f(a_1, b_2) \\ f(a_2, b_1) & f(a_2, b_2) \end{vmatrix} = f(a_1, b_1) \times f(a_2, b_2) - f(a_1, b_2) \times f(a_2, b_1)$$

$$\omega(P)$$

$$P$$

$$1$$

$$e(u, v)$$

$$u$$

$$v$$

$$P$$

$$\omega(P)$$

$$e(u,v) = \sum_{P:u\to v} \omega(P)$$

$$A$$

$$n$$

$$B$$

$$n$$

$$A \to B$$

$$S$$

$$S_i$$

$$A_i$$

$$B_{\sigma(S)_i}$$

$$\sigma(S)$$

$$i \neq j$$

$$S_i$$

$$S_j$$

$$t(\sigma)$$

$$\sigma$$

$$\sum_{S:A\to B}$$

$$A \to B$$

$$S$$

$$\prod_{i=1}^{n} \sum_{P:a_i\to b_{\sigma(i)}} \omega(P)$$

$$A$$

$$B$$

$$\sigma$$

$$P$$

$$\omega(P)$$

$$P$$

$$U$$

$$V$$

$$P$$

$$P_i : a_1 \to u \to b_1, P_j : a_2 \to u \to b_2$$

$$P_{(i)'} = a_1 \to u \to b_2, P_{(j)'} = a_2 \to u \to b_1$$

$$P'$$

$$P$$

$$\omega(P) = \omega(P'), t(P) = t(P') \pm 1$$

$$\sum_{V:A\to B} (-1)^{t(\sigma)} \prod_{i=1}^{n} \omega(V_i) = 0$$

$$\det(M) = \sum_{U:A\to B} (-1)^{t(U)} \prod_{i=1}^{n} \omega(U_i) = 0$$

$$n \times m$$

$$(x, y)$$

$$(x+1, y)$$

$$(x, y+1)$$

$$(1, 1)$$

$$(n, m)$$

$$10^9 + 7$$

$$2 \le n, m \le 3000$$

$$(1, 1)$$

$$A = \{(1, 2), (2, 1)\}$$

$$B = \{(n-1, m), (n, m-1)\}$$

$$A, B$$

$$f(a, b)$$

$$a \to b$$

$$O(nm)$$

$$f$$

$$O(nm)$$

$$n \times n$$

$$(x, y)$$

$$(x, y+1)$$

$$(x+1, y)$$

$$k$$

$$i$$

$$(1, a_i)$$

$$(n, b_i)$$

$$10^9 + 7$$

$$1 \leq n \leq 10^5$$

$$1 \leq k \leq 100$$

$$1 \leq a_1 < a_2 < ... < a_n \leq n$$

$$1 \leq b_1 < b_2 < ... < b_n \leq n$$

$$a_i$$

$$b_i$$

$$\sigma(S)_i = i$$

$$1$$

$$(1, a_i)$$

$$(n, b_j)$$

$$n - 1 + b_j - a_i$$

$$n - 1$$

$$e(A_i, B_j) = \binom{n - 1 + b_j - a_i}{n - 1}$$

$$O(n + k(k^2 + \log p))$$

$$\log p$$

$$S = \{v_1, v_2, ..., v_n\}$$

$$\mathrm{LCA}(v_1, v_2, ..., v_n)$$

$$\mathrm{LCA}(S)$$

$$\mathrm{LCA}(\{u\}) = u$$

$$u$$

$$v$$

$$\mathrm{LCA}(u, v) = u$$

$$u$$

$$v$$

$$v$$

$$u$$

$$u, v$$

$$\mathrm{LCA}(u, v)$$

$$\mathrm{LCA}(S)$$

$$S$$

$$\mathrm{LCA}(S)$$

$$S$$

$$\mathrm{LCA}(A \cup B) = \mathrm{LCA}(\mathrm{LCA}(A), \mathrm{LCA}(B))$$

$$d(u, v) = h(u) + h(v) - 2h(\mathrm{LCA}(u, v))$$

$$d$$

$$h$$

$$O(n)$$

$$\Theta(n)$$

$$\mathrm{fa}_{x,i}$$

$$\mathrm{fa}_{x,i}$$

$$x$$

$$2^i$$

$$\mathrm{fa}_{x,i}$$

$$u, v$$

$$u, v$$

$$y$$

$$y$$

$$y$$

$$y$$

$$i$$

$$0$$

$$0$$

$$\mathrm{fa}_{u,i} \neq \mathrm{fa}_{v,i}$$

$$u \leftarrow \mathrm{fa}_{u,i}, v \leftarrow \mathrm{fa}_{v,i}$$

$$\mathrm{fa}_{u,0}$$

$$O(n \log n)$$

$$O(\log n)$$

$$7$$

$$O(n)$$

$$m$$

$$O(m\alpha(m+n,n)+n)$$

$$O(m+n)$$

$$O(1)$$

$$O(m\alpha(m+n,n)+n)$$

$$O(m+n)$$

$$2n-1$$

$$u$$

$$pos(u)$$

$$u$$

$$E[1..2n-1]$$

$$pos(LCA(u,v)) = \min\{pos(k)|\ k \in E[pos(u)..pos(v)]\}$$

$$u$$

$$v$$

$$LCA(u,v)$$

$$LCA(u,v)$$

$$u$$

$$v$$

$$LCA(u,v)$$

$$O(n)$$

$$O(n)$$

$$O(n)$$

$$(u,v)$$

$$[\min\{pos[u],pos[v]\},\max\{pos[u],pos[v]\}]$$

$$O(n\log n)$$

$$O(1)$$

$$O(n)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(n) \sim O(1)$$

$$O(n) \sim O(1)$$

$$O(n) \sim O(1)$$

$$O(n) \sim O(1)$$

$$O(n)$$

$$n$$

$$m$$

$$s$$

$$t$$

$$k$$

$$x$$

$$f(x) = g(x) + h(x)$$

$$g(x)$$

$$h(x)$$

$$f(x)$$

$$x$$

$$k$$

$$h(x)$$

$$t$$

$$t$$

$$x$$

$$g(x)$$

$$(x, g(x))$$

$$(s, 0)$$

$$f(x) = g(x) + h(x)$$

$$x$$

$$k$$

$$g(x)$$

$$x$$

$$k$$

$$k$$

$$k$$

$$k$$

$$k$$

$$n$$

$$O(nk \log n)$$

$$s$$

$$k$$

$$t$$

$$x$$

$$t$$

$$dist_x$$

$$t$$

$$T$$

$$x$$

$$t$$

$$x$$

$$t$$

$$s$$

$$t$$

$$P$$

$$P$$

$$T$$

$$P'$$

$$P'$$

$$T$$

$$e$$

$$u$$

$$v$$

$$w$$

$$\Delta e = dist_v + w - dist_u$$

$$P$$

$$L_P = dist_s + \sum_{e \in P'} \Delta e$$

$$P$$

$$P'$$

$$s$$

$$t$$

$$P'$$

$$e_1, e_2$$

u_{e_2}

v_{e_1}

T

P

e_1, e_2

P'

S

$S' = P'$

2

P'

T

1

P'

L_P

2

P'

P'

P'

3

P'

P

L_P

k

2

P'

2

s

t

L_P

P'

P'

1

1

$$T$$
$$\Delta e$$
$$S$$
$$S$$
$$\Delta e$$
$$x$$
$$S$$
$$2$$
$$x$$
$$x$$
$$T$$
$$k-1$$
$$s$$
$$t$$
$$k$$
$$x$$
$$\Delta e$$
$$x$$
$$x$$
$$T$$
$$x$$
$$t$$
$$S$$
$$t$$
$$x$$
$$S$$
$$x$$
$$O(n+k)$$
$$T$$
$$O(nm\log m)$$
$$O(nm)$$
$$T$$
$$T$$
$$O((n+m)\log m + k\log k)$$

$$O(m + n\log m + k)$$

$$O(k)$$

$$O(k\log m)$$

TREE-BUILD (u, dep)

$u.hson \leftarrow 0$
$u.hson.size \leftarrow 0$
$u.deep \leftarrow dep$
$u.size \leftarrow 1$
foreach u ‘s son v
    $u.size \leftarrow u.size +$ TREE-BUILD $(v, dep + 1)$
    $v.father \leftarrow u$
    if $v.size > u.hson.size$
        $u.hson \leftarrow v$
return $u.size$

TREE-DECOMPOSITION (u, top)

$u.top \leftarrow top$
$tot \leftarrow tot + 1$
$u.dfn \leftarrow tot$
$rank(tot) \leftarrow u$
if $u.hson$ is not 0
    TREE-DECOMPOSITION $(u.hson, top)$
    foreach u ‘s son v
        if v is not $u.hson$
            TREE-DECOMPOSITION (v, v)

TREE-PATH-SUM (u, v)

$tot \leftarrow 0$
while $u.top$ is not $v.top$
    if $u.top.deep < v.top.deep$
        SWAP(u, v)
    $tot \leftarrow tot +$ sum of values between u and $u.top$
    $u \leftarrow u.top.father$
$tot \leftarrow tot +$ sum of values between u and v
return tot

$$T(n) \leq \begin{cases} 0 & n = 1 \\ T\left(\left\lfloor \frac{n-1}{2} \right\rfloor\right) + 1 & n \geq 2 \end{cases}$$

$$O(\log n)$$

$$O(\log n)$$

$$fa(x)$$

$$x$$

$$dep(x)$$

$$x$$

$$siz(x)$$

$$x$$

$$son(x)$$

$$x$$

$$top(x)$$

$$x$$

$$dfn(x)$$

$$x$$

$$rnk(x)$$

$$rnk(dfn(x)) = x$$

$$fa(x)$$

$$dep(x)$$

$$siz(x)$$

$$son(x)$$

$$top(x)$$

$$dfn(x)$$

$$rnk(x)$$

$$O(\log n)$$

$$O(\log n)$$

$$x$$

$$v$$

$$O(\log n)$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$\frac{\log n}{2}$$

$$O(\sqrt{n} \log n)$$

$$n$$

$$q$$

$$u$$

$$v$$

$$u$$

$$v$$

$$1 \leq n \leq 30000$$

$$0 \leq q \leq 200000$$

$$O(\log n)$$

$$O(\log n)$$

$$O(n \log n + q \log^2 n)$$

$$O(\log n)$$

$$1$$

$$3000$$

$$30000$$

$$n-1$$

$$u$$

$$k$$

$$k$$

$$u$$

$$k$$

$$d$$

$$dis(k, bot[u])$$

$$v$$

$$(dep[k] + dep[bot[u]] - d)/2$$

$$v$$

$$k$$

$$v$$

$$w$$

$$k$$

$$O(n^2)$$

$$O(n \log n)$$

$$T(n)$$

$$n$$

$$2999 + \sum_{i=1}^{2999} T(i) \leq 29940$$

$$21000$$

$$\sqrt{n}$$

$$D$$

$$D \neq 0$$

$$D - 1$$

$$2D - 1$$

$$D = 0$$

$$D^2$$

$$D = \sqrt{n}$$

$$f_{i,j}$$

$$O(n^2)$$

$$n$$

$$O(n)$$

$$k$$

$$2^i$$

$$2^i$$

$$2^i \le k < 2^{i+1}$$

$$d$$

$$1$$

$$d$$

$$k - 2^i \le 2^i \le d$$

$$O(1)$$

$$k$$

$$2^i$$

$$O(n \log n)$$

$$O(1)$$

$$G = \langle V, E \rangle$$

$$V$$

$$V_1$$

$$p(G - V_1) \le |\ V_1|$$

$$p(x)$$

$$x$$

$$G = \langle V, E \rangle$$

$$V$$

$$V_1$$

$$p(G-V_1) \leq |V_1|+1$$

$$p(x)$$

$$x$$

$$K_{2k+1}(k \geq 1)$$

$$k$$

$$k$$

$$K_{2k+1}$$

$$K_{2k}(k \geq 2)$$

$$k-1$$

$$K_{2k}$$

$$k-1$$

$$k$$

$$G$$

$$n(n \geq 2)$$

$$G$$

$$v_i, v_j$$

$$d(v_i)+d(v_j) \geq n-1$$

$$G$$

$$G$$

$$n(n \geq 3)$$

$$G$$

$$v_i, v_j$$

$$d(v_i)+d(v_j) \geq n$$

$$G$$

$$G$$

$$G$$

$$n(n \geq 3)$$

$$G$$

$$v_i$$

$$d(v_i) \geq \frac{n}{2}$$

$$G$$

$$G$$

$$D$$

$$n(n \geq 2)$$

$$D$$

$$D$$

$$n(n \geq 2)$$

$$D$$

$$D$$

$$n(n \geq 2)$$

$$D$$

$$P_{x,y} = \begin{cases} w_{(x,y)} & \text{if } (x,y) \in E \text{ and } x \neq t \\ 0 & \text{if } (x,y) \notin E \text{ or } x = t \end{cases}$$

$$\sum_{k \geq 0} k \times \left(P^k\right)_{s,t}$$

$$f(i,j) = \begin{cases} p_1 f(i-1,j) + p_2 f(i,j-1) + p_3 f(i+1,j) + p_4 f(i,j+1) + 1 & i^2 + j^2 \leq R \\ 0 & i^2 + j^2 > R \end{cases}$$

$$f(i+1,j) = \frac{f(i,j) - p_1 f(i-1,j) - p_2 f(i,j-1) - p_4 f(i,j+1) - 1}{p_3}$$

$$f(i,j) = \sum_{(k,j) \in E} \frac{f(i-1,k)}{\deg_k} (j \neq n)$$

$$f_{i,j} = 1 + \sum_{1 \leq k \leq n} p_{i,k} f_{k,j}$$

$$f_{i,i} = 1 - g_i + \sum_{1 \leq k \leq n} p_{i,k} f_{k,i}$$

$$F = J - G + PF$$

$$(I - P)F = J - G$$

$$G = PF + J - F$$

$$\pi G = \pi PF + \pi J - \pi F$$

$$\pi G = \pi J$$

$$\pi_i g_i = \sum_{j=1}^{n} \pi_j = 1$$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & 0 & \cdots & 0 & a_2 \\ 0 & 0 & 1 & \cdots & 0 & a_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & a_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} X = \begin{bmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \cdots & b_{1,n-1} & b_{1,n} \\ b_{2,1} & b_{2,2} & b_{2,3} & \cdots & b_{2,n-1} & b_{2,n} \\ b_{3,1} & b_{3,2} & b_{3,3} & \cdots & b_{3,n-1} & b_{3,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \cdots & b_{n-1,n-1} & b_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$G = (V, E)(V = \{v_1, v_2, \cdots, v_{\mid V \mid}\})$$

$$s \in V$$

$$t \in V$$

$$e = (x, y)$$

$$w_e$$

$$\forall x \in V \setminus \{t\}$$

$$\sum_{(x,y) \in E} w_{(x,y)} = 1$$

$$x$$

$$x$$

$$t$$

$$x$$

$$w_{(x,y)}$$

$$(x, y)$$

$$y$$

$$P$$

$$\left(P^k\right)_{s,t}$$

$$k$$

$$P$$

$$n$$

$$\mid V \mid$$

$$m$$

$$\mid E \mid$$

$$(0, 0)$$

$$(x, y)$$

$$p_1$$

$$(x - 1, y)$$

$$p_2$$

$$(x, y-1)$$

$$p_3$$

$$(x+1, y)$$

$$p_4$$

$$(x, y+1)$$

$$p_1 + p_2 + p_3 + p_4 = 1$$

$$R$$

$$0 \leq R \leq 50$$

$$p_1, p_2, p_3, p_4 > 0$$

$$10^9 + 7$$

$$f(i, j)$$

$$(i, j)$$

$$R$$

$$O(R^6)$$

$$O(R)$$

$$O(R^2)$$

$$O(R^2)$$

$$O(R^4)$$

$$O(R^2)$$

$$O(R)$$

$$2R + 1$$

$$(i, j)$$

$f(i, j)$□$f(i-1, j)$□$f(i, j-1)$□$f(i, j+1)$

$$f(i+1, j)$$

$$(i+1, j)$$

$$R$$

$$f(i+1, j) = 0$$

$$2R + 1$$

$$O(R^2)$$

$$O(R)$$

$$O(R)$$

$$O(R^3)$$

$$O(R^3)$$

$$O(n\sqrt{n})$$

$$O(\sqrt{n})$$

$$O(n^2)$$

$$0$$

$$0$$

$$0$$

$$O(R)$$

$$f(i,j) = p_1 f(i+1,j) + p_2 f(i,j+1) + p_3 f(\mathrm{pre}(i,j)) + 1$$

$$\mathrm{pre}(i,j) = (x,y)(x \le i, y \le j)$$

$$G = (V,E)$$

$$v_1$$

$$v_n$$

$$n \le 2000$$

$$p$$

$$p$$

$$[10^9, 1.01 \times 10^9]$$

$$p(A) = 0$$

$$p(\lambda)$$

$$A$$

$$I_n$$

$$n$$

$$n \times n$$

$$A$$

$$p(\lambda) = \det(\lambda I_n - A)$$

$$\det$$

$$n$$

$$A$$

$$n$$

$$n$$

$$n$$

$$E(t) = \sum_{i \geq 0} \Pr[t > i]$$

$$i$$

$$i \geq 0$$

$$f(i, j)$$

$$i$$

$$j$$

$$n$$

$$\deg_k$$

$$k$$

$$f$$

$$i$$

$$fi + 1 = f_i M$$

$$M$$

$$n$$

$$f$$

$$n$$

$$\Pr[t > i] = \sum_{j=1}^{n-1} f(i, j)$$

$$n$$

$$O(nm)$$

$$\Pr[t > 0], \Pr[t > 1], \cdots, \Pr[t > 3n]$$

$$O(n^2)$$

$$\Pr[t > i]$$

$$k$$

$$a$$

$$i \geq i_0$$

$$a_i = \sum_{j=1}^{k} c_j a_{i-j}$$

$$a$$

$$c$$

$$A(x)$$

$$C(x)$$

$$A(x) = A(x)C(x) + A_0(x)$$

$$A_0(x)$$

$$i < i_0$$

$$\Pr[t > i]$$

$$C(x)$$

$$A_0(x)$$

$$A(x) = \frac{A_0(x)}{1 - C(x)}$$

$$\sum_{i \geq 0} [x^i] A(x)$$

$$A(1)$$

$$x = 1$$

$$0$$

$$O(nm + n^2)$$

$$G$$

$$O(n^2)$$

$$G = (V, E)$$

$$1 \leq s \leq n$$

$$1 \leq t \leq n$$

$$s \neq t$$

$$v_s$$

$$v_t$$

$$3 \leq n \leq 400$$

$$p_{i,j}$$

$$i$$

$$(i, j)$$

$$j$$

$$0$$

$$f_{i,j}$$
$$i$$
$$j$$
$$f_{i,i} = 0$$
$$i \neq j$$
$$i = j$$
$$g_i$$
$$i$$
$$i$$
$$P$$
$$F$$
$$I$$
$$n$$
$$J$$
$$n$$
$$1$$
$$G$$
$$n$$
$$G_{i,i} = g_i$$
$$0$$
$$G$$
$$n$$
$$P$$
$$n$$
$$\pi$$
$$\sum_{i=1}^{n} \pi_i = 1$$
$$\pi P = \pi$$
$$\pi$$
$$[0, 1]$$
$$\pi_i$$
$$v_i$$
$$O(n^3)$$
$$\pi$$

$$\pi$$

$$G$$

$$1 \leq i \leq n$$

$$\pi_i g_i = 1$$

$$F = J - G + PF$$

$$\pi$$

$$\pi$$

$$\pi P = \pi$$

$$O(n^3)$$

$$G$$

$$(I - P)$$

$$G = (V, E)$$

$$r \in V$$

$$G$$

$$T = (V, A)$$

$$i \neq r$$

$$i$$

$$1$$

$$r$$

$$0$$

$$T$$

$$G$$

$$D$$

$$D_{i,i} = d_i$$

$$D_{i,j} = 0 (i \neq j)$$

$$d_i$$

$$i$$

$$A$$

$$r$$

$$D - A$$

$$r$$

$$r$$

$$G = (V, E)$$

P

$(I - P)$

$n - 1$

$(I - P)$

i

v_i

L

L

$n - 1$

L

0

L

L

n

L

G

L

i

i

v_i

0

L

i

i

L

i

L

$n - 1$

$AX = B$

A

B

X

A

A

$$B$$

$$A$$

$$n-1$$

$$n$$

$$0$$

$$X_{n,i}=0$$

$$Y$$

$$X_{n,i}=0$$

$$Y_{i,j}=1+Y_{j,j}+P_{i,k}X_{k,j}$$

$$X_{i,j}=Y_{i,j}-Y_{j,j}$$

$$O(n^3)$$

$$G=(V,E)$$

$$E$$

$$(u,v)$$

$$c(u,v)$$

$$(u,v)\notin E$$

$$c(u,v)=0$$

$$V$$

$$s$$

$$t$$

$$s\neq t$$

$$G=(V,E)$$

$$E$$

$$0\leq f(u,v)\leq c(u,v)$$

$$u$$

$$0$$

$$u$$

$$f(u)=\sum_{x\in V}f(u,x)-\sum_{x\in V}f(x,u)$$

$$G=(V,E)$$

$$f$$

$$f$$

$$| f |$$

$$s$$

$$f(s)$$

$$t$$

$$-f(t)$$

$$G = (V, E)$$

$$\{S, T\}$$

$$V$$

$$S \cup T = V$$

$$S \cap T = \emptyset$$

$$s \in S, t \in T$$

$$\{S, T\}$$

$$G$$

$$s$$

$$t$$

$$s$$

$$t$$

$$\{S, T\}$$

$$| \, | \, S, T \, | \, | = \sum_{u \in S} \sum_{v \in T} c(u, v)$$

$$G = (V, E)$$

$$f$$

$$f$$

$$f$$

$$G$$

$$G = (V, E)$$

$$s$$

$$t$$

$$\{S, T\}$$

$$\{S, T\}$$

$$\{S, T\}$$

$$G$$

$$G = (V, E)$$

$$w(u, v)$$

$$(u, v)$$

$$G$$

Input. The edges of the graph e, where each element in e is (u, v)
Output. The vertex of the Euler Road of the input graph.
Method.
Function Hierholzer (v)
    $circle \leftarrow$ Find a Circle in e Begin with v
    if $circle = \emptyset$
        return v
    $e \leftarrow e - circle$
    for each $v \in circle$
        $v \leftarrow$ Hierholzer(v)
    return $circle$
Endfunction
return Hierholzer(any vertex)

$$G$$

$$\Theta(m(n+m)) = \Theta(m^2)$$

$$O(nm + m^2)$$

$$\Theta(n + m \log m)$$

$$\Theta(n+m)$$

$$\Theta(n+m)$$

$$m$$

$$m^n$$

$$n$$

$$n$$

$$m^n$$

$$m$$

$$n$$

$$m^n$$

$$S = \{a_1, a_2, \cdots, a_m\}$$

$$D = \langle V, E \rangle$$

$$V = \{a_{i_1} a_{i_2} \cdots a_{i_{n-1}} \mid a_i \in S, 1 \le i \le n-1\}$$

$$E = \{a_{j_1} a_{j_2} \cdots a_{j_{n-1}} \mid a_j \in S, 1 \le j \le n\}$$

$$D$$

$$a_{i_1}a_{i_2}\cdots a_{i_{n-1}}$$

$$m$$

$$a_{i_1}a_{i_2}\cdots a_{i_{n-1}}a_r, r=1,2,\cdots,m$$

$$a_{j_1}a_{j_2}\cdots a_{j_{n-1}}$$

$$a_{j_2}a_{j_3}\cdots a_{j_n}$$

$$D$$

$$m$$

$$D$$

$$D$$

$$C$$

$$C$$

$$C$$

$$1\le m\le 1024$$

$$\Theta(nm)$$

$$x$$

$$x$$

$$x$$

$$x$$

$$rt$$

$$rt$$

$$O(n)$$

$$h$$

$$O(nh)$$

$$O(\log n)$$

$$O(n\log n)$$

$$O(n^2)$$

$$O(\log n)$$

$$O(\log n)$$

$$dep[x]$$

$$n$$

$$n\le 10^5, m\le 5\times 10^5$$

$$x$$

$$dist[x]$$
$$x$$
$$x$$
$$ch[x]$$
$$x$$
$$x$$
$$ch[x]$$
$$x$$
$$x$$
$$ans$$
$$dist[x], ch[x], ans$$
$$dist[x]$$
$$O(\log n)$$
$$ch[x], ans$$
$$dist[x]$$
$$x$$
$$u$$
$$dist[u]$$
$$dist(x, u)$$
$$ch[x], ans$$
$$ch[x]$$
$$0$$
$$n$$
$$v[x]$$
$$x$$
$$y$$
$$x$$
$$y$$
$$v[x] = y$$
$$dist[x]$$
$$x$$
$$x$$
$$ch[x]$$

$$x$$
$$x$$
$$x$$
$$y$$
$$dist[x]$$
$$0$$
$$y$$
$$x$$
$$u$$
$$v$$
$$d = dist(x, u)$$
$$x$$
$$v$$
$$dist[u]$$
$$0$$
$$y - d$$
$$v$$
$$ch[v]$$
$$0$$
$$y - d$$
$$dist[x]$$
$$ch[x]$$
$$n$$
$$1$$
$$u$$
$$c_u$$
$$u$$
$$u$$
$$n \leq 2 \times 10^5$$
$$O(1)$$
$$O(n^2)$$
$$n$$
$$n$$
$$O(n^2)$$

$$O(n)$$

$$i$$

$$u$$

$$u$$

$$u$$

$$u$$

$$u$$

$$O(n)$$

$$u$$

$$x$$

$$x$$

$$2x$$

$$n$$

$$\log n$$

$$y$$

$$y < n/2^x$$

$$n > 2^x$$

$$x < \log n$$

$$+1$$

$$+1$$

$$= \log n + 1$$

$$O(n(\log n + 1)) = O(n \log n)$$

$$O(m)$$

$$O(n^2 \log n)$$

$$O(n \log^2 n)$$

$$O(n\sqrt{m})$$

$$dom(u) = \{u\} \cup \left(\bigcap_{v \in pre(u)} dom(v) \right)$$

$$idom(u) = \begin{cases} sdom(u), \text{if } sdom(u) = sdom(v) \\ idom(v), & \text{otherwise} \end{cases}$$

$$s$$

$$u$$

$$s$$

u

v

v

u

v

u

$v\ dom\ u$

s

s

2

1

3

$1, 2$

$1, 2, 3$

$1, 2$

$u, v, w \neq s$

s

s

u

s

u

S

S

S

s

u

u

dom

v

v

dom

w

u

dom

w

w

$$v$$
$$v$$
$$u$$
$$w$$
$$u$$
$$u \ dom \ w$$
$$u \ dom \ v$$
$$v \ dom \ u$$
$$u = v$$
$$u \neq v$$
$$v$$
$$u$$
$$u$$
$$v$$
$$u \neq v \neq w$$
$$u \ dom \ w$$
$$v \ dom \ w$$
$$u \ dom \ v$$
$$v \ dom \ u$$
$$s \rightarrow ... \rightarrow u \rightarrow ... \rightarrow v \rightarrow ... \rightarrow w$$
$$u$$
$$v$$
$$u$$
$$s$$
$$v$$
$$s \rightarrow ... \rightarrow v \rightarrow ... \rightarrow w$$
$$u \ dom \ w$$
$$O(n^3)$$
$$pre(u)$$
$$u$$
$$O(n^2)$$
$$s$$
$$s$$
$$u$$
$$v$$
$$u$$

$$idom(u) = v$$

$$s$$

$$s$$

$$u$$

$$idom(u)$$

$$u$$

$$n$$

$$n - 1$$

$$\{s_1, s_2, ..., s_k\}$$

$$s \to ... \to s_1 \to ... \to s_2 \to ... \to ... \to s_k \to ... \to u$$

$$u$$

$$s_k$$

$$u$$

$$S$$

$$v \in S$$

$$\forall w \in S \setminus \{u, v\}, w \ dom \ v$$

$$idom(u) = v$$

$$v \ dom \ u$$

$$\forall w \in pre(u), v \ dom \ w$$

$$s$$

$$u$$

$$w \in pre(u)$$

$$v$$

$$s$$

$$u$$

$$v$$

$$v \ dom \ u$$

$$\exists w \in pre(u)$$

$$v$$

$$w$$

$$v$$

$$s \to \cdots \to w \to \cdots \to u$$

$$v$$

$$u$$

$$u$$

$$u$$

$$O(n\alpha(n, m))$$

$$s$$

$$T$$

$$dfn(u)$$

$$u$$

$$u < v$$

$$dfn(u) < dfn(v)$$

$$u$$

$$v$$

$$u, v$$

$$u$$

$$u$$

$$sdom(u)$$

$$sdom(u) = \min(v \mid \exists v = v_0 \to v_1 \to ... \to v_k = u, \forall 1 \le i \le k-1, v_i > u)$$

$$u$$

$$sdom(u) < u$$

$$u$$

$$T$$

$$fa(u)$$

$$fa(u) < u$$

$$u$$

$$u$$

$$idom(u)$$

$$T$$

$$T$$

$$s$$

$$u$$

$$idom(u)$$

$$u$$

$$sdom(u)$$

$$T$$

$$sdom(u)$$

$$u$$

$$fa(sdom(u))$$

$$fa(sdom(u)) < sdom(u)$$

$$sdom(u)$$

$$u$$

$$idom(u)$$

$$sdom(u)$$

$$s$$

$$sdom(u)$$

$$u$$

$$sdom(u)$$

$$u$$

$$u$$

$$idom(u)$$

$$sdom(u)$$

$$u \neq v$$

$$v$$

$$u$$

$$v$$

$$idom(u)$$

$$idom(u)$$

$$idom(v)$$

$$v$$

$$idom(v)$$

$$w$$

$$w$$

$$s$$

$$idom(v)$$

$$v$$

$$w$$

$$idom(u)$$

$$idom(u)$$

$$v$$

$$idom(v)$$

$$u$$

$$T$$

$$u$$

$$sdom(u) = \min(\{v \mid \exists v \to u, v < u)\} \cup \{sdom(w) \mid w > u \; and \; \exists w \to ... \to v \to u\})$$

$$x$$

$$sdom(u) \leq x$$

$$x$$

$$u$$

$$x = v_0 \to ... \to v_j = w$$

$$T$$

$$\forall i \in [j, k-1], v_i \geq w > u$$

$$w = v_j \to ... \to v_k = v$$

$$v \to u$$

$$sdom(u) \geq x$$

$$u$$

$$sdom(u) = v_0 \to v_1 \to ... \to v_k = u$$

$$k = 1$$

$$k > 1$$

$$k = 1$$

$$sdom(u) \to u$$

$$k > 1$$

$$j$$

$$j \geq 1$$

$$v_j$$

$$v_{k-1}$$

$$T$$

$$k$$

$$j$$

$$v_0 \to ... \to v_j$$

$$v_j$$

$$\forall i \in [1, j), v_i > v_j$$

$$i$$

$$v_i < v_j$$

$$v_i$$

$$v_i$$

$$v_j$$

$$j$$

$$sdom(v_j) \leq sdom(u)$$

$$sdom(u) \leq x$$

$$x = sdom(u)$$

$$T$$

$$u$$

$$sdom(u) \rightarrow u$$

$$G$$

$$u$$

$$T$$

$$sdom(u)$$

$$w$$

$$v$$

$$sdom(v) \geq sdom(w)$$

$$idom(u) = sdom(u)$$

$$idom(u)$$

$$sdom(u)$$

$$sdom(u)\ dom\ u$$

$$s$$

$$u$$

$$P$$

$$sdom(u)$$

$$P$$

$$v$$

$$P$$

$$v < sdom(u)$$

$$v$$

$$sdom(u) = idom(u) = s$$

$$w$$
$$P$$
$$v$$
$$sdom(u)$$
$$u$$
$$sdom(w) \leq v < sdom(v)$$
$$T$$
$$v$$
$$w$$
$$v = v_0 \rightarrow ...v_k = w$$
$$i \in [1, k-1], v_i < w$$
$$j \in [i, k-1]$$
$$v_j$$
$$w$$
$$v$$
$$sdom(u) \leq v_j$$
$$v_j$$
$$sdom(u)$$
$$u$$
$$w$$
$$sdom(w) \leq v < sdom(v)$$
$$y = sdom(u)$$
$$P$$
$$sdom(u)$$
$$u$$
$$T$$
$$sdom(u)$$
$$u$$
$$v$$
$$sdom(v) \leq sdom(u)$$
$$idom(v) = idom(u)$$
$$u$$
$$v$$

$$sdom(v) \leq sdom(u)$$

$$idom(u)$$

$$v$$

$$T$$

$$idom(u)$$

$$idom(v)$$

$$idom(v)$$

$$u$$

$$s$$

$$u$$

$$P$$

$$sdom(u)$$

$$P$$

$$x$$

$$P$$

$$x < sdom(u)$$

$$x$$

$$sdom(u) = idom(u) = s$$

$$y$$

$$P$$

$$x$$

$$sdom(u)$$

$$u$$

$$sdom(y) \leq x$$

$$sdom(y) \leq x < idom(v) \leq sdom(v)$$

$$v$$

$$y$$

$$sdom(u)$$

$$y$$

$$idom(v)$$

$$v$$

$$s$$

$$sdom(y)$$

$$y$$
$$v$$
$$idom(v)$$
$$y = idom(v)$$
$$P$$
$$idom(v)$$
$$sdom(u)$$
$$idom(u)$$
$$v$$
$$v$$
$$sdom(u)$$
$$u$$
$$sdom(v)$$
$$O(nm)$$
$$O(m + n \log n)$$
$$O(n)$$
$$v$$
$$v$$
$$v$$
$$v_0 \leftarrow v_1 \leftarrow ... \leftarrow v_k$$
$$v_i$$
$$v_o = a$$
$$a$$
$$v_k \leftarrow u$$
$$u$$
$$v_0, v_1, ..., v_k$$
$$v_{k+1} = u$$
$$u$$
$$v_i$$
$$v_i \leftarrow ... \leftarrow v_k \leftarrow v_i$$
$$c$$
$$P$$
$$a$$
$$a$$

$$b$$

$$b$$

$$a \leftarrow b$$

$$b$$

$$a$$

$$b$$

$$A$$

$$a$$

$$b$$

$$r$$

$$r$$

$$r$$

$$f_r$$

$$n$$

$$n$$

$$x_1, x_2, ..., x_n$$

$$m$$

$$x_i - x_j \leq c_k$$

$$1 \leq i, j \leq n, i \neq j, 1 \leq k \leq m$$

$$c_k$$

$$x_1 = a_1, x_2 = a_2, ..., x_n = a_n$$

$$x_i - x_j \leq c_k$$

$$x_i \leq x_j + c_k$$

$$dist[y] \leq dist[x] + z$$

$$x_i$$

$$x_i - x_j \leq c_k$$

$$j$$

$$i$$

$$c_k$$

$$\{a_1, a_2, ..., a_n\}$$

$$d$$

$$\{a_1 + d, a_2 + d, ..., a_n + d\}$$

$$d$$

$$dist[0] = 0$$

$$0$$

$$x_i = dist[i]$$

$$O(nm)$$

$$m$$

$$x_a - x_b \geq c_k$$

$$x_a - x_b \leq c_k$$

$$x_a = x_b$$

$$x_a - x_b \geq c$$

$$x_b - x_a \leq -c$$

$$x_a - x_b \leq c$$

$$x_a - x_b \leq c$$

$$x_a = x_b$$

$$x_a - x_b \leq 0,\ \ x_b - x_a \leq 0$$

$$\frac{x_i}{x_j} \leq c_k$$

$$x_i, x_j$$

$$c_k$$

$$\log$$

$$\log x_i - \log x_j \leq \log c_k$$

$$O(n+m)$$

$$O(n)$$

$$n$$

$$m$$

$$O(n)$$

$$O(1)$$

$$k$$

$$k$$

$$O(nm)$$

$$O(m \log m)$$

$$O(n+m)$$

$$dis_v = min(dis_v, dis_u + w_{u,v})$$

$$dis_v = max(dis_v, dis_u + w_{u,v})$$

$$u$$

$$v$$

$$u$$

$$low_v \geq dfn_u$$

$$u$$

$$G = \{V, E\}$$

$$e$$

$$e \in E$$

$$G - e$$

$$e$$

$$G$$

$$low_v > dfn_u$$

$$v$$

$$u$$

$$u$$

$$low_v = dfn_u$$

$$v$$

$$u - v$$

$$\kappa \leq \lambda \leq \delta$$

$$\lambda$$

$$\kappa$$

$$\delta$$

$$\lambda$$

$$\lambda$$

$$\delta$$

$$\delta + 1$$

$$\lambda$$

$$\lambda$$

$$\kappa$$

$$1$$

$$(s, t)$$

$$s$$

$$t$$

$$1$$

$$O(n^2)$$

$$O(\mid V \mid^3 \mid E \mid^2)$$

$$O(\mid V \mid^2 \mid E \mid \min(\mid V \mid^{2/3}, \mid E \mid^{1/2}))$$

$$O(\mid V \mid \mid E \mid + \mid V \mid^2 \log \mid V \mid)$$

$$O(\mid V \mid^3)$$

$$x$$

$$x_1$$

$$x_2$$

$$(x_1, x_2)$$

$$(u, v)$$

$$(u_2, v_1)$$

$$(v_2, u_1)$$

$$s$$

$$t$$

$$N(S) = \bigcup_{v \in S} N(v)$$

$$\sum_{v \in V} d^+(v) = \sum_{v \in V} d^-(v) = |E|$$

$$G = (V(G), E(G))$$

$$V(G)$$

$$V$$

$$E(G)$$

$$V(G)$$

$$G = (V, E)$$

$$V, E$$

$$G$$

$$V$$

$$E$$

$$G$$

$$G$$

E

(u, v)

$u, v \in V$

$e = (u, v)$

u

v

e

G

E

(u, v)

$u \to v$

$e = u \to v$

u

e

v

e

e

u

v

v

u

G

E

G

$e_k = (u_k, v_k)$

G

G

G

$|V(G)|$

G

$G = (V, E)$

v

e

v

e

$$u$$
$$v$$
$$(u,v)$$
$$u$$
$$v$$
$$v \in V$$
$$N(v)$$
$$S$$
$$S$$
$$N(S)$$
$$v$$
$$d(v)$$
$$(v,v)$$
$$d(v)$$
$$2$$
$$d(v) = |N(v)|$$
$$G = (V,E)$$
$$\sum_{v \in V} d(v) = 2|E|$$
$$d(v) = 0$$
$$v$$
$$d(v) = 1$$
$$v$$
$$2 \mid d(v)$$
$$v$$
$$2 \nmid d(v)$$
$$v$$
$$d(v) = |V| - 1$$
$$v$$
$$G$$
$$\delta(G)$$
$$\Delta(G)$$

$$\delta(G) = \min_{v \in G} d(v)$$

$$\Delta(G) = \max_{v \in G} d(v)$$

$$G = (V, E)$$

$$v$$

$$d^+(v)$$

$$v$$

$$d^-(v)$$

$$d^+(v) + d^-(v) = d(v)$$

$$G = (V, E)$$

$$G = (V, E)$$

$$k$$

$$G$$

$$k$$

$$k$$

$$E$$

$$e = (u, v)$$

$$u = v$$

$$e$$

$$E$$

$$e_1, e_2$$

$$(u, v)$$

$$(v, u)$$

$$u \to v$$

$$v \to u$$

$$w$$

$$e_1, e_2, ..., e_k$$

$$v_0, v_1, ..., v_k$$

$$e_i = (v_{i-1}, v_i)$$

$$i \in [1, k]$$

$$v_0 \to v_1 \to v_2 \to \cdots \to v_k$$

$$k$$

$$w$$

$$e_1, e_2, ..., e_k$$

$$w$$

$$w$$

$$w$$

$$w$$

$$v_0 = v_k$$

$$w$$

$$w$$

$$v_0 = v_k$$

$$w$$

$$G = (V, E)$$

$$H = (V', E')$$

$$V' \subseteq V$$

$$E' \subseteq E$$

$$H$$

$$G$$

$$H \subseteq G$$

$$H \subseteq G$$

$$\forall u, v \in V'$$

$$(u, v) \in E$$

$$(u, v) \in E'$$

$$H$$

$$G$$

$$V'$$

$$V' \subseteq V$$

$$V'$$

$$G[V']$$

$$H \subseteq G$$

$$V' = V$$

$$H$$

$$G$$

$$G$$

$$G$$

$$G$$
$$G$$
$$G$$
$$F$$
$$k$$
$$F$$
$$G$$
$$k$$
$$k$$
$$G = (V, E)$$
$$H = G[V^*]$$
$$\forall v \in V^*, (v, u) \in E$$
$$u \in V^*$$
$$H$$
$$G$$
$$G = (V, E)$$
$$u, v \in V$$
$$v_0 = u, v_k = v$$
$$u$$
$$v$$
$$G = (V, E)$$
$$G$$
$$G$$
$$H$$
$$G$$
$$F$$
$$H \subsetneq F \subseteq G$$
$$F$$
$$H$$
$$G$$
$$G = (V, E)$$
$$u, v \in V$$
$$v_0 = u, v_k = v$$

$$u$$
$$v$$
$$G = (V, E)$$
$$V' \subseteq V$$
$$G[V \setminus V']$$
$$G$$
$$V'$$
$$V'$$
$$G$$
$$G = (V, E)$$
$$k$$
$$| V | \geq k + 1$$
$$G$$
$$k - 1$$
$$G$$
$$k$$
$$k$$
$$k$$
$$G$$
$$\kappa(G)$$
$$K_n$$
$$n - 1$$
$$G = (V, E)$$
$$u, v \in V$$
$$u \neq v$$
$$u$$
$$v$$
$$u$$
$$v$$
$$V' \subseteq V$$
$$u, v \notin V'$$
$$G[V \setminus V']$$
$$u$$

$$v$$

$$V'$$

$$u$$

$$v$$

$$u$$

$$v$$

$$u$$

$$v$$

$$\kappa(u, v)$$

$$G = (V, E)$$

$$E' \subseteq E$$

$$G' = (V, E \setminus E')$$

$$G$$

$$E'$$

$$E'$$

$$G$$

$$G = (V, E)$$

$$k$$

$$G$$

$$k - 1$$

$$G$$

$$k$$

$$k$$

$$k$$

$$G$$

$$\lambda(G)$$

$$G = (V, E)$$

$$u, v \in V$$

$$u \neq v$$

$$u$$

$$v$$

$$E' \subseteq E$$

$$G' = (V, E \setminus E')$$

$$u$$
$$v$$
$$E'$$
$$u$$
$$v$$
$$u$$
$$v$$
$$u$$
$$v$$
$$\lambda(u, v)$$
$$2$$
$$2$$
$$2$$
$$2$$
$$2$$
$$G$$
$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$
$$O(\mid V \mid^2)$$
$$O(\mid E \mid)$$
$$O(\mid E \mid)$$
$$G = (V, E)$$
$$\bar{G}$$
$$V\left(\bar{G}\right) = V(G)$$
$$(u, v)$$
$$(u, v) \in E\left(\bar{G}\right)$$
$$(u, v) \notin E(G)$$
$$G = (V, E)$$
$$G$$
$$G' = (V, E')$$
$$E' = \{(v, u) \mid (u, v) \in E\}$$
$$G$$
$$G$$

n

K_n

G

G

n

$\overline{K}_n$

N_n

N_n

K_n

K_0

G

G

$G = (V, E)$

G

n

$n \geq 3$

C_n

2

$G = (V, E)$

v

G

$n + 1$

$n \geq 1$

S_n

$G = (V, E)$

v

G

$n + 1$

$n \geq 3$

W_n

$G = (V, E)$

G

$$n$$
$$P_n$$
$$1$$
$$1$$
$$n$$
$$m$$
$$K_{n,m}$$
$$K_5$$
$$K_{3,3}$$
$$G = (V, E)$$
$$V \geq 3$$
$$\mid E \mid \leq 3 \mid V \mid - 6$$
$$G$$
$$H$$
$$f : V(G) \to V(H)$$
$$(u, v) \in E(G)$$
$$(f(u), f(v)) \in E(H)$$
$$f$$
$$G$$
$$H$$
$$G$$
$$H$$
$$G \cong H$$
$$G \cong H$$
$$\mid V(G)| = \mid V(H)|, \mid E(G)| = \mid E(H)|$$
$$G$$
$$H$$
$$G$$
$$H$$
$$G = (V_1, E_1), H = (V_2, E_2)$$
$$G \cap H = (V_1 \cap V_2, E_1 \cap E_2)$$
$$G = (V_1, E_1), H = (V_2, E_2)$$

$$G \cup H = (V_1 \cup V_2, E_1 \cup E_2)$$

$$G = (V_1, E_1), H = (V_2, E_2)$$

$$H' \cong H$$

$$V(H') \cap V_1 = \emptyset$$

$$H'$$

$$H$$

$$G \cup H'$$

$$G$$

$$H$$

$$G + H$$

$$G \oplus H$$

$$G$$

$$H$$

$$G \cup H = G + H$$

$$G = (V, E)$$

$$V' \subseteq V$$

$$\forall v \in (V \setminus V')$$

$$(u, v) \in E$$

$$u \in V'$$

$$V'$$

$$G$$

$$G$$

$$\gamma(G)$$

$$G = (V, E)$$

$$V' \subseteq V$$

$$\forall v \in (V \setminus V')$$

$$(u, v) \in E$$

$$u \in V'$$

$$V'$$

$$G$$

$$G$$

$$\gamma^+(G)$$

$$\gamma^-(G)$$

$$G = (V, E)$$

$$E' \subseteq E$$

$$\forall e \in (E \setminus E')$$

$$E'$$

$$E'$$

$$G$$

$$G = (V, E)$$

$$V' \subseteq V$$

$$V'$$

$$V'$$

$$G$$

$$G$$

$$\alpha(G)$$

$$G = (V, E)$$

$$E' \in E$$

$$E'$$

$$E'$$

$$E'$$

$$G$$

$$G$$

$$\nu(G)$$

$$M$$

$$n$$

$$G$$

$$V' \subset V(G)$$

$$p_{\text{odd}}(G - V') \leq |\ V'|$$

$$p_{\text{odd}}$$

$$G = (V, E)$$

$$V' \subseteq V$$

$$\forall e \in E$$
$$e$$
$$V'$$
$$V'$$
$$G$$
$$K_{n,m}$$
$$\min(n, m)$$
$$G = (V, E)$$
$$E' \subseteq E$$
$$\forall v \in V$$
$$v$$
$$E'$$
$$E'$$
$$G$$
$$\rho(G)$$
$$\rho(G) + \nu(G) = |\ V(G)|$$
$$\nu(G) \leq \rho(G)$$
$$K_{n,m}$$
$$\max(n, m)$$
$$G = (V, E)$$
$$V' \subseteq V$$
$$V'$$
$$V'$$
$$G$$
$$\omega(G)$$
$$\omega(G) = \alpha(\bar{G})$$
$$k$$
$$(k-1)$$
$$\chi(G)$$
$$\chi(G) \leq \Delta(G) + 1$$
$$\Delta(G)$$

$$\chi(G) \leq \Delta(G)$$

$$|\ V(G)| = n$$

$$n \leq 3$$

$$n - 1$$

$$\Delta(G)$$

$$H := G - v$$

$$\chi(H) \leq \Delta(H) = \Delta(G)$$

$$\Delta := \Delta(G)$$

$$\Delta$$

$$c_1, c_2, ..., c_\Delta$$

$$\Delta$$

$$v_1, v_1, ..., v_\Delta$$

$$c_i$$

$$c_j$$

$$H_{i,j}$$

$$v_i$$

$$v_j$$

$$H_{i,j}$$

$$v_i$$

$$v_j$$

$$C_{i,j}$$

$$C_{i,j}$$

$$v_i$$

$$v_j$$

$$v_i$$

$$\Delta - 1$$

$$v_i$$

$$v_i$$

$$v_i$$

$$C_{i,j}$$

$$v_j$$

$$C_{i,j}$$

$$v_i$$

$$v_j$$

$$C_{i,j} \neq P$$

$$\Delta - 2$$

$$v_i$$

$$v_j$$

$$v_i$$

$$v_j$$

$$v_k$$

$$V(C_{i,j}) \cap V(C_{j,k}) = \{v_j\}$$

$$v_1$$

$$v_2$$

$$C_{1,2}$$

$$v_1$$

$$C_{1,3}$$

$$w \in V(C_{1,2}) \cap V(C_{2,3})$$

$$V(G) := \{v_1, v_2, ..., v_n\}$$

$$\deg(v_i) \geq \deg(v_{i+1}),\ \ \forall 1 \leq i \leq n-1$$

$$\max_{i=1}^{n} \min\{\deg(v_i) + 1, i\}$$

$$O\Big(n \max_{i=1}^{n} \min\{\deg(v_i) + 1, i\}\Big) = O(n^2)$$

$$\min\{\deg(v_i) + 1, i\}$$

$$C_3$$

$$V(G) := \{v_1, v_2, ..., v_n\}$$

$$\deg(v_i) \geq \deg(v_{i+1}),\ \ \forall 1 \leq i \leq n-1$$

$$V_0 = \emptyset$$

$$V(G) \setminus \bigcup_{i=0}^{m-1} V_i$$

$$V_m$$

$$v_{k_m} \in V_m$$

$$k_m = \min\{k : v_k \notin \bigcup_{i=0}^{m-1} V_i\}$$

$$\{v_{i_{m,1}}, v_{i_{m,2}}, ..., v_{i_{m,l_m}}\} \subset V_m,\ i_{m,1} < i_{m,2} < ... < i_{m,l_m}$$

$$v_j \in V_m$$

$$j > i_{m,l_m}$$

$$v_j$$

$$v_{i_{m,1}}, v_{i_{m,2}}, ..., v_{i_{m,l_m}}$$

$$V_i$$

$$V_1 \neq \emptyset$$

$$V_i \cap V_j = \emptyset \Longleftrightarrow i \neq j$$

$$\exists \alpha(G) \in \mathbb{N}^*, \forall i > \alpha(G),\ s.t.\ V_i = \emptyset$$

$$\bigcup_{i=1}^{\alpha(G)} V_i = V(G)$$

$$\chi(G) \leq \alpha(G) \leq \max_{i=1}^{n} \min\{\deg(v_i) + 1, i\}$$

$$v \notin \bigcup_{i=1}^{m} V_i$$

$$V_1, V_2, ..., V_m$$

$$\deg(v) \geq m$$

$$v_j \in \bigcup_{i=1}^{\deg(v_j)+1} V_i$$

$$\{V_i\}$$

$$v_j \in \bigcup_{i=1}^{j} V_i$$

$$(k-1)$$

$$\chi'(G)$$

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

$$\chi'(G) = \Delta(G)$$

$$n$$

$$n \neq 1$$

$$\chi'(K_n) = n$$

$$n$$

$$\chi'(K_n) = n - 1$$

$$(x, y)$$

$$x$$

$$y$$

$$l_x$$

$$l_y$$

$$l_x = l_y$$

$$l_x$$

$$l_x < l_y$$

$$y$$

$$l_x$$

$$l_y$$

$$y$$

$$l_x, l_y, \cdots$$

$$l_x$$

$$l_y$$

$$l_y$$

$$l_x$$

$$x$$

$$l_x$$

$$x$$

$$y$$

$$l_x$$

$$O(nm)$$

$$degree \bmod k \neq 0$$

$$degree/k$$

$$degree \bmod k$$

$$degree \bmod k = 0$$

$$degree/k$$

$$P(G, k)$$

$$P(K_n, k) = k(k-1)\cdots(k-n+1)$$

$$P(N_n, k) = k^n$$

$$e = (v_i, v_j) \notin E(G)$$

$$P(G, k) = P(G \cup e, k) + P(G \setminus e, k)$$

$$e = (v_i, v_j) \in E(G)$$

$$P(G, k) = P(G - e, k) - P(G \setminus e, k)$$

$$V_1$$

$$G[V_1]$$

$$\mid V_1 \mid$$

$$G - V_1$$

$$p (p \geq 2)$$

$$P(G, k) = \frac{\Pi_{i=1}^{p} (P(H_i, k))}{P(G[V_1], k)^{p-1}}$$

$$H_i = G[V_1 \cup V(G_i)]$$

$$\omega(G)$$

$$\chi(G)$$

$$\alpha(G)$$

$$\kappa(G)$$

$$3$$

$$\omega(G) \leq \chi(G)$$

$$\omega(G)$$

$$\alpha(G) \leq \kappa(G)$$

$$3$$

$$3$$

$$3$$

$$G$$

$$u, v$$

$$u, v$$

$$u, v$$

$$u, v$$

$$u$$

V_1

v

V_2

a

$N(a)$

V_1

V_2

$N(a)$

V_1

V_2

a

≤ 1

x, y

$N(x)$

V_1, V_2

x_1, x_2

y_1, y_2

$x_1 = y_1, x_2 = y_2$

V_1, V_2

x_1, y_1

x_2, y_2

x, y

V_1, V_2

$x - x_1 \sim y_1 - y, x - x_2 \sim y_2 - y$

$x - x_1 \sim y_1 - y - y_2 \sim x_2 - x$

≥ 4

V_1, V_2

x, y

$N(x)$

x

$\{x\} + N(x)$

x

≤ 3

≥ 4

u, v

$(u, v) \notin E$

I

u, v

A, B

I

u, v

A

$L = A + I$

L

u

L

I

A

$n = \mid V \mid$

$v_1, v_2, ..., v_n$

$1, 2, ..., n$

v_i

$\{v_i, v_{i+1}, ..., v_n\}$

1

n

$n - 1$

> 3

v

v

v_1, v_2

v_1, v_2

v

v

$O(n^4)$

$O(n + m)$

n

$$1$$

$$label_x$$

$$x$$

$$label$$

$$i$$

$$label_x = i$$

$$x$$

$$\sum_{i=1}^{n} label_i$$

$$2$$

$$O(n+m)$$

$$\alpha(x)$$

$$x$$

$$u, v, w$$

$$\alpha(u) < \alpha(v) < \alpha(w)$$

$$uw$$

$$vw$$

$$w$$

$$u$$

$$label$$

$$v$$

$$v$$

$$u$$

$$x$$

$$\alpha(v) < \alpha(x)$$

$$vx$$

$$ux$$

$$x$$

$$v$$

$$u$$

$$v_0, v_1, ..., v_k (k \geq 2)$$

$$v_i v_j$$

$$\mid i - j \mid = 1$$

$$\alpha(v_0) > \alpha(v_i)(i \in [1, k])$$

$$i \in [1, k-1]$$

$$\alpha(v_i) < \alpha(v_{i+1}) < \ldots < \alpha(v_k)$$

$$\alpha(v_i) < \alpha(v_{i-1}) < \ldots < \alpha(v_1) < \alpha(v_k) < \alpha(v_0)$$

$$\alpha(v_1) < \alpha(v_k) < \alpha(v_0)$$

$$v_1 v_0$$

$$v_k v_0$$

$$x$$

$$\alpha(v_k) < \alpha(x)$$

$$v_k x$$

$$v_1 x$$

$$j \in (1, k]$$

$$v_j x$$

$$v_0 x$$

$$v_0 v_1 \cdots v_j x$$

$$\geq 4$$

$$x < v_0$$

$$v_0, v_1, \ldots, v_j, x$$

$$v_0 < x$$

$$x, v_j, \ldots, v_1, v_0$$

$$\min(v_0, v_k)$$

$$u, v, w$$

$$\alpha(u) < \alpha(v) < \alpha(w)$$

$$uv$$

$$uw$$

$$vw$$

$$w, u, v$$

$$vw$$

$$v$$

$$\{v_i, v_{i+1}, \ldots, v_n\}$$

$$v_i$$

$$O(nm)$$

$$v_i$$

$$v_i, v_{i+1}, ..., v_n$$

$$\{v_{c_1}, v_{c_2}, ..., v_{c_k}\}$$

$$v_{c_1}$$

$$O(n+m)$$

$$O(n+m)$$

$$N(x)$$

$$x$$

$$x$$

$$\{x\} + N(x)$$

$$V$$

$$x$$

$$V \subseteq \{x\} + N(x)$$

$$V$$

$$V = \{x\} + N(x)$$

$$n$$

$$\{x\} + N(x)$$

$$A = \{x\} + N(x), B = \{y\} + N(y)$$

$$A \subsetneqq B$$

$$A$$

$$y$$

$$x$$

$$nxt_x$$

$$N(x)$$

$$y*$$

$$A \subseteq B$$

$$y$$

$$nxt_{y*} = x$$

$$y*$$

$$y* = nxt_{y*}$$

$$A \subsetneqq B$$

$$\mid A \mid + 1 \leq \mid B \mid$$

$$y$$

$$nxt_y = x$$

$$|\ N(x)| + 1 \leq |\ N(y)|$$

$$O(n+m)$$

$$O(m+n)$$

$$t$$

$$t \geq \chi(G)$$

$$t = \omega(G)$$

$$t = \omega(G) \leq \chi(G)$$

$$t = \chi(G) = \omega(G)$$

$$|\ \{x\} + N(x)|$$

$$\{v_1, v_2, ..., v_t\}$$

$$\{\{v_1 + N(v_1)\}, \{v_2 + N(v_2)\}, ..., \{v_t + N(v_t)\}\}$$

$$O(n+m)$$

$$t$$

$$t \leq \alpha(G), t \geq \kappa(G)$$

$$\alpha(G) \leq \kappa(G)$$

$$t = \alpha(G) = \kappa(G)$$

$$1$$

$$n+c$$

$$n$$

$$c$$

$$2n$$

$$n$$

$$k$$

$$k$$

$$u$$

$$u$$

$$u$$

$$v$$

$$i$$

$$1$$

2

3

4

5

6

7

8

9

low$[i]$

1

1

1

3

3

4

3

3

7

9

7

u

u

u

v

u

v

$\text{low}[v] = \text{dfn}[u]$

$u \to v$

u, v

u

$\text{low}[v] = \text{dfn}[u]$

u

v

u

$n + 1$

$\langle s, c, f \rangle$

s, c, f

s

c

f

u, v

w

u, v, c

u

v

c

2

3

c

2

u

v

1

$\langle x, y \rangle$

x

y

1

y

x

1

c

1

c

2

2

c

2

1

1

u

v

c

1

2

s

f

c

c

s, f

2

s, f

-1

2

$\sum$

212

$\mathcal{O}(n)$

q

S

$2 \leq | S | \leq n$

u

$u \notin S$

u

S

S

$| S |$

S

1

$O(n + m)$

$O(n)$

$O(n + m)$

(a, b)

(u, v)

$d_a[u] + 1 + d_b[v] = d_a[b]$

$$(a, b)$$

$$d_a[u] + d_b[v] = d_a[b]$$

$$(u, v)$$

$$((u, 0), (v, 1))$$

$$((u, 1), (v, 0))$$

$$(s, 0)$$

$$(t, 0)$$

$$n \times m$$

$$n$$

$$\log n$$

$$O(n^2)$$

$$O(n)$$

$$u$$

$$v$$

$$u$$

$$v$$

$$u$$

$$v$$

$$u$$

$$v$$

$$u$$

$$v$$

$$x, y$$

$$y, z$$

$$x, z$$

$$A, B$$

$$B, C$$

$$A, C$$

$$O(nm)$$

$$O(n)$$

$$u$$

$$u$$

$$u$$

$$0$$
$$u$$
$$u$$
$$O(n+m)$$
$$O(n+m)$$
$$k>2$$
$$k=2$$
$$n$$
$$\langle a,b\rangle$$
$$a$$
$$b$$
$$a$$
$$b$$
$$n$$
$$a$$
$$\neg a$$
$$b$$
$$\neg b$$
$$(a\vee b)$$
$$a,b$$
$$\neg a\rightarrow b\wedge\neg b\rightarrow a$$
$$a$$
$$b$$
$$b$$
$$a$$
$$a1,a2$$
$$b1,b2$$
$$a1$$
$$b2$$
$$(a1,b1)$$
$$(b2,a2)$$
$$a1$$
$$b1$$
$$b2$$

$a2$

x

$\neg x$

x

x

$\neg x$

x

$O(n+m)$

1

$2n$

2

n

$a1$

$a2$

$a1$

$a2$

$a2$

$a1$

$k(k>3)$

n

P

P

P

P

P

P

P

$O(n \log n)$

x

10

3

$6, 7, 9$

2

180

$$6$$

$$9$$

$$7$$

$$7$$

$$n$$

$$n$$

$$O(n \log n)$$

$$r+1$$

$$r-1$$

$$i$$

$$[l_i, r_i]$$

$$r = 1 \cdots n$$

$$r$$

$$l$$

$$l$$

$$r$$

$$\mathcal{O}((n+m) \log n)$$

$$n$$

$$m$$

$$[l, r]$$

$$[x, y]$$

$$l$$

$$r$$

$$l-1$$

$$[x, y]$$

$$a$$

$$r$$

$$[x, y]$$

$$b$$

$$b-a$$

$$i$$

$$[i+1, n]$$

$$[0, a_i]$$

$$10^9$$
$$n$$
$$m$$
$$i$$
$$i$$
$$i$$
$$j$$
$$a_i = a_j$$
$$pre_i = j$$
$$pre_i < l$$
$$l \le pre_i$$
$$[l, r]$$
$$pre_i < l$$
$$i$$
$$[l, r]$$
$$[1, r]$$
$$[1, l-1]$$
$$[l, r]$$
$$l-1$$
$$r$$
$$r$$
$$pre_i < l$$
$$l-1$$
$$pre_i < l$$
$$i$$
$$m$$
$$\mathcal{O}((n+m)\log n)$$
$$S \times (|\ \overrightarrow{AD} \cdot \overrightarrow{AB}| + |\ \overrightarrow{BC} \cdot \overrightarrow{BA}| - |\ \overrightarrow{AB} \cdot \overrightarrow{BA}|)/|\ \overrightarrow{AB} \cdot \overrightarrow{BA}|$$
$$n$$
$$2 \le n \le 50000, |\ x\ |, |\ y\ | \le 10^4$$
$$(i, i+1)$$
$$j+1$$

$$(i, i+1)$$

$$j$$

$$j$$

$$j$$

$$3 \leq n \leq 50000$$

$$S$$

$$T(n) = T(n-1) + g(n)$$

$$n$$

$$O$$

$$i-1$$

$$i$$

$$i$$

$$i$$

$$0$$

$$j$$

$$j$$

$$j$$

$$O(n)$$

$$O(n)$$

$$A$$

$$i$$

$$b$$

$$A = i + \frac{b}{2} - 1$$

$$A = 2 \times i + b - 2$$

$$P$$

$$A = i + \frac{b}{2} - \chi(P)$$

$$\chi(P)$$

$$P$$

$$V - E + F = 2$$

$$n$$

$$dx$$

$$dy$$

$$dx$$

$$dy$$

$$0$$

$$\gcd(dx, dy) + 1$$

$$\gcd(dx, dy)$$

$$dx, dy$$

$$dx$$

$$dy$$

$$0$$

$$dy$$

$$dx$$

$$A_1 = \{p_i \mid i = 0...m\}$$
$$A_2 = \{p_i \mid i = m + 1...n - 1\}$$

$$B = \{p_i \mid \mid x_i - x_m \mid < h\}$$

$$C(p_i) = \{p_j \mid p_j \in B,\, y_i - h < y_j \leq y_i\}$$

$$n$$

$$O(n \log n)$$

$$n$$

$$S_1, S_2$$

$$S_1$$

$$S_2$$

$$O(n)$$

$$T(n) = 2T(\frac{n}{2}) + O(n) = O(n \log n)$$

$$x_i$$

$$y_i$$

$$p_m(m = \left\lfloor \frac{n}{2} \right\rfloor)$$

$$A_1, A_2$$

$$h_1, h_2$$

$$h$$

$$A_1$$

$$A_2$$

$$h$$

$$x_m$$

$$h$$

$$B$$

$$m$$

$$m$$

$$A_1$$

$$A_2$$

$$B = \{p_i \mid \mid x_i - x_m \mid < h\}$$

$$B$$

$$B$$

$$p_i$$

$$B$$

$$h$$

$$y_i$$

$$p_j$$

$$y_i - y_j$$

$$h$$

$$C(p_i)$$

$$B$$

$$p_i$$

$$C(p_i) = \{p_j \mid p_j \in B, y_i - h < y_j \leq y_i\}$$

$$C$$

$$B$$

$$y_i$$

$$C(p_i)$$

$$p_i$$

$$B$$

$$B$$

$$y_i$$

$$O(n \log n)$$

$$O(n)$$

$$p_i \in B$$

$$p_j \in C(p_i)$$

$$(p_i, p_j)$$

$$y_i$$

$$|\ C(p_i)|$$

$$O(n)$$

$$7$$

$$C(p_i)$$

$$(y_i - h, y_i]$$

$$C(p_i)$$

$$p_i$$

$$(x_m - h, x_m + h)$$

$$2h \times h$$

$$h \times h$$

$$p_i$$

$$C(p_i) \cap A_1$$

$$C(p_i) \cap A_2$$

$$h$$

$$h \times h$$

$$\frac{h}{2} \times \frac{h}{2}$$

$$1$$

$$\frac{h}{\sqrt{2}}$$

$$h$$

$$4$$

$$8$$

$$p_i$$

$$\max(C(p_i)) = 7$$

$$x_i$$

$$r - l$$
$$h$$
$$B$$
$$d$$
$$x_m$$
$$\frac{d}{2}$$
$$B$$
$$d$$
$$\frac{d}{2}$$
$$O(n \log n)$$
$$x_i$$
$$y_i$$
$$y_i$$
$$x_i$$
$$i$$
$$x_i - x_j >= d$$
$$\mid y_i - y_j \mid < d$$
$$p_i$$
$$p_i$$
$$O(n \log n)$$
$$O(n \log n)$$
$$O(n)$$
$$i - 1$$
$$i$$
$$i - 1$$
$$s$$
$$s$$
$$i$$
$$O(1)$$
$$i - 1$$
$$s$$

$$i$$

$$i$$

$$O\left(\frac{1}{i}\right)$$

$$O(i)$$

$$i$$

$$O(1)$$

$$n$$

$$O(n)$$

$$r_2 = \frac{1}{2}\left(\frac{1}{\mid OA \mid - r_1} - \frac{1}{\mid OA \mid + r_1}\right)R^2$$

$$\mid OC \mid \cdot \mid OC'| = (\mid OA \mid + r_1) \cdot (\mid OB \mid - r_2) = R^2$$
$$\mid OD \mid \cdot \mid OD'| = (\mid OA \mid - r_1) \cdot (\mid OB \mid + r_2) = R^2$$

$$x_2 = x_0 + \frac{\mid OB \mid}{\mid OA \mid}(x_1 - x_0)$$

$$y_2 = y_0 + \frac{\mid OB \mid}{\mid OA \mid}(y_1 - y_0)$$

$$O$$

$$R$$

$$P$$

$$P'$$

$$P'$$

$$\overrightarrow{OP}$$

$$\mid OP \mid \cdot \mid OP'| = R^2$$

$$P$$

$$P'$$

$$P$$

$$O$$

$$O$$

$$O$$

$$O$$

$$A$$

$$O$$

$$A$$
$$r_1$$
$$B$$
$$r_2$$
$$\mid OB \mid$$
$$O$$
$$(x_0, y_0)$$
$$A$$
$$x_1, y_1$$
$$B$$
$$x_2, y_2$$
$$\mid OB \mid$$
$$r_2$$
$$O$$
$$A$$
$$O$$
$$A$$
$$O$$
$$O$$
$$O$$
$$3$$
$$2$$
$$4$$
$$\begin{cases} A_1x + B_1y + C \geq 0 \\ A_2x + B_2y + C \geq 0 \\ \cdots \end{cases}$$
$$Ax + By + C \geq 0$$
$$\theta \in (-\pi, \pi]$$
$$\theta = \arctan \frac{y}{x}$$
$$\vec{a} \to \vec{b} \to \vec{c}$$
$$\vec{a}$$
$$\vec{b}$$
$$D$$

$$\vec{c}$$
$$D$$
$$\vec{c}$$
$$D$$
$$\vec{c}$$
$$D$$
$$\vec{b}$$
$$\vec{f}$$
$$G$$
$$\vec{f}$$
$$\vec{a}$$
$$n$$
$$n$$
$$n$$
$$\{\vec{u}, \vec{v}\}$$
$$\vec{w}$$
$$N$$
$$\vec{v}$$
$$M$$
$$\vec{a}$$
$$M$$
$$\vec{a}$$
$$M$$
$$M$$
$$\vec{u}$$
$$\vec{v}$$
$$\vec{u}$$
$$\vec{v}$$
$$\vec{v}$$
$$M$$
$$\vec{a}$$
$$\vec{v}$$
$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$|AB| = \sqrt{(2-6)^2 + (2-5)^2} = \sqrt{4^2+3^2} = 5$$

$$\mid P \mid = \sqrt{x^2+y^2}$$

$$\begin{aligned} \therefore \quad \mid AB \mid &= \sqrt{\mid AC \mid^2 + \mid BC \mid^2} \\ &= \sqrt{\mid AD \mid^2 + \mid CD \mid^2 + \mid BC \mid^2} \end{aligned}$$

$$|AB| = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$$

$$\mid P \mid = \sqrt{x^2+y^2+z^2}$$

$$\begin{aligned} \| \overrightarrow{AB} \| &= \sqrt{(x_{11}-x_{21})^2 + (x_{12}-x_{22})^2 + \cdots + (x_{1n}-x_{2n})^2} \\ &= \sqrt{\sum_{i=1}^{n} (x_{1i}-x_{2i})^2} \end{aligned}$$

$$d(A,B) = \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid$$

$$d(A,B) = \mid 20-10 \mid + \mid 25-10 \mid = 10+15 = 25$$

$$\begin{aligned} d(A,B) &= \mid x_1 - y_1 \mid + \mid x_2 - y_2 \mid + \cdots + \mid x_n - y_n \mid \\ &= \sum_{i=1}^{n} \mid x_i - y_i \mid \end{aligned}$$

$$d(A,B) = \max(\mid x_1 - x_2 \mid, \mid y_1 - y_2 \mid)$$

$$\begin{aligned} d(x,y) &= \max\{\mid x_1 - y_1 \mid, \mid x_2 - y_2 \mid, \cdots, \mid x_n - y_n \mid\} \\ &= \max\{\mid x_i - y_i \mid\}(i \in [1,n]) \end{aligned}$$

$$d(A,B) = \max(\mid 20-10 \mid, \mid 25-10 \mid) = \max(10,15) = 15$$

$$\begin{aligned} y &= -x+1 (x \geq 0, y \geq 0) \\ y &= x+1 \quad (x \leq 0, y \geq 0) \\ y &= x-1 \quad (x \geq 0, y \leq 0) \\ y &= -x-1 (x \leq 0, y \leq 0) \end{aligned}$$

$$\begin{aligned} y &= 1 \quad\ (-1 \leq x \leq 1) \\ y &= -1 \ (-1 \leq x \leq 1) \\ x &= 1, \quad (-1 \leq y \leq 1) \\ x &= -1, (-1 \leq y \leq 1) \end{aligned}$$

$$\begin{aligned} d(A,B) &= \mid x_1 - x_2 \mid + \mid y_1 - y_2 \mid \\ &= \max\{x_1 - x_2 + y_1 - y_2, x_1 - x_2 + y_2 - y_1, x_2 - x_1 + y_1 - y_2, x_2 - x_1 + y_2 - y_1\} \\ &= \max(\mid (x_1+y_1) - (x_2+y_2) \mid, \mid (x_1-y_1) - (x_2-y_2) \mid) \end{aligned}$$

$$\begin{aligned} d(A,B) &= \max\{\mid x_1 - x_2 \mid, \mid y_1 - y_2 \mid\} \\ &= \max\left\{\left|\frac{x_1+y_1}{2} - \frac{x_2+y_2}{2}\right| + \left|\frac{x_1-y_1}{2} - \frac{x_2-y_2}{2}\right|\right\} \end{aligned}$$

$$A, B$$

$$A(x_1, y_1), B(x_2, y_2)$$

$$A(6,5), B(2,2)$$

$$A, B$$

$$P(x, y)$$

$$\triangle\ ADC$$

$$\angle ADC = 90^\circ$$

$$\triangle\ ACB$$

$$\angle ACB = 90^\circ$$

$$n$$

$$\vec{A}(x_{11}, x_{12}, \cdots, x_{1n}),\ \ \vec{B}(x_{21}, x_{22}, \cdots, x_{2n})$$

$$A(x_1, y_1), B(x_2, y_2)$$

$$A, B$$

$$A, B$$

$$A, B$$

$$A(25,20), B(10,10)$$

$$A, B$$

$$n$$

$$d(i,j) \geq 0$$

$$0$$

$$d(i,i) = 0$$

$$A$$

$$B$$

$$B$$

$$A$$

$$d(i,j) = d(j,i)$$

$$i$$

$$j$$

$$k$$

$$d(i,j) \leq d(i,k) + d(k,j)$$

$$|\ x_1 - x_2| + |\ y_1 - y_2|$$

$$x_1 - x_2 \geq 0$$

$$y_1 - y_2$$

$$(y_1 - y_2 \geq 0) \rightarrow |\ x_1 - x_2| + |\ y_1 - y_2| = x_1 + y_1 - (x_2 + y_2)$$

$$(y_1 - y_2 < 0) \rightarrow |\ x_1 - x_2| + |\ y_1 - y_2| = x_1 - y_1 - (x_2 - y_2)$$

$$x + y, x - y$$

$$A(x_1, y_1), B(x_2, y_2)$$

$$A, B$$

$$n$$

$$A, B$$

$$A(25, 20), B(10, 10)$$

$$1$$

$$|\ x\ | + |\ y\ | = 1$$

$$4$$

$$4$$

$$\sqrt{2}$$

$$1$$

$$1$$

$$\max(|\ x\ |, |\ y\ |) = 1$$

$$4$$

$$2$$

$$1$$

$$2$$

$$A(x_1, y_1), B(x_2, y_2)$$

$$A, B$$

$$(x_1 + y_1, x_1 - y_1), (x_2 + y_2, x_2 - y_2)$$

$$(x, y)$$

$$(x + y, x - y)$$

$$A, B$$

$$(\frac{x_1 + y_1}{2}, \frac{x_1 - y_1}{2}), (\frac{x_2 + y_2}{2}, \frac{x_2 - y_2}{2})$$

$$(x, y)$$

$$(\frac{x+y}{2},\frac{x-y}{2})$$

$$45^{\circ}$$

$$(x,y)$$

$$(x+y,x-y)$$

$$(x,y)$$

$$(\frac{x+y}{2},\frac{x-y}{2})$$

$$(x,y)$$

$$(x+y,x-y)$$

$$\max_{i,j\in n}\{\max\{\mid x_i - x_j \mid, \mid y_i - y_j \mid\}\}$$

$$x,y$$

$$L_m$$

$$A(x_1,y_1)$$

$$B(x_2,y_2)$$

$$L_m$$

$$d(L_m)=(\mid x_1-x_2\mid^m+\mid y_1-y_2\mid^m)^{\frac{1}{m}}$$

$$L_2$$

$$L_1$$

$$\sum_{i=1}^{ans}\left|\overrightarrow{h_i h_{i+1}}\right|$$

$$[0,\pi]$$

$$X$$

$$X$$

$$S$$

$$X$$

$$O(n\log n)$$

$$n$$

$$P$$

$$S_1,S_2$$

$$S_1$$

$$0$$

$$\overrightarrow{S_2S_1} \times \overrightarrow{S_1P} < 0$$

$$\overrightarrow{S_2S_1} \times \overrightarrow{S_1P} \geq 0$$

$$\overrightarrow{S_2S_1} \times \overrightarrow{S_1P} < 0$$

$$<$$

$$\leq$$

$$>$$

$$ans$$

$$1$$

$$h$$

$$ans + 1$$

$$O(n \log n)$$

$$P$$

$$P$$

$$P_1, P_2, P_3$$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} \geq 0$$

$$P$$

$$P_0$$

$$P_1, P_2$$

$$P_1$$

$$P_1$$

$$P_0$$

$$O$$

$$R$$

$$O$$

$$P$$

$$P'$$

$$OP \times OP' = R^2$$

$$P$$

$$P'$$

$$O$$

$$P$$

$$P$$

$$P$$

$$V$$

$$E$$

$$F$$

$$V - E + F = 2$$

$$O(n^2)$$

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

$$Ax + By + Cz - (Ax_0 + By_0 + Cz_0) = 0$$

$$P_0(x_0, y_0, z_0)$$

$$\boldsymbol{n}$$

$$\boldsymbol{n}$$

$$\boldsymbol{n}$$

$$\boldsymbol{n} = (A, B, C)$$

$$P(x, y, z)$$

$$\boldsymbol{n} \cdot \overrightarrow{PP_0} = 0$$

$$D = -(Ax_0 + By_0 + Cz_0)$$

$$Ax + By + Cz + D = 0$$

$$a$$

$$b$$

$$P$$

$$a' \parallel a$$

$$b' \parallel b'$$

$$a'$$

$$b'$$

$$a$$

$$b$$

$$a$$

$$\alpha$$

$$a$$

$$\alpha$$

$$A$$

$$a$$

$$P$$

$$\alpha$$

$$\alpha$$

$$O$$

$$a$$

$$PO$$

$$a \parallel \alpha$$

$$a \subset \alpha$$

$$0^\circ$$

$$\alpha$$

$$\beta$$

$$l$$

$$a, b$$

$$a \subset \alpha$$

$$b \subset \beta$$

$$l_1, l_2$$

$$s_1(m_1, n_1, p_1)$$

$$s_2(m_2, n_2, p_2)$$

$$\varphi$$

$$\cos\varphi = \frac{|m_1m_2 + n_1n_2 + p_1p_2|}{\sqrt{m_1^2 + n_1^2 + p_1^2}\sqrt{m_2^2 + n_2^2 + p_2^2}}$$

$$l_1 \perp l_2 \Longleftrightarrow m_1m_2 + n_1n_2 + p_1p_2 = 0$$

$$l_1 \parallel l_2 \Longleftrightarrow \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{p_1}{p_2}$$

$$\varphi$$

$$\varphi \in [0, \frac{\pi}{2}]$$

$$s(m, n, p)$$

$$f(a, b, c)$$

$$\sin\varphi = \frac{|am + bn + cp|}{\sqrt{a^2 + b^2 + c^2}\sqrt{m^2 + n^2 + p^2}}$$

$$\Longleftrightarrow am + bn + cp = 0$$

$$\Longleftrightarrow \frac{a}{m} = \frac{b}{n} = \frac{c}{p}$$

$$M□AB□N$$

$$\alpha$$

$$M$$

$$AC$$

$$AB$$

$$\beta$$

$$N$$

$$\gamma$$

$$\sin\gamma = \sin\alpha \cdot \sin\beta$$

$$O$$

$$B$$

$$BO$$

$$AO$$

$$OC$$

$\angle COB$□$\angle AOC$□$\angle AOB$

$$\cos\angle BOC = \cos\angle AOB \cdot \cos\angle AOC$$

$$\angle AOC$$

$$\angle AOB$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos A \\ b^2 &= a^2 + c^2 - 2ac\cos B \\ c^2 &= a^2 + b^2 - 2ab\cos C \end{aligned}$$

$$T = \frac{|\mathbf{u} \times \mathbf{b}|}{|\mathbf{a} \times \mathbf{b}|} = \frac{|\mathbf{u}|\sin\theta}{|\mathbf{a}|\sin\beta}$$

$$S = \frac{1}{2}\left|\sum_{i=1}^{n} \mathbf{v}_i \times \mathbf{v}_{(i \bmod n)+1}\right|$$

$$(5, 2)$$

$$(-1, 0)$$

$$Ax + By + C = 0$$

$$y = kx + b$$

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\triangle \text{ ABC}$$

$$A, B, C$$

a, b, c

R

$\triangle$ ABC

$\triangle$ ABC

A, B, C

a, b, c

P

$\mathbf{v}$

Q

$\overrightarrow{PQ} \times \mathbf{v}$

Q

0

Q

Q

a, b

b

a

a

b

a, b

b

a

a

b

a, b

0

0

0

AB, CD

E

l

l

l

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\beta$$

$$|\mathbf{u}\times\mathbf{b}| = |\mathbf{u}||\mathbf{b}|\sin\theta$$

$$\left|\frac{|\mathbf{u}|\sin\theta}{\sin\beta}\right| = l$$

$$\mathbf{a}$$

$$T$$

$$\mathbf{a}$$

$$B$$

$$T\mathbf{a}$$

$$p_1, p_2, \cdots, p_n$$

$$O$$

$$\mathbf{v}_i = p_i - O$$

$$S$$

$$n$$

$$6 \le n \le 3000$$

$$O$$

$$x$$

$$P$$

$$O$$

$$P$$

$$P$$

$$4$$

$$4$$

$$O(n^2\log n)$$

$$f(u,i,j) = \max_{v,k\le j,k\le\ siz\_v} f(u,i-1,j-k) + f(v,s_v,k)$$

$$n$$

$$1 \sim N$$

$$a_i$$

$$f(i,0/1)$$

$$i$$

$$i$$

$$i$$

$$x$$

$$i$$

$$f(i,0) = \sum \max\{f(x,1), f(x,0)\}$$

$$f(i,1) = \sum f(x,0) + a_i$$

$$n$$

$$i$$

$$a_i$$

$$m$$

$$n, m \leq 300$$

$$0$$

$$0$$

$$0$$

$$f(u,i,j)$$

$$u$$

$$u$$

$$i$$

$$j$$

$$u$$

$$v$$

$$v$$

$$u$$

$$x$$

$$s_x$$

$$x$$

$$siz\_x$$

$$f$$

$$j$$

$$O(nm)$$

$$n$$

$$u$$

$$v$$

$$s_i$$

$$i$$

$$s_u = 1 + \sum s_v$$

$$s_i$$

$$f_u$$

$$u$$

$$f_v \leftarrow f_u$$

$$u$$

$$v$$

$$v$$

$$u$$

$$v$$

$$s_v$$

$$v$$

$$n - s_v$$

$$f_v = f_u - s_v + n - s_v = f_u + n - 2 \times s_v$$

$$f_v = f_u + n - 2 \times s_v$$

$$f(i,j,l) = \sum f(i-1, x, l - sta(j))$$

$$\begin{cases} f(\emptyset) = 0, \\ f(S) = \min_{T \subseteq S;\ w(T) \leq W} \{t(T) + f(S \setminus T)\}. \end{cases}$$

$$N \times N$$

$$K$$

$$1 \leq N \leq 9, 1 \leq K \leq N \times N$$

$$8$$

$$f(i,j,l)$$

$$i$$

$$i$$

$$j$$

$$l$$

$$j$$

$$sit(j)$$

$$sit(j)$$

$$0$$

$$1$$

$$sta(j)$$

$$sit(j)$$

$$1$$

$$100101$$

$$sit(j) = 100101_{(2)} = 37, sta(j) = 3$$

$$j$$

$$x$$

$$f(i,j,l) = \sum f(i-1,x,l-sta(j))$$

$$x$$

$$x$$

$$n$$

$$i$$

$$w_i$$

$$t_i$$

$$W$$

$$100 \leq W \leq 400$$

$$1 \leq n \leq 16$$

$$1 \leq t_i \leq 50$$

$$10 \leq w_i \leq 100$$

$$S$$

$$t(S)$$

$$S$$

$$w(S)$$

$$S$$

$$f(S)$$

$$S$$

$$O(3^n)$$

$$\begin{aligned} f_{i,j} &= p_1 \cdot f_{i,j} + p_2 \cdot f_{i,j+1} + p_3 \cdot f_{i+1,j} + p_4 \cdot f_{i+1,j+1} + 1 \\ &= \frac{p_2 \cdot f_{i,j+1} + p_3 \cdot f_{i+1,j} + p_4 \cdot f_{i+1,j+1} + 1}{1 - p_1} \end{aligned}$$

$$f_{i,j,0} = min(f_{i-1,j,0} + w_{c_{i-1},c_i}, f_{i-1,j,1} + w_{d_{i-1},c_i} \cdot p_{i-1} + w_{c_{i-1},c_i} \cdot (1 - p_{i-1}))$$

$$w$$

$$b$$

$$f_{i,j}$$

$$i$$

$$j$$

$$f_{0,j} = 0$$

$$f_{i,0} = 1$$

$$f_{i,j}$$

$$\frac{i}{i+j}$$

$$\frac{j}{i+j} \cdot \frac{i}{i+j-1}$$

$$f_{i,j-3}$$

$$\frac{j}{i+j} \cdot \frac{j-1}{i+j-1} \cdot \frac{j-2}{i+j-2}$$

$$f_{i-1,j-2}$$

$$\frac{j}{i+j} \cdot \frac{j-1}{i+j-1} \cdot \frac{i}{i+j-2}$$

$$i, j$$

$$s$$

$$n$$

$$\frac{1}{s}$$

$$\frac{1}{n}$$

$$n$$

$$s$$

$$f_{i,j}$$

$$i$$

$$j$$

$$n$$

$$s$$

$$f_{n,s} = 0$$

$$f_{0,0}$$

$$f_{i,j}$$

$$f_{i,j}$$

$$i$$

$$j$$

$$p_1 = \frac{i}{n} \cdot \frac{j}{s}$$

$$f_{i,j+1}$$

$$i$$

$$p_2 = \frac{i}{n} \cdot (1 - \frac{j}{s})$$

$$f_{i+1,j}$$

$$j$$

$$p_3 = (1 - \frac{i}{n}) \cdot \frac{j}{s}$$

$$f_{i+1,j+1}$$

$$p_4 = (1 - \frac{i}{n}) \cdot (1 - \frac{j}{s})$$

$$n$$

$$i$$

$$c_i$$

$$d_i$$

$$p_i$$

$$m$$

$$i$$

$$i + 1$$

$$v$$

$$e$$

$$i$$

$$i + 1$$

$$p_i$$

$$d_i$$

$$d_i$$

$$m$$

$$1 - p_i$$

$$c_i$$

$$n$$

$$f_{i,j,0/1}$$

$$i$$

$$j$$

$$\min\{f_{n,i,0}, f_{n,i,1}\}, i \in [0, m]$$

$$f_{1,0,0} = f_{1,1,1} = 0$$

$$f_{i,j,0/1}$$

$$f_{i,j,0}$$

$$f_{i-1,j,0} + w_{c_{i-1},c_i}$$

$$f_{i-1,j,1} + w_{d_{i-1},c_i} \cdot p_{i-1} + w_{c_{i-1},c_i} \cdot (1 - p_{i-1})$$

$$f_{i,j,1}$$

$$(1 - p_i)$$

$$p_i$$

$$n \times m$$

$$x$$

$$y$$

$$m = 1$$

$$\frac{1}{2}$$

$$\frac{1}{2}$$

$$2 \cdot (n - x)$$

$$f_{i,j}$$

$$n$$

$$f_{n,j} = 0$$

$$f_{i,j}$$

$$f_{i,1} = \frac{1}{3} \cdot (f_{i+1,1} + f_{i,2} + f_{i,1}) + 1$$

$$f_{i,j} = \frac{1}{4} \cdot (f_{i,j} + f_{i,j-1} + f_{i,j+1} + f_{i+1,j}) + 1$$

$$f_{i,m} = \frac{1}{3} \cdot (f_{i,m} + f_{i,m-1} + f_{i+1,m}) + 1$$

$$2f_{i,1} - f_{i,2} = 3 + f_{i+1,1}$$

$$3f_{i,j} - f_{i,j-1} - f_{i,j+1} = 4 + f_{i+1,j}$$

$$2f_{i,m} - f_{i,m-1} = 3 + f_{i+1,m}$$

$$f_{i+1,j}$$

$$m$$

$$m$$

$$m$$

$$m$$

$$N \times M$$

$$1 \times 2$$

$$2 \times 1$$

$$n$$

$$m$$

$$dp(i,s)$$

$$i$$

$$i$$

$$s$$

$$s$$

$$dp(i,j,s)$$

$$i$$

$$j$$

$$s$$

$$O(1)$$

$$f_0$$

$$f_1$$

$$u = f_0(s)$$

$$n \times m$$

$$1 \times 2$$

$$2 \times 1$$

$$2^{nm}$$

$$N \times M$$

$$m$$

$$m+1$$

$$m$$

$$1$$

$$N \times M$$

$$m+1$$

$$0$$

$$0$$

$$N \times M$$

$$N \times N$$

$$N \leq 7$$

$$N \times M$$

$$4 \times N$$

$$n \times m$$

$$n \times n$$

$$m$$

$$2^m$$

$$N \times M$$

$$N, M \leq 8$$

$$m \times n$$

$$n \times m$$

$$N \times M$$

$$2 \times 2$$

$$m$$

$$1$$

$$2 \times 2$$

$$m+1$$

$$4$$

$$5$$

$$2$$

$$0$$

$$-1$$

$$2 \times 2$$

$$N \times N$$

$$K$$

$$k$$

$$N \times M$$

$$n \leq 800, m \leq 7$$

$$N \times M$$

$$1$$

$$r \times c$$

$$N \times M$$

$$9 \times 6$$

$$9 \times 6$$

$$S$$

$$T$$

$$N \times M$$

$$p$$

$$N \leq 5$$

$$M \leq 6$$

$$R \times C$$

$$(2 \leq R \times C \leq 15)$$

$$N \times M$$

$$K$$

$$N \times M$$

$$\min(N, M) \leq 5, \max(N, M) \leq 130$$

$$m = 2$$

$$m \leq 10, n \leq 10^9$$

$$n, m \leq 100$$

$$10^{18}$$

$$ans_{[l,r]} = ans_{[0,r]} - ans_{[0,l-1]}$$

$$a, b$$

$$[a, b]$$

$$i$$

$$dp_i$$

$$i$$

$$dp_i = 10 \times dp_{i-1} + 10^{i-1}$$

$$i - 1$$

$$i$$

$$dp$$

$$0$$

$$i$$

$$0$$

$$1$$

$$i-1$$

$$0$$

$$i-1$$

$$dp_i$$

$$i$$

$$dp_{pos,sta}$$

$$pos$$

$$sta$$

$$[l,r]$$

$$0$$

$$2$$

$$ans_i$$

$$[1,i]$$

$$ans_r - ans_{l-1}$$

$$n$$

$$n$$

$$n$$

$$f(i,st,op)$$

$$i$$

$$st$$

$$op$$

$$op=1$$

$$op=0$$

$$\gcd$$

$$f(i,st,op)=\sum_{k=1}^{maxx} f(i+1,k,op=1 \text{ and } k=maxx) \quad (\mid st-k \mid \geq 2)$$

$$k$$

$$maxx$$

$$op=1$$

$$f$$

$$[n,m]$$

$$n,m<10^{44},T<10^5$$

$$0,1,8$$

$$0,1,8$$

$$0,1,8$$

$$[n,m]=[1,m]-[1,n-1]$$

$$n$$

$$[n,m]=[1,m]-[1,n]+\text{check}(n)$$

$$x$$

$$S$$

$$S=\{22,333,0233\}$$

$$233233$$

$$23332333$$

$$2023320233$$

$$32233223$$

$$n$$

$$S$$

$$n$$

$$10^9+7$$

$$1\le n<10^{1201}□1\le m\le 100□1\le\sum_{i=1}^{m}|\ s_i|\le 1500□\min_{i=1}^{m}|\ s_i|\ge 1$$

$$|\ s_i|$$

$$s_i$$

$$n$$

$$0$$

$$s_i$$

$$0$$

$$f(i,j,0/1)$$

$$i$$

$$i$$

$$j$$

$$M$$

$$t_i$$

$$v_i$$

$$T$$

$$1 \le T \le 10^3$$

$$1 \le t_i, v_i, M \le 100$$

$$O(TM)$$

$$dp_i = \max\{dp_j + 1\} \quad (1 \le j < i \land a_j < a_i)$$

$$f_{i,j} = \max(f_{i-1,j}, f_{i-1,j-w_i} + v_i)$$

$$f_j = \max\left(f_j, f_{j-w_i} + v_i\right)$$

$$f_{i,j} = \max_{k=0}^{+\infty}(f_{i-1,j-k\times w_i} + v_i \times k)$$

$$f_{i,j} = \max(f_{i-1,j}, f_{i,j-w_i} + v_i)$$

$$f_{i,j} = \max_{k=0}^{k_i}(f_{i-1,j-k\times w_i} + v_i \times k)$$

$$dp_i = \sum(dp_i, dp_{i-c_i})$$

$$n$$

$$W$$

$$w_i$$

$$v_i$$

$$0$$

$$1$$

$$i$$

$$w_i$$

$$v_i$$

$$W$$

$$f_{i,j}$$

$$i$$

$$j$$

$$i-1$$

$$i$$

$$f_{i-1,j}$$

$$w_i$$

v_i

$f_{i-1,j-w_i} + v_i$

f_i

f_{i-1}

f_i

i

i

$f_{i,j}$

$j \geqslant w_i$

$f_{i,j}$

$f_{i,j-w_i}$

i

W

w_i

$f_{i,j}$

$f_{i,j-w_i}$

$f_{i,j}$

i

j

i

$O(n^3)$

$f_{i,j}$

$f_{i,j-w_i}$

$f_{i,j-w_i}$

$f_{i,j-2\times w_i}$

$f_{i,j-w_i}$

i

n

W

w_i

$$v_i$$

$$k_i$$

$$k_i$$

$$k_i$$

$$O(W\sum_{i=1}^{n}k_i)$$

$$O(nW)$$

$$O(\sum k_i)$$

$$A_{i,j}$$

$$i$$

$$j$$

$$\forall j \leq k_i$$

$$A_{i,j}$$

$$A_{i,1}, A_{i,2}$$

$$A_{i,2}, A_{i,3}$$

$$A_{i,j}(j \in [0, \lfloor \log_2(k_i+1) \rfloor - 1])$$

$$2^j$$

$$k_i + 1$$

$$2$$

$$k_i - 2^{\lfloor \log_2(k_i+1) \rfloor - 1}$$

$$6 = 1 + 2 + 3$$

$$8 = 1 + 2 + 4 + 1$$

$$18 = 1 + 2 + 4 + 8 + 3$$

$$31 = 1 + 2 + 4 + 8 + 16$$

$$\leq k_i$$

$$O(W\sum_{i=1}^{n}\log_2 k_i)$$

$$k$$

$$n$$

$$T$$

$$A_i$$

C_i

T

n

i

t_i

c_i

T

W

n

m

i

w_i

v_i

$t_{k,i}$

k

i

cnt_k

k

n

m

i

v_i

p_i

$v_i \times p_i$

V

v_i

$h(v_i)$

i, j

i

j

i

j

i

$$g_{i,v}$$

$$i$$

$$v$$

$$dp_0 = 1$$

$$0$$

$$dp$$

$$f_{i,j}$$

$$i$$

$$j$$

$$g_{i,j}$$

$$f_{i,j}$$

$$i$$

$$j$$

$$g_{i,j}$$

$$i$$

$$j$$

$$f_{i,j} = f_{i-1,j}$$

$$f_{i,j} \neq f_{i-1,j-v} + w$$

$$g_{i-1,j}$$

$$f_{i,j} \neq f_{i-1,j}$$

$$f_{i,j} = f_{i-1,j-v} + w$$

$$g_{i-1,j-v}$$

$$f_{i,j} = f_{i-1,j}$$

$$f_{i,j} = f_{i-1,j-v} + w$$

$$g_{i-1,j}$$

$$g_{i-1,j-v}$$

$$f_m$$

$$g_j$$

$$k$$

$$dp_{i,j,k}$$

$$i$$

$$j$$

$$k$$

$$dp_{i,j}$$

$$dp_{i,j} = \max(dp_{i-1,j}, dp_{i-1,j-v_i} + w_i)$$

$$dp_{i-1,j}$$

$$dp_{i-1,j-v_i} + w_i$$

$$k$$

$$k$$

$$dp_{i,j}$$

$$O(k)$$

$$O(nmk)$$

$$O(mk)$$

$$k$$

$$n \leq 100, v \leq 1000, k \leq 30$$

$$f(i,j)$$

$$i$$

$$j$$

$$f(i,j) = \max\{f(i,k) + f(k+1,j) + cost\}$$

$$cost$$

$$n$$

$$a_1, a_2, ..., a_n$$

$$n-1$$

$$f(i,j)$$

$$[i,j]$$

$$f(i,j) = \max\{f(i,k) + f(k+1,j) + \sum_{t=i}^{j} a_t\} \ (i \leq k < j)$$

$$sum_i$$

$$a$$

$$f(i,j) = \max\{f(i,k) + f(k+1,j) + sum_j - sum_{i-1}\}$$

$$f(i,j)$$

$$f(i,k)$$

$$f(k+1,j)$$

$$f(i,j)$$

$$len=j-i+1$$

$$len$$

$$i$$

$$len$$

$$i$$

$$j$$

$$k$$

$$O(n^3)$$

$$n$$

$$O(n^4)$$

$$2\times n$$

$$i$$

$$n+i$$

$$f(1,n),f(2,n+1),...,f(n-1,2n-2)$$

$$O(n^3)$$

$$C_{i,j}=\max_{k=1}^{n}(A_{i,k}+B_{k,j})$$

$$\begin{cases} f_{i,0}=\sum_{son}\max(f_{son,0},f_{son,1}) \\ f_{i,1}=w_i+\sum_{son}f_{son,0} \end{cases}$$

$$\begin{cases} f_{i,0}=g_{i,0}+\max(f_{son_i,0},f_{son_i,1}) \\ f_{i,1}=g_{i,1}+f_{son_i,0} \end{cases}$$

$$\begin{bmatrix} g_{i,0} & g_{i,0} \\ g_{i,1} & -\infty \end{bmatrix} \times \begin{bmatrix} f_{son_i,0} \\ f_{son_i,1} \end{bmatrix} = \begin{bmatrix} f_{i,0} \\ f_{i,1} \end{bmatrix}$$

$$n$$

$$m$$

$$x,y$$

$$x$$

$$y$$

$$A\times B=C$$

$$\max$$

$$f_{i,0}$$

$$i$$

$$f_{i,1}$$

$$i$$

$$\max(f_{root,0}, f_{root,1})$$

$$g_{i,0}$$

$$i$$

$$i$$

$$g_{i,1}$$

$$son_i$$

$$i$$

$$son_i$$

$$i$$

$$g_{i,0/1}$$

$$\max(f_{root,0}, f_{root,1})$$

$$g_{i,1}$$

$$f_{i,0/1}$$

$$g_{i,0/1}$$

$$End_i$$

$$i$$

$$g$$

$$g$$

$$g_{i,1}$$

$$i$$

$$top_i$$

$$fa_{top_i}$$

$$\max$$

$$d(i,r) = \max\{d(j,r') + h\}$$

$$n(n \leqslant 30)$$

$$j$$

$$i$$

$$i$$

$$j$$

(i, j)

j

$31, 41, 59$

$33, 83, 27$

$\{\}$

$d(i, r)$

i

r

j

i

r

r'

j

h

i

r

$$f(i,j) = \begin{cases} f(i-1, j-1) + 1 & A_i = B_j \\ \max(f(i-1, j), f(i, j-1)) & A_i \neq B_j \end{cases}$$

r

$r \leq 1000$

$7 \to 3 \to 8 \to 7 \to 5$

$O(2^r)$

$7 \to 3 \to 8 \to 7$

4

2

r

$r - 1$

n

A

m

B

$n, m \leq 5000$

A

$$B$$

$$f(i,j)$$

$$A$$

$$i$$

$$B$$

$$j$$

$$f(i,j)$$

$$f(n,m)$$

$$f(i,j)$$

$$A_i = B_j$$

$$A_i$$

$$B_j$$

$$O(nm)$$

$$O\left(\frac{nm}{w}\right)$$

$$n$$

$$A$$

$$n \leq 5000$$

$$A$$

$$f(i)$$

$$A_i$$

$$\max_{1 \leq i \leq n} f(i)$$

$$f(i)$$

$$A_i$$

$$f(i) = \max_{1 \leq j < i, A_j \leq A_i} (f(j) + 1)$$

$$O(n^2)$$

$$n$$

$$n \leq 10^5$$

$$O(n \log n)$$

$$a_1 ... a_n$$

$$d$$

$$len$$

$$d_{len}$$

$$len$$

$$d_1 = a_1, len = 1$$

$$i$$

$$n$$

$$i$$

$$d$$

$$len$$

$$a_i$$

$$d_{len}$$

$$d$$

$$d_{len}$$

$$a_i$$

$$x$$

$$x$$

$$x$$

$$x$$

$$1$$

$$4$$

$$5$$

$$D$$

$$4$$

$$5$$

$$D$$

$$\infty$$

$$\alpha \in (0, \frac{1}{2}]$$

$$\rho$$

$$\rho \in [\alpha, 1 - \alpha]$$

$$\alpha$$

$$\alpha$$

$$O(\log n)$$

$$\frac{1}{1-\alpha}$$

$$O(\log_{\frac{1}{1-\alpha}} n) = O(\log n)$$

$$\alpha$$

$$A$$

$$1$$

$$2$$

$$A$$

$$C$$

$$A$$

$$B$$

$$A$$

$$A$$

$$A$$

$$A$$

$$\rho_1$$

$$\rho_2$$

$$\gamma_1 = \rho_1 + (1-\rho_1)\rho_2$$

$$\gamma_2 = \frac{\rho_1}{\rho_1 + (1-\rho_1)\rho_2}$$

$$siz_A$$

$$\rho_2 \leq \frac{1-2\alpha}{1-\alpha}$$

$$\gamma_1, \gamma_2 \in [\alpha, 1-\alpha]$$

$$\rho_3, \gamma_3$$

$$\gamma_1 = \rho_1 + \rho_2\rho_3(1-\rho_3), \gamma_2 = \frac{\rho_1}{\rho 1 + (1-\rho_1)\rho 2\rho 3}, \gamma_3 = \frac{\rho_2(1-\rho_3)}{1-\rho_2\rho_3}$$

$$\alpha < 1 - \frac{\sqrt{2}}{2} \approx 0.292$$

$$\gamma_1, \gamma_2, \gamma_3 \in [\alpha, 1-\alpha]$$

$$\rho_2 \leq \frac{1-2\alpha}{1-\alpha}$$

$$\alpha = 0.25$$

$$O(n\sqrt{n} + \frac{nc}{32})$$

$$n\sqrt{n}$$

$$\frac{nc}{32}$$

$$O(n)$$

$$lca$$

$$lca$$

$$O(\sqrt{n}+\frac{c}{32})$$

$$\sqrt{n}$$

$$O(\frac{c}{32})$$

$$\text{mex}$$

$$O((n+m)(\sqrt{n}+\frac{c}{32}))$$

$$16$$

$$2^{16}$$

$$1$$

$$1$$

$$16$$

$$T_1 \leq val$$
$$T_2 > val$$

$$T_{1\ \text{left}} \leq val-1$$
$$T_{1\ \text{right}} > val-1\ \&\ T_{1\ \text{right}} \leq val$$

$$T_1 \leq val-1$$

$$val$$

$$val$$

$$priority$$

$$priority$$

$$val$$

$$priority$$

$$priority$$

$$\log_2 n$$

$$n$$

$$\log_2 n$$

$\log_2 n$

$priority$

$\log_2 n$

$priority$

$priority$

$priority$

$priority$

$priority$

$priority$

$priority$

cur

key

val

key

key

key

cur

cur

key

key

key

cur

cur

key

key

cur

cur

key

cur

key

key

cur

cur

key

cur

key

$+1$

cur

rk

rk

rk

cur

u

v

u

v

u

v

$priority$

$priority$

u

$priority$

v

u

v

u

u

v

u

v

v

val

val

T_1

T_2

$val - 1$

T_1

$T_{1\ \text{left}}$

T_1

$T_{1\ \text{right}}$

$\& \ T_{1\ \text{right}} \leq val$

T_1

$T_1 \leq val$

val

$T_{1\ \text{right}}$

val

$T_{1\ \text{right}}$

val

$+1$

$val - 1$

val

T_1

val

rk

val

val

val

val

val

val

val

1

val

n

$\{a_n\}$

n

v

v

v

v

$O(\log n)$

$O(n \log n)$

$O(n)$

$O(\log n)$

$O(\log n)$

$O(n)$

5 4 3 2 1

$[2, 4]$

5 2 3 4 1

100%

$1 \leq n$

m

$\leq 1e5$

1 2 3 4 5

1 2 3 4 5

$priority$

u

$priority$

$priority$

u

v

u

u

v

u

u

u

u

$1 \sim 5$

5

$[l, r]$

$[1, l-1]$, $[l, r]$, $[r+1, n]$

$[l, r]$

$[3, 4]$

$$[3,5]$$

$$[l,r]$$

$$r-l$$

$$10^5$$

$$O(n \times \log_2 n)$$

$$O(\log_2 n)$$

$$val$$

$$val$$

$$sz$$

$$sz$$

$$[1,l-1],\ [l,r],\ [r+1,n]$$

$$\leq 3(r_3(B)-r_3(\gamma))+1$$

$$\leq 3(q-1)(r'(X)-r(X))$$

$$x$$

$$x$$

$$y$$

$$z$$

$$yz$$

$$x$$

$$xz$$

$$xy$$

$$yz$$

$$x$$

$$x$$

$$y$$

$$y$$

$$z$$

$$x$$

$$xy$$

$$yz$$

$$T$$

$$T$$

$$T_x$$

$$T_x$$

T

T

T

T_x

T

T_x

x

y

$C(x, y)$

T

T

T_x

T

n

n

T_x

N_x

T_x

x

T

T

T

x

x

$O(n)$

x

T

(j, h, c, jh, hc)

$C(k, g)$

$(a, b, i, f, g, e, ig, \cdots)$

$C(k, g)$

$compress(x)$

x

$C(k, g)$

T

T

x

$compress(x)$

T

...

x

x

T

x

$compress(x)$

$compress(x)$

x

T

T

x

$compress(x)$

$O(\log n)$

$compress(x)$

$compress(x)$

x

x

$compress(x)$

x

$+1$

$expose$

$expose(x, y)$

x

T

y

x

y

$expose$

$$x$$

$$x$$

$$T$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$makeroot$$

$$y$$

$$access$$

$$y$$

$$x$$

$$y$$

$$n$$

$$x$$

$$\varphi(x) = \sum_{i=1}^{n} r(i)$$

$$r(i) = \lceil \log_2(\text{□}i \text{ □□□□□□□}) \rceil$$

$$3n \log n + 1$$

$$x$$

$$a \leq 3 \log n + 1$$

$$x$$

$$a = 1 + r'(\gamma) - 0 \leq \log n + 1$$

$$x$$

$$x$$

$$x$$

$$x$$

$$r_x(i)$$

$$i$$

$$x$$

$$r$$

$$x$$

$$a \leq 3(r_2(\gamma) - r_1(\gamma)) + 1$$

$$x$$

$$a \leq 3(r_3(B) - r_2(B)) + 1$$

$$a = r_4(\gamma) - r_3(\gamma) + 1$$

$$r_4(\gamma) \leq r_3(B)$$

$$a \leq 3r_3(B) + 3r_3(B) + 3r_2(\gamma) - 3r_3(\gamma) - 3r_2(B) - 3r_1(\gamma) + 3$$

$$X$$

$$B$$

$$r$$

$$r'(X)$$

$$r_3(\gamma), r_1(\gamma) \geq r_1(X)$$

$$r_3(B), r_2(\gamma) \leq r'(X)\square r_3(B) = r_2(B)$$

$$a \leq 9(r'(X) - r(X)) + 3$$

$$a' \leq 3 \log n + 1$$

$$a'$$

$$r'(X)$$

$$r(X)$$

$$r(X)$$

$$x$$

$$r(X)$$

$$a \leq 9(r'(x) - r(x)) + 3k + 1$$

$$k$$

$$a$$

$$3k + 1$$

$$3k + 1$$

$$3k + 1$$

$$k$$

$$x$$

$$k$$

$$\frac{k}{2}$$

$$a \leq 3 \log n + 1$$

$$a \leq 9(r'(X) - r(X)) + 3k + 1 + 18(r''(X) - r'(X)) - S + 1 + 3 \log n + 1, S \geq 3k$$

$$a \leq 18(r''(X) - r(X)) + 2 + 3 \log n + 1$$

$$a \leq 21(r''(X) - r(X)) + 3$$

$$a'$$

$$m$$

$$\sum_{i=1}^{m} a_{(i)'} + \sum_{i=1}^{m} a_i = \sum_{i=1}^{m} c_i + \varphi(x_n) - \varphi(x_0)$$

$$\sum_{i=1}^{m} c_i = \sum_{i=1}^{m} a_i + \sum_{i=1}^{m} a_{(i)'} - \varphi(x_n) + \varphi(x_0)$$

$$\sum_{i=1}^{m} c_i \leq \sum_{i=1}^{m} a_{(i)'} + 21m \log n + n \log n + 3m$$

$$m$$

$$m$$

$$n$$

$$m + n$$

$$a' \leq 3 \log n + 1$$

$$\sum_{i=1}^{m} c_i \leq 3(m + n) \log n + 21m \log n + n \log n + 4m + n$$

$$diam$$

$$len$$

$$maxs_{0/1}$$

$$maxs_0$$

$O(\log n)$

$O(\log n)$

$diam$

sum

$maxs$

sum

T

$compress(Y)$

sum

$compress(Z)$

A

X

α

sum

$compress(Y)$

sum

$compress(Y)$

$compress(Y)$

X

$compress(Z)$

sum

$compress(Y)$

A

X

β

sum

$compress(Z)$

sum

$compress(Z)$

$compress(Z)$

X

x

sum

$$sum$$

$$compress(Y)$$

$$A$$

$$X$$

$$Y$$

$$sum$$

$$sum$$

$$X$$

$$X$$

$$Y$$

$$Y$$

$$compress(Y)$$

$$Y$$

$$Y$$

$$P_1 = \{1, 3, 6\}, S_1 = \{6, 5, 3\}$$
$$P_2 = \{4, 9, 15\}, S_2 = \{15, 11, 6\}$$
$$P_3 = \{7, 15, 24\}, S_3 = \{24, 17, 9\}$$

$$B = \begin{bmatrix} 6 & 21 & 45 \\ 0 & 15 & 39 \\ 0 & 0 & 24 \end{bmatrix}$$

$$l = 39_{10} = 100111_2, r = 46_{10} = 101110_2$$

$$\langle a_i \rangle_{i=1}^{n}$$

$$\circ$$

$$\gcd, \min, \max, +, \text{and}, \text{or}, \text{xor}$$

$$[l, r]$$

$$a_l \circ a_{l+1} \circ \cdots \circ a_r$$

$$O(n \log \log n)$$

$$O(1)$$

$$O(\sqrt{n})$$

$$O(\sqrt{n})$$

$$P_i$$

$$S_i$$

$$\langle B_{i,j} \rangle$$

$$i$$

$$j$$

$$\circ$$

$$+$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\{1, 2, 3\}, \{4, 5, 6\}, \{7, 8, 9\}$$

$$B$$

$$i > j$$

$$O(n)$$

$$O(n)$$

$$O(1)$$

$$1$$

$$O(1)$$

$$1$$

$$2$$

$$k$$

$$O(\sqrt{k})$$

$$O(\log \log n)$$

$$O(n)$$

$$O(n \log \log n)$$

$$O(\log \log n)$$

$$[l, r]$$

$$u$$

$$u$$

$$[l, r]$$

$$[l, r]$$

$$u$$

$$O(1)$$

$$O(\log \log n)$$

$$O(\log \log n)$$

$$O(1)$$

$$O(\log\log\log n)$$

$$2$$

$$0$$

$$2$$

$$O(\sqrt{k})$$

$$O(n)$$

$$[l, r]$$

$$k = 4, l = 39, r = 46$$

$$2^k = 2^4 = 16$$

$$[0, 15]$$

$$[000000_2, 001111_2]$$

$$[16, 31]$$

$$[010000_2, 011111_2]$$

$$k$$

$$k = 4$$

$$l, r$$

$$k$$

$$k$$

$$l \oplus r \leq 2^k - 1$$

$$i \in [1, n]$$

$$i$$

$$1$$

$$[l, r]$$

$$l \oplus r$$

$$O(1)$$

$$a_x = val$$

$$l$$

$$\langle P_i \rangle, \langle S_i \rangle, \left\langle B_{i,j} \right\rangle$$

$$\langle P_i \rangle$$

$$\langle S_i \rangle$$

$$O(\sqrt{l})$$

$$\langle B_{i,j} \rangle$$

$$O(l)$$

$$O(l)$$

$$O(n + \sqrt{n} + \sqrt{\sqrt{n}} + \cdots) = O(n)$$

$$\langle B_{i,j} \rangle$$

$$\langle B_{i,j} \rangle$$

$$index$$

$$\langle B_{i,j} \rangle$$

$$index$$

$$\langle B_{i,j} \rangle$$

$$index$$

$$index$$

$$O(\sqrt{n})$$

$$\langle P_i \rangle$$

$$\langle S_i \rangle$$

$$index$$

$$O(n)$$

$$O(\sqrt{n})$$

$$O(\sqrt{n})$$

$$O(1)$$

$$index$$

$$O(\sqrt{n})$$

$$\mathrm{Update}(l, r, x)$$

$$[l, r]$$

$$x$$

$$O(\sqrt{n} \log \log n)$$

$$O(1)$$

$$O(\sqrt{n})$$

$$O(\log\log n)$$

$$O(\sqrt{n})$$

$$1$$

$$O(\sqrt{n})$$

$$O(\sqrt{n}\log\log n)$$

$$1$$

$$O(\sqrt{n}\log\log n)$$

$$\langle P_i \rangle$$

$$\langle S_i \rangle$$

$$O(\sqrt{n})$$

$$index$$

$$O(\sqrt{n}\log\log n)$$

$$index$$

$$index$$

$$O(1)$$

$$O(1)$$

$$O(\log\log n)$$

$$O(\sqrt{n})$$

$$\langle P_i \rangle$$

$$\langle S_i \rangle$$

$$O(\sqrt{n})$$

$$index$$

$$O(\sqrt{n})$$

$$\langle B_{i,j} \rangle$$

$$O(\sqrt{n} + \sqrt{\sqrt{n}} + \cdots) = O(\sqrt{n})$$

$$O(n\log\log n)$$

$$O(1)$$

$$O(\sqrt{n})$$

$$
\begin{aligned}
1 + w'(x) + w'(fa) - w(x) - w(fa) &\leq 1 + w'(fa) - w(x) \\
&\leq 1 + w'(x) - w(x)
\end{aligned}
$$

$$
\begin{aligned}
1 + w'(x) + w'(fa) + w'(g) - w(x) - w(fa) - w(g) &\leq 1 + w'(fa) + w'(g) - w(x) - w(fa) \\
&\leq 1 + w'(x) + w'(g) - 2w(x) \\
&\leq 3(w'(x) - w(x))
\end{aligned}
$$

$$
\begin{aligned}
1 + w'(x) + w'(fa) + w'(g) - w(x) - w(fa) - w(g) &\leq 1 + w'(fa) + w'(g) - w(x) - w(fa) \\
&\leq 1 + w'(g) + w'(fa) - 2w(x) \\
&\leq 2w'(x) - w'(g) - w'(fa) + w'(fa) + w'(g) - w(x) - w(fa) \\
&\leq 2(w'(x) - w(x))
\end{aligned}
$$

$$
\begin{aligned}
3\left(\sum_{i=0}^{n-2}\big(w^{(i+1)}(x_1) - w^{(i)}(x_1)\big) + w(n) - w^{(n-1)}(x_1)\right) + 1 &= 3(w(n) - w(x_1)) + 1 \\
&\leq \log n
\end{aligned}
$$

$$
\sum_{i=1}^{m}(\varphi(m-i+1) - \varphi(m-i)) + \varphi(0) = n\log n + m\log n
$$

$$O(\log N)$$

$$<$$

$$<$$

$$x$$

$$\text{size}$$

$$x$$

$$x$$

$$x$$

$$y$$

$$y$$

$$x$$

$$x$$

$$x$$

$$y$$

$$x$$

$$y$$

$$y$$

$$x$$

$$y$$

$$z$$

$$z$$

$$y$$

x
x
z
x
x
x
x
p
x
p
x
p
zig
x
x
p
x
p
x
p
p
g
x
p
g
x
p
x
p
p
x
x
g
x
6
n
m

$$w(x) = [\log(\text{size}(x))]$$

$$\varphi = \sum w(x)$$

$$\varphi(0) \leq n \log n$$

$$i$$

$$\varphi(i)$$

$$\leq 3(w'(x) - w(x))$$

$$w^{(n)}(x) = w'^{(n-1)}(x)$$

$$w^{(0)}(x) = w(x)$$

$$x_1, x_2, \cdots, x_n$$

$$x_1$$

$$n$$

$$O(\log n)$$

$$O(\log n)$$

$$k$$

$$k$$

$$x$$

$$x$$

$$x$$

$$size$$

$$cnt$$

$$x$$

$$1$$

$$k$$

$$k$$

$$size$$

$$k$$

$$k$$

$$0$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$x$$

$$y$$

$$x$$

$$y$$

$$x$$

$$y$$

$$x$$

$$\text{Splay}$$

$$y$$

$$x$$

$$cnt[x] > 1$$

$$x$$

$$cnt[x]$$

$$1$$

$$[L, R]$$

$$a_{L-1}$$

$$a_{R+1}$$

$$a_{R+1}$$

$$a[L, R]$$

$$[L, R]$$

$$0$$

$$n+1$$

$$L-1$$

$$R+1$$

$$0$$

$$O(n)$$

$$\text{kth}$$

$$\begin{cases} \texttt{Logn}[1] \leftarrow 0, \\ \texttt{Logn}[i] \leftarrow \texttt{Logn}\left[\frac{i}{2}\right] + 1. \end{cases}$$

$$\text{opt}$$

$$x \operatorname{opt} x = x$$

$$\max(x, x) = x$$

$$\gcd(x, x) = x$$

$$\mathrm{opt}$$

$$n$$

$$m$$

$$[l, r]$$

$$[l, r]$$

$$\Theta(n \log n)$$

$$\Theta(1)$$

$$2^i$$

$$\Theta(\log n)$$

$$\max(x, x) = x$$

$$\Theta(1)$$

$$f(i, j)$$

$$[i, i + 2^j - 1]$$

$$f(i, 0) = a_i$$

$$2^j - 1$$

$$f(i, j) = \max(f(i, j-1), f(i + 2^{j-1}, j-1))$$

$$[l, r]$$

$$[l, l + 2^s - 1]$$

$$[r - 2^s + 1, r]$$

$$s = \lfloor \log_2(r - l + 1) \rfloor$$

$$[l, r]$$

$$w$$

$$\Theta(\log w)$$

$$\Theta(\log n + \log w)$$

$$n$$

$$\Theta(\log n)$$

$$w$$

$$\Omega(\log w)$$

$$O(n \log n \log w)$$

$$\log_2(w^n) = \Theta(n \log w)$$

$$O(n \log w)$$

$$\Theta(n \log n)$$

$$O(n(\log w + \log x))$$

$$\Omega(n(\log w + \log x))$$

$$\Theta(n(\log w + \log x))$$

$$\Theta(\log w)$$

$$\Theta(n(\log n + \log w))$$

$$\Theta(\log w)$$

$$\Theta(n \log x)$$

$$\Theta(n(\log n + \log x))$$

$$A$$

$$\Phi(A) = \log_2\left(\prod_{i=1}^{n} A_i\right)$$

$$O(\log w)$$

$$\Phi(A)$$

$$1$$

$$\Phi(A)$$

$$\log_2(w^n) = \Theta(n \log w)$$

$$\Phi(A)$$

$$O(n(\log w + \log n))$$

$$\begin{aligned} &C(0) = 0 \\ &C(i) = (1-p)(1+C(i)) + p(1+C(i-1)) \end{aligned}$$

$$O(n)$$

$$O(\log n)$$

$$O(n)$$

$$i$$

$$p$$

$$i+1$$

$$p$$

$$\frac{1}{p}$$

$$L(n)$$

$$L(n) = \log_{\frac{1}{p}} n$$

$$L(n)$$

$$O(\log n)$$

$$i$$

$$p^{i-1}(1-p)$$

$$\sum_{i>=1} ip^{i-1}(1-p) = \frac{1}{1-p}$$

$$p$$

$$O(n)$$

$$O(n \log n)$$

$$L(n)$$

$$i$$

$$x$$

$$x$$

$$x$$

$$x$$

$$i$$

$$x$$

$$i$$

$$p$$

$$x$$

$$i$$

$$1-p$$

$$C(i)$$

$$i$$

$$C(i) = \frac{i}{p}$$

$$n$$

$$L(n)$$

$$\frac{L(n)-1}{p}$$

$$L(n)$$

$$L(n)$$

$$L(n)$$

$$\frac{1}{p}$$

$$L(n)$$

$$\frac{1}{p}$$

$$L(n)$$

$$\frac{1}{p}$$

$$\frac{L(n)-1}{p}+\frac{2}{p}$$

$$L(n)=\log_{\frac{1}{p}} n$$

$$O(\log n)$$

$$O(n)$$

$$\log_{\frac{1}{p}} n$$

$$O(\log n)$$

$$p$$

$$k$$

$$k$$

$$O(n)$$

$$A$$

$$B$$

$$A$$

$$a$$

$$B$$

$$b$$

$$(a<b)$$

$$A$$

$$B$$
$$b-a$$
$$k$$
$$k$$
$$k$$
$$k$$
$$O(\log n)$$
$$\alpha$$
$$(0.5,1)$$
$$0.7$$
$$0.8$$
$$\alpha$$
$$\alpha$$
$$O(\log N)$$
$$1$$
$$5$$
$$a=\{10,11,12,13,14\}$$
$$1$$
$$d$$
$$d_i$$
$$i$$
$$a$$
$$d_1$$
$$[1,5]$$
$$a_1,a_2,\cdots,a_5$$
$$d_1$$
$$a_1+a_2+\cdots+a_5$$
$$d_1=60$$
$$a_1+a_2+\cdots+a_5=60$$
$$d_i$$
$$d_{2\times i}$$
$$d_i$$

$$d_{2\times i+1}$$

$$d_i$$

$$[s,t]$$

$$d_i = a_s + a_{s+1} + \cdots + a_t$$

$$d_i$$

$$[s, \frac{s+t}{2}]$$

$$d_i$$

$$[\frac{s+t}{2}+1, t]$$

$$p$$

$$1$$

$$a$$

$$2p$$

$$p$$

$$2p+1$$

$$p$$

$$n$$

$$2^{\lceil \log n \rceil + 1}$$

$$\lceil \log n \rceil$$

$$2^{\lceil \log n \rceil}$$

$$2^{\lceil \log n \rceil + 1} - 1$$

$$4n$$

$$\frac{2^{\lceil \log n \rceil + 1} - 1}{n}$$

$$n = 2^x + 1 (x \in N_+)$$

$$2^{\lceil \log n \rceil + 1} - 1 = 2^{x+2} - 1 = 4n - 5$$

$$[l,r]$$

$$a_l + a_{l+1} + \cdots + a_r$$

$$[1,5]$$

$$d_1$$

$$60$$

$$[3,5]$$

$$[3,5]$$

$$[3,3]$$

$$[4,5]$$

$$[l,r]$$

$$O(\log n)$$

$$[l,r]$$

$$[l,r]$$

$$[l,r]$$

$$t_i$$

$$[3,5]$$

$$5$$

$$[3,3]$$

$$[4,5]$$

$$3$$

$$5$$

$$3$$

$$d_3$$

$$5 \times 2 = 10$$

$$[4,4]$$

$$[4,5]$$

$$6$$

$$7$$

$$4n$$

$$2p$$

$$2p+1$$

$$p$$

$$\text{ls}$$

$$\text{rs}$$

$$O(\log n)$$

$$m$$

$$O(m \log n)$$

$$2n-1$$

$$k$$

$$x$$

$$x$$

$$N$$

$$1$$

$$N$$

$$i$$

$$Pi$$

$$[L, R]$$

$$NewP$$

$$NewP$$

$$[L, R]$$

$$O(n \log n)$$

$$n$$

$$n \log n$$

$$O(n \log n)$$

$$[1, N]$$

$$[l, r]$$

$$[s, t]$$

$$[l, r]$$

$$[s, t]$$

$$[l, r]$$

$$[s, t]$$

$$[l, r]$$

$$\log n$$

$$O(\log \boxed{f()}\ n)$$

$$[x, y]$$

$$t$$

$$p$$

$$p$$

$$x$$
$$q$$
$$[x,y]$$
$$k$$
$$[2,4]$$
$$n$$
$$q$$
$$u \to v$$
$$w$$
$$i \in [l,r]$$
$$u \to i$$
$$w$$
$$i \in [l,r]$$
$$i \to u$$
$$w$$
$$s$$
$$1 \le n, q \le 10^5, 1 \le w \le 10^9$$
$$O(n)$$
$$O(\log n)$$
$$O(\log^2 w)$$
$$O(\log n)$$
$$O(n \log n)$$
$$O(1)$$
$$O(n \log^2 w)$$
$$[l,r]$$
$$[l,l]$$
$$[r,r]$$
$$p$$
$$[L,R]$$
$$[L,R]$$
$$[l,r]$$
$$l$$

$$r$$

$$[l, r]$$

$$[L, R]$$

$$p$$

$$l$$

$$r$$

$$p$$

$$l$$

$$r$$

$$p$$

$$r$$

$$l$$

$$l$$

$$[L, mid]$$

$$r$$

$$(mid, R]$$

$$O(1)$$

$$(l, r]$$

$$[l, r]$$

$$(l, mid]$$

$$(mid, r]$$

$$T(n) = 2T(n/2) + O(n) = O(n \log n)$$

$$O(n)$$

$$O(n \log n)$$

$$[l, r]$$

$$[l, l]$$

$$[r, r]$$

$$p$$

$$l, r$$

$$p$$

$$p$$

$$p$$

$$[l, r]$$

$$[l, mid] + (mid, r]$$

$$[l, r]$$

$$O(1)$$

$$O(1)$$

$$O(\log n)$$

$$O(\log \log n)$$

$$2$$

$$x$$

$$y$$

$$O(\log^2 w)$$

$$O(n \log n)$$

$$O(n \log^2 w + m \log^2 w \log n)$$

$$O(n \log n \log w + m \log^2 w)$$

$$1$$

$$n$$

$$1$$

$$n$$

$$\log n$$

$$\log n$$

$$\log^2 n$$

$$\log n$$

$$\log n$$

$$\log^2 n$$

$$\log^2 n$$

$$O(n \log n)$$

$$k$$

$$O(n \log^3 n)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(\log^2 n)$$

$$k$$

$$k$$

$$O(\log n)$$

$$O(\log^2 n)$$

$$O(n \log^2 n)$$

$$\forall i \in [1, n],\, B_i = \max(B_i, A_i)$$

$$\forall i \in [1, n],\, B_i = B_i + A_i$$

$$Pre_s = \max(Pre_s, Pre_u + Add_s), Add_s = Add_u + Add_s$$

$$[l, r]$$

$$x$$

$$\max$$

$$\min$$

$$a_i = \max(a_i, x)$$

$$a_i = \min(a_i, x)$$

$$a$$

$$\forall l \le i \le r,\, a_i = \min(a_i, t)$$

$$[l, r]$$

$$T \le 100, n \le 10^6, \sum m \le 10^6$$

$$t$$

$$t$$

$$Max$$

$$Se$$

$$Sum$$

$$Cnt$$

$$t$$

$$\min$$

$$Max \le t$$

$$t$$

$$Se < t \le Max$$

$$t$$

$$Cnt(t - Max)$$

$$Max$$

$$t$$

$$t \leq Se$$

$$O(m \log n)$$

$$n$$

$$x$$

$$x$$

$$\max$$

$$x$$

$$\min$$

$$N, M \leq 5 \times 10^5, |\ A_i| \leq 10^8$$

$$\max$$

$$\min$$

$$v$$

$$v$$

$$\max$$

$$\min$$

$$v$$

$$\min$$

$$\max$$

$$v$$

$$\max$$

$$\max$$

$$v$$

$$\max$$

$$O(m \log^2 n)$$

$$A, B$$

$$B$$

$$0$$

$$A$$

$$\min$$

$$A$$

$$\max$$

$$A$$

$$B$$

A_i

B_i

1

$n, m \leq 3 \times 10^5$

$x \neq 0$

B

B

B

A

B

B

B

A, B

A

$\min$

B

$\min$

A

B

$A_i + B_i$

$n, m \leq 3 \times 10^5$

A, B

A

B

B

A

A, B

$C_{1\sim4}, M_{1\sim4}, A_{\max}, B_{\max}$

A

B

$O(m \log^2 n)$

B

A

A

$\max$

B_i

A

$\min$

B_i

B

0

A

B

B_i

i

A, B

A

x

A

x

A

$\max$

B

$\max$

$\forall i \in [1, n],\ B_i = \max(B_i, A_i)$

$n, m \leq 10^5$

Add

$\max$

$\max$

Pre

Add

u

u

u

u

u

$$u$$

$$Add$$

$$u$$

$$s$$

$$(P_1, P_2)$$

$$P_1$$

$$P_2$$

$$O(m \log n)$$

$$x$$

$$O(\log n)$$

$$N$$

$$N$$

$$N$$

$$O(\log n)$$

$$n$$

$$S$$

$$N$$

$$N$$

$$S$$

$$P$$

$$S$$

$$O(\log n)$$

$$T3$$

$$T1$$

$$T2$$

$$\frac{1}{3}$$

$$f[i, j] = \max(f[i-1, j], f[i-1, (j - w_i) \bmod p] + v_i)$$

$$T(m) = 2 \cdot O(m) + T\left(\frac{m}{2}\right)$$

$$\max_{0 \le i < p} \left\{ f[i] + \max_{l \le i+j \le r} g_j \right\}$$

$$O(1)$$

$$\sum_{(w,v)\in S} w \bmod p \in [l,r]$$

$$\sum_{(w,v)\in S} v$$

$$m \leq 5\times 10^4, p \leq 500$$

$$f[S,j]$$

$$O(m\log m)$$

$$O(mp\log m)$$

$$f[i,j]$$

$$O(mp)$$

$$O(1)$$

$$O(mp)$$

$$O(m)$$

$$\Theta(m)$$

$$O(1)$$

$$O(k)$$

$$O(k)\geq O(1)$$

$$O(m)$$

$$\frac{m}{2}$$

$$O(\frac{m}{2})$$

$$O(\frac{m}{2})$$

$$T(m)=O(m)$$

$$O(mp)$$

$$f,g$$

$$O(p^2)$$

$$O(p)$$

$$m$$

$$S_i$$

$$1\sim n$$

$$O(n + \sum | \ S_i|)$$

$$O(nm)$$

$$n$$

$$1, 2, ..., n$$

$$S$$

$$u$$

$$u$$

$$fa_u$$

$$u$$

$$u$$

$$u$$

$$u$$

$$u$$

$$v$$

$$u$$

$$u$$

$$f$$

$$e$$

$$e$$

$$f$$

$$e$$

$$f$$

$$u$$

$$e$$

$$f$$

$$u$$

$$u$$

$$p$$

$$u$$

$$v$$

$$v$$

$$p$$

$$v$$

$$p$$

$$u$$

p

u

f

e

e

f

e

f

e

p

e

p

f

p

f

p

u

p_1, p_2

f

f

f

p_1

f

f

p_2

p_2

p_2

p_2

p_1

u

u

u

p

p_f

p

p_e

p

f

p

e

p

f

p_f

e

p_e

p

u

p

p

u

p_1, p_2

p_1, p_2

u

u

u

q_1

u

q_1

p

p

q_1

l, r

$[l+1, r-1]$

l

r

u

u

u

u

$$u$$

$$u$$

$$u$$

$$u$$

$$n$$

$$a$$

$$x$$

$$n$$

$$1$$

$$[l, r]$$

$$k$$

$$l \leq p \leq r$$

$$k \oplus \bigoplus_{i=p}^{n} a_i$$

$$s_x = \bigoplus_{i=1}^{x} a_i$$

$$s_{p-1} \oplus s_n \oplus k$$

$$s_n \oplus k$$

$$[l-1, r-1]$$

$$s_n \oplus k$$

$$n$$

$$a$$

$$[l, r]$$

$$k$$

$$k$$

$$[1, 8]$$

$$O(\log n)$$

$$x \times 2$$

$$x \times 2 + 1$$

$$[1, r]$$

$$k$$

$$[l, r]$$

$$k$$

$$O(1)$$

$$[l, r]$$

$$[1, r]$$

$$[1, l-1]$$

$$2n-1$$

$$n$$

$$\lceil \log_2 n \rceil + 1$$

$$n$$

$$2n-1+n(\lceil \log_2 n \rceil + 1)$$

$$n \le 10^5$$

$$\lceil \log_2 10^5 \rceil + 1 = 18$$

$$n$$

$$2 \times 10^5 - 1 + 18 \times 10^5$$

$$-1$$

$$20 \times 10^5$$

$$2^5 \times 10^5$$

$$k$$

$$O(\log n)$$

$$x, y$$

$$x, y$$

$$x + y$$

$$x, y$$

$$x$$

$$y$$

$$x$$

$$O(\log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$x, y$$

$$x + y$$

$$x, y$$

$$p$$

p

y

p

p

p

$O(\log n)$

m

$O(m \log n)$

X_a

X

a

X_{a+1}

X

$a+1$

Y

X_{a+1}

Y_a

Y

a

Y

Y_{a+1}

Y

$a+1$

X_{a+1}

Y_{a+1}

Y

$a+1$

x

k

x

$key \geq k$

x

x

$$x$$
$$x$$
$$x$$
$$x$$
$$x$$
$$-2\,147\,483\,647$$
$$x$$
$$x$$
$$2\,147\,483\,647$$
$$O(n)$$
$$O(\log n)$$
$$\Omega(\log\log n)$$
$$O(1)$$
$$O(\log n)$$
$$O(2^{2\sqrt{\log\log n}})$$
$$\{0, 11, 45, 81\}$$
$$14$$
$$0, 11$$
$$\{14, 45, 81\}$$
$$N$$
$$h_i$$
$$i$$
$$\geq h_i$$
$$c_i$$
$$\sum_{i=1}^{N} c_i$$
$$\leq r$$
$$\geq l$$
$$O(q\log q + q\log n)$$
$$O(n)$$
$$n$$
$$k$$
$$i \sim i + k - 1$$

$$O(n \times k)$$

$$k-1$$

$$k-1$$

$$k$$

$$100\%$$

$$n \leq 1000000$$

$$k$$

$$k$$

$$O(N)$$

$$i$$

$$N$$

$$y$$

$$x$$

$$x$$

$$x$$

$$D$$

$$x$$

$$N$$

$$D$$

$$W$$

$$1 \leq N \leq 100000, 1 \leq D \leq 1000000, 0 \leq x, y \leq 10^6$$

$$x$$

$$x$$

$$y$$

$$D$$

$$R$$

$$[L, R]$$

$$L$$

$$[L, R]$$

$$f(R) = \max[L, R] - \min[L, R]$$

$$f(R)$$

$$R$$

$$f(R) \geq D \Longrightarrow f(r) \geq D, R < r \leq N$$

$$L$$
$$R$$
$$L$$
$$R$$
$$[L, R]$$
$$R$$
$$L$$
$$L$$
$$R$$
$$O(n)$$
$$O(1)$$
$$O(n)$$
$$O(1)$$
$$O(n)$$
$$i$$
$$i$$
$$(x_0, y_0)$$
$$(x_1, y_1)$$
$$k$$
$$x = k$$
$$0$$
$$1 \leq n \leq 10^5$$
$$1 \leq k, x_0, x_1 \leq 39989$$
$$1 \leq y_0, y_1 \leq 10^9$$
$$[l, r]$$
$$k$$
$$k$$
$$x = k$$
$$x$$
$$(x, y_0)$$
$$(x, y_1)$$
$$y_0 < y_1$$

$$[x, x]$$

$$f(x) = 0 \cdot x + y_1$$

$$i$$

$$l_i$$

$$l_i$$

$$f$$

$$f$$

$$f$$

$$g$$

$$m$$

$$f$$

$$g$$

$$f$$

$$f$$

$$g$$

$$f$$

$$g$$

$$f$$

$$f$$

$$g$$

$$f$$

$$g$$

$$f$$

$$f$$

$$g$$

$$f$$

$$g$$

$$g$$

$$f$$

$$f$$

$$g$$

$$f$$

g

f

g

x

$O(\log n)$

$O(\log n)$

$O(\log n)$

$O(\log n)$

$O(\log^2 n)$

dist

1

dist

dist

0

dist

dist

1

n

dist

$\lceil \log(n+1) \rceil$

dist

x

$x-1$

$2^x - 1$

dist

dist

dist

dist

dist

dist

dist

dist

1

n

dist

$\lceil \log(n+1) \rceil$

n

m

$O(\log n + \log m)$

dist

dist

dist

dist

x

y

dist

dist

x

y

y

dist

x

dist

x

y

y

dist

dist

y

dist

x

dist

dist

dist

x

dist

$O(\log n)$

dist

$O(\log n)$

$O(n)$

$O(n)$

$O(\log n)$

$O(n \log n)$

n

$2n-1$

x

x

x

x

x

x

x

x

x, y

x, y

x

x

v

x, y

A

N

N

N

O

N

N

I

I

I

K

I

N

I
L
I
H
H
H
I
A
H
A
N
x
y
x
y
x
x
y
x
y
x
y
x
y
y
y
y
x
x, y
x, y
x
x, y
x
x
y
y

$$O(\log n)$$
$$n$$
$$1$$
$$q$$
$$u_1, v_1$$
$$u_2, v_2$$
$$u, v$$
$$c$$
$$u, v$$
$$c$$
$$u, v$$
$$51061$$
$$1 \leq n, q \leq 10^5, 0 \leq c \leq 10^4$$
$$u, v$$
$$x, y$$
$$n$$
$$m$$
$$u, v$$
$$u, v$$
$$u, v$$
$$n \leq 10^4, m \leq 2 \times 10^5$$
$$n$$
$$m$$
$$q$$
$$u, v$$
$$u, v$$
$$1 < n < 3 \times 10^4, 1 < m < 10^5, 0 \leq q \leq 4 \times 10^4$$
$$u, v$$
$$u$$
$$v$$
$$-1$$
$$n$$
$$m$$
$$1 \leq n \leq 5 \times 10^4, 1 \leq m \leq 2 \times 10^5, 1 \leq w_i \leq 10^4$$
$$n$$

$$q$$
$$x$$
$$y$$
$$(x, y)$$
$$(x, y)$$
$$1 \leq n, q, x, y \leq 10^5$$
$$(x, y)$$
$$x$$
$$y$$
$$(x, y)$$
$$x$$
$$y$$
$$(x, y)$$
$$x$$
$$x$$
$$x$$
$$siz2[x]$$
$$x$$
$$siz[x]$$
$$x$$
$$x$$
$$siz2[x]$$
$$siz2[x]$$
$$siz2[x]$$
$$siz2$$
$$T(n) = 2T(n/4) + O(1)$$
$$k$$
$$n$$
$$2^k$$
$$k = 2$$
$$k$$
$$k$$
$$k$$

$$k$$
$$n$$
$$k = 2$$
$$x$$
$$y$$
$$2, 3$$
$$k$$
$$k$$
$$2$$
$$\log n + O(1)$$
$$O(n \log^2 n)$$
$$n$$
$$O(n)$$
$$O(n \log n)$$
$$O(n)$$
$$O(n \log n)$$
$$R$$
$$R$$
$$R$$
$$R$$
$$R$$
$$O(h) = O(\log n)$$
$$\epsilon$$
$$R$$
$$R$$
$$u$$
$$u$$
$$u$$
$$n$$
$$T(n) = O(\sqrt{n})$$
$$k$$
$$T(n) = 2^{k-1} T(n/2^k) + O(1)$$
$$T(n) = O(n^{1-\frac{1}{k}})$$

$$k$$
$$k$$
$$\log n + O(1)$$
$$O(\log n)$$
$$B$$
$$B$$
$$O(n \log n / B)$$
$$O(B + n^{1-\frac{1}{k}})$$
$$B = O(\sqrt{n \log n})$$
$$O(\sqrt{n \log n})$$
$$O(\sqrt{n \log n} + n^{1-\frac{1}{k}})$$
$$2$$
$$n$$
$$1$$
$$2^0$$
$$O(n \log^2 n)$$
$$\log$$
$$O\left(\sum_{i \geq 0} (\frac{n}{2^i})^{1-\frac{1}{k}}\right) = O(n^{1-\frac{1}{k}})$$
$$0$$
$$n \times n$$
$$q$$
$$(x, y)$$
$$A$$
$$(x1, y1)$$
$$(x2, y2)$$
$$1 \leq n \leq 500000, 1 \leq q \leq 200000$$
$$O(n)$$
$$n$$
$$(x_i, y_i)$$
$$2 \leq n \leq 200000, 0 \leq x_i, y_i \leq 10^9$$

$$n$$

$$O(n)$$

$$ans$$

$$ans$$

$$n$$

$$(x_i, y_i)$$

$$k$$

$$n \le 100000, 1 \le k \le 100, 0 \le x_i, y_i < 2^{31}$$

$$k$$

$$k$$

$$k$$

$$2$$

$$WPL = \sum_{i=1}^{n} w_i l_i$$

$$WPL = 2*2 + 3*2 + 4*2 + 7*2 = 4 + 6 + 8 + 14 = 32$$

$$n$$

$$w_i$$

$$i$$

$$l_i$$

$$i$$

$$0$$

$$2$$

$$n$$

$$n$$

$$F$$

$$F$$

$$F$$

$$F$$

$$n$$

$$\lceil \log_2 n \rceil$$

$$d_1, d_2, ..., d_n$$

$$w_1, w_2, ..., w_n$$

$$d_1, d_2, ..., d_n$$

$$w_1, w_2, ..., w_n$$

$$0$$
$$1$$
$$0$$
$$1$$
$$O(1)$$
$$O(\log n)$$
$$O(\log n)$$
$$O(\log n)$$
$$O(1)$$
$$O(1)$$
$$O(1)$$
$$O(1)$$
$$O(1)$$
$$O(1)$$
$$O(\log n)$$
$$O(\log n)$$
$$O(\log n)$$
$$O(\log n)$$
$$O(\log n)$$
$$O(1)$$
$$O(n)$$
$$O(\log n)$$
$$O(\log n)$$
$$O(1)$$
$$o(\log n)$$
$$\Omega(\log \log n)$$
$$O(2^{2\sqrt{\log \log n}})$$
$$O(\log n)$$
$$O(\log n)$$
$$O(\log n)$$

$$O(1)$$

$$\times$$

$$\checkmark$$

$$\checkmark$$

$$\checkmark$$

$$\times$$

$$O(\log n)$$

$$k$$

$$O(1)$$

$$O(1)$$

$$10^9$$

$$f(x) = x \bmod M$$

$$127$$

$$n$$

$$s$$

$$x = s_0 \cdot 127^0 + s_1 \cdot 127^1 + s_2 \cdot 127^2 + ... + s_n \cdot 127^n$$

$$x$$

$$2^{64}$$

$$x$$

$$2^{64}$$

$$a, b$$

$$a$$

$$b$$

$$1...M$$

$$N$$

$$O(\frac{N}{M})$$

$$O(n \log^2 n)$$

$$O(n \log n)$$

$$O(n \log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(n \log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$0$$

$$1$$

$$2$$

$$1$$

$$2$$

$$3$$

$$3$$

$$1$$

$$4$$

$$a$$

$$a$$

$$n$$

$$Node^n$$

$$a$$

$$n$$

$$n$$

$$d$$

$$O(1)$$

$$O(\log n)$$

$$O(1)$$

$$O(n)$$

$$O(n)$$

$$O(1)$$

$$O(\log n)$$

$$O(1)$$

$$O(n)$$

$$O(1)$$

$$O(1)$$

$$O(1)$$

$$O(n)$$

$$O(1)$$

$$O(\log \min(l1, l2))$$

$$O(\log n)$$

$$O(n)$$

$$O(m + n)$$

$$O(\log \min(n, l - n))$$

$$O(\log n)$$

$$O(n)$$

$$O(1)$$

$$O(\log n)$$

$$O(\log n)$$

$$O(n)$$

$$O(n)$$

$$O(l)$$

$$O(l)$$

$$O(l)$$

$$O(l)$$

$$O(1)$$

$$O(1)$$

$$O(n)$$

$$O(n)$$

$$O(\log \min(n, l - n))$$

$$O(\log n)$$

$$O(n)$$

$$O(1)$$

$$O(1)$$

$$O(\log n)$$

$$\sum_{i=1}^{r} a_i$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{i} d_j$$

$$\begin{aligned} & \sum_{i=1}^{r} \sum_{j=1}^{i} d_j \\ = & \sum_{i=1}^{r} d_i \times (r - i + 1) \\ = & \sum_{i=1}^{r} d_i \times (r + 1) - \sum_{i=1}^{r} d_i \times i \end{aligned}$$

$$f(x, i) = \begin{cases} x & i = 0 \\ f(x, i-1) + \mathrm{lowbit}(f(x, i-1)) & i > 0 \end{cases}$$

$$g(x, i) = \begin{cases} x & i = 0 \\ g(x, i-1) - \mathrm{lowbit}(g(x, i-1)) & i, g(x, i-1) > 0 \\ 0 & \text{otherwise.} \end{cases}$$

$$d(i, j) = a(i, j) - a(i-1, j) - a(i, j-1) + a(i-1, j-1)□$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & -v \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -v & 0 & 0 & v \end{pmatrix}$$

$$\sum_{i=1}^{x} \sum_{j=1}^{y} \sum_{h=1}^{i} \sum_{k=1}^{j} d(h, k)$$

$$\begin{aligned} & \sum_{i=1}^{x} \sum_{j=1}^{y} \sum_{h=1}^{i} \sum_{k=1}^{j} d(h, k) \\ = & \sum_{i=1}^{x} \sum_{j=1}^{y} d(i, j) \times (x - i + 1) \times (y - j + 1) \\ = & \sum_{i=1}^{x} \sum_{j=1}^{y} d(i, j) \times (xy + x + y + 1) - d(i, j) \times i \times (y + 1) - d(i, j) \times j \times (x + 1) + d(i, j) \times i \times j \end{aligned}$$

$$a$$

$$x, y$$

$$a[x]$$

$$y$$

$$l, r$$

$$a[l...r]$$

$$l, r, x$$

$$a[l...r]$$

$$x$$

$$x$$

$$a[x]$$

$$1$$

$$(x \circ y) \circ z = x \circ (y \circ z)$$

$$\circ$$

$$x \circ y$$

$$x$$

$$y$$

$$\gcd$$

$$\max$$

$$\Theta(\log^2 n)$$

$$a[1...7]$$

$$a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7$$

$$7$$

$$A$$

$$B$$

$$C$$

$$A = a[1...4]$$

$$B = a[5...6]$$

$$C = a[7...7]$$

$$a[7]$$

$$A + B + C$$

$$3$$

$$[1, n]$$

$$\boldsymbol{\log n}$$

$$\log n$$

$$\log n$$

n

a

a

c

c

a

c_2

$a[1...2]$

c_4

$a[1...4]$

c_6

$a[5...6]$

c_8

$a[1...8]$

$c[x]$

$a[x]$

$a[x...x]$

1

$c[x]$

x

$a[1...7]$

c_7

c_7

a_7

c_6

c_6

$a[5...6]$

c_4

c_4

$a[1...4]$

c_0

c_0

c

$$c_7, c_6, c_4$$

$$a[1...7]$$

$$c_7 + c_6 + c_4$$

$$a[4...7]$$

$$c_7$$

$$c_6$$

$$c_4$$

$$a[1...4]$$

$$a[1...3]$$

$$a[1...3]$$

$$a[4...7]$$

$$a[1...7]$$

$$a[1...3]$$

$$c[x](x \geq 1)$$

$$c[x]$$

$$2^k$$

$$0$$

$$k$$

$$x$$

$$2^k$$

$$c[x]$$

$$x$$

$$c_{88}$$

$$88_{(10)} = 01011000_{(2)}$$

$$8$$

$$c_{88}$$

$$8$$

$$a$$

$$c_{88}$$

$$a[81...88]$$

$$x$$

$$\text{lowbit}(x)$$

$$c[x]$$

$$[x - \text{lowbit}(x) + 1, x]$$

$$\mathbf{lowbit}$$

$$\boldsymbol{k}$$

$$\boldsymbol{2^k}$$

$$6$$

$$a[4...7]$$

$$a[1...7]$$

$$a[1...3]$$

$$a[l...r]$$

$$a[1...r]$$

$$a[1...l-1]$$

$$l...r$$

$$1...r$$

$$1...l-1$$

$$a[1...7]$$

$$c_7$$

$$c_7$$

$$a_7$$

$$c_6$$

$$c_6$$

$$a[5...6]$$

$$c_4$$

$$c_4$$

$$a[1...4]$$

$$c_0$$

$$c_0$$

$$c$$

$$c_7, c_6, c_4$$

$$a[1...7]$$

$$c_7 + c_6 + c_4$$

$$c_6$$

$$a[5...6]$$

$$5-1=4$$

$$c_4$$

$$a[1...x]$$

$$c[x]$$

$$c[x]$$

$$a[x-\mathrm{lowbit}(x)+1...x]$$

$$x \leftarrow x-\mathrm{lowbit}(x)$$

$$x=0$$

$$c$$

$$c$$

$$\mathrm{ans}=0$$

$$c[x]$$

$$\mathrm{ans} \leftarrow \mathrm{ans}+c[x]$$

$$\mathrm{ans}$$

$$l(x)=x-\mathrm{lowbit}(x)+1$$

$$l(x)$$

$$c[x]$$

$$x$$

$$x$$

$$s \times 2^{k+1}+2^k$$

$$\mathrm{lowbit}(x)=2^k$$

$$c[x]$$

$$c[y]$$

$$c[x]$$

$$c[y]$$

$$[l(x), x]$$

$$[l(y), y]$$

$$c[x]$$

$$c[y]$$

$$\boldsymbol{x \le y}$$

$$\boldsymbol{c[x]}$$

$$\boldsymbol{c[y]}$$

$$\boldsymbol{c[x]}$$

$$\boldsymbol{c[y]}$$

$$c[x]$$

$$c[y]$$

$$[l(x), x]$$

$$[l(y), y]$$

$$l(y) \le x \le y$$

$$y$$

$$s \times 2^{k+1} + 2^k$$

$$l(y) = s \times 2^{k+1} + 1$$

$$x$$

$$s \times 2^{k+1} + b$$

$$1 \le b \le 2^k$$

$$\mathrm{lowbit}(x) = \mathrm{lowbit}(b)$$

$$b - \mathrm{lowbit}(b) \ge 0$$

$$l(x) = x - \mathrm{lowbit}(x) + 1 = s \times 2^{k+1} + b - \mathrm{lowbit}(b) + 1 \ge s \times 2^{k+1} + 1 = l(y)$$

$$l(y) \le l(x) \le x \le y$$

$$c[x]$$

$$c[y]$$

$$c[x]$$

$$c[y]$$

$$\mathbf{2}$$

$$\boldsymbol{c[x]}$$

$$\boldsymbol{c[x + \mathbf{lowbit}(x)]}$$

$$y = x + \mathrm{lowbit}(x)$$

$$x = s \times 2^{k+1} + 2^k$$

$$y = (s+1) \times 2^{k+1}$$

$$l(x) = s \times 2^{k+1} + 1$$

$$\text{lowbit}(y) \geq 2^{k+1}$$

$$l(y) = (s+1) \times 2^{k+1} - \text{lowbit}(y) + 1 \leq s \times 2^{k+1} + 1 = l(x)$$

$$l(y) \leq l(x) \leq x < y$$

$$c[x]$$

$$c[x + \text{lowbit}(x)]$$

$$3$$

$$\boldsymbol{x < y < x + \textbf{lowbit}(x)}$$

$$\boldsymbol{c[x]}$$

$$\boldsymbol{c[y]}$$

$$x = s \times 2^{k+1} + 2^k$$

$$y = x + b = s \times 2^{k+1} + 2^k + b$$

$$1 \leq b < 2^k$$

$$\text{lowbit}(y) = \text{lowbit}(b)$$

$$b - \text{lowbit}(b) \geq 0$$

$$l(y) = y - \text{lowbit}(y) + 1 = x + b - \text{lowbit}(b) + 1 > x$$

$$l(x) \leq x < l(y) \leq y$$

$$c[x]$$

$$c[y]$$

$$a$$

$$c$$

$$x$$

$$x + \text{lowbit}(x)$$

$$x + \text{lowbit}(x)$$

$$x$$

$$x \leq n$$

$$c[x]$$

$$n$$

$$fa[u]$$

$$u$$

$$u < fa[u]$$

$$u$$

$$u$$

$$u$$

$$u$$

$$\operatorname{lowbit}$$

$$fa[u]$$

$$\operatorname{lowbit}$$

$$y = x + \operatorname{lowbit}(x)$$

$$x = s \times 2^{k+1} + 2^k$$

$$y = (s+1) \times 2^{k+1}$$

$$\operatorname{lowbit}(y) \geq 2^{k+1} > \operatorname{lowbit}(x)$$

$$x$$

$$\log_2 \operatorname{lowbit}(x)$$

$$x$$

$$x$$

$$h(x)$$

$$x \bmod 2 = 1$$

$$h(x) = 0$$

$$h(x) = \max(h(y)) + 1$$

$$y$$

$$x$$

$$x$$

$$x - 1$$

$$1$$

$$0$$

$$c[u]$$

$$c[fa[u]]$$

$$2$$

$$c[u]$$

$$c[v]$$

$$v$$

$$u$$

$$c[u]$$

$c[v]$

v

u

u

v

$v' > u$

v'

u

$c[u]$

$c[v']$

u

u

v

$v < v' < fa[v]$

3

$c[v']$

$c[v]$

$c[v]$

$c[u]$

$c[v']$

$c[u]$

$v < u$

v

u

$c[u]$

$c[v]$

u

v'

$v > u$

v

u

$c[u]$

$c[v]$

$$u = s \times 2^{k+1} + 2^k$$

$$k = \log_2 \operatorname{lowbit}(u)$$

$$u - 2^t (0 \leq t < k)$$

$$k = 3$$

$$u$$

$$u$$

$$x$$

$$2^t$$

$$x$$

$$t$$

$$u$$

$$v$$

$$v + \operatorname{lowbit}(v) = u$$

$$v = u - 2^t$$

$$\operatorname{lowbit}(v) = 2^t$$

$$u = s \times 2^{k+1} + 2^k$$

$$\boldsymbol{0 \leq t < k}$$

$$u$$

$$t$$

$$0$$

$$v = u - 2^t$$

$$t$$

$$1$$

$$0$$

$$\operatorname{lowbit}(v) = 2^t$$

$$\boldsymbol{t = k}$$

$$v = u - 2^k$$

$$v$$

$$k$$

$$0$$

$$\operatorname{lowbit}(v) = 2^t$$

$$\boldsymbol{t > k}$$

$$v = u - 2^t$$

$$v$$

$$k$$

$$1$$

$$\text{lowbit}(v) = 2^k$$

$$\text{lowbit}(v) = 2^t$$

$$u$$

$$c$$

$$[l(u), u-1]$$

$$k = 3$$

$$u$$

$$u$$

$$[l(u), u-1]$$

$$u$$

$$u - 2^t (0 \le t < k)$$

$$t$$

$$u - 2^t$$

$$f(t) = u - 2^t$$

$$f(k-1), f(k-2), ..., f(0)$$

$$u$$

$$\text{lowbit}(f(t)) = 2^t$$

$$l(f(t)) = u - 2^t - 2^t + 1 = u - 2^{t+1} + 1$$

$$f(t+1)$$

$$f(t)$$

$$f(t+1) = u - 2^{t+1}$$

$$l(f(t)) = u - 2^{t+1} + 1$$

$$f(k-1)$$

$$l(f(k-1)) = u - 2^k + 1$$

$$l(u)$$

$$f(0)$$

$$u - 1$$

$$[l(u), u-1]$$

$$a[x]$$
$$c$$
$$a[x]$$
$$c[y]$$
$$c$$
$$a[x]$$
$$c[y]$$
$$c[x]$$
$$1$$
$$y$$
$$x$$
$$x$$
$$n$$
$$a$$
$$a[x]$$
$$x' = x$$
$$c[x']$$
$$x' \leftarrow x' + \mathrm{lowbit}(x')$$
$$x' > n$$
$$c[x']$$
$$c[x']$$
$$a[x]$$
$$p$$
$$c[x']$$
$$p$$
$$c[x']$$
$$a[x]$$
$$p$$
$$c[x']$$
$$p$$
$$c[x']$$
$$a[x]$$

$$p$$

$$a[x]$$

$$p - a[x]$$

$$a[x]$$

$$p$$

$$a[x]$$

$$a[x] \times p - a[x]$$

$$c$$

$$n$$

$$\Theta(n \log n)$$

$$a = (5, 1, 4)$$

$$a[1]$$

$$5$$

$$a[2]$$

$$1$$

$$a[3]$$

$$4$$

$$\Theta(n)$$

$$\Theta(n)$$

$$\Theta(n)$$

$$x \leftarrow x - \operatorname{lowbit}(x)$$

$$x$$

$$1$$

$$0$$

$$x$$

$$1$$

$$\operatorname{popcount}(x)$$

$$\Theta(\log n)$$

$$x \leq n$$

$$x$$

$$\log_2 \operatorname{lowbit}(x)$$

$$\log_2 n$$

$$c$$

$$\log n$$

$$\Theta(\log n)$$

$$a$$

$$d$$

$$d[i] = a[i] - a[i-1]$$

$$a_i = \sum_{j=1}^{i} d_j$$

$$a[1...r]$$

$$\sum_{i=1}^{r} a_i$$

$$d_j$$

$$r - j + 1$$

$$\sum_{i=1}^{r} d_i$$

$$\sum_{i=1}^{r} d_i \times i$$

$$d_i$$

$$d_i \times i$$

$$a[l...r]$$

$$x$$

$$d$$

$$d[i] = a[i] - a[i-1]$$

$$a[l]$$

$$v$$

$$a[l-1]$$

$$d[l]$$

$$v$$

$$a[r+1]$$

$$a[r]$$

$$v$$

$$d[r+1]$$

$$v$$

$$l$$

$$r+1$$

$$i$$

$$a[i]$$

$$a[i-1]$$

$$v$$

$$a[i]+v-(a[i-1]+v)$$

$$a[i]-a[i-1]$$

$$d[i]$$

$$d_i$$

$$l$$

$$v$$

$$r+1$$

$$-v$$

$$d_i \times i$$

$$l$$

$$v \times l$$

$$r+1$$

$$-v \times (r+1)$$

$$d_i$$

$$a[x]$$

$$d[1...x]$$

$$c(x,y)$$

$$a(x-\operatorname{lowbit}(x)+1, y-\operatorname{lowbit}(y)+1)...a(x,y)$$

$$a(x,y)$$

$$\operatorname{lowbit}(x)$$

$$\operatorname{lowbit}(y)$$

$$f(x,i)$$

$$x$$

$$i$$

$$0$$

$$c(f(x,i),f(y,j))$$

$$a(x,y)$$

$$a(x,y)$$

$$c(f(x,i),f(y,j))$$

$$f(x,i)\leq n$$

$$f(y,j)\leq m$$

$$c(p,q)$$

$$a(x,y)$$

$$p$$

$$q$$

$$n$$

$$c_1$$

$$a_1$$

$$m$$

$$c_2$$

$$a_2$$

$$c_1(p)$$

$$a_1[x]$$

$$c_2(q)$$

$$a_2[y]$$

$$p$$

$$x$$

$$q$$

$$y$$

$$p=f(x,i)$$

$$q=f(y,j)$$

$$c(g(x,i),g(y,j))$$

$$g(x,i),g(y,j)>0$$

$$\circ$$

$$\circ=+$$

$$c_1$$

$$c_1[g(x,0)]\circ c_1[g(x,1)]\circ c_1[g(x,2)]\circ\cdots$$

$$[1...x]$$

$$t(x) = c(x, g(y, 0)) \circ c(x, g(y, 1)) \circ c(x, g(y, 2)) \circ \cdots$$

$$t(x)$$

$$a(x - \mathrm{lowbit}(x) + 1, 1)...a(x, y)$$

$$t(g(x, 0)) \circ t(g(x, 1)) \circ t(g(x, 2)) \circ \cdots$$

$$a(1, 1)...a(x, y)$$

$$t(x)$$

$$s(i, j) = s(i - 1, j) + s(i, j - 1) - s(i - 1, j - 1) + a(i, j)$$

$$a$$

$$d$$

$$a(i, j) = a(i - 1, j) + a(i, j - 1) - a(i - 1, j - 1) + d(i, j)$$

$$d(i, j) = a(i, j) - a(i - 1, j) - a(i, j - 1) + a(i - 1, j - 1)$$

$$(x_1, y_1)$$

$$(x_2, y_2)$$

$$v$$

$$d(x_1, y_1)$$

$$d(x_2 + 1, y_2 + 1)$$

$$v$$

$$d(x_2 + 1, y_1)$$

$$d(x_1, y_2 + 1)$$

$$-v$$

$$d$$

$$0$$

$$a(2, 2)...a(3, 4)$$

$$v$$

$$a(2, 2)...a(3, 4)$$

$$2 \times 3$$

$$(x, y)$$

$$d(h, k)$$

$$(x - h + 1) \times (y - k + 1)$$

$$d(i, j)$$

$d(i,j) \times i$

$d(i,j) \times j$

$d(i,j) \times i \times j$

c_6

$a[5...6]$

a

b

$b[x]$

x

a

$a = (1, 3, 4, 3, 4)$

$b = (1, 0, 2, 2)$

b

a

k

k

k

k

a

b

a

$a[x]$

y

z

b

$b[y]$

1

$b[z]$

1

k

x

$[1, x]$

x_0

$$[1, x_0]$$

$$< k$$

$$[1, x_0 + 1]$$

$$\geq k$$

$$k$$

$$x_0 + 1$$

$$[1, 0]$$

$$0$$

$$\Theta(\log^2 n)$$

$$x = 0$$

$$\text{sum} = 0$$

$$i$$

$$\log_2 n$$

$$0$$

$$[x + 1 ... x + 2^i]$$

$$t$$

$$\text{sum} + t < k$$

$$x \leftarrow x + 2^i$$

$$\text{sum} \leftarrow \text{sum} + t$$

$$x$$

$$[1 ... x]$$

$$< k$$

$$x + 1$$

$$[x + 1 ... x + 2^i]$$

$$c[x + 2^i]$$

$$\text{lowbit}(x + 2^i)$$

$$2^i$$

$$x$$

$$2^j$$

$$j > i$$

$$c[x + 2^i]$$

$$[x+1...x+2^i]$$

$$\Theta(\log n)$$

$$n$$

$$a$$

$$a$$

$$i<j$$

$$a[i]>a[j]$$

$$(i,j)$$

$$a$$

$$b$$

$$n$$

$$1$$

$$i$$

$$j>i$$

$$a[j]<a[i]$$

$$a[i]=x$$

$$b[1...x-1]$$

$$a[i]$$

$$b[x]$$

$$1$$

$$b[1...x-1]$$

$$x=a[i]$$

$$i$$

$$j$$

$$i$$

$$a=(4,3,1,2,1)$$

$$i$$

$$5\to 1$$

$$a[5]=1$$

$$b[1...0]$$

$$0$$

$$b[1]$$

1

$b = (1, 0, 0, 0)$

$a[4] = 2$

$b[1...1]$

1

$b[2]$

1

$b = (1, 1, 0, 0)$

$a[3] = 1$

$b[1...0]$

0

$b[1]$

1

$b = (2, 1, 0, 0)$

$a[2] = 3$

$b[1...2]$

3

$b[3]$

1

$b = (2, 1, 1, 0)$

$a[1] = 4$

$b[1...3]$

4

$b[4]$

1

$b = (2, 1, 1, 1)$

$0 + 1 + 0 + 3 + 4 = 8$

i

$b[1...x - 1]$

$b[x]$

$b[x]$

$b[1...x-1]$

$b[x]$

$b[1...x-1]$

$i \leq j$

$a[i] > a[j]$

$i = j$

$a[i] > a[j]$

$i \leq j$

$i < j$

$i < j$

$a[i] \geq a[j]$

$b[1...x]$

$b[x]$

$b[1...x]$

$i \leq j$

$a[i] \geq a[j]$

$i = j$

$a[i] \geq a[j]$

$i \leq j$

$i < j$

$i \leq j$

$a[i] \geq a[j]$

j

$i < j$

$a[i] > a[j]$

$x = a[j]$

$b[x+1...V]$

V

b

a

$$b[x]$$

$$1$$

$$b[x+1...V]$$

$$x=a[j]$$

$$j$$

$$i$$

$$j$$

$$\Theta(\log^2 n)$$

$$\Theta(\log n)$$

$$r$$

$$\mathrm{lowbit}$$

$$l$$

$$c[x]$$

$$x-\mathrm{lowbit}(x)$$

$$l$$

$$l$$

$$\boldsymbol{a}[\boldsymbol{x}]$$

$$c[x-1]$$

$$l$$

$$c[x]$$

$$c[x-\mathrm{lowbit}(x)]$$

$$\Theta(\log^2 n)$$

$$r$$

$$l$$

$$r$$

$$1$$

$$l$$

$$0$$

$$r \geq l$$

$$r$$

$$1$$

$$r-\mathrm{lowbit}(r) \geq l$$

$$r$$

$$1$$

$$0$$

$$r$$

$$1$$

$$r$$

$$1$$

$$r \leftarrow r - \text{lowbit}(r)$$

$$r \leftarrow r - 1$$

$$r$$

$$1$$

$$0$$

$$r$$

$$\log n$$

$$r$$

$$l$$

$$\Theta(\log^2 n)$$

$$u = s \times 2^{k+1} + 2^k$$

$$k = \log_2 \text{lowbit}(u)$$

$$u - 2^t (0 \le t < k)$$

$$u$$

$$c$$

$$[l(u), u - 1]$$

$$a[x]$$

$$y$$

$$x$$

$$c[y]$$

$$a[x]$$

$$p$$

$$c[y]$$

$$\max(c[y], p)$$

$$(1, 2, 3, 4, 5)$$

$$5$$

$$4$$

$$4$$

$$5$$

$$c[y]$$

$$p$$

$$3$$

$$4$$

$$p$$

$$c[y]$$

$$c[y]$$

$$c[y]$$

$$c[y]$$

$$[l(y), y-1]$$

$$a[y]$$

$$[l(y), y]$$

$$c[y]$$

$$\log n$$

$$c$$

$$\Theta(\log^2 n)$$

$$n$$

$$\Theta(n \log^2 n)$$

$$\Theta(n)$$

$$\Theta(n)$$

$$\Theta(n)$$

$$c[i]$$

$$[i - \text{lowbit}(i) + 1, i]$$

$$\text{sum}$$

$$c$$

$$\text{tag}$$

$$\text{tag}$$

$$0$$

```
Input. A rooted tree T
Output. The dfs sequence of rooted tree T
ET(u)
      visit vertex u
      for all child v of u
            visit directed edge u → v
            ET(v)
            visit directed edge v → u
```

$$O(\log n)$$

$$O(\log n)$$

$$T$$

$$T$$

$$\mathrm{ETR}(T)$$

$$\mathrm{ETR}(T)$$

$$\mathrm{ETR}(T)$$

$$T$$

$$n$$

$$2n-2$$

$$\mathrm{ETR}(T)$$

$$3n-2$$

$$u$$

$$\mathrm{ETR}(T)$$

$$u$$

$$T$$

$$r$$

$$u$$

$$T$$

$$L$$

$$L$$

$$(u,u)$$

$$L^1$$

$$L^2$$

$$L$$

$$(u,u)$$

$$(u,u)$$

L^2

L^1

u

u

T_1

v

T_2

T

T_1

L_1

T_2

L_2

L_1

(u, u)

L_1^1

L_1^2

L_1

(u, u)

(u, u)

L_2

(v, v)

L_2^1

L_2^2

$L_1^2, L_1^1, [(u, v)], L_2^2, L_2^1, [(v, u)]$

T

L

(u, v)

(v, u)

T

L

T

$$L$$
$$L_1, [(u,v)], L_2, [(v,u)], L_3$$
$$L_2$$
$$L_1, L_3$$
$$L$$
$$[(u,v)]$$
$$[(v,u)]$$
$$u$$
$$v$$
$$u$$
$$L$$
$$L$$
$$u$$
$$L^1$$
$$L^2$$
$$L$$
$$u$$
$$u$$
$$O(\log n)$$
$$u$$
$$L$$
$$u$$
$$u$$
$$u$$
$$L$$
$$L$$
$$u$$
$$L^1$$
$$u$$
$$L^2$$
$$L$$
$$u$$
$$u$$
$$v$$
$$T$$

$$(u, u)$$

$$(v, v)$$

$$\mathrm{ETR}(T)$$

$$u$$

$$v$$

$$\mathrm{ETR}(T)$$

$$1$$

$$0$$

$$T$$

$$\mathrm{ETR}(T)$$

$$O(m\alpha(m, n))$$

$$O(m \log n)$$

$$O(m\alpha(m, n))$$

$$O(m \log n)$$

$$O(\alpha(n))$$

$$\alpha$$

$$A(m, n)$$

$$A(m,n) = \begin{cases} n+1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{otherwise} \end{cases}$$

$$\alpha(n)$$

$$m$$

$$A(m, m) \leqslant n$$

$$O(n)$$

$$A_k(j) = \begin{cases} j+1 & k = 0 \\ A_{k-1}^{(j+1)}(j) & k \geq 1 \end{cases}$$

$$0 \leq level(x) < \alpha(n)$$

$$1 \leq iter(x) \leq rnk(x)$$

$$\Phi(x) = \begin{cases} \alpha(n) \times rnk(x) & rnk(x) = 0 \text{ □ } x \text{ □□□□□□□} \\ (\alpha(n) - level(x)) \times rnk(x) - iter(x) & \text{otherwise} \end{cases}$$

$$\alpha(n) \times ((ij-1)/j+1) = \alpha(n) \times ((\left\lfloor \log_{\lfloor \frac{m}{n} \rfloor + 1} \frac{n}{2} \right\rfloor \times \left\lfloor \frac{m}{n} \right\rfloor - 1)/\left\lfloor \frac{m}{n} \right\rfloor + 1)$$

$$\alpha(n)$$

$$A_k(j)$$

$$A_k(j)$$

$$f^i(x)$$

$$f$$

$$x$$

$$i$$

$$f^0(x) = x$$

$$f^i(x) = f(f^{i-1}(x))$$

$$\alpha(n)$$

$$A_{\alpha(n)}(1) \geq n$$

$$A_{\alpha(n)}(\alpha(n)) \geq n$$

$$rnk(x)$$

$$fa(x)$$

$$rnk(x) + 1 \leq rnk(fa(x))$$

$$level(x)$$

$$level(x) = \max(k : rnk(fa(x)) \geq A_k(rnk(x)))$$

$$rnk(x) \geq 1$$

$$iter(x) = \max(i : rnk(fa(x)) \geq A^i_{level(x)}(rnk(x))$$

$$x$$

$$rnk(x) > 0$$

$$x$$

$$x$$

$$fa(x)$$

$$rnk(x) > 0$$

$$i, k$$

$$rnk(fa(x)) \geq A^i_k(rnk(x))$$

$$level(x) = \max(k)$$

$$iter(x) = \max(i)$$

$$level$$

$$A$$

$$iter$$

$$level(x)$$

$$level(x)$$

$$iter(x)$$

$$level(x)$$

$$iter(x)$$

$$A_k^j$$

$$\Phi(S) = \sum_{x \in S} \Phi(x)$$

$$S$$

$$x$$

$$\Phi(x)$$

$$\Theta(\alpha(n))$$

$$union(x, y)$$

$$x$$

$$y$$

$$find(x)$$

$$find(y)$$

$$0$$

$$\Theta(1)$$

$$rnk(x) \leq rnk(y)$$

$$x$$

$$y$$

$$x$$

$$y$$

$$y$$

$$y$$

$$c$$

$$1$$

$$c$$

$$\Phi(c)$$

$$\Phi(c')$$

$$c$$

$$rnk(c) > 0$$

$$iter(c)$$

$$level(c)$$

$$\Phi(c) = \Phi(c')$$

$$iter(c)$$

$$level(c)$$

$$iter(c)$$

$$\Phi(c') \leq \Phi(c) - 1$$

$$level(c)$$

$$iter(c)$$

$$0 < iter(c) \leq rnk(c)$$

$$iter(c)$$

$$rnk(c) - 1$$

$$level(c)$$

$$1$$

$$\Phi(c) = (\alpha(n) - level(c)) \times rnk(c) - iter(c)$$

$$\Phi(c') \leq \Phi(c) - 1$$

$$rnk(c)$$

$$rnk(fa(c))$$

$$x$$

$$y$$

$$x$$

$$rnk(x) = 0$$

$$\Phi(x) = \Phi(x') = 0$$

$$\alpha(x) \times rnk(x) \geq (\alpha(n) - level(x)) \times rnk(x) - iter(x)$$

$$\Phi(x') \leq \Phi(x)$$

$$y$$

$$y$$

$$\alpha(n)$$

$$union$$

$$\Theta(\alpha(n))$$

$$\Theta(s)$$

$$\Theta(s)$$

$$s-\alpha(n)$$

$$1$$

$$find(a)$$

$$\Theta(\alpha(n))$$

$$a$$

$$x$$

$$rnk$$

$$s-\alpha(n)$$

$$1$$

$$level(x)$$

$$iter(x)$$

$$1$$

$$s-\alpha(n)$$

$$level(x)$$

$$iter(x)$$

$$\Phi(x)=(\alpha(n)-level(x))\times rnk(x)-iter(x)$$

$$level(x)$$

$$iter(x)$$

$$rnk(fa(x))\geq A_{level(x)}^{iter(x)}(rnk(x))$$

$$root_x$$

$$x$$

$$rnk(root_x)\geq A_{level(x)}^{iter(x)+1}(rnk(x))$$

$$A_k^i$$

$$A_{level(x)}^{iter(x)+1}(rnk(x))=A_{level(x)}(A_{level(x)}^{iter(x)}(rnk(x)))$$

$$k(x)$$

$$level(x)$$

$$i(x)$$

$$iter(x)$$

$$rnk(root_x) \geq A_{k(x)}(A_{k(x)}^{i(x)}(x))$$

$$A_{k(x)}$$

$$y$$

$$y$$

$$x$$

$$k(y) = k(x)$$

$$x$$

$$x$$

$$y$$

$$y$$

$$x$$

$$\alpha(n) - 2$$

$$k$$

$$x$$

$$a$$

$$root_a$$

$$y$$

$$fa(x)$$

$$x$$

$$x$$

$$root_x$$

$$y$$

$$x$$

$$rnk(y) \geq rnk(fa(x))$$

$$rnk(fa(x)) \geq A_{k(x)}^{i(x)}(rnk(x))$$

$$k(x) = k(y)$$

$$k$$

$$rnk(fa(x)) \geq A_k^{i(x)}(rnk(x))$$

$$A_k$$

$$iter(y)$$

$$rnk(fa(y)) \geq A_k(rnk(y))$$

$$rnk(fa(y)) \geq A_k^{i(x)+1}(rnk(x))$$

$$rnk(x)$$

$$rnk(fa(y))$$

$$A_k$$

$$i(x) + 1$$

$$rnk(fa(y))$$

$$rnk(root_y) \geq rnk(fa(y))$$

$$rnk(x)$$

$$rnk(root_x) \geq A_k^{i(x)+1}(rnk(x))$$

$$iter(x)$$

$$rnk(x)$$

$$level(x)$$

$$\Phi(x)$$

$$x$$

$$s - \alpha(n) - 2$$

$$\Phi(S)$$

$$s - \alpha(n) - 2$$

$$\Theta(\alpha(n) + 2) = \Theta(\alpha(n))$$

$$\Theta(m\alpha(n))$$

$$rnk$$

$$rnk$$

$$rnk$$

$$rnk$$

$$rnk$$

$$rnk(fa(x)) \geq rnk(x) + 1$$

$$union(x, y)$$

$$\alpha(n)$$

$$\Omega(m \log_{1+\frac{m}{n}} n)$$

$$T_k$$

$$T_{k-1}$$

$$T_{k-j}$$

$$T_1$$

$$T_j$$

$$rnk(T_k) = r_k$$

$$r_k = (k-1)/j$$

$$j$$

$$(k-1)/j + 1$$

$$j = \left\lfloor \frac{m}{n} \right\rfloor$$

$$i = \left\lfloor \log_{j+1} \frac{n}{2} \right\rfloor$$

$$k = ij$$

$$\geq \alpha(n) \times (\log_{1+\frac{m}{n}} n - \frac{n}{m})$$

$$\Omega(m \log_{1+\frac{m}{n}} n - n) = \Omega(m \log_{1+\frac{m}{n}} n)$$

$$size(x)$$

$$size$$

$$size$$

$$rnk$$

$$rnk(fa(x)) \geq rnk(x) + 1$$

$$siz$$

$$x$$

$$y$$

$$siz(y)$$

$$siz(x)$$

$$\log_2 siz(x)$$

$$rnk(x)$$

$$siz$$

$$\log_2 siz(x)$$

$$\Theta(m\alpha(n))$$

$$K$$

$$K$$

$$N$$

$$l, r$$

$$O(n \log n)$$

$$K$$

$$K$$

$$3$$

$$7$$

$$2$$

$$3$$

$$[\text{left}, \text{right}]$$

$$[l, r]$$

$$O(\log n)$$

$$m$$

$$O(m \log n)$$

$$O(n \log n)$$

$$1482 \text{ ms}$$

$$889 \text{ ms}$$

$$n$$

$$n \leq 10^5$$

$$m \leq 5 \times 10^4$$

$$\Theta(n \log^2 n)$$

$$\Theta(n)$$

$$\Theta(n \log^2 n)$$

$$\Theta(n \log^2 n)$$

$$\Theta(n \log^2 n)$$

$$\Theta(n \log n)$$

$$[c \times 2^k + 1, (c + 1) \times 2^k]$$

$$lowbit(x) < 2^{lev}$$

$$[c \times 2^k, (c+1) \times 2^k)$$

$$\log n$$

$$k$$

$$\Theta(k)$$

$$\Theta(n \log n + m \log^2 n)$$

$$(\nexists\, x, z \in [l, r], y \notin [l, r],\, P_x < P_y < P_z)$$

$$\max_{l \le i \le r} P_i - \min_{l \le i \le r} P_i = r - l$$

$$\max_{l \le i \le r} P_i - \min_{l \le i \le r} P_i \ge r - l$$

$$Q_j = \max_{j \le k \le i} P_k - \min_{j \le k \le i} P_k - (i - j),\ \ 0 < j < i$$

$$\wedge$$

$$\vee$$

$$1 - n$$

$$\{5, 3, 4, 1, 2\}$$

$$[1, 1], [2, 2], [3, 3], [4, 4], [5, 5], [2, 3], [4, 5], [1, 3], [2, 5], [1, 5]$$

$$n$$

$$P$$

$$n$$

$$P_i$$

$$1, 2, \cdots, n$$

$$n$$

$$P$$

$$\mid P \mid = n$$

$$\forall i, P_i \in [1, n]$$

$$\nexists i, j \in [1, n], P_i = P_j$$

$$P$$

$$(P, [l, r])$$

$$[l, r]$$

$$P_{l \sim r}$$

$$P$$

$$[l, r]$$

$$l > r$$

$$(P, \emptyset)$$

$$P$$

$$I_P$$

$$(P, \emptyset) \in I_P$$

$$A = (P, [a, b]), B = (P, [x, y])$$

$$A, B \in I_P$$

$$A \subseteq B \iff x \le a \wedge b \le y$$

$$A = B \iff a = x \wedge b = y$$

$$A \cap B = (P, [\max(a, x), \min(b, y)])$$

$$A \cup B = (P, [\min(a, x), \max(b, y)])$$

$$A \setminus B = (P, \{i \mid i \in [a, b] \wedge i \notin [x, y]\})$$

$$A, B \in I_P, A \cap B \neq \emptyset, A \notin B, B \notin A$$

$$A \cup B, A \cap B, A \setminus B, B \setminus A \in I_P$$

$$O(n^2)$$

$$P$$

$$M$$

$$I_P$$

$$X \in I_P$$

$$\forall A \in I_P,\ X \cap A = (P, \emptyset) \vee X \subseteq A \vee A \subseteq X$$

$$M_P$$

$$(P, \emptyset) \in M_P$$

$$P$$

$$P$$

$$P = \{9, 1, 10, 3, 2, 5, 7, 6, 8, 4\}$$

$$[5, 8]$$

$$(P, [6, 9]) = \{5, 7, 6, 8\}$$

$$u$$

$$[u_l, u_r]$$

$$u$$

$$x$$

$$[x, x]$$

$$S_u$$

$$S_u$$

$$[5, 8]$$

$$\{[5, 5], [6, 7], [8, 8]\}$$

$$\{1, 2, 3\}$$

$$[4, 8]$$

$$\{2, 1\}$$

$$u$$

$$P_u$$

$$P_u = \{1, 2, \cdots, | S_u|\}$$

$$P_u = \{| S_u|, | S_u - 1 |, \cdots, 1\}$$

$$[1, 10]$$

$$[1, 10]$$

$$\{3, 1, 4, 2\}$$

$$u$$

$$u$$

$$\bigcup_{i=1}^{| S_u|} S_u[i] = [u_l, u_r]$$

$$u$$

$$\forall S_u[l \sim r]$$

$$\bigcup_{i=l}^{r} S_u[i] \in I_P$$

$$u$$

$$\forall S_u[l \sim r], l < r$$

$$\bigcup_{i=l}^{r} S_u[i] \notin I_P$$

$$1$$

$$u$$

$$S_u[l \sim r]$$

$$A = \bigcup_{i=l}^{r} S_u[i] \in I_P$$

$$A$$
$$A$$
$$O(n \log n)$$
$$i-1$$
$$P_i$$
$$l$$
$$L_i$$
$$i$$
$$< l$$
$$P_i$$
$$t$$
$$L_i$$
$$t_l = L_i$$
$$t'$$
$$t'_l = L_i$$
$$t’$$
$$L_i$$
$$(P, [l, r])$$
$$[l, r]$$
$$\max_{l \leq i \leq r} P_i - \min_{l \leq i \leq r} P_i - (r-l)$$
$$i$$
$$Q$$
$$[j, i]$$
$$1 \sim i-1$$
$$j$$
$$Q_j = 0$$
$$Q_{1 \sim i-1}$$
$$j$$
$$L_i$$
$$L_i = i$$
$$i$$

$$Q$$

$$[j, i]$$

$$[j, i+1]$$

$$P_{i+1} > \max$$

$$P_{i+1} < \min$$

$$Q_j$$

$$P_{i+1} > \max$$

$$Q_j$$

$$\max$$

$$P_{i+1}$$

$$Q_j$$

$$P_{i+1} < \min$$

$$Q_j = Q_j + \min - P_{i+1}$$

$$[x, y]$$

$$P_{x\sim i}, P_{x+1\sim i}, P_{x+2\sim i}, \cdots, P_{y\sim i}$$

$$\max$$

$$P_{x\sim i}, P_{x+1\sim i}, \cdots, P_{y\sim i}$$

$$\min$$

$$\max$$

$$\min$$

$$Q$$

$$j$$

$$P_j > P_{i+1}$$

$$P_{j+1\sim i}$$

$$P_{i+1}$$

$$Q_{j+1\sim i}$$

$$P_i, \max(P_i, P_{i-1}), \max(P_i, P_{i-1}, P_{i-2}), \cdots, \max(P_i, P_{i-1}, \cdots, P_{j+1})$$

$$\max$$

$$\min$$

$$Q_j$$

$$1$$

$$1$$

$$L_i$$

$$Q$$

$$Q$$

$$\max / \min$$

$$\max / \min$$

$$\max / \min$$

$$\underbrace{a_1, a_2, ..., a_s}_{b_1}, \underbrace{a_{s+1}, ..., a_{2s}}_{b_2}, ..., \underbrace{a_{(s-1)\times s+1}, ..., a_n}_{b_{\frac{n}{s}}}$$

$$n$$

$$\{a_i\}$$

$$n$$

$$a_l \sim a_r$$

$$x$$

$$\sum_{i=l}^{r} a_i$$

$$1 \leq n \leq 5 \times 10^4$$

$$s$$

$$b_i$$

$$n$$

$$s$$

$$l$$

$$r$$

$$s$$

$$O(s)$$

$$l$$

$$r$$

$$l$$

$$r$$

$$b_i$$

$$O(\frac{n}{s} + s)$$

$$l$$

$$r$$

$$s$$

$$O(s)$$

$$l$$

$$r$$

$$l$$

$$r$$

$$b_i$$

$$b_i$$

$$O(\frac{n}{s} + s)$$

$$\frac{n}{s} = s$$

$$s = \sqrt{n}$$

$$O(\sqrt{n})$$

$$\Omega(1), O(\sqrt{n})$$

$$O(\sqrt{n}) - O(1)$$

$$O(1)$$

$$O(T + \frac{n}{T})$$

$$O(1)$$

$$n$$

$$m$$

$$T$$

$$O(1)$$

$$O(T)$$

$$T$$

$$O(n)$$

$$O(mT + n\frac{m}{T})$$

$$T = \sqrt{n}$$

$$O(m\sqrt{n})$$

$$k$$

$\sqrt{n}$

(k, w)

k

w

k, w

k

w

w

k

k

w

k

k

u

w

x

$x_w < u_w$

u

x

x

u

$O(n)$

w

n

h_i

k

h_i

w

(i, h_i)

u

u_w

u

h

u

u

$$u$$

$$O(n)$$

$$n$$

$$O(\log n)$$

$$O(n)$$

$$O(\log n)$$

$$O(n)$$

$$O(h)$$

$$O(h)$$

$$1$$

$$O(h)$$

$$1$$

$$1$$

$$O(h)$$

$$O(h)$$

$$k$$

$$[k - count, k - 1]$$

$$k - count$$

$$O(h)$$

$$h$$

$$h$$

$$O(h)$$

$$O(\log n)$$

$$O(h)$$

$$O(n)$$

$$h$$

$$|\ height(T -> left) - height(T -> right)| \leq 1$$

$$A$$

$$A$$

$$B$$

$$A$$

$$A$$
$$B$$
$$B$$
$$A$$
$$B$$
$$A$$
$$B$$
$$T2$$
$$B$$
$$A$$
$$A$$
$$B$$
$$m$$
$$m$$
$$m$$
$$n$$
$$n-1$$
$$n$$
$$\left\lceil \frac{m}{2} \right\rceil$$
$$\left\lceil \frac{m}{2} \right\rceil$$
$$k$$
$$m$$
$$m/2$$
$$m/2+1$$
$$m-1$$
$$(m-1)/2$$
$$m-(m-1)/2$$
$$m/2$$
$$[7,11]$$
$$[7,10,11]$$
$$[2,3,5]$$
$$[2,3,4,5]$$

$$[2, 3]$$

$$[4, 5]$$

$$\left\lceil \frac{m}{2} \right\rceil$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$2 \times \sqrt{n}$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$n$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$10^6$$

$$\sqrt{n}$$

$$10^3$$

$$O(\log n)$$

$$O(\sqrt{n})$$

$$[x, y]$$

$$z$$

$$[x, y]$$

$$z$$

$$q$$

$$O(q\sqrt{n} \log n)$$

$$[x, y]$$

$$z$$

$$[x, y]$$

$$z$$

$$n$$

$$(x_i, y_i)$$

$$1 \le i \le n, 1 \le x_i, y_i \le n, 1 \le n \le 10^5$$

$$a, b, c, d$$

$$(a, b)$$

$$c, d$$

$$x, y$$

$$x$$

$$y$$

$$x_i \neq x_j (1 \le i, j \le n, i \neq j)$$

$$x_i \neq x_j (1 \le i, j \le n, i \neq j)$$

$$x$$

$$y$$

$$Y_i$$

$$i$$

$$\sqrt{n}$$

$$T_i$$

$$i$$

$$T_{i,j}$$

$$i$$

$$(j - lowbit(j), j]$$

$$a, b$$

$$1 \le x_i \le a, 1 \le y_i \le b$$

$$[1, a]$$

$$Y_i \le b$$

$$b$$

$$T$$

$$T$$

$$x$$

$$Y_x$$

$$y$$

$$T$$

$$x$$

$$y$$

$$\sqrt{n}$$

$$O(n)$$

$$O(n\sqrt{n})$$

$$O(\sqrt{n})$$

$$T$$

$$T_i$$

$$O(\log(\sqrt{n})\log n)$$

$$O(\sqrt{n}+\log(\sqrt{n})\log n)$$

$$O(\sqrt{n}+\log(\sqrt{n})\log n)$$

$$a$$

$$b$$

$$l_a, r_a, l_b, r_b$$

$$a[l_a...r_a]$$

$$b[l_b...r_b]$$

$$x, y$$

$$swap(b_x, b_y)$$

$$n$$

$$2 \leq n \leq 2 \cdot 10^5$$

$$q$$

$$1 \leq q \leq 2 \cdot 10^5$$

$$i$$

$$x_i$$

$$b$$

$$y_i$$

$$a$$

$$x$$

$$y$$

$$a$$

$$a$$

$$b$$

$$b$$

n

$1 \leq n \leq 10^5$

mex

1

$mex - 1$

mex

1

$n + 1$

i

i

$n + 2$

i

i

1

$i - 1$

i

i

Y_j

a_j

j

x

Y_j

y

a_j

z

1

$i - 1$

$[l, r]$

i

$l \leq j \leq r, Y_j \leq l - 1, a_j \leq i - 1$

$i - 1$

i

$[l, r]$

$$a_j \le i - 1$$

$$\begin{aligned}
\text{□□□□} &= n\log n - \log 1 - \log 2 - \cdots - \log n \\
&\le n\log n - 0 \times 2^0 - 1 \times 2^1 - \cdots - (\log n - 1) \times \frac{n}{2} \\
&= n\log n - (n-1) - (n-2) - (n-4) - \cdots - (n - \frac{n}{2}) \\
&= n\log n - n\log n + 1 + 2 + 4 + \cdots + \frac{n}{2} \\
&= n - 1 \\
&= O(n)
\end{aligned}$$

$$O(\log n)$$

$$O(\log n)$$

$$O(\log n)$$

$$h$$

$$h_i$$

$$h_{2i}$$

$$h_{2i+1}$$

$$1$$

$$n$$

$$O(n\log n)$$

$$k$$

$$O(k)$$

$$O(\log n)$$

$$\log 1 + \log 2 + \cdots + \log n = \Theta(n\log n)$$

$$O(\log n - k)$$

$$n/2$$

$$O(n)$$

$$k$$

$$k$$

$$k$$

$$k$$

$$k$$

$$k$$

$$k$$

k

k

k

k

$+1/-1$

k

k

$O(1)$

k

k

1

$O(\log n)$

k

$+1$

k

$O(\log n)$

$O((n+q)\log n)$

$O(\log n)$

$O(\log n)$

$O(\log^2 n)$

$O(\log^3 n)$

m

m

m

m

$\left\lceil \frac{m}{2} \right\rceil$

k

$k-1$

$k[i] < k[i+1]$

$2k-1$

k

o

$$v$$

$$v$$

$$v$$

$$h$$

$$m$$

$$m-1$$

$$m$$

$$U-1$$

$$U-1$$

$$U-1$$

$$U=2L$$

$$2L-1$$

$$L-1$$

$$L$$

$$U-1$$

$$U=2L-1$$

$$2L-2$$

$$L-1$$

$$2k-1$$

$$k-1$$

$$k-1$$

$$m/2-1$$

$$m/2$$

$$m/2-1$$

$$m/2-1$$

$$\left\lceil \frac{5}{2} \right\rceil$$

$$K$$

$$2 <= K <= 4$$

$$K=1$$

$$f_n = \begin{cases} 1 & (n=1) \\ 2 & (n=2) \\ f_{n-1}+f_{n-2}+1 & (n>2) \end{cases}$$

$$f_n = \frac{5+2\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-2\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n - 1$$

$$n < \log_{\frac{1+\sqrt{5}}{2}}(f_n+1) < \frac{3}{2}\log_2(f_n+1)$$

$$\begin{cases} h(B) = x+2 \\ h(A) = x+1 \\ x \le h(C) \le x+1 \end{cases}$$

$$\begin{cases} 0 \le h(C) - h(E) \le 1 \\ x+1 \le h'(D) = \max(h(C), h(E)) + 1 = h(C) + 1 \le x+2 \\ 0 \le h'(D) - h(A) \le 1 \end{cases}$$

$$\begin{cases} h(B) = x+2 \\ h(C) = x+1 \\ h(A) = x \end{cases}$$

$$\begin{cases} x-1 \le h'(rs_B), h'(ls_D) \le x \\ 0 \le h(A) - h'(rs_B) \le 1 \\ 0 \le h(E) - h'(ls_D) \le 1 \\ h'(B) = \max(h(A), h'(rs_B)) + 1 = x+1 \\ h'(D) = \max(h(E), h'(ls_D)) + 1 = x+1 \\ h'(B) - h'(D) = 0 \end{cases}$$

$$|\ h(ls) - h(rs)| \le 1$$

$$O(\log n)$$

$$f_n$$

$$n$$

$$\{f_n + 1\}$$

$$f_n$$

$$n$$

$$O(\log f_n)$$

$$f_n$$

$$h(B) - h(E) = 2$$

$$h(A)$$

$$h(C)$$

$$h(A) \ge h(C)$$

$$h(E) = x$$

$$h(C) \geq x$$

$$h(C)$$

$$h(A)$$

$$h(A) < h(C)$$

$$h(E) = x$$

```
function split(root)
    if root → right → right → level == root → level
        rotate_ left(root)
end function
```

```
function skew(root)
    if root → left → level == root → level
        rotate_ right(root)
end function
```

```
function insert(root, add)
    if root == NULL
        root ← add
    else if add → key < root → key        //□□□□□□ <=
        insert(root → left, add)
    else if add → key > root → key
        insert(root → right, add)
    end if
    //□□□□□□□□□□□ level □□□ skew □ split
    skew(root);
    split(root);
end function
```

```
//To rebalance the tree
if root->left->level < root->level − 1 or root->right->level < root->level − 1
{
    if root->right->level >− root->level
    {
        root->right->level ← root->level
    }
    skew(root)
    skew(root->right)
    skew(root->right->right)
    split(root)
    split(root->right)
}
```

$$O(\log N)$$

$$a$$

$$a$$

$$T$$

$$T$$

T

a

T

p

q

p

q

a

p

a

q

T

a

b

a

b

T

p

q

r

p

q

r

a

p

q

b

q

r

T

d

T

d

T

t

T

t

d

t

d

T

d

t

d

T

t

p

q

a

t

d

a

d

T

d

a

T

p

d

a

T

q

t

p

q

r

a

b

t

$a < b$

d

a

b

d

T

d

a

T

p

d

a

b

T

q

d

b

T

r

(6□7)

$O(\log n)$

$\log N$

$1/2 \log N$

25

$(10, 20)$

25

$(20, \infty)$

$(22, 24, 29)$

□$22, 24, 29$□

25

□$22, 24, 25, 29$□

24

$(10, 20, 24)$

30

31

$(25, 29, 30, 31)$

29

$(10, 20, 24, 29)$

$(10, 20, 24, 29)$

20

10

50

50

9□22□28□40

11□13□16□19

13

16

11

19

13

13

12

12

13

12

10

10

1

$(1, 2, 4, 6)$

2

6

4

4

10

12

36

18

$$20$$

$$18$$

$$25$$

$$36$$

$$20$$

$$25$$

$$20$$

$$25$$

$$25$$

$$36$$

$$15$$

$$O(n^2)$$

$$O(n)$$

$$O(n \log n)$$

$$O(\log n)$$

$$n$$

$$\frac{n}{2}$$

$$\infty$$

$$O(n \log n)$$

$$O(n)$$

$$O(\log n)$$

$$n$$

$$O(n)$$

```
do
    □□ run □□□
    if run□minRun□
        □□ run □□ min(minRun, nRemaining)
    push run □□ pending − runstack□
    if pending − runstack □□□□ 2 □ run □□□□
        □□ pending − runstack □□□□ 2 □ run
    start index ← start index + run □□□
    nRemaining ← nRemaining − run □□□
while nRemaining ≠ 0
```

$$O(n)$$

$$O(n \log n)$$

$$O(n \log n)$$
$$O(n)$$
$$nRemaining$$
$$minRun$$
$$getMinRunLength(nRemaining)$$
$$getMinRunLength$$
$$minRun$$
$$O(n)$$
$$O(n \log n)$$
$$O(n \log n)$$
$$O(n)$$
$$O(n \log(n)^2)$$
$$O(n \log n)$$
$$(last - first) \log(mid - first)$$
$$x \not< x$$
$$x < y$$
$$y \not< x$$
$$x < y, y < z$$
$$x < z$$
$$x \not< y, y \not< x, y \not< z, z \not< y$$
$$x \not< z, z \not< x$$
$$R$$
$$S$$
$$R$$
$$S$$
$$R$$
$$S$$
$$O$$
$$O(n \log n)$$
$$O(n \log n)$$
$$O(n + w)$$

$$w$$

$$n$$

$$u$$

$$d$$

$$\begin{gathered} A_1, A_{1+h}, A_{1+2h}, \ldots \\ A_2, A_{2+h}, A_{2+2h}, \ldots \\ \vdots \\ A_h, A_{h+h}, A_{h+2h}, \ldots \end{gathered}$$

$$y_1 \leq x_{n+1}, y_2 \leq x_{n+2}, \ldots, y_l \leq x_{n+l}$$

$$y_{k_1} \leq x_{n+k_1} \leq x'_{n+i}, y_{k_2} \leq x_{n+k_2} \leq x'_{n+i}, \ldots, y_{k_i} \leq x_{n+k_i} \leq x'_{n+i}$$

$$\begin{gathered} A_1, A_{1+h}, A_{1+2h}, \ldots \\ A_2, A_{2+h}, A_{2+2h}, \ldots \\ \vdots \\ A_h, A_{h+h}, A_{h+2h}, \ldots \end{gathered}$$

$$\begin{gathered} A_1, A_{1+k}, A_{1+2k}, \ldots \\ A_2, A_{2+k}, A_{2+2k}, \ldots \\ \vdots \\ A_k, A_{k+k}, A_{k+2k}, \ldots \end{gathered}$$

$$\begin{gathered} A_i, A_{i+k}, A_{i+2k}, \ldots \\ \ldots, A_{i+h}, A_{i+h+k}, A_{i+h+2k}, \ldots \end{gathered}$$

$$A_i \leq A_{i+h}, A_{i+k} \leq A_{i+h+k}, \ldots$$

$$by_0 = c - ax_0 \geq c - a(b-1) > ab - a - b - ab + a = -b$$

$$k = j - ah_{t+1} - bh_t$$

$$A_{j-bh_t} \leq A_{j-(b-1)h_t} \leq \ldots \leq A_{j-h_t} \leq A_j$$

$$A_{j-bh_t-ah_{t+1}} \leq A_{j-bh_t-(a-1)h_{t+1}} \leq \ldots \leq A_{j-bh_t-h_{t+1}} \leq A_{j-bh_t}$$

$$\sum_{j=h_{t-1}+1}^{n} O\left(\frac{h_{t+1}h_t}{h_{t-1}}\right) = O\left(\frac{nh_{t+1}h_t}{h_{t-1}}\right)$$

$$H(h_1 = 1, h_2 = 3, h_3 = 7, \ldots, h_{\lfloor \log_2 n \rfloor} = 2^{\lfloor \log_2 n \rfloor} - 1)$$

$$O\left(\frac{n^2}{h_t}\right) = O(n^{3/2})$$

$$O\left(\frac{n^2}{h_i}\right) = O(n^{3/2}/2^{i-k})\ (i > k)$$

$$\sum_{i=k}^{\lfloor \log_2 n \rfloor} O(n^{3/2}/2^{i-k}) = O(n^{3/2})$$

$$O\left(\frac{nh_{t+2}h_{t+1}}{h_t}\right) = O\left(\frac{nh_{t+2} \cdot h_{t+2}/2}{h_{t+2}/4}\right) = O(nh_{t+2})$$

$$2O(n^{3/2}) + \sum_{i=1}^{k-3} O(nh_{i+1}) = O(n^{3/2}) + \sum_{i=1}^{k-3} O(nh_{k-1}/2^{k-i-3}) = O(n^{3/2}) + O(nh_{k-1}) = O(n^{3/2})$$

$$1$$

$$O(n)$$

$$H$$

$$H$$

$$o(n^2)$$

$$1$$

$$H = \{2^k - 1 \mid k = 1, 2, ..., \lfloor \log_2 n \rfloor\}$$

$$O(n^{3/2})$$

$$2$$

$$H = \{k = 2^p \cdot 3^q \mid p, q \in \mathbb{N}, k \leq n\}$$

$$O(n \log^2 n)$$

$$1$$

$$\text{InsertionSort}(h)$$

$$\text{InsertionSort}$$

$$A$$

$$1$$

$$n, m$$

$$l$$

$$X(x_1, x_2, ..., x_{n+l}), Y(y_1, y_2, ..., y_{m+l})$$

$$X$$

$$X'(x'_1, ..., x'_{n+l})$$

$$Y$$

$$Y'(y'_1, ..., y'_{m+l})$$

$$1 \leq i \leq l$$

$$x'_{n+i}$$

$$X'$$

$$l - i$$

$$X$$
$$l-i$$
$$X$$
$$X'$$
$$\{x_{n+1}, ..., x_{n+l}\} \subset X$$
$$x'_{n+i}$$
$$l-i$$
$$x'_{n+i}$$
$$i$$
$$i$$
$$x_{n+k_1}, x_{n+k_2}, ..., x_{n+k_i}$$
$$x'_{n+i}$$
$$Y$$
$$Y'$$
$$i$$
$$y'_i \leq x'_{n+i} \, (1 \leq i \leq l)$$
$$\text{InsertionSort}(h)$$
$$\text{InsertionSort}(k)$$
$$h$$
$$\text{InsertionSort}(h)$$
$$\text{InsertionSort}(k)$$
$$i$$
$$(1 \leq i \leq \min(h, k))$$
$$...$$
$$i+h \geq k$$
$$1$$
$$X$$
$$Y$$
$$l$$
$$i+h$$
$$n$$
$$...$$
$$m$$

$$l$$

$$1$$

$$\mathrm{InsertionSort}(k)$$

$$A_i \le A_{i+h}\,(1 \le i \le \min(h,k))$$

$$i > \min(h,k)$$

$$w$$

$$(1 \le w \le \min(h,k))$$

$$k$$

$$i$$

$$\mathrm{InsertionSort}(k)$$

$$A_i \le A_{i+h}\,(1 \le i \le n-h)$$

$$H$$

$$h$$

$$i$$

$$1$$

$$2$$

$$a, b$$

$$\{ax + by \mid x, y \in \mathbb{N}\}$$

$$ab - a - b$$

$$ax + by = ab - a - b$$

$$x, y$$

$$(b-1, -1), (-1, a-1)$$

$$x = x_0 + tb, y = y_0 - ta$$

$$b - 1 - b = -1$$

$$t$$

$$x$$

$$y$$

$$c > ab - a - b$$

$$ax + by = c$$

$$(x_0, y_0)$$

$$0 \le x_0 < b$$

$$b(y_0 + 1) > 0$$

$$b > 0$$

$$y_0 + 1 > 0$$

$$y_0 \geq 0$$

$$(x_0, y_0)$$

$$2$$

$$1$$

$$2$$

$$\gcd(h_{t+1}, h_t) = 1$$

$$\text{InsertionSort}(h_{t+1})$$

$$\text{InsertionSort}(h_t)$$

$$\text{InsertionSort}(h_{t-1})$$

$$O\left(\frac{nh_{t+1}h_t}{h_{t-1}}\right)$$

$$j$$

$$i$$

$$O\left(\frac{h_{t+1}h_t}{h_{t-1}}\right)$$

$$j \leq h_{t+1}h_t$$

$$i$$

$$O\left(\frac{h_{t+1}h_t}{h_{t-1}}\right)$$

$$j > h_{t+1}h_t$$

$$k$$

$$1 \leq k \leq j - h_{t+1}h_t$$

$$h_{t+1}h_t - h_{t+1} - h_t < h_{t+1}h_t \leq j - k \leq j - 1$$

$$\gcd(h_{t+1}, h_t) = 1$$

$$2$$

$$a, b$$

$$ah_{t+1} + bh_t = j - k$$

$$1$$

$$A_k = A_{j-ah_{t+1}-bh_t} \leq A_j$$

$$1 \leq k \leq j - h_{t+1}h_t$$

$$A_k \leq A_j$$

$$i$$

$$h_{t-1}$$

$$O\left(\frac{h_{t+1}h_t}{h_{t-1}}\right)$$

$$i \leq j - h_{t+1}h_t$$

$$A_i \leq A_j$$

$$j$$

$$2$$

$$A$$

$$h$$

$$k$$

$$h$$

$$k$$

$$2$$

$$1$$

$$2$$

$$1$$

$$H$$

$$\text{InsertionSort}(h_{\lfloor \log_2 n \rfloor}), \text{InsertionSort}(h_{\lfloor \log_2 n \rfloor - 1}), ..., \text{InsertionSort}(h_2), \text{InsertionSort}(h_1)$$

$$h_t \geq \sqrt{n}$$

$$h_t$$

$$\text{InsertionSort}(h_t)$$

$$O\left(\frac{n^2}{h_t}\right)$$

$$\sqrt{n}$$

$$h_k$$

$$i > k$$

$$h_i$$

$$2h_i < h_{i+1}$$

$$\sqrt{n}$$

$$h_t < \sqrt{n}$$

$$O(n^{3/2})$$

$$h_t$$

$$2$$

$$2h_i < h_{i+1}$$

$$k$$

$$O(n^{3/2})$$

$$2$$

$$\text{InsertionSort}(2)$$

$$\text{InsertionSort}(3)$$

$$2 \cdot 3 - 2 - 3 = 1$$

$$2$$

$$i$$

$$\text{InsertionSort}(1)$$

$$O(n)$$

$$\text{InsertionSort}(4)$$

$$\text{InsertionSort}(6)$$

$$\text{InsertionSort}(2)$$

$$\text{InsertionSort}(3)$$

$$\text{InsertionSort}(2)$$

$$\text{InsertionSort}(1)$$

$$O(n)$$

$$2$$

$$\text{InsertionSort}(2h)$$

$$\text{InsertionSort}(3h)$$

$$\text{InsertionSort}(h)$$

$$O(n)$$

$$h_t > n/3$$

$$\text{InsertionSort}(h_t)$$

$$O(n^2/h_t)$$

$$n^2/h_t < 3n$$

$$O(n)$$

$$O(\log^2 n)$$

$$O(n \log^2 n)$$

$$h_t \leq n/3$$

$$3h_t \leq n$$

$$\text{InsertionSort}(2h_t)$$

$$\text{InsertionSort}(3h_t)$$

$$\text{InsertionSort}(h_t)$$

$$O(n)$$

$$O(\log^2 n)$$

$$O(n \log^2 n)$$

$$O(n \log^2 n)$$

$$O(1)$$

Input. An array A consisting of n elements.
Output. A will be sorted in nondecreasing order.
Method.
for $i \leftarrow 1$ **to**$n-1$
    $ith \leftarrow i$
    for$j \leftarrow i+1$ **to**n
        if$A[j] < A[ith]$
            $ith \leftarrow j$
    swap $A[i]$ and $A[ith]$

$$i$$

$$A_{i..n}$$

$$i$$

$$O(n^2)$$

Input. An array A consisting of n elements, where each element has k keys.
Output. Array A will be sorted in nondecreasing order stably.
Method.
for$i \leftarrow k$ **down to**1
    sort A into nondecreasing order by the i -th key stably

$$k$$

$$1$$

k

k

1

a_i

a

i

k

a

b

1

a_1

b_1

$a_1 < b_1$

$a < b$

$a_1 > b_1$

$a > b$

$a_1 = b_1$

2

a_2

b_2

$a_2 < b_2$

$a < b$

$a_2 > b_2$

$a > b$

$a_2 = b_2$

k

a_k

b_k

$a_k < b_k$

$a < b$

$a_k > b_k$

$a > b$

$a_k = b_k$

$a = b$

0

i

i

i

i

1

1

2

1

k

k

1

2

k

W

$\log_2 W$

$\log_2 W$

2^k

$O(n)$

≤ 1

$\leq B$

B

1

k

k

1

k

k

$k-1$

$k-2$

1

a_1

b_1

a

b

a_2

b_2

$a_1 = b_1$

a_2

b_2

a

b

a_3

b_3

$a_2 = b_2$

a_{k-1}

b_{k-1}

a

b

a_k

b_k

$a_{k-1} = b_{k-1}$

a_k

b_k

a_k

b_k

a_k

b_k

a_{k-1}

b_{k-1}

a_2

b_2

a_1

$$b_1$$

$$a_1$$

$$b_1$$

$$a$$

$$b$$

$$n$$

$$O(kn + \sum_{i=1}^{k} w_i)$$

$$w_i$$

$$i$$

$$O(nk \log n)$$

$$O(k + n)$$

$$X = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}$$

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} X_{i,j}\right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{i,j}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(a_i \text{ □ } a_j \text{ □□}) \end{aligned}$$

$$P(a_i\text{□}a_j \text{ □□}) = P(a_i\text{□}a_j \text{ □□□ } A_{i,j} \text{ □□□□□□□□□□□}) = \frac{2}{j-i+1}$$

$$\begin{aligned} E[X] &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(a_i \text{ □ } a_j \text{ □□}) \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1} \\ &= \sum_{i=1}^{n-1} \sum_{k=2}^{n-i+1} \frac{2}{k} \\ &= \sum_{i=1}^{n-1} O(\log n) \\ &= O(n \log n) \end{aligned}$$

$$T(n) \le T(\frac{n}{5}) + T(\frac{7n}{10}) + O(n)$$

$$
\begin{aligned}
T(n) &\leq T(\frac{n}{5}) + T(\frac{7n}{10}) + O(n) \\
&\leq \frac{cn}{5} + \frac{7cn}{10} + O(n) \\
&\leq \frac{9cn}{10} + O(n) \\
&= O(n)
\end{aligned}
$$

$$m$$

$$p$$

$$q$$

$$q$$

$$m$$

$$p$$

$$q$$

$$p$$

$$m$$

$$O(n \log n)$$

$$O(n^2)$$

$$T(n) = 2T(\frac{n}{2}) + \Theta(n)$$

$$T(n) = \Theta(n \log n)$$

$$T(n) = T(n-1) + \Theta(n)$$

$$T(n) = \Theta(n^2)$$

$$O(n \log n)$$

$$n$$

$$X$$

$$O(n + X)$$

$$n$$

$$O(n + X)$$

$$a_i$$

$$i$$

$$A_{i,j}$$

$$\{a_i, a_{i+1}, ..., a_j\}$$

$$X_{i,j}$$

0

1

a_i

a_j

a_i

a_j

a_i

a_j

$A_{i,j}$

a_i

a_j

$A_{i,j}$

a_i

a_j

a_i

a_j

$A_{i,j}$

x

$i < x < j$

a_x

$A_{i,j}$

a_x

a_i

a_j

a_i

a_j

a_i

a_j

a_i

a_j

a_i

a_j

$$A_{i,j}$$

$$a_i$$

$$a_j$$

$$a_i$$

$$A_{i,j}$$

$$A_{i,j}$$

$$A_{i,j}$$

$$a_i$$

$$a_i$$

$$a_i$$

$$a_j$$

$$P(a_i \text{ □ } a_j \text{ □□})$$

$$A_{i,j}$$

$$A_{i,j}$$

$$A_{i,j}$$

$$A_{i,j}$$

$$j - i + 1$$

$$O(n \log n)$$

$$O(n \log n)$$

$$O(n^2)$$

$$m$$

$$m$$

$$m$$

$$m$$

$$O(n)$$

$$O(n \log n)$$

$$\lfloor \log_2 n \rfloor$$

$$O(n^2)$$

$$k$$

$$k$$

$$k$$

$$k$$

$$O(n \log n)$$

$$A_p \cdots A_r$$

$$A_p \cdots A_q$$

$$A_{q+1} \cdots A_r$$

$$q - p + 1$$

$$k$$

$$O(n)$$

$$k$$

$$\left\lfloor \frac{n}{5} \right\rfloor$$

$$\left\lfloor \frac{n}{5} \right\rfloor$$

$$O(n)$$

$$T(n)$$

$$n$$

$$O(1)$$

$$\left\lfloor \frac{n}{5} \right\rfloor$$

$$O(n)$$

$$T(\frac{n}{5})$$

$$\frac{1}{2} \times \left\lfloor \frac{n}{5} \right\rfloor = \left\lfloor \frac{n}{10} \right\rfloor$$

$$3 \times \left\lfloor \frac{n}{10} \right\rfloor = \left\lfloor \frac{3n}{10} \right\rfloor$$

$$\left\lfloor \frac{3n}{10} \right\rfloor$$

$$\frac{7n}{10}$$

$$T(\frac{7n}{10})$$

$$T(n) = O(n)$$

$$T(n) \leq cn$$

$$c$$

$$T(n)$$

$$O(n)$$

for *state*
    $sum[state] \leftarrow f[state]$
for $i \leftarrow 0$ **to** D
    for $state'$ **in** lexicographical order
        $sum[state] \leftarrow sum[state] + sum[state']$

$$b_l \leftarrow b_l + k, b_{r+1} \leftarrow b_{r+1} - k$$

$$d_s \leftarrow d_s + 1$$
$$d_{lca} \leftarrow d_{lca} - 1$$
$$d_t \leftarrow d_t + 1$$
$$d_{f(lca)} \leftarrow d_{f(lca)} - 1$$

$$d_s \leftarrow d_s + 1$$
$$d_t \leftarrow d_t + 1$$
$$d_{lca} \leftarrow d_{lca} - 2$$

$$n$$

$$N$$

$$A$$

$$B$$

$$i$$

$$B[i]$$

$$A$$

$$0$$

$$i$$

$$i \geq 1$$

$$a$$

$$sum$$

$$sum_{x,y} = \sum_{i=1}^{x} \sum_{j=1}^{y} a_{i,j}$$

$$sum$$

$$sum_{i,j} = sum_{i-1,j} + sum_{i,j-1} - sum_{i-1,j-1} + a_{i,j}$$

$$sum_{i-1,j}$$

$$sum_{i,j-1}$$

$$sum_{i-1,j-1}$$

$$(x_1, y_1) - (x_2, y_2)$$

$$sum_{x_2,y_2} - sum_{x_1-1,y_2} - sum_{x_2,y_1-1} + sum_{x_1-1,y_1-1}$$

$$n \times m$$

$$0$$

$$1$$

$$0$$

$$U$$

$$D$$

$$f[\cdot]$$

$$\text{sum}[\cdot]$$

$$\text{sum}[i][\text{state}]$$

$$\text{state}$$

$$D - i$$

$$\text{state}$$

$$\text{sum}[0][\text{state}] = f[\text{state}]$$

$$\text{sum}[\text{state}] = \text{sum}[D][\text{state}]$$

$$\text{sum}[i][\text{state}] = \text{sum}[i-1][\text{state}] + \text{sum}[i][\text{state}]$$

$$\text{state}$$

$$i$$

$$\text{state}$$

$$1$$

$$O(D \times \mid U \mid)$$

$$\mid U \mid$$

$$U$$

$$sum_i$$

$$i$$

$$x, y$$

$$sum_x + sum_y - sum_{lca} - sum_{fa_{lca}}$$

$$x, y$$

$$sum_x + sum_y - 2 \cdot sum_{lca}$$

$$b_i = \begin{cases} a_i - a_{i-1} & i \in [2, n] \\ a_1 & i = 1 \end{cases}$$

$$a_i$$

$$b_i$$

$$a_n = \sum_{i=1}^{n} b_i$$

$$a_i$$

$$sum = \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \sum_{j=1}^{i} b_j = \sum_{i}^{n} (n - i + 1) b_i$$

$$[l, r]$$

$$k$$

$$b_l + k = a_l + k - a_{l-1}$$

$$b_{r+1} - k = a_{r+1} - (a_r + k)$$

$$\delta(s_1, t_1), \delta(s_2, t_2), \delta(s_3, t_3) \ldots$$

$$\delta(s, t)$$

$$\delta(s, t)$$

$$s$$

$$t$$

$$f(x)$$

$$x$$

$$d_i$$

$$a_i$$

$$lca$$

$$left$$

$$d_{lca} - 1 = a_{lca} - (a_{left} + 1)$$

$$d_{f(lca)} - 1 = a_{f(lca)} - (a_{lca} + 1)$$

$$N (2 \leq N \leq 50,000)$$

$$N - 1$$

$$1$$

$$N$$

$$K (1 \leq K \leq 100,000)$$

$$i$$

$$s_i$$

$$t_i$$

$$\Theta(n \log n)$$

$$\Theta(n)$$

$$\Theta(1)$$

$$1$$

$$1$$

$$mid = \left\lfloor \frac{l+r}{2} \right\rfloor$$

$$[l, r)$$

$$[l, mid)$$

$$[mid, r)$$

$$1$$

$$1$$

$$1$$

$$\leq 2$$

$$\leq 2$$

$$\leq 4$$

$$\leq 4$$

$$\leq 8$$

$$\leq n$$

$$= n$$

$$2^x$$

$$i < j$$

$$a_i > a_j$$

$$(i, j)$$

$$\Theta(n \log n)$$

$$O(n \log n)$$

$$\Theta(n)$$

Input. An array A consisting of n elements.
Output. A will be sorted in nondecreasing order stably.
Method.
for $i \leftarrow 2$ **to**n
    $key \leftarrow A[i]$
    $j \leftarrow i - 1$
    while$j > 0$ **and**$A[j] > key$
        $A[j+1] \leftarrow A[j]$
        $j \leftarrow j - 1$
    $A[j+1] \leftarrow key$

$$O(n)$$

$$O(n^2)$$

$$n - 1$$

$$O(n \log n)$$

$$\max\left(\frac{s}{b_i}, \frac{s \cdot a_i}{b_{i+1}}\right) < \max\left(\frac{s}{b_{i+1}}, \frac{s \cdot a_{i+1}}{b_i}\right)$$

$$\max\left(\frac{1}{b_i}, \frac{a_i}{b_{i+1}}\right) < \max\left(\frac{1}{b_{i+1}}, \frac{a_{i+1}}{b_i}\right)$$

$$\max(b_{i+1}, a_i \cdot b_i) < \max(b_i, a_{i+1} \cdot b_{i+1})$$

$$n = 1$$

$$F_1$$

$$n$$

$$F_{n+1}$$

$$F_n$$

$$i$$

$$a_i, b_i$$

$$s$$

$$i$$

$$a_i$$

$$i$$

$$\frac{s}{b_i}$$

$$i + 1$$

$$\frac{s \cdot a_i}{b_{i+1}}$$

$$i$$

$$i+1$$

$$i$$

$$\frac{s}{b_{i+1}}$$

$$i+1$$

$$\frac{s \cdot a_{i+1}}{b_i}$$

$$s$$

$$0$$

$$10^9$$

$$1$$

$$N$$

$$N(1 \leq N \leq 10^5)$$

$$i$$

$$D_i(1 \leq D_i \leq 10^9)$$

$$P_i(1 \leq P_i \leq 10^9)$$

$$n$$

$$n$$

$$a$$

$$n$$

$$O(n)$$

$$O(n \log n)$$

$$O(n \log n)$$

$$O(n \log n)$$

$$0$$

$$2$$

$$j$$

$$a$$

$$n$$

$$O(n)$$

$$O(n + \max\{\mid x \mid : x \in a\})$$

Input. An array A consisting of n positive integers no greater than w.
Output. Array A after sorting in nondecreasing order stably.
Method.
for $i \leftarrow 0$ **to** w
    $cnt[i] \leftarrow 0$
for $i \leftarrow 1$ **to** n
    $cnt[A[i]] \leftarrow cnt[A[i]] + 1$
for $i \leftarrow 1$ **to** w
    $cnt[i] \leftarrow cnt[i] + cnt[i-1]$
for $i \leftarrow n$ **downto** 1
    $B[cnt[A[i]]] \leftarrow A[i]$
    $cnt[A[i]] \leftarrow cnt[A[i]] - 1$
return B

$$C$$

$$i$$

$$A$$

$$i$$

$$C$$

$$A$$

$$C$$

$$A$$

$$C$$

$$C$$

$$A$$

$$A$$

$$O(n+w)$$

$$w$$

$$0, 1, -1, 2, -2, \cdots$$

$$x, y, z$$

$$n$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{2}{n}$$

$$n, n+1, n(n+1)$$

$$n = 1$$

$$n+1$$

$$n(n+1)$$

$$n$$

$$1 \ldots n$$

$$n$$
$$n$$
$$n$$
$$1...n$$
$$n$$
$$n$$
$$n$$
$$n$$
$$n, 1, n-2, 3, \cdots$$
$$n$$
$$n$$
$$n$$
$$4$$
$$n$$
$$4$$
$$1, \frac{2}{1}, \frac{3}{2}, \cdots, \frac{n-1}{n-2}, n$$
$$n$$
$$p, q$$
$$p \times q \equiv 0 \pmod n$$
$$(3 \times 6)\%9 = 0$$
$$p, q$$
$$0$$
$$4 = 2 \times 2$$
$$p, q$$
$$1$$
$$1$$
$$n$$
$$n$$
$$0$$
$$1, 2, 3, \cdots, n$$
$$n$$
$$1 \cdots n-2$$
$$+1$$
$$N$$
$$N$$

$$1...N$$

$$S$$

$$S$$

$$n = 3, 4, 5$$

$$k$$

$$k$$

$$S$$

$$\frac{(k-1)\sum_{i=1}^{n} i}{k}$$

$$n$$

$$\{1, n\}, \{2, n-1\}\cdots$$

$$n$$

$$n$$

$$n-1$$

$$\{n\}, \{1, n-1\}, \{2, n-2\}\cdots$$

$$n \geq 3$$

$$n$$

$$v_1, v_2, ..., v_n$$

$$w$$

$$P$$

$$w$$

$$P$$

$$k$$

$$n$$

$$v$$

$$n$$

$$1$$

$$\frac{w}{2}$$

$$x$$

$$\binom{n}{w-x}$$

$$f_{i,j}$$

$$i$$

$$1$$

$$j$$

$$f$$

$$i \leq 20$$

$$T(n) = aT\left(\frac{n}{b}\right) + f(n) \qquad \forall n > b$$

$$T(n) = \begin{cases} \Theta(n^{\log_b a}) & f(n) = O(n^{\log_b a - \epsilon}), \epsilon > 0 \\ \Theta(f(n)) & f(n) = \Omega(n^{\log_b a + \epsilon}), \epsilon \geq 0 \\ \Theta(n^{\log_b a} \log^{k+1} n) & f(n) = \Theta(n^{\log_b a} \log^k n), k \geq 0 \end{cases}$$

$$\sum_{i=1}^{m} p_i + F(S_0) - F(S_m)$$

$$\sum_{i=1}^{m} p_i \geq \sum_{i=1}^{m} c_i$$

$$n$$

$$100n$$

$$n$$

$$n^2$$

$$100$$

$$\Theta$$

$$O$$

$$\Omega$$

$$O$$

$$o$$

$$O$$

$$o$$

$$f(n)$$

$$g(n)$$

$$f(n) = \Theta(g(n))$$

$$\exists c_1, c_2, n_0 > 0$$

$$\forall n \geq n_0, 0 \leq c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)$$

$$f(n) = \Theta(g(n))$$

$$c_1, c_2$$

$$f(n)$$

$$c_1 \cdot g(n)$$

$$c_2 \cdot g(n)$$

$$3n^2 + 5n - 3 = \Theta(n^2)$$

$$c_1, c_2, n_0$$

$$2, 4, 100$$

$$n\sqrt{n} + n\log^5 n + m\log m + nm = \Theta(n\sqrt{n} + m\log m + nm)$$

$$c_1, c_2, n_0$$

$$1, 2, 100$$

$$\Theta$$

$$O$$

$$f(n) = O(g(n))$$

$$\exists c, n_0$$

$$\forall n \geq n_0, 0 \leq f(n) \leq c \cdot g(n)$$

$$O$$

$$O$$

$$\Theta$$

$$\Theta$$

$$O$$

$$O$$

$$O$$

$$\Omega$$

$$f(n) = \Omega(g(n))$$

$$\exists c, n_0$$

$$\forall n \geq n_0, 0 \leq c \cdot g(n) \leq f(n)$$

$$O$$

$$o$$

$$o$$

$$o$$

$$f(n) = o(g(n))$$

$$c$$

$$\exists n_0$$

$$\forall n \geq n_0, 0 \leq f(n) < c \cdot g(n)$$

$$\Omega$$

$$\omega$$

$$f(n) = \omega(g(n))$$

$$c$$

$$\exists n_0$$

$$\forall n \geq n_0, 0 \leq c \cdot g(n) < f(n)$$

$$f(n) = \Theta(g(n)) \iff f(n) = O(g(n)) \land f(n) = \Omega(g(n))$$

$$f_1(n) + f_2(n) = O(\max(f_1(n), f_2(n)))$$

$$f_1(n) \times f_2(n) = O(f_1(n) \times f_2(n))$$

$$\forall a \neq 1, \log_a n = O(\log_2 n)$$

$$n$$

$$m$$

$$\Theta(n^2 m)$$

$$n$$

$$m$$

$$\Theta(n + m)$$

$$N$$

$$O(1)$$

$$1$$

$$O(1)$$

$$af(n/b) \leq cf(n)$$

$$c < 1$$

$$n$$

$$n$$

$$a$$

$$(\frac{n}{b})$$

$$f(n)$$

$$0$$

$$f(n)$$

$$1$$

$$a$$

$$f\left(\frac{n}{b}\right)$$

$$af\left(\frac{n}{b}\right)$$

$$\log_b n$$

$$n^{\log_b a}$$

$$T(n) = \Theta(n^{\log_b a}) + g(n)$$

$$g(n) = \sum_{j=0}^{\log_b n-1} a^j f(n/b^j)$$

$$f(n) = O(n^{\log_b a-\epsilon})$$

$$g(n) = O(n^{\log_b a})$$

$$g(n) = \Omega(f(n))$$

$$af(\frac{n}{b}) \leq cf(n)$$

$$c$$

$$n$$

$$g(n) = O(f(n)$$

$$g(n) = \Theta(f(n))$$

$$f(n) = \Theta(n^{\log_b a})$$

$$g(n) = O(n^{\log_b a} \log n)$$

$$T(n)$$

$$g(n)$$

$$T(n) = T\left(\frac{n}{2}\right) + 1$$

$$a = 1, b = 2, \log_2 1 = 0$$

$$\epsilon$$

$$0$$

$$T(n) = \Theta(\log n)$$

$$T(n) = T\left(\frac{n}{2}\right) + n$$

$$a = 1, b = 2, \log_2 1 = 0$$

$$\epsilon$$

$$0.5$$

$$T(n) = \Theta(n)$$

$$T(n) = T\left(\frac{n}{2}\right) + \log n$$

$$a = 1, b = 2, \log_2 1 = 0$$

$$k$$

$$1$$

$$T(n) = \Theta(\log^2 n)$$

$$O(n)$$

$$O(n \log n)$$

$$O(\log n)$$

$$S$$

$$S_0$$

$$F$$

$$S$$

$$F(S) \geq F(S_0)$$

$$S_1, S_2, \cdots, S_m$$

$$S_0$$

$$m$$

$$c_i$$

$$i$$

$$p_i = c_i + F(S_i) - F(S_{i-1})$$

$$m$$

$$F(S) \geq F(S_0)$$

$$p_i = O(T(n))$$

$$O(T(n))$$

$$O(n + n^2/k + k)$$

$$n$$

$$k \approx n$$

$$O(n)$$

$$O(n^2)$$

Input. An array A consisting of n elements.
Output. A will be sorted in nondecreasing order stably.
Method.
$flag \leftarrow True$
while$flag$
    $flag \leftarrow False$
    for$i \leftarrow 1$ **to**$n-1$
        if$A[i] > A[i+1]$
            $flag \leftarrow True$
            Swap $A[i]$ and $A[i+1]$

$$i$$

$$i$$

$$i$$

$$n-1$$

$$O(n)$$

$$\frac{(n-1)n}{2}$$

$$O(n^2)$$

$$O(n^2)$$

Input. A range $[l, r]$ meaning that the domain of $f(x)$.
Output. The maximum value of $f(x)$ and the value of x at that time .
Method.
while $r - l > \varepsilon$
    $mid \leftarrow \frac{l+r}{2}$
    $lmid \leftarrow mid - \varepsilon$
    $rmid \leftarrow mid + \varepsilon$
    if $f(lmid) < f(rmid)$
        $r \leftarrow mid$
    else
        $l \leftarrow mid$

$$O(1)$$

$$O(\log n)$$

$$n$$

$$O(\log n)$$

$$O(1)$$

$$O(\log n)$$

$$n$$

$$n \geq 0$$

$$1$$

$$0$$

$$x$$

$$x+1$$

$$x-1$$

$$M$$

$$H$$

$$H$$

$$H$$

$$H$$

$$20,\ 15,\ 10,\ 17$$

$$15$$

$$15,\ 15,\ 10,\ 15$$

$$1$$

$$5$$

$$4$$

$$2$$

$$7$$

$$H$$

$$M$$

$$1$$

$$M$$

$$1$$

$$10^9$$

$$1$$

$$10^9$$

$$[1,\ 10^9]$$

$$[l,r]$$

$$lmid, rmid(lmid < rmid)$$

$$f(lmid) < f(rmid)$$

$$[rmid, r]$$

$$lmid$$

$$rmid$$

$$lmid$$

$$rmid$$

$$mid - \varepsilon$$

$$mid + \varepsilon$$

$$N$$

$$[l, r]$$

$$[l, x]$$

$$[x, r]$$

$$x$$

$$N$$

$$[l, r]$$

$$c_i$$

$$d_i$$

$$\frac{\sum c_i}{\sum d_i}$$

$$[0, 31]$$

$$[0, 31]$$

$$[0, 127]$$

$$[0, 1023]$$

$$[0, 1048575]$$

$$n$$

$$k$$

$$i$$

$$(i + k) \bmod n + 1$$

$$m$$

$$a_i$$

$$m$$

$$10^9 + 7$$

$$1 \leq n \leq 10^6$$

$$1 \leq m \leq 10^{18}$$

$$1 \leq k \leq n$$

$$0 \leq a_i \leq 10^9$$

$$m$$

$$m$$

$$10^{18}$$

$$2^i$$

$$x$$

$$2^i$$

$$x$$

$$2^i$$

$$2^i$$

$$2^{i-1}$$

$$2^{i-1}$$

$$2^{i-1}+2^{i-1}=2^i$$

$$m \leq 10^{18}$$

$$65$$

$$i$$

$$\Theta(n \log m)$$

$$\Theta(\log m)$$

LATEX

LATEX

LATEX

$$0.0015$$

$$0.0015$$

$$0.00149999 \cdots$$

$$ans$$

$$\frac{\mid x-ans \mid}{\max(1, ans)} < 10^{-9}$$

$$x$$

$$ans$$

$$5 \cdot 10^5$$

$$k=1$$

$$30\%$$

$$n \leq 1000$$

$$40\%$$

$$m \leq 100$$

$$70\%$$

$$40\%$$

$$[1, n]$$

$$l$$

$$[l, n]$$

$$r$$

$$[1, n]$$

$$[0, n]$$

$$x$$

$$x = 0$$

$$[1, n]$$

$$y$$

$$[y, y]$$

$$2 \sim n$$

$$i$$

$$[1, i - 1]$$

$$O(\log n)$$

$$[i \cdot low, i \cdot high]$$

$$i$$

$$low$$

$$high$$

$$O(\sqrt{n})$$

$$2$$

$$\sqrt{n}$$

$$d$$

$$d > 1$$

$$d - 1$$

$$d - 1$$

$$n \times m$$

$$(1, 1)$$

$$(n, m)$$

$$(n, m)$$

$$N$$

$$a_1, a_2, \cdots, a_N (0 \le a_1 < a2 < \cdots < a_N \le 10^{18})$$

$$a_{i+1} - a_i (0 \le i \le N-1)$$

$$a_{i+1} - a_i (0 \le i \le N-1)$$

$$10^6$$

$$\left\lfloor \frac{n}{2} \right\rfloor$$

$$5 \times 10^4$$

$$1999$$

$$n < 2000$$

$$n \ge 2000$$

$$x$$

$$x$$

$$[1, 10^{18}]$$

$$\frac{N+1}{2}$$

$$[s, t]$$

$$mn, mx$$

$$[mn+1, mx-1]$$

$$3N$$

$$O(N^2)$$

$$O(N^2)$$

$$\left\lfloor \frac{a_n - a_1}{N-1} \right\rfloor$$

$$i$$

$$ans$$

$$[i, i+ans]$$

$$ans$$

$$ans$$

$$i$$

$$h \le 7$$

$$\le 16$$

$$h \le 4$$

$$h > 4$$

$$1-(\frac{2^h-2}{2^h-1})$$

$$1 \leq k \leq 3$$

$$k=3$$

$$k=2$$

$$k=1$$

$$k=1$$

$$k=2$$

$$k=3$$

$$k=1$$

$$k=3$$

$$h=7$$

$$\frac{(1+7)\times 7}{2}=28$$

$$k$$

$$2^k-1$$

$$2^k-1$$

$$2^k-2$$

$$h=7$$

$$2^3-1=7$$

$$2^3-2=6$$

$$k \leq 5$$

$$9+7+5+3$$

$$121-40$$

$$\sum_{k=\frac{n}{m}(i-1)+1}^{\frac{ni}{m}} f^2(k)$$

$$f(x)$$

$$x$$

$$1$$

$$n$$

$$n \leq 10^9$$

$$\sum_{i=1}^{n} f^2(i)$$

$$n$$

$$f(n)$$

$$m$$

$$i$$

$$f(x)$$

$$[l, r]$$

$$4^2 = 16$$

$$2 + 2 \times 2 + 2 = 8$$

$$\Theta(n)$$

$$O(1)$$

$$n$$

$$n$$

$$n$$

$$[1, n]$$

$$[1, 10^9]$$

$$50$$

$$[1, 10^9]$$

$$\mathrm{wnext}(i, t) = \begin{cases} \mathrm{next}(i) & t = 0 \\ \max(\mathrm{next}(i), \mathrm{wnext}(i, t-1)) & t > 0 \\ \min(\mathrm{next}(i), \mathrm{wnext}(i, t+1)) & t < 0 \end{cases}$$

$$[1, n]$$

$$[1, 10]$$

$$[0, 4)$$

$$[4, 100]$$

$$[0, 10.0)$$

$$t$$

$$t < 0$$

$$-t$$

$$t = 0$$

$$n = 10, t = 1000$$

$$n = 10, t = -1000$$

$$0$$

$$100$$

$$[L, R]$$

$$N$$

$$[L, R]$$

$$[L, R]$$

$$N$$

$$[L, R]$$

$$[L, R]$$

$$[L, R]$$

$$[minPrecision, maxPrecision]$$

$$[1, 5]$$

$$[-9999, 9999]$$

$$x_i \neq y_i$$

$$a, b$$

$$-1000 \leq a, b \leq 1000$$

$$s$$

$$t$$

$$10^{-4}$$

$$\{\omega \in \Omega : X(\omega) \leq t\} \in \mathcal{F}$$

$$I_A(\omega) = \begin{cases} 1, \omega \in A \\ 0, \omega \notin A \end{cases}$$

$$F(x) = P(X \leq x)$$

$$P(l < x \leq l + \Delta x) = F(l + \Delta x) - F(l)$$

$$F(x) = \int_{-\infty}^{x} f(x) dx$$

$$P(X \leq x, Y \leq y) = P(X \leq x) P(Y \leq y)$$

$$(\Omega, \mathcal{F}, P)$$

$$\Omega$$

$$X : \Omega \to \mathbb{R}$$

$$t \in \mathbb{R}$$

$$X$$

$$\Omega$$

$$A$$

$$I_A$$

$$A$$

$$X$$

$$X$$

$$X \sim F(x)$$

$$F(x) = F(x+0)$$

$$\mathbb{R}$$

$$F(-\infty) = 0$$

$$F(+\infty) = 1$$

$$X$$

$$x_1, x_2, \cdots$$

$$P\{X = x_i\} = p_i$$

$$X$$

$$X$$

$$P\{X = x\}$$

$$0$$

$$0$$

$$X$$

$$\frac{1}{2}$$

$$0$$

$$\frac{1}{2}$$

$$(0, 1)$$

$$X$$

$$r \in (0, 1)$$

$$P\{X = r\} = 0$$

$$P\{X = 0\} = \frac{1}{2}$$

$$X \sim F(x)$$

$$\lim_{\Delta x \to 0^+} \frac{F(l + \Delta x) - F(l)}{\Delta x}$$

$$X$$

$$l$$

$$f(x)$$

$$f(x)$$

$$X$$

$$X, Y$$

$$x, y \in \mathbb{R}$$

$$X, Y$$

$$P(X = \alpha)$$

$$0$$

$$X$$

$$Y$$

$$f, g$$

$$f(X)$$

$$g(Y)$$

$$X$$

$$Y$$

$$f(X, Y)$$

$$XY^2$$

$$X$$

$$Y$$

$$Y$$

$$y$$

$$f(X, y)$$

$$f(X, Y)$$

$$\sum x_i p_i$$

$$\int_{\mathbb{R}} x f(x) dx$$

$$\int_{\mathbb{R}} x dF(x)$$

$$P\left\{X=(-1)^k\frac{2^k}{k}\right\}=\frac{1}{2^k},\quad k=1,2,\cdots$$

$$f(y)=\frac{1}{\pi}\cdot\frac{1}{1+y^2},\quad y\in(-\infty,+\infty)$$

$$E(XY)=EX\cdot EY$$

$$I_A(\omega)=\begin{cases}1,\omega\in A\\0,\omega\notin A\end{cases}$$

$$S=\sum_{k=1}^{n}k\cdot I_k$$

$$ES=E\left(\sum_{k=1}^{n}k\cdot I_k\right)=\sum_{k=1}^{n}k\cdot E[I_k]=\sum_{k=1}^{n}k\cdot p_k$$

$$P(X=x_i\mid Y=y)\qquad f_{X\mid Y}(x\mid y)$$

$$E[E[X\mid Y]]=EX$$

$$g(k)=\begin{cases}1, & k\le d\\1+f(k), & k>d\end{cases}$$

$$f(x)=Eg(k)=1+\frac{1}{x}\cdot\int_d^x f(t)dt$$

$$f(x)=1+\ln\frac{x}{d}$$

$$E(X-EX)^2$$

$$E((X-EX)(Y-EY))$$

$$\mathrm{Cov}(X,Y)=E((X-EX)(Y-EY))=E(X-EX)E(Y-EY)=0$$

$$\frac{\mathrm{Cov}(X,Y)}{\sigma(X)\sigma(Y)}$$

$$X$$

$$p_i=P\{X=x_i\}$$

$$X$$

$$EX$$

$$X$$

$$f(x)$$

$$X$$

$$EX$$

$$X$$
$$F(x)$$
$$X$$
$$EX$$
$$X$$
$$\sum x_i p_i$$
$$-\ln 2$$
$$X$$
$$Y$$
$$Y$$
$$X, Y$$
$$a, b$$
$$E(aX + b) = a \cdot EX + b$$
$$E(X + Y) = EX + EY$$
$$X$$
$$Y$$
$$X$$
$$Y$$
$$X$$
$$Y$$
$$X$$
$$[-1, 1]$$
$$Y = X^2$$
$$A$$
$$I_A$$
$$EI_A = P(A)$$
$$n$$
$$\{a_i\}$$
$$a_k$$
$$p_k$$
$$k$$
$$1 - p_k$$

$$0$$

$$S = \sum_{i=1}^{n} a_i$$

$$S$$

$$I_k$$

$$a_k = k$$

$$X$$

$$Y$$

$$Y = y$$

$$X$$

$$X$$

$$E[X \mid Y = y]$$

$$E[X \mid Y]$$

$$Y$$

$$L$$

$$d$$

$$f(x)$$

$$x$$

$$x \leq d$$

$$x > d$$

$$k$$

$$k \sim U[0, x]$$

$$X$$

$$EX$$

$$X$$

$$DX$$

$$Var(x)$$

$$\sigma(X) = \sqrt{DX}$$

$$X$$

$$a, b$$

$$D(aX + b) = a^2 \cdot DX$$

$$DX = E(X^2) - (EX)^2$$

$$D(X+Y)=DX+DY$$

$$D(X+Y)$$

$$DX+DY$$

$$D(X+Y)$$

$$DX+DY$$

$$X, Y$$

$$X$$

$$Y$$

$$\operatorname{Cov}(X, Y)$$

$$X, Y, Z$$

$$\operatorname{Cov}(X, Y)=\operatorname{Cov}(Y, X)$$

$$a, b$$

$$\operatorname{Cov}(aX+bY, Z)=a \cdot \operatorname{Cov}(X, Z)+b \cdot \operatorname{Cov}(Y, Z)$$

$$DX=\operatorname{Cov}(X, X)$$

$$D(X+Y)=DX+2\operatorname{Cov}(X, Y)+DY$$

$$D(X+Y)=DX+DY$$

$$\operatorname{Cov}(X, Y)=0$$

$$X$$

$$Y$$

$$\operatorname{Cov}(X, Y)=0$$

$$X$$

$$Y$$

$$X, Y$$

$$X$$

$$Y$$

$$\rho_{X,Y}$$

$$|\ \rho_{X,Y}|$$

$$X$$

$$Y$$

$$|\ \rho_{X,Y}| \leq 1$$

$$|\ \rho_{X,Y}|=1$$

$$a$$
$$b$$
$$P(X = a + bY) = 1$$
$$\rho_{X,Y} = 1$$
$$a$$
$$b$$
$$P(X = a + bY) = 1$$
$$\rho_{X,Y} = -1$$
$$\rho_{X,Y} = 0$$
$$X$$
$$Y$$
$$X$$
$$Y$$
$$X, Y$$
$$\mathrm{Cov}(X, Y) = 0$$
$$X$$
$$Y$$
$$1$$
$$X, Y$$
$$P(B \mid A) = \frac{P(AB)}{P(A)} \quad \forall B \in \mathcal{F}$$
$$P(AB) = P(A)P(B \mid A)$$
$$P(B) = \sum_{i=1}^{n} P(A_i)P(B \mid A_i)$$
$$P(A_i | \ B) = \frac{P(A_i B)}{P(B)} = \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{n} P(A_j)P(B \mid A_j)}$$
$$P(AB) = P(A)P(B)$$
$$P(A_{i_1} A_{i_2} \cdots A_{i_r}) = \prod_{k=1}^{r} P(A_{i_k})$$
$$50$$
$$A$$
$$B$$

$$P(B \mid A)$$

$$(\Omega, \mathcal{F}, P)$$

$$A \in \mathcal{F}$$

$$P(A) > 0$$

$$P(\cdot \mid A)$$

$$P(\cdot \mid A)$$

$$(\Omega, \mathcal{F})$$

$$(\Omega, \mathcal{F}, P)$$

$$P(A) > 0$$

$$B$$

$$(\Omega, \mathcal{F}, P)$$

$$A_1, \cdots, A_n$$

$$\Omega$$

$$B$$

$$B$$

$$A_1, A_2, \cdots, A_n$$

$$P(A_i)$$

$$P(B \mid A_i)$$

$$B$$

$$B$$

$$P(B \mid A) = P(B)$$

$$B$$

$$A$$

$$B$$

$$A$$

$$A$$

$$B$$

$$A$$

$$B$$

$$A_1, A_2, \cdots, A_n$$

$$\{A_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}$$

$$A$$

$$B$$

$$C$$

$$P(A) = P(B) = P(C) = \frac{1}{2}$$

$$P(AB) = P(BC) = P(CA) = P(ABC) = \frac{1}{4}$$

$$A, B, C$$

$$P(ABC) \neq P(A)P(B)P(C)$$

$$A, B, C$$

$$P\left\{\bigcup_{i=1}^{m} A_i\right\} \leq \sum_{i=1}^{m} P\{A_i\}$$

$$P\{X \geq a\} \leq \frac{EX}{a}$$

$$I \leq \frac{X}{a}$$

$$P\{X \geq a\} = EI \leq E\left[\frac{X}{a}\right] = \frac{EX}{a}$$

$$P\{\mid X - EX \mid \geq a\} \leq \frac{DX}{a^2}$$

$$P\{\mid X - EX \mid \geq k\sigma\} \leq \frac{1}{k^2}$$

$$P\{\mid X - EX \mid \geq a\} = P\{(X - EX)^2 \geq a^2\}$$

$$P\{(X - EX)^2 \geq a^2\} \leq \frac{E(X - EX)^2}{a^2} = \frac{DX}{a^2}$$

$$P\{X \geq a\} = P\{\mathrm{e}^{tX} > \mathrm{e}^{ta}\} \leq \frac{E\mathrm{e}^{tX}}{\mathrm{e}^{ta}}$$

$$P\{X \leq a\} = P\{\mathrm{e}^{tX} > \mathrm{e}^{ta}\} \leq \frac{E\mathrm{e}^{tX}}{\mathrm{e}^{ta}}$$

$$P\{X = i\} = \begin{cases} p_i, & i = 1 \\ 1 - p_1, & i = 0 \end{cases}$$

$$P\{\mid X - \mu \mid \geq \epsilon\mu\} \leq 2\exp\left(-\frac{1}{3}\mu\epsilon^2\right)$$

$$P\{\mid X-EX\mid\geq\epsilon\}\leq 2\exp\left(\frac{-2\epsilon^2}{\sum_{i=1}^{n}(b_i-a_i)^2}\right)$$

$$P\left\{\left|\frac{4X}{n}-\pi\right|\geq\epsilon\pi\right\}\leq\delta$$

$$P\left\{\left|X-\frac{\pi}{4}n\right|\geq\epsilon\cdot\frac{\pi}{4}n\right\}\leq\delta$$

$$2\exp\left(-\frac{1}{3}\epsilon^2\cdot\frac{\pi}{4}n\right)\leq\delta$$

$$n\geq\frac{12}{\pi}\epsilon^{-2}\ln\frac{2}{\delta}$$

$$\left(1-\frac{1}{n}\right)^{n\log\epsilon^{-1}}\leq e^{\log\epsilon}=\epsilon$$

$$E\mid X-EX\mid=\int_0^\infty P\{\mid X-E[X]|\geq t\}\mathrm{d}t\leq 2\int_0^\infty\exp\left(-\frac{t^2}{n}\right)\mathrm{d}t=\sqrt{\pi n}$$

$$A_1,\cdots,A_m$$

$$X$$

$$a$$

$$I$$

$$X\geq a$$

$$X$$

$$a>0$$

$$a$$

$$k\sigma$$

$$\sigma$$

$$X$$

$$(X-EX)^2$$

$$\mathrm{e}^{tX}$$

$$X$$

$$t>0$$

$$t<0$$

$$0$$

$$1$$

$$X$$

$$n$$

$$X_1, X_2, \cdots, X_n$$

$$X = \sum_{i=1}^{n} X_i$$

$$\mu = EX$$

$$0 < \epsilon < 1$$

$$X_1, \cdots, X_n$$

$$X_i \in [a_i, b_i]$$

$$X = \sum_{i=1}^{n} X_i$$

$$EX$$

$$a_1 + \cdots + a_n$$

$$\pi$$

$$[-1, 1]^2$$

$$n$$

$$x^2 + y^2 \leq 1$$

$$m$$

$$\frac{4m}{n}$$

$$\pi$$

$$(1 - \delta)$$

$$\epsilon$$

$$n$$

$$X_i$$

$$i$$

$$X = \sum_{i=1}^{n} X_i$$

$$n$$

$$n = \Omega(\epsilon^{-2} \ln \frac{1}{\delta})$$

$$n$$

$$k$$

$$(1 - \epsilon)$$

$$M = n \log \epsilon^{-1}$$

$$M$$

$$k > 1$$

$$\frac{\epsilon}{k}$$

$$n \log \frac{k}{\epsilon}$$

$$n$$

$$\frac{n}{2}$$

$$n$$

$$n$$

$$\frac{n}{2}$$

$$\frac{n}{2}$$

$$\frac{1}{n}$$

$$\leq n$$

$$n^2$$

$$2\lceil \log_2 n \rceil$$

$$n$$

$$\lceil \log_2 n \rceil$$

$$n$$

$$\frac{n}{2}$$

$$n\lceil \log_2 n \rceil$$

$$\frac{n}{2}$$

$$\frac{n}{2}$$

$$\frac{n}{2}$$

$$\frac{n}{2}$$

$$X_i$$

$$i$$

$$X := X_1 + \cdots + X_n$$

$$\big|X - \mathrm{E}[X]\big|$$

$$t = c \cdot \sqrt{n}$$

$$c$$

$$\Pr\Big[\big|X - \mathrm{E}[X]\big| \geq t\Big] \leq 2\mathrm{e}^{-c^2}$$

$$\Theta(\sqrt{n})$$

$$n + \Theta(\sqrt{n}\log n)$$

$$n$$

$$\Theta(\sqrt{n}\log n)$$

$$\Theta(\sqrt{n})$$

$$\sqrt{\pi n} \cdot 2\lceil \log_2 n \rceil$$

$$n + 2\sqrt{\pi n}\lceil \log_2 n \rceil$$

$$n$$

$$n$$

$$1 - \frac{1}{n}$$

$$\Omega\left(\frac{\log n}{\log\log n}\right)$$

$$P(A) = \frac{\#(A)}{\#(\Omega)}$$

$$\Omega$$

$$\mathcal{F}$$

$$P$$

$$\Omega$$

$$\Omega$$

$$A, B, C, \cdots$$

$$\omega$$

$$A$$

$$A$$

$$\omega \in A$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$A$$

$$4$$

$$A = \{5, 6\}$$

$$\omega = 3$$

$$\omega \notin A$$

$$A$$

$$\Omega$$

$$A \cup B$$

$$A + B$$

$$A \cap B$$

$$AB$$

$$\mathcal{F} \subset 2^{\Omega}$$

$$2^{\Omega}$$

$$\Omega$$

$$\mathcal{F} = 2^{\Omega}$$

$$\Omega$$

$$2^{\Omega}$$

$$\Omega$$

$$2^{\Omega}$$

$$\mathcal{F} = 2^{\Omega}$$

$$2^{\Omega}$$

$$\mathcal{F}$$

$$\emptyset \in \mathcal{F}$$

$$A \in \mathcal{F}$$

$$\bar{A} \in \mathcal{F}$$

$$A_n \in \mathcal{F}, n = 1, 2, 3...$$

$$\bigcup A_n \in \mathcal{F}$$

$$\mathcal{F}$$

$$\emptyset$$

$$\mathcal{F}$$

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

$$\mathcal{F}_1 = \{\emptyset, \Omega\}$$

$$\mathcal{F}_2 = \{\emptyset, \{1, 3, 5\}, \{2, 4, 6\}, \Omega\}$$

$$\mathcal{F}_3 = \{\emptyset, \{1\}, \Omega\}$$

$$\mathcal{F}_4 = \{\{1,3,5\}, \{2,4,6\}\}$$

$$\emptyset$$

$$\Omega$$

$$A$$

$$\#(\cdot)$$

$$\Omega$$

$$P$$

$$\mathcal{F}$$

$$[0,1]$$

$$\Omega$$

$$1$$

$$P(\Omega) = 1$$

$$A_1, A_2, \cdots$$

$$P\left(\bigcup_{i\geq 1} A_i\right) = \sum_{i\geq 1} P(A_i)$$

$$A, B \in \mathcal{F}$$

$$A \subset B$$

$$P(A) \leq P(B)$$

$$P(A+B) = P(A) + P(B) - P(AB)$$

$$P(A-B) = P(A) - P(AB)$$

$$A - B$$

$$\Omega$$

$$\mathcal{F}$$

$$P$$

$$(\Omega, \mathcal{F}, P)$$

$$\Omega$$

$$A(z) = \sum_{\alpha\in\mathcal{A}} z^{|\ \alpha\ |} = \sum_{n\geq 0} a_n z^n$$

$$\mathcal{A} \cong \mathcal{E} \times \mathcal{A} \cong \mathcal{A} \times \mathcal{E}$$

$$\mathcal{A} + \mathcal{B} = (\mathcal{E}_1 \times \mathcal{A}) + (\mathcal{E}_2 \times \mathcal{B})$$

$$A(z) + B(z)$$

$$A(z)+B(z)=\sum_{\alpha\in\mathcal{A}} z^{|\,\alpha\,|}+\sum_{\beta\in\mathcal{B}} z^{|\,\beta\,|}=\sum_{n\geq 0}(a_n+b_n)z^n$$

$$\mathcal{A}\times\mathcal{B}=\{(\alpha,\beta)\mid \alpha\in\mathcal{A},\beta\in\mathcal{B}\}$$

$$A(z)\cdot B(z)$$

$$\gamma=(\alpha_1,\alpha_2,...,\alpha_n)\Longrightarrow|\,\gamma\,|=|\,\alpha_1|+|\,\alpha_2|+\cdots+|\,\alpha_n|$$

$$A(z)\cdot B(z)=\left(\sum_{\alpha\in\mathcal{A}} z^{|\,\alpha\,|}\right)\left(\sum_{\beta\in\mathcal{B}} z^{|\,\beta\,|}\right)=\sum_{(\alpha,\beta)\in(\mathcal{A}\times\mathcal{B})} z^{|\,\alpha\,|+|\,\beta\,|}=\sum_{n\geq 0}\sum_{i+j=n} a_i b_j z^n$$

$$\begin{aligned}\mathrm{SEQ}(\{a\})&=\{\epsilon\}+\{a\}+\{(a,a)\}+\{(a,a,a)\}+\cdots\\ \mathrm{SEQ}(\{a,b\})&=\{\epsilon\}+\{a,b\}+\{(a,b)\}+\{(b,a)\}+\{(a,a)\}+\{(b,b)\}\\ &\quad+\{(a,b,a)\}+\{(a,b,b)\}+\{(a,a,b)\}\\ &\quad+\{(b,b,a)\}+\{(b,a,b)\}+\{(b,b,b)\}+\{(a,a,a)\}+\{(b,a,a)\}\\ &\quad+\cdots\end{aligned}$$

$$\mathrm{SEQ}(\mathcal{A})=\mathcal{E}+\mathcal{A}+(\mathcal{A}\times\mathcal{A})+(\mathcal{A}\times\mathcal{A}\times\mathcal{A})+\cdots$$

$$Q(A(z))=1+A(z)+A(z)^2+A(z)^3+\cdots=\frac{1}{1-A(z)}$$

$$\mathcal{T}=\{\bullet\}\times\mathrm{SEQ}(\mathcal{T})$$

$$T(z)=\frac{z}{1-T(z)}$$

$$\begin{aligned}\mathrm{MSET}(\{a\})&=\{\epsilon\}+\{a\}+\{(a,a)\}+\{(a,a,a)\}+\cdots\\ \mathrm{MSET}(\{a,b\})&=\{\epsilon\}+\{a\}+\{(a,a)\}+\{(a,a,a)\}+\cdots\\ &\quad+\{b\}+\{(a,b)\}+\{(a,a,b)\}+\cdots\\ &\quad+\{(b,b)\}+\{(a,b,b)\}+\{(a,a,b,b)\}+\cdots\\ &\quad+\cdots\end{aligned}$$

$$\mathrm{MSET}(\{\alpha_0,\alpha_1,...,\alpha_n\})=\mathrm{MSET}(\{\alpha_0,\alpha_1,...,\alpha_{n-1}\})\times\mathrm{SEQ}(\{\alpha_n\})$$

$$\mathrm{MSET}(\mathcal{A})=\prod_{\alpha\in\mathcal{A}}\mathrm{SEQ}(\{\alpha\})$$

$$\mathrm{MSET}(\mathcal{A})=\mathrm{SEQ}(\mathcal{A})/\mathbf{R}$$

$$\mathrm{Exp}(A(z))=\prod_{\alpha\in\mathcal{A}}\left(1-z^{|\,\alpha\,|}\right)^{-1}=\prod_{n\geq 1}\left(1-z^n\right)^{-a_n}$$

$$\ln(1+z)=\frac{z}{1}-\frac{z^2}{2}+\frac{z^3}{3}-\cdots=\sum_{n\geq 1}\frac{(-1)^{n-1}z^n}{n}$$

$$\begin{aligned}\operatorname{Exp}(A(z)) &= \exp\left(\sum_{n\geq 1} -a_n \cdot \ln(1-z^n)\right)\\ &= \exp\left(\sum_{n\geq 1} -a_n \cdot \sum_{m\geq 1} \frac{-z^{nm}}{m}\right)\\ &= \exp\left(\frac{A(z)}{1} + \frac{A(z^2)}{2} + \frac{A(z^3)}{3} + \cdots\right)\end{aligned}$$

$$\mathrm{MSET}(\mathcal{I})$$

$$\mathcal{T} = \{\bullet\} \times \mathrm{MSET}(\mathcal{T})$$

$$T(z) - \frac{1}{2}T^2(z) + \frac{1}{2}T(z^2)$$

$$\begin{aligned}\mathrm{PSET}(\{a\}) &= \{\epsilon\} + \{a\}\\ \mathrm{PSET}(\{a,b\}) &= \{\epsilon\} + \{a\} + \{b\} + \{(a,b)\}\\ \mathrm{PSET}(\{a,b,c\}) &= \{\epsilon\} + \{a\} + \{b\} + \{(a,b)\} + \{c\} + \{(a,c)\} + \{(b,c)\} + \{(a,b,c)\}\end{aligned}$$

$$\mathrm{PSET}(\{\alpha_0, \alpha_1, ..., \alpha_n\}) = \mathrm{PSET}(\{\alpha_0, \alpha_1, ..., \alpha_{n-1}\}) \times (\{\epsilon\} + \{\alpha_n\})$$

$$\mathrm{PSET}(\mathcal{A}) \cong \prod_{\alpha\in\mathcal{A}} (\{\epsilon\} + \{\alpha\})$$

$$\begin{aligned}\overline{\operatorname{Exp}}(A(z)) &= \prod_{\alpha\in\mathcal{A}} \left(1 + z^{\mid \alpha \mid}\right) = \prod_{n\geq 1} (1+z^n)^{a_n}\\ &= \exp\left(\sum_{n\geq 1} a_n \cdot \ln(1+z^n)\right)\\ &= \exp\left(\sum_{n\geq 1} a_n \cdot \sum_{m\geq 1} \frac{(-1)^{m-1} z^{nm}}{m}\right)\\ &= \exp\left(\frac{A(z)}{1} - \frac{A(z^2)}{2} + \frac{A(z^3)}{3} - \cdots\right)\end{aligned}$$

$$\mathrm{CYC}(\mathcal{A}) = (\mathrm{SEQ}(\mathcal{A}) \setminus \{\epsilon\})/\mathbf{S}$$

$$\mathrm{CYC}(\{\mathtt{a}, \mathtt{b}\})_3 = \{\mathtt{aaa}\} + \{\mathtt{aab}\} + \{\mathtt{abb}\} + \{\mathtt{bbb}\}$$

$$\mathrm{CYC}(\{\mathtt{a}, \mathtt{b}\})_4 = \{\mathtt{aaaa}\} + \{\mathtt{aaab}\} + \{\mathtt{aabb}\} + \{\mathtt{abbb}\} + \{\mathtt{bbbb}\} + \{\mathtt{abab}\}$$

$$\operatorname{Log}(A(z)) = \sum_{n\geq 1} \frac{\varphi(n)}{n} \ln \frac{1}{1-A(z^n)}$$

$$\mathrm{SEQ}_{=k}(\mathcal{B}), \quad \mathrm{SEQ}_{\geq k}(\mathcal{B}), \quad \mathrm{SEQ}_{1..k}(\mathcal{B})$$

$$\mathcal{A} = \mathfrak{K}_k(\mathcal{B})$$

$$\alpha = \{(\beta_1, \beta_2, ..., \beta_k) \mid \beta \in \mathcal{B}\}$$

$$A_{n,k} = \mathrm{card}\{\alpha \in \mathcal{A} \mid \mid \alpha \mid = n, \chi(\alpha) = k\}$$

$$A(z,u)=\sum_{n,k}A_{n,k}u^kz^n=\sum_{\alpha\in\mathcal{A}}z^{|\ \alpha\ |}u^{\chi(\alpha)}$$

$$A(z,u)=\sum_{k\geq 0}u^kB(z)^k=\frac{1}{1-uB(z)}$$
$$\implies A(z)=B(z)^k$$

$$\mathcal{A}=\mathrm{SEQ}_{\geq k}(\mathcal{B})\implies A(z)=\frac{B(z)^k}{1-B(z)}$$

$$A(z,u)=\prod_n(1-uz^n)^{-b_n}$$
$$\implies A(z)=[u^k]\exp\left(\frac{u}{1}B(z)+\frac{u^2}{2}B(z^2)+\frac{u^3}{3}B(z^3)+\cdots\right)$$

$$A(z,u)=\prod_n(1+uz^n)^{b_n}$$
$$\implies A(z)=[u^k]\exp\left(\frac{u}{1}B(z)-\frac{u^2}{2}B(z^2)+\frac{u^3}{3}B(z^3)-\cdots\right)$$

$$[u^3]A(z,u)=\frac{1}{0!}([u^3]1)+\frac{1}{1!}\left([u^3]\left(\frac{u}{1}B(z)+\frac{u^2}{2}B(z^2)+\frac{u^3}{3}B(z^3)+\cdots\right)\right)$$
$$+\frac{1}{2!}\left([u^3]\left(\frac{u}{1}B(z)+\frac{u^2}{2}B(z^2)+\cdots\right)^2\right)$$
$$+\frac{1}{3!}\left([u^3]\left(\frac{u}{1}B(z)+\cdots\right)^3\right)$$
$$=\frac{B(z)^3}{6}+\frac{B(z)B(z^2)}{2}+\frac{B(z)^3}{3}$$

$$[u^4]A(z,u)=\frac{1}{0!}([u^4]1)+\frac{1}{1!}\left([u^4]\left(\frac{u}{1}B(z)+\frac{u^2}{2}B(z^2)+\frac{u^3}{3}B(z^3)+\frac{u^4}{4}B(z^4)+\cdots\right)\right)$$
$$+\frac{1}{2!}\left([u^4]\left(\frac{u}{1}B(z)+\frac{u^2}{2}B(z^2)+\frac{u^3}{3}B(z^3)+\cdots\right)^2\right)$$
$$+\frac{1}{3!}\left([u^4]\left(\frac{u}{1}B(z)+\frac{u^2}{2}B(z^2)+\cdots\right)^3\right)$$
$$+\frac{1}{4!}\left([u^4]\left(\frac{u}{1}B(z)+\cdots\right)^4\right)$$
$$=\frac{B(z^4)}{4}+\frac{1}{2!}\left(\frac{B(z^2)^2}{4}+\frac{2B(z)B(z^3)}{3}\right)+\frac{1}{3!}\left(\frac{3B(z)^2B(z^2)}{2}\right)+\frac{B(z)^4}{4!}$$
$$=\frac{B(z)^4}{24}+\frac{B(z)^2B(z^2)}{4}+\frac{B(z)B(z^3)}{3}+\frac{B(z^2)^2}{8}+\frac{B(z^4)}{4}$$

$$\mathrm{PSET}_2(\mathcal{A}): \quad \frac{A(z)^2}{2} - \frac{A(z^2)}{2}$$

$$\mathrm{MSET}_2(\mathcal{A}): \quad \frac{A(z)^2}{2} + \frac{A(z^2)}{2}$$

$$\mathrm{CYC}_2(\mathcal{A}): \quad \frac{A(z)^2}{2} + \frac{A(z^2)}{2}$$

$$\mathrm{PSET}_3(\mathcal{A}): \quad \frac{A(z)^3}{6} - \frac{A(z)A(z^2)}{2} + \frac{A(z^3)}{3}$$

$$\mathrm{MSET}_3(\mathcal{A}): \quad \frac{A(z)^3}{6} + \frac{A(z)A(z^2)}{2} + \frac{A(z^3)}{3}$$

$$\mathrm{CYC}_3(\mathcal{A}): \quad \frac{A(z)^3}{3} + \frac{2A(z^3)}{3}$$

$$\mathrm{PSET}_4(\mathcal{A}): \quad \frac{A(z)^4}{24} - \frac{A(z)^2A(z^2)}{4} + \frac{A(z)A(z^3)}{3} + \frac{A(z^2)^2}{8} - \frac{A(z^4)}{4}$$

$$\mathrm{MSET}_4(\mathcal{A}): \quad \frac{A(z)^4}{24} + \frac{A(z)^2A(z^2)}{4} + \frac{A(z)A(z^3)}{3} + \frac{A(z^2)^2}{8} + \frac{A(z^4)}{4}$$

$$\mathrm{CYC}_4(\mathcal{A}): \quad \frac{A(z)^4}{4} + \frac{A(z^2)^2}{4} + \frac{A(z^4)}{2}$$

$$\mathcal{T} = \{\bullet\} \times \mathrm{MSET}_{0,1,2,3}(\mathcal{T})$$

$$\hat{\mathcal{T}} = \{\epsilon\} + \{\bullet\} \times \mathrm{MSET}_3(\hat{\mathcal{T}})$$

$$(\mathcal{A}, |\cdot|)$$

$$\mathcal{A}$$

$$|\cdot|$$

$$\{0,1\}$$

$$0$$

$$\mathcal{A}$$

$$a_n$$

$$[z^n]A(z)$$

$$A(z)$$

$$z^n$$

$$\mathcal{A}_n$$

$$\mathcal{A}$$

$$n$$

$$a_n = \mathrm{card}(\mathcal{A}_n)$$

$$\mathrm{card}$$

$$\epsilon$$

$$\mathcal{E} = \{\epsilon\}$$

$$0$$

$$E(z) = 1$$

$$\circ$$

$$\bullet$$

$$\mathcal{Z}_\circ = \{\circ\}$$

$$\mathcal{Z}_\bullet = \{\bullet\}$$

$$\mathcal{Z}$$

$$1$$

$$Z(z) = z$$

$$\mathcal{A}$$

$$\mathcal{B}$$

$$\mathcal{A} = \mathcal{B}$$

$$\mathcal{A} \cong \mathcal{B}$$

$$\times$$

$$\mathcal{A}$$

$$\mathcal{B}$$

$$\mathcal{A}$$

$$\mathcal{B}$$

$$\mathcal{A}$$

$$\mathcal{B}$$

$$(\alpha, \beta)$$

$$\{(a, b)\}, \{(b, a)\}$$

$$\mathcal{A}_0 = \emptyset$$

$$\mathcal{A}$$

$$0$$

$$Q$$

$$\mathcal{T}$$

$$\{(b, a)\}, \{(a, b, a)\}$$

$$\mathrm{SEQ}(\{a, b\})$$

$$\mathrm{MSET}(\{a, b\})$$

$$\mathcal{A}_0 = \emptyset$$

$$\mathbf{R}$$

$$(\alpha_1, ..., \alpha_n)\mathbf{R}(\beta_1, ..., \beta_n)$$

$$\sigma$$

$$j$$

$$\beta_j = \alpha_{\sigma(j)}$$

$$A(z) = \exp(\ln(A(z)))$$

$$\mathrm{Exp}$$

$$f(n)$$

$$n$$

$$f(1), f(2), ..., f(10^5)$$

$$998244353$$

$$\mathcal{I}$$

$$\mathcal{I} = \mathrm{SEQ}_{\geq 1}(\mathcal{Z}) = \mathcal{Z} \times \mathrm{SEQ}(\mathcal{Z})$$

$$\geq 1$$

$$n$$

$$v_1, ..., v_n$$

$$m$$

$$1, 2, ..., m$$

$$998244353$$

$$1 \leq n, m \leq 10^5$$

$$1 \leq v_i \leq m$$

$$\mathcal{A}$$

$$\mathrm{MSET}(\mathcal{A})$$

$$n$$

$$998244353$$

$$1 \leq n \leq 2 \times 10^5$$

$$\mathcal{T}$$

$$\mathcal{A}_0 = \emptyset$$

$$\overline{\mathrm{Exp}}$$

$$\mathrm{PSET}(\mathcal{A}) \subset \mathrm{MSET}(\mathcal{A})$$

$$\mathbf{S}$$

$$(\alpha_1, ..., \alpha_n)\mathbf{S}(\beta_1, ..., \beta_n)$$

$$\tau$$

$$j$$

$$\beta_j = \alpha_{\tau(j)}$$

$$\mathtt{a}, \mathtt{b}$$

$$1$$

$$3$$

$$4$$

$$\mathtt{aabSbaaSaba}$$

$$\mathtt{abbSbabSbba}$$

$$\mathtt{aaabSbaaaSabaaSaaba}$$

$$\mathtt{aabbSbaabSbbaaSabba}$$

$$\mathtt{abbbSbabbSbbabSbbba}$$

$$\mathtt{ababSbaba}$$

$$\varphi$$

$$\mathrm{Log}$$

$$\mathrm{SEQ}$$

$$\mathrm{SEQ}_{=k}(\mathcal{B})$$

$$\mathrm{SEQ}_k(\mathcal{B})$$

$$\mathrm{SEQ}_{1..k}(\mathcal{B})$$

$$[1..k]$$

$$\mathfrak{K}$$

$$\mathrm{SEQ}, \mathrm{PSET}, \mathrm{MSET}, \mathrm{CYC}$$

$$\alpha \in \mathcal{A}$$

$$\chi$$

$$\chi(\alpha) = k$$

$$u^k$$

$$\mathcal{A} = \mathrm{SEQ}_k(\mathcal{B})$$

$$\mathrm{MSET}_k(\mathcal{B})$$

$$\mathrm{PSET}_k(\mathcal{B})$$

$$\mathrm{CYC}_k(\mathcal{B})$$

$$\mathrm{MSET}_3(\mathcal{B})$$

$$\mathrm{MSET}_4(\mathcal{B})$$

$$\mathcal{A} = \mathrm{MSET}_3(\mathcal{B})$$

$$\mathcal{A} = \mathrm{MSET}_4(\mathcal{B})$$

$$\mathcal{A} = \mathfrak{K}_k(\mathcal{B})$$

$$A(z)$$

$$B(z), B(z^2), ..., B(z^k)$$

$$\mathfrak{K}_k(\mathcal{B})$$

$$\mathcal{B}_0 = \emptyset$$

$$n$$

$$3$$

$$4$$

$$998244353$$

$$1 \le n \le 10^5$$

$$\mathcal{T}$$

$$\hat{\mathcal{T}} = \mathcal{T} + \{\epsilon\}$$

$$f(x) = f_0(x) + x^{\lfloor n/2 \rfloor} f_1(x)$$

$$f(x+c) = f_0(x+c) + (x+c)^{\lfloor n/2 \rfloor} f_1(x+c)$$

$$T(n) = 2T(n/2) + O(n \log n) = O(n \log^2 n)$$

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$f(x+c) = f(c) + \frac{f'(c)}{1!}x + \frac{f''(c)}{2!}x^2 + \cdots + \frac{f^{(n)}(c)}{n!}x^n$$

$$\begin{aligned} t![x^t]f(x+c) &= f^{(t)}(c) \\ &= \sum_{i=t}^{n} f_i i! \frac{c^{i-t}}{(i-t)!} \\ &= \sum_{i=0}^{n-t} f_{i+t}(i+t)! \frac{c^i}{i!} \end{aligned}$$

$$A_0(x) = \sum_{i=0}^{n} f_{n-i}(n-i)! x^i$$

$$B_0(x) = \sum_{i=0}^{n} \frac{c^i}{i!} x^i$$

$$
\begin{aligned}
[x^{n-t}](A_0(x)B_0(x)) &= \sum_{i=0}^{n-t}([x^{n-t-i}]A_0(x))([x^i]B_0(x)) \\
&= \sum_{i=0}^{n-t} f_{i+t}(i+t)!\frac{c^i}{i!} \\
&= t![x^t]f(x+c)
\end{aligned}
$$

$$
\begin{aligned}
f(x+c) &= \sum_{i=0}^{n} f_i(x+c)^i \\
&= \sum_{i=0}^{n} f_i\left(\sum_{j=0}^{i}\binom{i}{j}x^j c^{i-j}\right) \\
&= \sum_{i=0}^{n} f_i i!\left(\sum_{j=0}^{i}\frac{x^j}{j!}\frac{c^{i-j}}{(i-j)!}\right) \\
&= \sum_{i=0}^{n} \frac{x^i}{i!}\left(\sum_{j=i}^{n} f_j j!\frac{c^{j-i}}{(j-i)!}\right)
\end{aligned}
$$

$$
\begin{aligned}
f(x) &= \sum_{0\le i\le n} f(i)\prod_{0\le j\le n\,\wedge\, j\ne i}\frac{x-j}{i-j} \\
&= \sum_{0\le i\le n} f(i)\frac{x!}{(x-n-1)!(x-i)}\frac{(-1)^{n-i}}{i!(n-i)!} \\
&= \frac{x!}{(x-n-1)!}\sum_{0\le i\le n}\frac{f(i)}{(x-i)}\frac{(-1)^{n-i}}{i!(n-i)!}
\end{aligned}
$$

$$
\begin{aligned}
A_0(x) &= \sum_{0\le i\le n}\frac{f(i)(-1)^{n-i}}{i!(n-i)!}x^i \\
B_0(x) &= \sum_{i\ge 0}\frac{1}{c-n+i}x^i
\end{aligned}
$$

$$
\begin{aligned}
[x^{n+t}](A_0(x)B_0(x)) &= \sum_{i=0}^{n+t}([x^i]A_0(x))([x^{n+t-i}]B_0(x)) \\
&= \sum_{i=0}^{n}\frac{f(i)(-1)^{n-i}}{i!(n-i)!}\frac{1}{c+t-i} \\
&= \frac{(c+t-n-1)!}{(c+t)!}f(c+t)
\end{aligned}
$$

$$
x^{\overline{n}} = \sum_{i=0}^{n}\begin{bmatrix}n\\i\end{bmatrix}x^i, \quad n\ge 0
$$

$$
f_{2n}(x) = x^{\overline{n}}\cdot(x+n)^{\overline{n}} = f_n(x)f_n(x+n)
$$

$$
T(n) = T(n/2) + O(n\log n) = O(n\log n)
$$

$$
n! \equiv \left(\prod_{i=0}^{v-1} g(iv)\right)\cdot\prod_{i=v^2+1}^{n} i \pmod p
$$

$$g_{2d}(x) = g_d(x)g_d(x+d)$$

$$g_{d+1}(x) = (x+d+1)\cdot g_d(x)$$

$$T(n) = T(n/2) + O(n\log n) = O(n\log n)$$

$$[n!] = \left(\prod_{i=0}^{n-1}[i+1]\right)[1]$$

$$\begin{bmatrix}\binom{n}{m+1}\\ \sum_{i=0}^{m}\binom{n}{i}\end{bmatrix} = \begin{bmatrix}(n-m)/(m+1) & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}\binom{n}{m}\\ \sum_{i=0}^{m-1}\binom{n}{i}\end{bmatrix}$$

$$\begin{aligned}\begin{bmatrix}\binom{n}{m+1}\\ \sum_{i=0}^{m}\binom{n}{i}\end{bmatrix} &= \left(\prod_{i=0}^{m}\begin{bmatrix}(n-i)/(i+1) & 0\\ 1 & 1\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}\\ &= \frac{1}{(m+1)!}\left(\prod_{i=0}^{m}\begin{bmatrix}n-i & 0\\ i+1 & i+1\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}\end{aligned}$$

$$\begin{aligned}M_d(x) &= \prod_{i=1}^{d}\begin{bmatrix}-x+n+1-i & 0\\ x+i & x+i\end{bmatrix}\\ &= \begin{bmatrix}f_d(x) & 0\\ g_d(x) & h_d(x)\end{bmatrix}\end{aligned}$$

$$\begin{aligned}M_{2d}(x) &= \prod_{i=1}^{2d}\begin{bmatrix}-x+n+1-i & 0\\ x+i & x+i\end{bmatrix}\\ &= \left(\prod_{i=1}^{d}\begin{bmatrix}-x-d+n+1-i & 0\\ x+d+i & x+d+i\end{bmatrix}\right)\left(\prod_{i=1}^{d}\begin{bmatrix}-x+n+1-i & 0\\ x+i & x+i\end{bmatrix}\right)\\ &= \begin{bmatrix}f_d(x+d) & 0\\ g_d(x+d) & h_d(x+d)\end{bmatrix}\begin{bmatrix}f_d(x) & 0\\ g_d(x) & h_d(x)\end{bmatrix}\\ &= \begin{bmatrix}f_d(x+d)f_d(x) & 0\\ g_d(x+d)f_d(x)+h_d(x+d)g_d(x) & h_d(x+d)h_d(x)\end{bmatrix}\end{aligned}$$

$$\prod_{i=0}^{m}\begin{bmatrix}n-i & 0\\ i+1 & i+1\end{bmatrix} = \left(\prod_{i=(k+1)v}^{m}\begin{bmatrix}n-i & 0\\ i+1 & i+1\end{bmatrix}\right)\begin{bmatrix}f_v(kv) & 0\\ g_v(kv) & h_v(kv)\end{bmatrix}\cdots\begin{bmatrix}f_v(0) & 0\\ g_v(0) & h_v(0)\end{bmatrix}$$

$$\begin{bmatrix}(n+1)!\\ (n+1)!H_{n+1}\end{bmatrix} = \begin{bmatrix}n+1 & 0\\ 1 & n+1\end{bmatrix}\begin{bmatrix}n!\\ n!H_n\end{bmatrix}$$

$$\begin{bmatrix}\begin{bmatrix}n+1\\1\end{bmatrix}\\ \begin{bmatrix}n+1\\2\end{bmatrix}\end{bmatrix} = \begin{bmatrix}n!\\ n!H_n\end{bmatrix} = \left(\prod_{i=0}^{n-1}\begin{bmatrix}i+1 & 0\\ 1 & i+1\end{bmatrix}\right)\begin{bmatrix}1\\0\end{bmatrix}$$

$$-\frac{1}{P_0(n)}\begin{bmatrix} P_1(n) & P_2(n) & P_3(n) & \cdots & P_{m-1}(n) & P_m(n) \\ -P_0(n) & & & & & \\ & -P_0(n) & & & & \\ & & -P_0(n) & & & \\ & & & \ddots & & \\ & & & & -P_0(n) & \end{bmatrix}\begin{bmatrix} a_{n-1} \\ a_{n-2} \\ a_{n-3} \\ \vdots \\ a_{n-m+1} \\ a_{n-m} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n-1} \\ a_{n-2} \\ \vdots \\ a_{n-m+2} \\ a_{n-m+1} \end{bmatrix}$$

$$B(\lambda) = \begin{bmatrix} P_1(\lambda) & P_2(\lambda) & P_3(\lambda) & \cdots & P_{m-1}(\lambda) & P_m(\lambda) \\ -P_0(\lambda) & & & & & \\ & -P_0(\lambda) & & & & \\ & & -P_0(\lambda) & & & \\ & & & \ddots & & \\ & & & & -P_0(\lambda) & \end{bmatrix}$$

$$f(x) = \sum_{i=0}^{n} f_i x^i$$

$$c$$

$$f(x+c)$$

$$f(x) \mapsto f(x+c)$$

$$(x+c)^{\lfloor n/2 \rfloor}$$

$$O(n \log n)$$

$$f(x)$$

$$c$$

$$t \geq 0$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$$

$$n$$

$$f$$

$$f(0), f(1), ..., f(n)$$

$$998244353$$

$$f(c), f(c+1), ..., f(c+n)$$

$$1 \leq n \leq 10^5, n < m \leq 10^8$$

$$x - i \neq 0$$

$$c > n$$

$$B_0(x)$$

$$t \geq 0$$

$$B_0(x)$$

$$d$$

$$f(d), f(d+k), ..., f(d+nk)$$

$$f(c+d), f(c+d+k), ..., f(c+d+nk)$$

$$g(x) = f(d+kx)$$

$$g(0), g(1), ..., g(n)$$

$$g(c/k), g(c/k+1), ..., g(c/k+n)$$

$$167772161$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}, \begin{bmatrix} n \\ 1 \end{bmatrix}, ..., \begin{bmatrix} n \\ n \end{bmatrix}$$

$$1 \le n < 262144$$

$$x^{\overline{n}} = x \cdot (x+1) \cdots (x+n-1)$$

$$f_n(x) = x^{\overline{n}}$$

$$O(n \log n)$$

$$f_n(x+n)$$

$$f_n(x)$$

$$n! \bmod p$$

$$p$$

$$1 \le n < p \le 2^{31} - 1$$

$$v = \lfloor \sqrt{n} \rfloor$$

$$g(x) = \prod_{i=1}^{v} (x+i)$$

$$\prod_{i=v^2+1}^{n} i$$

$$O(\sqrt{n})$$

$$g(x)$$

$$O(n \log n)$$

$$g(0), g(v), g(2v), ..., g(v^2 - v)$$

$$O(\sqrt{n} \log^2 n)$$

$$g_d(x) = \prod_{i=1}^{d} (x+i)$$

$$d+1$$

$$g_d(0), g_d(v), ..., g_d(dv)$$

$$d$$

$$2d+1$$

$$g_{2d}(x)$$

$$g_d((d+1)v), g_d((d+2)v), ..., g_d(2dv)$$

$$h(x) = g_d(vx)$$

$$h(0), h(1), ..., h(d)$$

$$h(d+1), h(d+2), ..., h(2d)$$

$$g_d(d), g_d(v+d), ..., g_d(2dv+d)$$

$$h(x) = g_d(vx)$$

$$h(0), h(1), ..., h(d)$$

$$h(d/v), h(d/v+1), h(d/v+2), ..., h(d/v+2d)$$

$$g_{2d}(0), g_{2d}(v), ..., g_{2d}(2dv)$$

$$g_d(0), g_d(v), ..., g_d(dv)$$

$$g_{d+1}(0), g_{d+1}(v), ..., g_{d+1}(dv), g_{d+1}((d+1)v)$$

$$g_1(0) = 1, g_1(v) = v+1$$

$$\sqrt{n}$$

$$O(\sqrt{n}\log n)$$

$$\sum_{i=0}^{m}\binom{n}{i} \bmod 998244353$$

$$0 \le m \le n \le 9 \times 10^8$$

$$n! = n \cdot (n-1)!$$

$$v = \lfloor \sqrt{m} \rfloor$$

$$M_d(0), M_d(v), ..., M_d(dv)$$

$$f_d(0), f_d(v), ..., f_d(dv)$$

$$h_d(0), ..., h_d(dv)$$

$$g_d(0), ..., g_d(dv)$$

$$O(\sqrt{m}\log m)$$

$$\sum_{i=1}^{n} i^{-1} \bmod p$$

$$p$$

$$1 \leq n < p < 2^{30}$$

$$H_n = \sum_{k=1}^{n} k^{-1}$$

$$\begin{bmatrix} n+1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}$$

$$a$$

$$\forall n \geq m, \sum_{k=0}^{m} a_{n-k} P_k(n) = 0$$

$$P_k$$

$$d$$

$$P_k$$

$$a_0, a_1, ..., a_{m-1}$$

$$a_n$$

$$998244353$$

$$n \leq 6 \times 10^8$$

$$1 \leq m, d \leq 7$$

$$7s$$

$$\lambda$$

$$n!$$

$$\prod_{i=0}^{T-1} (aT + i)$$

$$m$$

$$1$$

$$m$$

$$m \times m$$

$$-\frac{1}{P_0(n)}$$

$$\prod_{i=0}^{T-1} B(aT + m + i)$$

$$B_T(\lambda) = \prod_{i=0}^{T-1} B(\lambda + i)$$

$$dT$$

$$\lambda$$

$$dT + 1$$

$$B_T(m)$$

$$B_T(m + T)$$

$$B_T(m + 2T)$$

$$\dots$$

$$B_T(m + (dT - 1)T)$$

$$B_T(m + dT^2)$$

$$\lambda$$

$$t = \log_2 T$$

$$1$$

$$O(m^2 dT \log(dT))$$

$$B_T(p + dT^2)$$

$$B_T(p + (dT + 1)T)$$

$$B_T(p + (dT + 2)T)$$

$$\dots$$

$$B_T(p + (2dT - 1)T)$$

$$B_T(p + (2dT)dT)$$

$$O(m^2 dT \log(dT))$$

$$B_T(p + 2dT^2)$$

$$B_T(p + (2dT + 1)T)$$

$$B_T(p + (2dT + 2)T)$$

$$\dots$$

$$B_T(p + (3dT - 1)T)$$

$$B_T(p + (3dT)dT)$$

$$O(m^2 dT \log(dT))$$

$$B_T(p + 3dT^2)$$

$$B_T(p + (3dT + 1)T)$$

$$B_T(p+(3dT+2)T)$$

$$\ldots$$

$$B_T(p+(4dT-1)T)$$

$$B_T(p+(4dT)dT)$$

$$B_{2T}(v)=B_T(v+T)B_T(v)$$

$$O(m^2 dT\log(dT))$$

$$\Theta(dT)$$

$$O(m^3 dT)$$

$$T\geq\sqrt{n/d}$$

$$-\frac{1}{P_0(n)}$$

$$\Theta(\sqrt{nd}(m^3+m^2\log(nd)))$$

$$\Theta(n/T)$$

$$O(T)$$

$$\Theta(\sqrt{nd}(m^3+m^2\log(nd)))$$

$$m,d$$

$$\Theta(\sqrt{n}\log n)$$

$$F(x)=\sum_n a_n x^n$$

$$F(x)\pm G(x)=\sum_n (a_n\pm b_n)x^n$$

$$F(x)G(x)=\sum_n x^n\sum_{i=0}^{n} a_i b_{n-i}$$

$$F(x)x+1=F(x)$$

$$F(x)=\frac{1}{1-x}$$

$$F(x)px+1=F(x)$$

$$F(x)=\frac{1}{1-px}$$

$$F(x)=\sum_{n\geq1} x^n=\frac{x}{1-x}$$

$$
\begin{aligned}
F(x) &= \sum_{n\geq 0} x^{2n} \\
&= \sum_{n\geq 0} (x^2)^n \\
&= \frac{1}{1-x^2}
\end{aligned}
$$

$$
\begin{aligned}
F(x) &= \sum_{n\geq 0} (n+1)x^n \\
&= \sum_{n\geq 1} nx^{n-1} \\
&= \sum_{n\geq 0} (x^n)' \\
&= \left(\frac{1}{1-x}\right)' \\
&= \frac{1}{(1-x)^2}
\end{aligned}
$$

$$F(x) = \sum_{n\geq 0} \binom{m}{n} x^n = (1+x)^m$$

$$F(x) = \sum_{n\geq 0} \binom{m+n}{n} x^n = \frac{1}{(1-x)^{m+1}}$$

$$
\begin{aligned}
\frac{1}{(1-x)^{m+1}} &= \frac{1}{(1-x)^m}\frac{1}{1-x} \\
&= \left(\sum_{n\geq 0} \binom{m+n-1}{n} x^n\right)\left(\sum_{n\geq 0} x^n\right) \\
&= \sum_{n\geq 0} x^n \sum_{i=0}^{n} \binom{m+i-1}{i} \\
&= \sum_{n\geq 0} \binom{m+n}{n} x^n
\end{aligned}
$$

$$F(x) = xF(x) + x^2F(x) - a_0x + a_1x + a_0$$

$$F(x) = \frac{x}{1-x-x^2}$$

$$
\begin{aligned}
F(x) &= \frac{x}{1-(x+x^2)} \\
&= \sum_{n\geq 0}(x+x^2)^n \\
&= \sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}x^{2i}x^{n-i} \\
&= \sum_{n\geq 0}\sum_{i=0}^{n}\binom{n}{i}x^{n+i} \\
&= \sum_{n\geq 0}x^n\sum_{i=0}^{n}\binom{n-i}{i}
\end{aligned}
$$

$$\frac{A}{1-ax}+\frac{B}{1-bx}=\frac{x}{1-x-x^2}$$

$$\frac{A-Abx+B-aBx}{(1-ax)(1-bx)}=\frac{x}{1-x-x^2}$$

$$\begin{cases} A+B=0 \\ -Ab-aB=1 \\ a+b=1 \\ ab=-1 \end{cases}$$

$$\begin{cases} A=\frac{1}{\sqrt{5}} \\ B=-\frac{1}{\sqrt{5}} \\ a=\frac{1+\sqrt{5}}{2} \\ b=\frac{1-\sqrt{5}}{2} \end{cases}$$

$$\frac{x}{1-x-x^2}=\sum_{n\geq 0}x^n\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

$$G(x)=\frac{1}{(1-x)(1-2x)^2}$$

$$G(x)=\frac{c_0}{1-x}+\frac{c_1}{1-2x}+\frac{c_2}{(1-2x)^2}$$

$$\begin{cases} c_0=1 \\ c_1=-2 \\ c_2=2 \end{cases}$$

$$[x^n]G(x)=1-2^{n+1}+(n+1)\cdot 2^{n+1}$$

$$\binom{r}{k}=\frac{r^{\underline{k}}}{k!}\quad (r\in\mathbf{C},k\in\mathbf{N})$$

$$(1+x)^\alpha=\sum_{n\geq 0}\binom{\alpha}{n}x^n$$

$$F(x) = \frac{(1+x)(1-x^3)x(1-x^4)(1+x)}{(1-x^2)(1-x)(1-x^2)(1-x^4)(1-x)(1-x^3)} = \frac{x}{(1-x)^4}$$

$$F(x) = \sum_{n\ge 1}\binom{n+2}{n-1}x^n$$

$$F_i(x) = \sum_{j=0}^{m_i} x^j = \frac{1-x^{m_i+1}}{1-x}$$

$$G(x) = \prod_{i=1}^{n} F_i(x) = (1-x)^{-n}\prod_{i=1}^{n}(1-x^{m_i+1})$$

$$\begin{aligned}(1-x)^{-n} &= \sum_{i\ge 0}\binom{-n}{i}(-x)^i \\ &= \sum_{i\ge 0}\binom{n-1+i}{i}x^i\end{aligned}$$

$$c_k\sum_{i=a-k}^{b-k}\binom{n-1+i}{i} = c_k\left(\binom{n+b-k}{b-k} - \binom{n+a-k-1}{a-k-1}\right)$$

$$a$$

$$a$$

$$a$$

$$0$$

$$a = \langle 1,2,3\rangle$$

$$1+2x+3x^2$$

$$a = \langle 1,1,1,\cdots\rangle$$

$$\sum_{n\ge 0}x^n$$

$$a = \langle 1,2,4,8,16,\cdots\rangle$$

$$\sum_{n\ge 0}2^n x^n$$

$$a = \langle 1,3,5,7,9,\cdots\rangle$$

$$\sum_{n\ge 0}(2n+1)x^n$$

$$a$$

$$a,b$$

$$F(x),G(x)$$

$$F(x)\pm G(x)$$

$$\langle a_n \pm b_n\rangle$$

$$F(x)G(x)$$

$$\langle\sum_{i=0}^{n} a_i b_{n-i}\rangle$$

$$\langle 1, 1, 1, \cdots\rangle$$

$$F(x) = \sum_{n\geq 0} x^n$$

$$\sum_{n\geq 0} x^n$$

$$\langle 1, p, p^2, p^3, p^4, \cdots\rangle$$

$$F(x) = \sum_{n\geq 0} p^n x^n$$

$$a = \langle 0, 1, 1, 1, 1, \cdots\rangle$$

$$a = \langle 1, 0, 1, 0, 1, \cdots\rangle$$

$$a = \langle 1, 2, 3, 4, \cdots\rangle$$

$$a_n = \binom{m}{n}$$

$$m$$

$$n \geq 0$$

$$a_n = \binom{m+n}{n}$$

$$m$$

$$n \geq 0$$

$$m = 0$$

$$F(x) = \frac{1}{1-x}$$

$$m > 0$$

$$a_0 = 0, a_1 = 1, a_n = a_{n-1} + a_{n-2}\ (n > 1)$$

$$F(x)$$

$$F(x)$$

$$x + x^2$$

$$a_n$$

$$\frac{x}{1-x-x^2}$$

$$P(x), Q(x)$$

$$\frac{P(x)}{Q(x)}$$

$$Q(x)$$

$$\prod(1-p_ix)^{d_i}$$

$$r$$

$$\alpha \in \mathbf{C}$$

$$n$$

$$n$$

$$n$$

$$10007$$

$$a_n$$

$$n$$

$$n$$

$$n$$

$$\sum_{n\geq 0} x^{2n} = \frac{1}{1-x^2}$$

$$1+x$$

$$1+x+x^2 = \frac{1-x^3}{1-x}$$

$$\frac{x}{1-x^2}$$

$$\sum_{n\geq 0} x^{4n} = \frac{1}{1-x^4}$$

$$1+x+x^2+x^3 = \frac{1-x^4}{1-x}$$

$$1+x$$

$$\frac{1}{1-x^3}$$

$$\binom{n+2}{n-1} = \binom{n+2}{3}$$

$$n$$

$$i$$

$$m_i$$

$$a$$

$$b$$

$$n \leq 10, 0 \leq a \leq b \leq 10^7, m_i \leq 10^6$$

$$i$$

$$j$$

$$i$$

$$\sum_{i=a}^{b} [x^i] G(x)$$

$$n \leq 10$$

$$\prod_{i=1}^{n} (1 - x^{m_i+1})$$

$$2^n$$

$$(1-x)^{-n}$$

$$\prod_{i=1}^{n} (1 - x^{m_i+1})$$

$$x^k$$

$$c_k$$

$$(1-x)^{-n}$$

$$O(b)$$

$$O(2^n + b)$$

$$p = 1004535809 = 479 \times 2^{21} + 1, g = 3$$

$$p = 998244353 = 7 \times 17 \times 2^{23} + 1, g = 3$$

$$F = \mathbb{Z}/p$$

$$p$$

$$n$$

$$p-1$$

$$n$$

$$p = \xi n + 1$$

$$\xi$$

$$p = qn + 1, (n = 2^m)$$

$$g$$

$$g^{qn} \equiv 1 \pmod p$$

$$g_n = g^q \pmod p$$

$$\omega_n$$

$$g_n^n \equiv 1 \pmod p, g_n^{n/2} \equiv -1 \pmod p$$

$$N$$

$$n$$

$$n$$

$$N$$

$$n$$

$$\frac{qN}{n}$$

$$q$$

$$g^{qn}$$

$$\mathrm{e}^{2\pi n}$$

$$l$$

$$g_l = g^{\frac{p-1}{l}}$$

$$\omega_n = g_l = g_N^{\frac{N}{l}} = g_N^{\frac{p-1}{l}}$$

$$2$$

$$p$$

$$0$$

$$p-1$$

$$p$$

$$2$$

$$1$$

$$2$$

$$4$$

$$p$$

$$\alpha$$

$$p^\alpha$$

$$2p^\alpha$$

$$g$$

$$g(f(x)) \equiv 0 \pmod{x^n}$$

$$\sum_{i=0}^{+\infty} \frac{g^{(i)}(f_0(x))}{i!}(f(x)-f_0(x))^i \equiv 0 \pmod{x^n}$$

$$\forall 2 \leqslant i : (f(x)-f_0(x))^i \equiv 0 \pmod{x^n}$$

$$\sum_{i=0}^{+\infty}\frac{g^{(i)}(f_0(x))}{i!}(f(x)-f_0(x))^i \equiv g(f_0(x))+g'(f_0(x))(f(x)-f_0(x)) \equiv 0 \pmod{x^n}$$

$$f(x) \equiv f_0(x)-\frac{g(f_0(x))}{g'(f_0(x))} \pmod{x^n}$$

$$g(f(x))=\frac{1}{f(x)}-h(x) \equiv 0 \pmod{x^n}$$

$$\begin{aligned} f(x) &\equiv f_0(x)-\frac{\frac{1}{f_0(x)}-h(x)}{-\frac{1}{f_0^2(x)}} \pmod{x^n} \\ &\equiv 2f_0(x)-f_0^2(x)h(x) \pmod{x^n} \end{aligned}$$

$$T(n)=T\left(\frac{n}{2}\right)+O(n\log n)=O(n\log n)$$

$$g(f(x))=f^2(x)-h(x) \equiv 0 \pmod{x^n}$$

$$\begin{aligned} f(x) &\equiv f_0(x)-\frac{f_0^2(x)-h(x)}{2f_0(x)} \pmod{x^n} \\ &\equiv \frac{f_0^2(x)+h(x)}{2f_0(x)} \pmod{x^n} \end{aligned}$$

$$T(n)=T\left(\frac{n}{2}\right)+O(n\log n)=O(n\log n)$$

$$g(f(x))=\ln f(x)-h(x) \pmod{x^n}$$

$$\begin{aligned} f(x) &\equiv f_0(x)-\frac{\ln f_0(x)-h(x)}{\frac{1}{f_0(x)}} \pmod{x^n} \\ &\equiv f_0(x)(1-\ln f_0(x)+h(x)) \pmod{x^n} \end{aligned}$$

$$T(n)=T\left(\frac{n}{2}\right)+O(n\log n)=O(n\log n)$$

$$g(x)$$

$$f(x)$$

$$x^n$$

$$f(x)$$

$$n=1$$

$$[x^0]g(f(x))=0$$

$$x^{\lceil \frac{n}{2} \rceil}$$

$$f_0(x)$$

$$x^n$$

$$f(x)$$

$$g(f(x))$$

$$f_0(x)$$

$$f(x) - f_0(x)$$

$$\left\lceil \frac{n}{2} \right\rceil$$

$$h(x)$$

$$h(x)$$

$$h(x)$$

$$f(x_1), f(x_2), ..., f(x_n)$$

$$X_0 = \left\{x_1, x_2, ..., x_{\lfloor \frac{n}{2} \rfloor}\right\}$$

$$X_1 = \left\{x_{\lfloor \frac{n}{2} \rfloor + 1}, x_{\lfloor \frac{n}{2} \rfloor + 2}, ..., x_n\right\}$$

$$g_0(x) = \prod_{x_i \in X_0} (x - x_i)$$

$$f_0(x) \equiv f(x) \pmod{g_0(x)}$$

$$T(n) = 2T\left(\frac{n}{2}\right) + O(n \log n) = O(n \log^2 n)$$

$$X = \{(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)\}$$

$$f(x) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} y_i$$

$$\prod_{j \neq i} (x_i - x_j) = \lim_{x \to x_i} \frac{\prod_{j=1}^{n} (x - x_j)}{x - x_i} = M'(x_i)$$

$$f(x) = \sum_{i=1}^{n} \frac{y_i}{M'(x_i)} \prod_{j \neq i} (x - x_j)$$

$$f_0(x) = \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} v_i \prod_{j \neq i \land j \leq \lfloor \frac{n}{2} \rfloor} (x - x_j)$$

$$M_0(x) = \prod_{i=1}^{\lfloor \frac{n}{2} \rfloor} (x - x_i)$$

$$f_1(x) = \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} v_i \prod_{j \neq i \land \lfloor \frac{n}{2} \rfloor < j \leq n} (x - x_j)$$

$$M_1(x) = \prod_{i=\lfloor \frac{n}{2} \rfloor + 1}^{n} (x - x_i)$$

$$f(x)$$

$$n$$

$$x_1, x_2, ..., x_n$$

$$\forall x \in X_0 : g_0(x) = 0$$

$$f(x)$$

$$g_0(x)Q(x) + f_0(x)$$

$$\forall x \in X_0 : f(x) = g_0(x)Q(x) + f_0(x) = f_0(x)$$

$$X_1$$

$$n + 1$$

$$n$$

$$f(x)$$

$$\forall (x, y) \in X : f(x) = y$$

$$M(x) = \prod_{i=1}^{n} (x - x_i)$$

$$M(x)$$

$$O(n \log^2 n)$$

$$M'(x_i)$$

$$v_i = \frac{y_i}{M'(x_i)}$$

$$f(x)$$

$$n = 1$$

$$f(x) = v_1, M(x) = x - x_1$$

$$f(x) = f_0(x)M_1(x) + f_1(x)M_0(x), M(x) = M_0(x)M_1(x)$$

$$O(n \log^2 n)$$

$$v_t = \begin{bmatrix} f_t \\ f_{t+1} \\ \vdots \\ f_{t+k-1} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} f_{t+1} \\ f_{t+2} \\ \vdots \\ f_{t+k} \end{bmatrix}}_{v_{t+1}} = \underbrace{\begin{bmatrix} & 1 & & \\ & & \ddots & \\ & & & 1 \\ a_k & a_{k-1} & \cdots & a_1 \end{bmatrix}}_{M} \times \underbrace{\begin{bmatrix} f_t \\ f_{t+1} \\ \vdots \\ f_{t+k-1} \end{bmatrix}}_{v_t}$$

$$v_{t+k} = \sum_{i=1}^{k} a_i v_{t+k-i}$$

$$M^k v_t = \sum_{i=1}^{k} a_i M^{k-i} v_t$$

$$\Gamma_2(x) = \det\left(\begin{bmatrix} x & -1 \\ -a_2 & x - a_1 \end{bmatrix}\right) = x^2 - a_1 x - a_2$$

$$\begin{aligned} \Gamma_3(x) &= \det\left(\begin{bmatrix} x & -1 & 0 \\ 0 & x & -1 \\ -a_3 & -a_2 & x - a_1 \end{bmatrix}\right) \\ &= x \cdot (-1)^{1+1} \cdot \Gamma_2(x) + (-1) \cdot (-1)^{1+2} \cdot \det\left(\begin{bmatrix} 0 & -1 \\ -a_3 & x - a_1 \end{bmatrix}\right) \\ &= x^3 - a_1 x^2 - a_2 x - a_3 \end{aligned}$$

$$\Gamma(x) = \Gamma_k(x) = x^k - \sum_{i=1}^{k} a_i x^{k-i}$$

$$M^n v_0 = \sum_{i=0}^{k-1} g_i M^i v_0$$

$$v_n = \sum_{i=0}^{k-1} g_i v_i$$

$$\{f_i\}$$

$$k$$

$$f_0 ... f_{k-1}$$

$$f_n = \sum_{i=1}^{k} f_{n-i} a_i$$

$$a_i$$

$$f_n$$

$$F(\sum c_i x^i) = \sum c_i f_i$$

$$F(x^n)$$

$$f_n = \sum_{i=1}^{k} f_{n-i} a_i$$

$$F(x^n) = F(\sum_{i=1}^{k} a_i x^{n-i})$$

$$F(x^n - \sum_{i=1}^{k} a_i x^{n-i}) = F(x^{n-k}(x^k - \sum_{i=0}^{k-1} a_{k-i} x^i)) = 0$$

$$G(x) = x^k - \sum_{i=0}^{k-1} a_{k-i} x^i$$

$$F(A(x) + x^n G(x)) = F(A(x)) + F(x^n G(x)) = F(A(x))$$

$$A(x)$$

$$x^n G(x)$$

$$A(x)$$

$$F(A(x) \bmod G(x))$$

$$A(x) \bmod G(x)$$

$$k-1$$

$$f_{0..k-1}$$

$$x^n \bmod G(x)$$

$$O(k \log k \log n)$$

$$t \geq 0$$

$$v_t$$

$$v_{t+k}$$

$$v_{t+k}$$

$$v_t$$

$$\Gamma(x) = x^k - \sum_{i=1}^{k} a_i x^{k-i}$$

$$\Gamma(M) = O$$

$$O$$

$$k \times k$$

$$M^n$$

$$f(x) = x^n$$

$$f(M) = M^n$$

$$f(x)$$

$$f(x) = Q(x)\Gamma(x) + R(x)$$

$$f(M) = R(M)$$

$$M$$

$$\Gamma(x) = \det(xI - M)$$

$$I$$

$$k \times k$$

$$\Gamma(M) = O$$

$$M$$

$$2 \times 2$$

$$3 \times 3$$

$$g(x) = f(x) \bmod \Gamma(x)$$

$$g(M) = M^n$$

$$\deg(g(x)) < k$$

$$g(x) = g_0 + g_1 x + \cdots + g_{k-1} x^{k-1}$$

$$v_0, v_1, \ldots, v_{k-1}$$

$$f_0, f_1, \ldots, f_{k-1}$$

$$f_n$$

$$O(k)$$

$$g(x)$$

$$f =< f_0, f_1, f_2, \cdots, f_n > \quad (f_0, f_1, f_2, \cdots, f_n \in R)$$

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots + f_n x^n$$

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

$$f(x) = a_0 + a_1 x + \ldots + a_n x^n \tag{1}$$

$$g(x) = b_0 + b_1 x + \ldots + b_m x^m \tag{2}$$

$$\boxed{Q(x) = \sum_{i=0}^{n} \sum_{j=0}^{m} a_i b_j x^{i+j}} = c_0 + c_1 x + \ldots + c_{n+m} x^{n+m}$$

$$f^1 = f, f^k = f^{k-1} \times f$$

$$(f \circ g)(x) = f(g(x)) = f_0 + \sum_{k=1}^{+\infty} f_k g^k(x)$$

$$\left(\sum_{k=0}^{+\infty} f_k x^k \right)' = \sum_{k=1}^{+\infty} k f_k x^{k-1}$$

$$k f_k = \underbrace{f_k + f_k + \cdots + f_k}_{k\square f_k}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots$$

$$f \times f^{-1} = f^{-1} \times f = 1$$

$$f_0^{-1}=\frac{1}{f_0},f_n^{-1}=\frac{-1}{f_0}\sum_{k=0}^{n-1}f_k^{-1}f_{n-k}$$

$$\mathrm{e}^x=1+x+\frac{1}{2!}x^2+...+\frac{1}{n!}x^n+...$$

$$(1+x)^a=1+ax+\frac{a(a-1)}{2!}x^2+...+\frac{a(a-1)...(a-n+1)}{n!}x^n+...$$

$$\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+...+\frac{(-1)^n}{(2n)!}x^{2n}+...$$

$$\sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5+...+\frac{(-1)^n}{(2n+1)!}x^{2n+1}+...$$

$$\frac{1}{1+x}=1-x+x^2+...+(-1)^nx^n+...$$

$$\ln(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3+...+\frac{(-1)^{n-1}}{n}x^n+...$$

$$\frac{1}{1+x^2}=1-x^2+x^4+...+(-1)^nx^{2n}+...$$

$$\arctan x=x-\frac{1}{3}x^3+\frac{1}{5}x^5+...+\frac{(-1)^n}{2n+1}x^{2n+1}+...$$

$$\frac{1}{\sqrt{1+x}}=1-\frac{1}{2}x+\frac{3}{8}x^2+...+(-1)^n\frac{(2n)!}{(n!)^2 4^n}x^n+...$$

$$\frac{1}{\sqrt{1-x^2}}=1+\frac{1}{2}x^2+\frac{3}{8}x^4+...+\frac{(2n)!}{(n!)^2 4^n}x^{2n}+...$$

$$\arcsin x=x+\frac{1}{6}x^3+\frac{3}{40}x^5+...+\frac{(2n)!}{(n!)^2(2n+1)4^n}x^{2n+1}+...$$

$$n[x^n]f^k=k[x^{n-k}]\left(\frac{x}{g}\right)^n$$

$$f(x)=g(x)h(x)$$

$$f(x)=Q(x)g(x)+R(x)$$
$$\deg R<\deg g$$

$$f(x)\equiv R(x)\pmod{g(x)}$$

$$f(x_0)=R(x_0)$$

$$f^t(x_0)=R^t(x_0)$$

$$1+x+x^2+x^3+...\equiv 1+x+...+x^{n-1}\pmod{x^n}$$

$$f(x_1),f(x_2),...,f(x_n)$$

$$(x_0, y_0), (x_1, y_1), ..., (x_n, y_n)$$

$$g(x) = (x - x_0)...(x - x_n)$$

$$f(x) \equiv R(x) \pmod{g(x)}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + ...$$

$$f(x) = 0$$

$$a(x - x_1)^{c_1}(x - x_2)^{c_2} \cdots (x - x_m)^{c_m}$$

$$c_1 + c_2 + \cdots + c_m = n, x_1, x_2, \cdots, x_m$$ □□□□

$$\gcd(f, 0) = f, \gcd(f, g) = \gcd(g, f \bmod g)$$

$$f(x)P(x) + g(x)Q(x) = \gcd(f(x), g(x))$$

$$f(x)P(x) + g(x)Q(x) = h(x)$$

$$f(x)g(x) \equiv 1 \pmod{h(x)}$$

$$IDFT\left(\frac{DFT(1)}{DFT(f(x))}\right)$$

$$F(x) = \sum_n a_n k_n(x)$$

$$\sum a_n x^n$$

$$f(x) = \sum_{n=0}^{m} a_n x^n$$

$$\sum_{n=0}^{\infty} a_n x^n$$

$$R$$

$$R$$

$$R[x]$$

$$f$$

$$R$$

$$x$$

$$R[[x]]$$

$$f$$

$$f(x)$$

$$\deg f$$

$$f(x)$$

$$g(x)$$

$$Q(x) = f(x) \cdot g(x)$$

$$R$$

$$R$$

$$2^n$$

$$O(n2^n)$$

$$O(2^{2n})$$

$$2^n$$

$$R[[x]]$$

$$f$$

$$R[[x]]$$

$$f, g$$

$$f \circ g$$

$$f$$

$$g_0 = 0$$

$$R$$

$$\circ$$

$$(f \circ g) \circ h$$

$$f \circ (g \circ h)$$

$$R$$

$$\circ$$

$$1 \times x$$

$$O((n \log n)^{1.5})$$

$$O(n^2)$$

$$R$$

$$0$$

$$0$$

$$f$$

$$f_0 \neq 0$$

$$f^{-1}$$

$$f^{-1}$$

$$n$$

$$O(n^2)$$

$$O(n \log n)$$

$$f(x)$$

$$\frac{1}{f(x)}$$

$$x = 0$$

$$0$$

$$0$$

$$f_0 = 0$$

$$f_1 \neq 0$$

$$f$$

$$g(f(x)) = f(g(x)) = x$$

$$g$$

$$n, k$$

$$[x^k]f(x)$$

$$f(x)$$

$$x^k$$

$$f(x)$$

$$g(x)$$

$$h(x)$$

$$g(x)$$

$$f(x)$$

$$g(x)$$

$$f(x)$$

$$g(x)$$

$$f(x)$$

$$g(x)$$

$$f(x)$$

$$f(x), g(x)$$

$$Q(x), R(x)$$

$$\deg f \geq \deg g$$

$$\deg Q = \deg f - \deg g$$

$$Q(x) = 0$$

$$Q(x)$$

$$g(x)$$

$$f(x)$$

$$R(x)$$

$$g(x)$$

$$f(x)$$

$$f(x)$$

$$R(x)$$

$$g(x)$$

$$g(x)$$

$$x_0$$

$$f(x)$$

$$R(x)$$

$$x_0$$

$$g(x)$$

$$k$$

$$(x - x_0)^k$$

$$g(x)$$

$$0$$

$$k$$

$$t$$

$$t$$

$$x^n$$

$$x^n$$

$$n$$

$$f(x)$$

$$n$$

$$x_1, x_2, ..., x_n$$

$$n + 1$$

$$n$$

$$f(x)$$

$$n + 1$$

$$f(x)$$

$$n + 1$$

$$1$$

$$n + 1$$

$$f(x)$$

$$x_0$$

$$x_n$$

$$R(x)$$

$$R(x)$$

$$g(x)$$

$$R(x)$$

$$x = 1$$

$$1 + x + x^2 + x^3 + ...$$

$$x - 1$$

$$0$$

$$x^n$$

$$x^n - 1$$

$$n$$

$$f$$

$$n$$

$$f(x)$$

$$P$$

$$P[x]$$

$$(P(x), Q(x))$$

$$O(n \log^2 n)$$

$$h(x)$$

$$f(x)$$

$$f(x)$$

$$h(x)$$

$$f(x)$$

$$g(x)$$

$$g(x)$$

$$f(x)$$

$$h(x)$$

$$\gcd(f, g) = 1$$

$$h(x)$$

$$f(x)$$

$$h(x)$$

$$g(x)$$

$$f(x)$$

$$x^n$$

$$f^{-1}(x)$$

$$x^n$$

$$x^n - 1$$

$$x^n$$

$$0$$

$$k_n(x)$$

$$k_n(x) = x^n$$

$$k_n(x) = \frac{x^n}{n!}$$

$$k_n(x) = \frac{1}{n^x}$$

$$FWT[A] = merge(FWT[A_0], FWT[A_0] + FWT[A_1])$$

$$UFWT[A'] = merge(UFWT[A_{(0)'}], UFWT[A_{(1)'}] - UFWT[A_{(0)'}])$$

$$FWT[A] = merge(FWT[A_0] + FWT[A_1], FWT[A_1])$$

$$UFWT[A'] = merge(UFWT[A_{(0)'}] - UFWT[A_{(1)'}], UFWT[A_{(1)'}])$$

$$FWT[A] = merge(FWT[A_0] + FWT[A_1], FWT[A_0] - FWT[A_1])$$

$$UFWT[A'] = merge(\frac{UFWT[A_{(0)'}] + UFWT[A_{(1)'}]}{2}, \frac{UFWT[A_{(0)'}] - UFWT[A_{(1)'}]}{2})$$

$$FWT[A] = merge(FWT[A_1] - FWT[A_0], FWT[A_1] + FWT[A_0])$$

$$UFWT[A'] = merge(\frac{UFWT[A_{(1)'}] - UFWT[A_{(0)'}]}{2}, \frac{UFWT[A_{(1)'}] + UFWT[A_{(0)'}]}{2})$$

$$A$$

$$FWT[A]$$

$$C$$

$$A$$

$$B$$

$$C = A \cdot B$$

$$FWT[A], FWT[B]$$

$$FWT[C] = FWT[A] \cdot FWT[B]$$

$$O(n)$$

$$FWT[C]$$

$$C$$

$$O(n \log n)$$

$$C_i = \sum_{i=j \bigoplus k} A_j B_k$$

$$\bigoplus$$

$$*$$

$$k = i \mid j$$

$$i$$

$$1$$

$$j$$

$$1$$

$$k$$

$$1$$

$$FWT[C] = FWT[A]*FWT[B]$$

$$FWT[A] = A' = \sum_{i=i \mid j} A_j$$

$$j$$

$$1$$

$$i$$

$$C_i = \sum_{i=j \mid k} A_j * B_k \Longrightarrow FWT[C] = FWT[A] * FWT[B]$$

$$FWT[A]$$

$$O(n^2)$$

$$A_0$$

$$A$$

$$A_1$$

$$A_0$$

$$0$$

$$0$$

$$A_1$$

$$1$$

$$0$$

$$+$$

$$O(\log n)$$

$$O(n \log n)$$

$$FWT[A]$$

$$A_0$$

$$A_0 = FWT[A_0]$$

$$A_1$$

$$FWT[A_0] + FWT[A_1] \quad A_1 = A_{(0)'} + A_{(1)'}$$

$$i \& j$$

$$1$$

$$i$$

$$j$$

$$i$$

$$k$$

$$j$$

$$k$$

$$i \operatorname{xor} j$$

$$k$$

$$FWT[A]$$

$$A_i = \sum_{C_1} A_j - \sum_{C_2} A_j$$

$$C_1$$

$$i \& j$$

$$0$$

$$C_2$$

$$i \& j$$

$$1$$

$$A_i = \sum_{C_1} A_j - \sum_{C_2} A_j$$

$$C_1$$

$$i \mid j$$

$$0$$

$$C_2$$

$$i \mid j$$

$$1$$

$$E$$

$$[1, 4 \cdot 10^6]$$

$$\sum_{(x,y) \in E} (x \oplus y)^{33} x^{-2} y^{-1} \bmod 10^9 + 7$$

$$f(x) = a_n (x - x_1)^{k_1} (x - x_2)^{k_2} ... (x - x_t)^{k_t}$$

$$(x - a - b\mathrm{i})(x - a + b\mathrm{i}) = x^2 - 2ax + a^2 + b^2$$

$$f(x) = a_n (x - x_1)^{k_1} (x - x_2)^{k_2} ... (x - x_t)^{k_t} (x^2 + p_1 x + q_1)^{l_1} (x^2 + p_2 x + q_2)^{l_2} ... (x^2 + p_s x + q_s)^{l_s}$$

$$f(x) = (x^2 + p_1 x + q_1) g(x)$$

$$f(x) = (x^2 + px + q) g(x) + rx + s$$

$$p_1 = p + dp$$

$$q_1 = q + dq$$

$$dr = \frac{\partial r}{\partial p} dp + \frac{\partial r}{\partial q} dq$$

$$ds = \frac{\partial s}{\partial p}dp + \frac{\partial s}{\partial q}dq$$

$$0 = xg(x) + \frac{\partial g(x)}{\partial p}(x^2 + px + q) + \frac{\partial r}{\partial p}x + \frac{\partial s}{\partial p}$$

$$0 = g(x) + \frac{\partial g(x)}{\partial q}(x^2 + px + q) + \frac{\partial r}{\partial q}x + \frac{\partial s}{\partial q}$$

$$xg(x) = -\frac{\partial g(x)}{\partial p}(x^2 + px + q) - \frac{\partial r}{\partial p}x - \frac{\partial s}{\partial p}$$

$$g(x) = -\frac{\partial g(x)}{\partial q}(x^2 + px + q) - \frac{\partial r}{\partial q}x - \frac{\partial s}{\partial q}$$

$$-\frac{\partial r}{\partial p}dp - \frac{\partial r}{\partial q}dq = r$$

$$-\frac{\partial s}{\partial p}dp - \frac{\partial s}{\partial q}dq = s$$

$$n$$

$$n$$

$$1$$

$$n$$

$$n$$

$$n$$

$$n$$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$$

$$k_1 + k_2 + ... + k_t = n$$

$$a + b\mathrm{i}$$

$$a - b\mathrm{i}$$

$$x^2 - 2ax + a^2 + b^2 = x^2 + px + q$$

$$c$$

$$x - c$$

$$a \pm b\mathrm{i}$$

$$x^2 - 2ax + a^2 + b^2$$

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$$

$$k_1 + k_2 + ... + k_t + 2(l_1 + l_2 + ... + l_s) = n$$

$$2$$

$$(r, s)$$

$$(p, q)$$

$$f(x)$$

$$g(x)$$

$$rx + s$$

$$x^2 + px + q$$

$$x$$

$$s$$

$$r$$

$$ds$$

$$dr$$

$$0$$

$$ds$$

$$dr$$

$$s$$

$$r$$

$$p$$

$$q$$

$$dp$$

$$dq$$

$$\begin{aligned} A &= 5x^2 + 3x + 7 \\ B &= 7x^2 + 2x + 1 \end{aligned}$$

$$\begin{aligned} C &= A \times B \\ &= 35x^4 + 31x^3 + 60x^2 + 17x + 7 \end{aligned}$$

$$F(\omega) = \mathbb{F}[f(t)] = \int_{-\infty}^{\infty} f(t) \mathrm{e}^{-\mathrm{i}\omega t} dt$$

$$f(t) = \mathbb{F}^{-1}[F(\omega)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \mathrm{e}^{\mathrm{i}\omega t} d\omega$$

$$X_k = \sum_{n=0}^{N-1} x_n \mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}kn}$$

$$\hat{x} = \mathcal{F} x$$

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \mathrm{e}^{\mathrm{i}\frac{2\pi}{N}kn}$$

$$x = \mathcal{F}^{-1} \hat{x}$$

$$\binom{n}{3}+\binom{n}{7}+\binom{n}{11}+\binom{n}{15}+...$$

$$f(x)=(1+x)^n=\binom{n}{0}x^0+\binom{n}{1}x^1+\binom{n}{2}x^2+\binom{n}{3}x^3+...$$

$$f(\mathrm{i})=(1+\mathrm{i})^n=\binom{n}{0}+\binom{n}{1}\mathrm{i}-\binom{n}{2}-\binom{n}{3}\mathrm{i}+...$$

$$f(1)+\mathrm{i}f(\mathrm{i})-f(-1)-\mathrm{i}f(-\mathrm{i})=4\binom{n}{3}+4\binom{n}{7}+4\binom{n}{11}+4\binom{n}{15}+...$$

$$\binom{n}{3}+\binom{n}{7}+\binom{n}{11}+\binom{n}{15}+...=\frac{2^n+\mathrm{i}(1+\mathrm{i})^n-\mathrm{i}(1-\mathrm{i})^n}{4}$$

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ \vdots \\ X_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-1} \\ 1 & \alpha^2 & \alpha^4 & \cdots & \alpha^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha^{n-1} & \alpha^{2(n-1)} & \cdots & \alpha^{(n-1)(n-1)} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$f(x)=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+a_7x^7$$

$$\begin{aligned} f(x)&=(a_0+a_2x^2+a_4x^4+a_6x^6)+(a_1x+a_3x^3+a_5x^5+a_7x^7)\\ &=(a_0+a_2x^2+a_4x^4+a_6x^6)+x(a_1+a_3x^2+a_5x^4+a_7x^6) \end{aligned}$$

$$\begin{aligned} G(x)&=a_0+a_2x+a_4x^2+a_6x^3\\ H(x)&=a_1+a_3x+a_5x^2+a_7x^3 \end{aligned}$$

$$f(x)=G(x^2)+x\times H(x^2)$$

$$\begin{aligned} f(\omega_n^k)&=G((\omega_n^k)^2)+\omega_n^k\times H((\omega_n^k)^2)\\ &=G(\omega_n^{2k})+\omega_n^k\times H(\omega_n^{2k})\\ &=G(\omega_{n/2}^k)+\omega_n^k\times H(\omega_{n/2}^k) \end{aligned}$$

$$\begin{aligned} f(\omega_n^{k+n/2})&=G(\omega_n^{2k+n})+\omega_n^{k+n/2}\times H(\omega_n^{2k+n})\\ &=G(\omega_n^{2k})-\omega_n^k\times H(\omega_n^{2k})\\ &=G(\omega_{n/2}^k)-\omega_n^k\times H(\omega_{n/2}^k) \end{aligned}$$

$$R(x)=\left\lfloor\frac{R\left(\left\lfloor\frac{x}{2}\right\rfloor\right)}{2}\right\rfloor+(x\bmod 2)\times\frac{len}{2}$$

$$\begin{aligned} f(\omega_n^k)&=G(\omega_{n/2}^k)+\omega_n^k\times H(\omega_{n/2}^k)\\ f(\omega_n^{k+n/2})&=G(\omega_{n/2}^k)-\omega_n^k\times H(\omega_{n/2}^k) \end{aligned}$$

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_n^1 & \omega_n^2 & \omega_n^3 & \cdots & \omega_n^{n-1} \\ 1 & \omega_n^2 & \omega_n^4 & \omega_n^6 & \cdots & \omega_n^{2(n-1)} \\ 1 & \omega_n^3 & \omega_n^6 & \omega_n^9 & \cdots & \omega_n^{3(n-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_n^{n-1} & \omega_n^{2(n-1)} & \omega_n^{3(n-1)} & \cdots & \omega_n^{(n-1)^2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{n-1} \end{bmatrix}$$

$$\frac{1}{\omega_k} = \omega_k^{-1} = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{k}} = \cos\left(\frac{2\pi}{k}\right) + \mathrm{i}\sin\left(-\frac{2\pi}{k}\right)$$

$$A(x) = \sum_{i=0}^{n-1} y_i x^i$$

$$\begin{aligned} A(b_k) &= \sum_{i=0}^{n-1} f(\omega_n^i)\omega_n^{-ik} = \sum_{i=0}^{n-1} \omega_n^{-ik} \sum_{j=0}^{n-1} a_j (\omega_n^i)^j \\ &= \sum_{i=0}^{n-1}\sum_{j=0}^{n-1} a_j \omega_n^{i(j-k)} = \sum_{j=0}^{n-1} a_j \sum_{i=0}^{n-1} \left(\omega_n^{j-k}\right)^i \end{aligned}$$

$$S(\omega_n^a) = \sum_{i=0}^{n-1} (\omega_n^a)^i$$

$$\omega_n^a S(\omega_n^a) = \sum_{i=1}^{n} (\omega_n^a)^i$$

$$S(\omega_n^a) = \frac{(\omega_n^a)^n - (\omega_n^a)^0}{\omega_n^a - 1} = 0$$

$$S(\omega_n^a) = \begin{cases} n, a = 0 \\ 0, a \neq 0 \end{cases}$$

$$A(b_k) = \sum_{j=0}^{n-1} a_j S\left(\omega_n^{j-k}\right) = a_k \cdot n$$

$$\begin{aligned} &\{(b_0, A(b_0)), (b_1, A(b_1)), \cdots, (b_{n-1}, A(b_{n-1}))\} \\ = &\{(b_0, a_0 \cdot n), (b_1, a_1 \cdot n), \cdots, (b_{n-1}, a_{n-1} \cdot n)\} \end{aligned}$$

$$O(n \log n)$$

$$n$$

$$O(n^2)$$

$$A$$

$$B$$

$$C = A \times B$$

$$O(n^2)$$

$$n$$

$$A$$

$$B$$

$$C$$

$$c_i$$

$$c_i = \sum_{j=0}^{i} a_j b_{i-j}$$

$$O(n)$$

$$O(n)$$

$$O(n^2)$$

$$O(n \log n)$$

$$f(t)$$

$$t$$

$$\omega$$

$$f(t)$$

$$2\pi$$

$$2\pi$$

$$\{x_n\}_{n=0}^{N-1}$$

$$\mathrm{e}$$

$$i$$

$$\mathcal{F}$$

$$1$$

$$\frac{1}{N}$$

$$\frac{1}{\sqrt{N}}$$

$$x_n$$

$$f(x)$$

$$x^n$$

$$X_k$$

$$f(x)$$

$$\mathrm{e}^{\frac{-2\pi \mathrm{i} k}{N}}$$

$$f(\mathrm{e}^{\frac{-2\pi \mathrm{i} k}{N}})$$

$$f(x)$$

$$f(\mathrm{i})$$

$$\alpha=\mathrm{e}^{-\mathrm{i}\frac{2\pi}{N}n}$$

$$x=\omega_n^k$$

$$f(x)$$

$$8$$

$$x$$

$$f(x)$$

$$\omega_n^i=-\omega_n^{i+n/2}$$

$$G(x^2)$$

$$H(x^2)$$

$$\omega_n^i$$

$$\omega_n^{i+n/2}$$

$$G(x^2)$$

$$H(x^2)$$

$$G(\omega_{n/2}^k)$$

$$H(\omega_{n/2}^k)$$

$$f(\omega_n^k)$$

$$f(\omega_n^{k+n/2})$$

$$G$$

$$H$$

$$2^m(m\in\mathbf{N}^*)$$

$$2^m(m\in\mathbf{N}^*)$$

$$0$$

$$2^m-1$$

$$n$$

$$\omega_n^0,\omega_n^1,\omega_n^2,\cdots,\omega_n^{n-1}(n=2^m(m\in\mathbf{N}^*))$$

$$2^m$$

$$\omega_n$$

$$\omega_n$$

$$n$$

$$n = 2^k, k \in \mathbf{N}^*$$

$$O(n \log n)$$

$$O(1)$$

$$8$$

$$\{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$$

$$\{x_0, x_2, x_4, x_6\}, \{x_1, x_3, x_5, x_7\}$$

$$\{x_0, x_4\}\{x_2, x_6\}, \{x_1, x_5\}, \{x_3, x_7\}$$

$$\{x_0\}\{x_4\}\{x_2\}\{x_6\}\{x_1\}\{x_5\}\{x_3\}\{x_7\}$$

$$x_1$$

$$O(n \log n)$$

$$O(n \log n)$$

$$O(n)$$

$$len = 2^k$$

$$k$$

$$R(x)$$

$$k$$

$$x$$

$$0$$

$$R(0), R(1), \cdots, R(n-1)$$

$$R(0) = 0$$

$$R(x)$$

$$R(x)$$

$$R\left(\left\lfloor \frac{x}{2} \right\rfloor\right)$$

$$x$$

$$2$$

$$x$$

$$0$$

$$0$$

$$1$$

$$1$$

$$\frac{len}{2} = 2^{k-1}$$

$$k = 5$$

$$len = (100000)_2$$

$$(11001)_2$$

$$(1100)_2$$

$$R((1100)_2) = R((01100)_2) = (00110)_2$$

$$(00011)_2$$

$$1$$

$$(10000)_2 = 2^{k-1}$$

$$0$$

$$O(n)$$

$$G(\omega_{n/2}^k)$$

$$H(\omega_{n/2}^k)$$

$$f(\omega_n^k)$$

$$f(\omega_n^{k+n/2})$$

$$n, k$$

$$G(\omega_{n/2}^k)$$

$$k$$

$$H(\omega_{n/2}^k)$$

$$k + \frac{n}{2}$$

$$f(\omega_n^k)$$

$$k$$

$$f(\omega_n^{k+n/2})$$

$$k + \frac{n}{2}$$

$$k$$

$$k + \frac{n}{2}$$

$$\frac{n}{2}$$

$$s = \frac{n}{2}$$

$$\{G(\omega_{n/2}^k)\}$$

$$l_g = 0, 2s, 4s, \cdots, N - 2s$$

$$\{H(\omega_{n/2}^k)\}$$

$$l_h = s, 3s, 5s, \cdots, N - s$$

$$k = 0, 1, 2, \cdots, s - 1$$

$$G(\omega_{n/2}^k)$$

$$l_g + k$$

$$H(\omega_{n/2}^k)$$

$$l_h + k$$

$$f(\omega_n^k)$$

$$f(\omega_n^{k+n/2})$$

$$x$$

$$n$$

$$0$$

$$\omega_k$$

$$\mathrm{e}^{-\frac{2\pi\mathrm{i}}{k}}$$

$$n$$

$$1$$

$$-1$$

$$\pi$$

$$1$$

$$-1$$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1} = \sum_{i=0}^{n-1} a_i x^i$$

$$y_i = f(\omega_n^i), i \in \{0, 1, \cdots, n-1\}$$

$$\{a_0, a_1, \cdots, a_{n-1}\}$$

$$\{y_0, y_1, y_2, \cdots, y_{n-1}\}$$

$$A$$

$$b_i = \omega_n^{-i}$$

$$A$$

$$x = b_0, b_1, \cdots, b_{n-1}$$

$$\{A(b_0), A(b_1), \cdots, A(b_{n-1})\}$$

$$A(x)$$

$$A(b_k)$$

$$S(\omega_n^a) = \sum_{i=0}^{n-1} (\omega_n^a)^i$$

$$a = 0 \pmod n$$

$$S(\omega_n^a) = n$$

$$a \neq 0 \pmod n$$

$$b_i = \omega_n^{-i}$$

$$A$$

$$\{y_0, y_1, y_2, \cdots, y_{n-1}\}$$

$$n$$

$$f(x)$$

$$\omega_n^i$$

$$A(x)$$

$$A(\omega_n^k) = \sum_{j=0}^{n-1} a_j S\big(\omega_n^{j+k}\big)$$

$$j + k = 0 \pmod n$$

$$S\big(\omega_n^{j+k}\big) = n$$

$$0$$

$$A(\omega_n^k) = a_{n-k} \cdot n$$

$$\{y_0, y_1, y_2, \cdots, y_{n-1}\}$$

$$n$$

$$n - 1$$

$$f(x)$$

$$\frac{\mathrm{e}^{\tan x}}{1 + x^2} \sin\Big(\sqrt{1 + \ln^2 x}\Big)$$

$$-\mathrm{i} \ln\Big(x + \mathrm{i}\sqrt{1 - x^2}\Big)$$

$$\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x \exp(-t^2)\mathrm{d}t$$

$$[x^0]f^{-1}(x) = \left([x^0]f(x)\right)^{-1}$$

$$\begin{aligned}
f(x)f_0^{-1}(x) &\equiv 1 \pmod{x^{\lceil\frac{n}{2}\rceil}}\\
f(x)f^{-1}(x) &\equiv 1 \pmod{x^{\lceil\frac{n}{2}\rceil}}\\
f^{-1}(x) - f_0^{-1}(x) &\equiv 0 \pmod{x^{\lceil\frac{n}{2}\rceil}}
\end{aligned}$$

$$f^{-2}(x) - 2f^{-1}(x)f_0^{-1}(x) + f_0^{-2}(x) \equiv 0 \pmod{x^n}$$

$$f^{-1}(x) \equiv f_0^{-1}(x)(2 - f(x)f_0^{-1}(x)) \pmod{x^n}$$

$$T(n) = T\left(\frac{n}{2}\right) + O(n\log n) = O(n\log n)$$

$$\begin{aligned}
f^{-1}(x) \bmod x^{2n} &= f(-x)(f(x)f(-x))^{-1} \bmod x^{2n}\\
&= f(-x)g^{-1}(x^2) \bmod x^{2n}
\end{aligned}$$

$$f^2(x) \equiv g(x) \pmod{x^n}$$

$$[x^0]f(x) = \sqrt{[x^0]g(x)}$$

$$\begin{aligned}
f_0^2(x) &\equiv g(x) && \pmod{x^{\lceil\frac{n}{2}\rceil}}\\
f_0^2(x) - g(x) &\equiv 0 && \pmod{x^{\lceil\frac{n}{2}\rceil}}\\
\left(f_0^2(x) - g(x)\right)^2 &\equiv 0 && \pmod{x^n}\\
\left(f_0^2(x) + g(x)\right)^2 &\equiv 4f_0^2(x)g(x) && \pmod{x^n}\\
\left(\frac{f_0^2(x) + g(x)}{2f_0(x)}\right)^2 &\equiv g(x) && \pmod{x^n}\\
\frac{f_0^2(x) + g(x)}{2f_0(x)} &\equiv f(x) && \pmod{x^n}\\
2^{-1}f_0(x) + 2^{-1}f_0^{-1}(x)g(x) &\equiv f(x) && \pmod{x^n}
\end{aligned}$$

$$T(n) = T\left(\frac{n}{2}\right) + O(n\log n) = O(n\log n)$$

$$f^R(x) = x^{\deg f} f\left(\frac{1}{x}\right)$$

$$\begin{aligned}
x^n f\left(\frac{1}{x}\right) &= x^{n-m}Q\left(\frac{1}{x}\right)x^m g\left(\frac{1}{x}\right) + x^{n-m+1}x^{m-1}R\left(\frac{1}{x}\right)\\
f^R(x) &= Q^R(x)g^R(x) + x^{n-m+1}R^R(x)
\end{aligned}$$

$$f^R(x) \equiv Q^R(x)g^R(x) \pmod{x^{n-m+1}}$$

$$f^k(x) = \exp(k\ln f(x))$$

$$f^k(x) = f_i^k x^{ik} \exp\left(k \ln \frac{f(x)}{f_i x^i}\right)$$

$$\sin x = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}$$

$$\cos x = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2}$$

$$\sin f(x) = \frac{\exp(\mathrm{i}f(x)) - \exp(-\mathrm{i}f(x))}{2\mathrm{i}}$$

$$\cos f(x) = \frac{\exp(\mathrm{i}f(x)) + \exp(-\mathrm{i}f(x))}{2}$$

$$\begin{aligned} &\mathrm{i} = \sqrt{-1} \equiv \sqrt{998244352} \pmod{998244353} \\ \implies &\mathrm{i} \equiv 86583718 \pmod{998244353} \\ \text{or} \quad &\mathrm{i} \equiv 911660635 \pmod{998244353} \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin x = \frac{1}{\sqrt{1 - x^2}}$$

$$\arcsin x = \int \frac{1}{\sqrt{1 - x^2}} \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \arccos x = -\frac{1}{\sqrt{1 - x^2}}$$

$$\arccos x = -\int \frac{1}{\sqrt{1 - x^2}} \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan x = \frac{1}{1 + x^2}$$

$$\arctan x = \int \frac{1}{1 + x^2} \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \arcsin f(x) = \frac{f'(x)}{\sqrt{1 - f^2(x)}}$$

$$\arcsin f(x) = \int \frac{f'(x)}{\sqrt{1 - f^2(x)}} \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \arccos f(x) = -\frac{f'(x)}{\sqrt{1 - f^2(x)}}$$

$$\arccos f(x) = -\int \frac{f'(x)}{\sqrt{1 - f^2(x)}} \mathrm{d}x$$

$$\frac{\mathrm{d}}{\mathrm{d}x} \arctan f(x) = \frac{f'(x)}{1 + f^2(x)}$$

$$\arctan f(x) = \int \frac{f'(x)}{1 + f^2(x)} \mathrm{d}x$$

$$F$$

$$u \to \partial u$$

$$\partial(u+v) = \partial u + \partial v$$

$$\partial(uv) = u\partial v + v\partial u$$

$$F$$

$$u$$

$$u$$

$$u$$

$$F$$

$$u$$

$$F$$

$$a \in F$$

$$\partial u = u\partial a$$

$$u$$

$$F$$

$$a \in F$$

$$\partial u = \frac{\partial a}{a}$$

$$P$$

$$P(f(x)) = 0$$

$$f(x)$$

$$2x + 1$$

$$\sqrt{x}$$

$$(1 + x^2)^{-1}$$

$$\mid x \mid$$

$$f(x)$$

$$f^{-1}(x)$$

$$f(x)$$

$$x^{\lceil \frac{n}{2} \rceil}$$

$$f_0^{-1}(x)$$

$$f(x)$$

$$f^{-1}(x) \bmod x^{2n}$$

$$g^{-1}(x) \bmod x^n$$

$$g^{-1}(x^2) \bmod x^{2n}$$

$$f(x)f(-x)$$

$$g(x)$$

$$f(x)$$

$$[x^0]g(x)$$

$$0$$

$$[x^0]g(x)$$

$$g(x)$$

$$[x^0]g(x)$$

$$f(x)$$

$$g(x)$$

$$x^{\lceil \frac{n}{2} \rceil}$$

$$f_0(x)$$

$$f(x)$$

$$f^{-1}(x)$$

$$[x^0]g(x) \neq 1$$

$$[x^0]f(x)$$

$$f_0(x)$$

$$0$$

$$[x^0]g(x) = 0$$

$$g(x)$$

$$x^k h(x)$$

$$[x^0]h(x) \neq 0$$

$$k$$

$$g(x)$$

$$k$$

$$h(x)$$

$$\sqrt{h(x)}$$

$$f(x) \equiv x^{k/2}\sqrt{h(x)} \pmod{x^n}$$

$$f(x), g(x)$$

$$g(x)$$

$$f(x)$$

$$Q(x)$$

$$R(x)$$

$$R(x)$$

$$f(x)$$

$$n = \deg f, m = \deg g$$

$$f(x) = Q(x)g(x) + R(x)$$

$$x$$

$$\frac{1}{x}$$

$$x^n$$

$$R^R(x)$$

$$x^{n-m+1}$$

$$x^{n-m+1}$$

$$R^R(x)$$

$$Q^R(x)$$

$$(n-m) < (n-m+1)$$

$$Q^R(x)$$

$$Q(x)$$

$$R(x)$$

$$O(n \log n)$$

$$f(x)$$

$$x^n$$

$$\ln f(x)$$

$$\exp f(x)$$

$$f(x)$$

$$\ln f(x)$$

$$f(x)$$

$$\exp f(x)$$

$$O(n\log n)$$

$$\exp$$

$$f^k(x)$$

$$O(n\log n\log k)$$

$$[x^0]f(x)=1$$

$$[x^0]f(x)\neq 1$$

$$f(x)$$

$$f_ix^i$$

$$O(n\log n)$$

$$f(x)$$

$$x^n$$

$$\sin f(x),\cos f(x)$$

$$\tan f(x)$$

$$(\mathrm{e}^{\mathrm{i}x}=\cos x+\mathrm{i}\sin x)$$

$$f(x)$$

$$x^n$$

$$\sin f(x)$$

$$\cos f(x)$$

$$\tan f(x)=\frac{\sin f(x)}{\cos f(x)}$$

$$\tan f(x)$$

$$\mathbb{Z}_{998244353}$$

$$\mathrm{i}$$

$$86583718$$

$$911660635$$

$$f(x)$$

$$x^n$$

$$\arcsin f(x),\arccos f(x)$$

$$\arctan f(x)$$

$$\ln$$

$$f(x)$$

$$\hat{F}(x) = \sum_{n} a_n \frac{x^n}{n!}$$

$$\begin{aligned}
\hat{F}(x)\hat{G}(x) &= \sum_{i\geq 0} a_i \frac{x^i}{i!} \sum_{j\geq 0} b_j \frac{x^j}{j!} \\
&= \sum_{n\geq 0} x^n \sum_{i=0}^{n} a_i b_{n-i} \frac{1}{i!(n-i)!} \\
&= \sum_{n\geq 0} \frac{x^n}{n!} \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}
\end{aligned}$$

$$\left\langle \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \right\rangle$$

$$\hat{F}(x) = \sum_{n\geq 0} \frac{x^n}{n!} = \mathrm{e}^x$$

$$\hat{F}(x) = \sum_{n\geq 0} \frac{p^n x^n}{n!} = \mathrm{e}^{px}$$

$$F(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$

$$\hat{H}(x) = \hat{F}(x)\hat{G}(x) = \sum_{n\geq 0} \left[\sum_{i=0}^{n} \binom{n}{i} f_i g_{n-i} \right] \frac{x^n}{n!}$$

$$F_k(n) = \frac{n!}{k!} \sum_{\sum_i^k a_i = n} \prod_{j=1}^{k} \frac{f_{a_j}}{a_{(j)!}}$$

$$\hat{F}(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}$$

$$\begin{aligned}
G_k(x) &= \sum_{n\geq 0} F_k(n) \frac{x^n}{n!} \\
&= \sum_{n\geq 0} x^n \frac{1}{k!} \sum_{\sum_i^k a_i = n} \prod_{j=1}^{k} \frac{f_{a_j}}{a_{(j)!}} \\
&= \frac{1}{k!} \sum_{n\geq 0} \sum_{\sum_i^k a_i = n} \prod_{j=1}^{k} \frac{f_{a_j} x^{a_j}}{a_{(j)!}} \\
&= \frac{1}{k!} \hat{F}^k(x)
\end{aligned}$$

$$\sum_{k\geq 0} G_k(x) = \sum_{k\geq 0} \frac{\hat{F}^k(x)}{k!} = \exp \hat{F}(x)$$

$$
\begin{aligned}
H_k(x) &= \sum_{n\ge 0}\frac{x^n}{n!}F_k(n)\\
&= \sum_{n\ge 0}\frac{x^n}{n!}\sum_{i=1}^{n-k+1}\binom{n}{i}F_{k-1}(n-i)\times g_i\times\frac{1}{k}\\
&= \frac{1}{k}\sum_{n\ge 0}\frac{x^n}{n!}\sum_{i=0}^{n}\binom{n}{i}F_{k-1}(n-i)\times g_i\\
&= \frac{1}{k}\cdot H_{k-1}(x)G(x)
\end{aligned}
$$

$$
\begin{aligned}
H_k(x) &= \frac{1}{k}\cdot H_{k-1}(x)G(x)\\
&= \frac{1}{k}\cdot\frac{1}{k-1}\cdot H_{k-2}(x)G^2(x)\\
&= \cdots\\
&= \frac{1}{k}\cdot\frac{1}{k-1}\cdots\frac{1}{2}\cdot H_1(x)G^{k-1}(x)\\
&= \frac{1}{k!}G^k(x)
\end{aligned}
$$

$$
\sum_{k\ge 0}H_k(x) = \sum_{k\ge 0}\frac{G^k(x)}{k!} = \exp G(x)
$$

$$
\hat{P}(x) = \sum_{n\ge 0}\frac{n!x^n}{n!} = \sum_{n\ge 0}x^n = \frac{1}{1-x}
$$

$$
\hat{Q}(x) = \sum_{n\ge 1}\frac{(n-1)!x^n}{n!} = \sum_{n\ge 1}\frac{x^n}{n} = -\ln(1-x) = \ln\left(\frac{1}{1-x}\right)
$$

$$
\exp\hat{F}(x) = \sum_{n\ge 0}2^{\binom{n}{2}}\frac{x^n}{n!}
$$

$$
\sum_{n\ge 2}\frac{x^n}{n} = -\ln(1-x) - x
$$

$$
\underbrace{f\circ f\circ\cdots\circ f}_{k} = \underbrace{f\circ f\circ\cdots\circ f}_{k-1}
$$

$$
\widehat{F_k}(x) = \sum_{n\ge 0}f_{n,k}\frac{x^n}{n!}
$$

$$
\widehat{F_k}(x) = x\exp\hat{F}_{k-1}(x)
$$

$$
s = \prod_{i=1}^{n}a_i - \prod_{i=1}^{n}(a_i - b_i)
$$

$$
\frac{k!}{b_{(1)!}b_{(2)!}\cdots b_{(n)!}}
$$

$$F_j(x)=\sum_{i\geq 0}(a_j-i)\frac{x^i}{i!}$$

$$[x^k]\prod_{j=1}^{n}F_j(x)$$

$$\begin{aligned}F_j(x)&=\sum_{i\geq 0}a_j\frac{x^i}{i!}-\sum_{i\geq 1}\frac{x^i}{(i-1)!}\\&=a_j\mathrm{e}^x-x\mathrm{e}^x\\&=(a_j-x)\mathrm{e}^x\end{aligned}$$

$$\prod_{j=1}^{n}F_j(x)=\mathrm{e}^{nx}\prod_{j=1}^{n}(a_j-x)$$

$$\begin{aligned}\prod_{j=1}^{n}F_j(x)&=\left(\sum_{i\geq 0}\frac{n^ix^i}{i!}\right)\left(\sum_{i=0}^{n}c_ix^i\right)\\&=\sum_{i\geq 0}\sum_{j=0}^{i}c_jx^j\frac{n^{i-j}x^{i-j}}{(i-j)!}\\&=\sum_{i\geq 0}\frac{x^i}{i!}\sum_{j=0}^{i}n^{i-j}i^{\underline{j}}c_j\end{aligned}$$

$$a$$

$$a,b$$

$$\hat{F}(x),\hat{G}(x)$$

$$\hat{F}(x)\hat{G}(x)$$

$$\langle 1,1,1,\cdots\rangle$$

$$\mathrm{e}^x$$

$$x=0$$

$$\langle 1,p,p^2,\cdots\rangle$$

$$a$$

$$F(x)$$

$$\left\langle\frac{a_n}{n!}\right\rangle$$

$$f^n(x)$$

$$f$$

$$[x^k]\hat{H}(x)$$

$$[x^k]\hat{H}(x)$$

$$x^{a_i}$$

$$\sum_i a_i = k$$

$$n$$

$$k > 0$$

$$k = 0$$

$$\exp$$

$$\exp$$

$$f(x)$$

$$0$$

$$\exp(f(x))$$

$$f^k(x)$$

$$k$$

$$f(x)$$

$$\frac{1}{k!}$$

$$F_k(n)$$

$$n$$

$$k$$

$$\exp$$

$$f_i$$

$$i$$

$$i$$

$$F_k(n)$$

$$f_n$$

$$\hat{F}(x)$$

$$F_k(n)$$

$$G_k(x)$$

$$k \geq 0$$

$$\exp(f(x))$$

$$f(x)$$

$$F_k(n)$$

$$n$$
$$k$$
$$g_i$$
$$i$$
$$f_i$$
$$G(x)$$
$$\{g_i\}$$
$$H_k(x)$$
$$\{F_k(n)\}$$
$$n$$
$$i$$
$$g_i$$
$$n-i$$
$$k-1$$
$$F_{k-1}(n-i)$$
$$k$$
$$k$$
$$n-(k-1)\geq i$$
$$k-1$$
$$F_{k-1}(n-i)$$
$$0$$
$$k=1$$
$$H_1(x)=G(x)$$
$$g_0=0$$
$$g_0=1$$
$$[x^n]G^k$$
$$[x^n]G^y, y>k$$
$$G$$
$$\exp$$
$$n$$
$$1,2,\cdots,n$$
$$n$$

$$(n-1)!$$

$$n$$

$$\exp \hat{Q}(x) = \hat{P}(x)$$

$$\exp$$

$$p = [4, 3, 2, 5, 1]$$

$$p_i$$

$$i$$

$$[5, 3, 2, 1, 4]$$

$$n$$

$$1, 2, \cdots, n$$

$$n$$

$$1, 2, \cdots, n$$

$$\exp$$

$$n$$

$$\hat{F}(x)$$

$$n$$

$$\exp \hat{F}(x)$$

$$n$$

$$n$$

$$\hat{F}(x)$$

$$n$$

$$\exp \hat{F}(x)$$

$$\ln$$

$$n$$

$$p_i \neq i$$

$$1$$

$$\exp(-\ln(1-x) - x)$$

$$f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$$

$$nk \leq 2 \times 10^6, 1 \leq k \leq 3$$

$$i$$

$$f(i)$$

$$i$$

$$k$$
$$k-1$$
$$k$$
$$1$$
$$n$$
$$k$$
$$n$$
$$k$$
$$\widehat{F}_k(x)$$
$$k$$
$$k-1$$
$$\exp \hat{F}_k(x)$$
$$n$$
$$n$$
$$a_1, a_2, \cdots, a_n$$
$$0$$
$$s$$
$$k$$
$$1, 2, \cdots, n$$
$$x$$
$$s$$
$$\prod_{i \neq x} a_i$$
$$a_x$$
$$k$$
$$s$$
$$1 \leq n \leq 5000, 1 \leq k \leq 10^9, 0 \leq a_i \leq 10^9$$
$$k$$
$$a_i$$
$$b_i$$
$$k$$
$$\prod_{i=1}^{n} (a_i - b_i)$$

$$\prod_{i=1}^{n}(a_i - b_i)$$

$$n^k$$

$$k$$

$$i$$

$$b_i$$

$$a_j$$

$$F_j(x)$$

$$\prod_{j=1}^{n}(a_j - x)$$

$$n$$

$$\sum_{i=0}^{n} c_i x^i$$

$$x^k$$

$$\tilde{F}(x) = \sum_{i\geq 1} \frac{f_i}{i^x}$$

$$\tilde{F}(x) = \prod_{p\in\mathcal{P}} \left(1 + \frac{f_p}{p^x} + \frac{f_{p^2}}{p^{2x}} + \frac{f_{p^3}}{p^{3x}} + \cdots\right)$$

$$\tilde{F}(x)\tilde{G}(x) = \sum_i \sum_j \frac{f_i g_j}{(ij)^x} = \sum_i \frac{1}{i^x} \sum_{d\mid i} f_d g_{\frac{i}{d}}$$

$$\zeta(x) = \prod_{p\in\mathcal{P}} \left(1 + \frac{1}{p^x} + \frac{1}{p^{2x}} + \dots\right) = \prod_{p\in\mathcal{P}} \frac{1}{1 - p^{-x}}$$

$$\widetilde{M}(x) = \prod_{p\in\mathcal{P}} \left(1 - \frac{1}{p^x}\right) = \prod_{p\in\mathcal{P}} (1 - p^{-x})$$

$$\tilde{\Phi}(x) = \prod_{p\in\mathcal{P}} \left(1 + \frac{p-1}{p^x} + \frac{p(p-1)}{p^{2x}} + \frac{p^2(p-1)}{p^{3x}} + \dots\right) = \prod_{p\in\mathcal{P}} \frac{1 - p^{-x}}{1 - p^{1-x}}$$

$$\tilde{I}_k(x) = \prod_{p\in\mathcal{P}} \left(1 + \frac{p^k}{p^x} + \frac{p^{2k}}{p^{2x}} + \dots\right) = \prod_{p\in\mathcal{P}} \frac{1}{1 - p^{k-x}} = \zeta(x - k)$$

$$h(x) = \sum_{d|x} f(d) g\left(\frac{x}{d}\right) = \sum_{ab=x} f(a) g(b)$$

$$h = f * g$$

$$\tilde{F}(x)\tilde{G}(x) = \sum_i \sum_j \frac{f_i g_j}{(ij)^x} = \sum_i \frac{1}{i^x} \sum_{d \mid i} f_d g_{\frac{i}{d}}$$

$$\begin{aligned} r(y) &= \sum_{d|y} (f(d) - g(d)) h\left(\frac{y}{d}\right) \\ &= (f(y) - g(y))h(1) \\ &\neq 0 \end{aligned}$$

$$g(x) = \frac{\varepsilon(x) - \sum_{d|x, d \neq 1} f(d) g\left(\frac{x}{d}\right)}{f(1)}$$

$$h(a) = \sum_{d_1|a} f(d_1) g\left(\frac{a}{d_1}\right), h(b) = \sum_{d_2|b} f(d_2) g\left(\frac{b}{d_2}\right),$$

$$\begin{aligned} h(a)h(b) &= \sum_{d_1|a} f(d_1) g\left(\frac{a}{d_1}\right) \sum_{d_2|b} f(d_2) g\left(\frac{b}{d_2}\right) \\ &= \sum_{d|ab} f(d) g\left(\frac{ab}{d}\right) \\ &= h(ab) \end{aligned}$$

$$g(nm) = -\sum_{d|nm, d \neq 1} f(d) g\left(\frac{nm}{d}\right) = -\sum_{a|n, b|m, ab \neq 1} f(ab) g\left(\frac{nm}{ab}\right)$$

$$\begin{aligned} g(nm) &= -\sum_{a|n, b|m, ab \neq 1} f(ab) g\left(\frac{nm}{ab}\right) \\ &= -\sum_{a|n, b|m, ab \neq 1} f(a) f(b) g\left(\frac{n}{a}\right) g\left(\frac{m}{b}\right) \\ &= f(1)f(1)g(n)g(m) - \sum_{a|n, b|m} f(a) f(b) g\left(\frac{n}{a}\right) g\left(\frac{m}{b}\right) \\ &= g(n)g(m) - \sum_{a|n} f(a) g\left(\frac{n}{a}\right) \sum_{b|m} f(b) g\left(\frac{m}{b}\right) \\ &= g(n)g(m) - \varepsilon(n) - \varepsilon(m) \\ &= g(n)g(m) \end{aligned}$$

$$\varepsilon = \mu * 1 \Longleftrightarrow \varepsilon(n) = \sum_{d|n} \mu(d)$$

$$d = 1 * 1 \Longleftrightarrow d(n) = \sum_{d|n} 1$$

$$\sigma = \mathrm{id} * 1 \Longleftrightarrow \sigma(n) = \sum_{d|n} d$$

$$\varphi = \mu * \mathrm{id} \Longleftrightarrow \varphi(n) = \sum_{d|n} d \cdot \mu(\frac{n}{d})$$

$$\tilde{F}(x) = \prod_{p \in \mathcal{P}} \left(1 + \sum_{k \geq 1} \frac{p^{3k-1}(p-1)}{p^{kx}} \right) = \prod_{p \in \mathcal{P}} \frac{1 - p^{2-x}}{1 - p^{3-x}} = \frac{\zeta(x-3)}{\zeta(x-2)}$$

$$\mathcal{P}$$

$$f_1, f_2, ...$$

$$f$$

$$\forall i \perp j,\, f_{ij} = f_i f_j$$

$$f, g$$

$$[1, 1, 1, ...]$$

$$\sum_{i \geq 1} \frac{1}{i^x} = \zeta(x)$$

$$\zeta$$

$$[1, 1, 1, ...]$$

$$\mu$$

$$\zeta(x) \widetilde{M}(x) = 1$$

$$\widetilde{M}(x) = \frac{1}{\zeta(x)}$$

$$\varphi$$

$$\tilde{\Phi}(x) = \frac{\zeta(x-1)}{\zeta(x)}$$

$$I_k(n) = n^k$$

$$\varphi * 1 = I$$

$$*$$

$$\tilde{\Phi}(x)\zeta(x) = \zeta(x-1)$$

$$\sigma_k(n) = \sum_{d \,|\, n} d^k$$

$$\tilde{S}(x) = \zeta(x-k)\zeta(x)$$

$$u(n) = |\ \mu(n)|$$

$$\tilde{U}(x) = \prod_{p \in \mathcal{P}} (1 + p^{-x}) = \frac{\zeta(x)}{\zeta(2x)}$$

$$f(x)$$

$$g(x)$$

$$h(x)$$

$$f, g$$

$$f^*g = g^*f$$

$$(f^*g)^*h = f^*(g^*h)$$

$$(f+g)^*h = f^*h + g^*h$$

$$f = g$$

$$f^*h = g^*h$$

$$h(x)$$

$$h(1) \neq 0$$

$$x$$

$$f(x) \neq g(y)$$

$$y \in \mathbb{N}$$

$$f(y) \neq g(y)$$

$$r = f^*h - g^*h = (f-g)^*h$$

$$f^*h$$

$$g^*h$$

$$y$$

$$f^*h \neq g^*h$$

$$\varepsilon$$

$$f$$

$$f^*\varepsilon = f$$

$$1$$

$$1$$

$$\zeta$$

$$f(x) \neq 0$$

$$g(x)$$

$$f^*g = \varepsilon$$

$$g(x)$$

$$f(x)$$

$$g(x)$$

$$f(x)$$

$$g(x)$$

$$h = f^*g$$

$$\gcd(a, b) = 1$$

$$f^*g = \varepsilon$$

$$f(1) = 1$$

$$nm = 1$$

$$g(nm) = g(1) = 1$$

$$nm > 1(\gcd(n, m) = 1)$$

$$xy < nm(\gcd(x, y) = 1)$$

$$g(xy) = g(x)g(y)$$

$$\frac{nm}{ab} < nm$$

$$f$$

$$g$$

$$f * g$$

$$g$$

$$f_i = i^2\varphi(i)$$

$$g$$

$$f$$

$$\tilde{F}(x)\zeta(x-2) = \zeta(x-3)$$

$$\zeta(x-2)$$

$$I_2$$

$$g = I_2$$

$$f * g = I_3$$

$$
\begin{aligned}
[x^{n+t}](A_0(x)B_0(x)) &= \sum_{i=0}^{n+t}([x^i]A_0(x))([x^{n+t-i}]B_0(x)) \\
&= \sum_{i=0}^{n+t} a_i c^{i^2-(i-t)^2} \\
&= c^{-t^2}\sum_{i=0}^{n+t} a_i c^{2it} \\
&= c^{-t^2} A(c^{2t})
\end{aligned}
$$

$$ki = \binom{i+k}{2} - \binom{i}{2} - \binom{k}{2}$$

$$A(c^k) = c^{-\binom{k}{2}}\sum_{i=0}^{n} a_i c^{\binom{i+k}{2}-\binom{i}{2}}$$

$$
\begin{aligned}
[x^{n+t}](A_0(x)B_0(x)) &= \sum_{i=0}^{n+t}([x^{n+t-i}]A_0(x))([x^i]B_0(x)) \\
&= \sum_{i=0}^{n+t} a_{i-t} c^{\binom{i}{2}-\binom{i-t}{2}} \\
&= \sum_{i=-t}^{n} a_i c^{\binom{i+t}{2}-\binom{i}{2}} \\
&= c^{\binom{t}{2}} \cdot A(c^t)
\end{aligned}
$$

$$A(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$$

$$c \in \mathbb{C} \setminus \{0\}$$

$$A(1), A(c), A(c^2), ...$$

$$c$$

$$A_0(x) = \sum_{i \geq 0} a_i c^{i^2} x^i$$

$$\forall j > n$$

$$a_j = 0$$

$$B_0(x) = \sum_{i \geq 0} c^{-(i-n)^2} x^i$$

$$t \geq 0$$

$$c^{t^2}[x^{n+t}](A_0(x)B_0(x))$$

$$A(1), A(c^2), ...$$

$$A(c), A(c^3), ...$$

$$A(cx)$$

$$x^n$$

$$k$$

$$i$$

$$\binom{a}{b} = \frac{a(a-1)\cdots(a-b+1)}{b!}$$

$$A_0(x) = \sum_i a_{n-i} c^{-\binom{n-i}{2}} x^i$$

$$\forall j > n$$

$$\forall j < 0$$

$$a_j = 0$$

$$B_0(x) = \sum_{i \geq 0} c^{\binom{i}{2}} x^i$$

$$t \geq 0$$

$$c^{-\binom{t}{2}} [x^{n+t}] (A_0(x) B_0(x))$$

$$A(1), A(c), \dots$$

$$\forall i \geq 0$$

$$c^{\binom{i+1}{2}} = c^{\binom{i}{2}} \cdot c^i$$

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

$$x_{i+1} = x_i - \frac{x_i^2 - n}{2x_i} = \frac{x_i + \frac{n}{x_i}}{2}$$

$$[a, b]$$

$$f(x)$$

$$f(x) = 0$$

$$f(x)$$

$$x_0$$

$$x_i$$

$$f(x)$$

$$(x_i, f(x_i))$$

$$l$$

$$l$$

$$x$$

$$x_{i+1}$$

$$f(x)$$

$$x_i$$

$$f(x) = x^2 - n$$

$$\sqrt{n}$$

$$x^2 \leq n$$

$$x$$

$$x_0$$

$$x_0 = 2^{\lfloor \frac{1}{2} \log_2 n \rfloor}$$

$$x_0$$

$$n = 10^{1000}$$

$$x_0$$

$$x_0 = 1$$

$$\begin{array}{ccccccccccccc}
1 & & 5 & & 14 & & 30 & & 55 & & 91 \\
& 4 & & 9 & & 16 & & 25 & & 36 & \\
& & 5 & & 7 & & 9 & & 11 & & \\
& & & 2 & & 2 & & 2 & & &
\end{array}$$

$$f(k) = \sum_{i=1}^{n} \binom{k-1}{i-1} \sum_{j=1}^{i} (-1)^{i+j} \binom{i-1}{j-1} f(j)$$

$$f(x) \equiv f(a) \pmod{(x-a)}$$

$$\begin{cases} f(x) \equiv y_1 \pmod{(x-x_1)} \\ f(x) \equiv y_2 \pmod{(x-x_2)} \\ \vdots \\ f(x) \equiv y_n \pmod{(x-x_n)} \end{cases}$$

$$M(x) = \prod_{i=1}^{n} (x - x_i),$$

$$m_i(x) = \frac{M}{x - x_i}$$

$$m_i(x_i)^{-1} = \prod_{j \neq i} (x_i - x_j)^{-1}$$

$$f(x) \equiv \sum_{i=1}^{n} y_i (m_i(x))(m_i(x_i)^{-1}) \pmod{M(x)}$$

$$\equiv \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} \pmod{M(x)}$$

$$f(x) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$$f(k) = \sum_{i=1}^{n} y_i \prod_{j \neq i} \frac{k - x_j}{x_i - x_j}$$

$$f(x) = \sum_{i=1}^{n+1} y_i \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$$= \sum_{i=1}^{n+1} y_i \prod_{j \neq i} \frac{x - j}{i - j}$$

$$\frac{\prod_{j=1}^{n+1} (x - j)}{x - i}$$

$$(-1)^{n+1-i} \cdot (i-1)! \cdot (n+1-i)!$$

$$f(x) = \sum_{i=1}^{n+1} y_i \cdot \frac{\prod_{j=1}^{n+1} (x - j)}{(x - i) \cdot (-1)^{n+1-i} \cdot (i-1)! \cdot (n+1-i)!}$$

$$f(x) = \sum_{i=1}^{n+1} \binom{x-1}{i-1} \sum_{j=1}^{i} (-1)^{i+j} \binom{i-1}{j-1} y_j = \sum_{i=1}^{n+1} y_i \cdot \frac{\prod_{j=1}^{n+1} (x - j)}{(x - i) \cdot (-1)^{n+1-i} \cdot (i-1)! \cdot (n+1-i)!}$$

$$n$$

$$(x_i, y_i)$$

$$k$$

$$\forall i, j$$

$$i \neq j \Longleftrightarrow x_i \neq x_j$$

$$f(x_i) \equiv y_i \pmod{998244353}$$

$$\deg(f(x)) < n$$

$$\deg(0) = -\infty$$

$$f(k) \bmod 998244353$$

$$x_i = i$$

$$f(x) = \sum_{i=0}^{3} a_i x^i$$

$$f(1)$$

$$f(6)$$

$$1, 5, 14, 30, 55, 91$$

$$f(x)$$

$$n$$

$$f(x)$$

$$i-1$$

$$\sum_{j=1}^{i} (-1)^{i+j} \binom{i-1}{j-1} f(j)$$

$$i-1$$

$$f(k)$$

$$\binom{k-1}{i-1}$$

$$O(n^2)$$

$$f(x) = \sum_{i=0}^{n-1} a_i x^i$$

$$x_i$$

$$f(x)$$

$$f(x_i) = y_i$$

$$n$$

$$n$$

$$a_i$$

$$f(x)$$

$$O(n^3)$$

$$f(x) - f(a) = (a_0 - a_0) + a_1(x^1 - a^1) + a_1(x^2 - a^2) + \cdots + a_n(x^n - a^n)$$

$$(x-a)$$

$$f(x)$$

$$m_i(x)$$

$$(x - x_i)$$

$$\deg(f(x)) < n$$

$$M(x)$$

$$f(x)$$

$$f(x)$$

$$P_1(x_1, y_1), P_2(x_2, y_2), \cdots, P_n(x_n, y_n)$$

$$i$$

$$x$$

$$P_i'(x_i, 0)$$

$$n$$

$$f_1(x), f_2(x), \cdots, f_n(x)$$

$$i$$

$$f_i(x)$$

$$\begin{cases} P_j'(x_j, 0), (j \neq i) \\ P_i(x_i, y_i) \end{cases}$$

$$f(x) = \sum_{i=1}^{n} f_i(x)$$

$$f_i(x) = a \cdot \prod_{j \neq i} (x - x_j)$$

$$P_i(x_i, y_i)$$

$$a = \frac{y_i}{\prod_{j \neq i} (x_i - x_j)}$$

$$f_i(x) = y_i \cdot \frac{\prod_{j \neq i} (x - x_j)}{\prod_{j \neq i} (x_i - x_j)} = y_i \cdot \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$$f(x) = \sum_{i=1}^{n} y_i \cdot \prod_{j \neq i} \frac{x - x_j}{x_i - x_j}$$

$$998244353$$

$$f(k)$$

$$k$$

$$O(n^2)$$

$$O(n)$$

$$n$$

$$f(x)$$

$$f(1), \cdots, f(n+1)$$

$$1 \le i \le n+1$$

$$i-j$$

$$1, \cdots, n+1$$

$$(x-i)$$

$$O(n)$$

$$n, k$$

$$\sum_{i=1}^{n} i^k$$

$$10^9+7$$

$$k+1$$

$$1^i, \cdots, (k+2)^i$$

$$O(n)$$

$$\int_l^r f(x)\mathrm{d}x = \frac{(r-l)(f(l)+f(r)+4f(\frac{l+r}{2}))}{6}$$

$$\begin{aligned}
\int_l^r f(x)\mathrm{d}x &= F(r)-F(l) \\
&= \frac{a}{3}(r^3-l^3)+\frac{b}{2}(r^2-l^2)+c(r-l) \\
&= (r-l)(\frac{a}{3}(l^2+r^2+lr)+\frac{b}{2}(l+r)+c) \\
&= \frac{r-l}{6}(2al^2+2ar^2+2alr+3bl+3br+6c) \\
&= \frac{r-l}{6}((al^2+bl+c)+(ar^2+br+c)+4(a(\frac{l+r}{2})^2+b(\frac{l+r}{2})+c)) \\
&= \frac{r-l}{6}(f(l)+f(r)+4f(\frac{l+r}{2}))
\end{aligned}$$

$$-\tfrac{1}{90}\left(\tfrac{r-l}{2}\right)^5 f^{(4)}(\xi)$$

$$f(x)$$

$$[l, r]$$

$$\int_l^r f(x)\mathrm{d}x$$

$$f(x)$$

$$[l, r]$$

$$x$$

$$x$$

$$x$$

$$f(x) = ax^2 + bx + c$$

$$f(x) = ax^2 + bx + c$$

$$F(x) = \int_0^x f(x)\mathrm{d}x = \frac{a}{3}x^3 + \frac{b}{2}x^2 + cx + D$$

$$n$$

$$[l, r]$$

$$2n$$

$$x$$

$$x_i = l + ih, \quad i = 0...2n,$$

$$h = \frac{r-l}{2n}.$$

$$[x_{2i-2}, x_{2i}]$$

$$i = 1...n$$

$$[x_{2i-2}, x_{2i}]$$

$$i = 1...n$$

$$(x_{2i-2}, x_{2i-1}, x_{2i})$$

$$P(x)$$

$$P(x)$$

$$\int_{x_{2i-2}}^{x_{2i}} f(x)\ dx \approx \int_{x_{2i-2}}^{x_{2i}} P(x)\ dx = (f(x_{2i-2}) + 4f(x_{2i-1}) + (f(x_{2i}))\frac{h}{3}$$

$$\int_l^r f(x)dx \approx (f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + ... + 4f(x_{2N-1}) + f(x_{2N}))\frac{h}{3}$$

$$\xi$$

$$[l, r]$$

$$n$$

$$\begin{cases} 4x + y = 100 \\ x - y \quad = 100 \end{cases}$$

$$\begin{cases} 2x_1 + 5x_3 + 6x_4 = 9 \\ x_3 + x_4 \qquad\qquad\quad = -4 \\ 2x_3 + 2x_4 \qquad\qquad = -8 \end{cases}$$

$$\left(\begin{array}{cccc|c} 2 & 0 & 5 & 6 & 9 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 2 & 2 & -8 \end{array}\right)$$

$$\xrightarrow{r_3-2r_2} \left(\begin{array}{cccc|c} 2 & 0 & 5 & 6 & 9 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{\frac{r_1}{2}} \left(\begin{array}{cccc|c} 1 & 0 & 2.5 & 3 & 4.5 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{r_1-r_2\times 2.5} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0.5 & 14.5 \\ 0 & 0 & 1 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{cases} x_1 + 0.5x_4 &= 14.5 \\ x_3 + x_4 &= -4 \end{cases}$$

$$\begin{cases} x_1 = -0.5x_4 + 14.5 \\ x_3 = -x_4 - 4 \end{cases}$$

$$\begin{cases} x_1 = -0.5x_4 + 14.5 \\ x_2 = x_2 \\ x_3 = -x_4 - 4 \\ x_4 = x_4 \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} x_2 + \begin{pmatrix} -0.5 \\ 0 \\ -1 \\ 1 \end{pmatrix} x_4 + \begin{pmatrix} 14.5 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} C_1 + \begin{pmatrix} -0.5 \\ 0 \\ -1 \\ 1 \end{pmatrix} C_2 + \begin{pmatrix} 14.5 \\ 0 \\ -4 \\ 0 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$$

$$\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}$$

$$\begin{cases} a_{1,1}x_1 \oplus a_{1,2}x_2 \oplus \cdots \oplus a_{1,n}x_n &= b_1 \\ a_{2,1}x_1 \oplus a_{2,2}x_2 \oplus \cdots \oplus a_{2,n}x_n &= b_2 \\ \cdots & \cdots \\ a_{m,1}x_1 \oplus a_{m,2}x_2 \oplus \cdots \oplus a_{m,n}x_n &= b_1 \end{cases}$$

$$y$$

$$5x = 200$$

$$x = 40$$
$$x = 40$$
$$y = -60$$
$$k$$
$$k$$
$$A$$
$$b$$
$$(A \mid b)$$
$$x_1$$
$$x_3$$
$$x_1$$
$$x_3$$
$$x_2$$
$$x_4$$
$$x_2 = x_2, x_4 = x_4$$
$$C_1$$
$$C_2$$
$$x_2$$
$$x_4$$
$$C_1$$
$$C_2$$
$$N \times N$$
$$S_n$$
$$n$$
$$\sigma$$
$$\sigma$$
$$\operatorname{sgn}(\sigma) = 1$$
$$\operatorname{sgn}(\sigma) = -1$$
$$k$$
$$O(n^3)$$
$$A$$
$$A^{-1}$$

$$A \times A^{-1} = A^{-1} \times A = I$$

$$A$$

$$A^{-1}$$

$$n$$

$$A$$

$$n \times 2n$$

$$(A, I_n)$$

$$(I_n, A^{-1})$$

$$A$$

$$A^{-1}$$

$$I_n$$

$$A$$

$$\oplus$$

$$a_{i,j}$$

$$b_i$$

$$0$$

$$1$$

$$0$$

$$1$$

$$01$$

$$0$$

$$1$$

$$O(\frac{n^2 m}{\omega})$$

$$n$$

$$m$$

$$\omega$$

$$32$$

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{n} \left[\exists d \in (\sqrt{n}, n] \cap \mathbb{P}, d \mid i\right] f(i) + \sum_{i=1}^{n} \left[\forall d \in (\sqrt{n}, n] \cap \mathbb{P}, d \nmid i\right] f(i)$$

$$\sum_{i=1}^{n} f(i) = \sum_{i=1}^{\sqrt{n}} f(i) \cdot \left(\sum_{d=\lfloor \sqrt{n} \rfloor + 1}^{\lfloor \frac{n}{i} \rfloor} [d \in \mathbb{P}] f(d) \right) + \sum_{i=1}^{n} \left[\forall d \in (\sqrt{n}, n] \cap \mathbb{P}, d \nmid i\right] f(i)$$

$$\sum_{i=1}^{n} f(i)$$

$$\mathbb{P}$$

$$p_i$$

$$i$$

$$m$$

$$\sqrt{n}$$

$$p \in \mathbb{P}, c \in \mathbb{N}$$

$$f(p^c)$$

$$p$$

$$[1, n]$$

$$> \sqrt{n}$$

$$\left\lfloor \frac{n}{i} \right\rfloor (i \in [1, n] \cap \mathbb{N})$$

$$\sqrt{n}$$

$$[1, n]$$

$$> \sqrt{n}$$

$$\sum_{i=1}^{\sqrt{n}} f(i) \cdot \left(\sum_{d=\lfloor \sqrt{n} \rfloor + 1}^{\lfloor \frac{n}{i} \rfloor} [d \in \mathbb{P}] f(d) \right)$$

$$i$$

$$\mathcal{O}(1)$$

$$g(t, l) = \sum_{i=1}^{l} [\forall j \in [1, t], \gcd(i, p_j) = 1] f(i)$$

$$[1, l]$$

$$p_1, p_2, ..., p_t$$

$$f$$

$$\sum_{i=1}^{\sqrt{n}} f(i) \cdot g\left(m, \left\lfloor \frac{n}{i} \right\rfloor\right)$$

$$g(0, l) = \sum_{i=1}^{l} f(i)$$

$$g(t, l) = g(t-1, l) - f(p_t) \cdot g\left(t-1, \left\lfloor \frac{l}{p_t} \right\rfloor\right)$$

$$l$$

$$\sqrt{n}$$

$$\mathcal{O}\left(\frac{\sqrt{n}}{\ln\sqrt{n}}\cdot\sqrt{n}\right)=\mathcal{O}\left(\frac{n}{\log n}\right)$$

$$p_{t+1}^2>l$$

$$1$$

$$g(t,l)=f(1)=1$$

$$p_t^2>l$$

$$g(t,l)=g(t-1,l)-f(p_t)$$

$$p_t^2>l$$

$$t$$

$$t_l$$

$$\forall t>t_l, g(t,l)=g(t_l,l)-\sum_{i=t_l}^{t-1}f(p_i)$$

$$f$$

$$g$$

$$\mathcal{O}\left(\frac{n^{\frac{3}{4}}}{\log n}\right)$$

$$\sum_{i=1}^{n}\left[\forall d\in(\sqrt{n},n]\cap\mathbb{P}, d\nmid i\right]f(i)$$

$$h(t,l)=\sum_{i=1}^{l}\left[i=\prod_{j=t}^{m}p_j^{c_j}, c_j\in\mathbb{N}\right]f(i)$$

$$[1,l]$$

$$p_t, p_{t+1}, \ldots, p_m$$

$$f$$

$$h(0,n)$$

$$h(m+1,l)=1$$

$$h(t,l)=h(t+1,l)+\sum_{c\in\mathbb{N}^*}f(p_t^c)\cdot h\left(t+1,\left\lfloor\frac{l}{p_t^c}\right\rfloor\right)$$

$$l$$

$$\sqrt{n}$$

$$\mathcal{O}\left(\sqrt{n}\cdot\frac{\sqrt{n}}{\ln\sqrt{n}}\right)=\mathcal{O}\left(\frac{n}{\log n}\right)$$

$$g$$

$$p_t > l$$

$$p_t, p_{t+1}, ..., p_m$$

$$1$$

$$h(t,l)=f(1)=1$$

$$\forall p_t^2 > l, h(t,l) = h(t-1,l)+f(p_t)$$

$$p_t^2 > l$$

$$t$$

$$t_l$$

$$h$$

$$h$$

$$\sum_{i=p_{t_l}}^{\min(l,\sqrt{n})}[i\in\mathbb{P}]f(i)$$

$$\mathcal{O}\left(\frac{n^{\frac{3}{4}}}{\log n}\right)$$

$$\sum_{i=1}^{n}f(i)$$

$$0\equiv a^2-1\equiv(a+1)(a-1)\pmod p$$

$$(7!)_p\equiv 1\cdot 2\cdot\underbrace{1}_{3}\cdot 4\cdot 5\underbrace{2}_{6}\cdot 7\equiv 2\bmod 3$$

$$\begin{aligned}(n!)_p &= 1\cdot 2\cdot 3\cdot ...\cdot(p-2)\cdot(p-1)\cdot\underbrace{1}_{p}\cdot(p+1)\cdot(p+2)\cdot ...\cdot(2p-1)\cdot\underbrace{2}_{2p}\\ &\quad\cdot(2p+1)\cdot ...\cdot(p^2-1)\cdot\underbrace{1}_{p^2}\cdot(p^2+1)\cdot ...\cdot n\pmod p\\ &= 1\cdot 2\cdot 3\cdot ...\cdot(p-2)\cdot(p-1)\cdot\underbrace{1}_{p}\cdot 1\cdot 2\cdot ...\cdot(p-1)\cdot\underbrace{2}_{2p}\cdot 1\cdot 2\\ &\quad\cdot ...\cdot(p-1)\cdot\underbrace{1}_{p^2}\cdot 1\cdot 2\cdot ...\cdot(n\bmod p)\pmod p\end{aligned}$$

$$\begin{aligned}(n!)_p = &\underbrace{1\cdot 2\cdot 3\cdot ...\cdot(p-2)\cdot(p-1)\cdot 1}_{1\text{ st}}\cdot\underbrace{1\cdot 2\cdot 3\cdot ...\cdot(p-2)\cdot(p-1)\cdot 2}_{2\text{ nd}}\cdot ...\\ &\cdot\underbrace{1\cdot 2\cdot 3\cdot ...\cdot(p-2)\cdot(p-1)\cdot 1}_{p\text{ th}}\cdot ...\cdot\underbrace{1\cdot 2\cdot\cdot ...\cdot(n\bmod p)}_{\text{tail}}\pmod p.\end{aligned}$$

$$(p-1)! \equiv -1 \pmod p$$

$$(n!)_p = \underbrace{\dots \cdot 1} \cdot \underbrace{\dots \cdot 2} \cdot \dots \cdot \underbrace{\dots \cdot (p-1)} \cdot \underbrace{\dots \cdot 1} \cdot \underbrace{\dots \cdot 1} \cdot \underbrace{\dots \cdot 2} \cdots$$

$$\left(\left\lfloor \frac{n}{p} \right\rfloor !\right)_p$$

$$v_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - S_p(n)}{p-1}$$

$$v_p\left(\binom{m}{n}\right) = \frac{S_p(n) + S_p(m-n) - S_p(m)}{p-1}$$

$$(p^{(q)!})_p \equiv \begin{cases} 1, & (p=2) \wedge (q \geq 3), \\ -1, \text{otherwise.} \end{cases}$$

$$(n!)_p \equiv (\pm 1)^{\lfloor n/p^q \rfloor} (N_{(0)!})_p \pmod{p^q}$$

$$\begin{aligned} (n!)_p &= \prod_{(1 \leq r \leq n)'} r \\ &= \left(\prod_{i=0}^{\lfloor n/p^q \rfloor - 1} \prod_{(1 \leq j \leq p^q)'} (ip^q + j) \right) \left(\prod_{(1 \leq j \leq N_0)'} (\lfloor n/p^q \rfloor p^q + j) \right) \\ &\equiv ((p^{(q)!})_p)^{\lfloor n/p^q \rfloor} (N_{(0)!})_p \\ &\equiv (\pm 1)^{\lfloor n/p^q \rfloor} (N_{(0)!})_p \pmod{p^q} \end{aligned}$$

$$\frac{n!}{p^{\sum_{j \geq 1} \left\lfloor \frac{n}{p^j} \right\rfloor}} \equiv (\pm 1)^{\sum_{j \geq q} \left\lfloor \frac{n}{p^j} \right\rfloor} \prod_{j \geq 0} (N_{(j)!})_p \pmod{p^q}$$

$$\frac{(\pm 1)^{\sum_{j \geq q} (\lfloor n/p^j \rfloor - \lfloor m/p^j \rfloor - \lfloor r/p^j \rfloor)}}{p^{\nu(n!) - \nu(m!) - \nu(r!)}} \binom{n}{m} \equiv \frac{n!/p^{\nu(n!)}}{(m!/p^{\nu(m!)})(r!/p^{\nu(r!)})} \pmod{p^q}$$

$$\sum_{k=1}^{n} \left\lfloor \frac{(3k+6)!+1}{3k+7} - \left\lfloor \frac{(3k+6)!}{3k+7} \right\rfloor \right\rfloor$$

$$(3k+6)! \equiv -1 \pmod{3k+7}$$

$$\left\lfloor \frac{(3k+6)!+1}{3k+7} - \left\lfloor \frac{(3k+6)!}{3k+7} \right\rfloor \right\rfloor = \left\lfloor k - \left\lfloor k - \frac{1}{3k+7} \right\rfloor \right\rfloor = 1$$

$$(3k+6)! \equiv 0 \pmod{3k+7}$$

$$\left\lfloor \frac{(3k+6)!+1}{3k+7} - \left\lfloor \frac{(3k+6)!}{3k+7} \right\rfloor \right\rfloor = \left\lfloor k + \frac{1}{3k+7} - k \right\rfloor = 0$$

$$\sum_{k=1}^{n} \left\lfloor \frac{(3k+6)!+1}{3k+7} - \left\lfloor \frac{(3k+6)!}{3k+7} \right\rfloor \right\rfloor = \sum_{k=1}^{n} [3k+7 \text{ is prime}]$$

p

$$(p-1)! \equiv -1 \pmod p$$

$$p = 2$$

$$p \geq 3$$

$$\mathbf{Z}_p$$

$$\overline{-1}$$

$$\mathbf{Z}_p$$

$$a$$

$$a^{-1}$$

$$\mathbf{Z}_p$$

$$\overline{1}$$

$$a$$

$$a^{-1}$$

$$a = a^{-1}$$

$$a^2 \equiv 1 \pmod p$$

$$a \equiv 1 \pmod p$$

$$a \equiv -1 \pmod p$$

$$\mathbf{Z}_p \setminus \{\overline{0}, \overline{1}, \overline{-1}\}$$

$$\overline{1}$$

$$\mathbf{Z}_p$$

$$\overline{-1}$$

$$n$$

$$(n!)_p$$

$$n$$

$$p$$

$$(n!)_p = n!/(\lfloor n/p \rfloor ! p^{\lfloor n/p \rfloor})$$

$$(p!)_p = (p-1)! \equiv -1 \pmod p$$

$$p$$

$$p$$

$$p!$$

$$p$$

$$n! \bmod p$$

$$p$$

$$p$$

$$(n!)_p$$

$$p$$

$$p$$

$$(p-1)! \bmod p$$

$$\left\lfloor \frac{n}{p} \right\rfloor$$

$$\left\lfloor \frac{n}{p} \right\rfloor$$

$$-1$$

$$-1$$

$$1$$

$$-1$$

$$p$$

$$O(p)$$

$$(n!)_p$$

$$O(p)$$

$$(\left\lfloor \frac{n}{p} \right\rfloor !)_p$$

$$O(\log_p n)$$

$$O(p \log_p n)$$

$$0!, 1!, 2!, ..., (p-1)!$$

$$p$$

$$O(\log_p n)$$

$$0!, 1!, ..., (p-1)!$$

$$O(p + \log_p n)$$

$$(n!)_p$$

$$O(\log_p n)$$

$$0!,\ 1!,\ 2!,\ ...,\ (p-1)!$$

$$p$$

$$n$$

$$p$$

$$p$$

$$n!$$

$$p$$

$$v_p(n!)$$

$$S_p(n)$$

$$p$$

$$n$$

$$v_2(n!) = n - S_2(n)$$

$$n!$$

$$1 \times 2 \times \cdots \times p \times \cdots \times 2p \times \cdots \times \lfloor n/p \rfloor p \times \cdots \times n$$

$$p$$

$$p \times 2p \times \cdots \times \lfloor n/p \rfloor p = p^{\lfloor n/p \rfloor} \lfloor n/p \rfloor !$$

$$\lfloor n/p \rfloor !$$

$$p$$

$$O(\log n)$$

$$\nu(n!) = \sum_{j \geq 1} \lfloor n/p^j \rfloor$$

$$p$$

$$p$$

$$p$$

$$\binom{m}{n}$$

$$p$$

$$m$$

$$n$$

$$2$$

$$v_2\left(\binom{m}{n}\right) = S_2(n) + S_2(m-n) - S_2(m)$$

$$p$$

$$q$$

$$(p^{(q)!})_p \equiv \pm 1 \pmod{p^q}$$

$$m$$

$$m^2 \equiv 1 \pmod{p^q}$$

$$p^q = 2$$

$$p = 2$$

$$q \geq 3$$

$$\pm 1, 2^{q-1} \pm 1$$

$$\pm 1$$

$$\pm 1$$

$$\pm 1$$

$$p^q$$

$$8$$

$$2$$

$$1$$

$$-1$$

$$p$$

$$q$$

$$n$$

$$N_0 = n \bmod p^q$$

$$\prod{}'$$

$$p$$

$$1 \times 2 \times 3 \times \cdots \times n$$

$$(0 \times p^q + 1) \times (0 \times p^q + 2) \times \cdots \times (\lfloor n/p^q \rfloor p^q + N_0)$$

$$p$$

$$q$$

$$n$$

$$N_j = \lfloor n/p^j \rfloor \bmod p^q$$

$$\pm 1$$

$$r = n - m$$

$$n > m$$

$$p^q$$

$$\pm 1$$

$$n$$

$$3k + 7$$

$$(3k + 6)! + 1 = k(3k + 7)$$

$$3k + 7$$

$$(3k+7) \mid (3k+6)!$$

$$(3k+6)! = k(3k+7)$$

$$\frac{0}{1}, \frac{1}{0}$$

$$\frac{0}{1}, \quad \frac{1}{1}, \quad \frac{1}{0}$$

$$\frac{0}{1}, \quad \frac{1}{2}, \quad \frac{1}{1}, \quad \frac{2}{1}, \quad \frac{1}{0}$$

$$\frac{0}{1}, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{1}{1}, \quad \frac{3}{2}, \quad \frac{2}{1}, \quad \frac{3}{1}, \quad \frac{1}{0}$$

$$\frac{a}{b} \leq \frac{a+c}{b+d} \leq \frac{c}{d}$$

$$\begin{aligned} &\frac{a}{b} \leq \frac{c}{d} \\ \implies &ad \leq bc \\ \implies &ad + ab \leq bc + ab \\ \implies &\frac{a}{b} \leq \frac{a+c}{b+d} \end{aligned}$$

$$bc - ad = 1$$

$$a(b+d) - b(a+c) = ad - bc = 1$$

$$F_1 = \left\{ \frac{0}{1}, \qquad \frac{1}{1} \right\}$$

$$F_2 = \left\{ \frac{0}{1}, \qquad \frac{1}{2}, \qquad \frac{1}{1} \right\}$$

$$F_3 = \left\{ \frac{0}{1}, \qquad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{2}{3}, \qquad \frac{1}{1} \right\}$$

$$F_4 = \left\{ \frac{0}{1}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{1}{2}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{1}{1} \right\}$$

$$F_5 = \left\{ \frac{0}{1}, \quad \frac{1}{5}, \quad \frac{1}{4}, \quad \frac{1}{3}, \quad \frac{2}{5}, \quad \frac{1}{2}, \quad \frac{3}{5}, \quad \frac{2}{3}, \quad \frac{3}{4}, \quad \frac{4}{5}, \quad \frac{1}{1} \right\}$$

$$L_i = L_{i-1} + \varphi(i)$$

$$L_i = 1 + \sum_{k=1}^{i} \varphi(k)$$

$$F_{k,n} = \frac{np_k + p_{k-1}}{nq_k + q_{k-1}}$$

$$F_{0,n} = \frac{np_0 + p_{-1}}{nq_0 + q_{-1}} = \frac{na_0 + 1}{n} = a_0 + \frac{1}{n}$$

$$x = \frac{r_{k+1}p_k + p_{k-1}}{r_{k+1}q_k + q_{k-1}}$$

$$F_{k,n+1} - F_{k,n} = \frac{(p_{k-1}q_k - p_k q_{k-1})n + (p_k q_{k-1} - p_{k-1}q_k)(n+1)}{((n+1)q_k + q_{k-1})(nq_k + q_{k-1})} = \frac{(-1)^{k-1}}{((n+1)q_k + q_{k-1})(nq_k + q_{k-1})}$$

$$\frac{1}{q_k(q_k + q_{k+1})} < \left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}}$$

$$\left| x - \frac{p_k}{q_k} \right| = \frac{1}{q_k(r_{k+1}q_k + q_{k-1})}$$

$$a_{k+1} < \lfloor r_{k+1} \rfloor < a_{k+1} + 1$$

$$q_{k+1} = a_{k+1}q_k + q_{k-1} < r_{k+1}q_k + q_{k-1} < a_{k+1}q_k + q_k + q_{k-1} = q_k + q_{k+1}$$

$$\left| \frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} \right| = \frac{1}{q_{k+1}q_k}$$

$$\frac{p_{k-1}}{q_{k-1}} = F_{k,0} \ Fk,1 \ \frac{p_{k+1}}{q_{k+1}} \ x \ \frac{p_k}{q_k}$$

$$\left| x - \frac{p_{k-1}}{q_{k-1}} \right| > | \ F_{k,0} - F_{k,1}| = \frac{1}{q_{k-1}(q_k + q_{k+1})}$$

$$\frac{1}{2q_{k+1}^2} < \frac{1}{q_k(q_k + q_{k+1})} < \left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k q_{k+1}} < \frac{1}{q_k^2}$$

$$| \ qx - p \ | \leq \frac{1}{2}$$

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{2q}$$

$$\left| x - \frac{a}{b} \right| > \left| x - \frac{p}{q} \right|$$

$$[0,2] = \frac{1}{2}, [0,2,4] = \frac{4}{9}, [0,2,4,2] = \frac{9}{20} ...$$

$$[0,1] = 1, [0,2] = \frac{1}{2}$$

$$[0,2,1] = \frac{1}{3}, [0,2,2] = \frac{2}{5}, [0,2,3] = \frac{3}{7}, [0,2,4] = \frac{4}{9}$$

$$[0,2,4,1] = \frac{5}{11}, [0,2,4,2] = \frac{9}{20}, ...$$

$$F_{1,0} < F_{1,1} \leq F_{1,a_2} = F_{3,0} < ... < x < ... < F_{2,0} = F_{0,a_1} \leq F_{0,1}$$

$$F_{k,r} \ \frac{a}{b} \ F_{k,r+1} \ x$$

$$\left| F_{k,r} - \frac{a}{b} \right| < | \ F_{k,r+1} - F_{k,r}| = \frac{1}{((r+1)q_k + q_{k-1})(rq_k + q_{k-1})}$$

$$\left| F_{k,r} - \frac{a}{b} \right| = \frac{|\ a(rq_k + q_{k-1}) - b(rp_k + p_{k-1})|}{b(rq_k + q_{k-1})} \geq \frac{1}{b(rq_k + q_{k-1})}$$

$$\left| x - \frac{a}{b} \right| > |\ x - F_{k,r+1}|$$

$$|\ bx - a\ | > |\ qx - p\ |$$

$$\left| x - \frac{a}{b} \right| > \frac{q}{b} \left| x - \frac{p}{q} \right|$$

$$\frac{p_{k-1}}{q_{k-1}}\ \frac{a}{b}\ \frac{p_{k+1}}{q_{k+1}}\ x\ \frac{p_k}{q_k}$$

$$\left| x - \frac{a}{b} \right| > \left| \frac{p_{k+1}}{q_{k+1}} - \frac{a}{b} \right|$$

$$|\ q_s x - p_s| > \frac{1}{q_s + q_{s+1}} \leq \frac{1}{q_{k+1}} > |\ q_k x - p_k|$$

$$x = \frac{2a_0 + 1}{2b_0}$$

$$b_1 = \frac{2b_0}{d} < b_0$$

$$a_1 = \frac{2a_0 + 1}{d}$$

$$p_N = 2a_0 + 1$$

$$q_N = 2b_0 = a_N q_{N-1} + q_{N-2}$$

$$|\ q_{N-1} x - p_{N-1}| = \frac{1}{q_N} = \frac{1}{2b_0} \leq \frac{1}{2}$$

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$$

$$|\ bx - a\ | \leq |\ qx - p\ | < \frac{1}{2q}$$

$$\left| x - \frac{a}{b} \right| < \frac{1}{2qb}$$

$$\left| \frac{p}{q} - \frac{a}{b} \right| = \frac{|\ pb - aq\ |}{qb} \geq \frac{1}{qb}$$

$$\left| \frac{p}{q} - \frac{a}{b} \right| = \left| \frac{p}{q} - x + x - \frac{a}{b} \right| \leq \left| \frac{p}{q} - x \right| + \left| x - \frac{a}{b} \right| < \frac{b + q}{2bq^2}$$

$$\frac{1}{qb} \leq \left| \frac{p}{q} - \frac{a}{b} \right| < \frac{b + q}{2bq^2}$$

$$\frac{1}{0}$$

$$\infty$$

$$\frac{a}{b}, \frac{c}{d}$$

$$\frac{a+c}{b+d}$$

$$i$$

$$i-1$$

$$\frac{a}{b} \leq \frac{c}{d}$$

$$\frac{0}{1}, \frac{1}{0}$$

$$\frac{a}{b}, \frac{c}{d}$$

$$bc - ad = 1$$

$$\frac{a}{b}, \frac{a+c}{b+d}, \frac{c}{d}$$

$$\gcd(a, b) = \gcd(c, d) = 1$$

$$\frac{a}{b} < \frac{c}{d}$$

$$\frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}$$

$$\frac{p_k}{q_k}$$

$$[a_0; a_1, ..., a_k, 1]$$

$$a_0$$

$$a_1$$

$$a_2$$

$$a_k$$

$$[a_0; a_1, ..., a_k, 1]$$

$$a_k > 1$$

$$[a_0; a_1, ..., a_k - 1, 1]$$

$$a_k = 1$$

$$[a_0; a_1, ..., a_{k-1}, 1]$$

$$[a_0; a_1, ..., a_k, 1]$$

$$[a_0; a_1, ..., a_k + 1, 1]$$

$$[a_0; a_1, ..., a_k, 1, 1]$$

$$1$$

$$v$$

$$v$$

$$1$$

$$10$$

$$1$$

$$11$$

$$\frac{5}{2} = [2; 2] = [2; 1, 1]$$

$$1011_2$$

$$[2; 1, 1]$$

$$\frac{2}{5} = [0; 2, 2] = [0; 2, 1, 1]$$

$$1100_2$$

$$[0; 2, 2]$$

$$\frac{p}{q}$$

$$p$$

$$q$$

$$A = [a_0; a_1, ..., a_n]$$

$$B = [b_0; b_1, ..., b_m]$$

$$A$$

$$B$$

$$a_0$$

$$a_1$$

$$a_k$$

$$b_k$$

$$a_k = b_k$$

$$a_{k+1}$$

$$b_{k+1}$$

$$a_k < b_k$$

$$a_k > b_k$$

$$a < b$$

$$A$$

$$B$$

$$(a_0, -a_1, a_2, -a_3, ...) < (b_0, -b_1, b_2, -b_3, ...)$$

$$A < B$$

$$\infty$$

$$A - \varepsilon$$

$$A + \varepsilon$$

$$A$$

$$A = [a_0; a_1, ..., a_n]$$

$$[a_0; a_1, ..., a_n, \infty]$$

$$[a_0; a_1, ..., a_n - 1, 1, \infty]$$

$$A - \varepsilon$$

$$A + \varepsilon$$

$$n$$

$$\frac{0}{1} \leq \frac{p_0}{q_0} < \frac{p_1}{q_1} \leq \frac{1}{0}$$

$$\frac{p}{q}$$

$$(q; p)$$

$$\frac{p_0}{q_0} < \frac{p}{q} < \frac{p_1}{q_1}$$

$$\frac{p_0}{q_0}$$

$$\frac{p_1}{q_1}$$

$$\frac{p_0}{q_0}$$

$$\frac{p_1}{q_1}$$

$$\frac{p_0}{q_0} = [a_0; a_1, ..., a_{k-1}, a_k, ...]$$

$$\frac{p_1}{q_1} = [a_0; a_1, ..., a_{k-1}, b_k, ...]$$

$$[a_0; a_1, ..., \min(a_k, b_k) + 1]$$

$$r_0$$

$$r_1$$

$$r_0$$

$$r_1$$

$$r_0 + \varepsilon$$

$$r_1 - \varepsilon$$

$$N$$

$$(C_i, J_i)$$

$$(x, y)$$

$$C_i x + J_i y$$

$$A_i x + B_i y$$

$$i$$

$$A_i = C_i - C_{i-1}$$

$$B_i = J_i - J_{i-1}$$

$$A_i x + B_i y > 0$$

$$A_i, B_i > 0$$

$$x, y > 0$$

$$A_i, B_i \leq 0$$

$$A_i > 0$$

$$B_i \leq 0$$

$$\frac{y}{x} < \frac{A_i}{-B_i}$$

$$A_i \leq 0$$

$$B_i > 0$$

$$\frac{y}{x} > \frac{-A_i}{B_i}$$

$$\frac{p_0}{q_0}$$

$$\frac{-A_i}{B_i}$$

$$\frac{p_1}{q_1}$$

$$\frac{A_i}{-B_i}$$

$$\frac{p_0}{q_0} < \frac{p_1}{q_1}$$

$$\frac{p}{q}$$

$$(q; p)$$

$$\frac{p_0}{q_0} < \frac{p}{q} < \frac{p_1}{q_1}$$

$$\frac{1}{1}$$

$$\frac{p}{q}$$

$$\frac{p}{p+q}$$

$$\frac{p+q}{q}$$

$$p > q$$

$$\frac{p}{q}$$

$$\frac{p-q}{q}$$

$$p < q$$

$$\frac{p}{q-p}$$

$$s_k = \frac{p}{q}$$

$$s_{k+1} = \frac{q}{p \bmod q} = \frac{q}{p - \lfloor p/q \rfloor \cdot q}$$

$$p > q$$

$$s_k = \frac{p}{q}$$

$$\frac{p \bmod q}{q} = \frac{1}{s_{k+1}}$$

$$s_{k+2}$$

$$\frac{1}{s_{k+3}}$$

$$a_0 > 0$$

$$[a_0; a_1, ..., a_k]$$

$$\frac{p-q}{q} = [a_0 - 1; a_1, ..., a_k]$$

$$a_0 = 0$$

$$a_1 > 1$$

$$\frac{p}{q-p} = [0; a_1 - 1, a_2, ..., a_k]$$

$$a_0 = 0$$

$$a_1 = 1$$

$$\frac{p}{q-p} = [a_2; a_3, ..., a_k]$$

$$\frac{p}{q} = [a_0; a_1, ..., a_k]$$

$$\frac{p+q}{q} = 1 + \frac{p}{q}$$

$$[a_0 + 1; a_1, ..., a_k]$$

$$\frac{p}{p+q} = \frac{1}{1 + \frac{q}{p}}$$

$$a_0 > 0$$

$$[0, 1, a_0, a_1, ..., a_k]$$

$$a_0 = 0$$

$$[0, a_1 + 1, a_2, ..., a_k]$$

$$1$$

$$v$$

$$2v$$

$$2v + 1$$

$$i$$

$$F_i$$

$$i$$

$$F_i$$

$$i - 1$$

$$\frac{a}{b}, \frac{x}{y}, \frac{c}{d}$$

$$x = a + c, y = b + d$$

$$F_i$$

$$L_i$$

$$0 \leq n \leq a_{k+1}$$

$$\frac{p_{-1}}{q_{-1}}$$

$$\frac{p_0}{q_0} = a_0$$

$$k$$

$$0$$

$$F_{k,0} = \frac{p_{k-1}}{q_{k-1}}$$

$$F_{k,0} = \frac{p_{k-1}}{q_{k-1}}$$

$$F_{k,n} = [a_0, ..., a_k, n]$$

$$k$$

$$n$$

$$x$$

$$\frac{p_k}{q_k}$$

$$x$$

$$\left| x - \frac{p_k}{q_k} \right|$$

$$\frac{1}{q_{k+1} q_k}$$

$$k$$

$$q_k \leq q_{k+1}$$

$$\left| x - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}$$

$$\theta$$

$$\left| \frac{p}{q} - \theta \right| \leqslant \frac{1}{q^2}$$

$$qx$$

$$\frac{1}{2}$$

$$p$$

$$q$$

$$\frac{1}{q^2}$$

$$\mid qx - p \mid < \frac{1}{q}$$

$$x$$

$$\mid qx - p \mid$$

$$x$$

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}$$

$$x$$

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}$$

$$\sqrt{5}$$

$$\frac{1+\sqrt{5}}{2} = [1, 1, 1, ...]$$

$$1$$

$$x$$

$$n$$

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{n^2+4}q^2}$$

$$\frac{n+\sqrt{n^2+4}}{2} = [n, n, n, ...]$$

$$\theta$$

$$\alpha$$

$$\varepsilon$$

$$n$$

$$m$$

$$|n\theta - m - \alpha| < \varepsilon$$

$$(0, 1)$$

$$x$$

$$\frac{p}{q}$$

$$a$$

$$0 < b \leq q$$

$$\frac{a}{b} \neq \frac{p}{q}$$

$$\frac{p}{q}$$

$$x$$

$$[0, 2, 4, 2]$$

$$\sqrt{6}$$

$$2$$

$$a_0$$

$$a_0 + 1$$

$$x$$

$$1$$

$$F_{1,0}$$

$$F_{0,1}$$

$$x = n + \frac{1}{2}$$

$$x$$

$$1$$

$$n$$

$$F_{1,0}$$

$$F_{0,1}$$

$$F_{k,a_{k+1}} = F_{k+2,0}$$

$$x$$

$$\frac{a}{b}$$

$$\frac{a}{b}$$

$$F_{k,r}$$

$$b > (r+1)q_k + q_{k-1}$$

$$F_{k,r}$$

$$F_{k,r+1}$$

$$F_{k,r+1}$$

$$F_{k,r+1}$$

$$x$$

$$\frac{a}{b}$$

$$x$$

$$\frac{p}{q}$$

$$a$$

$$0 < b \leq q$$

$$\frac{a}{b} \neq \frac{p}{q}$$

$$|\ q_k x - p_k| < \frac{1}{q_{k+1}}$$

$$F_{k,n} = \frac{a}{b}$$

$$|\ bx - a\ | \geq \frac{1}{q_{k+1}}$$

$$b = nq_k + q_{k-1} > q_k$$

$$F_{k,n} = \frac{a}{b}$$

$$\frac{p_k}{q_k}$$

$$x$$

$$s < k$$

$$|\ q_s x - p_s| > |\ q_k x - p_k|$$

$$q_{k+1} \geq q_s + q_{s+1}$$

$$q_k$$

$$b$$

$$\min|\ bx - a\ | \leq \frac{1}{2}$$

$$a$$

$$b$$

$$0 < b \leq q_k$$

$$b$$

$$b$$

$$b_0$$

$$a$$
$$a_0$$
$$a_0$$
$$\frac{a_0}{b_0}$$
$$\frac{p_s}{q_s}$$
$$s < k$$
$$s = k$$
$$a_0$$
$$b_0$$
$$\mid b_0 x - a \mid$$
$$a$$
$$\mid b_0 x - a \mid = \frac{1}{2}$$
$$a_0$$
$$d = \gcd(2a_0 + 1, 2b_0) \geq 3$$
$$\mid b_1 x - a_1 \mid = \mid b_0 x - a_0 \mid = \frac{1}{2}$$
$$b_0$$
$$x$$
$$a_N \geq 2$$
$$q_{N-2} \geq 1$$
$$q_{N-1} < b_0$$
$$q_{N-1}$$
$$b_0$$
$$x$$
$$\frac{p}{q}$$
$$\frac{p}{q}$$
$$x$$
$$\frac{p}{q}$$
$$0 < b \leq q$$

$$\frac{a}{b} \neq \frac{p}{q}$$

$$b > q$$

$$\forall a, b, c \in \mathbb{Z}, \left\lfloor \frac{a}{bc} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{a}{b} \right\rfloor}{c} \right\rfloor$$

$$\frac{a}{b} = \left\lfloor \frac{a}{b} \right\rfloor + r(0 \leq r < 1)$$

$$\implies \left\lfloor \frac{a}{bc} \right\rfloor = \left\lfloor \frac{a}{b} \cdot \frac{1}{c} \right\rfloor = \left\lfloor \frac{1}{c}\left(\left\lfloor \frac{a}{b} \right\rfloor + r\right) \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{a}{b} \right\rfloor}{c} + \frac{r}{c} \right\rfloor = \left\lfloor \frac{\left\lfloor \frac{a}{b} \right\rfloor}{c} \right\rfloor$$

$\square$

$$\forall n \in \mathbb{N}_+, \left|\left\{ \left\lfloor \frac{n}{d} \right\rfloor \mid d \in \mathbb{N}_+, d \leq n \right\}\right| \leq \lfloor 2\sqrt{n} \rfloor$$

$$\left\lfloor \frac{n}{i} \right\rfloor = \left\lfloor \frac{n}{j} \right\rfloor$$

$$\therefore \left\lfloor \frac{n}{k} \right\rfloor \geq \left\lfloor \frac{n}{\frac{n}{i}} \right\rfloor = \lfloor i \rfloor = i$$

$$\therefore j = \max \text{□□□□□□□} \ i = i_{\max} = \left\lfloor \frac{n}{k} \right\rfloor = \left\lfloor \frac{n}{\left\lfloor \frac{n}{i} \right\rfloor} \right\rfloor \square$$

□□ $f(i)$ □□□□□□□□□ $s(i)$.
$l \leftarrow 1$
$r \leftarrow 0$
$result \leftarrow 0$
while $l \leq n$ **do** :
    $r \leftarrow \left\lfloor \frac{n}{\lfloor n/l \rfloor} \right\rfloor$
    $result \leftarrow result + [s(r) - s(l-1)] \times \left\lfloor \frac{n}{l} \right\rfloor$
    $l \leftarrow r + 1$
end while

$$\sum_{i=1}^{n} f(i) g(\left\lfloor \frac{n}{i} \right\rfloor)$$

$$O(1)$$

$$f(r) - f(l)$$

$$f$$

$$O(\sqrt{n})$$

$$\left\lfloor \frac{n}{i} \right\rfloor$$

$$\int \int | \ f(x,y)| \ dxdy$$

$$5$$

$$5$$

$$5$$

$$\blacksquare$$

$$\square$$

$$| \ V \ |$$

$$V$$

$$d \leq \lfloor \sqrt{n} \rfloor$$

$$\left\lfloor \frac{n}{d} \right\rfloor$$

$$\lfloor \sqrt{n} \rfloor$$

$$d > \lfloor \sqrt{n} \rfloor$$

$$\left\lfloor \frac{n}{d} \right\rfloor \leq \lfloor \sqrt{n} \rfloor$$

$$\lfloor \sqrt{n} \rfloor$$

$$n$$

$$i \leq j \leq n$$

$$j$$

$$\left\lfloor \frac{n}{\lfloor \frac{n}{i} \rfloor} \right\rfloor$$

$$\left\lfloor \frac{n}{i} \right\rfloor$$

$$\left\lfloor \frac{n}{\lfloor \frac{n}{i} \rfloor} \right\rfloor$$

$$k = \left\lfloor \frac{n}{i} \right\rfloor$$

$$k \leq \frac{n}{i}$$

$$\sum_{i=1}^{n} f(i) \left\lfloor \frac{n}{i} \right\rfloor$$

$$\left\lfloor \frac{n}{i} \right\rfloor$$

$$f(i)$$

$$s(i) = \sum_{j=1}^{i}$$

$$[l, r] = [l, \left\lfloor \frac{n}{\lfloor \frac{n}{i} \rfloor} \right\rfloor]$$

$$result$$

$$T$$

$$n$$

$$\sum_{i=1}^{n} \left\lfloor \frac{n}{i} \right\rfloor$$

$$\left\lfloor \frac{n}{i} \right\rfloor$$

$$O(T\sqrt{n})$$

$$\left\lfloor \frac{a_1}{i} \right\rfloor$$

$$\left\lfloor \frac{a_2}{i} \right\rfloor \cdots \left\lfloor \frac{a_n}{i} \right\rfloor$$

$$\left\lfloor \frac{n}{i} \right\rfloor$$

$$\min_{j=1}^{n} \{ \left\lfloor \frac{a_j}{i} \right\rfloor \}$$

$$[n = 1] = \sum_{d \mid n} \mu(n)$$

$$O\left(\sum_{k=1}^{\pi(n)} \frac{n}{p_k} \right) = O\left(n \sum_{k=1}^{\pi(n)} \frac{1}{p_k} \right)$$

$$\sum_{k=1}^{\pi(n)} \frac{1}{p_k} = \log \log n + B_1 + O\left(\frac{1}{\log n} \right)$$

$$\begin{aligned} \sum_{k=1}^{\pi(n)} \frac{1}{p_k} &= O\left(\sum_{k=2}^{\pi(n)} \frac{1}{k \log k} \right) \\ &= O\left(\int_2^{\pi(n)} \frac{\mathrm{d}x}{x \log x} \right) \\ &= O(\log \log \pi(n)) = O(\log \log n) \end{aligned}$$

$$\varphi(n) = n \times \prod_{i=1}^{s} \frac{p_i - 1}{p_i}$$

$$= p_1 \times n' \times \prod_{i=1}^{s} \frac{p_i - 1}{p_i}$$

$$= p_1 \times \varphi(n')$$

$$\varphi(n) = \varphi(p_1) \times \varphi(n')$$

$$= (p_1 - 1) \times \varphi(n')$$

$$\mu(n) = \begin{cases} 0 & n' \bmod p_1 = 0 \\ -\mu(n') & \text{otherwise} \end{cases}$$

$$g_n = \begin{cases} g_x \cdot p x \bmod p = 0 \\ p \quad \text{otherwise} \end{cases}$$

$$n$$

$$n$$

$$1$$

$$n$$

$$x$$

$$x > 1$$

$$O(n \log\log n)$$

$$p_k$$

$$k$$

$$\pi(n)$$

$$\leq n$$

$$\sum_{k=1}^{\pi(n)}$$

$$\pi(n)$$

$$\frac{n}{p_k}$$

$$B_1$$

$$O(n \log\log n)$$

$$\sum_{k \leq \pi(n)} 1/p_k = O(\log\log n)$$

$$\pi(n) = \Theta(n/\log n)$$

$$n$$

$$\Theta(n \log n)$$

$$n$$

$$\sqrt{n}$$

$$n \ln \ln \sqrt{n} + o(n)$$

$$1$$

$$8$$

$$1$$

$$n$$

$$\frac{n}{8}$$

$$n$$

$$n$$

$$\frac{n}{8}$$

$$\sqrt{n}$$

$$s$$

$$\left\lceil \frac{n}{s} \right\rceil$$

$$k$$

$$k = 0 ... \left\lfloor \frac{n}{s} \right\rfloor$$

$$[ks, ks + s - 1]$$

$$k$$

$$1$$

$$\sqrt{n}$$

$$[1, \sqrt{n}]$$

$$0$$

$$1$$

$$n$$

$$n$$

$$O(\sqrt{n} + S)$$

$$[1, \sqrt{n}]$$

$$S$$

$$S$$

$$10^4$$

$$10^5$$

$$O(n)$$

$$p_1$$

$$n$$

$$n' = \frac{n}{p_1}$$

$$n$$

$$n' \times p_1$$

$$n' \bmod p_1$$

$$n' \bmod p_1 = 0$$

$$n'$$

$$n$$

$$n' \bmod p_1 \neq 0$$

$$n'$$

$$p_1$$

$$n$$

$$p_1$$

$$n$$

$$n' = \frac{n}{p_1}$$

$$n$$

$$\mu(n) = -1$$

$$d_i$$

$$i$$

$$num_i$$

$$i$$

$$n = \prod_{i=1}^{m} p_i^{c_i}$$

$$d_i = \prod_{i=1}^{m} (c_i + 1)$$

$$p_i^{c_i}$$

$$p_i^0, p_i^1, ..., p_i^{c_i}$$

$$c_i + 1$$

$$n$$

$$\prod_{i=1}^{m}(c_i + 1)$$

$$d_i$$

$$i$$

$$num_i \leftarrow 1, d_i \leftarrow 2$$

$$q = \left\lfloor \frac{i}{p} \right\rfloor$$

$$p$$

$$i$$

$$p$$

$$q$$

$$num_i \leftarrow num_q + 1, d_i \leftarrow \frac{d_q}{num_i} \times (num_i + 1)$$

$$p, q$$

$$num_i \leftarrow 1, d_i \leftarrow d_q \times (num_i + 1)$$

$$f_i$$

$$i$$

$$g_i$$

$$i$$

$$p^0 + p^1 + p^2 + ...p^k$$

$$f$$

$$p$$

$$k$$

$$O(1)$$

$$f(p^k)$$

$$O(n)$$

$$f(1), f(2), ..., f(n)$$

$$n$$

$$\prod_{i=1}^{k} p_i^{\alpha_i}$$

$$p_1 < p_2 < \ldots < p_k$$

$$g_n = p_1^{\alpha_1}$$

$$n$$

$$x \cdot p$$

$$p$$

$$g$$

$$n = g_n$$

$$n$$

$$O(1)$$

$$f(n)$$

$$f(n) = f(\frac{n}{g_n}) \cdot f(g_n)$$

$$x^k \equiv a \pmod m$$

$$a^{\frac{\varphi(m)}{d}} \equiv 1 \pmod m$$

$$a \equiv g^{di} \pmod m, \qquad \left(0 \le i < \frac{\varphi(m)}{d}\right)$$

$$g^{ky} \equiv g^{\operatorname{ind}_g a} \pmod m$$

$$ky \equiv \operatorname{ind}_g a \pmod{\varphi(m)}$$

$$\begin{aligned} a^{\frac{\varphi(m)}{d}} \equiv 1 \pmod m &\iff g^{\frac{\varphi(m)}{d} \operatorname{ind}_g a} \equiv 1 \pmod m \\ &\iff \varphi(m) \mid \frac{\varphi(m)}{d} \operatorname{ind}_g a \\ &\iff \varphi(m) d \mid \varphi(m) \operatorname{ind}_g a \\ &\iff d \mid \operatorname{ind}_g a \end{aligned}$$

$$\operatorname{ind}_g a \equiv di \pmod{\varphi(m)}, \qquad \left(0 \le i < \frac{\varphi(m)}{d}\right)$$

$$a \equiv g^{di} \pmod m, \qquad \left(0 \le i < \frac{\varphi(m)}{d}\right)$$

$$k \ge 2$$

$$a$$

$$m$$

$$(a, m) = 1$$

x

a

m

k

a

m

k

k

$k \geq 2$

a

m

$(a, m) = 1$

m

g

$d = (k, \varphi(m))$

a

m

k

$d \mid \operatorname{ind}_g a$

(1)

m

d

m

k

$\frac{\varphi(m)}{d}$

$x = g^y$

(1)

y

$d = (k, \varphi(m)) \mid \operatorname{ind}_g a$

$d \mid \operatorname{ind}_g a$

(2)

$\varphi(m)$

d

$$(1)$$

$$m$$

$$d$$

$$a$$

$$m$$

$$k$$

$$d \mid \operatorname{ind}_g a$$

$$m$$

$$k$$

$$\frac{\varphi(m)}{d}$$

$$a+b\sqrt{d}$$

$$N(a+b\sqrt{d})=a^2-db^2$$

$$N(a_1+b_1\sqrt{d})N(a_2+b_2\sqrt{d})=N((a_1+b_1\sqrt{d})(a_2+b_2\sqrt{d}))$$

$$\frac{N(a_1+b_1\sqrt{d})}{N(a_2+b_2\sqrt{d})}=N\left(\frac{a_1+b_1\sqrt{d}}{a_2+b_2\sqrt{d}}\right)$$

$$a+b\sqrt{d}=\frac{a-b\sqrt{d}}{N(a+b\sqrt{d})}$$

$$\frac{-p\pm\sqrt{p^2-4q}}{2}$$

$$x^2-ax+\frac{a^2-db^2}{4}=0$$

$$10=2*5=(-2)*(-5)$$

$$\chi(n,k_1k_2)=\chi(n,k_1)$$

$$\chi(n,k_1k_2)=\chi(n,k_1)\chi(n,k_2)$$

$$\chi(g,p^a)=\mathrm{e}^{\frac{2\pi \mathrm{i} l}{\varphi(p^a)}}$$

$$\chi(g,p^a,l)=\mathrm{e}^{\frac{2\pi \mathrm{i} l}{\varphi(p^a)}}$$

$$a+b\sqrt{d}=e+f\sqrt{d}$$

$$a=e \quad b=f$$

$$\begin{pmatrix} a & b \\ db & a \end{pmatrix}$$

$$\begin{pmatrix} a_1 & b_1 \\ db_1 & a_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ db_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + db_1b_2 & a_1b_2 + b_1a_2 \\ d(a_1b_2 + b_1a_2) & a_1a_2 + db_1b_2 \end{pmatrix}$$

$$N(a + b\sqrt{d}) = \begin{vmatrix} a & b \\ db & a \end{vmatrix}$$

$$\begin{pmatrix} a & b \\ db & a \end{pmatrix}^* = \begin{pmatrix} a & -b \\ -db & a \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$x^2 + y^2 = z^2$$

$$N(x + y\mathrm{i}) = z^2$$

$$(u + v\mathrm{i})^2 = x + y\mathrm{i}$$

$$N(u + v\mathrm{i}) = z$$

$$u^2 - v^2 = x$$

$$2uv = y$$

$$u^2 + v^2 = z$$

$$x^4 + y^4 = z^4$$

$$x^4 + y^4 = z^2$$

$$N(x + y\mathrm{i}) = n$$

$$N((1 + \mathrm{i})^a)N(u_1 + v_1\mathrm{i})N(u_2 + v_2\mathrm{i}) = 2^a pq$$

$$f(n) = 4 \sum_{d \mid n} \chi(n, 4, 1)$$

$$x^2 + 3y^2 = z^3$$

$$u^2 + 3v^2 = z$$

$$x = u^3 - 9uv^2$$

$$y = 3u^2v - 3v^3$$

$$x^2 + 3y^2 = n$$

$$x^2 - xy + y^2 = n$$

$$x^2 + xy + y^2 = n$$

$$x^2 - xy + y^2 = n$$

$$f(n) = 6 \sum_{d \mid n} \chi(n, 3, 1)$$

$$x^2 + 3y^2 = n$$

$$12 \sum_{d \mid n} \chi(n, 3, 1)$$

$$36 \sum_{d \mid n} \chi(n, 3, 1)$$

$$a$$

$$b$$

$$d$$

$$d$$

$$d$$

$$0$$

$$0$$

$$2$$

$$d$$

$$0$$

$$0$$

$$0$$

$$1$$

$$x^2 + px + q = 0$$

$$d$$

$$Z(\sqrt{d})$$

$$d$$

$$d$$

$$a + b\sqrt{d}$$

$$d$$

$$4$$

$$1$$

$$a$$

$$b$$

$$\frac{a + b\sqrt{d}}{2}$$

$$1$$

$$4$$

$$1$$

d

d

d

d

i

1

-1

1

-1

-1

-2

-3

-7

-11

2

3

5

6

7

11

13

17

19

21

29

33

37

41

57

73

16

1

1

5

-19

-43

-67

-163

-1

-3

-1

2

5

-2

-5

10

-2

5

2

-5

-2

-5

k

$\chi(n)$

0

0

$\chi(n) = 0$

$\gcd(n, k) > 1$

k

$\chi(n + k) = \chi(n)$

m

n

$\chi(mn) = \chi(m)\chi(n)$

$\chi(n)$

k

k

$\chi(n, k)$

$$1$$
$$1$$
$$\chi(1) = 1$$
$$-1$$
$$\pm 1$$
$$\chi(-1) = \pm 1$$
$$\gcd(n, k) = 1$$
$$(\chi(n))^{\varphi(k)} = \chi(n^{\varphi(k)}) = \chi(1) = 1$$
$$n$$
$$k$$
$$k$$
$$1$$
$$\varphi(k)$$
$$k$$
$$\gcd(n, k) = 1$$
$$\chi(n)$$
$$1$$
$$k$$
$$k$$
$$\chi^0(n, k)$$
$$1$$
$$2$$
$$\pm 1$$
$$3$$
$$4$$
$$k$$
$$k$$
$$k$$
$$\gcd(k_1, k_2) = 1$$
$$k_1$$
$$\chi(n, k_1)$$
$$n \equiv 1 \pmod{k}_2$$

$$\gcd(k_1, k_2) = 1$$

$$k_1$$

$$\chi(n, k_1)$$

$$k_2$$

$$\chi(n, k_2)$$

$$n$$

$$p^a$$

$$g$$

$$g$$

$$g$$

$$p^a$$

$$\varphi(p^a)$$

$$l$$

$$p^a$$

$$\phi(p^a)$$

$$\gcd(n, p) = 1$$

$$p^a$$

$$g$$

$$p^a$$

$$l = 0$$

$$l = 0$$

$$l = \frac{\varphi(p^a)}{2}$$

$$2^a$$

$$2^a$$

$$\varphi(2^a)$$

$$k$$

$$\varphi(k)$$

$$k$$

$$k$$

$$k$$

$$1$$

$$\gcd(a,k)=1$$

$$a$$

$$1$$

$$k$$

$$k$$

$$\chi$$

$$\chi(a)$$

$$1$$

$$k$$

$$k$$

$$k$$

$$\left(\frac{p}{q}\right)$$

$$q$$

$$3$$

$$\left(\frac{n}{3}\right)$$

$$4$$

$$\left(\frac{-1}{n}\right)=\left(\frac{-4}{n}\right)$$

$$\sqrt{d}$$

$$Q(\sqrt{d})$$

$$Q(\sqrt{d})$$

$$0$$

$$\sqrt{d}$$

$$1$$

$$\sqrt{d}$$

$$d$$

$$d$$

$$d$$

$$d$$

$$Q(\sqrt{d})$$

$$d$$

-1

d

-1

-3

1

-1

d

-1

-3

1

-1

d

-1

4

1

-1

i

$-\mathrm{i}$

d

-3

6

1

-1

$\frac{1+\sqrt{3}\mathrm{i}}{2}$

$\frac{1-\sqrt{3}\mathrm{i}}{2}$

$\frac{-1+\sqrt{3}\mathrm{i}}{2}$

$\frac{-1-\sqrt{3}\mathrm{i}}{2}$

d

-1

-3

i

$$\frac{1+\sqrt{3}\mathrm{i}}{2}$$

$$1$$

$$-1$$

$$Z(\mathrm{i})$$

$$Z(\sqrt{3}\mathrm{i})$$

$$d$$

$$-1$$

$$Q(\mathrm{i})$$

$$Z(\mathrm{i})$$

$$1+\mathrm{i}$$

$$2$$

$$4k+3$$

$$4k+1$$

$$\pi$$

$$\pi \equiv 1 \bmod 2(1+\mathrm{i})$$

$$2(1+\mathrm{i})$$

$$4$$

$$1+\mathrm{i}$$

$$1+\mathrm{i}$$

$$1+\mathrm{i}$$

$$4$$

$$4$$

$$1$$

$$4$$

$$3$$

$$P$$

$$AP$$

$$n$$

$$n$$

$$4k+3$$

$$n$$

$$4k+3$$

$$n$$

$$4k+3$$

$$2$$

$$1+\mathrm{i}$$

$$4k+1$$

$$4k+1$$

$$2$$

$$2$$

$$4$$

$$\chi$$

$$\chi(n,4,1)=\left(\frac{-1}{n}\right)=\left(\frac{-4}{n}\right)$$

$$Q(\sqrt{3}\mathrm{i})$$

$$Z(\sqrt{3}\mathrm{i})$$

$$\frac{3+\sqrt{3}\mathrm{i}}{2}$$

$$3$$

$$3k+2$$

$$2$$

$$6k+5$$

$$3k+1$$

$$6k+1$$

$$\pi$$

$$\pi\equiv 1 \bmod 3$$

$$3$$

$$6$$

$$\frac{3+\sqrt{3}\mathrm{i}}{2}$$

$$3$$

$$3$$

$$9$$

$$6$$

$$\frac{3+\sqrt{3}\mathrm{i}}{2}$$

$$\frac{3+\sqrt{3}\mathrm{i}}{2}$$

$$x^2+3y^2=z^2$$

$$x^2-xy+y^2=z^2$$

$$x^2+xy+y^2=z^2$$

$$z$$

$$\gcd(x,y)=1$$

$$u$$

$$v$$

$$\gcd(u,3v)=1$$

$$x^3+y^3=z^3$$

$$n$$

$$n$$

$$\chi$$

$$\chi(n,3,1)=\left(\frac{n}{3}\right)$$

$$n=2^l m$$

$$m$$

$$l$$

$$l=0$$

$$l$$

$$x^2\equiv a \pmod p$$

$$a^{\frac{p-1}{2}}\equiv\begin{cases}1 \pmod p, & (\exists x\in\mathbf{Z}),\quad a\equiv x^2 \pmod p,\\ -1 \pmod p, \text{otherwise.}\end{cases}$$

$$\left(a^{\frac{p-1}{2}}+1\right)\left(a^{\frac{p-1}{2}}-1\right)\equiv 0 \pmod p$$

$$x^{p-1}-a^{\frac{p-1}{2}}=(x^2)^{\frac{p-1}{2}}-a^{\frac{p-1}{2}}=(x^2-a)P(x)$$

$$\begin{aligned}x^p-x&=x\left(x^{p-1}-a^{\frac{p-1}{2}}\right)+x\left(a^{\frac{p-1}{2}}-1\right)\\&=(x^2-a)xP(x)+\left(a^{\frac{p-1}{2}}-1\right)x\end{aligned}$$

$$\left(\frac{a}{p}\right)=\begin{cases}0, & p\mid a,\\ 1, & (p\nmid a)\wedge((\exists x\in\mathbf{Z}),\quad a\equiv x^2 \pmod p)),\\ -1, \text{otherwise.}\end{cases}$$

$$a^{\frac{p-1}{2}} \equiv \left(\frac{a}{p}\right) \pmod p$$

$$\left(\frac{1}{p}\right) = 1$$

$$\begin{aligned}\left(\frac{-1}{p}\right) &= (-1)^{\frac{p-1}{2}} \\ &= \begin{cases} 1, & p \equiv 1 \pmod 4, \\ -1, p \equiv 3 \pmod 4. \end{cases}\end{aligned}$$

$$\left(\frac{a_1 a_2}{p}\right) = \left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)$$

$$\left(\frac{ab^2}{p}\right) = \left(\frac{a}{p}\right)$$

$$\begin{aligned}\left(\frac{2}{p}\right) &= (-1)^{\frac{p^2-1}{8}} \\ &= \begin{cases} 1, & p \equiv \pm 1 \pmod 8 \\ -1, p \equiv \pm 3 \pmod 8 \end{cases}\end{aligned}$$

$$a_1 \equiv a_2 \pmod p \Longleftrightarrow \left(\frac{a_1}{p}\right) \equiv \left(\frac{a_2}{p}\right) \pmod p$$

$$a_1 \equiv a_2 \pmod p \Longleftrightarrow \left(\frac{a_1}{p}\right) = \left(\frac{a_2}{p}\right)$$

$$\left(\frac{a_1 a_2}{p}\right) \equiv a_1^{\frac{p-1}{2}} a_2^{\frac{p-1}{2}} \equiv \left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right) \pmod p$$

$$\left(\frac{a_1 a_2}{p}\right) = \left(\frac{a_1}{p}\right)\left(\frac{a_2}{p}\right)$$

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}$$

$$\left(\frac{n}{p}\right) = (-1)^m$$

$$\begin{aligned}\left(\frac{2}{p}\right) &= (-1)^{\frac{p^2-1}{8}} \\ &= \begin{cases} 1, & p \equiv \pm 1 \pmod 8 \\ -1, p \equiv \pm 3 \pmod 8 \end{cases}\end{aligned}$$

$$\begin{aligned}\left(a^{(p+1)/4}\right)^2 &\equiv a^{(p+1)/2} &&\pmod p\\ &\equiv x^{p+1} &&\pmod p\\ &\equiv (x^2)(x^{p-1}) &&\pmod p\\ &\equiv x^2 &&\pmod p \quad (\because \text{Fermat's little theorem})\end{aligned}$$

$$\begin{aligned}\mathrm{i}^2 &\equiv \left(2ab^2\right)^2 &&\pmod p\\ &\equiv \left(2a\cdot(2a)^{(p-5)/4}\right)^2 &&\pmod p\\ &\equiv \left((2a)^{(p-1)/4}\right)^2 &&\pmod p\\ &\equiv (2a)^{\frac{p-1}{2}} &&\pmod p\\ &\equiv -1 &&\pmod p\end{aligned}$$

$$\begin{aligned}(ab(\mathrm{i}-1))^2 &\equiv a^2\cdot(2a)^{(p-5)/4}\cdot(-2\mathrm{i}) &&\pmod p\\ &\equiv a\cdot(-\mathrm{i})\cdot(2a)^{(p-1)/4} &&\pmod p\\ &\equiv a &&\pmod p\end{aligned}$$

$$\begin{aligned}x^p &\equiv x(x^2)^{\frac{p-1}{2}} &&\pmod{f(x)}\\ &\equiv x(r^2-a)^{\frac{p-1}{2}} &&\pmod{f(x)} \quad (\because x^2\equiv r^2-a \pmod{f(x)})\\ &\equiv -x &&\pmod{f(x)} \quad (\because r^2-a \text{ is quadratic non-residue})\end{aligned}$$

$$\begin{aligned}(a+b)^p &= \sum_{i=0}^{p}\binom{p}{i}a^ib^{p-i}\\ &= \sum_{i=0}^{p}\frac{p!}{i!(p-i)!}a^ib^{p-i}\\ &\equiv a^p+b^p \pmod p\end{aligned}$$

$$\begin{aligned}(r-x)^p &\equiv r^p-x^p &&\pmod{f(x)}\\ &\equiv r+x &&\pmod{f(x)}\end{aligned}$$

$$\begin{aligned}(a_0+a_1x)^2 &= a_0^2+2a_0a_1x+a_1^2x^2\\ &\equiv (r-x)^{p+1} &&\pmod{f(x)}\\ &\equiv (r-x)^p(r-x) &&\pmod{f(x)}\\ &\equiv (r+x)(r-x) &&\pmod{f(x)}\\ &\equiv r^2-x^2 &&\pmod{f(x)}\\ &\equiv a &&\pmod{f(x)}\end{aligned}$$

$$\begin{aligned}(a_0+a_1x)^2 &= a_0^2+2a_0a_1x+a_1^2x^2\\ &\equiv a_0^2+2a_0a_1x+a_1^2(r^2-a) \pmod{f(x)}\end{aligned}$$

$$b=\left[x^{(p+1)/2}\right]\frac{1}{1-tx+ax^2}$$

$$[x^n]\frac{k_0+k_1x}{1+k_2x+k_3x^2}=\begin{cases}[x^{(n-1)/2}]\dfrac{k_1-k_0k_2+k_1k_3x}{1+(2k_3-k_2^2)x+k_3^2x^2}, \text{if } n \bmod 2 = 1\\ [x^{n/2}]\dfrac{k_0+(k_0k_3-k_1k_2)x}{1+(2k_3-k_2^2)x+k_3^2x^2}, \quad \text{else if } n \neq 0\end{cases}$$

$$(r-b)^{\frac{p-1}{2}}(r+b)^{\frac{p-1}{2}} \equiv -1 \pmod p$$

$$\begin{aligned}\phi : \mathbb{F}_p[x]/(x^2-a) &\to \mathbb{F}_p \times \mathbb{F}_p\\ x &\mapsto (b,-b)\end{aligned}$$

$$\begin{aligned}(a_0+a_1b, a_0-a_1b) &= \phi(a_0+a_1x)\\ &= \phi(r-x)^{\frac{p-1}{2}}\\ &= ((r-b)^{\frac{p-1}{2}}, (r+b)^{\frac{p-1}{2}})\\ &= (\pm 1, \mp 1)\end{aligned}$$

$$\begin{aligned}g^{2^n} &\equiv r^{2^n\cdot m} &\pmod p\\ &\equiv r^{p-1} &\pmod p\\ &\equiv 1 &\pmod p\end{aligned}$$

$$\begin{aligned}g^{2^{n-1}} &\equiv r^{2^{n-1}\cdot m} &\pmod p\\ &\equiv r^{\frac{p-1}{2}} &\pmod p\\ &\equiv -1 &\pmod p\end{aligned}$$

$$\begin{aligned}g^{2^{n-1}\cdot e} &\equiv (-1)^e &\pmod p\\ &\equiv a^{2^{n-1}\cdot m} &\pmod p\\ &\equiv a^{\frac{p-1}{2}} &\pmod p\\ &\equiv 1 &\pmod p\end{aligned}$$

$$\begin{aligned}\left(abg^{-e/2}\right)^2 &\equiv a^2b^2g^{-e} &\pmod p\\ &\equiv a^{m+1}g^{-e} &\pmod p\\ &\equiv a &\pmod p\end{aligned}$$

$$\left(g^eg^{-(e \bmod 2^k)}\right)^{2^{n-1-k}} \equiv g^{2^{n-1}\cdot e_k} \equiv \begin{cases}1 \pmod p, & \text{if } e_k = 0\\ -1 \pmod p, & \text{else if } e_k = 1\end{cases}$$

$$a$$

$$p$$

$$(a,p)=1$$

$$x$$

$$a$$

$$p$$

$$a$$

$$p$$

$$p$$

$$p$$

$$p$$

$$(a, p) = 1$$

$$a$$

$$p$$

$$a$$

$$a$$

$$p$$

$$a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$$

$$a$$

$$p$$

$$a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$$

$$a^{p-1} \equiv 1 \pmod{p}$$

$$(a, p) = 1$$

$$a$$

$$a^{p-1} \equiv \pm 1 \pmod{p}$$

$$p$$

$$P(x)$$

$$a$$

$$p$$

$$a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$$

$$a$$

$$p$$

$$a^{\frac{p-1}{2}} \equiv -1 \pmod{p}$$

$$p$$

$$a$$

$$a$$

$$a_1 \equiv a_2 \pmod{p} \Longleftrightarrow \left(\frac{a_1}{p}\right) = \left(\frac{a_2}{p}\right)$$

$$a_1, a_2$$

$$a, b$$

$$p \nmid b$$

$$\left| \left(\frac{a_1}{p} \right) - \left(\frac{a_2}{p} \right) \right| \leq 2$$

$$p > 2$$

$$\left| \left(\frac{a_1 a_2}{p} \right) - \left(\frac{a_1}{p} \right) \left(\frac{a_2}{p} \right) \right| \leq 2$$

$$p > 2$$

$$p$$

$$q$$

$$(n, p) = 1$$

$$k \, \left(1 \leq k \leq \frac{p-1}{2} \right)$$

$$r_k$$

$$nk$$

$$p$$

$$m$$

$$r_k$$

$$\frac{p}{2}$$

$$p$$

$$n$$

$$p$$

$$n$$

$$p$$

$$p$$

$$\frac{p-1}{2}$$

$$a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$$

$$\frac{p-1}{2} \mid (p-1)$$

$$a^{\frac{p-1}{2}} \equiv 1 \pmod{p}$$

$$\frac{p-1}{2}$$

$$p$$

$$\frac{p-1}{2}$$

$$x^2 \equiv a \pmod{p}$$

$$p$$

$$a$$

$$p \bmod 4 = 3$$

$$a^{(p+1)/4} \bmod p$$

$$p \bmod 8 = 5$$

$$b \equiv (2a)^{(p-5)/8} \pmod{p}$$

$$\mathrm{i} \equiv 2ab^2 \pmod{p}$$

$$\mathrm{i}^2 \equiv -1 \pmod{p}$$

$$ab(\mathrm{i}-1) \bmod p$$

$$x^2 \equiv a \pmod{p}$$

$$p$$

$$a$$

$$r$$

$$r^2 - a$$

$$(r-x)^{(p+1)/2} \bmod (x^2 - (r^2 - a))$$

$$\mathbf{C}$$

$$x^2 + 1 \in \mathbf{R}[x]$$

$$\mathbf{R}[x]$$

$$x^2 + 1$$

$$\mathbf{R}[x]/(x^2+1)$$

$$a_0 + a_1 x$$

$$a_0, a_1 \in \mathbf{R}$$

$$x^2 \equiv -1 \pmod{(x^2+1)}$$

$$\mathbf{R}[x]/(x^2+1)$$

$$\mathbf{C}$$

$$\mathbb{F}_p$$

$$\mathbb{F}_p[x]$$

$$x^2 - (r^2 - a) \in \mathbb{F}_p[x]$$

$$\mathbb{F}_p[x]/(x^2 - (r^2 - a))$$

$$r$$

$$a \equiv 0 \pmod p$$

$$r^2 - a$$

$$x \equiv 0 \pmod p$$

$$a \not\equiv 0 \pmod p$$

$$r^2 - a$$

$$\frac{p-3}{2}$$

$$r^2 \equiv a \pmod p$$

$$\frac{p-1}{2}$$

$$r^2 - a$$

$$r$$

$$f(x) = x^2 - (r^2 - a) \in \mathbb{F}_p[x]$$

$$a_0 + a_1 x = (r - x)^{(p+1)/2} \bmod (x^2 - (r^2 - a))$$

$$a_0^2 \equiv a \pmod p$$

$$a_1 \equiv 0 \pmod p$$

$$i = 0$$

$$i = p$$

$$p$$

$$p$$

$$p$$

$$a_1 \not\equiv 0 \pmod p$$

$$x$$

$$a_0 \equiv 0 \pmod p$$

$$r^2 - a \equiv a/a_1^2 \pmod p$$

$$a_1 \equiv 0 \pmod p$$

$$a_0^2 \equiv a \pmod p$$

$$b = x^{(p+1)/2} \bmod (x^2 - tx + a)$$

$$b^2 \equiv a \pmod p$$

$$x^2 - tx + a \in \mathbb{F}_p[x]$$

$$t$$

$$n = 0$$

$$[x^0]\frac{k_0 + k_1 x}{1 + k_2 x + k_3 x^2} = k_0$$

$$x^2 \equiv a \pmod p$$

$$p$$

$$a$$

$$r$$

$$r^2 - a$$

$$a_0 + a_1 x = (r - x)^{\frac{p-1}{2}} \bmod (x^2 - a)$$

$$a_0 \equiv 0 \pmod p$$

$$a_1^{-2} \equiv a \pmod p$$

$$b$$

$$b^2 \equiv a \pmod p$$

$$(r - b)(r + b) = r^2 - a$$

$$2a_0 = (\pm 1) + (\mp 1) = 0$$

$$2a_1 b = (\pm 1) - (\mp 1) = \pm 2$$

$$x^2 \equiv a \pmod p$$

$$p$$

$$a$$

$$p$$

$$p - 1 = 2^n \cdot m$$

$$m$$

$$r \in \mathbb{F}_p$$

$$r$$

$$g \equiv r^m \pmod p$$

$$b \equiv a^{(m-1)/2} \pmod p$$

$$e \in \{0, 1, 2, ..., 2^n - 1\}$$

$$ab^2 \equiv g^e \pmod p$$

$$a$$

$$e$$

$$\left(abg^{-e/2}\right)^2 \equiv a \pmod p$$

$$g$$

$$2^n$$

$$ab^2 \equiv a^m \pmod p$$

$$x^{2^n} \equiv 1 \pmod p$$

$$a^m$$

$$g$$

$$a^m \equiv g^e \pmod p$$

$$a$$

$$e$$

$$e$$

$$e$$

$$e$$

$$e = e_0 + 2e_1 + 4e_2 + \cdots$$

$$e_k \in \{0, 1\}$$

$$a$$

$$e_0 = 0$$

$$e_1$$

$$e_2$$

$$a^r \equiv a^r (a^{\delta_m(a)})^q \equiv a^n \equiv 1 \pmod m$$

$$\delta_m(ab) = \delta_m(a)\delta_m(b)$$

$$(\delta_m(a), \delta_m(b)) = 1$$

$$(ab)^{[\delta_m(a), \delta_m(b)]} \equiv 1 \pmod m$$

$$\delta_m(ab) \mid [\delta_m(a), \delta_m(b)]$$

$$\delta_m(a)\delta_m(b) \mid [\delta_m(a), \delta_m(b)]$$

$$1 \equiv (ab)^{\delta_m(ab)\delta_m(b)} \equiv a^{\delta_m(ab)\delta_m(b)} \pmod m$$

$$\delta_m(a) \mid \delta_m(ab)$$

$$\delta_m(b) \mid \delta_m(ab)$$

$$\delta_m(a)\delta_m(b) \mid \delta_m(ab)$$

$$(ab)^{\delta_m(a)\delta_m(b)} \equiv (a^{\delta_m(a)})^{\delta_m(b)} \times (b^{\delta_m(b)})^{\delta_m(a)} \equiv 1 \pmod m$$

$$\delta_m(ab) \mid \delta_m(a)\delta_m(b)$$

$$\delta_m(ab) = \delta_m(a)\delta_m(b)$$

$$\delta_m(a^k) = \frac{\delta_m(a)}{(\delta_m(a), k)}$$

$$\begin{aligned} a^{k\delta_m(a^k)} = (a^k)^{\delta_m(a^k)} \equiv 1 \pmod m \\ \implies \delta_m(a) \mid k\delta_m(a^k) \\ \implies \frac{\delta_m(a)}{(\delta_m(a), k)} \mid \delta_m(a^k) \end{aligned}$$

$$(a^k)^{\frac{\delta_m(a)}{(\delta_m(a),k)}} = (a^{\delta_m(a)})^{\frac{k}{(\delta_m(a),k)}} \equiv 1 \pmod m$$

$$\delta_m(a^k) \mid \frac{\delta_m(a)}{(\delta_m(a), k)}$$

$$\delta_m(a^k) = \frac{\delta_m(a)}{(\delta_m(a), k)}$$

$$1 \equiv g^{kt} \equiv g^{x\varphi(m)+(t,\varphi(m))} \equiv g^{(t,\varphi(m))} \pmod m$$

$$\delta_m(g^k) = \frac{\delta_m(g)}{(\delta_m(g), k)} = \frac{\varphi(m)}{(\varphi(m), k)}$$

$$\left(\delta_m(a) = \prod_{i=1}^{k} p_i^{\alpha_i}, \delta_m(b) = \prod_{i=1}^{k} p_i^{\beta_i}\right)$$

$$\delta_m(a) = XY, \delta_m(b) = ZW$$

$$\begin{aligned} \delta_m(a^X) &= \frac{\delta_m(a)}{(\delta_m(a), X)} \\ &= \frac{XY}{X} \\ &= Y \end{aligned}$$

$$\delta_m(b^Z) = W$$

$$\begin{aligned} \delta_m(a^X b^Z) &= \delta_m(a^X)\delta_m(b^Z) \\ &= YW \\ &= \left[\delta_p(a), \delta_p(b)\right] \end{aligned}$$

$$\delta_p(g) = \left[\delta_p(1), \delta_p(2), \cdots, \delta_p(p-1)\right]$$

$$x^{\delta_p(g)} \equiv 1 \pmod p$$

$$\begin{aligned}(g+p)^{p-1} &\equiv \binom{p-1}{0}g^{p-1}+\binom{p-1}{1}pg^{p-2} \pmod{p^2}\\ &\equiv g^{p-1}+p(p-1)g^{p-2} \pmod{p^2}\\ &\equiv 1-pg^{p-2} \pmod{p^2}\\ &\not\equiv 1 \pmod{p^2}\end{aligned}$$

$$g^{\varphi(p^\beta)}=1+p^\beta k_\beta$$

$$\begin{aligned}g^{\varphi(p^{\beta+1})} &= \left(g^{\varphi(p^\beta)}\right)^p\\ &= \left(1+p^\beta k_\beta\right)^p\\ &\equiv 1+p^{\beta+1}k_\beta \pmod{p^{\beta+2}}\end{aligned}$$

$$g^{\varphi(p^\beta)}\not\equiv 1 \pmod{p^{\beta+1}} \Longrightarrow g^\delta \not\equiv 1 \pmod{p^{\beta+1}}$$

$$\delta_{p^\alpha}(g)=p^{\alpha-1}(p-1)=\varphi(p^\alpha)$$

$$G^{\delta_{2p^\alpha}(G)}\equiv 1 \pmod{p^\alpha}$$

$$\begin{aligned}a^{2^{\alpha-2}} &= (2k+1)^{2^{\alpha-2}}\\ &\equiv 1+\binom{2^{\alpha-2}}{1}(2k)+\binom{2^{\alpha-2}}{2}(2k)^2 \pmod{2^\alpha}\\ &\equiv 1+2^{\alpha-1}k+2^{\alpha-1}(2^{\alpha-2}-1)k^2 \pmod{2^\alpha}\\ &\equiv 1+2^{\alpha-1}(k+(2^{\alpha-2}-1)k^2) \pmod{2^\alpha}\\ &\equiv 1 \pmod{2^\alpha}\end{aligned}$$

$$a^{\varphi(r)}\equiv 1 \pmod{r}\ , \quad a^{\varphi(t)}\equiv 1 \pmod{t}$$

$$a^{\frac{1}{2}\varphi(r)\varphi(t)}\equiv 1 \pmod{rt}$$

$$\delta_m(a)\le \frac{1}{2}\varphi(r)\varphi(t)=\frac{1}{2}\varphi(rt)=\frac{1}{2}\varphi(m)<\varphi(m)$$

$$a\in \mathbf{Z}$$

$$m\in \mathbf{N}^*$$

$$(a,m)=1$$

$$a^{\varphi(m)}\equiv 1 \pmod{m}$$

$$a^n\equiv 1 \pmod{m}$$

$$n$$

$$n$$

$$a$$

$$m$$

$$\delta_m(a)$$

$$\mathrm{ord}_m(a)$$

$$m$$

$$a$$

$$\delta$$

$$\delta^-$$

$$a^n \equiv -1 \pmod m$$

$$a, a^2, \cdots, a^{\delta_m(a)}$$

$$m$$

$$i \neq j$$

$$a^i \equiv a^j \pmod m$$

$$a^{\mid i-j \mid} \equiv 1 \pmod p$$

$$0 < \mid i-j \mid < \delta_m(a)$$

$$a^n \equiv 1 \pmod m$$

$$\delta_m(a) \mid n$$

$$n$$

$$\delta_m(a)$$

$$n = \delta_m(a)q + r, 0 \leq r < \delta_m(a)$$

$$r > 0$$

$$\delta_m(a)$$

$$r = 0$$

$$\delta_m(a) \mid n$$

$$a^p \equiv a^q \pmod m$$

$$p \equiv q \pmod{\delta_m(a)}$$

$$m \in \mathbf{N}^*$$

$$a, b \in \mathbf{Z}$$

$$(a, m) = (b, m) = 1$$

$$a^{\delta_m(a)} \equiv 1 \pmod m$$

$$b^{\delta_m(b)} \equiv 1 \pmod m$$

$$\delta_m(ab) = \delta_m(a)\delta_m(b)$$

$$(\delta_m(a), \delta_m(b)) = 1$$

$$(ab)^{\delta_m(ab)} \equiv 1 \pmod m$$

$$\delta_m(a) \mid \delta_m(ab)\delta_m(b)$$

$$(\delta_m(a), \delta_m(b)) = 1$$

$$k \in \mathbf{N}$$

$$m \in \mathbf{N}^*$$

$$a \in \mathbf{Z}$$

$$(a, m) = 1$$

$$a^{\delta_m(a)} \equiv 1 \pmod m$$

$$m \in \mathbf{N}^*$$

$$g \in \mathbf{Z}$$

$$(g, m) = 1$$

$$\delta_m(g) = \varphi(m)$$

$$g$$

$$m$$

$$g$$

$$\delta_m(g) = \left|\mathbf{Z}_m^*\right| = \varphi(m)$$

$$m$$

$$g^i \bmod m, 0 < i < m$$

$$m$$

$$m$$

$$m \geqslant 3, (g, m) = 1$$

$$g$$

$$m$$

$$\varphi(m)$$

$$p$$

$$g^{\frac{\varphi(m)}{p}} \not\equiv 1 \pmod m$$

$$\varphi(m)$$

$$p$$

$$g^{\frac{\varphi(m)}{p}} \not\equiv 1 \pmod m$$

$$g$$

$$m$$

$$g$$

$$m$$

$$t < \varphi(m)$$

$$g^t \equiv 1 \pmod{m}$$

$$k, x$$

$$kt = x\varphi(m) + (t, \varphi(m))$$

$$g^{\varphi(m)} \equiv 1 \pmod{m}$$

$$(t, \varphi(m)) \mid \varphi(m)$$

$$(t, \varphi(m)) \leqslant t < \varphi(m)$$

$$\varphi(m)$$

$$p$$

$$(t, \varphi(m)) \mid \frac{\varphi(m)}{p}$$

$$g^{\frac{\varphi(m)}{p}} \equiv g^{(t,\varphi(m))} \equiv 1 \pmod{m}$$

$$m$$

$$\varphi(\varphi(m))$$

$$m$$

$$g$$

$$(k, \varphi(m)) = 1$$

$$\delta_m(g^k) = \varphi(m)$$

$$g^k$$

$$m$$

$$(\varphi(m), k) = 1$$

$$1 \leq k \leq \varphi(m)$$

$$k$$

$$\varphi(\varphi(m))$$

$$\varphi(\varphi(m))$$

$$m$$

$$m = 2, 4, p^\alpha, 2p^\alpha$$

$$p$$

$$\alpha \in \mathbf{N}^*$$

$$m = 2, 4$$

$$m = p^{\alpha}$$

$$m = 2p^{\alpha}$$

$$m \neq 2, 4, p, p^{\alpha}$$

$$m = 2, 4$$

$$m = p^{\alpha}$$

$$p$$

$$\alpha \in \mathbf{N}^*$$

$$p$$

$$p$$

$$a$$

$$b$$

$$p$$

$$c \in \mathbf{Z}$$

$$\delta_p(c) = \left[\delta_p(a), \delta_p(b)\right]$$

$$\delta_m(a), \delta_m(b)$$

$$Y = \prod_{i=1}^{k} p_i^{[\alpha_i > \beta_i]\alpha_i}$$

$$X = \frac{\delta_m(a)}{Y}$$

$$W = \prod_{i=1}^{k} p_i^{[\alpha_i \leq \beta_i]\beta_i}$$

$$Z = \frac{\delta_m(b)}{W}$$

$$(Y, W) = 1$$

$$YW = \left[\delta_p(a), \delta_p(b)\right]$$

$$c = a^X b^Z$$

$$1 \sim (p - 1)$$

$$g \in \mathbf{Z}$$

$$\delta_p(j) \mid \delta_p(g)(j = 1, 2, \cdots, p - 1)$$

$$j = 1, 2, \cdots, p - 1$$

$$\delta_p(g) \geq p - 1$$

$$\delta_p(g) \leq p-1$$

$$\delta_p(g) = p-1 = \varphi(p)$$

$$g$$

$$p$$

$$p$$

$$\alpha \in \mathbf{N}^*$$

$$p^\alpha$$

$$p$$

$$p$$

$$g$$

$$g^{p-1} \not\equiv 1 \pmod{p^2}$$

$$p$$

$$g$$

$$g$$

$$g+p$$

$$g+p$$

$$p$$

$$g$$

$$\alpha \in \mathbf{N}^*$$

$$g$$

$$p^\alpha$$

$$\beta \in \mathbf{N}^*$$

$$p \nmid k_\beta$$

$$\beta = 1$$

$$g$$

$$\beta$$

$$p \nmid k_\beta$$

$$\beta + 1$$

$$\beta \in \mathbf{N}^*$$

$$\delta = \delta_{p^\alpha}(g)$$

$$\delta \mid p^{\alpha-1}(p-1)$$

$$g$$
$$p$$
$$g^\delta \equiv 1 \pmod{p^\alpha}$$
$$\delta = p^{\beta-1}(p-1)$$
$$1 \le \beta \le \alpha$$
$$g^\delta \equiv 1 \pmod{p^\alpha}$$
$$\beta \ge \alpha$$
$$\beta = \alpha$$
$$g$$
$$p^\alpha$$
$$m = 2p^\alpha$$
$$p$$
$$\alpha \in \mathbf{N}^*$$
$$p$$
$$\alpha \in \mathbf{N}^*$$
$$2p^\alpha$$
$$g$$
$$p^\alpha$$
$$g + p^\alpha$$
$$p^\alpha$$
$$g$$
$$g + p^\alpha$$
$$G$$
$$(G, 2p^\alpha) = 1$$
$$\delta_{2p^\alpha}(G) \mid \varphi(2p^\alpha)$$
$$G^{\delta_{2p^\alpha}(G)} \equiv 1 \pmod{2p^\alpha}$$
$$G$$
$$p^\alpha$$
$$\varphi(p^\alpha) \mid \delta_{2p^\alpha}(G)$$
$$\varphi(p^\alpha) = \varphi(2p^\alpha)$$
$$G$$

$$2p^\alpha$$

$$m \neq 2, 4, p^\alpha, p^\alpha$$

$$p$$

$$\alpha \in \mathbf{N}^*$$

$$m \neq 2, 4$$

$$p$$

$$\alpha \in \mathbf{N}^*$$

$$m = p^\alpha, 2p^\alpha$$

$$m$$

$$m = 2^\alpha$$

$$\alpha \in \mathbf{N}^*, \alpha \geq 3$$

$$a = 2k + 1$$

$$k$$

$$(2^{\alpha-2} - 1)k^2$$

$$m$$

$$2$$

$$m$$

$$m = rt$$

$$2 < r < t$$

$$(r, t) = 1$$

$$(a, m) = 1$$

$$n > 2$$

$$\varphi(n)$$

$$m$$

$$4$$

$$p$$

$$g_p = O(p^{0.25+\epsilon})$$

$$\epsilon > 0$$

$$p$$

$$g_p = \Omega(\log p)$$

$$a^m \equiv 1 \pmod{n}$$

$$\lambda(n) = \max\{\delta_n(a) : (a, n) = 1\}$$

$$\lambda(p^r) = \begin{cases} \frac{1}{2}\varphi(p^r), p = 2 \land r \geq 3, \\ \varphi(p^r), \quad \text{otherwise.} \end{cases}$$

$$\lambda(2^r) \mid 2^{r-2}$$

$$\lambda(p^r) = 2^{r-2} = \frac{1}{2}\varphi(p^r)$$

$$\lambda(n) = \left[\lambda(p_1^{r_1}), \lambda(p_2^{r_2}), ..., \lambda\left(p_k^{r_k}\right)\right]$$

$$x$$

$$\pi(x)$$

$$x$$

$$\pi(x) \sim \frac{x}{\ln(x)}$$

$$x$$

$$a$$

$$\frac{a}{x}$$

$$a$$

$$(x, \frac{a}{x})$$

$$[1, \sqrt{a}]$$

$$1$$

$$[2, n-1]$$

$$a$$

$$a^{n-1} \equiv 1 \pmod{n}$$

$$a^{n-1} \equiv 1 \pmod{n}$$

$$n$$

$$n$$

$$a$$

$$a^{n-1} \equiv 1 \pmod{n}$$

$$n$$

$$n = 341$$

$$a = 2$$

$$341 = 11 \cdot 31$$

$$2^{340} \equiv 1 \pmod{341}$$

$$341$$

$$a^{n-1} \equiv 1 \pmod{n}$$

$$n$$

$$n$$

$$n \nmid a$$

$$a$$

$$a^{n-1} \equiv 1 \pmod{n}$$

$$n$$

$$(a, n) = 1$$

$$a$$

$$m$$

$$p$$

$$r$$

$$n = p^r$$

$$\lambda(n) = \varphi(n)$$

$$\lambda(n) = \frac{1}{2}\varphi(n)$$

$$p^r$$

$$p = 2$$

$$r \geq 3$$

$$5^{2^{r-3}} \equiv 1 + 2^{r-1} \pmod{2^{r-2}}$$

$$\lambda(2^r) > 2^{r-3}$$

$$n$$

$$\lambda(n) \mid \varphi(n)$$

$$a$$

$$b$$

$$a \mid b \Longrightarrow \lambda(a) \mid \lambda(b)$$

$$n$$

$$n = \prod_{i=1}^{k} p_i^{r_i}$$

$$a$$

$$b$$

$$\lambda([a, b]) = [\lambda(a), \lambda(b)]$$

$$n$$

$$a$$

$$a$$

$$n$$

$$a^{n-1} \equiv 1 \pmod{n}$$

$$n$$

$$561 = 3 \times 11 \times 17$$

$$n$$

$$n$$

$$n$$

$$n$$

$$p$$

$$p - 1 \mid n - 1$$

$$n$$

$$\lambda(n) \mid n - 1$$

$$\lambda(n)$$

$$3$$

$$C(n)$$

$$n$$

$$C(n) > n^{2/7}$$

$$C(n) < n \exp\left(-c \frac{\ln n \ln \ln \ln n}{\ln \ln n}\right)$$

$$c$$

$$C(10^9) = 646$$

$$C(10^{18}) = 1\ 401\ 644$$

$$n$$

$$2^n - 1$$

$$561 = 3 \cdot 11 \cdot 17$$

$$2^{561} - 1$$

$$23 \cdot 89 = 2^{11} - 1 \mid 2^{561} - 1$$

$$2^{561} - 1$$

$$22$$

$$88$$

$$2^{561} - 2$$

$$v_2(2^{561}-2)=1<v_2(88)=3$$

$$88 \nmid 2^{561}-2$$

$$2^{561}-1$$

$$O(k\log^3 n)$$

$$O(k\log^2 n\log\log n\log\log\log n)$$

$$p$$

$$x^2 \equiv 1 \pmod p$$

$$x \equiv 1 \pmod p$$

$$x \equiv p-1 \pmod p$$

$$(x+1)(x-1) \equiv 0 \bmod p$$

$$p^e$$

$$a^{n-1} \equiv 1 \pmod n$$

$$n-1$$

$$n-1=u\times 2^t$$

$$a$$

$$v=a^u \bmod n$$

$$t$$

$$a^{u\times 2^s} \equiv n-1 \pmod n$$

$$1$$

$$a^{u\times 2^s} \not\equiv n-1 \pmod n$$

$$1$$

$$t$$

$$n$$

$$[2,\min\{n-2,\lfloor 2\ln^2 n\rfloor\}]$$

$$n$$

$$[1,2^{64})$$

$$[1,2^{32})$$

$$\{2,7,61\}$$

$$[1,2^{64})$$

$$\{2,325,9375,28178,450775,9780504,1795265022\}$$

$$\{2,3,5,7,11,13,17,19,23,29,31,37\}$$

$$12$$
$$[1, 2^{64})$$
$$a$$
$$n$$
$$n$$
$$a$$
$$a \bmod n$$
$$a \equiv 0 \pmod n$$
$$n$$
$$n$$
$$n$$
$$n$$
$$n = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$$
$$p$$
$$k$$
$$(k_1 + 1) \times (k_2 + 1) \times (k_3 + 1) \cdots \times (k_n + 1)$$
$$2$$
$$n = p_1^{k_1} p_2^{k_2} \cdots p_n^{k_n}$$
$$k_1 \geq k_2 \geq k_3 \geq \cdots \geq k_n$$
$$2$$
$$n$$
$$2$$
$$n$$
$$n$$
$$n$$
$$n$$
$$n = p_1 p_2 \cdots p_n$$
$$n$$
$$n$$
$$2^{64} - 1$$
$$2^{64} - 1$$
$$1$$
$$ans$$
$$10^{18}$$

$$10$$

$$1$$

$$2$$

$$5$$

$$10$$

$$\frac{1}{3}$$

$$\frac{1}{4}$$

$$\frac{1}{6}$$

$$12$$

$$24$$

$$60$$

$$6$$

$$8$$

$$12$$

$$n$$

$$10^{20}$$

$$\int_1^{\sqrt{n}} \sqrt[3]{\frac{n}{x^2}}\mathrm{d}x = O(\sqrt{n})$$

$$\begin{aligned} F(n) &= \sum_{i=1}^{n} f(i) \\ &= \sum_{i=1}^{n} \sum_{d \mid i} h(d) g\left(\frac{i}{d}\right) \\ &= \sum_{d=1}^{n} \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} h(d) g(i) \\ &= \sum_{d=1}^{n} h(d) \sum_{i=1}^{\lfloor \frac{n}{d} \rfloor} g(i) \\ &= \sum_{d=1}^{n} h(d) G\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \\ &= \sum_{\substack{d=1 \\ d \text{ is PN}}}^{n} h(d) G\left(\left\lfloor \frac{n}{d} \right\rfloor\right) \end{aligned}$$

$$
\begin{aligned}
& f(p^k) = \sum_{i=0}^{k} g(p^{k-i})h(p^i) \\
\Longleftrightarrow\ & p^k(p^k-1) = \sum_{i=0}^{k} p^{k-i}\varphi(p^{k-i})h(p^i) \\
\Longleftrightarrow\ & p^k(p^k-1) = \sum_{i=0}^{k} p^{2k-2i-1}(p-1)h(p^i) \\
\Longleftrightarrow\ & p^k(p^k-1) = h(p^k) + \sum_{i=0}^{k-1} p^{2k-2i-1}(p-1)h(p^i) \\
\Longleftrightarrow\ & h(p^k) = p^k(p^k-1) - \sum_{i=0}^{k-1} p^{2k-2i-1}(p-1)h(p^i) \\
\Longleftrightarrow\ & h(p^k) - p^2h(p^{k-1}) = p^k(p^k-1) - p^{k+1}(p^{k-1}-1) - p(p-1)h(p^{k-1}) \\
\Longleftrightarrow\ & h(p^k) - ph(p^{k-1}) = p^{k+1} - p^k \\
\Longleftrightarrow\ & \frac{h(p^k)}{p^k} - \frac{h(p^{k-1})}{p^{k-1}} = p-1
\end{aligned}
$$

$$
f(n) = \begin{cases} 1 & n = 1 \\ p \oplus c & n = p^c \\ f(a)f(b) n = ab \text{ and } a \perp b \end{cases}
$$

$$
f(p) = \begin{cases} p+1\ p = 2 \\ p-1 \text{ otherwise} \end{cases}
$$

$$
g(n) = \begin{cases} 3\varphi(n) 2 \mid n \\ \varphi(n) \ \text{ otherwise} \end{cases}
$$

$$
\begin{aligned}
G(n) &= \sum_{i=1}^{n} [i \bmod 2 = 1]\varphi(i) + 3\sum_{i=1}^{n}[i \bmod 2 = 0]\varphi(i) \\
&= \sum_{i=1}^{n} \varphi(i) + 2\sum_{i=1}^{n}[i \bmod 2 = 0]\varphi(i) \\
&= \sum_{i=1}^{n} \varphi(i) + 2\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} \varphi(2i)
\end{aligned}
$$

$$
\begin{aligned}
S_2(n) &= \sum_{i=1}^{n} \varphi(2i) \\
&= \sum_{i=1}^{\frac{n}{2}} (\varphi(2(2i-1)) + \varphi(2(2i))) \\
&= \sum_{i=1}^{\frac{n}{2}} (\varphi(2i-1) + 2\varphi(2i)) \\
&= \sum_{i=1}^{\frac{n}{2}} (\varphi(2i-1) + \varphi(2i)) + \sum_{i=1}^{\frac{n}{2}} \varphi(2i) \\
&= \sum_{i=1}^{n} \varphi(i) + S_2\left(\frac{n}{2}\right) \\
&= S_1(n) + S_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right)
\end{aligned}
$$

$$
\begin{aligned}
S_2(n) &= S_2(n-1) + \varphi(2n) \\
&= S_2(n-1) + \varphi(n) \\
&= \sum_{i=1}^{n-1} \varphi(i) + S_2\left(\frac{n-1}{2}\right) + \varphi(n) \\
&= S_1(n) + S_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right)
\end{aligned}
$$

$$g$$

$$g$$

$$g$$

$$p$$

$$g(p) = f(p)$$

$$f$$

$$F(n) = \sum_{i=1}^{n} f(i)$$

$$n$$

$$n$$

$$n = \prod_{i=1}^{m} p_i^{e_i}$$

$$n$$

$$\forall 1 \le i \le m, e_i > 1$$

$$a^2 b^3$$

$$e_i$$

$$p_i^{e_i}$$

$$a^2$$
$$e_i$$
$$p_i^3$$
$$b^3$$
$$p_i^{e_i-3}$$
$$a^2$$
$$n$$
$$O(\sqrt{n})$$
$$a$$
$$b$$
$$n$$
$$\sqrt{n}$$
$$n$$
$$O(\sqrt{n})$$
$$O(\sqrt{n})$$
$$g$$
$$p$$
$$g(p) = f(p)$$
$$G(n) = \sum_{i=1}^{n} g(i)$$
$$h = f/g$$
$$/$$
$$h$$
$$h(1) = 1$$
$$f = g * h$$
$$*$$
$$p$$
$$f(p) = g(1)h(p) + g(p)h(1) = h(p) + g(p) \implies h(p) = 0$$
$$h(p) = 0$$
$$h$$
$$n$$
$$h(n) = 0$$

$$h$$

$$f = g*h$$

$$O(\sqrt{n})$$

$$h$$

$$h$$

$$h(p^c)$$

$$h$$

$$h$$

$$d$$

$$h(d)G\left(\left\lfloor\frac{n}{d}\right\rfloor\right)$$

$$F(n)$$

$$h(p^c)$$

$$h(p^c)$$

$$p, c$$

$$h(p^c)$$

$$f = g*h$$

$$f(p^c) = \sum_{i=0}^{c} g(p^i)h(p^{c-i})$$

$$h(p^c) = f(p^c) - \sum_{i=1}^{c} g(p^i)h(p^{c-i})$$

$$p$$

$$c$$

$$h(p^c)$$

$$g$$

$$G$$

$$h(p^c)$$

$$h(p^c)$$

$$h(p^c)$$

$$O(\sqrt{n})$$

$$O\left(\frac{\sqrt{n}}{\log n}\right)$$

$$p$$

$$c$$

$$\log n$$

$$h(p^c)$$

$$(c-1)$$

$$O\left(\frac{\sqrt{n}}{\log n} \cdot \log n \cdot \log n\right) = O(\sqrt{n} \log n)$$

$$n$$

$$O(\sqrt{n})$$

$$O(\sqrt{n})$$

$$G$$

$$G\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

$$O(1)$$

$$O(\sqrt{n})$$

$$G\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

$$O(n^{\frac{2}{3}})$$

$$G(n)$$

$$G\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

$$h(p^c)$$

$$a$$

$$a_{i,j}$$

$$h(p_i^j)$$

$$O\left(\frac{\sqrt{n}}{\log n} \cdot \log n\right) = O(\sqrt{n})$$

$$f(p^k) = p^k(p^k - 1)$$

$$\sum_{i=1}^{n} f(i)$$

$$f(p) = p(p-1) = \mathrm{id}(p)\varphi(p)$$

$$g(n) = \mathrm{id}(n)\varphi(n)$$

$$G(n)$$

$$(\mathrm{id} \cdot \varphi) * \mathrm{id} = \mathrm{id}_2$$

$$G(n) = \sum_{i=1}^{n} i^2 - \sum_{i=2}^{n} d \cdot G\left(\left\lfloor \frac{n}{d} \right\rfloor\right)$$

$$h(p^k)$$

$$h(p^k)$$

$$p, k$$

$$h(p) = 0$$

$$h(p^k) = (k-1)(p-1)p^k$$

$$f(n)$$

$$g$$

$$g(p) = f(p)$$

$$g$$

$$G(n)$$

$$S_1(n) = \sum_{i=1}^{n} \varphi(i)$$

$$S_2(n) = \sum_{i=1}^{n} \varphi(2i)$$

$$G(n) = S_1(n) + 2S_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$

$$2 \mid n$$

$$2 \nmid n$$

$$S_2(n) = S_1(n) + S_2\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$

$$S_1$$

$$S_2$$

$$G$$

$$P(A) = \prod_{i=0}^{k-1} \frac{n-i}{n}$$

$$P(A) = \prod_{i=0}^{k-1} \frac{n-i}{n} \leq \frac{1}{2}$$

$$P(A) \leq \prod_{i=1}^{k-1} \exp\left(-\frac{i}{n}\right) = \exp\left(-\frac{k(k-1)}{2n}\right)$$

$$\exp\left(-\frac{k(k-1)}{2n}\right) \leq \frac{1}{2} \Longrightarrow P(A) \leq \frac{1}{2}$$

$$1, 3, 11, 23, 31, 11, 23, 31, \ldots$$

$$\begin{aligned} f(x) &= x^2 + c - kn \\ &= (k_1 p + a)^2 + c - kn \\ &= k_1^2 p^2 + 2k_1 pa + a^2 + c - kn \\ &\equiv a^2 + c \pmod p \end{aligned}$$

$$N \in \mathbf{N}_+$$

$$N$$

$$[2, \sqrt{N}]$$

$$[\sqrt{N} + 1, N)$$

$$[2, \sqrt{N}]$$

$$O(\sqrt{N})$$

$$N \geq 10^{18}$$

$$N$$

$$O(1)$$

$$N \geq 10^{18}$$

$$\frac{1}{10^{18}}$$

$$10^{18}$$

$$[2, \sqrt{N}]$$

$$[2, \sqrt{N}]$$

$$N$$

$$N$$

$$i$$

$$i$$

$$\frac{N}{A}$$

$$p$$

$$pi = \frac{N}{A}$$

$$piA = N$$

$$A$$

$$N$$

$$i$$

$$i$$

$$R_1, R_2, ..., R_n$$

$$= \frac{N}{R_1^{q_1} \cdot R_2^{q_2} \cdot \dots \cdot R_n^{q_n}}$$

$$A$$

$$i$$

$$N$$

$$K$$

$$K = K_1^{e_1} \cdot K_2^{e_2} \cdot \dots \cdot K_3^{e_3}$$

$$K_1 < K$$

$$K$$

$$K_1$$

$$K$$

$$K$$

$$O(\sqrt{N})$$

$$O(\sqrt{\frac{N}{\ln N}})$$

$$50\%$$

$$n$$

$$k$$

$$1, 2, ..., k$$

$$n$$

$$k$$

$$A$$

$$A$$

$$P(\overline{A}) = 1 - P(A)$$

$$P(\overline{A}) \geq \frac{1}{2}$$

$$1 + x \leq \mathrm{e}^x$$

$$n = 365$$

$$k \geq 23$$

$$23$$

$$50\%$$

$$k > 56$$

$$n = 365$$

$$99\%$$

$$n$$

$$\frac{1}{2}(\sqrt{8n \ln 2 + 1} + 1) \approx \sqrt{2n \ln 2}$$

$$50\%$$

$$n$$

$$[1, 1000]$$

$$i$$

$$i = 1$$

$$k$$

$$i = 1$$

$$\frac{1}{1000}$$

$$i = 2$$

$$\frac{1}{500}$$

$$k_2$$

$$k_1 - k$$

$$k_1 + k$$

$$i$$

$$n$$

$$n$$

$$\forall k \in \mathbf{N}_+, \gcd(k, n) \mid n$$

$$k$$

$$1 < \gcd(k, n) < n$$

$$n$$

$$\gcd(k, n)$$

$$k$$

$$n$$

$$f(x) = (x^2 + c) \bmod n$$

$$\{x_i\}$$

$$x_1$$

$$x_2 = f(x_1), x_3 = f(x_2), ..., x_i = f(x_{i-1})$$

$$c$$

$$N = 50, c = 2, x_1 = 1$$

$$f(x)$$

$$3$$

$$11, 23, 31$$

$$\rho$$

$$f(x) = (x^2 + c) \bmod n$$

$$\forall x \equiv y \pmod p, f(x) \equiv f(y) \pmod p$$

$$p \mid n$$

$$x \equiv y \pmod p$$

$$x = k_1 p + a$$

$$y = k_2 p + a$$

$$k_1, k_2, a \in \mathbb{Z}, a \in [0, p)$$

$$f(x) = (x^2 + c) \bmod n$$

$$f(x) = x^2 + c - kn$$

$$k \in \mathbb{Z}$$

$$f(y) \equiv a^2 + c \pmod p$$

$$f(x) \equiv f(y) \pmod p$$

$$O(\sqrt{n})$$

$$m$$

$$n$$

$$m \leq \sqrt{n}$$

$$\{x_i\}$$

$$m$$

$$\{y_i\}$$

$$\{x_i \bmod m\}$$

$$O(\sqrt{m}) \leq O(n^{\frac{1}{4}})$$

$$O(n^{\frac{1}{4}})$$

$$i, j$$

$$x_i \neq x_j \wedge y_i = y_j$$

$$n \nmid |\ x_i - x_j| \wedge m \mid |\ x_i - x_j|$$

$$\gcd(n, |\ x_i - x_j|)$$

$$n$$

$$\forall x \equiv y \pmod{p}, f(x) \equiv f(y) \pmod{p}$$

$$f(x)$$

$$x^2 + c$$

$$c$$

$$O(\sqrt{m})$$

$$i$$

$$n$$

$$\gcd(|\ x_i - x_j|, n)$$

$$O(\sqrt{m})$$

$$n$$

$$O(\sqrt{m}) \leq O(n^{\frac{1}{4}})$$

$$O(n^{\frac{1}{4}})$$

$$O(\sqrt{m})$$

$$a = f(1), b = f(f(1))$$

$$a = f(a), b = f(f(b))$$

$$d = \gcd(|\ x_i - x_j|, n)$$

$$1 < d < n$$

$$d$$

$$x_i$$

$$c$$

$$\gcd$$

$$O(\log N)$$

$$\gcd$$

$$1 < \gcd(a, b)$$

$$1 < \gcd(ac, b)$$

$$c \in \mathbf{N}_+$$

$$1 < \gcd(ac \bmod b, b) = \gcd(ac, b)$$

$$\gcd$$

$$s = \prod |\ x_0 - x_j| \bmod n$$

$$s = 0$$

$$n$$

$$2^k - 1$$

$$1 < \gcd(s, n) < n$$

$$k = 7$$

$$n$$

$$p$$

$$n$$

$$p$$

$$n$$

$$p$$

$$\frac{a_0 \frac{a_1 x + b_1}{c_1 x + d_1} + b_0}{c_0 \frac{a_1 x + b_1}{c_1 x + d_1} + d_0} = \frac{a_0(a_1 x + b_1) + b_0(c_1 x + d_1)}{c_0(a_1 x + b_1) + d_0(c_1 x + d_1)} = \frac{(a_0 a_1 + b_0 c_1)x + (a_0 b_1 + b_0 d_1)}{(c_0 a_1 + d_0 c_1)x + (c_0 b_1 + d_0 d_1)}$$

$$y = \frac{ax + b}{cx + d} \iff y(cx + d) = ax + b \iff x = -\frac{dy - b}{cy - a}$$

$$[a_0; a_1, ..., a_k, x] = [a_0; [a_1; [...; [a_k; x]]]] = (L_0 \circ L_1 \circ ... \circ L_k)(x) = \frac{p_k x + p_{k-1}}{q_k x + q_{k-1}}$$

$$(L_l \circ L_{l+1} \circ ... \circ L_r)(\infty)$$

$$(L \circ L_{a_0} \circ ... \circ L_{a_k})(x) = L\left(\frac{p_k x + p_{k-1}}{q_k x + q_{k-1}}\right) = \frac{a_k x + b_k}{c_k x + d_k}$$

$$L(x, y) \mapsto \frac{1}{L(x, y) - c_k}$$

$$x = [a_0; a_1, ..., a_k, x]$$

$$x = \frac{p_k x + p_{k-1}}{q_k x + q_{k-1}}$$

$$q_k x^2 + (q_{k-1} - p_k)x - p_{k-1} = 0$$

$$x = (L_0 \circ L_1)(y) = (L_0 \circ L_1 \circ L_0^{-1})(x)$$

$$\alpha = [a_0; a_1, ..., a_{k-1}, s_k] = \frac{s_k p_{k-1} + p_{k-2}}{s_k q_{k-1} + q_{k-2}}$$

$$s_k = -\frac{\alpha q_{k-1} - p_{k-1}}{\alpha q_k - p_k} = -\frac{q_{k-1} y\sqrt{n} + (x q_{k-1} - z p_{k-1})}{q_k y\sqrt{n} + (x q_k - z p_k)}$$

$$s_k = \frac{x_k + y_k\sqrt{n}}{z_k}$$

$$s_{k+1} = \frac{1}{s_k - a_k} = \frac{z_k}{(x_k - z_k a_k) + y_k\sqrt{n}} = \frac{z_k(x_k - y_k a_k) - y_k z_k\sqrt{n}}{(x_k - y_k a_k)^2 - y_k^2 n}$$

$$\begin{aligned} x*k+1 &= \quad z_k t_k \\ y*k+1 &= -y*k z_k \\ z*k+1 &= t_k^2 - y_k^2 n \end{aligned}$$

$$x = [\overline{a_0, a_1, ..., a_{l-1}}]$$

$$x' = [\overline{a_{l-1}, a_{l-2}, ..., a_0}]$$

$$y = -\frac{1}{x'}$$

$$\frac{p_{l-1}}{p_{l-2}} = [a_{l-1}, a_{l-2}, ..., a_0] = \frac{p'{}_{l-1}}{q'{}_{l-1}}$$

$$\frac{q_{l-1}}{q_{l-2}} = [a_{l-1}, a_{l-2}, ..., a_1] = \frac{p'{}_{l-2}}{q'{}_{l-2}}$$

$$x = \frac{x p_{l-1} + p_{l-2}}{x q_{l-1} + q_{l-2}}$$

$$q_{l-1} x^2 + (q_{l-2} - p_{l-1})x - p_{l-2} = 0$$

$$\sqrt{d} = [\left\lfloor \sqrt{d} \right\rfloor, \overline{a_1, a_2, ..., a_{l-1}, 2\left\lfloor \sqrt{d} \right\rfloor}]$$

$$a_k = a_{l-k}$$

$$\left\lfloor \sqrt{d} \right\rfloor + \sqrt{d}$$

$$\sqrt{d} = [\left\lfloor \sqrt{d} \right\rfloor, \overline{a_1, a_2, ..., a_{l-1}, 2\left\lfloor \sqrt{d} \right\rfloor}]$$

$$\frac{1}{-\left\lfloor \sqrt{d} \right\rfloor + \sqrt{d}} = [\overline{a_{l-1}, a_{l-2}, ..., a_1, 2\left\lfloor \sqrt{d} \right\rfloor}] = [\overline{a_1, a_2, ..., a_{l-1}, 2\left\lfloor \sqrt{d} \right\rfloor}]$$

$$\frac{\sqrt{d}}{2} = [\left\lfloor \frac{\sqrt{d}-1}{2} \right\rfloor + \frac{1}{2}, \overline{a_1, a_2, ..., a_{l-1}, 2\left\lfloor \frac{\sqrt{d}-1}{2} \right\rfloor + 1}]$$

$$a_k = a_{l-k}$$

$$\begin{aligned}
\sqrt{74} = 8 + (-8) + \sqrt{74} &= \left[8, \frac{8+\sqrt{74}}{10}\right] \\
&= \left[8, 1 + \frac{-2+\sqrt{74}}{10}\right] = \left[8, 1, \frac{2+\sqrt{74}}{7}\right] \\
&= \left[8, 1, 1 + \frac{-5+\sqrt{74}}{7}\right] = \left[8, 1, 1, \frac{5+\sqrt{74}}{7}\right] \\
&= \left[8, 1, 1, 1 + \frac{-2+\sqrt{74}}{7}\right] = \left[8, 1, 1, 1, \frac{2+\sqrt{74}}{10}\right] \\
&= \left[8, 1, 1, 1, 1 + \frac{-8+\sqrt{74}}{10}\right] = \left[8, 1, 1, 1, 1, 8 + \sqrt{74}\right] \\
&= \left[8, 1, 1, 1, 1, 16 + (-8) + \sqrt{74}\right] = \left[8, \overline{1, 1, 1, 1, 16}\right]
\end{aligned}$$

$$\begin{aligned}
r_1 &= \tfrac{8+\sqrt{74}}{10} &&= \left[\overline{1, 1, 1, 1, 16}\right] \\
r_2 &= \tfrac{2+\sqrt{74}}{7} &&= \left[\overline{1, 1, 1, 16, 1}\right] \\
r_3 &= \tfrac{5+\sqrt{74}}{7} &&= \left[\overline{1, 1, 16, 1, 1}\right] \\
r_4 &= \tfrac{2+\sqrt{74}}{10} &&= \left[\overline{1, 16, 1, 1, 1}\right] \\
r_5 &= 8 + \sqrt{74} &&= \left[\overline{16, 1, 1, 1, 1}\right]
\end{aligned}$$

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \frac{ax+b}{cx+d}$$

$$a, b, c, d \in \mathbb{R}$$

$$L_0(x) = \frac{a_0 x + b_0}{c_0 x + d_0}$$

$$L_1(x) = \frac{a_1 x + b_1}{c_1 x + d_1}$$

$$(L_0 \circ L_1)(x) = L_0(L_1(x))$$

$$a_1, ..., a_n$$

$$m$$

$$[a_l; a_{l+1}, ..., a_r]$$

$$[a_0; a_1, ..., a_k, b_0, b_1, ..., b_k] = [a_0; a_1, ..., a_k, [b_1; b_2, ..., b_k]]$$

$$L_k(x) = [a_k; x] = a_k + \frac{1}{x} = \frac{a_k \cdot x + 1}{1 \cdot x + 0}$$

$$L_k(\infty) = a_k$$

$$L(x) = \frac{ax + b}{cx + d}$$

$$A = [a_0; a_1, ..., a_n]$$

$$L(A)$$

$$[b_0; b_1, ..., b_m]$$

$$\frac{p}{q}$$

$$A + \frac{p}{q} = \frac{qA + p}{q}$$

$$A \cdot \frac{p}{q} = \frac{pA}{q}$$

$$[a_0; a_1, ..., a_k] = (L_{a_0} \circ L_{a_1} \circ ... \circ L_{a_k})(\infty)$$

$$L([a_0; a_1, ..., a_k]) = (L \circ L_{a_0} \circ L_{a_1} \circ ...L_{a_k})(\infty)$$

$$L_{a_0}$$

$$L_{a_1}$$

$$L(x)$$

$$x$$

$$x \geq 0$$

$$L(\frac{p_k x + p_{k-1}}{q_k x + q_{k-1}})$$

$$L(\frac{p_k}{q_k}) = \frac{a_k}{c_k}$$

$$L(\frac{p_{k-1}}{q_{k-1}}) = \frac{b_k}{d_k}$$

$$x = [a_{k+1}; ..., a_n]$$

$$L(A)$$

$$b_0 = \lfloor L(A) \rfloor$$

$$\left\lfloor L(\frac{p_k}{q_k}) \right\rfloor$$

$$\left\lfloor L(\frac{p_{k-1}}{q_{k-1}}) \right\rfloor$$

$$b_0$$

$$L(A) = (L_{b_0} \circ L_{b_1} \circ ... \circ L_{b_m})(\infty)$$

$$b_0$$

$$L_{b_0}^{-1}$$

$$L_{a_{k+1}}$$

$$L_{a_{k+2}}$$

$$b_1$$

$$[b_0; b_1, ..., b_m]$$

$$A = [a_0; a_1, ..., a_n]$$

$$B = [b_0; b_1, ..., b_m]$$

$$A + B$$

$$A \cdot B$$

$$L(x) = \frac{ax + b}{cx + d}$$

$$L(x, y) = \frac{axy + bx + cy + d}{exy + fx + gy + h}$$

$$L(x, y) \mapsto L(L_{a_k}(x), y)$$

$$L(x, y) \mapsto L(x, L_{b_k}(y))$$

$$L(x) \mapsto L(L_{a_k}(x))$$

$$\left\lfloor \frac{a}{e} \right\rfloor = \left\lfloor \frac{b}{f} \right\rfloor = \left\lfloor \frac{c}{g} \right\rfloor = \left\lfloor \frac{d}{h} \right\rfloor$$

$$c_k$$

$$l$$

$$r_k$$

$$k$$

$$x = [a_0; a_1, ..., a_k, x]$$

$$x = [a_0; a_1, ...]$$

$$x = [a_0; a_1, ..., a_k, y]$$

$$y$$

$$x = [a_0; a_1, ...]$$

$$[a_0, a_1, a_2, a_3, a_0, a_1, a_2, a_3, ...] = [\overline{a_0, a_1, a_2, a_3}]$$

$$[a_0, a_1, a_2, a_3, a_1, a_2, a_3, ...] = [a_0, \overline{a_1, a_2, a_3}]$$

$$a_0$$

$$a_0$$

$$x = [1; 1, 1, ...]$$

$$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$

$$x$$

$$k + 1$$

$$x$$

$$x$$

$$x = [a_0; a_1, ..., a_k, y]$$

$$y = [b_0; b_1, ..., b_k, y]$$

$$x = L_0(y)$$

$$y = L_1(y)$$

$$L_0$$

$$L_1$$

$$ax^2 + bx + c = 0$$

$$x$$

$$\xi$$

$$\frac{1}{q}\left(c + \sqrt{d}\right) \quad q \mid N(c - \sqrt{d})$$

$$a$$

$$c$$

$$a_k = \frac{c_{k+1} + c_k}{q_k}$$

$$c$$

$$\sqrt{d}$$

$$q$$

$$q$$

$$q_k q_{k+1} = -N(c_{k+1} - \sqrt{d})$$

$$\sqrt{d}$$

$$c$$

$$r$$

$$c$$

$$\sqrt{d}$$

$$c$$

$$q$$

$$c$$

$$q$$

$$\alpha = \frac{x + y\sqrt{n}}{z}$$

$$x, y, z, n \in \mathbb{Z}$$

$$n > 0$$

$$k$$

$$s_k$$

$$(xq_k - zp_k) - q_k y\sqrt{n}$$

$$\sqrt{n}$$

$$s_{k+1}$$

$$s_k$$

$$a_k = \lfloor s_k \rfloor = \left\lfloor \frac{x_k + y_k \lfloor \sqrt{n} \rfloor}{z_k} \right\rfloor$$

$$t_k = x_k - y_k a_k$$

$$x_{k+1}, y_{k+1}, z_{k+1}$$

$$\alpha = \sqrt{n}$$

$$x$$

$$y$$

$$z$$

$$\frac{x + y\sqrt{n}}{z}$$

$$-1$$

$$0$$

$$x$$

$$1$$

$$-1$$

$$0$$

$$r_{k+1}$$

$$-1$$

$$0$$

$$-1$$

$$0$$

$$a_k$$

$$r_{k+1}$$

$$\sqrt{74}$$

$$r_k$$

$$r_{l+1-k}$$

$$l$$

$$l$$

$$d$$

$$l$$

$$-1$$

$$x$$

$$k$$

$$x$$

$$\sqrt{x} = [a_0; a_1, ...]$$

$$\frac{p_k}{q_k} = [a_0; a_1, ..., a_k]$$

$$0 \leq k \leq 10^9$$

$$\sqrt{x}$$

$$a_k$$

$$T$$

$$\left\lfloor \frac{k-1}{T} \right\rfloor$$

$$\left| \frac{x}{y} - \sqrt{D} \right| \leqslant \frac{1}{y^2}$$

$$\left| N(x + y\sqrt{D}) \right| = |\ x - y\sqrt{D}|\ |\ x + y\sqrt{D}| \leqslant \frac{1}{y}(\frac{1}{y} + 2y\sqrt{D}) \leqslant 2\sqrt{D} + 1$$

$$\left| \frac{x}{y} - \sqrt{D} \right| = \frac{s}{y(x + y\sqrt{D})} < \frac{s}{2y^2\sqrt{D}} < \frac{1}{2y^2}$$

$$p_k^2 - Dq_k^2 = (-1)^{k+1} Q_{k+1}$$

$$r_{k+1} = \frac{P_{k+1} + \sqrt{D}}{Q_{k+1}}$$

$$\sqrt{D} = \frac{r_{k+1} p_k + p_{k-1}}{r_{k+1} q_k + q_{k-1}}$$

$$p_k P_{k+1} + p_{k-1} Q_{k+1} + p_k \sqrt{D} = q_k D + (q_k P_{k+1} + q_{k-1} Q_{k+1}) \sqrt{D}$$

$$p_k = q_k P_{k+1} + q_{k-1} Q_{k+1}$$

$$q_k D = p_k P_{k+1} + p_{k-1} Q_{k+1}$$

$$p_k^2 - Dq_k^2 = (p_k q_{k-1} - p_{k-1} q_k) Q_{k+1} = (-1)^{k-1} Q_{k+1} = (-1)^{k+1} Q_{k+1}$$

$$\Delta + \sqrt{D}$$

$$r_1(\sqrt{D}) = r_1(\Delta + \sqrt{D})$$

$$Q_0(\sqrt{D}) = Q_0(\Delta + \sqrt{D}) = 1$$

$$r_k = P_k + \sqrt{D} > 1$$

$$-1 < P_k - \sqrt{D} < 0$$

$$P_k < \sqrt{D} < P_k + 1$$

$$P_k = \left\lfloor \sqrt{D} \right\rfloor = \Delta$$

$$r_k = \Delta + \sqrt{D} = r_0$$

$$X = x_1 x_2 + y_1 y_2 D$$

$$Y = x_1 y_2 + x_2 y_1$$

$$X + Y\sqrt{D} = (x_1 + y_1 \sqrt{D})(x_2 + y_2 \sqrt{D})$$

$$x_n + y_n \sqrt{D} = (x_1 + y_1 \sqrt{D})^n$$

$$(x_1 + y_1 \sqrt{D})^n < X + Y\sqrt{D} < (x_1 + y_1 \sqrt{D})^{n+1}$$

$$1 < S + T\sqrt{D} < x_1 + y_1 \sqrt{D}$$

$$0 < S - T\sqrt{D} = \frac{1}{S + T\sqrt{D}} < 1$$

$$2S = S + T\sqrt{D} + S - T\sqrt{D} > 0$$

$$2T\sqrt{D} = S + T\sqrt{D} - (S - T\sqrt{D}) > 0$$

$$x_n + y_n \sqrt{D} = (x_1 + y_1 \sqrt{D})^n$$

$$\left(\frac{P_v - \sqrt{D}}{Q_v}\right) = -\frac{1}{\left(\frac{P_v + \sqrt{D}}{Q_v}\right)}$$

$$D = P_v^2 + Q_v^2$$

$$\frac{\sqrt{5}}{2} = [\frac{1}{2}, \overline{1}]$$

$$\frac{1}{2} = \frac{1}{2}\frac{1}{1}$$

$$\frac{\sqrt{17}}{2} = [\frac{3}{2}, \overline{1, 1, 3}]$$

$$\frac{3}{2} + \frac{1}{1 + \frac{1}{1}} = \frac{3}{2} + \frac{1}{2} = \frac{1}{2}\frac{4}{1}$$

$$a_n + pa_{n-1} + qa_{n-2} = 0$$
$$b_n + pb_{n-1} + qb_{n-2} = 0$$

$$a + b\sqrt{D}$$

$$D$$

$$a$$

$$b$$

$$D \equiv 2 \pmod 4$$

$$D \equiv 3 \pmod 4$$

$$a$$

$$b$$

$$a$$

$$b$$

$$D \equiv 1 \pmod 4$$

$$a + b\sqrt{D}$$

$$a + b\sqrt{D}$$

$$a^2 - Db^2$$

$$1$$

$$-1$$

$$Z(\sqrt{D})$$

$$\sqrt{D}$$

$$\sqrt{D}$$

$$\pm 1$$

$$\sqrt{D}$$

$$\sqrt{D}$$

$$\pm 1$$

$$\pm 1$$

$$-1$$

$$\frac{1}{0}$$

$$1$$

$$-1$$

$$x^2 - Dy^2 = s$$

$$\mid s \mid < \sqrt{D}$$

$$\frac{x}{y}$$

$$\sqrt{D}$$

$$s > 0$$

$$x^2 - Dy^2 > 0$$

$$x > y\sqrt{D}$$

$$\frac{x}{y}$$

$$\sqrt{D}$$

$$s < 0$$

$$x^2 - Dy^2 = s$$

$$y^2 - \frac{1}{D}x^2 = -\frac{s}{D}$$

$$\frac{y}{x}$$

$$\frac{1}{\sqrt{D}}$$

$$\frac{x}{y}$$

$$\sqrt{D}$$

$$Q_{k+1}$$

r_k

r

p_k

q_k

$k = nL$

n

L

$Q_k = 1$

$\sqrt{D}$

$\Delta + \sqrt{D}$

1

0

Q_k

$Q_0 = 1$

$Q_{nL} = 1$

$Q_k = 1$

L

$k = nL$

(x_1, y_1)

(x_2, y_2)

$x^2 - Dy^2 = 1$

$x^2 - Dy^2 = 1$

$x^2 - Dy^2 = -1$

D

$(\pm 1, 0)$

D

1

-1

4

-4

$\xi_0 = \sqrt{D}$

l

$$\frac{p_n}{q_n}$$

$$l$$

$$x = p_{jl-1}, y = q_{jl-1}, j = 1, 2, 3, \cdots$$

$$l$$

$$x = p_{jl-1}, y = q_{jl-1}, j = 2, 4, 6, \cdots$$

$$x = p_{jl-1}, y = q_{jl-1}, j = 1, 3, 5, \cdots$$

$$l$$

$$x + y\sqrt{D} = \pm(p_{l-1} \pm q_{l-1}\sqrt{D})^j, j = 0, 1, 2, \cdots$$

$$l$$

$$x + y\sqrt{D} = \pm(p_{l-1} \pm q_{l-1}\sqrt{D})^j, j = 0, 2, 4, \cdots$$

$$x + y\sqrt{D} = \pm(p_{l-1} \pm q_{l-1}\sqrt{D})^j, j = 1, 3, 5, \cdots$$

$$x^2 - Dy^2 = 1$$

$$x + y\sqrt{D}$$

$$(x_1, y_1)$$

$$(X, Y)$$

$$x_1 + y_1\sqrt{D} > 1$$

$$x_n - y_n\sqrt{D}$$

$$(x_1 + y_1\sqrt{D})^n$$

$$(S, T)$$

$$(S, T)$$

$$(x_1, y_1)$$

$$\sqrt{D}$$

$$x^2 - Dy^2 = -1$$

$$(x_1, y_1)$$

$$x^2 - Dy^2 = \pm 1$$

$$(x_{2n}, y_{2n})$$

$$x^2 - Dy^2 = 1$$

$$(x_{2n-1}, y_{2n-1})$$

$$x^2 - Dy^2 = -1$$

$$x^2 - Dy^2 = 1$$

$$x^2 - Dy^2 = -1$$

$$\sqrt{D}$$

$$D$$

$$D = r^2 + s^2$$

$$x^2 - Dy^2 = r$$

$$x^2 - Dy^2 = s$$

$$4k + 3$$

$$4k + 1$$

$$r_k$$

$$r_{l+1-k}$$

$$v$$

$$D$$

$$v$$

$$P_v = r$$

$$Q_v = s$$

$$P_v = s$$

$$Q_v = r$$

$$v - 1$$

$$\left\lfloor \frac{P_{v-1} + \sqrt{D}}{Q_{v-1}} \right\rfloor$$

$$v$$

$$Q_v = Q_{v-1}$$

$$x^2 - Dy^2 = (-1)^v Q_v$$

$$v$$

$$v - 1$$

$$x^2 - Dy^2 = -1$$

$$8$$

$$D \equiv 1 or 2 \pmod 8$$

$$D$$

$$D = r^2 + s^2$$

$$r, s \equiv \pm 3 \pmod 8$$

$$x^2 - Dy^2 = -1$$

$$34$$

$$34 = 3^2 + 5^2$$

$$x^2 - 34y^2 = -1$$

$$x^2 - Dy^2 = -1$$

$$x^2 - Dy^2 = r$$

$$x^2 - Dy^2 = s$$

$$8$$

$$D$$

$$4k + 1$$

$$D$$

$$4k + 1$$

$$D$$

$$4k + 1$$

$$\xi_0 = \frac{\sqrt{D}}{2}$$

$$2$$

$$D$$

$$5$$

$$\frac{\sqrt{5}}{2}$$

$$\frac{1}{2} + \frac{\sqrt{5}}{2}$$

$$D$$

$$17$$

$$\frac{\sqrt{17}}{2}$$

$$4 + \sqrt{17}$$

$$1$$

$$100$$

$$5$$

$$13$$

$$29$$

$$53$$

$$61$$

$$85$$

$$2$$

$$17$$

$$37$$

$$41$$

$$65$$

$$73$$

$$89$$

$$97$$

$$n$$

$$n$$

$$a$$

$$b$$

$$a$$

$$b$$

$$a + b\sqrt{D}$$

$$x^2 + px + q = 0$$

$$\frac{1}{2} + \frac{1}{2}\sqrt{5}$$

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & n \text{ □□□□□□} \\ (-1)^k k\text{□}n \text{ □□□□□□□□□□□} & \end{cases}$$

$$\sum_{d|n} \mu(d) = \begin{cases} 1 n = 1 \\ 0 n \neq 1 \end{cases}$$

$$\varphi * 1 = \mathrm{id}$$

$$
\begin{aligned}
\varphi * 1 &= \sum_{d|n} \varphi(\frac{n}{d}) \\
&= \sum_{i=0}^{c} \varphi(p^i) \\
&= 1 + p^0 \cdot (p-1) + p^1 \cdot (p-1) + \cdots + p^{c-1} \cdot (p-1) \\
&= p^c \\
&= \mathrm{id}
\end{aligned}
$$

$$
\sum_{d|n} \mu(d) f(\frac{n}{d}) = \sum_{d|n} \mu(d) \sum_{k|\frac{n}{d}} g(k) = \sum_{k|n} g(k) \sum_{d|\frac{n}{k}} \mu(d) = g(n)
$$

$$
\begin{aligned}
&\sum_{n \mid d} \mu(\frac{d}{n}) f(d) \\
&= \sum_{k=1}^{+\infty} \mu(k) f(kn) = \sum_{k=1}^{+\infty} \mu(k) \sum_{kn \mid d} g(d) \\
&= \sum_{n \mid d} g(d) \sum_{k \mid \frac{d}{n}} \mu(k) = \sum_{n \mid d} g(d) \epsilon(\frac{d}{n}) \\
&= g(n)
\end{aligned}
$$

$$
\sum_{i=x}^{n} \sum_{j=y}^{m} [\gcd(i,j) = k] \qquad (1 \leqslant T, x, y, n, m, k \leqslant 5 \times 10^4)
$$

$$
\sum_{i=1}^{n} \sum_{j=1}^{m} [\gcd(i,j) = k]
$$

$$
\sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{k} \rfloor} [\gcd(i,j) = 1]
$$

$$
\sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{k} \rfloor} \varepsilon(\gcd(i,j))
$$

$$
\sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} \sum_{j=1}^{\lfloor \frac{m}{k} \rfloor} \sum_{d| \gcd(i,j)} \mu(d)
$$

$$
\sum_{d=1} \mu(d) \sum_{i=1}^{\lfloor \frac{n}{k} \rfloor} [d \mid i] \sum_{j=1}^{\lfloor \frac{m}{k} \rfloor} [d \mid j]
$$

$$
\sum_{d=1}^{\min(\lfloor \frac{n}{k} \rfloor, \lfloor \frac{m}{k} \rfloor)} \mu(d) \left\lfloor \frac{n}{kd} \right\rfloor \left\lfloor \frac{m}{kd} \right\rfloor
$$

$$
\sum_{i=1}^{n} \mathrm{lcm}(i,n) \quad \text{s.t.}\ 1 \leqslant T \leqslant 3 \times 10^5, 1 \leqslant n \leqslant 10^6
$$

$$\sum_{i=1}^{n} \frac{i \cdot n}{\gcd(i,n)}$$

$$\frac{1}{2} \cdot \left(\sum_{i=1}^{n-1} \frac{i \cdot n}{\gcd(i,n)} + \sum_{i=n-1}^{1} \frac{i \cdot n}{\gcd(i,n)} \right) + n$$

$$\frac{1}{2} \cdot \left(\sum_{i=1}^{n-1} \frac{i \cdot n}{\gcd(i,n)} + \sum_{i=n-1}^{1} \frac{i \cdot n}{\gcd(n-i,n)} \right) + n$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n-1} \frac{n^2}{\gcd(i,n)} + n$$

$$\frac{1}{2} \cdot \sum_{i=1}^{n} \frac{n^2}{\gcd(i,n)} + \frac{n}{2}$$

$$\frac{1}{2} \cdot \sum_{d|n} \frac{n^2 \cdot \varphi(\frac{n}{d})}{d} + \frac{n}{2}$$

$$\frac{1}{2} n \cdot \left(\sum_{d'|n} d' \cdot \varphi(d') + 1 \right)$$

$$\mathrm{g}(p_j^k) = \sum_{w=0}^{k} p_j^w \cdot \varphi(p_j^w)$$

$$\sum_{w=0}^{k} p_j^{2w-1} \cdot (p_j - 1)$$

$$\mathrm{g}(p_j^{k+1}) = \mathrm{g}(p_j^k) + p_j^{2k+1} \cdot (p_j - 1)$$

$$\mathrm{g}(i \cdot p_j) = \mathrm{g}(a) \cdot \mathrm{g}(p_j^{w+1})$$

$$\mathrm{g}(i) = \mathrm{g}(a) \cdot \mathrm{g}(p_j^w)$$

$$\mathrm{g}(i \cdot p_j) - \mathrm{g}(i) = \mathrm{g}(a) \cdot p_j^{2w+1} \cdot (p_j - 1)$$

$$\mathrm{g}(i) - \mathrm{g}(\frac{i}{p_j}) = \mathrm{g}(a) \cdot p_j^{2w-1} \cdot (p_j - 1)$$

$$\mathrm{g}(i \cdot p_j) = \mathrm{g}(i) + \left(\mathrm{g}(i) - \mathrm{g}(\frac{i}{p_j}) \right) \cdot p_j^2$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \mathrm{lcm}(i,j) \qquad (n, m \leqslant 10^7)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{m} \frac{i \cdot j}{\gcd(i,j)}$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{d|i,d|j,\,\gcd(\frac{i}{d},\frac{j}{d})=1}\frac{i\cdot j}{d}$$

$$\sum_{d=1}^{n} d\cdot\sum_{i=1}^{\lfloor\frac{n}{d}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{d}\rfloor}[\gcd(i,j)=1]\ i\cdot j$$

$$\operatorname{sum}(n,m)=\sum_{i=1}^{n}\sum_{j=1}^{m}[\gcd(i,j)=1]\ i\cdot j$$

$$\sum_{d=1}^{n}\sum_{d|i}^{n}\sum_{d|j}^{m}\mu(d)\cdot i\cdot j$$

$$\sum_{d=1}^{n}\mu(d)\cdot d^2\cdot\sum_{i=1}^{\lfloor\frac{n}{d}\rfloor}\sum_{j=1}^{\lfloor\frac{m}{d}\rfloor} i\cdot j$$

$$g(n,m)=\sum_{i=1}^{n}\sum_{j=1}^{m} i\cdot j=\frac{n\cdot(n+1)}{2}\times\frac{m\cdot(m+1)}{2}$$

$$\operatorname{sum}(n,m)=\sum_{d=1}^{n}\mu(d)\cdot d^2\cdot g(\left\lfloor\frac{n}{d}\right\rfloor,\left\lfloor\frac{m}{d}\right\rfloor)$$

$$\sum_{d=1}^{n} d\cdot\operatorname{sum}(\left\lfloor\frac{n}{d}\right\rfloor,\left\lfloor\frac{m}{d}\right\rfloor)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{m} d(i\cdot j)\qquad (n,m,T\le 5\times 10^4)$$

$$d(i\cdot j)=\sum_{x|i}\sum_{y|j}[\gcd(x,y)=1]$$

$$\begin{aligned}
d(i\cdot j)&=\sum_{x|i}\sum_{y|j}[\gcd(x,y)=1]\\
&=\sum_{x|i}\sum_{y|j}\sum_{p|\gcd(x,y)}\mu(p)\\
&=\sum_{p=1}^{min(i,j)}\sum_{x|i}\sum_{y|j}[p\mid\gcd(x,y)]\cdot\mu(p)\\
&=\sum_{p|i,p|j}\mu(p)\sum_{x|i}\sum_{y|j}[p\mid\gcd(x,y)]\\
&=\sum_{p|i,p|j}\mu(p)\sum_{x|\frac{i}{p}}\sum_{y|\frac{j}{p}}1\\
&=\sum_{p|i,p|j}\mu(p)d\left(\frac{i}{p}\right)d\left(\frac{j}{p}\right)
\end{aligned}$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{m} d(i \cdot j) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{p|i,p|j} \mu(p) d\left(\frac{i}{p}\right) d\left(\frac{j}{p}\right) \\
&= \sum_{p=1}^{min(n,m)} \sum_{i=1}^{n}\sum_{j=1}^{m} [p \mid i, p \mid j] \cdot \mu(p) d\left(\frac{i}{p}\right) d\left(\frac{j}{p}\right) \\
&= \sum_{p=1}^{min(n,m)} \sum_{i=1}^{\left\lfloor \frac{n}{p} \right\rfloor}\sum_{j=1}^{\left\lfloor \frac{m}{p} \right\rfloor} \mu(p) d(i) d(j) \\
&= \sum_{p=1}^{min(n,m)} \mu(p) \sum_{i=1}^{\left\lfloor \frac{n}{p} \right\rfloor} d(i) \sum_{j=1}^{\left\lfloor \frac{m}{p} \right\rfloor} d(j) \\
&= \sum_{p=1}^{min(n,m)} \mu(p) S\left(\left\lfloor \frac{n}{p} \right\rfloor\right) S\left(\left\lfloor \frac{m}{p} \right\rfloor\right) \left(S(n) = \sum_{i=1}^{n} d(i)\right)
\end{aligned}
$$

$$
f(n) = \sum_{i=1}^{n} t(i) g\left(\left\lfloor \frac{n}{i} \right\rfloor\right)
$$

$$
\Longleftrightarrow
$$

$$
g(n) = \sum_{i=1}^{n} \mu(i) t(i) f\left(\left\lfloor \frac{n}{i} \right\rfloor\right)
$$

$$
\begin{aligned}
g(n) &= \sum_{i=1}^{n} \mu(i)t(i)f\left(\left\lfloor \frac{n}{i} \right\rfloor\right) \\
&= \sum_{i=1}^{n} \mu(i)t(i) \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} t(j)g\left(\left\lfloor \frac{\lfloor \frac{n}{i} \rfloor}{j} \right\rfloor\right) \\
&= \sum_{i=1}^{n} \mu(i)t(i) \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} t(j)g\left(\left\lfloor \frac{n}{ij} \right\rfloor\right) \\
&= \sum_{T=1}^{n} \sum_{i=1}^{n} \mu(i)t(i) \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} [ij=T]t(j)g\left(\left\lfloor \frac{n}{T} \right\rfloor\right) && \text{□□□□ ij □□□} \\
&= \sum_{T=1}^{n} \sum_{i|T} \mu(i)t(i)t\left(\frac{T}{i}\right)g\left(\left\lfloor \frac{n}{T} \right\rfloor\right) && \text{□}\sum_{j=1}^{\lfloor \frac{n}{i} \rfloor}[ij=T] \text{ □□□□□□□ 1□□□□□□} \\
&= \sum_{T=1}^{n} g\left(\left\lfloor \frac{n}{T} \right\rfloor\right) \sum_{i|T} \mu(i)t(i)t\left(\frac{T}{i}\right) \\
&= \sum_{T=1}^{n} g\left(\left\lfloor \frac{n}{T} \right\rfloor\right) \sum_{i|T} \mu(i)t(T) && \text{□t □□□□□□□□} \\
&= \sum_{T=1}^{n} g\left(\left\lfloor \frac{n}{T} \right\rfloor\right) t(T) \sum_{i|T} \mu(i) \\
&= \sum_{T=1}^{n} g\left(\left\lfloor \frac{n}{T} \right\rfloor\right) t(T)\varepsilon(T) && \text{□}\mu * 1 = \varepsilon\text{□} \\
&= g(n)t(1) && \text{□□□□□ T=1}, \varepsilon(T) = 1 \text{ □□} \\
&= g(n) && \square
\end{aligned}
$$

$$f(n)$$

$$g(n)$$

$$f(n)$$

$$\mu$$

$$n = \prod_{i=1}^{k} p_i^{c_i}$$

$$p_i$$

$$c_i \geq 1$$

$$n = 1$$

$$\mu(n) = 1$$

$$n \neq 1$$

$$i \in [1, k]$$

$$c_i > 1$$

$$\mu(n) = 0$$

$$\mu(n)$$

$$0$$

$$i \in [1, k]$$

$$c_i = 1$$

$$\mu(n) = (-1)^k$$

$$n = \prod_{i=1}^{k} p_i$$

$$\{p_i\}_{i=1}^{k}$$

$$\mu(n)$$

$$-1$$

$$k$$

$$k$$

$$\sum_{d|n} \mu(d) = \varepsilon(n)$$

$$\mu^* 1 = \varepsilon$$

$$n = \prod_{i=1}^{k} p_i^{c_i}, n' = \prod_{i=1}^{k} p_i$$

$$\sum_{d|n} \mu(d) = \sum_{d|n'} \mu(d) = \sum_{i=0}^{k} \binom{k}{i} \cdot (-1)^i = (1 + (-1))^k$$

$$k = 0$$

$$n = 1$$

$$1$$

$$0$$

$$\sum_{d|n} \mu(d) = [n = 1] = \varepsilon(n)$$

$$\mu * 1 = \varepsilon$$

$$\zeta$$

$$1$$

$$[\gcd(i, j) = 1] = \sum_{d | \gcd(i,j)} \mu(d)$$

$$\gcd(i, j) = 1$$

$$n$$

$$1$$

$$1$$

$$[\gcd(i,j)=1]$$

$$0$$

$$\varepsilon$$

$$[\gcd(i,j)=1]=\varepsilon(\gcd(i,j))$$

$$\varepsilon$$

$$\mu$$

$$n$$

$$n=\prod_{i=1}^{k} p_i^{c_i}$$

$$\varphi$$

$$n'=p^c$$

$$\varphi * 1=\sum_{d|n'} \varphi(\frac{n'}{d})=\mathrm{id}$$

$$p$$

$$d=p^0,p^1,p^2,\cdots,p^c$$

$$\mu$$

$$\varphi(n)=\sum_{d|n} d\cdot\mu(\frac{n}{d})$$

$$f(n),g(n)$$

$$f(n)=\sum_{d|n} g(d)$$

$$g(n)=\sum_{d|n} \mu(d)f(\frac{n}{d})$$

$$f(n)$$

$$g(n)$$

$$g(n)$$

$$f(n)$$

$$g(n)$$

$$g(n)$$

$$1$$

$$g(n)$$

$$g(n)$$

$$\zeta$$

$$f(n) = \sum_{n \mid d} g(d)$$

$$g(n) = \sum_{n \mid d} \mu(\frac{d}{n}) f(d)$$

$$\sum_{d|n} g(d)$$

$$f(\frac{n}{d})$$

$$\sum_{d|n} \mu(d) = [n = 1]$$

$$\frac{n}{k} = 1$$

$$1$$

$$n = k$$

$$\sum_{k|n} [n = k] \cdot g(k) = g(n)$$

$$f = g * 1$$

$$g = f * \mu$$

$$f * \mu = g*1*\mu \implies f * \mu = g$$

$$1 * \mu = \varepsilon$$

$$d$$

$$kn$$

$$f$$

$$k$$

$$kn$$

$$n$$

$$k$$

$$\epsilon$$

$$d = n$$

$$4$$

$$\gcd(i, j) = 1$$

$$\varepsilon(\gcd(i,j))$$

$$\varepsilon(n)$$

$$n=1$$

$$1$$

$$0$$

$$\varepsilon$$

$$d \mid \gcd(i,j)$$

$$1 \sim \left\lfloor \frac{n}{k} \right\rfloor$$

$$d$$

$$\left\lfloor \frac{n}{kd} \right\rfloor$$

$$\Theta(N+T\sqrt{n})$$

$$\gcd(i,n)=\gcd(n-i,n)$$

$$\gcd(i,n)$$

$$\gcd(i,n)=d$$

$$\gcd(i,n)=d$$

$$\gcd(\frac{i}{d},\frac{n}{d})=1$$

$$\gcd(i,n)=d$$

$$\varphi(\frac{n}{d})$$

$$d'=\frac{n}{d}$$

$$\mathrm{g}(n)=\sum_{d|n} d\cdot\varphi(d)$$

$$\mathrm{g}$$

$$\Theta(n)$$

$$\Theta(1)$$

$$\mathrm{g}(p_j^k)$$

$$p_j^0,p_j^1,\cdots,p_j^k$$

$$\varphi(p_j^w)=p_j^{w-1}\cdot(p_j-1)$$

$$\mathrm{g}(i\cdot p_j)(p_j|\ i)$$

$$i = a \cdot p_j^w (\gcd(a, p_j) = 1)$$

$$\Theta(n + T)$$

$$20101009$$

$$d$$

$$d$$

$$\gcd$$

$$\text{sum}(n, m)$$

$$[\gcd(i, j) = 1]$$

$$\varepsilon(\gcd(i, j))$$

$$i = i' \cdot d$$

$$j = j' \cdot d$$

$$\Theta(1)$$

$$\left\lfloor \frac{n}{\lfloor \frac{n}{d} \rfloor} \right\rfloor$$

$$\text{sum}(n, m)$$

$$\text{sum}(n, m)$$

$$\text{sum}$$

$$n \leqslant m$$

$$\Theta(n + m)$$

$$d(n) = \sum_{i|n} 1$$

$$d(n)$$

$$n$$

$$d$$

$$O(n)$$

$$\mu, d$$

$$O(\sqrt{n})$$

$$O(n + T\sqrt{n})$$

$$f, g$$

$$t$$

$$t(1) = 1$$

$$
\begin{aligned}
F_k(n) &= \sum_{i=2}^{n} [p_k \le \operatorname{lpf}(i)] f(i) \\
&= \sum_{\substack{k \le i \\ p_i^2 \le n}} \sum_{\substack{c \ge 1 \\ p_i^c \le n}} f(p_i^c)([c > 1] + F_{i+1}(n/p_i^c)) + \sum_{\substack{k \le i \\ p_i \le n}} f(p_i) \\
&= \sum_{\substack{k \le i \\ p_i^2 \le n}} \sum_{\substack{c \ge 1 \\ p_i^c \le n}} f(p_i^c)([c > 1] + F_{i+1}(n/p_i^c)) + F_{\text{prime}}(n) - F_{\text{prime}}(p_{k-1}) \\
&= \sum_{\substack{k \le i \\ p_i^2 \le n}} \sum_{\substack{c \ge 1 \\ p_i^{c+1} \le n}} \left(f(p_i^c) F_{i+1}(n/p_i^c) + f\left(p_i^{c+1}\right)\right) + F_{\text{prime}}(n) - F_{\text{prime}}(p_{k-1})
\end{aligned}
$$

$$
G_k(n) = G_{k-1}(n) - [p_k^2 \le n] g(p_k) (G_{k-1}(n/p_k) - G_{k-1}(p_{k-1}))
$$

$$
\begin{aligned}
T(n) &= \sum_{i^2 \le n} O\left(\pi\left(\sqrt{i}\right)\right) + \sum_{i^2 \le n} O\left(\pi\left(\sqrt{\frac{n}{i}}\right)\right) \\
&= \sum_{i^2 \le n} O\left(\frac{\sqrt{i}}{\ln \sqrt{i}}\right) + \sum_{i^2 \le n} O\left(\frac{\sqrt{\frac{n}{i}}}{\ln \sqrt{\frac{n}{i}}}\right) \\
&= O\left(\int_1^{\sqrt{n}} \frac{\sqrt{\frac{n}{x}}}{\log \sqrt{\frac{n}{x}}} \mathrm{d}x\right) \\
&= O\left(\frac{n^{\frac{3}{4}}}{\log n}\right)
\end{aligned}
$$

$$
f(n) = \begin{cases} 1 & n = 1 \\ p \operatorname{xor} c & n = p^c \\ f(a)f(b) n = ab \land a \perp b \end{cases}
$$

$$
O\left(\frac{n^{\frac{3}{4}}}{\log n}\right)
$$

$$
\Theta\left(n^{1-\epsilon}\right)
$$

$$
f(p)
$$

$$
p
$$

$$
f(p^c)
$$

$$
p
$$

$$
x/y := \left\lfloor \frac{x}{y} \right\rfloor
$$

$$
\operatorname{isprime}(n) := [| \ \{d : d \mid n\} \ | = 2]
$$

$$
n
$$

$$
1
$$

$$
0
$$

$$p_k$$

$$k$$

$$p_1 = 2, p_2 = 3$$

$$p_0 = 1$$

$$\operatorname{lpf}(n) := [1 < n] \min\{p : p \mid n\} + [1 = n]$$

$$n$$

$$n = 1$$

$$1$$

$$F_{\text{prime}}(n) := \sum_{2 \le p \le n} f(p)$$

$$F_k(n) := \sum_{i=2}^{n} [p_k \le \operatorname{lpf}(i)] f(i)$$

$$F_k(n)$$

$$F_1(n) + f(1) = F_1(n) + 1$$

$$F_k(n)$$

$$i$$

$$p_i^c \le n < p_i^{c+1}$$

$$c$$

$$p_i^{c+1} > n \iff n/p_i^c < p_i < p_{i+1}$$

$$F_{i+1}(n/p_i^c) = 0$$

$$F_k(n) = 0 (p_k > n)$$

$$F_{\text{prime}}(n)$$

$$F_k(n)$$

$$p$$

$$p^2 < n$$

$$F_{\text{prime}}(n)$$

$$F_k(n)$$

$$F_{\text{prime}}$$

$$1, 2, ..., \lfloor \sqrt{n} \rfloor, n/\sqrt{n}, ..., n/2, n$$

$$O(\sqrt{n})$$

$$f(p)$$

$$p$$

$$f(p) = \sum a_i p^{c_i}$$

$$p^{c_i}$$

$$F_{\text{prime}}(n)$$

$$a_i \sum_{2 \le p \le n} p^{c_i}$$

$$p^{c_i}$$

$$n, s, g(p) = p^s$$

$$m = n/i$$

$$\sum_{p \le m} g(p)$$

$$g(p) = p^s$$

$$G_k(n) := \sum_{i=1}^{n} [p_k < \text{lpf}(i) \vee \text{isprime}(i)] g(i)$$

$$k$$

$$g$$

$$x$$

$$\text{lpf}(x) \le \sqrt{x}$$

$$F_{\text{prime}} = G_{\lfloor \sqrt{n} \rfloor}$$

$$G_{\lfloor \sqrt{n} \rfloor}$$

$$G$$

$$G_0(n) = \sum_{i=2}^{n} g(i)$$

$$p_0 = 1$$

$$n < p_k^2$$

$$G$$

$$G_k(n) = G_{k-1}(n)$$

$$p_k^2 \le n$$

$$p_k$$

$$-g(p_k) G_{k-1}(n/p_k)$$

$$p_k^2 \le n \iff p_k \le n/p_k$$

$$\operatorname{lpf}(i) < p_k$$

$$i$$

$$g(p_k)G_{k-1}(p_{k-1})$$

$$F_k(n)$$

$$O(n^{1-\epsilon})$$

$$O\left(\frac{n^{\frac{3}{4}}}{\log n}\right)$$

$$F_{\text{prime}}(n)$$

$$m = n/i$$

$$p_k^2 \le m$$

$$p_k$$

$$F_k$$

$$F_{\text{prime}}$$

$$n/i$$

$$O(\sqrt{n})$$

$$O(\sqrt{n})$$

$$O(\sqrt{n})$$

$$\text{lis}$$

$$v$$

$$\text{id}(v)$$

$$v$$

$$\text{lis}$$

$$v$$

$$\text{id}(v) \le \sqrt{n}$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$\sqrt{n}$$

$$v$$

$$\text{le}$$

$$\text{id}(v)$$

$$\mathrm{le}_v = \mathrm{id}(v)$$

$$\sqrt{n}$$

$$v$$

$$\mathrm{ge}$$

$$\mathrm{id}(v)$$

$$v$$

$$v' = n/v < \sqrt{n}$$

$$\mathrm{id}(v)$$

$$\mathrm{ge}_{v'} = \mathrm{id}(v)$$

$$O(\sqrt{n})$$

$$\mathrm{id}$$

$$O(1)$$

$$F_k$$

$$F_{\mathrm{prime}}$$

$$\mathrm{id}$$

$$O(\sqrt{n})$$

$$F_k(n)$$

$$F_{\mathrm{prime}}(n)$$

$$p_k^2 \leq n$$

$$s_k := F_{\mathrm{prime}}(p_k)$$

$$F_k$$

$$F_{\mathrm{prime}}(p_{k-1})$$

$$G$$

$$G_{k-1}(p_{k-1}) = \sum_{i=1}^{k-1} g(p_i)$$

$$f$$

$$\sqrt{n}$$

$$f$$

$$f(p)$$

$$f(p^c)$$

$$[1, \sqrt{n}]$$

$$\sqrt{n}$$

$$f$$

$$f(p)$$

$$G$$

$$F_{\text{prime}}$$

$$O(\sqrt{n})$$

$$F_k$$

$$F_1(n)$$

$$\sum_{i=1}^{n} \mu(i)$$

$$f(p) = -1$$

$$g(p) = -1, G_0(n) = \sum_{i=2}^{n} g(i) = -n + 1$$

$$F_{\text{prime}}$$

$$O(\sqrt{n})$$

$$\sum_{i=1}^{n} \varphi(i)$$

$$f(p) = p - 1$$

$$f(p)$$

$$(p)$$

$$g(p) = p, G_0(n) = \sum_{i=2}^{n} g(i) = \frac{(n+2)(n-1)}{2}$$

$$f(p)$$

$$(-1)$$

$$g(p) = -1, G_0(n) = \sum_{i=2}^{n} g(i) = -n + 1$$

$$F_{\text{prime}}$$

$$O(\sqrt{n})$$

$$f(n)$$

$$f(p) = p - 1 + 2[p = 2]$$

$$\varphi$$

$$2$$

$$\phi(x,a) = \#\{n \le x \mid n \bmod p = 0 \Longrightarrow p > p_a\}$$

$$P_k(x,a) = \#\{n \le x \mid n = q_1 q_2 \cdots q_k \Longrightarrow \forall i, q_i > p_a\}$$

$$\phi(x,a) = P_0(x,a) + P_1(x,a) + \cdots + P_k(x,a) + \cdots$$

$$\pi(x) = \phi(x,a) + a - 1 - P_2(x,a)$$

$$P_2(x,a) = \sum_{y < p \le \sqrt{x}} \left(\pi\left(\frac{x}{p}\right) - \pi(p) + 1\right)$$

$$\phi(u,0) = [u]$$

$$\phi(x,b) = \phi(x,b-1) - \phi\left(\frac{x}{p_b}, b-1\right)$$

$$\phi(x,a) = \sum_{\substack{1 \le n \le x \\ P^+(n) \le y}} \mu(n)[x/n]$$

$$\begin{array}{cccc}
 & & \phi(x,a) & \\
 & \swarrow & & \searrow \\
 & \phi(x,a-1) & & -\phi\left(\frac{x}{p_a}, a-1\right) \\
\swarrow & \downarrow & \downarrow & \searrow \\
\phi(x,a-2) & \phi\left(\frac{x}{p_{a-1}}, a-2\right) & -\phi\left(\frac{x}{p_a}, a-2\right) & \phi\left(\frac{x}{p_a p_{a-1}}, a-2\right) \\
 & & \vdots &
\end{array}$$

$$\phi(x,a) = S_0 + S$$

$$S_0 = \sum_{n \le y} \mu(n) \left[\frac{x}{n}\right]$$

$$S = \sum_{n/\delta(n) \le y \le n} \mu(n) \phi\left(\frac{x}{n}, \pi(\delta(n)) - 1\right)$$

$$S = -\sum_{p \le y} \sum_{\substack{\delta(m) > p \\ m \le y < mp}} \mu(m) \phi\left(\frac{x}{mp}, \pi(p) - 1\right)$$

$$S = S_1 + S_2 + S_3$$

$$S_1 = -\sum_{x^{1/3} < p \le y} \sum_{\substack{\delta(m) > p \\ m \le y < mp}} \mu(m) \phi\left(\frac{x}{mp}, \pi(p) - 1\right)$$

$$S_2 = -\sum_{x^{1/4}<p\le x^{1/3}} \sum_{\substack{\delta(m)>p \\ m\le y<mp}} \mu(m)\phi\left(\frac{x}{mp}, \pi(p)-1\right)$$

$$S_3 = -\sum_{p\le x^{1/4}} \sum_{\substack{\delta(m)>p \\ m\le y<mp}} \mu(m)\phi\left(\frac{x}{mp}, \pi(p)-1\right)$$

$$S_1 = \sum_{x^{1/3}<p\le y} \sum_{p<q\le y} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$S_2 = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{p<q\le y} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$\frac{x}{pq} < x^{1/3} < p$$

$$\phi\left(\frac{x}{pq}, \pi(p)-1\right) = 1$$

$$S_1 = \frac{\left(\pi(y)-\pi\left(x^{1/3}\right)\right)\left(\pi(y)-\pi\left(x^{1/3}\right)-1\right)}{2}$$

$$S_2 = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{p<q\le y} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$S_2 = U + V$$

$$U = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{\substack{p<q<y \\ q>x/p^2}} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$V = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{\substack{p<q<y \\ q\le x/p^2}} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$U = \sum_{\sqrt{x/y}<p\le x^{1/3}} \sum_{\substack{p<q\le y \\ q>x/p^2}} \phi\left(\frac{x}{pq}, \pi(p)-1\right)$$

$$U = \sum_{\sqrt{x/y}<p\le x^{1/3}} \#\left\{q \mid \frac{x}{p^2} < q \le y\right\}$$

$$U = \sum_{\sqrt{x/y}<p\le x^{1/3}} \left(\pi(y) - \pi\left(\frac{x}{p^2}\right)\right)$$

$$\phi\left(\frac{x}{pq}, \pi(p)-1\right) = 1 + \pi\left(\frac{x}{pq}\right) - (\pi(p)-1) = 2 - \pi(p) + \pi\left(\frac{x}{pq}\right)$$

$$V = V_1 + V_2$$

$$V_1 = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{p<q\le \min(x/p^2,y)} (2-\pi(p))$$

$$V_2 = \sum_{x^{1/4}<p\le x^{1/3}} \sum_{p<q\le \min(x/p^2,y)} \pi\left(\frac{x}{pq}\right)$$

$$V_2 = \sum_{x^{1/4}<p\le \sqrt{x/y}} \sum_{p<q\le y} \pi\left(\frac{x}{pq}\right) + \sum_{\sqrt{x/y}<p\le x^{1/3}} \sum_{p<q\le x/p^2} \pi\left(\frac{x}{pq}\right)$$

$$V_2 = W_1 + W_2 + W_3 + W_4 + W_5$$

$$W_1 = \sum_{x^{1/4}<p\le x/y^2} \sum_{p<q\le y} \pi\left(\frac{x}{pq}\right)$$

$$W_2 = \sum_{x/y^2<p\le \sqrt{x/y}} \sum_{p<q\le \sqrt{x/p}} \pi\left(\frac{x}{pq}\right)$$

$$W_3 = \sum_{x/y^2<p\le \sqrt{x/y}} \sum_{\sqrt{x/p}<q\le y} \pi\left(\frac{x}{pq}\right)$$

$$W_4 = \sum_{\sqrt{x/y}<p\le x^{1/3}} \sum_{p<q\le \sqrt{x/p}} \pi\left(\frac{x}{pq}\right)$$

$$W_5 = \sum_{\sqrt{x/y}<p\le x^{1/3}} \sum_{\sqrt{x/p}<q\le x/p^2} \pi\left(\frac{x}{pq}\right)$$

$$\pi\left(\frac{x}{y^2}\right)\pi(y) = O\left(\frac{x}{y\log^2 x}\right)$$

$$O\left(\sum_{x/y^2<p\le \sqrt{x/y}} \pi\left(\sqrt{\frac{x}{p}}\right)\right) = O\left(\frac{x^{3/4}}{y^{1/4}\log^2 x}\right)$$

$$O\left(\sum_{x/y^2<p\le \sqrt{x/y}} \pi\left(\sqrt{\frac{x}{p}}\right)\right) = O\left(\frac{x^{3/4}}{y^{1/4}\log^2 x}\right)$$

$$O\left(\sum_{\sqrt{x/y}<p\le x^{1/3}} \pi\left(\sqrt{\frac{x}{p}}\right)\right) = O\left(\frac{x^{2/3}}{\log^2 x}\right)$$

$$O\left(\sum_{\sqrt{x/y}<p\le x^{1/3}} \pi\left(\sqrt{\frac{x}{p}}\right)\right) = O\left(\frac{x^{2/3}}{\log^2 x}\right)$$

$$O\left(\frac{x}{y}\log x\log\log x + yx^{1/4}\right)$$

$$O\left(\frac{x}{y}\log\log x + \frac{x}{y}\log x\log\log x + x^{1/4}y + \frac{x^{2/3}}{\log^2 x}\right)$$

$$O\left(\frac{x}{z}\log x \log\log\log x + \frac{yx^{1/4}}{\log x} + z^{3/2}\right)$$

$$1 \sim n$$

$$[x]$$

$$x$$

$$p_k$$

$$k$$

$$p_1 = 2$$

$$\pi(x)$$

$$1 \sim x$$

$$\mu(x)$$

$$S$$

$$\#S$$

$$S$$

$$\delta(x)$$

$$x$$

$$P^+(n)$$

$$x$$

$$\phi(x, a)$$

$$x$$

$$p_a$$

$$P_k(x, a)$$

$$x$$

$$k$$

$$p_a$$

$$P_0(x, a) = 1$$

$$p_a^k > x$$

$$P_k(x, a) = 0$$

$$y$$

$$x^{1/3} \le y \le x^{1/2}$$

$$a = \pi(y)$$

$$k \ge 3$$

$$P_1(x,a) = \pi(x) - a$$

$$P_k(x,a) = 0$$

$$\pi(x)$$

$$\phi(x,a)$$

$$P_2(x,a)$$

$$(2)$$

$$P_2(x,a)$$

$$y < p \leq q$$

$$pq \leq x$$

$$(p,q)$$

$$p \in \left[y+1, \sqrt{x}\right]$$

$$p$$

$$q \in [p, x/p]$$

$$p \in \left[y+1, \sqrt{x}\right]$$

$$\frac{x}{p} \in \left[1, \frac{x}{y}\right]$$

$$\left[1, \frac{x}{y}\right]$$

$$p \in \left[y+1, \sqrt{x}\right]$$

$$\pi\left(\frac{x}{p}\right) - \pi(p) + 1$$

$$L$$

$$L = y$$

$$O\left(\frac{x}{y} \log \log x\right)$$

$$O(y)$$

$$P_2(x,a)$$

$$b \leq a$$

$$x$$

$$p_{b-1}$$

$$p_b$$

p_b

1

$\phi\left(\frac{x}{p_b}, b-1\right)$

$\phi(x, b)$

5.1

ϕ

$\phi(x, a)$

(7)

$\phi(u, 0)$

$\phi(x, a)$

1

$\phi(x, a)$

$\phi(x, a)$

$y \geq x^{1/3}$

3

y

$\frac{x}{\log^3 x}$

1

$b = 0$

$\mu(n)\phi\left(\frac{x}{n}, b\right)$

(6)

2

2

$\mu(n)\phi\left(\frac{x}{n}, b\right)$

(6)

$b = 0$

$n \leq y$

$n > y$

2

$$\mu(n)\phi\left(\frac{x}{n}, b\right)$$

$$n \le y$$

$$\mu(n)\phi\left(\frac{x}{n}, b\right)$$

$$n > y$$

$$n = mp_b (m \le y)$$

$$5.2$$

$$S_0$$

$$S$$

$$S_0$$

$$O(y \log\log x)$$

$$S$$

$$S_1, S_2$$

$$m$$

$$\delta(m) > p > x^{1/4}$$

$$m > p^2 > \sqrt{x}$$

$$m \le y$$

$$mp > x^{1/2} \ge y$$

$$y \le mp$$

$$S_1$$

$$1$$

$$(p, q)$$

$$x^{1/3} < p < q \le y$$

$$O(1)$$

$$S_1$$

$$S_2$$

$$q > \frac{x}{p^2}$$

$$q \le \frac{x}{p^2}$$

$$q > \frac{x}{p^2}$$

$$p^2 > \frac{x}{q} \leq \frac{x}{y}, p > \sqrt{\frac{x}{y}}$$

$$\frac{x}{p^2} < y$$

$$\pi(t)(t \leq y)$$

$$O(y)$$

$$U$$

$$V$$

$$p \leq \frac{x}{pq} < x^{1/2} < p^2$$

$$V$$

$$\pi(t)(t \leq y)$$

$$O(x^{1/3})$$

$$V_1$$

$$V_2$$

$$q$$

$$\pi\left(\frac{x}{pq}\right)$$

$$\pi\left(\frac{x}{pq}\right)$$

$$V_2$$

$$q \leq \min\left(\frac{x}{p^2}, y\right)$$

$$y < \frac{x}{pq} < x^{1/2}$$

$$\pi\left(\frac{x}{pq}\right)$$

$$[1, \sqrt{x}]$$

$$(p, q)$$

$$\pi\left(\frac{x}{pq}\right)$$

$$p$$

$$q$$

$$\pi\left(\frac{x}{pq}\right)$$

$$O(1)$$

$$q$$

$$\pi(t)$$

$$t \leq y$$

$$\pi\left(\frac{x}{pq}\right)$$

$$y$$

$$\pi(t) < \pi(t+1) = \pi\left(\frac{x}{pq}\right)$$

$$t$$

$$\pi\left(\frac{x}{pq}\right)$$

$$q$$

$$W_3$$

$$W_4$$

$$q$$

$$\pi\left(\frac{x}{pq}\right)$$

$$W_3$$

$$W_4$$

$$(p, q)$$

$$W_4$$

$$W_3$$

$$W_5$$

$$x^{1/4}$$

$$\left[1, \frac{x}{y}\right]$$

$$p_k$$

$$m$$

$$\delta(m) > p_k$$

$$-\mu(m)\phi\left(\frac{x}{mp_k}, k-1\right)$$

$$O(\log x)$$

$$3$$

$$P_2(x, a)$$

$$W_1, W_2, W_3, W_4, W_5$$

$$S_3$$

$$O\left(\frac{x}{y}\log\log x\right)$$

$$O(y)$$

$$W_1, W_2$$

$$y$$

$$O\big(\sqrt{x}\log\log x\big)$$

$$O(y)$$

$$W_1$$

$$W_2$$

$$W_3$$

$$W_4$$

$$W_5$$

$$\phi(u, b)$$

$$O(1)$$

$$O(\log x)$$

$$O\left(\frac{x}{y}\log x\log\log x\right)$$

$$S_3$$

$$O(\log x)$$

$$\pm\phi\left(\frac{x}{mp_b}, b-1\right)$$

$$m \le y, b < \pi(x^{1/4})$$

$$O\big(y\pi\big(x^{1/4}\big)\big)$$

$$S_3$$

$$O(y)$$

$$y = x^{1/3}\log^3 x\log\log x$$

$$O\left(\frac{x^{2/3}}{\log^2 x}\right)$$

$$O\big(x^{1/3}\log^3 x\log\log x\big)$$

$$2$$

$$z$$

$$y$$

$$z$$

$$z > y$$

$$S_3$$

$$z$$

$$x^{1/4}$$

$$S$$

$$p \le \frac{x}{pq} < p^2$$

$$2, 3, 5$$

$$\pi(x)$$

$$\binom{n}{m} \bmod p = \binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor} \cdot \binom{n \bmod p}{m \bmod p} \bmod p$$

$$\begin{aligned}
(a+b)^p &= \sum_{n=0}^{p}\binom{p}{n}a^n b^{p-n} \\
&\equiv \sum_{n=0}^{p}[n = 0 \vee n = p]a^n b^{p-n} \\
&\equiv a^p + b^p \pmod p
\end{aligned}$$

$$\begin{aligned}
(ax^n + bx^m)^p &\equiv a^p x^{pn} + b^p x^{pm} \\
&\equiv ax^{pn} + bx^{pm} \\
&\equiv f(x^p)
\end{aligned}$$

$$\begin{aligned}
(1+x)^n &\equiv (1+x)^{p\lfloor n/p \rfloor}(1+x)^{n \bmod p} \\
&\equiv (1+x^p)^{\lfloor n/p \rfloor}(1+x)^{n \bmod p}
\end{aligned}$$

$$M = p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdots p_r^{\alpha_r} = \prod_{i=1}^{r} p_i^{\alpha_i}$$

$$\begin{cases} a_1 \equiv \binom{n}{m} \pmod{p_1^{\alpha_1}} \\ a_2 \equiv \binom{n}{m} \pmod{p_2^{\alpha_2}} \\ \qquad \cdots \\ a_r \equiv \binom{n}{m} \pmod{p_r^{\alpha_r}} \end{cases}$$

$$\frac{\frac{n!}{p^x}}{\frac{m!}{p^y}\frac{(n-m)!}{p^z}} p^{x-y-z} \bmod p^{\alpha}$$

$$\frac{n!}{q^x} \bmod q^k$$

$$n! = q^{\left\lfloor\frac{n}{q}\right\rfloor} \cdot \left(\left\lfloor\frac{n}{q}\right\rfloor\right)! \cdot \left(\prod_{i,(i,q)=1}^{q^k} i\right)^{\left\lfloor\frac{n}{q^k}\right\rfloor} \cdot \left(\prod_{i,(i,q)=1}^{n \bmod q^k} i\right)$$

$$\frac{n!}{q^{\left\lfloor\frac{n}{q}\right\rfloor}} = \left(\left\lfloor\frac{n}{q}\right\rfloor\right)! \cdot \left(\prod_{i,(i,q)=1}^{q^k} i\right)^{\left\lfloor\frac{n}{q^k}\right\rfloor} \cdot \left(\prod_{i,(i,q)=1}^{n \bmod q^k} i\right)$$

$$\frac{n!}{q^x} = \frac{\left(\left\lfloor\frac{n}{q}\right\rfloor\right)!}{q^{x'}} \cdot \left(\prod_{i,(i,q)=1}^{q^k} i\right)^{\left\lfloor\frac{n}{q^k}\right\rfloor} \cdot \left(\prod_{i,(i,q)=1}^{n \bmod q^k} i\right)$$

$$p$$

$$n \bmod p$$

$$m \bmod p$$

$$p$$

$$\binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor}$$

$$p$$

$$10^5$$

$$m = 0$$

$$1$$

$$O(f(p) + g(n)\log n)$$

$$f(n)$$

$$g(n)$$

$$\binom{p}{n} \bmod p$$

$$\binom{p}{n} = \frac{p!}{n!(p-n)!}$$

$$p$$

$$1$$

$$n = 0$$

$$n = p$$

$$n!(p-n)!$$

$$p$$

$$\binom{p}{n} \bmod p = [n = 0 \vee n = p]$$

$$f^p(x) = (ax^n + bx^m)^p \bmod p$$

$$(1+x)^n \bmod p$$

$$\binom{n}{m}$$

$$x^m$$

$$p$$

$$n \bmod p \leq p - 1$$

$$p$$

$$\binom{n}{m} \bmod p = \binom{\lfloor n/p \rfloor}{\lfloor m/p \rfloor} \cdot \binom{n \bmod p}{m \bmod p} \bmod p$$

$$p$$

$$p$$

$$\binom{n}{m} \bmod M$$

$$M$$

$$\binom{n}{m} \bmod p^\alpha$$

$$p$$

$$\alpha$$

$$M$$

$$i, j$$

$$p_i^{\alpha_i}$$

$$p_j^{\alpha_j}$$

$$r$$

$$a_i$$

$$\binom{n}{m}$$

$$a_i = \binom{n}{m} \bmod p_i^{\alpha_i}$$

$$\binom{n}{m} \bmod p^\alpha$$

$$p$$

$$\binom{n}{m} = \frac{n!}{m!(n-m)!}$$

$$\binom{n}{m} \bmod p^\alpha = \frac{n!}{m!(n-m)!} \bmod p^\alpha$$

$$p^\alpha$$

$$ax \equiv 1 \pmod p$$

$$\gcd(a, p) = 1$$

$$\mathrm{inv}_{m!}$$

$$\mathrm{inv}_{(n-m)!}$$

$$x$$

$$n!$$

$$p$$

$$y, z$$

$$n! \bmod q^k$$

$$n = 22, q = 3, k = 2$$

$$22! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12$$
$$\times 13 \times 14 \times 15 \times 16 \times 17 \times 18 \times 19 \times 20 \times 21 \times 22$$

$$q$$

$$22! = 3^7 \times (1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7) \times (1 \times 2 \times 4 \times 5 \times 7 \times 8 \times 10 \times 11 \times 13 \times 14 \times 16 \times 17 \times 19 \times 20 \times 22)$$

$$3$$

$$\left\lfloor \frac{n}{q} \right\rfloor$$

$$7!$$

$$\left\lfloor \frac{n}{q} \right\rfloor !$$

$$q$$

$$n!$$

$$q$$

$$1 \times 2 \times 4 \times 5 \times 7 \times 8 \equiv 10 \times 11 \times 13 \times 14 \times 16 \times 17 \pmod{3^2}$$

$$\prod_{i,(i,q)=1}^{q^k} i \equiv \prod_{i,(i,q)=1}^{q^k} (i + tq^k) \pmod{q^k}$$

$$t$$

$$\prod_{i,(i,q)=1}^{q^k} i$$

$$\left\lfloor \frac{n}{q^k} \right\rfloor$$

$$\prod_{i,(i,q)=1}^{q^k} i$$

$$\left\lfloor \frac{n}{q^k} \right\rfloor$$

$$\prod_{i,(i,q)=1}^{n \bmod q^k} i$$

$$19 \times 20 \times 22$$

$$q^k$$

$$n!$$

$$\frac{n!}{q^x} \bmod q^k$$

$$\left(\left\lfloor \frac{n}{q} \right\rfloor \right)!$$

$$x$$

$$x'$$

$$q$$

$$q$$

$$\prod_{j \geq 0} (N_{(j)!})_p$$

$$O(p \log p)$$

$$x$$

$$p$$

$$\frac{n!}{n\text{□□□}p\text{□□□□□□□□}} \bmod p$$

$$O(p + \log p)$$

$$O(p + T \log p)$$

$$ax \equiv b \pmod{n}$$

$$x \equiv ba^{-1} \pmod{n}$$

$$a'x \equiv b' \pmod{n'}$$

$$x_i \equiv (x' + i \cdot n') \pmod{n} \quad \text{for } i = 0...g-1$$

$$ax + nk = b$$

$$a\frac{b}{\gcd(a,n)}x_0 + n\frac{b}{\gcd(a,n)}k_0 = b$$

$$x = x_0 + nt$$

$$k = k_0 - at$$

$$x = (x \bmod t + t) \bmod t$$

$$t = \frac{n}{\gcd(a,n)}$$

$$a$$

$$b$$

$$n$$

$$x$$

$$[0, n-1]$$

$$x$$

$$a$$

$$n$$

$$\gcd(a,n) = 1$$

$$a$$

$$a$$

$$a$$

$$n$$

$$\gcd(a,n) \neq 1$$

$$2x \equiv 1 \pmod{4}$$

$$g = \gcd(a,n)$$

$$a$$

$$n$$

$$a$$

$$n$$

$$b$$

$$g$$

$$x$$
$$ax \equiv b \pmod{n}$$
$$g$$
$$g$$
$$g$$
$$b$$
$$a$$
$$b$$
$$n$$
$$g$$
$$a'$$
$$n'$$
$$x'$$
$$x$$
$$x'$$
$$g$$
$$g = \gcd(a, n)$$
$$0$$
$$ax \equiv b \pmod{n}$$
$$ax \equiv b \pmod{n}$$
$$x$$
$$k$$
$$\gcd(a, n) \mid b$$
$$ax + nk = b$$
$$x_0, k_0$$
$$ax_0 + nk_0 = \gcd(a, n)$$
$$\gcd(a, n)$$
$$b$$
$$\gcd(a, n) = 1$$
$$x_0$$
$$k_0$$
$$ax + nk = b$$
$$t$$

$$v_p(x^n - y^n) = v_p(x-y)$$

$$v_p(x^n + y^n) = v_p(x+y)$$

$$\sum_{i=0}^{n-1} x^i y^{n-1-i} \equiv nx^{n-1} \not\equiv 0 \pmod p$$

$$v_p(x^n - y^n) = v_p(x-y) + v_p(n)$$

$$v_p(x^n + y^n) = v_p(x+y) + v_p(n)$$

$$\begin{aligned}\sum_{i=0}^{p-1} x^{p-1-i} y^i &= \sum_{i=0}^{p-1} x^{p-1-i} \sum_{j=0}^{i} \binom{i}{j} x^j (kp)^{i-j} \\ &\equiv px^{p-1} \pmod{p^2}\end{aligned}$$

$$v_p(x^n - y^n) = v_p(x-y) + 1$$

$$v_p(x^n - y^n) = v_p(x-y) + a$$

$$v_p(x^n - y^n) = v_p(x-y)$$

$$v_p(x^n - y^n) = v_p(x-y) + v_p(x+y) + v_p(n) - 1$$

$$\begin{aligned}v_p(x^n - y^n) &= v_p(x^{2^a} - y^{2^a}) \\ &= v_p\left((x-y)(x+y)\prod_{i=1}^{a-1}\left(x^{2^i} + y^{2^i}\right)\right)\end{aligned}$$

$$v_p(x^n - y^n) = v_p(x-y) + v_p(x+y) + v_p(n) - 1$$

$$v_p(n)$$

$$n$$

$$p$$

$$v_p(n)$$

$$p^{v_p(n)} \mid n$$

$$p^{v_p(n)+1} \nmid n$$

$$p$$

$$x, y$$

$$p \nmid x$$

$$p \nmid y$$

$$n$$

$$p$$

$$(n, p) = 1$$

$$n$$

$$p \mid x - y$$

$$p \mid x + y$$

$$n$$

$$p \mid x - y$$

$$p \mid x - y \Longleftrightarrow x \equiv y \pmod p$$

$$x^n - y^n = (x - y) \sum_{i=0}^{n-1} x^i y^{n-1-i}$$

$$p \mid x + y$$

$$p$$

$$p \mid x - y$$

$$p \mid x + y$$

$$n$$

$$p \mid x - y$$

$$y = x + kp$$

$$p \mid n$$

$$n = p$$

$$n = p^a$$

$$p \mid x + y$$

$$p = 2$$

$$p \mid x - y$$

$$n$$

$$n$$

$$x, y, n$$

$$4 \mid x - y$$

$$v_2(x + y) = 1$$

$$v_2(x^n - y^n) = v_2(x - y) + v_2(n)$$

$$n$$

$$p \nmid \binom{p}{2}$$

$$n = 2^a b$$

$$a = v_p(n)$$

$$2 \nmid b$$

$$2 \mid x-y \Longrightarrow 4 \mid x^2-y^2$$

$$(\forall i \geq 1), \quad x^{2^i}+y^{2^i} \equiv 2 \pmod 4$$

$$i^{-1} \equiv \begin{cases} 1, & \text{if } i=1, \\ -\left\lfloor \frac{p}{i} \right\rfloor (p \bmod i)^{-1}, & \text{otherwise.} \end{cases} \pmod p$$

$$ax \equiv 1 \pmod b$$

$$x$$

$$a \bmod b$$

$$a^{-1}$$

$$ax \equiv 1 \pmod b$$

$$ax \equiv a^{b-1} \pmod b$$

$$x \equiv a^{b-2} \pmod b$$

$$b$$

$$\gcd(a,b)=1$$

$$1,2,...,n$$

$$p$$

$$O(n)$$

$$1^{-1} \equiv 1 \pmod p$$

$$\forall p \in \mathbf{Z}$$

$$1 \times 1 \equiv 1 \pmod p$$

$$p$$

$$1$$

$$1$$

$$i^{-1}$$

$$k = \left\lfloor \frac{p}{i} \right\rfloor$$

$$j = p \bmod i$$

$$p = ki+j$$

$$\bmod p$$

$$ki+j \equiv 0 \pmod p$$

$$i^{-1} \times j^{-1}$$

$$kj^{-1} + i^{-1} \equiv 0 \pmod p$$

$$i^{-1} \equiv -kj^{-1} \pmod p$$

$$j = p \bmod i$$

$$p = ki + j$$

$$i^{-1} \equiv -\left\lfloor \frac{p}{i} \right\rfloor (p \bmod i)^{-1} \pmod p$$

$$p \bmod i < i$$

$$p$$

$$j^{-1}, j < i$$

$$p - \left\lfloor \frac{p}{i} \right\rfloor$$

$$i \mid p$$

$$i \mid p$$

$$i$$

$$i^{-1}$$

$$i$$

$$p$$

$$10^9 + 7$$

$$i^{-1} \equiv -kj^{-1} \pmod p$$

$$j^{-1}$$

$$j = 1$$

$$1$$

$$1, 2, ..., n$$

$$O(n)$$

$$O(n^{\frac{1}{3}})$$

$$1$$

$$n$$

$$n$$

$$1 \leq a_i < p$$

$$n$$

$$s_i$$

$$s_n$$

$$sv_n$$

$$sv_n$$

$$n$$

$$a_n$$

$$a_n$$

$$a_1$$

$$a_{n-1}$$

$$sv_{n-1}$$

$$sv_i$$

$$a_i^{-1}$$

$$s_{i-1} \times sv_i$$

$$O(n + \log p)$$

$$n$$

$$\begin{cases} ax + by & = a \\ ax_1 + by_1 & = b \end{cases}$$

$$(a, b) \to (b, a - qb)$$

$$\begin{cases} ax + by & = a \\ ax_1 + by_1 & = b \end{cases}$$
$$\to \begin{cases} ax_1 + by_1 & = b \\ a(x - qx_1) + b(y - qy_1) & = a - qb \end{cases}$$

$$\begin{bmatrix} b \\ a \bmod b \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} \gcd(a, b) \\ 0 \end{bmatrix} = \left(\cdots \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \cdots \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \gcd(a, b) \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\pm 1$$

$$a$$

$$b$$

$$a > b$$

$$b$$

$$a$$

$$b$$

$$a = b \times q + r$$

$$r < b$$

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

$$a = bk + c$$

$$c = a \bmod b$$

$$d \mid a,\ \ d \mid b$$

$$c = a - bk, \frac{c}{d} = \frac{a}{d} - \frac{b}{d}k$$

$$\frac{c}{d}$$

$$d \mid c$$

$$a, b$$

$$b, a \bmod b$$

$$d \mid b,\ \ d \mid (a \bmod b)$$

$$\frac{a \bmod b}{d} = \frac{a}{d} - \frac{b}{d}k,\ \ \frac{a \bmod b}{d} + \frac{b}{d}k = \frac{a}{d}$$

$$\frac{a}{d}$$

$$d \mid a$$

$$b, a \bmod b$$

$$a, b$$

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

$$\gcd(a, b) = \gcd(b, r)$$

$$a$$

$$b$$

$$\gcd(a, b) = 1$$

$$a$$

$$b$$

$$n$$

$$O(n)$$

$$a, b$$

$$O(\log \max(a, b))$$

$$\gcd(a, b)$$

$$a < b$$

$$\gcd(a, b) = \gcd(b, a)$$

$$a \geq b$$

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

$$a$$

$$a$$

$$O(\log a) = O(n)$$

$$O(n)$$

$$a$$

$$b$$

$$\gcd(a, b)$$

$$a \geq b$$

$$a = b$$

$$\gcd(a, b) = a = b$$

$$\forall d \mid a, d \mid b$$

$$d \mid a - b$$

$$a$$

$$b$$

$$a - b$$

$$b$$

$$\gcd(a, b) = \gcd(a - b, b)$$

$$a \gg b$$

$$O(n)$$

$$2 \mid a, 2 \mid b$$

$$\gcd(a, b) = 2 \gcd\left(\frac{a}{2}, \frac{b}{2}\right)$$

$$2 \mid a$$

$$2 \mid b$$

$$2 \mid b$$

$$2 \nmid b$$

$$\gcd(a, b) = \gcd\left(\frac{a}{2}, b\right)$$

$$O(\log n)$$

$$2 \mid a$$

$$2 \mid b$$

$$a, b$$

$$2 \mid a - b$$

$$O(\log n)$$

$$a = p_1^{k_{a_1}} p_2^{k_{a_2}} \cdots p_s^{k_{a_s}}$$

$$b = p_1^{k_{b_1}} p_2^{k_{b_2}} \cdots p_s^{k_{b_s}}$$

$$a$$

$$b$$

$$p_1^{\min(k_{a_1}, k_{b_1})} p_2^{\min(k_{a_2}, k_{b_2})} \cdots p_s^{\min(k_{a_s}, k_{b_s})}$$

$$p_1^{\max(k_{a_1}, k_{b_1})} p_2^{\max(k_{a_2}, k_{b_2})} \cdots p_s^{\max(k_{a_s}, k_{b_s})}$$

$$k_a + k_b = \max(k_a, k_b) + \min(k_a, k_b)$$

$$\gcd(a, b) \times \operatorname{lcm}(a, b) = a \times b$$

$$\gcd$$

$$O(1)$$

$$\gcd$$

$$\gcd$$

$$ax + by = \gcd(a, b)$$

$$ax_1 + by_1 = \gcd(a, b)$$

$$bx_2 + (a \bmod b)y_2 = \gcd(b, a \bmod b)$$

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

$$ax_1 + by_1 = bx_2 + (a \bmod b)y_2$$

$$a \bmod b = a - (\left\lfloor \frac{a}{b} \right\rfloor \times b)$$

$$ax_1 + by_1 = bx_2 + (a - (\left\lfloor \frac{a}{b} \right\rfloor \times b))y_2$$

$$ax_1 + by_1 = ay_2 + bx_2 - \left\lfloor \frac{a}{b} \right\rfloor \times by_2 = ay_2 + b(x_2 - \left\lfloor \frac{a}{b} \right\rfloor y_2)$$

$$a = a, b = b$$

$$x_1 = y_2, y_1 = x_2 - \left\lfloor \frac{a}{b} \right\rfloor y_2$$

$$x_2, y_2$$

$$\gcd$$

$$\gcd$$

$$x, y$$

$$ax + by = \gcd(a, b)$$

$$b \neq 0$$

$$\mid x \mid \leq b, \mid y \mid \leq a$$

$$\gcd(a, b) = b$$

$$a \bmod b = 0$$

$$x_1 = 0, y_1 = 1$$

$$a, b \geq 1 \geq \mid x_1 \mid, \mid y_1 \mid$$

$$\gcd(a, b) \neq b$$

$$\mid x_2 \mid \leq (a \bmod b), \mid y_2 \mid \leq b$$

$$x_1 = y_2, y_1 = x_2 - \left\lfloor \frac{a}{b} \right\rfloor y_2$$

$$\mid x_1 \mid = \mid y_2 \mid \leq b, \mid y_1 \mid \leq \mid x_2 \mid + \mid \left\lfloor \frac{a}{b} \right\rfloor y_2 \mid \leq (a \bmod b) + \left\lfloor \frac{a}{b} \right\rfloor \mid y_2 \mid$$

$$\leq a - \left\lfloor \frac{a}{b} \right\rfloor b + \left\lfloor \frac{a}{b} \right\rfloor \mid y_2 \mid \leq a - \left\lfloor \frac{a}{b} \right\rfloor (b - \mid y_2 \mid)$$

$$a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor b \leq a - \left\lfloor \frac{a}{b} \right\rfloor (b - \mid y_2 \mid) \leq a$$

$$\mid x_1 \mid \leq b, \mid y_1 \mid \leq a$$

$$x = 1$$

$$y = 0$$

$$x_1 = 0$$

$$y_1 = 1$$

$$a \bmod b = a - (\left\lfloor \frac{a}{b} \right\rfloor \times b)$$

$$q = \left\lfloor \frac{a}{b} \right\rfloor$$

$$a$$

$$ax + by = a$$

$$b$$

$$ax_1 + by_1 = b$$

$$a_1$$

$$b_1$$

$$x \cdot a + y \cdot b = a_1$$

$$x_1 \cdot a + y_1 \cdot b = b_1$$

$$\gcd$$

$$a_1$$

$$\gcd$$

$$x \cdot a + y \cdot b = g$$

$$a$$

$$b$$

$$\gcd(a, b) = \gcd(b, a \bmod b)$$

$$\lfloor c \rfloor$$

$$c$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & -\lfloor a/b \rfloor \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$a \cdot x_1 + b \cdot x_2 = \gcd(a, b)$$

$$2 \times 2$$

$$\prod_{i=1}^{n} A_i \equiv \prod_{i=1}^{n} (A_i \times a) \pmod p$$

$$\because (A_i, p) = 1, (A_i \times a, p) = 1$$

$$a^{p-1} \times f \equiv f \pmod p$$

$$a^{p-1} \equiv 1 \pmod p$$

$$(a+1)^p = a^p + \binom{p}{1} a^{p-1} + \binom{p}{2} a^{p-2} + \cdots + \binom{p}{p-1} a + 1$$

$$a^b \equiv \begin{cases} a^{b \bmod \varphi(m)}, & \gcd(a, m) = 1, \\ a^b, & \gcd(a, m) \neq 1, b < \varphi(m), \\ a^{(b \bmod \varphi(m)) + \varphi(m)}, & \gcd(a, m) \neq 1, b \geq \varphi(m). \end{cases} \pmod m$$

$$b > r \Longrightarrow a^b \equiv a^{r + ((b-r) \bmod \varphi(m))} \pmod m$$

$$a^r \equiv a^{r + \varphi(m)} \equiv a^{r \bmod \varphi(m) + \varphi(m)} \pmod m$$

$$
\begin{aligned}
a^b &\equiv a^{r+(b-r) \bmod \varphi(m)} \\
&\equiv a^{r \bmod \varphi(m)+\varphi(m)+(b-r) \bmod \varphi(m)} \pmod m \\
&\equiv a^{\varphi(m)+b \bmod \varphi(m)}
\end{aligned}
$$

$$p$$

$$\gcd(a,p)=1$$

$$a^{p-1} \equiv 1 \pmod p$$

$$a$$

$$a^p \equiv a \pmod p$$

$$p$$

$$p$$

$$a$$

$$A=\{1,2,3...,p-1\}$$

$$A_i \times a \pmod p$$

$$A_i \times a \pmod p < p$$

$$A_i \times a$$

$$A_i$$

$$f=(p-1)!$$

$$f \equiv a \times A_1 \times a \times A_2 \times a \times A_3 ... \times A_{p-1} \pmod p$$

$$1^p \equiv 1 \pmod p$$

$$a^p \equiv a \pmod p$$

$$\binom{p}{k}=\frac{p(p-1)\cdots(p-k+1)}{k!}$$

$$1 \le k \le p-1$$

$$p$$

$$\binom{p}{1} \equiv \binom{p}{2} \equiv \cdots \equiv \binom{p}{p-1} \equiv 0 \pmod p$$

$$(a+1)^p \equiv a^p+1 \pmod p$$

$$a^p \equiv a \pmod p$$

$$(a+1)^p \equiv a+1 \pmod p$$

$$\gcd(a,m)=1$$

$$a^{\varphi(m)} \equiv 1 \pmod m$$

$$m$$

$$r_1, r_2, \cdots, r_{\varphi(m)}$$

$$m$$

$$ar_1, ar_2, \cdots, ar_{\varphi(m)}$$

$$m$$

$$r_1 r_2 \cdots r_{\varphi(m)} \equiv ar_1 \cdot ar_2 \cdots ar_{\varphi(m)} \equiv a^{\varphi(m)} r_1 r_2 \cdots r_{\varphi(m)} \pmod m$$

$$r_1 r_2 \cdots r_{\varphi(m)}$$

$$a^{\varphi(m)} \equiv 1 \pmod m$$

$$m$$

$$\varphi(m) = m - 1$$

$$b < \varphi(m)$$

$$m$$

$$b < \varphi(m)$$

$$b \geq \varphi(m)$$

$$\pmod m$$

$$a^b \bmod m$$

$$[0, m)$$

$$a^i \bmod m = (a^{i-1} \bmod m) \times a \bmod m$$

$$a$$

$$a^i \bmod n$$

$$a$$

$$0$$

$$1$$

$$b$$

$$m$$

$$r$$

$$s$$

$$r$$

$$a^0 \bmod m$$

$$a^{r-1} \bmod m$$

$$r$$

$$s$$

$$r$$

$$a$$

$$m$$

$$s$$

$$r$$

$$0$$

$$\forall i \geq r, a^i \equiv a^{i+s} \pmod m$$

$$a$$

$$m$$

$$a$$

$$a$$

$$\gcd(a, m) = 1$$

$$m$$

$$a$$

$$r$$

$$m'$$

$$m = a^r m'$$

$$\gcd(a, m') = 1$$

$$a^{\varphi(m\prime)} \equiv 1 \pmod{m'}$$

$$\gcd(a^r, m') = 1$$

$$\varphi(m) = \varphi(m') \times (a-1)a^{r-1}$$

$$\varphi(m') \mid \varphi(m)$$

$$a^{\varphi(m\prime)} \equiv 1 \pmod{m'}, \varphi(m') \mid \varphi(m) \implies a^{\varphi(m)} \equiv 1 \pmod{m'}$$

$$a^{\varphi(m)} = km' + 1$$

$$a^r$$

$$a^{r+\varphi(m)} = km + a^r$$

$$m = a^r m'$$

$$m$$

$$a$$

$$r$$

$$a^r \equiv a^{r+\varphi(m)} \pmod m$$

$$m = a^r m'$$

$$\varphi(m) = \varphi(a^r)\varphi(m') \geq \varphi(a^r) = a^{r-1}(a-1) \geq r$$

$$a$$

$$2$$

$$r \geq 1$$

$$\varphi(m) \geq r$$

$$a^b \equiv a^{b \bmod \varphi(m) + \varphi(m)} \pmod m$$

$$a$$

$$a = p^k$$

$$\forall r, a^r \equiv a^{r + \varphi(m)} \pmod m$$

$$s$$

$$a^s \equiv 1 \pmod m$$

$$p^{\operatorname{lcm}(s,k)} \equiv 1 \pmod m$$

$$s' = \frac{s}{\gcd(s,k)}$$

$$p^{s'k} \equiv 1 \pmod m$$

$$s' \mid s$$

$$s \mid \varphi(m)$$

$$r' = \left\lceil \frac{r}{k} \right\rceil \leq r \leq \varphi(m)$$

$$r', s'$$

$$\varphi(m)$$

$$a^b \equiv a^{b \bmod \varphi(m) + \varphi(m)} \pmod m$$

$$a$$

$$a$$

$$a = a_1 a_2$$

$$a_i = p_i^{k_i}$$

$$a_i$$

$$s_i$$

$$s \mid \operatorname{lcm}(s_1, s_2)$$

$$s_1 \mid \varphi(m), s_2 \mid \varphi(m)$$

$$\operatorname{lcm}(s_1, s_2) \mid \varphi(m)$$

$$s \mid \varphi(m)$$

$$r = \max(\left\lceil \frac{r_i}{k_i} \right\rceil) \leq \max(r_i) \leq \varphi(m)$$

$$r, s$$

$$\varphi(m)$$

$$a^b \equiv a^{b \bmod \varphi(m) + \varphi(m)} \pmod m$$

$$\begin{aligned}
\varphi(n) &= \prod_{i=1}^{s} \varphi(p_i^{k_i}) \\
&= \prod_{i=1}^{s} (p_i - 1) \times p_i^{k_i - 1} \\
&= \prod_{i=1}^{s} p_i^{k_i} \times (1 - \frac{1}{p_i}) \\
&= n \ \prod_{i=1}^{s} (1 - \frac{1}{p_i}) \qquad \square
\end{aligned}$$

$$a^b \equiv \begin{cases} a^{b \bmod \varphi(p)}, & \gcd(a,p) = 1 \\ a^b, & \gcd(a,p) \neq 1, b < \varphi(p) \\ a^{b \bmod \varphi(p) + \varphi(p)}, & \gcd(a,p) \neq 1, b \geq \varphi(p) \end{cases} \pmod p$$

$$\varphi(n)$$

$$n$$

$$n$$

$$\varphi(1) = 1$$

$$\varphi(n) = n - 1$$

$$\gcd(a, b) = 1$$

$$\varphi(a \times b) = \varphi(a) \times \varphi(b)$$

$$n$$

$$\varphi(2n) = \varphi(n)$$

$$n = \sum_{d|n} \varphi(d)$$

$$\gcd(k, n) = d$$

$$\gcd(\frac{k}{d}, \frac{n}{d}) = 1, (k < n)$$

$$f(x)$$

$$\gcd(k, n) = x$$

$$n = \sum_{i=1}^{n} f(i)$$

$$f(x) = \varphi(\frac{n}{x})$$

$$n = \sum_{d|n} \varphi(\frac{n}{d})$$

$$d$$

$$\frac{n}{d}$$

$$n = \sum_{d|n} \varphi(d)$$

$$n = p^k$$

$$p$$

$$\varphi(n) = p^k - p^{k-1}$$

$$n = \prod_{i=1}^{s} p_i^{k_i}$$

$$p_i$$

$$\varphi(n) = n \times \prod_{i=1}^{s} \frac{p_i - 1}{p_i}$$

$$p$$

$$\varphi(p^k) = p^{k-1} \times (p-1)$$

$$p^k$$

$$p^{k-1}$$

$$p$$

$$p^k$$

$$\varphi(p^k) = p^k - p^{k-1} = p^{k-1} \times (p-1)$$

$$\varphi(n) = n \times \prod_{i=1}^{s} \frac{p_i - 1}{p_i}$$

$$\varphi(x)$$

$$\gcd(a, m) = 1$$

$$a^{\varphi(m)} \equiv 1 \pmod m$$

$$f(a, b, c, n) = \sum_{i=0}^{n} \left\lfloor \frac{ai + b}{c} \right\rfloor$$

$$
\begin{aligned}
f(a,b,c,n) &= \sum_{i=0}^{n} \left\lfloor \frac{ai+b}{c} \right\rfloor \\
&= \sum_{i=0}^{n} \left\lfloor \frac{\left(\left\lfloor \frac{a}{c} \right\rfloor c + a \bmod c\right)i + \left(\left\lfloor \frac{b}{c} \right\rfloor c + b \bmod c\right)}{c} \right\rfloor \\
&= \frac{n(n+1)}{2}\left\lfloor \frac{a}{c} \right\rfloor + (n+1)\left\lfloor \frac{b}{c} \right\rfloor + \sum_{i=0}^{n} \left\lfloor \frac{(a \bmod c)i + (b \bmod c)}{c} \right\rfloor \\
&= \frac{n(n+1)}{2}\left\lfloor \frac{a}{c} \right\rfloor + (n+1)\left\lfloor \frac{b}{c} \right\rfloor + f(a \bmod c, b \bmod c, c, n)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=0}^{n} \left\lfloor \frac{ai+b}{c} \right\rfloor &= \sum_{i=0}^{n} \sum_{j=0}^{\left\lfloor \frac{ai+b}{c} \right\rfloor - 1} 1 \\
&= \sum_{j=0}^{\left\lfloor \frac{an+b}{c} \right\rfloor - 1} \sum_{i=0}^{n} \left[j < \left\lfloor \frac{ai+b}{c} \right\rfloor \right]
\end{aligned}
$$

$$
j < \left\lfloor \frac{ai+b}{c} \right\rfloor \Longleftrightarrow j+1 \leq \left\lfloor \frac{ai+b}{c} \right\rfloor \Longleftrightarrow j+1 \leq \frac{ai+b}{c}
$$

$$
j+1 \leq \frac{ai+b}{c} \Longleftrightarrow jc+c \leq ai+b \Longleftrightarrow jc+c-b-1 < ai
$$

$$
jc+c-b-1 < ai \Longleftrightarrow \left\lfloor \frac{jc+c-b-1}{a} \right\rfloor < i
$$

$$
\begin{aligned}
f(a,b,c,n) &= \sum_{j=0}^{m-1} \sum_{i=0}^{n} \left[i > \left\lfloor \frac{jc+c-b-1}{a} \right\rfloor \right] \\
&= \sum_{j=0}^{m-1} (n - \left\lfloor \frac{jc+c-b-1}{a} \right\rfloor) \\
&= nm - f(c, c-b-1, a, m-1)
\end{aligned}
$$

$$
g(a,b,c,n) = \sum_{i=0}^{n} i \left\lfloor \frac{ai+b}{c} \right\rfloor
$$

$$
h(a,b,c,n) = \sum_{i=0}^{n} \left\lfloor \frac{ai+b}{c} \right\rfloor^2
$$

$$
g(a,b,c,n) = g(a \bmod c, b \bmod c, c, n) + \left\lfloor \frac{a}{c} \right\rfloor \frac{n(n+1)(2n+1)}{6} + \left\lfloor \frac{b}{c} \right\rfloor \frac{n(n+1)}{2}
$$

$$
\begin{aligned}
g(a,b,c,n) &= \sum_{i=0}^{n} i \left\lfloor \frac{ai+b}{c} \right\rfloor \\
&= \sum_{j=0}^{m-1} \sum_{i=0}^{n} \left[j < \left\lfloor \frac{ai+b}{c} \right\rfloor \right] \cdot i
\end{aligned}
$$

$$
\begin{aligned}
g(a,b,c,n) &= \sum_{j=0}^{m-1}\sum_{i=0}^{n}[i>t]\cdot i \\
&= \sum_{j=0}^{m-1}\frac{1}{2}(t+n+1)(n-t) \\
&= \frac{1}{2}\left[mn(n+1)-\sum_{j=0}^{m-1}t^2-\sum_{j=0}^{m-1}t\right] \\
&= \frac{1}{2}[mn(n+1)-h(c,c-b-1,a,m-1)-f(c,c-b-1,a,m-1)]
\end{aligned}
$$

$$
\begin{aligned}
h(a,b,c,n) = {} & h(a \bmod c, b \bmod c, c, n) \\
& +2\left\lfloor\frac{b}{c}\right\rfloor f(a \bmod c, b \bmod c, c, n)+2\left\lfloor\frac{a}{c}\right\rfloor g(a \bmod c, b \bmod c, c, n) \\
& +\left\lfloor\frac{a}{c}\right\rfloor^2\frac{n(n+1)(2n+1)}{6}+\left\lfloor\frac{b}{c}\right\rfloor^2(n+1)+\left\lfloor\frac{a}{c}\right\rfloor\left\lfloor\frac{b}{c}\right\rfloor n(n+1)
\end{aligned}
$$

$$
n^2 = 2\frac{n(n+1)}{2}-n=\left(2\sum_{i=0}^{n}i\right)-n
$$

$$
\begin{aligned}
h(a,b,c,n) &= \sum_{i=0}^{n}\left\lfloor\frac{ai+b}{c}\right\rfloor^2 \\
&= \sum_{i=0}^{n}\left[\left(2\sum_{j=1}^{\left\lfloor\frac{ai+b}{c}\right\rfloor}j\right)-\left\lfloor\frac{ai+b}{c}\right\rfloor\right] \\
&= \left(2\sum_{i=0}^{n}\sum_{j=1}^{\left\lfloor\frac{ai+b}{c}\right\rfloor}j\right)-f(a,b,c,n)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=0}^{n}\sum_{j=1}^{\left\lfloor\frac{ai+b}{c}\right\rfloor}j &= \sum_{i=0}^{n}\sum_{j=0}^{\left\lfloor\frac{ai+b}{c}\right\rfloor-1}(j+1) \\
&= \sum_{j=0}^{m-1}(j+1)\sum_{i=0}^{n}\left[j<\left\lfloor\frac{ai+b}{c}\right\rfloor\right] \\
&= \sum_{j=0}^{m-1}(j+1)\sum_{i=0}^{n}[i>t] \\
&= \sum_{j=0}^{m-1}(j+1)(n-t) \\
&= \frac{1}{2}nm(m+1)-\sum_{j=0}^{m-1}(j+1)\left\lfloor\frac{jc+c-b-1}{a}\right\rfloor \\
&= \frac{1}{2}nm(m+1)-g(c,c-b-1,a,m-1)-f(c,c-b-1,a,m-1)
\end{aligned}
$$

$$
h(a,b,c,n)=nm(m+1)-2g(c,c-b-1,a,m-1)-2f(c,c-b-1,a,m-1)-f(a,b,c,n)
$$

$$a, b, c, n$$

$$O(\log n)$$

$$a \geq c$$

$$b \geq c$$

$$a, b$$

$$c$$

$$a < c, b < c$$

$$i$$

$$i$$

$$f(a, b, c, n) = \sum_{i=0}^{n} \left\lfloor \frac{ai + b}{c} \right\rfloor$$

$$0 \leq i \leq n$$

$$\left\lfloor \frac{ai + b}{c} \right\rfloor$$

$$j$$

$$i$$

$$j$$

$$j$$

$$n$$

$$i$$

$$i$$

$$j$$

$$j$$

$$i, j$$

$$n$$

$$j$$

$$j$$

$$i$$

$$i, j$$

$$n$$

$$i, j$$

$$i$$

$$m = \left\lfloor \frac{an+b}{c} \right\rfloor$$

$$a, c$$

$$O(\log n)$$

$$g$$

$$a < c, b < c$$

$$m = \left\lfloor \frac{an+b}{c} \right\rfloor$$

$$t = \left\lfloor \frac{jc+c-b-1}{a} \right\rfloor$$

$$a < c, b < c$$

$$m = \left\lfloor \frac{an+b}{c} \right\rfloor, t = \left\lfloor \frac{jc+c-b-1}{a} \right\rfloor$$

$$j$$

$$\sum \times \sum$$

$$3$$

$$O(\log n)$$

$$\begin{aligned} \sum_{i=1}^{n} (f * g)(i) &= \sum_{i=1}^{n} \sum_{d \mid i} g(d) f\left(\frac{i}{d}\right) \\ &= \sum_{i=1}^{n} g(i) S\left(\left\lfloor \frac{n}{i} \right\rfloor\right) \end{aligned}$$

$$\begin{aligned} \sum_{i=1}^{n} \sum_{d \mid i} g(d) f\left(\frac{i}{d}\right) &= \sum_{i=1}^{n} \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} g(i) f(j) \\ &= \sum_{i=1}^{n} g(i) \sum_{j=1}^{\lfloor \frac{n}{i} \rfloor} f(j) \\ &= \sum_{i=1}^{n} g(i) S\left(\left\lfloor \frac{n}{i} \right\rfloor\right) \end{aligned}$$

$$\begin{aligned} g(1) S(n) &= \sum_{i=1}^{n} g(i) S\left(\left\lfloor \frac{n}{i} \right\rfloor\right) - \sum_{i=2}^{n} g(i) S\left(\left\lfloor \frac{n}{i} \right\rfloor\right) \\ &= \sum_{i=1}^{n} (f * g)(i) - \sum_{i=2}^{n} g(i) S\left(\left\lfloor \frac{n}{i} \right\rfloor\right) \end{aligned}$$

$$\sum_{k=1}^{n} (f * g)(k) = \mathrm{i} \frac{n(n+1)}{2}$$

$$S(n)=\sum_{i=2}^{n}S\left(\left\lfloor\frac{n}{i}\right\rfloor\right)$$

$$T(n)=O(\sqrt{n})+O\left(\sum_{i=2}^{\sqrt{n}}T\left(\left\lfloor\frac{n}{i}\right\rfloor\right)\right)$$

$$\begin{aligned}T\left(\left\lfloor\frac{n}{i}\right\rfloor\right)&=O\left(\sqrt{\frac{n}{i}}\right)+O\left(\sum_{j=2}^{\sqrt{\frac{n}{i}}}T\left(\left\lfloor\frac{n}{ij}\right\rfloor\right)\right)\\&=O\left(\sqrt{\frac{n}{i}}\right)\end{aligned}$$

$$\begin{aligned}T(n)=O(\sqrt{n})+O\left(\sum_{i=2}^{\sqrt{n}}\sqrt{\frac{n}{i}}\right)&=O\left(\sum_{i=2}^{\sqrt{n}}\sqrt{\frac{n}{i}}\right)\\&=O\left(\int_0^{\sqrt{n}}\sqrt{\frac{n}{x}}\mathrm{d}x\right)\\&=O\left(n^{\frac{3}{4}}\right)\end{aligned}$$

$$\begin{aligned}T'(n)&=O\left(\sum_{i=2}^{\frac{n}{k}}\sqrt{\frac{n}{i}}\right)\\&=O\left(\int_0^{\frac{n}{k}}\sqrt{\frac{n}{x}}\mathrm{d}x\right)\\&=O\left(\frac{n}{\sqrt{k}}\right)\end{aligned}$$

$$T(n)=O(k)+T'(n)=O(k)+O\left(\frac{n}{\sqrt{k}}\right)$$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}i\cdot j\cdot\gcd(i,j)\pmod p$$

$$\sum_{d=1}^{n}F^2\left(\left\lfloor\frac{n}{d}\right\rfloor\right)\cdot d^2\varphi(d)$$

$$f(n)=n^2\varphi(n)=(\mathrm{id}^2\varphi)(n)$$

$$S(n)=\sum_{i=1}^{n}f(i)=\sum_{i=1}^{n}(\mathrm{id}^2\varphi)(i)$$

$$S(n)=\sum_{i=1}^{n}\left(\left(\mathrm{id}^2\varphi\right)*\mathrm{id}^2\right)(i)-\sum_{i=2}^{n}\mathrm{id}^2(i)S\left(\left\lfloor\frac{n}{i}\right\rfloor\right)$$

$$
\begin{aligned}
((\mathrm{id}^2 \varphi)^* \mathrm{id}^2)(i) &= \sum_{d|i} \left(\mathrm{id}^2 \varphi\right)(d)\, \mathrm{id}^2 \left(\frac{i}{d}\right) \\
&= \sum_{d|i} d^2 \varphi(d) \left(\frac{i}{d}\right)^2 \\
&= \sum_{d|i} i^2 \varphi(d) = i^2 \sum_{d|i} \varphi(d) \\
&= i^2 (\varphi^* 1)(i) = i^3
\end{aligned}
$$

$$
\begin{aligned}
S(n) &= \sum_{i=1}^{n} \left((\mathrm{id}^2 \varphi)^* \mathrm{id}^2\right)(i) - \sum_{i=2}^{n} \mathrm{id}^2(i) S\left(\left\lfloor \frac{n}{i} \right\rfloor\right) \\
&= \sum_{i=1}^{n} i^3 - \sum_{i=2}^{n} i^2 S\left(\left\lfloor \frac{n}{i} \right\rfloor\right) \\
&= \left(\frac{1}{2} n(n+1)\right)^2 - \sum_{i=2}^{n} i^2 S\left(\left\lfloor \frac{n}{i} \right\rfloor\right)
\end{aligned}
$$

$$f$$

$$S(n) = \sum_{i=1}^{n} f(i)$$

$$S(n)$$

$$S\left(\left\lfloor \frac{n}{i} \right\rfloor\right)$$

$$g$$

$$f^* g$$

$$f$$

$$g$$

$$g(d) f\left(\frac{i}{d}\right)$$

$$i \le n$$

$$d$$

$$\frac{i}{d}$$

$$i, j$$

$$g$$

$$\sum_{i=1}^{n} (f^* g)(i)$$

$$g$$

$$\sum_{i=2}^{n} g(i) S\left(\left\lfloor \frac{n}{i} \right\rfloor\right)$$

$$g(1)S(n)$$

$$f$$

$$g$$

$$f$$

$$f(n) = \mathrm{i}\varphi(n)$$

$$f$$

$$g(n) = 1$$

$$\sum_{k} (f * g)(k)$$

$$g$$

$$O(1)$$

$$\sum_{i=1}^{n} (f * g)(i)$$

$$g(n)$$

$$O(1)$$

$$S(n)$$

$$T(n)$$

$$S(n)$$

$$\left\lfloor \frac{n}{i} \right\rfloor$$

$$O(\sqrt{n})$$

$$O\left(\sum_{j=2}^{\sqrt{\frac{n}{i}}} T\left(\left\lfloor \frac{n}{ij} \right\rfloor\right)\right)$$

$$S(1)$$

$$S(k)$$

$$\sum_{i} S\left(\left\lfloor \frac{n}{i} \right\rfloor\right)$$

$$k < \left\lfloor \frac{n}{i} \right\rfloor \leq n$$

$$T'(n)$$

$$k = \Theta\left(n^{\frac{2}{3}}\right)$$

$$T(n)$$

$$O\left(n^{\frac{2}{3}}\right)$$

$$S_1(n) = \sum_{i=1}^{n} \mu(i)$$

$$S_2(n) = \sum_{i=1}^{n} \varphi(i)$$

$$1 \le n < 2^{31}$$

$$\varphi(x)$$

$$n \le 10^{10}, 5 \times 10^8 \le p \le 1.1 \times 10^9$$

$$p$$

$$\varphi * 1 = \mathrm{id}$$

$$F(n) = \frac{1}{2} n(n+1)$$

$$\sum_{d=1}^{n} F\left(\left\lfloor \frac{n}{d} \right\rfloor\right)^2$$

$$d^2 \varphi(d)$$

$$g$$

$$f \times g$$

$$g$$

$$\varphi$$

$$\varphi * 1$$

$$f$$

$$\mathrm{id}^2$$

$$\mathrm{id}^2$$

$$S(n)$$

$$g^k \equiv a \pmod{m}$$

$$\begin{aligned} \operatorname{ind}_g a = \operatorname{ind}_g b &\iff \operatorname{ind}_g a \equiv \operatorname{ind}_g b \pmod{\varphi(m)} \\ &\iff g^{\operatorname{ind}_g a} \equiv g^{\operatorname{ind}_g b} \pmod{m} \\ &\iff a \equiv b \pmod{m} \end{aligned}$$

$$a^x \equiv b \pmod{m}$$

$$x^a \equiv b \pmod{p}$$

$$(g^a)^c \equiv b \pmod{p}$$

$$g^{ac} \equiv g^t \pmod{p}$$

$$ac \equiv t \pmod{\varphi(p)}$$

$$\forall\ t \in \mathbf{Z},\, x^a \equiv g^{c \cdot a + t \cdot \varphi(p)} \equiv b \pmod{p}$$

$$\forall\ t \in \mathbf{Z}, a \mid t \cdot \varphi(p),\, x \equiv g^{c + \frac{t \cdot \varphi(p)}{a}} \pmod{p}$$

$$\forall\ i \in \mathbf{Z}, x \equiv g^{c + \frac{\varphi(p)}{(a, \varphi(p))} \cdot i} \pmod{p}$$

$$a^x \equiv b \pmod{m}$$

$$\frac{a}{d_1} \cdot a^{x-1} \equiv \frac{b}{d_1} \pmod{\frac{m}{d_1}}$$

$$\frac{a^2}{d_1 d_2} \cdot a^{x-2} \equiv \frac{b}{d_1 d_2} \pmod{\frac{m}{d_1 d_2}}$$

$$\frac{a^k}{D} \cdot a^{x-k} \equiv \frac{b}{D} \pmod{\frac{m}{D}}$$

$$m$$

$$g$$

$$(a, m) = 1$$

$$a$$

$$0 \le k < \varphi(m)$$

$$k$$

$$g$$

$$m$$

$$k = \operatorname{ind}_g a$$

$$\operatorname{ind} a$$

$$\operatorname{ind}_g 1 = 0$$

$$\operatorname{ind}_g g = 1$$

$$g$$

$$m$$

$$(a, m) = (b, m) = 1$$

$$\operatorname{ind}_g (ab) \equiv \operatorname{ind}_g a + \operatorname{ind}_g b \pmod{\varphi(m)}$$

$$(\forall n \in \mathbf{N}), \quad \operatorname{ind}_g a^n \equiv n \operatorname{ind}_g a \pmod{\varphi(m)}$$

$$g_1$$

$$m$$

$$\operatorname{ind}_g a \equiv \operatorname{ind}_{g_1} a \cdot \operatorname{ind}_g g_1 \pmod{\varphi(m)}$$

$$a \equiv b \pmod m \iff \operatorname{ind}_g a = \operatorname{ind}_g b$$

$$g^{\operatorname{ind}_g(ab)} \equiv ab \equiv g^{\operatorname{ind}_g a} g^{\operatorname{ind}_g b} \equiv g^{\operatorname{ind}_g a + \operatorname{ind}_g b} \pmod m$$

$$x = \operatorname{ind}_{g_1} a$$

$$a \equiv g_1^x \pmod m$$

$$y = \operatorname{ind}_g g_1$$

$$g_1 \equiv g^y \pmod m$$

$$a \equiv g^{xy} \pmod m$$

$$\operatorname{ind}_g a \equiv xy \equiv \operatorname{ind}_{g_1} a \cdot \operatorname{ind}_g g_1 \pmod{\varphi(m)}$$

$$a, b, m \in \mathbf{Z}^+$$

$$O(\sqrt{m})$$

$$a \perp m$$

$$x$$

$$0 \le x < m$$

$$m$$

$$x = A\lceil\sqrt{m}\rceil - B$$

$$0 \le A, B \le \lceil\sqrt{m}\rceil$$

$$a^{A\lceil\sqrt{m}\rceil - B} \equiv b \pmod m$$

$$a^{A\lceil\sqrt{m}\rceil} \equiv ba^B \pmod m$$

$$a, b$$

$$ba^B$$

$$B$$

$$a^{A\lceil\sqrt{m}\rceil}$$

$$A$$

$$ba^B$$

$$x$$

$$x = A\lceil\sqrt{m}\rceil - B$$

$$A, B$$

$$\lceil \sqrt{m} \rceil$$

$$\Theta(\sqrt{m})$$

$$\log$$

$$a$$

$$m$$

$$A, B$$

$$a^{A\lceil \sqrt{m} \rceil} \equiv ba^B \pmod m$$

$$a^{A\lceil \sqrt{m} \rceil - B} \equiv b \pmod m$$

$$a^B$$

$$a^B \perp m$$

$$a \perp m$$

$$a, b \in \mathbf{Z}^+$$

$$p \in \mathbf{P}$$

$$p$$

$$p$$

$$g$$

$$p$$

$$x \ (1 \le x < p)$$

$$i \ (0 \le i < p - 1)$$

$$x = g^i$$

$$x = g^c$$

$$g$$

$$p$$

$$g$$

$$c$$

$$(g^c)^a \equiv b \pmod p$$

$$O(\sqrt{p})$$

$$c$$

$$x_0 \equiv g^c \pmod p$$

$$x = g^c$$

$$b = g^t$$

$$g^t \equiv b \pmod p$$

$$t$$

$$c$$

$$x$$

$$x_0 \equiv g^c \pmod p$$

$$x_0 \equiv g^c \pmod p$$

$$g^{\varphi(p)} \equiv 1 \pmod p$$

$$\frac{a}{(a, \varphi(p))} \mid t$$

$$t = \frac{a}{(a, \varphi(p))} \cdot i$$

$$a, b, m \in \mathbf{Z}^+$$

$$a, m$$

$$(a, m) = 1$$

$$m$$

$$a$$

$$d_1 = (a, m)$$

$$d_1 \nmid b$$

$$d_1$$

$$a$$

$$\frac{m}{d_1}$$

$$d_2 = \left(a, \frac{m}{d_1}\right)$$

$$d_2 \nmid \frac{b}{d_1}$$

$$d_2$$

$$a \perp \frac{m}{d_1 d_2 \cdots d_k}$$

$$D = \prod_{i=1}^{k} d_i$$

$$a \perp \frac{m}{D}$$

$$\frac{a^k}{D} \perp \frac{m}{D}$$

$$\frac{a^k}{D}$$

$$x - k$$

$$k$$

$$k$$

$$\Theta(k)$$

$$a^i \equiv b \pmod{m}$$

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ x \equiv a_2 \pmod{n_2} \\ \quad \vdots \\ x \equiv a_k \pmod{n_k} \end{cases}$$

$$\begin{aligned} x &\equiv \sum_{j=1}^{k} a_j c_j &\pmod{n_i} \\ &\equiv a_i c_i &\pmod{n_i} \\ &\equiv a_i \cdot m_i \cdot (m_i^{-1} \bmod n_i) &\pmod{n_i} \\ &\equiv a_i &\pmod{n_i} \end{aligned}$$

$$\begin{cases} a \equiv a_1 \pmod{p_1} \\ a \equiv a_2 \pmod{p_2} \\ \quad \vdots \\ a \equiv a_k \pmod{p_k} \end{cases}$$

$$a = x_1 + x_2 p_1 + x_3 p_1 p_2 + ... + x_k p_1 ... p_{k-1}$$

$$p_i \cdot r_{i,j} \equiv 1 \pmod{p_j}$$

$$a_1 \equiv x_1 \pmod{p_1}$$

$$a_2 \equiv x_1 + x_2 p_1 \pmod{p_2}$$

$$\begin{aligned} a_2 - x_1 &\equiv x_2 p_1 &\pmod{p_2} \\ (a_2 - x_1) r_{1,2} &\equiv x_2 &\pmod{p_2} \\ x_2 &\equiv (a_2 - x_1) r_{1,2} &\pmod{p_2} \end{aligned}$$

$$x_k = (...((a_k - x_1) r_{1,k} - x_2) r_{2,k}) - ...) r_{k-1,k} \bmod p_k$$

Chinese Remainder Algorithm $\mathrm{cra}(\mathbf{v}, \mathbf{m})$:
Input : $\mathbf{m} = (m_0, m_1, ..., m_{n-1}), m_i \in \mathbb{Z}^+ \wedge \gcd(m_i, m_j) = 1$ for all $i \neq j$,
$\mathbf{v} = (v_0, ..., v_{n-1})$ where $v_i = x \bmod m_i$.
Output : $x \bmod \prod_{i=0}^{n-1} m_i$.

```
for i from 1 to (n − 1) do
    C_i ← (∏_{j=0}^{i−1} m_j)^{−1} mod m_i
x ← v_0
for i from 1 to (n − 1) do
    u ← (v_i − x) · C_i mod m_i
    x ← x + u ∏_{j=0}^{i−1} m_j
return(x)
```

$$G^{\sum_{k|n}\binom{n}{k}} \bmod 999\ 911\ 659$$

$$G^{\sum_{k|n}\binom{n}{k} \bmod 999\ 911\ 658} \bmod 999\ 911\ 659$$

$$\sum_{k|n}\binom{n}{k} \bmod 999\ 911\ 658$$

$$\begin{cases} x \equiv a_1 \pmod 2 \\ x \equiv a_2 \pmod 3 \\ x \equiv a_3 \pmod{4679} \\ x \equiv a_4 \pmod{35617} \end{cases}$$

$$3$$

$$2$$

$$5$$

$$3$$

$$7$$

$$2$$

$$2 \times 70 + 3 \times 21 + 2 \times 15 = 233 = 2 \times 105 + 23$$

$$23$$

$$n_1, n_2, \cdots, n_k$$

$$n$$

$$i$$

$$m_i = \frac{n}{n_i}$$

$$m_i$$

$$n_i$$

$$m_i^{-1}$$

$$c_i = m_i m_i^{-1}$$

$$n_i$$

$$n$$

$$x = \sum_{i=1}^{k} a_i c_i \pmod{n}$$

$$x$$

$$i = 1, 2, \cdots, k$$

$$x \equiv a_i \pmod{n_i}$$

$$i \neq j$$

$$m_j \equiv 0 \pmod{n_i}$$

$$c_j \equiv m_j \equiv 0 \pmod{n_i}$$

$$c_i \equiv m_i \cdot (m_i^{-1} \bmod n_i) \equiv 1 \pmod{n_i}$$

$$i = 1, 2, \cdots, k$$

$$x$$

$$x \equiv a_i \pmod{n_i}$$

$$a_i$$

$$\{a_i\}$$

$$x$$

$$x \neq y$$

$$i$$

$$x$$

$$y$$

$$n_i$$

$$\{a_i\}$$

$$x$$

$$n = 3 \times 5 \times 7 = 105$$

$$n_1 = 3, m_1 = n/n_1 = 35, m_1^{-1} \equiv 2 \pmod{3}$$

$$c_1 = 35 \times 2 = 70$$

$$n_2 = 5, m_2 = n/n_2 = 21, m_2^{-1} \equiv 1 \pmod{5}$$

$$c_2 = 21 \times 1 = 21$$

$$n_3 = 7, m_3 = n/n_3 = 15, m_3^{-1} \equiv 1 \pmod{7}$$

$$c_3 = 15 \times 1 = 15$$

$$x \equiv 2 \times 70 + 3 \times 21 + 2 \times 15 \equiv 233 \equiv 23 \pmod{105}$$

$$a$$

$$a < \prod_{i=1}^{k} p_i$$

$$p_i$$

$$a$$

$$a$$

$$x_1, ..., x_k$$

$$r_{ij}$$

$$p_i$$

$$p_j$$

$$a$$

$$x_1$$

$$p_1$$

$$O(k^2)$$

$$G, n$$

$$1 \leq G, n \leq 10^9$$

$$G = 999\ 911\ 659$$

$$0$$

$$999\ 911\ 658$$

$$\forall x \in [1, 999\ 911\ 657]$$

$$x$$

$$999\ 911\ 658 = 2 \times 3 \times 4679 \times 35617$$

$$\sum_{k|n} \binom{n}{k}$$

$$2$$

$$3$$

$$4679$$

$$35617$$

$$x \equiv a_1 \pmod{m_1}$$

$$x \equiv a_2 \pmod{m_2}$$

$$x = m_1 p + a_1 = m_2 q + a_2$$

$$p, q$$

$$m_1 p - m_2 q = a_2 - a_1$$

$$a_2 - a_1$$

$$\gcd(m_1, m_2)$$

$$(p, q)$$

$$x \equiv b \pmod{M}$$

$$b = m_1 p + a_1$$

$$M = \operatorname{lcm}(m_1, m_2)$$

$$[a_0, a_1, a_2, a_3] = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3}}}$$

$$r = [a_0; a_1, ..., a_k, 1] = [a_0; a_1, ..., a_k + 1]$$

$$r = a_0 + \cfrac{1}{a_1 + \frac{1}{... + \frac{1}{a_k}}}$$

$$r = [1; 1, 1, 1] = 1 + \cfrac{1}{1 + \frac{1}{1 + \frac{1}{1}}},$$

$$r = [1; 1, 2] = 1 + \cfrac{1}{1 + \frac{1}{2}}.$$

$$r = a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + ...}} = \lim_{k \to \infty} r_k$$

$$r_k = \frac{\phi^{k+2} - \psi^{k+2}}{\phi^{k+1} - \psi^{k+1}}$$

$$r = 1 + \cfrac{1}{1 + \frac{1}{1 + ...}} = \lim_{k \to \infty} r_k = \phi = \frac{1 + \sqrt{5}}{2}.$$

$$r = 1 + \frac{1}{r} \implies r^2 = r + 1$$

$$r = [a_0; a_1, ..., a_{k-1}, s_k]$$

$$s_k = [a_k; s_{k+1}] = a_k + \frac{1}{s_{k+1}}$$

$$s_k = \lfloor s_k \rfloor + \frac{1}{s_{k+1}}$$

$$s_{k+1} = (s_k - \lfloor s_k \rfloor)^{-1}$$

$$s_{k+1} = \left(\frac{p}{q} - \left\lfloor \frac{p}{q} \right\rfloor\right)^{-1} = \frac{q}{p - q \cdot \left\lfloor \frac{p}{q} \right\rfloor} = \frac{q}{p \bmod q}.$$

$$\frac{p_k}{q_k} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}$$

$$p_k = a_k p_{k-1} + p_{k-2}$$

$$q_k = a_k q_{k-1} + q_{k-2}$$

$$p_{-1} = 1 \quad p_0 = a_0$$

$$q_{-1} = 0 \quad q_0 = 1$$

$$\frac{q_k}{q_{k-1}} = [a_k, a_{k-1}, ..., a_1]$$

$$\frac{p_k}{p_{k-1}} = [a_k, a_{k-1}, ..., a_0]$$

$$\frac{p_k}{p_{k-1}} = [a_k, a_{k-1}, ..., a_2]$$

$$\frac{p_k}{p_{k-1}} = a_k + \frac{1}{\frac{p_{k-1}}{p_{k-2}}}$$

$$\frac{q_k}{q_{k-1}} = a_k + \frac{1}{\frac{q_{k-1}}{q_{k-2}}}$$

$$p_{k+1}q_k - q_{k+1}p_k = (a_{k+1}p_k + p_{k-1})q_k - (a_{k+1}q_k + q_{k-1})p_k = -(p_k q_{k-1} - q_k p_{k-1})$$

$$p_{k+1}q_k - q_{k+1}p_k = (-1)^k$$

$$\frac{p_{k+1}}{q_{k+1}} - \frac{p_k}{q_k} = \frac{(-1)^k}{q_{k+1}q_k}$$

$$p_{k+1}q_k - q_{k+1}p_k = (-1)^k$$

$$ax - by = 1$$

$$x = [a_0, a_1, a_2, ...]$$

$$\frac{1}{x} = \frac{1}{[a_0, a_1, a_2, ...]} = [0, a_0, a_1, a_2, ...]$$

$$\begin{gathered} r_0 = [a_0] \\ r_1 = [a_0, a_1] \\ \cdots \\ r_k = [a_0, a_1, ..., a_k] \end{gathered}$$

$$r_k = \frac{P_k(a_0, a_1, ..., a_k)}{P_{k-1}(a_1, ..., a_k)} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-1} + q_{k-2}}$$

$$P_k(x_0, x_1, ..., x_k) = \det \begin{bmatrix} x_k & 1 & 0 & ... & 0 \\ -1 & x_{k-1} & 1 & ... & 0 \\ 0 & -1 & x_2 & . & \vdots \\ \vdots & \vdots & . & \ddots & 1 \\ 0 & 0 & ... & -1 & x_0 \end{bmatrix}.$$

$$r_k = \frac{P_k(a_0, a_1, ..., a_k)}{Q_k(a_0, a_1, ..., a_k)}$$

$$r_k = a_0 + \frac{1}{[a_1; a_2, ..., a_k]} = a_0 + \frac{Q_{k-1}(a_1, ..., a_k)}{P_{k-1}(a_1, ..., a_k)} = \frac{a_0 P_{k-1}(a_1, ..., a_k) + Q_{k-1}(a_1, ..., a_k)}{P_{k-1}(a_1, ..., a_k)}$$

$$P_k(a_0, ..., a_k) = a_0 P_{k-1}(a_1, ..., a_k) + P_{k-2}(a_2, ..., a_k)$$

$$P_0(a_0) = a_0,$$

$$P_1(a_0, a_1) = a_0 a_1 + 1.$$

$$T_k = \det \begin{bmatrix} a_0 & b_0 & 0 & ... & 0 \\ c_0 & a_1 & b_1 & ... & 0 \\ 0 & c_1 & a_2 & . & \vdots \\ \vdots & \vdots & . & \ddots & c_{k-1} \\ 0 & 0 & ... & b_{k-1} & a_k \end{bmatrix}$$

$$P_k = \det \begin{bmatrix} x_k & 1 & 0 & ... & 0 \\ -1 & x_{k-1} & 1 & ... & 0 \\ 0 & -1 & x_2 & . & \vdots \\ \vdots & \vdots & . & \ddots & 1 \\ 0 & 0 & ... & -1 & x_0 \end{bmatrix}.$$

$$P_k(a_0, ..., a_k) = a_k P_{k-1}(a_0, ..., a_{k-1}) + P_{k-2}(a_0, ..., a_{k-2})$$

$$\left| \frac{p_k}{q_k} - r \right| \leq \frac{1}{q_k q_{k+1}} \leq \frac{1}{q_k^2}$$

$$|\ p_k - q_k r\ | \leq \frac{1}{q_{k+1}}$$

$$x = [a_0, a_1, ..., a_k, r_{k+1}]$$

$$x = \frac{r_{k+1} p_k + p_{k-1}}{r_{k+1} q_k + q_{k-1}}$$

$$x - \frac{p_k}{q_k} = \frac{(-1)^k}{q_k(r_{k+1} q_k + q_{k-1})}$$

$$r_k = a_0 + \sum_{i=1}^{k} \frac{(-1)^{i-1}}{q_i q_{i-1}}$$

$$r_k - r_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$$

$$\left| r - \frac{p_k}{q_k} \right| \leq \frac{1}{q_{k+1} q_k} \leq \frac{1}{q_k^2}$$

$$| \ r - r_k| = | \ r_k - r_{k+1}| - | \ r - r_{k+1}| \leq | \ r_k - r_{k+1}|$$

$$\frac{p_k}{q_k} - \frac{p_{k-1}}{q_{k-1}} = \frac{p_k q_{k-1} - p_{k-1} q_k}{q_k q_{k-1}}$$

$$\begin{aligned} p_k q_{k-1} - p_{k-1} q_k &= (a_k p_{k-1} + p_{k-2}) q_{k-1} - p_{k-1}(a_k q_{k-1} + q_{k-2}) \\ &= p_{k-2} q_{k-1} - p_{k-1} q_{k-2}, \end{aligned}$$

$$r_1 - r_0 = \left(a_0 + \frac{1}{a_1}\right) - a_0 = \frac{1}{a_1}$$

$$r_k - r_{k-1} = \frac{(-1)^{k-1}}{q_k q_{k-1}}$$

$$r_k = (r_k - r_{k-1}) + ... + (r_1 - r_0) + r_0 = a_0 + \sum_{i=1}^{k} \frac{(-1)^{i-1}}{q_i q_{i-1}}$$

$$r = \lim_{k \to \infty} r_k = a_0 + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{q_i q_{i-1}}$$

$$r - r_k = \sum_{i=k+1}^{\infty} \frac{(-1)^{i-1}}{q_i q_{i-1}}$$

$$| \ r - r_k| = | \ r_k - r_{k+1}| - | \ r - r_{k+1}| \leq | \ r_k - r_{k+1}|$$

$$\left| r - \frac{p_k}{q_k} \right| \leq \frac{1}{q_k q_{k+1}} \leq \frac{1}{q_k^2}$$

$$A q_{k-1} - B p_{k-1} = (-1)^{k-1} g$$

$$\frac{p}{q} = \frac{t p_{k-1} + p_{k-2}}{t q_{k-1} + q_{k-2}}$$

$$\vec{r}_k = a_k \vec{r}_{k-1} + \vec{r}_{k-2}$$

$$s_k = -\frac{\vec{r}_{k-2} \times \vec{r}}{\vec{r}_{k-1} \times \vec{r}} = \left| \frac{\vec{r}_{k-2} \times \vec{r}}{\vec{r}_{k-1} \times \vec{r}} \right|$$

$$\vec{r}_k = \vec{r}_{k-2} + \left\lfloor \left| \frac{\vec{r} \times \vec{r}_{k-2}}{\vec{r} \times \vec{r}_{k-1}} \right| \right\rfloor \vec{r}_{k-1}$$

$$a_k = \left\lfloor \left| \frac{q_{k-1} r - p_{k-1}}{q_{k-2} r - p_{k-2}} \right| \right\rfloor$$

$$r = [a_0; a_1, ..., a_{k-1}, s_k] = \frac{s_k p_{k-1} + p_{k-2}}{s_k q_{k-1} + q_{k-2}}$$

$$\vec{r} \parallel s_k \vec{r}_{k-1} + \vec{r}_{k-2}$$

$$0 = s_k (\vec{r}_{k-1} \times \vec{r}) + (\vec{r}_{k-2} \times \vec{r})$$

$$s_k = -\frac{\vec{r}_{k-2} \times \vec{r}}{\vec{r}_{k-1} \times \vec{r}}$$

$$\mid \vec{r}_{k-2} \times \vec{r}_k \mid = \mid \vec{r}_{k-2} \times (\vec{r}_{k-2} + a_k \vec{r}_{k-1}) \mid = a_k \mid \vec{r}_{k-2} \times \vec{r}_{k-1} \mid = a_k$$

$$\sum_{i=1}^{n} \left(p \cdot i - \left\lfloor \frac{p \cdot i}{q} \right\rfloor q \right) = \frac{pn(n+1)}{2} - q \sum_{i=1}^{n} \left\lfloor \frac{p \cdot i}{q} \right\rfloor$$

$$4$$

$$0$$

$$r = \frac{p}{q}$$

$$k$$

$$k = O(\log \min(p, q))$$

$$a_0, a_1, ..., a_k \in \mathbb{Z}$$

$$a_1, a_2, ..., a_k \geq 1$$

$$r$$

$$r = [a_0; a_1, a_2, ..., a_k]$$

$$r = \frac{5}{3}$$

$$1$$

$$1$$

$$[a_0, a_1, a_2, a_3] = [a_0, a_1, a_2, a_3 - 1, 1]$$

$$1$$

$$1$$

$$0$$

$$0$$

$$0$$

$$1$$

$$0$$

$$1$$

$$1$$

$$1$$

$$0$$

$$a_0, a_1, a_2, ...$$

$$a_1, a_2, ... \geq 1$$

$$r_k = [a_0; a_1, ..., a_k]$$

$$r$$

$$r = [a_0; a_1, a_2, ...]$$

$$r = [a_0; a_1, ...]$$

$$k$$

$$r + k = [a_0 + k; a_1, ...]$$

$$a_0 > 0$$

$$\frac{1}{r} = [0; a_0, a_1, ...]$$

$$a_1 \square 0$$

$$\frac{1}{r} = [a_1; a_2, ...]$$

$$r_0, r_1, r_2, ...$$

$$r$$

$$r_k = [a_0; a_1, ..., a_k] = \frac{p_k}{q_k}$$

$$r$$

$$k$$

$$r = [1; 1, 1, 1, ...]$$

$$r_k = \frac{F_{k+2}}{F_{k+1}}$$

$$F_k$$

$$F_0 = 0$$

$$F_1 = 1$$

$$F_k = F_{k-1} + F_{k-2}$$

$$\phi = \frac{1 + \sqrt{5}}{2} \approx 1.618$$

$$\psi = \frac{1 - \sqrt{5}}{2} = -\frac{1}{\phi} \approx -0.618$$

$$r$$

$$r_k = [a_0; a_1, ..., a_{k-1}, a_k]$$

$$1 \leq t \leq a_k$$

$$[a_0; a_1, ..., a_{k-1}, t]$$

$$r$$
$$r$$
$$s_k = [a_k; a_{k+1}, a_{k+2}, ...]$$
$$s_k$$
$$r$$
$$k$$
$$a_k$$
$$s_k \geq 1$$
$$k \geq 1$$
$$[a_0; a_1, ..., a_k]$$
$$a_i$$
$$r = [s_0] = s_0$$
$$s_k$$
$$s_k$$
$$a_k = \lfloor s_k \rfloor$$
$$s_{k+1} = (s_k - a_k)^{-1}$$
$$a_0, a_1, ...$$
$$s_k = a_k$$
$$r$$
$$r$$
$$(0, 1)$$
$$0$$
$$0$$
$$1$$
$$0$$
$$1$$
$$s_k$$
$$s_{k+1}$$
$$s_{k+1}$$
$$s_k = \frac{p}{q}$$
$$r = \frac{p}{q}$$
$$p$$

$$q$$

$$\frac{p_k}{q_k} = [a_0; a_1, ..., a_k]$$

$$\gcd(p_k, q_k) = 1$$

$$0$$

$$r_k = \frac{p_k}{q_k}$$

$$\frac{p_{-1}}{q_{-1}} = \frac{1}{0}$$

$$\frac{p_{-2}}{q_{-2}} = \frac{0}{1}$$

$$-1$$

$$p_k$$

$$q_k$$

$$a_k$$

$$b_k$$

$$a_k$$

$$b_k$$

$$a_k$$

$$b_k$$

$$a_0 \neq 0$$

$$a_0 = 0$$

$$p_{k+1}q_k - q_{k+1}p_k$$

$$\frac{a}{b}$$

$$\frac{1}{x}$$

$$\frac{p}{q}$$

$$\left| r - \frac{p}{q} \right|$$

$$x$$

$$q \leq x$$

$$\frac{p}{q}$$

$$r$$

$$r$$

$$r = [a_0, a_1, a_2, ...]$$

$$r$$

$$k$$

$$r_k = \frac{p_k}{q_k}$$

$$P_k(a_0, ..., a_k)$$

$$r_k$$

$$r_{k-1}$$

$$r_{k-2}$$

$$r_{-1} = \frac{1}{0}$$

$$r_{-2} = \frac{0}{1}$$

$$r_k$$

$$a_0, a_1, ..., a_k$$

$$Q_k(a_0, ..., a_k) = P_{k-1}(a_1, ..., a_k)$$

$$r_0 = \frac{a_0}{1}$$

$$r_1 = \frac{a_0 a_1 + 1}{a_1}$$

$$P_{-1} = 1$$

$$P_{-2} = 0$$

$$r_{-1} = \frac{1}{0}$$

$$r_{-2} = \frac{0}{1}$$

$$T_k = a_k T_{k-1} - b_{k-1} c_{k-1} T_{k-2}$$

$$P_k$$

$$p_{-2}, p_{-1}, p_0, p_1, ..., p_k$$

$$q_{-2}, q_{-1}, q_0, q_1, ..., q_k$$

$$r_k = \frac{p_k}{q_k}$$

$$r$$

$$q_k$$

$$q_k$$

$$r_k$$

$$r = \frac{1+\sqrt{5}}{2}$$

$$r = \frac{1+\sqrt{5}}{2}$$

$$r$$

$$k$$

$$r_k = \frac{p_k}{q_k}$$

$$r_k$$

$$r_{k+1}$$

$$r$$

$$|\ r - r_k|$$

$$p_k$$

$$q_k$$

$$r_k - r_{k-1}$$

$$r_{k-1} - r_{k-2}$$

$$r_k$$

$$q_k$$

$$q_i q_{i-1}$$

$$(-1)^k$$

$$r_k$$

$$r$$

$$r_k$$

$$r$$

$$r_k$$

$$r_k$$

$$r_{k+1}$$

$$A, B, C \in \mathbb{Z}$$

$$x, y \in \mathbb{Z}$$

$$Ax + By = C$$

$$\frac{A}{B} = [a_0; a_1, ..., a_k]$$

$$p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$$

$$p_k$$

$$q_k$$

$$A$$

$$B$$

$$g = \gcd(A, B)$$

$$C$$

$$g$$

$$x = (-1)^{k-1} \frac{C}{g} q_{k-1}$$

$$y = (-1)^{k} \frac{C}{g} p_{k-1}$$

$$y = rx$$

$$(q_k; p_k)$$

$$(q_k; p_k)$$

$$(q; p)$$

$$0 \le t \le a_k$$

$$r = \frac{9}{7}$$

$$r_k = \frac{p_k}{q_k}$$

$$\vec{r}_k = (q_k; p_k)$$

$$\vec{r} = (1; r)$$

$$(x; y)$$

$$\frac{y}{x}$$

$$(x_1; y_1) \times (x_2; y_2) = x_1 y_2 - x_2 y_1$$

$$r_{k-1}$$

$$r_{k-2}$$

$$r$$

$$\vec{r}_{k-1}$$

$$\vec{r}_{k-2}$$

$$\vec{r}$$

$$a_k = \lfloor s_k \rfloor$$

$$\vec{r}_k$$

$$\vec{r}_k \times r = (q;p) \times (1;r) = qr - p$$

$$a_k = \lfloor s_k \rfloor$$

$$s_k = [a_k; a_{k+1}, a_{k+2}, ...]$$

$$\vec{r}$$

$$s_k \vec{r}_{k-1} + \vec{r}_{k-2}$$

$$\vec{r}$$

$$\vec{r}_{k-1}$$

$$\vec{p}$$

$$\vec{p} \times \vec{r}$$

$$\vec{r}_{k-1} \times \vec{r}$$

$$a_k = \lfloor s_k \rfloor$$

$$\vec{r}_{k-1}$$

$$\vec{r}_{k-2}$$

$$\vec{r}$$

$$a_k$$

$$\vec{r}_{k-1}$$

$$\vec{r}_{k-2}$$

$$\vec{r}$$

$$\vec{r}_2 = (4;3)$$

$$\vec{r}_1 = (1;1)$$

$$\vec{r}_0 = (1;0)$$

$$y = rx$$

$$\vec{r}_1$$

$$\vec{r}_0$$

$$\vec{r}_2$$

$$\vec{r}_1$$

$$\vec{r}_3 = (9;7)$$

$$\vec{r}_{k-2}$$

$$\vec{r}_k$$

$$\vec{0}$$

$$\vec{0}$$

$$\vec{r}_{k-2} + t \cdot \vec{r}_{k-1}$$

$$t$$

$$0 \le t \le a_k$$

$$k$$

$$\vec{r}_k$$

$$y = rx$$

$$x \ge 0$$

$$\vec{r}_k$$

$$y = rx$$

$$x > 0$$

$$(x;y)$$

$$r = [a_0; a_1, ..., a_k] = \frac{p_k}{q_k}$$

$$0 \le x \le N$$

$$0 \le y \le rx$$

$$0 \le x$$

$$y = rx$$

$$x \le N$$

$$t = \left\lfloor \frac{N}{q_k} \right\rfloor$$

$$1 \le \alpha \le t$$

$$(0;0)$$

$$t$$

$$\alpha \cdot (q_k; p_k)$$

$$(t+1)(q_k; p_k)$$

$$(t+1)q_k$$

$$N$$

$$(x;y)$$

$$y=rx$$

$$x\leq N$$

$$(x;y)$$

$$(x';y')$$

$$x'\leq N$$

$$(x';y')-(x;y)=(\Delta x;\Delta y)$$

$$y=rx$$

$$(\Delta x;\Delta y)$$

$$\Delta x\leq N-x$$

$$\Delta y\leq r\Delta x$$

$$r\Delta x-\Delta y$$

$$y=rx$$

$$(\Delta x;\Delta y)$$

$$r$$

$$i$$

$$0\leq t<a_i$$

$$(\Delta x;\Delta y)$$

$$(q_{i-1};p_{i-1})+t\cdot(q_i;p_i)$$

$$i$$

$$i$$

$$i$$

$$t=\left\lfloor\frac{N-x-q_{i-1}}{q_i}\right\rfloor$$

$$N-x-q_{i-1}\geq 0$$

$$(\Delta x;\Delta y)=(q_{i-1};p_{i-1})+t\cdot(q_i;p_i)$$

$$\Delta y\leq r\Delta x$$

$$t<a_i$$

$$i+2$$

$$x+q_{i-1}+a_iq_i=x+q_{i+1}$$

$$N$$

$$(\Delta x; \Delta y)$$

$$(x; y)$$

$$k = \left\lfloor \frac{N - x}{\Delta x} \right\rfloor$$

$$N$$

$$A$$

$$B$$

$$N$$

$$x \geq 0$$

$$y \geq 0$$

$$Ax + By \leq N$$

$$Ax + By$$

$$1 \leq A, B, N \leq 2 \cdot 10^9$$

$$O(\sqrt{N})$$

$$O(\log N)$$

$$x \mapsto \left\lfloor \frac{N}{A} \right\rfloor - x$$

$$x$$

$$(x; y)$$

$$0 \leq x \leq \left\lfloor \frac{N}{A} \right\rfloor$$

$$By - Ax \leq N \bmod A$$

$$By - Ax$$

$$x$$

$$y$$

$$\left\lfloor \frac{Ax + (N \bmod A)}{B} \right\rfloor$$

$$0 \leq x \leq N$$

$$y = \left\lfloor \frac{Ax + B}{C} \right\rfloor$$

$$x \leq N$$

$$y = \frac{Ax + B}{C}$$

$$t$$

$$\sum_{x=1}^{N} \lfloor \mathrm{e} x \rfloor$$

$$\mathrm{e} = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, ..., 1, 2n, 1, ...]$$

$$N \leq 10^{4000}$$

$$(x; y)$$

$$1 \leq x \leq N$$

$$1 \leq y \leq \mathrm{e} x$$

$$y = \mathrm{e} x$$

$$p$$

$$q$$

$$n$$

$$\sum_{i=1}^{n} [p \cdot i \bmod q]$$

$$a \bmod b = a - \left\lfloor \frac{a}{b} \right\rfloor b$$

$$x$$

$$1$$

$$N$$

$$\lfloor rx \rfloor$$

$$N$$

$$M$$

$$A$$

$$B$$

$$\sum_{i=0}^{N-1} \left\lfloor \frac{A \cdot i + B}{M} \right\rfloor$$

$$y = \frac{Ax + B}{M}$$

$$B = 0$$

$$[0, N-1]$$

$$[0, N-1]$$

$$[0, N-1]$$

$$y = \frac{Ax + B}{M}$$

$$[0, N - 1]$$

$$B = 0$$

$$\frac{p}{q}$$

$$1 \le p, q \le 10^9$$

$$m$$

$$pq^{-1}$$

$$m \sim 10^9$$

$$\frac{p}{q}$$

$$1 \le x \le N$$

$$Ax \bmod M$$

$$x$$

$$m$$

$$r$$

$$(p, q)$$

$$p \equiv qr \pmod{m}$$

$$(p_1, q_1)$$

$$(p_2, q_2)$$

$$p_1 q_2 \equiv p_2 q_1 \pmod{m}$$

$$\frac{p_1}{q_1} \ne \frac{p_2}{q_2}$$

$$|\, p_1 q_2 - p_2 q_1|$$

$$m$$

$$1 \le p, q \le 10^9$$

$$p_1, q_1$$

$$p_2, q_2$$

$$10^9$$

$$10^{18}$$

$$m > 10^{18}$$

$$\frac{p}{q}$$

$$1 \le p, q \le 10^9$$

$$r$$

$$m$$

$$q$$

$$1 \le q \le 10^9$$

$$qr \bmod m \le 10^9$$

$$1 \le q \le 10^9$$

$$q$$

$$q$$

$$qr \bmod m$$

$$qr = km + b$$

$$(q, m)$$

$$1 \le q \le 10^9$$

$$qr - km \ge 0$$

$$m$$

$$q$$

$$1 \le q \le 10^9$$

$$\frac{r}{m} q - k \ge 0$$

$$\frac{k}{q}$$

$$\frac{r}{m}$$

$$\frac{r}{m}$$

$$f(x) \equiv 0 \pmod m$$

$$f(x) \equiv 0 \pmod{p^a}$$

$$f(x) \equiv 0 \pmod{p^{a-1}}$$

$$f(x) \equiv 0 \pmod{p^{a-1}}$$

$$x = x_0 + p^{a-1} t$$

$$f(x) \equiv 0 \pmod{p^a}$$

$$f(x_0 + p^{a-1} t) \equiv 0 \pmod{p^a}$$

$$f(x_0) + p^{a-1} t f'(x_0) \equiv 0 \pmod{p^a}$$

$$t f'(x_0) \equiv -\frac{f(x_0)}{p^{a-1}} \pmod{p}$$

$$f(x) \equiv 0 \pmod{p}$$

$$f(x) \equiv g(x) \prod_{i=1}^{k} (x - x_i) \pmod{p}$$

$$(\forall i = 2, 3, ..., k), \quad 0 \equiv f(x_i) \equiv (x_i - x_1) h(x_i) \pmod{p}$$

$$h(x) \equiv g(x) \prod_{i=2}^{k} (x - x_i) \pmod{p}$$

$$f(x) \equiv a_n \prod_{i=1}^{n} (x - x_i) \pmod{p}$$

$$0 \equiv f(x_{n+1}) \equiv a_n \prod_{i=1}^{n} (x_{n+1} - x_i) \pmod{p}$$

$$(\forall i = 0, 1, ..., n), \quad p \mid b_i$$

$$f(x) = g(x)(x^p - x) + r(x)$$

$$x^n + \sum_{i=0}^{n-1} a_i x^i \equiv 0 \pmod{p}$$

$$x^p - x = f(x) q(x) + p r(x)$$

$$x^p - x = f(x) q(x) + r_1(x)$$

$$0 \equiv x^p - x \equiv f(x) q(x) \pmod{p}$$

$$x^n \equiv a \pmod{p}$$

$$a^{\frac{p-1}{n}} \equiv 1 \pmod{p}$$

$$a^{\frac{p-1}{n}} \equiv (x_0^n)^{\frac{p-1}{n}} \equiv 1 \pmod{p}$$

$$\begin{aligned} x^p - x &= x(x^{p-1} - 1) \\ &= x\left((x^n)^{\frac{p-1}{n}} - a^{\frac{p-1}{n}} + a^{\frac{p-1}{n}} - 1\right) \\ &= (x^n - a) P(x) + x\left(a^{\frac{p-1}{n}} - 1\right) \end{aligned}$$

$$x^n + \sum_{i=0}^{n-1} a_i x^i \equiv 0 \pmod{p}$$

$$x^n + \sum_{i=0}^{n-2} a_i x^i \equiv 0 \pmod{p}$$

$$x^n \equiv a \pmod{p}$$

$$m$$

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

$$x \in \mathbf{Z}_m$$

$$x$$

$$m$$

$$a_n \not\equiv 0 \pmod{m}$$

$$n$$

$$m$$

$$m = p^a \ (p \in \mathbf{P}, a \in \mathbf{Z}_{>1})$$

$$x_0$$

$$x_0$$

$$p$$

$$a > 1$$

$$f(x) = \sum_{i=0}^{n} a_i x^i \ (p^a \nmid a_n)$$

$$f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$$

$$x_0$$

$$f'(x_0) \not\equiv 0 \pmod{p}$$

$$t$$

$$f'(x_0) \equiv 0 \pmod{p}$$

$$f(x_0) \equiv 0 \pmod{p^a}$$

$$t = 0, 1, ..., p-1$$

(3)

$$x$$

(4)

$$f'(x_0) \equiv 0 \pmod{p}$$

$$f(x_0) \not\equiv 0 \pmod{p^a}$$

(3)

(4)

$$(3)$$

$$(4)$$

$$f'(x_0) \not\equiv 0 \pmod p$$

$$t$$

$$(5)$$

$$t_0$$

$$(3)$$

$$(4)$$

$$f'(x_0) \equiv 0 \pmod p$$

$$f(x_0) \equiv 0 \pmod{p^a}$$

$$t$$

$$(5)$$

$$(3)$$

$$(4)$$

$$f'(x_0) \equiv 0 \pmod p$$

$$f(x_0) \not\equiv 0 \pmod{p^a}$$

$$(5)$$

$$(3)$$

$$(4)$$

$$p$$

$$a$$

$$f(x)$$

$$x_0$$

$$s$$

$$f(x) \equiv 0 \pmod p$$

$$f'(a) \not\equiv 0 \pmod p$$

$$x_s \in \mathbf{Z}_{p^a}$$

$$x_s \equiv s \pmod p$$

$$x_s$$

$$(4)$$

$$f(x) \equiv 0 \pmod p$$

$$f'(a) \equiv 0 \pmod p$$

(4)

$$f(x) \equiv 0 \pmod p$$

$$p \in \mathbf{P}$$

$$f(x) = \sum_{i=0}^{n} a_i x^i$$

$$p \nmid a_n$$

$$x \in \mathbf{Z}_p$$

$$k$$

$$x_1, x_2, ..., x_k \ (k \leq n)$$

$$\deg g = n - k$$

$$[x^{n-k}]g(x) = a_n$$

$$k$$

$$k = 1$$

$$f(x) = (x - x_1)g(x) + r$$

$$r \in \mathbf{Z}$$

$$f(x_1) \equiv 0 \pmod p$$

$$r \equiv 0 \pmod p$$

$$f(x) \equiv (x - x_1)g(x) \pmod p$$

$$k - 1$$

$$k > 1$$

$$f(x)$$

$$k$$

$$x_1, x_2, ..., x_k$$

$$f(x) \equiv (x - x_1)h(x) \pmod p$$

$$h(x)$$

$$k - 1$$

$$x_2, x_3, ..., x_k$$

$$\deg g = n - k$$

$$[x^{n-k}]g(x) = a_n$$

$$p$$

$$(\forall x \in \mathbf{Z}), \quad x^{p-1} - 1 \equiv \prod_{i=1}^{p-1}(x-i) \pmod p$$

$$(p-1)! \equiv -1 \pmod p$$

$$(6)$$

$$n$$

$$f(x)$$

$$n+1$$

$$x_1, x_2, ..., x_{n+1}$$

$$x_1, x_2, ..., x_n$$

$$x = x_{n+1}$$

$$p$$

$$\sum_{i=0}^{n} b_i x^i \equiv 0 \pmod p$$

$$n$$

$$(6)$$

$$p$$

$$\deg r < p$$

$$r(x)$$

$$f(x) \equiv 0 \pmod p$$

$$r(x) \equiv 0 \pmod p$$

$$n \geq p$$

$$f(x)$$

$$\deg r < p$$

$$x$$

$$x^p \equiv x \pmod p$$

$$r(x) \equiv 0 \pmod p$$

$$f(x)$$

$$p$$

$$r(x) \not\equiv 0 \pmod p$$

$$f(x) \equiv r(x) \pmod p$$

$$f(x)$$

$$r(x)$$

$$n \le p$$

$$n$$

$$q(x)$$

$$r(x) \ (\deg r < n)$$

$$q(x)$$

$$r_1(x) \ (\deg r_1 < n)$$

$$(7)$$

$$n$$

$$r_1 \equiv 0 \pmod p$$

$$n$$

$$r(x)$$

$$r_1(x) = pr(x)$$

$$(8)$$

$$x$$

$$f(x)q(x) \equiv 0 \pmod p$$

$$p$$

$$(7)$$

$$s$$

$$s \le n$$

$$\deg q = p - n$$

$$q(x) \equiv 0 \pmod p$$

$$p - n$$

$$f(x)q(x) \equiv 0 \pmod p$$

$$f(x) \equiv 0 \pmod p$$

$$q(x) \equiv 0 \pmod p$$

$$s + (p - n) \ge p$$

$$s \ge n$$

$$s = n$$

$$\mathbf{Z}_p$$

$$n \nmid p - 1$$

$$p \nmid a$$

$$(9)$$

$$n$$

$$(9)$$

$$(9)$$

$$x_0$$

$$a^{\frac{p-1}{n}} \equiv 1 \pmod p$$

$$P(x)$$

$$(9)$$

$$n$$

$$m$$

$$p$$

$$n < p$$

$$x$$

$$x - \frac{a_{n-1}}{n}$$

$$x^{n-1}$$

$$p$$

$$n < p$$

$$n = 1$$

$$n = 2$$

$$(10)$$

$$\gcd(a_1, b_1) = \gcd(b_1, r_1) = \gcd(r_1, r_2) = \cdots = (r_{n-1}, r_n) = 1$$

$$\begin{aligned}
a_1 &= q_1 b_1 + r_1 & \qquad (0 \le r_1 < b_1) \\
b_1 &= q_2 r_1 + r_2 & \qquad (0 \le r_2 < r_1) \\
r_1 &= q_3 r_2 + r_3 & \qquad (0 \le r_3 < r_2) \\
&\cdots \\
r_{n-3} &= q_{n-1} r_{n-2} + r_{n-1} \\
r_{n-2} &= q_n r_{n-1} + r_n \\
r_{n-1} &= q_{n+1} r_n
\end{aligned}$$

$$r_{n-2} = x_n r_{n-1} + 1$$

$$1 = r_{n-2} - x_n r_{n-1}$$

$$1 = (1 + x_n x_{n-1}) r_{n-2} - x_n r_{n-3}$$

$$f_{i,j} = \min_{\gcd(k,l_i)=j} f_{i-1,k} + c_i.$$

$$ax + by = n$$

$$\begin{cases} x = x_0 + bt \\ y = y_0 - at \end{cases}$$

$$ax = n - by > ab - a - b - by \geqslant ab - a - b - b(a-1) = -a$$

$$ab = a(x+1) + b(y+1)$$

$$ab = a(x+1) + b(y+1) \geqslant ab + ab = 2ab$$

$$ab - a - b - ax - by = a(-x-1) + b(a-1-y)$$

$$\sum_{i=0}^{\left[\frac{n}{a}\right]} \left[\frac{n-ia}{b}\right]$$

$$y = -\frac{a}{b}x + \frac{n}{b}$$

$$a, b$$

$$x, y$$

$$\gcd(a,b) \mid ax + by$$

$$x, y$$

$$ax + by = \gcd(a,b)$$

$$\gcd(a,b) \mid a, \gcd(a,b) \mid b$$

$$\gcd(a,b) \mid ax, \gcd(a,b) \mid by$$

$$x, y$$

$$\gcd(a,b) \mid ax + by$$

$$0$$

$$\gcd(a,b) = a$$

$$a, b$$

$$0$$

$$\gcd(a,b) = \gcd(a,-b)$$

$$a, b$$

$$0$$

$$a \geq b, \gcd(a,b) = d$$

$$ax + by = d$$

$$d$$

$$a_1 x + b_1 y = 1$$

$$(a_1, b_1) = 1$$

$$a_1 x + b_1 y = 1$$

$$\gcd(a, b) \to \gcd(b, a \bmod b) \to \dots$$

$$r$$

$$r_n = 1$$

$$q$$

$$x$$

$$r_{n-1} = r_{n-3} - x_{n-1} r_{n-2}$$

$$r_{n-2}, \cdots, r_1$$

$$1 = a_1 x + b_1 y$$

$$d = 1$$

$$a, b$$

$$d > 0$$

$$a, b$$

$$x, y$$

$$ax + by = d$$

$$d = \gcd(a, b)$$

$$a, b$$

$$x, y$$

$$ax + by = 1$$

$$a, b$$

$$n$$

$$a_1, a_2, \dots, a_n$$

$$x_1, x_2, \dots, x_n$$

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = \gcd(a_1, a_2, \dots, a_n)$$

$$a_1, a_2, \dots, a_n$$

$$d > 0$$

$$a_1, a_2, \dots, a_n$$

$$x_1, x_2, \dots, x_n$$

$$a_1 x_1 + a_2 x_2 + \cdots + a_n x_n = d$$

$$d = \gcd(a_1, a_2, \dots, a_n)$$

n

l_i

c_i

c_i

i

l_i

-1

$l_{i_1}, ..., l_{i_k}$

1

$x_1, ..., x_n$

$l_{i_1}x_1 + \cdots + l_{i_k}x_k = 1$

$l_1, ..., l_n$

0

$\gcd$

0

$\gcd$

1

1

$\gcd(0, x) = x$

10^9

$l_1, ..., l_n$

$f_{i,j}$

i

j

$f_{n,1}$

j

j

$O(n + m)$

a

b

n

a

$$b$$

$$C = ab - a - b$$

$$C - n$$

$$[0, C]$$

$$a - 1$$

$$y_0$$

$$C - n$$

$$ax + by = ab - a - b$$

$$y + 1$$

$$x + 1$$

$$y + 1$$

$$x + 1$$

$$C - n$$

$$a - 1$$

$$-x - 1$$

$$a - 1 - y$$

$$C - n$$

$$ax + by = n$$

$$n < ab$$

$$(0,0)(b,0)(0,a)$$

$$ax + by = n$$

$$0, a, \cdots, (b-1)a$$

$$\pmod{b}$$

$$n < ab$$

$$ax + by = n$$

$$O(\log \max(a, b))$$

$$a = p_1 p_2 \cdots p_s$$

$$a = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}, p_1 < p_2 < \cdots < p_s$$

$$h(x) = f(x^p)$$
$$h(x) = f^p(x)$$
$$h(x) = f(x)g(x)$$
$$h(x) = \sum_{d|x} f(d)g\left(\frac{x}{d}\right)$$

$$a, b \in \mathbf{Z}$$

$$a \neq 0$$

$$\exists q \in \mathbf{Z}$$

$$b = aq$$

$$b$$
$$a$$

$$a \mid b$$

$$b$$
$$a$$

$$a \nmid b$$

$$a \mid b \iff -a \mid b \iff a \mid -b \iff \mid a \mid \mid \mid b \mid$$

$$a \mid b \wedge b \mid c \implies a \mid c$$

$$a \mid b \wedge a \mid c \iff \forall x, y \in \mathbf{Z}, a \mid (xb + yc)$$

$$a \mid b \wedge b \mid a \implies b = \pm a$$

$$m \neq 0$$

$$a \mid b \iff ma \mid mb$$

$$b \neq 0$$

$$a \mid b \implies \mid a \mid \leq \mid b \mid$$

$$a \neq 0, b = qa + c$$

$$a \mid b \iff a \mid c$$

$$a \mid b$$

$$b$$
$$a$$
$$a$$
$$b$$
$$0$$
$$0$$

$$b \neq 0$$

$$b$$

$$b \neq 0$$

$$\pm 1$$

$$\pm b$$

$$b$$

$$b = \pm 1$$

$$b$$

$$b \neq 0$$

$$b$$

$$b \neq 0$$

$$d$$

$$b$$

$$\frac{b}{d}$$

$$b$$

$$b > 0$$

$$d$$

$$b$$

$$\frac{b}{d}$$

$$b$$

$$a, b$$

$$a \neq 0$$

$$d$$

$$q$$

$$r$$

$$b = qa + r, d \leq r < \mid a \mid + d$$

$$d$$

$$r$$

$$a \mid b$$

$$a \mid r$$

$$d$$

$$0$$

$$b = qa + r, 0 \leq r < \mid a \mid$$

$$r$$

$$d$$

$$a$$

$$b = qa + r, -\frac{\mid a \mid}{2} \leq r < \mid a \mid - \frac{\mid a \mid}{2}$$

$$d$$

$$1$$

$$b = qa + r, 1 \leq r < \mid a \mid + 1$$

$$a$$

$$0$$

$$(a - 1)$$

$$a$$

$$a$$

$$a$$

$$a$$

$$a$$

$$\gcd(a_1, a_2) = 1$$

$$a_1$$

$$a_2$$

$$\gcd(a_1, ..., a_k) = 1$$

$$a_1, ..., a_k$$

$$6$$

$$10$$

$$15$$

$$p \neq 0, \pm 1$$

$$p$$

$$p$$

$$a \neq 0, \pm 1$$

$$a$$

$$a$$

$$p$$

$$-p$$

$$1$$
$$a$$
$$a$$
$$d$$
$$e$$
$$1 < d, e < a$$
$$p$$
$$1$$
$$d$$
$$d = p$$
$$1$$
$$a$$
$$a$$
$$p \leq \sqrt{a}$$
$$p \mid a$$
$$3$$
$$6n \pm 1$$
$$p$$
$$p \mid a_1 a_2$$
$$p \mid a_1$$
$$p \mid a_2$$
$$a$$
$$p_j (1 \leq j \leq s)$$
$$a$$
$$m \neq 0$$
$$m \mid (a - b)$$
$$m$$
$$a$$
$$b$$
$$m$$
$$b$$
$$a$$
$$m$$
$$a \equiv b \pmod{m}$$

$$a$$

$$b$$

$$m$$

$$b$$

$$a$$

$$m$$

$$a \not\equiv b \pmod{m}$$

$$m$$

$$a \equiv b \pmod{(-m)}$$

$$b$$

$$a$$

$$m$$

$$b$$

$$a$$

$$m$$

$$a \equiv a \pmod{m}$$

$$a \equiv b \pmod{m}$$

$$b \equiv a \pmod{m}$$

$$a \equiv b \pmod{m}, b \equiv c \pmod{m}$$

$$a \equiv c \pmod{m}$$

$$a, b, c, d \in \mathbf{Z}, m \in \mathbf{N}^*, a \equiv b \pmod{m}, c \equiv d \pmod{m}$$

$$a \pm c \equiv b \pm d \pmod{m}$$

$$a \times c \equiv b \times d \pmod{m}$$

$$a, b \in \mathbf{Z}, k, m \in \mathbf{N}^*, a \equiv b \pmod{m}$$

$$ak \equiv bk \pmod{mk}$$

$$a, b \in \mathbf{Z}, d, m \in \mathbf{N}^*, d \mid a, d \mid b, d \mid m$$

$$a \equiv b \pmod{m}$$

$$\frac{a}{d} \equiv \frac{b}{d} \left(\bmod \frac{m}{d}\right)$$

$$a, b \in \mathbf{Z}, d, m \in \mathbf{N}^*, d \mid m$$

$$a \equiv b \pmod{m}$$

$$a \equiv b \pmod{d}$$

$$a, b \in \mathbf{Z}, d, m \in \mathbf{N}^*$$

$$a \equiv b \pmod m$$

$$\gcd(a, m) = \gcd(b, m)$$

$$d$$

$$m$$

$$a, b$$

$$d$$

$$a, b$$

$$f(n)$$

$$f(1) = 1$$

$$\forall x, y \in \mathbf{N}^*, \gcd(x, y) = 1$$

$$f(xy) = f(x)f(y)$$

$$f(n)$$

$$f(n)$$

$$f(1) = 1$$

$$\forall x, y \in \mathbf{N}^*$$

$$f(xy) = f(x)f(y)$$

$$f(n)$$

$$f(x)$$

$$g(x)$$

$$x = \prod p_i^{k_i}$$

$$F(x)$$

$$F(x) = \prod F(p_i^{k_i})$$

$$F(x)$$

$$F(x) = \prod F(p_i)^{k_i}$$

$$\varepsilon(n) = [n = 1]$$

$$\mathrm{id}_k(n) = n^k$$

$$\mathrm{id}_1(n)$$

$$\mathrm{id}(n)$$

$$1(n) = 1$$

$$\sigma_k(n) = \sum_{d|n} d^k$$

$$\sigma_0(n)$$

$$d(n)$$

$$\tau(n)$$

$$\sigma_1(n)$$

$$\sigma(n)$$

$$\varphi(n) = \sum_{i=1}^{n} [\gcd(i,n) = 1]$$

$$\mu(n) = \begin{cases} 1 & n = 1 \\ 0 & \exists d > 1, d^2 \mid n \\ (-1)^{\omega(n)} & \text{otherwise} \end{cases}$$

$$\omega(n)$$

$$n$$

$$f$$

$$a, b$$

$$f(ab) = f(a) + f(b)$$

$$\lambda(\mu \boldsymbol{a}) = (\lambda\mu)\boldsymbol{a}$$
$$(\lambda + \mu)\boldsymbol{a} = \lambda\boldsymbol{a} + \mu\boldsymbol{a}$$
$$\lambda(\boldsymbol{a} + \boldsymbol{b}) = \lambda\boldsymbol{a} + \lambda\boldsymbol{b}$$

$$(-\lambda)\boldsymbol{a} = -(\lambda\boldsymbol{a}) = -\lambda(\boldsymbol{a})$$
$$\lambda(\boldsymbol{a} - \boldsymbol{b}) = \lambda\boldsymbol{a} - \lambda\boldsymbol{b}$$

$$\boldsymbol{a} + \boldsymbol{b} = (m + p, n + q)$$
$$\boldsymbol{a} - \boldsymbol{b} = (m - p, n - q)$$
$$k\boldsymbol{a} = (km, kn)$$

$$\boldsymbol{b} = (l(\cos\theta\cos\alpha - \sin\theta\sin\alpha), l(\sin\theta\cos\alpha + \cos\theta\sin\alpha))$$

$$\boldsymbol{b} = (l\cos\theta\cos\alpha - l\sin\theta\sin\alpha, l\sin\theta\cos\alpha + l\cos\theta\sin\alpha)$$

$$\boldsymbol{b} = (x\cos\alpha - y\sin\alpha, y\cos\alpha + x\sin\alpha)$$

$$\vec{a}$$

$$\boldsymbol{a}$$

$$\overrightarrow{AB}$$

$$|\overrightarrow{AB}|$$

$$|\boldsymbol{a}|$$

$$0$$

$$\vec{0}$$

$$\mathbf{0}$$

$$1$$

$$\vec{e}$$

$$\boldsymbol{e}$$

$$\boldsymbol{a} \parallel \boldsymbol{b}$$

$$\boldsymbol{a}, \boldsymbol{b}$$

$$\overrightarrow{OA} = \boldsymbol{a}, \overrightarrow{OB} = \boldsymbol{b}$$

$$\theta = \angle AOB$$

$$\boldsymbol{a}$$

$$\boldsymbol{b}$$

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle$$

$$\theta = 0$$

$$\theta = \pi$$

$$\theta = \frac{\pi}{2}$$

$$\boldsymbol{a} \perp \boldsymbol{b}$$

$$\theta \in [0, \pi]$$

$$A$$

$$B$$

$$C$$

$$\overrightarrow{AB} + \overrightarrow{BC}$$

$$A$$

$$C$$

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

$$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{a} + (-\boldsymbol{b})$$

$$A, B$$

$$\overrightarrow{AB}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$\lambda$$
$$\boldsymbol{a}$$
$$\lambda \boldsymbol{a}$$
$$|\ \lambda \boldsymbol{a}| = |\ \lambda\ |\ |\ \boldsymbol{a}|$$
$$\lambda > 0$$
$$\lambda \boldsymbol{a}$$
$$\boldsymbol{a}$$
$$\lambda = 0$$
$$\lambda \boldsymbol{a} = \boldsymbol{0}$$
$$\lambda < 0$$
$$\lambda \boldsymbol{a}$$
$$\boldsymbol{a}$$
$$\boldsymbol{a}$$
$$\boldsymbol{b}$$
$$\Longleftrightarrow$$
$$\lambda$$
$$\boldsymbol{b} = \lambda \boldsymbol{a}$$
$$\boldsymbol{a}$$
$$\lambda$$
$$\boldsymbol{b} = \lambda \boldsymbol{a}$$
$$\boldsymbol{a} \parallel \boldsymbol{b}$$
$$\boldsymbol{a} \parallel \boldsymbol{b}$$
$$\boldsymbol{a} \neq \boldsymbol{0}$$
$$|\ \boldsymbol{b}| = \mu\ |\ \boldsymbol{a}|$$
$$\boldsymbol{a}$$
$$\boldsymbol{b}$$
$$\boldsymbol{b} = \mu \boldsymbol{a}$$
$$\boldsymbol{b} = -\mu \boldsymbol{a}$$
$$\boldsymbol{e_1}, \boldsymbol{e_2}$$
$$(x, y)$$
$$\boldsymbol{e_1}, \boldsymbol{e_2}$$
$$\boldsymbol{p}$$
$$\mathbf{p} = x\boldsymbol{e_1} + y\boldsymbol{e_2}$$

$$i, j$$

$$(x, y)$$

$$(x, y)$$

$$\overrightarrow{OP} = \boldsymbol{p}$$

$$P(x, y)$$

$$\boldsymbol{a} = (m, n)$$

$$\boldsymbol{b} = (p, q)$$

$$A(a, b), B(c, d)$$

$$\overrightarrow{AB} = (c - a, d - b)$$

$$P$$

$$\overrightarrow{OP}$$

$$A, B, C$$

$$\overrightarrow{OB} = \lambda\overrightarrow{OA} + (1 - \lambda)\overrightarrow{OC}$$

$$ABC$$

$$D$$

$$BC$$

$$n$$

$$n\ BD = k\ DC$$

$$\overrightarrow{AD} = \frac{n}{k+n}\overrightarrow{AB} + \frac{k}{k+n}\overrightarrow{AC}$$

$$\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$$

$$(x, y, z)$$

$$\boldsymbol{p}$$

$$\mathbf{p} = x\boldsymbol{e_1} + y\boldsymbol{e_2} + z\boldsymbol{e_3}$$

$$\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$$

$$(x, y, z)$$

$$\boldsymbol{x}, \boldsymbol{y}$$

$$\boldsymbol{p}$$

$$\boldsymbol{x}, \boldsymbol{y}$$

$$(a, b)$$

$$\boldsymbol{p} = a\boldsymbol{x} + b\boldsymbol{y}$$

$$ABCD$$

$$\boldsymbol{n}$$

$$\overrightarrow{AB}, \overrightarrow{AD}$$

$$\overrightarrow{AB} \cdot \boldsymbol{n} = \mathbf{0}$$

$$\overrightarrow{AD} \cdot \boldsymbol{n} = \mathbf{0}$$

$$\boldsymbol{a} = (x, y)$$

$$\theta$$

$$l = \sqrt{x^2 + y^2}$$

$$x = l\cos\theta, y = l\sin\theta$$

$$\alpha$$

$$\boldsymbol{b} = (l\cos(\theta + \alpha), l\sin(\theta + \alpha))$$

$$x, y$$

$$\beta = k_1 a_1 + k_2 a_2 + \cdots + k_r a_r = (a_1, a_2, \cdots, a_r)\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_r \end{pmatrix}$$

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n = b$$

$$S = n - r(A)$$

$$R(A) = \operatorname{span}\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$$

$$y = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n = (\alpha_1, \alpha_2, \cdots, \alpha_n)\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix} = A\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

$$V$$

$$\mathbb{P}$$

$$+$$

$$(V, +)$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$V$$

$$\cdot : \mathbb{P} \times V \mapsto V$$

$$p \cdot v$$

$$pv$$

$$p$$

$$\mathbb{P}$$

$$v$$

$$V$$

$$V$$

$$\mathbf{u}, \mathbf{v} \in V, a \in \mathbb{P}$$

$$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$$

$$a, b \in \mathbb{P}, \mathbf{u} \in V$$

$$(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$$

$$a, b \in \mathbb{P}, \mathbf{u} \in V$$

$$a(b\mathbf{u}) = (ab)\mathbf{u}$$

$$1 \in \mathbb{P}$$

$$\mathbb{P}$$

$$u \in V$$

$$1\mathbf{u} = \mathbf{u}$$

$$(V, +, \cdot, \mathbb{P})$$

$$V$$

$$+, \cdot$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$V$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$\mathbf{0}$$

$$\boldsymbol{\theta}$$

$$V$$

$$(V, +, \cdot, \mathbb{P})$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$(V, +, \cdot, \mathbb{P})$$

$$\theta$$

$$\forall \alpha \in V$$

$$-\alpha$$

$$\exists 0 \in \mathbb{P}$$

$$\forall \alpha \in V$$

$$0\alpha = \theta$$

$$\forall k \in \mathbb{P}$$

$$k\theta = \theta$$

$$(-1)\alpha = -\alpha,\ \ \forall \alpha \in V$$

$$\forall \alpha \in V, k \in \mathbb{P}$$

$$k\alpha = \theta \Longrightarrow k = 0 \vee \alpha = \theta$$

$$\forall \alpha, \beta, \gamma \in V$$

$$\alpha + \beta = \alpha + \gamma \Longrightarrow \beta = \gamma$$

$$\mathbb{P}^n$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$\mathbb{R}$$

$$\mathbb{C}$$

$$\mathbb{N}_p$$

$$p$$

$$\mathbb{P}$$

$$n \times m$$

$$\mathbb{P}^{n\times m}$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$\mathbb{P}[x]$$

$$\mathbb{P}$$

$$[a, b]$$

$$C[a, b]$$

$$(V, +, \cdot, \mathbb{P})$$

$$a_1, a_2, ..., a_n \in V$$

$$V$$

$$k_1, k_2, ..., k_n \in \mathbb{P}$$

$$\sum_{i=1}^{n} k_i a_i$$

$$a_1, a_2, ..., a_n$$

$$\beta \in V$$

$$a_1, a_2, ..., a_n$$

$$\beta$$

$$a_1, a_2, ..., a_n$$

$$k_1, k_2, ..., k_n \in \mathbb{P}$$

$$a_1, a_2, ..., a_n$$

$$\sum_{i=1}^{n} k_i a_i = \theta \Longleftrightarrow k_i = 0, i = 1, 2, ..., n$$

$$a_1, a_2, ..., a_n$$

$$a_1, a_2, ..., a_n$$

$$a$$

$$k$$

$$V$$

$$\beta$$

$$(a_1, a_2 \cdots, a_r)$$

$$(V, +, \cdot, \mathbb{P})$$

$$\theta$$

$$\beta$$

$$a_1, a_2, ..., a_n$$

$$a_1, a_2, ..., a_n$$

$$a_1, a_2, ..., a_n$$

$$\beta$$

$$a_1, a_2, ..., a_n$$

$$a_1, a_2, ..., a_n, \beta$$

$$(V, +, \cdot, \mathbb{P})$$

$$b_1, b_2, ..., b_m$$

$$\{a_1, a_2, ..., a_n\} \subseteq \{b_1, b_2, ..., b_m\}$$

$$a_1, a_2, ..., a_n$$

$$\forall \beta \in \{b_1, b_2, ..., b_m\} \setminus \{a_1, a_2, ..., a_n\}$$

$$a_1, a_2, ..., a_n, \beta$$

$$a_1, a_2, ..., a_n$$

$$b_1, b_2, ..., b_m$$

$$V$$

$$\theta, \theta, ..., \theta$$

$$0$$

$$1$$

$$b_1, b_2, ..., b_m$$

$$\mathrm{rank}\{b_1, b_2, ..., b_m\}$$

$$\mathrm{rank}\{\theta, \theta, ..., \theta\} = 0$$

$$a_1, a_2, ..., a_n$$

$$b_1, b_2, ..., b_m$$

$$b_1, b_2, ..., b_m$$

$$a_1, a_2, ..., a_n$$

$$a_1, a_2, ..., a_n$$

$$b_1, b_2, ..., b_m$$

$$b_1, b_2, ..., b_m$$

$$a_1, a_2, ..., a_n$$

$$\{a_1, a_2, ..., a_n\} \cong \{b_1, b_2, ..., b_m\}$$

$$(V, +, \cdot, \mathbb{P})$$

$$a_1, a_2, ..., a_n$$

$$b_1, b_2, ..., b_m$$

$$n > m$$

$$a_1, a_2, ..., a_n$$

$$a_1, a_2, ..., a_n$$

$$n \leq m$$

$$a_1, a_2, ..., a_n$$

$$b_1, b_2, ..., b_m$$

$$\mathrm{rank}\{a_1, a_2, ..., a_n\} \leq \mathrm{rank}\{b_1, b_2, ..., b_m\}$$

$$(V, +, \cdot, \mathbb{P})$$

$$\left\{ v=\sum_{i=1}^{n} k_i a_i : a_i \in V, k_i \in \mathbb{P}, i=1,2,...,n \right\}$$

$$a_1, a_2, ..., a_n$$

$$\text{span}\{a_1, a_2, ..., a_n\}$$

$$n$$

$$a$$

$$(V, +, \cdot, \mathbb{P})$$

$$(V_1, +, \cdot, \mathbb{P})$$

$$\emptyset \neq V_1$$

$$V_1 \subseteq V$$

$$V_1$$

$$+, \cdot$$

$$\mathbb{P}$$

$$V_1$$

$$V$$

$$V_1 \leq V$$

$$V$$

$$V$$

$$\subseteq$$

$$\subset$$

$$V_1$$

$$V$$

$$V_1 < V$$

$$V$$

$$V_1$$

$$V_1$$

$$\forall u, v \in V_1$$

$$u+v \in V_1$$

$$\forall v \in V_1$$

$$\forall k \in \mathbb{P}$$

$$kv \in V_1$$

$$(V_1, +, \cdot, \mathbb{P})$$

$$(V_2, +, \cdot, \mathbb{P})$$

$$V_1 \cap V_2$$

$$V_1 \cap V_2$$

$$V_1$$

$$V_2$$

$$\bigcap_{i=1}^{m} V_i$$

$$V$$

$$V = \{u + v \mid u \in V_1, v \in V_2\}$$

$$V$$

$$V_1$$

$$V_2$$

$$V = V_1 + V_2$$

$$V_1 + V_2$$

$$V_1 \cup V_2$$

$$\sum_{i=1}^{m} V_i$$

$$V = V_1 + V_2$$

$$V$$

$$v$$

$$v_1, v_2$$

$$v = v_1 + v_2$$

$$V$$

$$V_1$$

$$V_2$$

$$V_1 \oplus V_2$$

$$\bigoplus_{i=1}^{m} V_i$$

$$V_1$$

$$V_2$$

$$V_1 \times V_2$$

$$\mathbb{P}$$

$$+ : (V_1 \times V_2) \times (V_1 \times V_2) \mapsto V_1 \times V_2; ((u_1, v_1), (u_2, v_2)) \to (u_1 + u_2, v_1 + v_2)$$

$$\cdot : \mathbb{P} \times (V_1 \times V_2) \mapsto V_1 \times V_2; (k, (u, v)) \to (ku, kv)$$

$$\prod_{i=1}^{m} V_i$$

$$V = \mathbb{R}^3$$

$$V_1 := \{(x, 0, 0) |\ x \in \mathbb{R}\}$$

$$V_2 := \{(x, y, 0) |\ x, y \in \mathbb{R}\}$$

$$V_3 := \{(0, y, z) |\ y, z \in \mathbb{R}\}$$

$$V_4 := \{(x, 0, z) |\ x, z \in \mathbb{R}\}$$

$$V_1 < V_2 < V$$

$$V_3 < V$$

$$V_2 = V_1 + V_2$$

$$V = V_1 \oplus V_3 = V_2 + V_3$$

$$V_2 \oplus V_3 = V_4$$

$$V_2 \oplus V_4 = V_3$$

$$V_3 \oplus V_4 = V_2$$

$$V_2 + V_3 \leq V$$

$$V_1, V_2, V_3$$

$$\mathbb{P}$$

$$V_1 \cap V_2 = V_2 \cap V_1$$

$$V_1 \cap (V_2 \cap V_3) = (V_1 \cap V_2) \cap V_3$$

$$V_1, V_2, V_3$$

$$\mathbb{P}$$

$$V_1 + V_2 = V_2 + V_1$$

$$V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$$

$$V_1, V_2, V_3$$

$$\mathbb{P}$$

$$V_1 \cap (V_2 + V_3) \supseteq (V_1 \cap V_2) + (V_1 \cap V_3)$$

$$V_1 + (V_2 \cap V_3) \subseteq (V_1 + V_2) \cap (V_1 + V_3)$$

$$\text{span}\{a_1, a_2, ..., a_n\} + \text{span}\{b_1, b_2, ..., b_m\} = \text{span}\{a_1, a_2, ..., a_n, b_1, b_2, ..., b_m\}$$

$$V_1, V_2$$

$$\mathbb{P}$$

$$V_1 + V_2 = V_1 \oplus V_2$$

$$\exists \beta \in V_1 + V_2$$

$$V_1$$

$$V_2$$

$$\rightarrow$$

$$\theta$$

$$V_1$$

$$V_2$$

$$V_1 \cap V_2 = \{\theta\}$$

$$1 \Longrightarrow 2$$

$$2 \Longrightarrow 3$$

$$\beta = \beta_1 + \beta_2$$

$$\beta_1 \in V_1, \beta_2 \in V_2$$

$$\theta = \alpha_1 + \alpha_2$$

$$\theta \neq \alpha_1 \in V_1, \alpha_2 \in V_2$$

$$\beta = \beta + \theta = (\beta_1 + \alpha_1) + (\beta_2 + \alpha_2)$$

$$\beta_1 \neq \beta_1 + \alpha_1$$

$$3 \Longrightarrow 4$$

$$V_1$$

$$V_2$$

$$\alpha$$

$$\theta = \alpha + (-\alpha) = (-\alpha) + \alpha$$

$$4 \Longrightarrow 1$$

$$V_1 + V_2$$

$$\beta \in V_1 + V_2$$

$$\beta = \beta_1 + \beta_2 = \gamma_1 + \gamma_2$$

$$\beta_1, \gamma_1 \in V_1, \beta_2, \gamma_2 \in V_2$$

$$\beta_1, \beta_2, \gamma_1, \gamma_2)$$

$$\theta \neq \beta_1 - \gamma_1 = \gamma_2 - \beta_2 \in V_1 \cap V_2$$

$$V, V'$$

$$\mathbb{P}$$

$$\sigma : V \mapsto V'$$

$$\forall u, v \in V$$

$$\forall k \in \mathbb{P}$$

$$\sigma(u+v) = \sigma(u) + \sigma(v)$$

$$\sigma(ku) = k\sigma(u)$$

$$\sigma$$

$$V$$

$$V'$$

$$V$$

$$V'$$

$$V \cong V'$$

$$\sigma$$

$$\sigma$$

$$\mathbb{P}$$

$$\mathbb{P}$$

$$n$$

$$\mathbb{P}^n$$

$$A$$

$$A$$

$$x$$

$$A$$

$$b$$

$$x$$

$$Ax = b$$

$$A$$

$$b$$

$$n$$

$$A$$

$$r(A)$$

S

S

$Ax = 0$

A

W

$Ax = 0$

x

W

W

A

W

A

$N(A)$

A

$N(A)$

$Ax = 0$

A

A

$N(A)$

A

n

α

n

α

A

A

$R(A)$

y

$R(A)$

x

Ax

A

A

$$R(A^T)$$

$$R(A)$$

$$R(A^T)$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\ \boldsymbol{a}|\ |\ \boldsymbol{b}| \cos\theta$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + dhc + gbf - ahf - dbi - gec$$

$$\begin{vmatrix} i & j & k \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$$

$$|\ \boldsymbol{a} \times \boldsymbol{b}| = |\ \boldsymbol{a}|\ |\ \boldsymbol{b}| \sin\langle \boldsymbol{a}, \boldsymbol{b} \rangle$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = |\ \boldsymbol{a}|\ |\ \boldsymbol{b}| \cos\theta$$

$$(\boldsymbol{a} \times \boldsymbol{b})^2 = \boldsymbol{a}^2 \boldsymbol{b}^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2$$

$$(a, b, c) = (a \times b)c = \begin{vmatrix} a_x & b_x & c_x \\ a_y & b_y & c_y \\ a_z & b_z & c_z \end{vmatrix} = a_x b_y c_z + a_y b_z c_x + a_z b_x c_y - a_z b_y c_x - a_y b_x c_z - a_x b_z c_y$$

$$V = \frac{1}{6} |\ [ABACAD]|$$

$$(a, b, c) = (b, c, a) = (c, a, b) = -(b, a, c) = -(a, c, b) = -(c, b, a)$$

$$(a \times b)c = a(b \times c)$$

$$(a \times b) \times a = (a^2)b - (ab)a$$

$$(a \times b) \times a = \lambda a + \mu b$$

$$\lambda a^2 + \mu(ab) = 0$$

$$\lambda(ab) + \mu b^2 = (b, a \times b, a) = (a \times b)^2$$

$$(a \times b)^2 = a^2 b^2 - (ab)^2$$

$$\lambda = -ab$$

$$\mu = a^2$$

$$(a \times b) \times c = (ac)b - (bc)a$$

$$c = \alpha a + \beta b + \gamma(a \times b)$$

$$(a \times b) \times c = (a \times b) \times (\alpha a + \beta b + \gamma(a \times b)) = \alpha(a \times b) \times a + \beta(a \times b) \times b$$

$$(a \times b) \times a = (a^2)b - (ab)a$$

$$(a \times b) \times b = -(b \times a) \times b = -(b^2)a + (ab)b$$

$$(a \times b) \times c = \alpha((a^2)b - (ab)a) + \beta((ab)b - (b^2)a) = (\alpha(-ab) + \beta(-b^2))a + (\alpha a^2 + \beta ab)b = (ac)b - (bc)a$$

$$(a \times b) \times c = (ac)b - (bc)a$$

$$a \times (b \times c) = (ac)b - (ab)c$$

$$(a \times b)(c \times d) = (ac)(bd) - (ad)(bc)$$

$$(a \times b)(c \times d) = (c, d, a \times b) = (a \times b, c, d) = ((a \times b) \times c)d = (b(ac) - a(bc))d = (ac)(bd) - (ad)(bc)$$

$$(a \times b)^2 = a^2 b^2 - (ab)^2$$

$$\boldsymbol{a}, \boldsymbol{b}$$

$$\theta$$

$$|\ \boldsymbol{a}| \cos\theta$$

$$\boldsymbol{a}$$

$$\boldsymbol{b}$$

$$\boldsymbol{a} \cdot \boldsymbol{b}$$

$$\boldsymbol{a}$$

$$\boldsymbol{b}$$

$$\boldsymbol{a}$$

$$2$$

$$2$$

$$2$$

$$4$$

$$2$$

$$0$$

$$0$$

$$\boldsymbol{a} \perp \boldsymbol{b}$$

$$\iff$$

$$\boldsymbol{a} \cdot \boldsymbol{b} = 0$$

$$\boldsymbol{a} = \lambda \boldsymbol{b}$$

$$\iff$$

$$|\ \boldsymbol{a} \cdot \boldsymbol{b}| = |\ \boldsymbol{a}|\ |\ \boldsymbol{b}|$$

$$\boldsymbol{a}=(m,n),\boldsymbol{b}=(p,q),$$

$$\boldsymbol{a}\cdot\boldsymbol{b}=mp+nq$$

$$|\ \boldsymbol{a}|=\sqrt{m^2+n^2}$$

$$\cos\theta=\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\ \boldsymbol{a}|\ |\ \boldsymbol{b}|}$$

$$\boldsymbol{a},\boldsymbol{b}$$

$$\boldsymbol{a}\times\boldsymbol{b}$$

$$|\ \boldsymbol{a}\times\boldsymbol{b}|=|\ \boldsymbol{a}|\ |\ \boldsymbol{b}|\sin\langle\boldsymbol{a},\boldsymbol{b}\rangle$$

$$\boldsymbol{a}\times\boldsymbol{b}$$

$$\boldsymbol{a},\boldsymbol{b}$$

$$\boldsymbol{a},\boldsymbol{b},\boldsymbol{a}\times\boldsymbol{b}$$

$$S=\frac{1}{2}ab\sin C$$

$$|\ \boldsymbol{a}\times\boldsymbol{b}|$$

$$\boldsymbol{a},\boldsymbol{b}$$

$$a=(x_1,y_1,z_1)$$

$$b=(x_2,y_2,z_2)$$

$$c$$

$$c=a\times b$$

$$i,j,k$$

$$x,y,z$$

$$c=(y_1z_2-y_2z_1,x_2z_1-x_1z_2,x_1y_2-x_2y_1)$$

$$\boldsymbol{a}=(m,n),\boldsymbol{b}=(p,q)$$

$$(m,n)$$

$$(p,q)$$

$$(m,n,0)$$

$$(p,q,0)$$

$$(0,0,mq-np)$$

$$mq-np$$

$$\boldsymbol{b}$$

$$\boldsymbol{a}$$

$$0$$

0

0

a, b, c

$(a \times b)c$

a, b, c

$[abc]$

(a, b, c)

(abc)

$\mid (a \times b)c \mid$

a, b, c

(a, b, c)

$a \times b$

c

a

b

a

b

c

a

b

c

$(a, b, c) = 0$

$a \times b$

a

b

$a \times b$

a

b

a

b

$a \times b$

a

b

a

$$b$$

$$a \times b$$

$$\begin{cases} 7x_1 + 8x_2 + 9x_3 = 13 \\ 4x_1 + 5x_2 + 6x_3 = 12 \\ x_1 + 2x_2 + 3x_3 = 11 \end{cases}$$

$$\begin{pmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 13 \\ 12 \\ 11 \end{pmatrix}$$

$$Ax = b$$

$$\mathrm{diag}\{\lambda_1, \cdots, \lambda_n\}$$

$$C_{i,j} = \sum_{k=1}^{M} A_{i,k} B_{k,j}$$

$$\begin{pmatrix} 7 \\ 4 \\ 1 \end{pmatrix} x_1 + \begin{pmatrix} 8 \\ 5 \\ 2 \end{pmatrix} x_2 + \begin{pmatrix} 9 \\ 6 \\ 3 \end{pmatrix} x_3 = \begin{pmatrix} 13 \\ 12 \\ 11 \end{pmatrix}$$

$$[F_{n-1} \quad F_{n-2}] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = [F_n \quad F_{n-1}]$$

$$f_1 = f_2 = 0$$
$$f_n = 7f_{n-1} + 6f_{n-2} + 5n + 4 \times 3^n$$

$$[f_n \quad f_{n-1} \quad n \quad 3^n \quad 1]$$

$$[f_{n+1} \quad f_n \quad n+1 \quad 3^{n+1} \quad 1]$$

$$\begin{bmatrix} 7 & 1 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 \\ 12 & 0 & 0 & 3 & 0 \\ 5 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$[A \quad B \quad C \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ v & 0 & 0 & 1 \end{bmatrix} = [A + v \quad B \quad C \quad 1]$$

$$[A \quad B \quad C \quad 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [A \quad B \cdot v \quad C \quad 1]$$

$$[k \quad t \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ d & 0 & 1 \end{bmatrix} = [k + d \quad t \quad 1]$$

$$[k \quad t \quad 1] \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [k \quad t + d \times k \quad 1]$$

$$C_{k+1}[i,j] = \sum_{p=1}^{n} C_k[i,p] \cdot G[p,j]$$

$$C_{k+1} = C_k \cdot G$$

$$C_k = \underbrace{G \cdot G \cdots G}_{k□} = G^k$$

$$L_{k+1}[i,j] = \min_{1 \leq p \leq n} \{L_k[i,p] + G[p,j]\}$$

$$A \odot B = C \quad \Longleftrightarrow \quad C[i,j] = \min_{1 \leq p \leq n} \{A[i,p] + B[p,j]\}$$

$$L_{k+1} = L_k \odot G$$

$$L_k = \underbrace{G \odot \ldots \odot G}_{k□} = G^{\odot k}$$

$$A$$

$$A_{i,i}$$

$$I$$

$$n$$

$$n$$

$$i$$

$$j$$

$$i$$

$$j$$

$$j$$

$$i$$

$$0$$

$$\lambda_1, \cdots, \lambda_n$$

$$1$$

$$I$$

$$0$$

$$0$$

$$0$$

$$A$$

$$1$$

$$A$$

$$A$$

$$1$$

$$A$$
$$A$$
$$P \times M$$
$$B$$
$$M \times Q$$
$$C$$
$$A$$
$$B$$
$$C$$
$$i$$
$$j$$
$$C$$
$$i$$
$$j$$
$$A$$
$$i$$
$$M$$
$$B$$
$$j$$
$$M$$
$$i$$
$$j$$
$$A$$
$$i$$
$$B$$
$$j$$
$$A$$
$$P$$
$$A \times P = I$$
$$F_1 = F_2 = 1$$
$$F_i = F_{i-1} + F_{i-2} (i \geq 3)$$
$$n$$
$$n$$

$$10^{18}$$

$$\text{ans} = [F_2 \ \ F_1] = [1 \ \ 1], \text{base} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$F_n$$

$$\text{ans base}^{n-2}$$

$$[1 \ \ 1] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^{n-2} [1 \ \ 1]$$

$$n \leq 2$$

$$1$$

$$\text{base}$$

$$n - 2$$

$$n$$

$$F_1, F_2$$

$$F_3$$

$$n - 2$$

$$n$$

$$10^9 + 7$$

$$f_n$$

$$f_{n-1}, f_{n-2}, n$$

$$[f_n \ \ f_{n-1} \ \ n]$$

$$+1$$

$$n$$

$$n$$

$$A_i, B_i, C_i$$

$$i$$

$$1$$

$$m$$

$$[l, r]$$

$$3$$

$$7$$

$$A_i = A_i + B_i$$

$$B_i = B_i + C_i$$

$$C_i = C_i + A_i$$

$$A_i = A_i + B_i$$

$$B_i$$

$$v$$

$$A_i = A_i + v$$

$$B_i = B_i \cdot v$$

$$C_i = v$$

$$998244353$$

$$A_i = A_i + v$$

$$B_i = B_i \cdot v$$

$$n$$

$$1$$

$$k_i, t_i$$

$$0$$

$$\mathrm{Add}(x, d)$$

$$x$$

$$k_i \leftarrow k_i + d$$

$$\mathrm{Mul}(x, d)$$

$$x$$

$$t_i \leftarrow t_i + d \times k_i$$

$$\mathrm{Query}(x)$$

$$x$$

$$t_x$$

$$n,\ m \leq 100000,\ -10 \leq d \leq 10$$

$$n$$

$$1$$

$$k$$

$$(u, v)$$

$$u$$

$$v$$

$$k$$

$$G$$
$$(u \to v)$$
$$G[u,v] = 1$$
$$0$$
$$G[u,v]$$
$$k = 1$$
$$k$$
$$C_k$$
$$C_{k+1}$$
$$O(n^3 \log k)$$
$$n$$
$$k$$
$$(u,v)$$
$$u$$
$$v$$
$$k$$
$$G$$
$$G[i,j]$$
$$i$$
$$j$$
$$i,j$$
$$G[i,j] = \infty$$
$$k = 1$$
$$k$$
$$L_k$$
$$k+1$$
$$\odot$$
$$O(n^3 \log k)$$
$$k$$
$$n$$
$$1$$
$$k$$

$$(u, v)$$

$$u$$

$$v$$

$$k$$

$$1$$

$$k$$

$$k$$

$$0$$

$$\text{base}$$

$$\text{base}$$

$$3 \times 3$$

$$T(kx + ly) = kTx + lTy$$

$$T\alpha_j = a_{1j}\beta_1 + \cdots + a_{mj}\beta_n$$

$$T(\alpha_1, \cdots, \alpha_n) = (T\alpha_1, \cdots, T\alpha_n) = (\beta_1, \cdots, \beta_m)A$$

$$N(T) = \{x \in V \mid Tx = 0\}$$

$$R(T) = Im(T) = \{y \in W \mid y = Tx, Vx \in V\}$$

$$\dim N(T) + \dim R(T) = \dim V$$

$$T\alpha_j = a_{1j}\alpha_1 + \cdots + a_{nj}\alpha_n$$

$$T(\alpha_1, \cdots, \alpha_n) = (T\alpha_1, \cdots, T\alpha_n) = (\alpha_1, \cdots, \alpha_n)A$$

$$(T_1 + T_2)x = T_1x + T_2x$$

$$(kT_1)x = k(T_1x)$$

$$(T_1T_2)x = T_2(T_1x)$$

$$(T_1T_2)x = T_1(T_2x) = x$$

$$T_2 = T_1^{-1}$$

$$T_1T_2 = T_2T_1 = T_e$$

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n = (x_1, x_2, \cdots, x_n)\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

$$\xi = (\alpha_1, \cdots, \alpha_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$T\xi = T(\alpha_1, \cdots, \alpha_n)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$T\xi = T(\alpha_1, \cdots, \alpha_n)\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = (\alpha_1, \cdots, \alpha_n)A\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$y_i = (x_1, x_2, \cdots, x_n)\begin{pmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{ni} \end{pmatrix}$$

$$(y_1, y_2, \cdots, y_n) = (x_1, x_2, \cdots, x_n)A$$

$$z = (x_1, x_2, \cdots, x_n)\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = (y_1, y_2 \cdots, y_n)\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

$$(y_1, y_2 \cdots, y_n) = (x_1, x_2 \cdots, x_n)A$$

$$\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} = A\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix}$$

$$\begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} = A^{-1}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix}$$

$$Iy = Xa$$

$$T(\beta) = T(\alpha)C = \alpha AC$$

$$T(\beta) = \beta B = \alpha CB$$

$$B = C^{-1}AC$$

$$V$$

$$W$$

$$F$$

$$T$$

V

W

W

x

y

F

k

l

T

V

W

$W = V$

T

V

T_e

T_0

$L(V, W)$

V

W

$L(V, V)$

$L(V)$

V

n

V

$\alpha_1, \cdots, \alpha_n$

W

m

W

$\beta_1, \cdots, \beta_m$

T

V

W

α

T

β

A

T

T

V

W

$N(T)$

V

$R(T)$

W

$N(T)$

$R(T)$

V

$N(T)$

T

$R(T)$

T

T

V

W

V

$N(T)$

$R(T)$

T

V

V

n

V

$\alpha_1, \cdots, \alpha_n$

T

V

A

T

T

T

$T\alpha_1, \cdots, T\alpha_n$

T

A

V

n

$\alpha_1, \cdots, \alpha_n$

V

n

A

V

V

T

T

A

$L(V, V)$

n

$L(V)$

$L(V)$

$L(V)$

T_1

T_2

V

x

F

k

$L(V)$

F

$L(V)$

T_1

T_2

T_1

T_2

T_1T_2

(T_1T_2)

$L(V)$

$L(V)$

T_1

$L(V)$

T_2

V

x

T_2

T_1

V

n

$\alpha_1, \cdots, \alpha_n$

V

T_1

A

T_2

B

$T_1 + T_2$

$A + B$

kT_1

kA

T_1T_2

AB

T_1

A^{-1}

n

x

$$n$$
$$V$$
$$V$$
$$y$$
$$y$$
$$y$$
$$x_1, x_2, \cdots, x_n$$
$$V$$
$$n$$
$$L(V)$$
$$T$$
$$T$$
$$\alpha_1, \cdots, \alpha_n$$
$$A$$
$$V$$
$$V$$
$$x$$
$$I$$
$$x = Ix$$
$$T$$
$$T$$
$$V$$
$$T$$
$$V$$
$$V$$
$$T$$
$$n$$
$$x$$
$$n$$
$$y$$
$$V$$
$$1 \leq i \leq n$$
$$y_i$$
$$x_1, x_2, \cdots, x_n$$
$$n$$

y

n

A

A

$x_1, x_2\cdots, x_n$

$y_1, y_2\cdots, y_n$

$y_1, y_2\cdots, y_n$

$x_1, x_2\cdots, x_n$

A^{-1}

n

x

n

y

V

V

z

x

I

A

I

A

A

A

y

I

y

a

X

I

X

T

α

β

T

A

$$\beta = \alpha A$$

$$T$$

$$b$$

$$b$$

$$V$$

$$\beta b$$

$$V$$

$$\alpha A b$$

$$V$$

$$T$$

$$V$$

$$\alpha$$

$$T$$

$$\alpha$$

$$\alpha$$

$$T(\alpha)$$

$$\alpha$$

$$A$$

$$T(\alpha) = \alpha A$$

$$T$$

$$\beta$$

$$V$$

$$B$$

$$\alpha$$

$$V$$

$$A$$

$$C$$

$$\beta = \alpha C$$

$$B$$

$$A$$

$$T$$

$$T$$

$$\beta$$

$$\beta$$

$$\alpha$$

$$C$$

$$C$$

$$L(V)$$

$$T$$

$$T$$

$$A$$

$$B$$

$$C$$

$$B = C^{-1}AC$$

$$A$$

$$B$$

$$\beta = \alpha C$$

$$T(\alpha) = \alpha A$$

$$T(\beta) = \beta B$$

$$T(\beta) = T(\alpha)C$$

$$m_A(\lambda) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_k)^{r_k}$$

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

$$T_i = u_i(T)\frac{m_A(T)}{(T - \lambda_i T_e)^{r_i}}$$

$$T_1 + T_2 + \cdots + T_k = T_e$$

$$T_D = \lambda_1 T_1 + \lambda_2 T_2 + \cdots + \lambda_k T_k$$

$$T_N = T - T_D$$

$$T_N^{r_i}(\xi_i) = T - T_D^{r_i}(\xi_i) = T - \lambda_i T_i^{r_i}(\xi_i) = 0$$

$$T = T_D + T_N$$

$$T_D T_N = T_N T_D$$

$$T = T_D + T_N$$

$$T_D T_N = T_N T_D$$

$$A = D + N$$

$$DN = ND$$

$$\begin{pmatrix} D(\lambda) & 0 \\ 0 & 0 \end{pmatrix}$$

$$D(\lambda) = \begin{pmatrix} d_1(\lambda) & & \\ & \ddots & \\ & & d_r(\lambda) \end{pmatrix}$$

$$d_i(\lambda) = (\lambda - \lambda_1)^{e_{i1}} (\lambda - \lambda_2)^{e_{i2}} \cdots (\lambda - \lambda_S)^{e_{iS}}$$

$$d_i(\lambda) |\ d_{i+1}(\lambda)$$

$$\mathrm{diag}\{f_1(\lambda), f_2(\lambda), \cdots, f_r(\lambda), 0, \cdots, 0\}$$

$$\begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}$$

$$\begin{pmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix}$$

$$B_i = \begin{pmatrix} J_{i1} & & & 0 \\ & J_{i2} & & \\ & & \ddots & \\ 0 & & & J_{is_i} \end{pmatrix}$$

$$m_A(\lambda) = (\lambda - \lambda_1)^{r_1} (\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_k)^{r_k}$$

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

$$V_i = N((A - \lambda_i I)^{r_i})$$

$$S_i = \lambda_i T_e + T_i$$

$$V_i = W_{i1} \oplus W_{i2} \oplus \cdots \oplus W_{is_i}$$

$$N_i = \begin{pmatrix} N_{i1} & & & 0 \\ & N_{i2} & & \\ & & \ddots & \\ 0 & & & N_{is_i} \end{pmatrix}$$

$$B_i = \begin{pmatrix} \lambda_i & & & 0 \\ & \lambda_i & & \\ & & \ddots & \\ 0 & & & \lambda_i \end{pmatrix} + \begin{pmatrix} N_{i1} & & & 0 \\ & N_{i2} & & \\ & & \ddots & \\ 0 & & & N_{is_i} \end{pmatrix} = \begin{pmatrix} J_{i1} & & & 0 \\ & J_{i2} & & \\ & & \ddots & \\ 0 & & & J_{is_i} \end{pmatrix}$$

$$\begin{pmatrix} J_1 & & & 0 \\ & J_2 & & \\ & & \ddots & \\ 0 & & & J_m \end{pmatrix}$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda_i \end{pmatrix}$$

$$\mathrm{diag}\{d_1(\lambda), d_2(\lambda), \cdots, d_n(\lambda)\}$$

$$T$$

$$n$$

$$V$$

$$T$$

$$V$$

$$V_i = N((A - \lambda_i I)^{r_i})$$

$$A$$

$$T$$

$$T$$

$$T_i$$

$$V$$

$$V_i$$

$$u_i(T)$$

$$T_e$$

$$V$$

$$T_i$$

$$V_i$$

$$T_i|_{V_i}$$

$$V_i$$

$$i$$

$$j$$

$$T_i$$

$$V_j$$

$T_i|_{V_j}$

V_j

T_i

V

ξ

V_i

ξ_i

T_i

T

T_D

T

V_i

T_D

T_D

V_i

$T_D|_{V_i}$

V_i

λ_i

T_D

T_N

T

V_i

T_N

V_i

ξ_i

r

r_i

V

ξ

T_N

r

ξ

T_N

V

T

T_D

T_N

T_D

T_N

T

T_1

T_2

V

$T_1T_2 = T_2T_1$

T_1

T_2

T

n

V

T_D

T_N

T

T

T

T

T_D

T

T_N

T

A

n

D

N

A

A

A

A

D

A

N

A

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λ

λ

λ

λ

$A(\lambda)$

λ

A

$\lambda I - A$

λ

λ

λ

λ

λ

$\varphi(\lambda)$

λ

$A(\lambda)$

$B(\lambda)$

$A(\lambda)$

$B(\lambda)$

λ

λ

r

$A(\lambda)$

$d_i(\lambda)$

1

$d_i(\lambda)|\ d_{i+1}(\lambda)$

$d_i(\lambda)$

0

$A(\lambda)$

$B(\lambda)$

$A(\lambda)$

$B(\lambda)$

$A(\lambda)$

$d_1(\lambda), d_2(\lambda), \cdots, d_m(\lambda)$

$\lambda_1, \cdots, \lambda_S$

$e_{1j}, e_{2j}, \cdots, e_{mj}$

$d_m(\lambda)$

$A(\lambda)$

j

e_{ij}

$(\lambda - \lambda_j)^{e_{ij}}$

$A(\lambda)$

$B(\lambda)$

$A(\lambda)$

$B(\lambda)$

λ

$A(\lambda)$

$A(\lambda)$

$f_1(\lambda), f_2(\lambda), \cdots, f_r(\lambda)$

$(\lambda - \lambda_j)^{e_{ij}}$

$A(\lambda)$

1

r

A

B

λ

A

B

$\lambda I - A$

$\lambda I - B$

$\lambda I - A$

n

λ

n

A

B

$\lambda I - A$

$\lambda I - B$

$\lambda I - A$

$\lambda I - A$

n

n

$\lambda I - A$

n

λ

1

0

λ

λ

0

T

n

V

$\lambda_1, \cdots, \lambda_k$

T

T

$J_{i1}, \cdots, J_{is_i}$

λ_i

A

T

S_i

T

V_i

$T \mid_{V_i}$

S_i

T_e

V

T_i

S_i

T_i

V_i

$T - \lambda_i T_e$

V_i

$(T - \lambda_i T_e)|_{V_i}$

V_i

T_i

W_{ij}

V_i

T_i

N_{ij}

V_i

S_i

$J_{i1}, J_{i2}, \cdots, J_{is_i}$

λ_i

V_i

V

T

n

J_i

n

A

A

A

B_i

B_i

B_i

0

A

$\lambda I - A$

$(\lambda - \lambda_i)^{n_i}$

$\lambda I - A$

A

A

$\lambda I - A$

A

A

$\lambda I - A$

n

n

A

$\lambda I - A$

$d_n(\lambda)$

A

$m_A(\lambda)$

A

$m_A(\lambda)$

$\lambda I - A$

$\lambda I - A$

k, x_0

$k \sin(x - x_0)$

$$k_1 \sin x + k_2 \cos x$$

$$D_i(k) = \mathrm{diag}\{1, \cdots, 1, k, 1, \cdots, 1\}$$

$$P_{ij} = \begin{pmatrix} I_{i-1} & & & & \\ & 0 & & 1 & \\ & & I_{j-i-1} & & \\ & 1 & & 0 & \\ & & & & I_{n-j} \end{pmatrix}$$

$$T_{ij}(k) = \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & 1 & \cdots & k & & \\ & & & \ddots & \vdots & & \\ & & & & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{pmatrix}$$

$$|\ D_i(k)| = k$$

$$|\ P_{ij}| = -1$$

$$|\ T_{ij}(k)| = 1$$

$$D_i(k)^{-1} = D_i\left(\frac{1}{k}\right)$$

$$P_{ij}^{-1} = P_{ij}$$

$$T_{ij}(k)^{-1} = T_{ij}(-k)$$

$$i$$

$$k$$

$$k$$

$$0$$

$$1$$

$$k$$

$$1$$

$$D_i(1)$$

$$I$$

$$1$$

$$0$$

$$1$$

$$i$$

$$j$$

0

i

j

j

i

1

i

j

I

i

j

k

i

j

k

0

$T_{ij}(0)$

I

A

i

k

$B \mapsto D_i(k)B$

i

j

$B \mapsto P_{ij}B$

j

k

i

$B \mapsto T_{ij}(k)B$

I

A

A

A

A

A

$D_i(k)$

i

k

$D_i(k)$

i

k

k

0

0

0

$D_i(k)$

k

k

A

P_{ij}

i

j

P_{ij}

i

j

1

0

I

I

I

I

I

-1

-1

$(-1)^p$

p

$T_{ij}(k)$

j

k

i

$T_{ij}(k)$

i

k

j

$T_{ij}(k)$

I

1

A

1

I

I

0

I

0

0

0

0

0

0

0

I

0

1

1

A

n

n

B

$AB = BA = I$

A

B

A

A^{-1}

A

A

A

A

A^{-1}

A^{-1}

A

A

B

AB

$B^{-1}A^{-1}$

A

A^T

I

I

A

A

I

A

A

n

A

0

E_{ij}

i

j

1

$n \times n$

$$D_i(k) = I_n + (k-1)E_{ii}$$

$$P_{ij} = I_n - E_{ii} - E_{jj} + E_{ij} + E_{ji}$$

$$T_{ij}(k) = I_n + kE_{ij}$$

$$(\lambda_0 I - A)X = 0$$

$$r(\lambda_i I - A) + \dim N(\lambda_i I - A) = n$$

$$\dim E(\lambda_i) = n - r(\lambda_i I - A)$$

$$m_A(\lambda) = (\lambda - \lambda_1)^{r_1} \cdots (\lambda - \lambda_S)^{r_S}$$

$$W_i = N((\lambda_i I - A)^{r_i})$$

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_S$$

$$\mathrm{diag}\{A_1, A_2, \cdots, A_S\}$$

$$\dim E(\lambda_1) + \cdots + \dim E(\lambda_m) = n$$

$$T^r(\xi_0) = 0$$

$$T^{r-1}(\xi_0) \neq 0$$

$$T^s(\xi) = 0$$

$$T^{s-1}(\xi) \neq 0$$

$$T^r(\xi_0) = 0$$

$$N_r = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

$$h(x) = a_0 + a_1 x + \cdots + a_m x^m$$

$$T^{r-k}(\xi) = 0$$

$$\xi = T^k(\eta)$$

$$V = W_1 \oplus W_2$$

$$V = W_1 \oplus W_2 \oplus \cdots \oplus W_S$$

$$N = \begin{pmatrix} N_{r_1} & & & 0 \\ & N_{r_2} & & \\ & & \cdots & \\ 0 & & & N_{r_S} \end{pmatrix}$$

$$A$$

$$\lambda_0$$

$$A$$

$$E(\lambda_0)$$

$$E(\lambda_i) = N(\lambda_i I - A)$$

$$E(\lambda_i)$$

$$\lambda_i$$

$$T$$

$$V$$

$$T$$

$$V$$

$$F$$

$$W$$

$$V$$

$$T$$

$$V$$

$$W$$

$$x$$

$$T(x)$$

$$W$$

$$W$$

$$T$$

$$V$$

$$V$$

$$T$$

$$V$$

$$T$$

$$W$$

$$T$$

$$T$$

$$W$$

$$W$$

$$T$$

$$W$$

$$T\mid_W$$

$$V$$

$$T$$

$$R(T)$$

$$N(T)$$

$$T$$

$$V$$

$$0$$

$$V$$

$$T$$

$$T$$

$$T$$

$$\lambda$$

$$A$$

$$A$$

$$W_i$$

$$\lambda_i$$

$$E(\lambda_i) = N(\lambda_i I - A)$$

$$W_i$$

$$E(\lambda_i)$$

$$r_i$$

$$r_i$$

$$A$$

$$T$$

$$W_i$$

$$T_i = T\mid_{W_i}$$

$$T_i$$

$$(x-\lambda_i)^{r_i}$$

$$V$$

$$F$$

$$T$$

$$V$$

V

T

W_i

T

A_i

T_i

n

A

A

T

T

A

$\lambda_1, \cdots, \lambda_m$

A

A

n

A

A

λ

n

A

n

A

A

n

T

V

r

T^r

T

V

r

$N^r = 0$

r

T^r

T

x^r

ξ_0

T

V

ξ

V

s

$\xi, T(\xi), \cdots, T^{s-1}(\xi)$

T

V

W

V

ξ_0

r

$\xi_0, T(\xi_0), \cdots, T^{r-1}(\xi_0)$

W

W

T

T

ξ_0

W

$\xi_0, T(\xi_0), \cdots, T^{r-1}(\xi_0)$

W

T

W

T

W

ξ

$T^r(\xi) = 0$

r

W

T

T

W

$T\ |_W$

W

$T\ |_W$

W

$T^{r-1}(\xi_0), T^{r-2}(\xi_0), \cdots, \xi_0$

r

N_r

r

r

T

n

V

V

T

$r_1 \geq \cdots \geq r_S$

T

n

A

A

N

$r_1 \geq \cdots \geq r_S$

A

T

V

$a_0 \neq 0$

$h(T)$

$h(T)$

$h(T)$

T

T

V

W

r

T

ξ

W

k

W

η

T

n

V

x^r

T

W_1

r

T

W_1

W_2

W_2

T

T

n

V

V

T

n

N_{r_i}

r_i

T

W_i

r_i

$r_1 \geq \cdots \geq r_S$

$$V$$

$$T$$

$$T$$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$a_{i_1 j_1} a_{i_2 j_2} \cdots a_{i_n j_n}$$

$$s = \pi(i_1 i_2 \cdots i_n)$$

$$t = \pi(j_1 j_2 \cdots j_n)$$

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_{i1} + c_{i1} & b_{i2} + c_{i2} & \cdots & b_{in} + c_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$\det A = a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in}$$

$$\det A = a_{1j} A_{1j} + a_{2j} A_{2j} + \cdots + a_{nj} A_{nj}$$

$$a_{i1} A_{j1} + a_{i2} A_{j2} + \cdots + a_{in} A_{jn} = 0$$

$$a_{1i} A_{1j} + a_{2i} A_{2j} + \cdots + a_{ni} A_{nj} = 0$$

$$A$$

$$\det A$$

$$A$$

$$\pi(j_1 j_2 \cdots j_n)$$

$$j_1 j_2 \cdots j_n$$

$$n$$

$$n!$$

$$A$$

$$n$$

$$a_{1 j_1} a_{2 j_2} \cdots a_{n j_n}$$

$$a_{1 j_1} a_{2 j_2} \cdots a_{n j_n}$$

$$(-1)^{\pi(j_1 j_2 \cdots j_n)}$$

$$j_1 j_2 \cdots j_n$$

$$j_1 j_2 \cdots j_n$$

$$n$$

$i_1, i_2, \cdots, i_n$

$j_1, j_2, \cdots, j_n$

$i_1, i_2, \cdots, i_n$

$j_1, j_2, \cdots, j_n$

$1, 2, \cdots, n$

n

$(-1)^{s+t}$

$\det A$

i

$\det A_1$

$\det A_2$

A_1

i

$b_{i1}, b_{i2}, \cdots, b_{in}$

A_2

i

$c_{i1}, c_{i2}, \cdots, c_{in}$

A_1

A_2

A

n

$\det A$

k

k

k

k

n

$\det A$

a_{ij}

M_{ij}

a_{ij}

$n-1$

$$n$$
$$\det A$$
$$a_{ij}$$
$$M_{ij}$$
$$(-1)^{i+j}$$
$$a_{ij}$$
$$A_{ij}$$
$$n$$
$$\det A$$
$$i$$
$$j$$
$$a_{ij}$$
$$0$$
$$a_{ij}$$
$$A_{ij}$$
$$\det A$$
$$\det A$$
$$0$$
$$i \neg j$$
$$O(n^3)$$
$$n$$
$$A$$
$$k$$
$$k$$
$$0$$
$$0$$
$$0$$
$$0$$
$$T\xi = \lambda\xi$$
$$T(\alpha_1, \alpha_2, \cdots, \alpha_n) = (\alpha_1, \alpha_2, \cdots, \alpha_n)A$$
$$\xi = (\alpha_1, \alpha_2, \cdots, \alpha_n)X$$
$$T\xi = \lambda_0\xi$$

$$T(\alpha_1, \alpha_2, \cdots, \alpha_n)X = \lambda_0(\alpha_1, \alpha_2, \cdots, \alpha_n)X$$

$$(\alpha_1, \alpha_2, \cdots, \alpha_n)AX = \lambda_0(\alpha_1, \alpha_2, \cdots, \alpha_n)X$$

$$AX = \lambda_0 X$$

$$(A - \lambda_0 I)X = 0$$

$$p_A(\lambda) = \det(\lambda I_n - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

$$\xi = (\alpha_1, \cdots, \alpha_n)X$$

$$f(\lambda) = \mid \lambda I - A \mid = (\lambda - \lambda_1)^{d_1} \cdots (\lambda - \lambda_m)^{d_m}$$

$$k_1 X_1 + k_2 X_2 + \cdots + k_t X_t$$

$$\xi_i = (\alpha_1, \cdots, \alpha_n)X_i$$

$$k_1 \xi_1 + k_2 \xi_2 + \cdots + k_t \xi_t$$

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ & a_{2,2} & \cdots & a_{2,n} \\ & & \ddots & \vdots \\ & & & a_{n,n} \end{bmatrix}$$

$$\begin{aligned} p_A(x) &= \det(xI_n - A) \\ &= \begin{bmatrix} x - a_{1,1} & -a_{1,2} & \cdots & -a_{1,n} \\ & x - a_{2,2} & \cdots & -a_{2,n} \\ & & \ddots & \vdots \\ & & & x - a_{n,n} \end{bmatrix} \\ &= \prod_{i=1}^{n} (x - a_{i,i}) \end{aligned}$$

$$B = P^{-1}AP$$

$$\begin{aligned} \det(xI_n - P^{-1}AP) &= \det(xP^{-1}I_nP - P^{-1}AP) \\ &= \det(P^{-1}xI_nP - P^{-1}AP) \\ &= \det(P^{-1}) \cdot \det(P) \cdot \det(xI_n - A) \\ &= \det(xI_n - A) \\ &= p_A(x) \end{aligned}$$

$$-(\lambda_1 + \cdots + \lambda_n) = -(a_{11} + \cdots + a_{nn}) = -trA$$

$$(-1)^n \mid A \mid = (-1)^n (\lambda_1 \cdots \lambda_n)$$

$$\lambda^n \mid \lambda I_m - AB \mid = \lambda^m \mid \lambda I_n - BA \mid$$

$$\phi(\lambda_1), \cdots, \phi(\lambda_n)$$

$$H = \begin{bmatrix} \alpha_1 & h_{12} & \cdots & \cdots & h_{1n} \\ \beta_2 & \alpha_2 & h_{23} & \cdots & \vdots \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & h_{(n-1)n} \\ & & & \beta_n & \alpha_n \end{bmatrix}$$

$$H_0 = [], \quad p_0(x) = 1$$

$$H_1 = [\alpha_1], \quad p_1(x) = \det(xI_1 - H_1) = x - \alpha_1$$

$$H_2 = \begin{bmatrix} \alpha_1 & h_{12} \\ \beta_2 & \alpha_2 \end{bmatrix}, \quad p_2(x) = \det(xI_2 - H_2) = (x - \alpha_2)p_1(x) - \beta_2 h_{12} p_0(x)$$

$$\begin{aligned} p_3(x) &= \det(xI_3 - H_3) \\ &= \begin{vmatrix} x - \alpha_1 & -h_{12} & -h_{13} \\ -\beta_2 & x - \alpha_2 & -h_{23} \\ & -\beta_3 & x - \alpha_3 \end{vmatrix} \\ &= (x - \alpha_3) \cdot (-1)^{3+3} p_2(x) - \beta_3 \cdot (-1)^{3+2} \begin{vmatrix} x - \alpha_1 & -h_{13} \\ -\beta_2 & -h_{23} \end{vmatrix} \\ &= (x - \alpha_3)p_2(x) - \beta_3(h_{23}p_1(x) + \beta_2 h_{13} p_0(x)) \end{aligned}$$

$$p_i(x) = (x - \alpha_i)p_{i-1}(x) - \sum_{m=1}^{i-1} h_{i-m,i} \left(\prod_{j=i-m+1}^{i} \beta_j \right) p_{i-m-1}(x)$$

$$\begin{aligned} p_A(A) &= A^n + c_1 A^{n-1} + \cdots + c_{n-1}A + c_n I \\ &= O \end{aligned}$$

$$\begin{aligned} f_{km-1}x^{km-1} + \cdots + f_1 x + f_0 = (\cdots(&f_{km-1}x^{k-1} + \cdots + f_{k(m-1)})x^k \\ &+ f_{k(m-1)-1}x^{k-1} + \cdots + f_{k(m-2)})x^k \\ &+ \cdots \\ &+ f_{k-1}x^{k-1} + \cdots + f_1 x + f_0 \end{aligned}$$

$$A$$

$$n$$

$$n$$

$$A$$

$$I$$

$$A$$

$$A$$

$$A$$

$$V$$

$$F$$

$$T$$
$$V$$
$$F$$
$$\lambda$$
$$V$$
$$\xi$$
$$\lambda$$
$$T$$
$$\xi$$
$$T$$
$$\lambda$$
$$\alpha_1, \alpha_2, \cdots, \alpha_n$$
$$V$$
$$T$$
$$A$$
$$\lambda_0$$
$$T$$
$$\xi$$
$$T$$
$$\lambda_0$$
$$X$$
$$0$$
$$n \times n$$
$$A$$
$$n \geq 0 \wedge n \in \mathbb{Z}$$
$$\lambda$$
$$\lambda I - A$$
$$A$$
$$A$$
$$n$$
$$A$$
$$p_A(\lambda)$$
$$I_n$$

$$n \times n$$
$$p_A(\lambda) = \det(A - \lambda I_n)$$
$$(-1)^n$$
$$p_A(\lambda)$$
$$n$$
$$0 \times 0$$
$$1$$
$$(\lambda_0 I - A)X = 0$$
$$X$$
$$A$$
$$\lambda_0$$
$$T$$
$$\lambda_0$$
$$A$$
$$\lambda_0$$
$$T$$
$$\xi$$
$$A$$
$$X$$
$$d_i$$
$$\lambda_i$$
$$n$$
$$\mid \lambda I - A \mid$$
$$f(\lambda) = \mid \lambda I - A \mid$$
$$F$$
$$A$$
$$A$$
$$\lambda$$
$$(\lambda I - A)X = 0$$
$$X_1, \cdots, X_t$$
$$A$$
$$\lambda$$

$$k_i$$

$$T$$

$$\lambda$$

$$\lambda$$

$$k_i$$

$$V$$

$$n \times n$$

$$A$$

$$A$$

$$n \times n$$

$$A$$

$$B$$

$$n \times n$$

$$P$$

$$A$$

$$B$$

$$A \mapsto P^{-1}AP$$

$$A$$

$$P^{-1}AP$$

$$A \mapsto PAP^{-1}$$

$$p_A(0) = (-1)^n \cdot \det(A)$$

$$p_A(0) = \det(-1 \cdot I_n A) = \det(-1 \cdot I_n) \cdot \det(A)$$

$$\det(A) = \det(P^{-1}AP)$$

$$A$$

$$f(\lambda) = \mid \lambda I - A \mid$$

$$n - 1$$

$$trA$$

$$A$$

$$A$$

$$A$$

$$B$$

$$AB$$

$$BA$$

A

m

n

B

n

m

AB

BA

n

A

P

$P^{-1}AP$

A

A

n

$\lambda_1, \cdots, \lambda_n$

$\phi(x)$

$\phi(A)$

n

kA

$k\lambda_1, \cdots, k\lambda_n$

A^m

$\lambda_1^m, \cdots, \lambda_n^m$

$n \times n$

B

$A \mapsto T_{ij}(k)AT_{ij}(-k)$

$A \mapsto T_{ij}(k)A$

i

j

$T_{ij}(-k)$

A

i

$-k$

j

$n > 2$

β

$n \times n$

$O(n^3)$

H_i

H

i

i

$p_i(x) = \det(xI_i - H_i)$

$2 \leq i \leq n$

n

A

$f(\lambda) = \mid \lambda I - A \mid$

$f(A) = 0$

T

$f(\lambda)$

T

$f(T)$

A

V

n

n^2

n^2

T

0

n

$n^2 + 1$

f

$f(T)$

T

f

$$T$$

$$f$$

$$A$$

$$A$$

$$m_A(\lambda)$$

$$A$$

$$A$$

$$f(\lambda)$$

$$m_A(\lambda)$$

$$A$$

$$(\mathbb{Z}/m\mathbb{Z})^{n\times n}$$

$$m$$

$$m$$

$$\mathbb{R}^{n\times n}$$

$$O$$

$$n \times n$$

$$A \in \mathbb{C}^{n\times n}$$

$$p_A(x) = x^n + \sum_{i=1}^{n} c_i x^{n-i} \in \mathbb{C}[x]$$

$$A$$

$$A^K$$

$$K$$

$$f(x) = x^K \bmod p_A(x)$$

$$f(A) = A^K$$

$$\deg(f(x)) < n$$

$$f(x) = \sum_{i=0}^{n-1} f_i x^i$$

$$n = km$$

$$k = \sqrt{n}$$

$$f(A)$$

$$O(\sqrt{n})$$

$$n = \operatorname{rank}\{b_1, b_2, ..., b_n\} \leq \operatorname{rank}\{a_1, a_2, ..., a_n\} \leq n$$

$$V$$

$$V$$

$$\{\theta\}$$

$$V$$

$$\dim V$$

$$V$$

$$n$$

$$V$$

$$n+1$$

$$V$$

$$n$$

$$V$$

$$V$$

$$a_1, a_2, ..., a_n$$

$$V$$

$$V$$

$$b_1, b_2, ..., b_n$$

$$b_1, b_2, ..., b_n$$

$$a_1, a_2, ..., a_n$$

$$\operatorname{rank}\{a_1, a_2, ..., a_n\} = n$$

$$V$$

$$a_1, a_2, ..., a_m$$

$$V$$

$$V_1, V_2$$

$$\mathbb{P}$$

$$V_1 + V_2$$

$$V_1 \cap V_2$$

$$\dim V_1 + \dim V_2 = \dim(V_1 + V_2) + \dim(V_1 \cap V_2)$$

$$\dim V_1 = n_1$$

$$\dim V_2 = n_2$$

$$\dim(V_1 \cap V_2) = m$$

$$V_1 \cap V_2$$

$$a_1, a_2, ..., a_m$$

$$V_1$$

$$V_2$$

$$a_1, a_2, ..., a_m, b_1, b_2, ..., b_{n_1-m}$$

$$a_1, a_2, ..., a_m, c_1, c_2, ..., c_{n_2-m}$$

$$a_1, a_2, ..., a_m, b_1, b_2, ..., b_{n_1-m}, c_1, c_2, ..., c_{n_2-m}$$

$$\sum_{i=1}^{m} r_i a_i + \sum_{i=1}^{n_1-m} s_i b_i + \sum_{i=1}^{n_2-m} t_i c_i = \theta$$

$$\sum_{i=1}^{n_2-m} t_i c_i = -\sum_{i=1}^{m} r_i a_i - \sum_{i=1}^{n_1-m} s_i b_i$$

$$V_2$$

$$V_1$$

$$V_1 \cap V_2$$

$$\sum_{i=1}^{n_2-m} t_i c_i = \sum_{i=1}^{m} k_i a_i$$

$$t_1 = t_2 = ... = t_{n_2-m} = k_1 = k_2 = ... = k_m = 0$$

$$r_1 = r_2 = ... = r_m = s_1 = s_2 = ... = s_{n_1-m} = t_1 = t_2 = ... = t_{n_2-m} = 0$$

$$V_1, V_2$$

$$\mathbb{P}$$

$$V_1 + V_2$$

$$V_1 \cap V_2$$

$$V_1 + V_2 = V_1 \oplus V_2$$

$$\dim V_1 + \dim V_2 = \dim(V_1 + V_2)$$

$$a_1, a_2, ..., a_n$$

$$V_1$$

$$b_1, b_2, ..., b_m$$

$$V_2$$

$$a_1, a_2, ..., a_n, b_1, b_2, ..., b_m$$

$$V_1 + V_2$$

$$\mathbb{R}^2$$

$$u, v$$

$$u, v$$

$$u, v$$

$$u = -v$$

$$u, v, w$$

$$u + 4v + 6w = \theta$$

$$\mathbb{Z}_2^n$$

$$\mathbb{R}^n$$

$$p$$

$$x$$

$$1$$

$$a_x$$

$$a_x \leftarrow p$$

$$p \leftarrow p \ \text{ xor } a_x$$

$$\text{xor}$$

$$\text{xor}$$

$$a_x$$

$$a_x$$

$$i$$

$$i$$

$$1$$

$$i$$

$$\text{xor}$$

$$p$$

$$0$$

$$0$$

$$k$$

$$n$$

$$m$$

$$O(nm)$$

$$O(n^2 m)$$

$$T$$

$$\bigoplus_{i=1}^n a_i$$

$$a_i = 1$$

$$a_i > 1$$

$$T \neq 0$$

$$a_i > 1$$

$$T$$

$$\neq 0$$

$$a_i = 1$$

$$T$$

$$2$$

$$a_i > 1$$

$$T \neq 0$$

$$T = 0$$

$$T = 0$$

$$2$$

$$a_i > 1, T \neq 0$$

$$2$$

$$a_i > 1, T \neq 0$$

$$T = 0$$

$$2$$

$$a_i > 1, T \neq 0$$

$$\operatorname{mex}(S) = \min\{x\} \quad (x \notin S, x \in N)$$

$$\operatorname{SG}(x) = \operatorname{mex}\{\operatorname{SG}(y_1), \operatorname{SG}(y_2), ..., \operatorname{SG}(y_k)\}$$

$$n$$

$$a_i$$

$$n = 3$$

$$2, 5, 4$$

$$1$$

$$2$$

$$0, 5, 4$$

$$2$$

$$4$$

$$2, 1, 4$$

$$0, 0, 5$$

$$3$$

$$5$$

$$(i, j, k)$$

$$i, j, k$$

$$O(N + M)$$

$$N$$

$$M$$

$$O(\prod_{i=1}^{n} a_i)$$

$$= a_1 \oplus a_2 \oplus ... \oplus a_n$$

$$0$$

$$a_1 \oplus a_2 \oplus ... \oplus a_n \neq 0$$

$$a_1 \oplus a_2 \oplus ... \oplus a_n = 0$$

$$a_1 \oplus a_2 \oplus ... \oplus a_n = 0$$

$$a_1 \oplus a_2 \oplus ... \oplus a_n = 0$$

$$0$$

$$a_1 \oplus a_2 \oplus ... \oplus a_n = 0$$

$$a_1 \oplus a_2 \oplus ... a_n = k \neq 0$$

$$a_i$$

$$a_{(i)'}$$

$$a_{(i)'} = a_i \oplus k$$

$$k$$

$$1$$

$$d$$

$$2^d \leq k < 2^{d+1}$$

$$a_i$$

$$d$$

$$a_i$$

$$a_i > a_i \oplus k$$

$$a_i$$

$$a_{(i)'}$$

$$a_i = a_{(i)'}$$

$$\mathrm{mex}$$

$$S$$

$$\mathrm{mex}(\{0,2,4\}) = 1$$

$$\mathrm{mex}(\{1,2\}) = 0$$

$$x$$

$$k$$

$$y_1, y_2, ..., y_k$$

$$\mathrm{SG}$$

$$n$$

$$s_1, s_2, ..., s_n$$

$$\mathrm{SG}(s_1) \oplus \mathrm{SG}(s_2) \oplus ... \oplus \mathrm{SG}(s_n) \neq 0$$

$$x$$

$$x'$$

$$s_{(1)'}, s_{(2)'}, ..., s_{(n)'}$$

$$i$$

$$s_{(i)'} < s_i$$

$$\mathrm{SG}(s_1)' = \mathrm{SG}(s_2)' = ... \mathrm{SG}(s_n)' = 0$$

$$x$$

$$s_{(1)'}, s_{(2)'}, ..., s_{(n)'}$$

$$s_i$$

$$\mathrm{SG}$$

$$s_i$$

$$\mathrm{SG}$$

$$\mathrm{SG}$$

$$\mathrm{SG}(s_i)$$

$$x'$$

$$\mathrm{SG}(x') = \mathrm{SG}(s_{(1)'}) \oplus \mathrm{SG}(s_{(2)'}) \oplus ... \oplus \mathrm{SG}(s_{(n)'}) < \mathrm{SG}(X)$$

$$x'$$

$$\mathrm{SG}(s_1) \oplus \mathrm{SG}(s_2) \oplus ... \oplus \mathrm{SG}(s_n)$$

$$x$$

$$\mathrm{mex}$$

$$x$$

$$x$$

$$y < x$$

$$x$$

$$y$$

$$n$$

$$n$$

$$\mathrm{SG}(x) = x$$

$$\sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}$$

$$\begin{aligned}
\sum_{k=0}^{n+m} \binom{n+m}{k} x^k &= (x+1)^{n+m} \\
&= (x+1)^n (x+1)^m \\
&= \sum_{r=0}^{n} \binom{n}{r} x^r \sum_{s=0}^{m} \binom{m}{s} x^s \\
&= \sum_{k=0}^{n+m} \sum_{r=0}^{k} \binom{n}{r} \binom{m}{k-r} x^k
\end{aligned}$$

$$\binom{n+m}{k} = \sum_{r=0}^{k} \binom{n}{r} \binom{m}{k-r}$$

$$\sum_{i=-r}^{s} \binom{n}{r+i} \binom{m}{s-i} = \binom{n+m}{r+s}$$

$$\sum_{i=1}^{n} \binom{n}{i} \binom{n}{i-1} = \binom{2n}{n-1}$$

$$\sum_{i=1}^{n} \binom{n}{i} \binom{n}{i-1} = \sum_{i=0}^{n-1} \binom{n}{i+1} \binom{n}{i} = \sum_{i=0}^{n-1} \binom{n}{n-1-i} \binom{n}{i} = \binom{2n}{n-1}$$

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}$$

$$\sum_{i=0}^{n} \binom{n}{i}^2 = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i} = \binom{2n}{n}$$

$$\sum_{i=0}^{m} \binom{n}{i} \binom{m}{i} = \binom{n+m}{m}$$

$$\sum_{i=0}^{m} \binom{n}{i} \binom{m}{i} = \sum_{i=0}^{m} \binom{n}{i} \binom{m}{m-i} = \binom{n+m}{m}$$

$$n+m$$

$$k$$

$$n+m$$

$$n$$

$$m$$

$$n$$

$$i$$

$$m$$

$$k-i$$

$$i$$

$$\binom{n+m}{m}$$

$$(0,0)$$

$$(n,m)$$

$$n+m$$

$$(0,0)$$

$$n$$

$$m$$

$$\binom{n+m}{m}$$

$$n+m$$

$$n$$

$$m$$

$$n$$

$$i$$

$$m$$

$$m-i$$

$$\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k \left\{ {n-1 \atop k} \right\}$$

$$\left\{ {n \atop m} \right\} = \sum_{i=0}^{m} \frac{(-1)^{m-i} i^n}{i!(m-i)!}$$

$$\begin{aligned} G_i &= i^n \\ G_i &= \sum_{j=0}^{i} \binom{i}{j} F_j \end{aligned}$$

$$\begin{aligned} F_i &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} G_j \\ &= \sum_{j=0}^{i} (-1)^{i-j} \binom{i}{j} j^n \\ &= \sum_{j=0}^{i} \frac{i!(-1)^{i-j} j^n}{j!(i-j)!} \end{aligned}$$

$$\left\{ {n \atop m} \right\} = \frac{F_m}{m!} = \sum_{i=0}^{m} \frac{(-1)^{m-i} i^n}{i!(m-i)!}$$

$$\left[{n \atop k} \right] = \left[{n-1 \atop k-1} \right] + (n-1) \left[{n-1 \atop k} \right]$$

$$x^{\overline{n}} = \sum_k \left[{n \atop k} \right] x^k$$

$$x^n = \sum_k \left\{ {n \atop k} \right\} (-1)^{n-k} x^{\overline{k}}$$

$$x^n = \sum_k \left\{ {n \atop k} \right\} x^{\underline{k}}$$

$$x^{\underline{n}} = \sum_k \left[{n \atop k} \right] (-1)^{n-k} x^k$$

$$f(x) = \sum_{i=0}^{n} b_i x^{\underline{i}}$$

$$(i, a_i), i = 0..n$$

$$a_k = \sum_{i=0}^{n} b_i k^{\underline{i}}$$

$$a_k = \sum_{i=0}^{n} \frac{b_i k!}{(k-i)!}$$

$$\frac{a_k}{k!} = \sum_{i=0}^{k} b_i \frac{1}{(k-i)!}$$

$$\left\{ {n \atop k} \right\}$$

$$S(n, k)$$

$$n$$

$$k$$

$$\left\{ {n \atop 0} \right\} = [n = 0]$$

$$\left\{ {n-1 \atop k-1} \right\}$$

$$k \left\{ {n-1 \atop k} \right\}$$

$$n$$

$$i$$

$$G_i$$

$$n$$

$$i$$

$$F_i$$

$$F_i$$

$$\left\{ {n \atop i} \right\}$$

$$F_i$$

$$\left\{ {n \atop i} \right\}$$

$$i!$$

$$i$$

$$n$$

$$\left\{ {n \atop i} \right\}$$

$$i = 0..n$$

$$n$$

$$i$$

$$O(n \log n)$$

$$i$$

$$k$$

$$\left\{ {i \atop k} \right\}$$

$$i = 0..n$$

$$i$$

$$k$$

$$i$$

$$[i > 0]$$

$$F(x) = \sum_{i=1}^{+\infty} \frac{x^i}{i!} = \mathrm{e}^x - 1$$

$$F^k(x)$$

$$i$$

$$k$$

$$k!$$

$$i$$

$$k$$

$$\left\{ {i \atop k} \right\} = \frac{\left[\frac{x^i}{i!}\right] F^k(x)}{k!}$$

$$O(n \log n)$$

$$\exp F(x) = \sum_{i=0}^{+\infty} \frac{F^i(x)}{i!}$$

$$i$$

$$i!$$

$$\left[{n \atop k} \right]$$

$$s(n, k)$$

$$n$$

$$k$$

$$[A, B, C, D]$$

$$[A, B, C, D] = [B, C, D, A] = [C, D, A, B] = [D, A, B, C]$$

$$[A, B, C, D] \neq [D, C, B, A]$$

$$\left[{n \atop 0} \right] = [n = 0]$$

$$\left[{n-1 \atop k-1} \right]$$

$$(n-1) \left[{n-1 \atop k} \right]$$

$$F_n(x) = \sum_{i=0}^{n} \left[{n \atop i} \right] x^i$$

$$F_n(x) = (n-1)F_{n-1}(x) + xF_{n-1}(x)$$

$$F_n(x) = \prod_{i=0}^{n-1} (x+i) = \frac{(x+n-1)!}{(x-1)!}$$

$$x$$

$$n$$

$$x^{\overline{n}}$$

$$O(n \log^2 n)$$

$$O(n \log n)$$

$$F(x) = \sum_{i=1}^{n} \frac{(i-1)! x^i}{i!} = \sum_{i=1}^{n} \frac{x^i}{i}$$

$$k$$

$$\begin{bmatrix} i \\ k \end{bmatrix}$$

$$O(n \log n)$$

$$x^{\overline{n}} = \prod_{k=0}^{n-1} (x+k)$$

$$x^{\underline{n}} = \frac{x!}{(x-n)!} = \prod_{k=0}^{n-1} (x-k)$$

$$n+1$$

$$b$$

$$a$$

$$O(n \log n)$$

$$n = r_1 + r_2 + ... + r_k \quad r_1 \geq r_2 \geq ... \geq r_k \geq 1$$

$$n - k = y_1 + y_2 + ... + y_k \quad y_1 \geq y_2 \geq ... \geq y_k \geq 0$$

$$p(n,k) = \sum_{j=0}^{k} p(n-k, j)$$

$$p(n,k) = p(n-1, k-1) + p(n-k, k)$$

$$\frac{1}{1-x^k} = 1 + x^k + x^{2k} + x^{3k} + ...$$

$$1 + p_1 x + p_2 x^2 + p_3 x^3 + ... = \frac{1}{1-x} \frac{1}{1-x^2} \frac{1}{1-x^3} ...$$

$$\sum_{n,k=0}^{\infty} p(n,k) x^n y^k = \frac{1}{1-xy} \frac{1}{1-x^2 y} \frac{1}{1-x^3 y} ...$$

$$n = r_1 + r_2 + ... + r_k \quad r_1 > r_2 > ... > r_k \geq 1$$

$$n - k = y_1 + y_2 + ... + y_k \quad y_1 > y_2 > ... > y_k \geq 0$$

$$pd(n,k) = pd(n-k, k-1) + pd(n-k, k)$$

$$\prod_{i=1}^{\infty} (1 + x^i) = \frac{\prod_{i=1}^{\infty} (1 - x^{2i})}{\prod_{i=1}^{\infty} (1 - x^i)} = \prod_{i=1}^{\infty} \frac{1}{1 - x^{2i-1}}$$

$$po_n = pd_n$$

$$pd_n = pde_n + pdo_n$$

$$\prod_{i=1}^{\infty} (1 - x^i)$$

$$\sum_{i=0}^{\infty}(pde_n - pdo_n)x^n = \prod_{i=1}^{\infty}(1-x^i)$$

$$n = b + (b+1) + ... + (b+b-1) = \frac{b(3b-1)}{2}$$

$$\prod_{i=0}^{b-1}(-x)^{b+i} = (-1)^b \prod_{i=0}^{b-1} x^{b+i} = (-1)^b x^n$$

$$n = (s+1) + (s+2) + ... + (s+s) = \frac{s(3s-1)}{2}$$

$$\prod_{i=1}^{s}(-x)^{s+i} = (-1)^s \prod_{i=1}^{s} x^{s+i} = (-1)^s x^n$$

$$(1-x)(1-x^2)(1-x^3)... = ... + x^{26} - x^{15} + x^7 - x^2 + 1 - x + x^5 - x^{12} + x^{22} - ... = \sum_{k=-\infty}^{+\infty}(-1)^k x^{\frac{k(3k-1)}{2}}$$

$$(1 + p_1x + p_2x^2 + p_3x^3 + ...)(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - ...) = 1$$

$$p_n = p_{n-1} + p_{n-2} - p_{n-5} - p_{n-7} + ...$$

$$n$$

$$p_n$$

$$n$$

$$0$$

$$p_n$$

$$n$$

$$k$$

$$k$$

$$p(n,k)$$

$$k$$

$$p(n,k)$$

$$j$$

$$p(n-k,j)$$

$$p(0,k)$$

$$p(1,k)$$

$$p(2,k)$$

$$p(3,k)$$

$$p(4,k)$$

$$p(5,k)$$

$p(6,k)$

$p(7,k)$

$p(8,k)$

k

$p(n,k)$

n

10000

k

1000

1000007

k

$12 = 5 + 4 + 2 + 1$

$12 = 5 + 4 + 2 + 1$

$12 = 4 + 3 + 2 + 2 + 1$

k

n

k

k

k

$p(n,k)$

pd_n

n

pd_n

k

$pd(n,k)$

k

j

0

$pd(n-k,j)$

j

k

$k-1$

$pd(0,k)$

$pd(1,k)$

$pd(2,k)$

$pd(3,k)$

$pd(4,k)$

$pd(5,k)$

$pd(6,k)$

$pd(7,k)$

$pd(8,k)$

pd_n

n

50000

1000007

po_n

n

k

k

pde_n

n

pdo_n

n

k

0

1

-1

b

45

s

1

b

s

b

s

$$n$$

$$n$$

$$0$$

$$b = s$$

$$n$$

$$b = s + 1$$

$$n$$

$$0$$

$$0$$

$$1$$

$$p_n$$

$$n$$

$$50000$$

$$1000007$$

$$| A \cup B \cup C | = | A | + | B | + | C | - | A \cap B | - | B \cap C | - | C \cap A | + | A \cap B \cap C |$$

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_i | S_i| - \sum_{i<j} | S_i \cap S_j| + \sum_{i<j<k} | S_i \cap S_j \cap S_k| - \cdots$$
$$+ (-1)^{m-1} \sum_{a_i < a_{i+1}} \left| \bigcap_{i=1}^{m} S_{a_i} \right| + \cdots + (-1)^{n-1} | S_1 \cap \cdots \cap S_n|$$

$$\left| \bigcup_{i=1}^{n} S_i \right| = \sum_{m=1}^{n} (-1)^{m-1} \sum_{a_i < a_{i+1}} \left| \bigcap_{i=1}^{m} S_{a_i} \right|$$

$$\begin{aligned} Cnt = &| \{T_i\} | - | \{T_i \cap T_j | \; i < j\} | + \cdots + (-1)^{k-1} \left| \left\{ \bigcap_{i=1}^{k} T_{a_i} | \; a_i < a_{i+1} \right\} \right| \\ &+ \cdots + (-1)^{m-1} | \{T_1 \cap \cdots \cap T_m\} | \\ &= \binom{m}{1} - \binom{m}{2} + \cdots + (-1)^{m-1} \binom{m}{m} \\ &= \binom{m}{0} - \sum_{i=0}^{m} (-1)^i \binom{m}{i} \\ &= 1 - (1-1)^m = 1 \end{aligned}$$

$$\left| \bigcap_{i=1}^{n} S_i \right| = | U | - \left| \bigcup_{i=1}^{n} \overline{S_i} \right|$$

$$\left| \bigcap_{a_i < a_{i+1}}^{1 \le i \le k} S_{a_i} \right|$$

$$\sum_{i=1}^{n} x_i = m - \sum_{i=1}^{k}(b_{a_i} + 1)$$

$$\sum_{i=1}^{4} C_i x_i = S - \sum_{i=1}^{k} C_{a_i}(D_{a_i} + 1)$$

$$Ans = \sum_{S \subseteq E} (-1)^{|\ S\ |-1} F(S)$$

$$F(S) = \left| \bigcap_{(i,j) \in S} Q_{i,j} \right|$$

$$F(S) \Longleftrightarrow F(\{k_i\}) = \left| \bigcap_{k_i} Q_{k_i} \right|$$

$$\begin{aligned} Ans &= \sum_{K \subseteq M} (-1)^{|\ K\ |-1} \left| \bigcap_{k_i \in K} Q_{k_i} \right| \\ &= \sum_{i} |\ Q_i| - \sum_{i<j} |\ Q_i \cap Q_j| + \sum_{i<j<k} |\ Q_i \cap Q_j \cap Q_k| - \cdots + (-1)^{T-1} \left| \bigcap_{i=1}^{T} Q_i \right| \end{aligned}$$

$$Ans = \left| \bigcup_{i=1}^{T} Q_i \right|$$

$$Ans = m^n - A_m^n$$

$$f(k) = \lfloor (N/k) \rfloor^2 - \sum_{i=2}^{i*k \le N} f(i*k)$$

$$n = \prod_{i=1}^{k} p_i^{c_i}$$

$$\varphi(n) = \left| \bigcap_{i=1}^{k} S_i \right| = |\ U\ | - \left| \bigcup_{i=1}^{k} \overline{S_i} \right|$$

$$\left| \bigcap_{a_i < a_{i+1}} S_{a_i} \right| = \frac{n}{\prod p_{a_i}}$$

$$\begin{aligned} \varphi(n) &= n - \sum_{i} \frac{n}{p_i} + \sum_{i<j} \frac{n}{p_i p_j} - \cdots + (-1)^k \frac{n}{p_1 p_2 \cdots p_n} \\ &= n \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right) \\ &= n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) \end{aligned}$$

$$f(S) = \sum_{T \subseteq S} g(T)$$

$$g(S)=\sum_{T\subseteq S}(-1)^{|\,S\,|-|\,T\,|}f(T)$$

$$\begin{aligned}&\sum_{T\subseteq S}(-1)^{|\,S\,|-|\,T\,|}f(T)\\&=\sum_{T\subseteq S}(-1)^{|\,S\,|-|\,T\,|}\sum_{Q\subseteq T}g(Q)\\&=\sum_{Q}g(Q)\sum_{Q\subseteq T\subseteq S}(-1)^{|\,S\,|-|\,T\,|}\end{aligned}$$

$$=\sum_{Q}g(Q)\sum_{T\subseteq(S\setminus Q)}(-1)^{|\,S\setminus Q\,|-|\,T\,|}$$

$$\begin{aligned}F(P)&=\sum_{T\subseteq P}(-1)^{|\,P\,|-|\,T\,|}\\&=\sum_{i=0}^{|\,P\,|}\binom{|\,P\,|}{i}(-1)^{|\,P\,|-i}=\sum_{i=0}^{|\,P\,|}\binom{|\,P\,|}{i}1^i(-1)^{|\,P\,|-i}\\&=(1-1)^{|\,P\,|}=0^{|\,P\,|}\end{aligned}$$

$$\sum_{Q}g(Q)\sum_{T\subseteq(S\setminus Q)}(-1)^{|\,S\setminus Q\,|-|\,T\,|}=\sum_{Q}g(Q)F(S\setminus Q)=\sum_{Q}g(Q)\cdot 0^{|\,S\setminus Q\,|}$$

$$\sum_{Q}g(Q)\cdot 0^{|\,S\setminus Q\,|}=g(S)$$

$$f(S)=\sum_{S\subseteq T}g(T)$$

$$g(S)=\sum_{S\subseteq T}(-1)^{|\,T\,|-|\,S\,|}f(T)$$

$$f[i,j]=\binom{i}{j}\sum_{k=1}^{i-j}(2^j-1)^k2^{(i-j-k)j}f[i-j,k]$$

$$f[i]=\sum_{j=1}^{i}(-1)^{j-1}\binom{i}{j}2^{(i-j)j}f[i-j]$$

$$\max_{i\in S}x_i=\sum_{T\subseteq S}(-1)^{|\,T\,|-1}\min_{j\in T}x_j$$

$$\min_{i\in S}x_i=\sum_{T\subseteq S}(-1)^{|\,T\,|-1}\max_{j\in T}x_j$$

$$\begin{aligned}\left|f\left(\max_{i\in S}x_i\right)\right|&=\left|\bigcup_{i\in S}f(x_i)\right|\\&=\sum_{T\subseteq S}(-1)^{|\,T\,|-1}\left|\bigcap_{j\in T}f(x_j)\right|\\&=\sum_{T\subseteq S}(-1)^{|\,T\,|-1}\left|f\left(\min_{j\in T}x_j\right)\right|\end{aligned}$$

$$E\left(\max_{i\in S} x_i\right) = \sum_{T\subseteq S}(-1)^{\mid T\mid -1}E\left(\min_{j\in T} x_j\right)$$

$$E\left(\min_{i\in S} x_i\right) = \sum_{T\subseteq S}(-1)^{\mid T\mid -1}E\left(\max_{j\in T} x_j\right)$$

$$E\left(\max_{i\in S} x_i\right) = \sum_{y} P(y=x)\max_{j\in S} y_j$$

$$\begin{aligned}E\left(\max_{i\in S} x_i\right) &= \sum_{y} P(y=x)\max_{j\in S} y_j\\ &= \sum_{y} P(y=x)\sum_{T\subseteq S}(-1)^{\mid T\mid -1}\min_{j\in T} y_j\end{aligned}$$

$$\begin{aligned}E\left(\max_{i\in S} x_i\right) &= \sum_{y} P(y=x)\sum_{T\subseteq S}(-1)^{\mid T\mid -1}\min_{j\in T} y_j\\ &= \sum_{T\subseteq S}(-1)^{\mid T\mid -1}\sum_{y} P(y=x)\min_{j\in T} y_j\\ &= \sum_{T\subseteq S}(-1)^{\mid T\mid -1}E\left(\min_{j\in T} y_j\right)\end{aligned}$$

$$\underset{i\in S}{\mathrm{kthmax}}\, x_i = \sum_{T\subseteq S}(-1)^{\mid T\mid -k}\binom{\mid T\mid -1}{k-1}\min_{j\in T} x_j$$

$$\underset{i\in S}{\mathrm{kthmin}}\, x_i = \sum_{T\subseteq S}(-1)^{\mid T\mid -k}\binom{\mid T\mid -1}{k-1}\max_{j\in T} x_j$$

$$E\left(\underset{i\in S}{\mathrm{kthmax}}\, x_i\right) = \sum_{T\subseteq S}(-1)^{\mid T\mid -k}\binom{\mid T\mid -1}{k-1}E\left(\min_{j\in T} x_j\right)$$

$$E\left(\underset{i\in S}{\mathrm{kthmin}}\, x_i\right) = \sum_{T\subseteq S}(-1)^{\mid T\mid -k}\binom{\mid T\mid -1}{k-1}E\left(\max_{j\in T} x_j\right)$$

$$\begin{aligned}\sum_{T\subseteq S}(-1)^{\mid T\mid -k}\binom{\mid T\mid -1}{k-1}\min_{j\in T} x_j &= \sum_{i\in S} x_i \sum_{T\subseteq S}(-1)^{\mid T\mid -k}\binom{\mid T\mid -1}{k-1}\left[x_i = \min_{j\in T} x_j\right]\\ &= \sum_{i\in S} x_i \sum_{j=k}^{n}\binom{n-i}{j-1}\binom{j-1}{k-1}(-1)^{j-k}\end{aligned}$$

$$\begin{aligned}\sum_{T\subseteq S}(-1)^{\mid T\mid -k}\binom{\mid T\mid -1}{k-1}\min_{j\in T} x_j &= \sum_{i\in S} x_i \sum_{j=k}^{n}\binom{n-i}{j-1}\binom{j-1}{k-1}(-1)^{j-k}\\ &= \sum_{i\in S} x_i \sum_{j=k}^{n}\binom{n-i}{k-1}\binom{n-i-k+1}{j-k}(-1)^{j-k}\\ &= \sum_{i\in S}\binom{n-i}{k-1}x_i \sum_{j=k}^{n}\binom{n-i-k+1}{j-k}(-1)^{j-k}\\ &= \sum_{i\in S}\binom{n-i}{k-1}x_i \sum_{j=0}^{n-i-k+1}\binom{n-i-k+1}{j}(-1)^{j}\end{aligned}$$

$$\binom{n-i}{k-1}\sum_{j=0}^{n-i-k+1}\binom{n-i-k+1}{j}(-1)^j=1$$

$$\binom{n-i}{k-1}\sum_{j=0}^{n-i-k+1}\binom{n-i-k+1}{j}(-1)^j=0$$

$$\sum_{i\in S}\binom{n-i}{k-1}x_i\sum_{j=0}^{n-i-k+1}\binom{n-i-k+1}{j}(-1)^j=\operatorname*{kthmax}_{i\in S}x_i$$

$$\operatorname*{lcm}_{i\in S}x_i=\prod_{T\subseteq S}\left(\gcd_{j\in T}x_j\right)^{(-1)^{|T|-1}}$$

$$E\left(\max_{i\in S}x_i\right)$$

$$E\left(\max_{i\in S}x_i\right)=E\left(\sum_{T\subseteq S}(-1)^{|T|-1}\min_{i\in T}x_i\right)=\sum_{T\subseteq S}(-1)^{|T|-1}E\left(\min_{i\in T}x_i\right)$$

$$f(i)=1+\frac{1}{\deg(i)}\sum_{e=1}^{k}\left(A_{j_e}+B_{j_e}f(i)\right)+\frac{f(p_i)}{\deg(i)}$$

$$f(i)=\frac{\deg(i)+\sum_{e=1}^{k}A_{j_e}}{\deg(i)-\sum_{e=1}^{k}B_{j_e}}+\frac{f(p_i)}{\deg(i)-\sum_{e=1}^{k}B_{j_e}}$$

$$f(1)=\frac{\deg(1)+\sum_{e=1}^{k}A_{j_e}}{\deg(1)-\sum_{e=1}^{k}B_{j_e}}$$

$$10$$

$$15$$

$$21$$

$$10+15+21=46$$

$$A,B,C$$

$$|A\cup B\cup C|$$

$$|A|,|B|,|C|$$

$$|A\cap B|,|B\cap C|,|C\cap A|$$

$$|A\cap B\cap C|$$

$$P_i$$

$$P_i$$

$$S_i$$

$$T_1,T_2,\cdots,T_m$$

$$\sum_{i=1}^{n} x_i = m$$

$$n$$

$$x_i \leq b_i$$

$$m, b_i \in \mathbb{N}$$

$$x_i < b_i$$

$$\sum_{i=1}^{n} x_i = m$$

$$\binom{m+n-1}{n-1}$$

$$m$$

$$n$$

$$n-1$$

$$m+n-1$$

$$n-1$$

$$n-1$$

$$n$$

$$n$$

$$m+n-1$$

$$n-1$$

$$\binom{m+n-1}{n-1}$$

$$\sum_{i=1}^{n} x_i = m$$

$$x_i$$

$$x_i$$

$$x_i$$

$$x_i \leq b_i$$

$$|\bigcap_{i=1}^{n} S_i|$$

$$\left|\bigcap_{i=1}^{n} S_i\right| = |\ U\ | - \left|\bigcup_{i=1}^{n} \overline{S_i}\right|$$

$$|\ U\ |$$

$$\overline{S_{a_i}}$$

$$\overline{S_{a_i}}$$

$$x_{a_i} \geq b_{a_i} + 1$$

$$0$$

$$0$$

$$k$$

$$a$$

$$C_i$$

$$n$$

$$D_i$$

$$S$$

$$n \leq 10^3, S \leq 10^5$$

$$O(4nS)$$

$$\sum_{i=1}^{4} C_i x_i = S, x_i \leq D_i$$

$$x_i$$

$$x_i \leq D_i$$

$$O(4S + 2^4 n)$$

$$n$$

$$G = (V, E)$$

$$F(S)$$

$$S \subseteq E$$

$$F(S)$$

$$G' = (V, S)$$

$$m$$

$$\mid S \mid$$

$$F(S)$$

$$F(S)$$

$$G' = (V, S)$$

$$x_i = x_j$$

$$x_i = x_j$$

$$Q_{i,j}$$

$$G'$$

$$G'$$

$$Q$$

$$(i, j)$$

$$x_i = x_j$$

$$F(S)$$

$$(i, j)$$

$$T = \frac{n(n+1)}{2}$$

$$(i, j)$$

$$k, 1 \leq k \leq T$$

$$Q_{i,j}$$

$$Q_k$$

$$x_i = x_j$$

$$P_k$$

$$S \Longleftrightarrow K = \{k_1, k_2, \cdots, k_m\}$$

$$M = \left\{1, 2, \cdots, \frac{n(n+1)}{2}\right\}$$

$$1 \sim T$$

$$U$$

$$\mid U \mid = m^n$$

$$A_m^n = \frac{m!}{n!}$$

$$F(S)$$

$$k$$

$$1 \leq x, y \leq N$$

$$f(k)$$

$$k$$

$$(x, y)$$

$$f(1)$$

$$f(N)$$

$$k$$

$$k$$

$$k$$

$$f(k) =$$

$$k$$

$$-$$

$$k$$

$$k$$

$$k$$

$$k > N/2$$

$$f(k) = \lfloor (N/k) \rfloor^2$$

$$f(N)$$

$$f(1)$$

$$O(\sum_{i=1}^{N} N/i) = O(N \sum_{i=1}^{N} 1/i) = O(N \log N)$$

$$\varphi(n)$$

$$\varphi(n) = \mid \{1 \le x \le n \mid \gcd(x, n) = 1\} \mid$$

$$O(n \log n)$$

$$O(n)$$

$$O(n^{\frac{2}{3}})$$

$$p_i$$

$$x$$

$$p_i$$

$$p_i \nmid x$$

$$S_i$$

$$\mid U \mid = n$$

$$\overline{S_i}$$

$$p_i \mid x$$

$$\mid \overline{S_i} \mid = \frac{n}{p_i}$$

$$f(S), g(S)$$

$$Q$$

$$Q$$
$$P$$
$$F(P) = \sum_{T \subseteq P} (-1)^{|\ P\ | - |\ T\ |}$$
$$|\ S \setminus Q\ | = 0$$
$$0^0 = 1$$
$$Q = S$$
$$g(S)$$
$$0^{|\ S \setminus Q\ |} = 0$$
$$U$$
$$f(S), g(S)$$
$$n$$
$$10^9 + 7$$
$$n \leq 5 \times 10^3$$
$$f[i, j]$$
$$i$$
$$j$$
$$0$$
$$j$$
$$k$$
$$0$$
$$k$$
$$j$$
$$2^j - 1$$
$$j$$
$$k$$
$$2^{i-j-k}$$
$$O(n^3)$$
$$j$$
$$0$$
$$j$$
$$0$$

$$f[i]$$

$$i$$

$$j$$

$$j$$

$$i-j$$

$$\left(2^{i-j}\right)^{j}=2^{(i-j)j}$$

$$O(n^{2})$$

$$\{x_i\}$$

$$n$$

$$S=\{1,2,3,\cdots,n\}$$

$$X$$

$$X$$

$$a,b,c\in X$$

$$a\leq b$$

$$b\leq a$$

$$a=b$$

$$a\leq b$$

$$b\leq c$$

$$a\leq c$$

$$a\leq b$$

$$b\leq a$$

$$x\in S$$

$$x$$

$$k$$

$$f:x\mapsto\{1,2,\cdots,k\}$$

$$x,y\in S$$

$$f(\min(x,y))=f(x)\cap f(y)$$

$$f(\max(x,y))=f(x)\cup f(y)$$

$$\left|f\left(\max_{i\in S}x_i\right)\right|$$

$$\max_{i\in S}x_i$$

$$\min$$

$$y$$

$$n$$

$$\max$$

$$\min$$

$$n < m$$

$$\binom{n}{m} = 0$$

$$\forall 1 \le i < n, x_i \le x_{i+1}$$

$$\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$$

$$i = n - k + 1$$

$$\operatorname{lcm}, \gcd, a^1, a^{-1}$$

$$\max, \min, +, -$$

$$n$$

$$x$$

$$Q$$

$$S$$

$$x$$

$$S$$

$$x$$

$$998244353$$

$$1 \le n \le 18, 1 \le Q \le 5000, 1 \le \mid S \mid \le n$$

$$x_i$$

$$i$$

$$T \in [n]$$

$$F(T) = E(\min_{i \in T} x_i)$$

$$E(\min_{i \in T} x_i)$$

$$T$$

$$f(i)$$

$$i$$

$$T$$

$$i \in T$$

$$f(i) = 0$$

$$i \notin T$$

$$f(i) = 1 + \frac{1}{\deg(i)} \sum_{(i,j) \in E} f(j)$$

$$O(n^3)$$

$$T$$

$$F(T)$$

$$O(2^n n^3)$$

$$1$$

$$u$$

$$p_u$$

$$i$$

$$f(i)$$

$$i$$

$$f(i) = 0$$

$$f(i)$$

$$f(i) = A_i + B_i f(p_i)$$

$$A_i, B_i$$

$$i$$

$$j_1, \cdots, j_k$$

$$f(j_e) = A_{j_e} + B_{j_e} f(i)$$

$$f(i)$$

$$A_i + B_i f(p_i)$$

$$f(1)$$

$$f(i)$$

$$F(T) = f(x)$$

$$O(n)$$

$$T$$

$$F(T)$$

$$O(2^n n)$$

$$E(\max_{i \in S} x_i) = \sum_{T \subseteq S} (-1)^{|\ T\ | - 1} F(T)$$

$$F'(T) = (-1)^{|\,T\,|-1} F(T)$$

$$E(\max_{i \in S} x_i) = \sum_{T \subseteq S} F'(T)$$

$$O(2^n n)$$

$$S$$

$$E(\max_{i \in S} x_i)$$

$$O(1)$$

$$\sum_{i=1}^{n} \binom{n-1}{i-1} c_i g_{n-i} = g_n$$

$$c_n = g_n - \sum_{i=1}^{n-1} \binom{n-1}{i-1} c_i g_{n-i}$$

$$\sum_{i=1}^{n} \binom{n-1}{i-1} c_i g_{n-i} = g_n$$

$$\sum_{i=1}^{n} \frac{c_i}{(i-1)!} \frac{g_{n-i}}{(n-i)!} = \frac{g_n}{(n-1)!}$$

$$C(x) = \sum_{n=1} \frac{c_n}{(n-1)!} x^n$$

$$G(x) = \sum_{n=0} \frac{g_n}{n!} x^n$$

$$H(x) = \sum_{n=1} \frac{g_n}{(n-1)!} x^n$$

$$\exp(C(x)) = G(x)$$
$$C(x) = \ln(G(x))$$

$$g_n = \sum_{i=0}^{n} \binom{n}{i} 2^{i(n-i)}$$

$$b_n = \sum_{i=1}^{n} c_{n,i}$$

$$g_n = \sum_{i=1}^{n} c_{n,i} 2^i$$

$$\sum_{i=0}^{n} \binom{n}{i} 2^{i(n-i)} = \sum_{i=1}^{n} c_{n,i} 2^i$$

$$c_{n,i} = \sum_{i=0} n-1 \binom{n-1}{i-1} c_{n,1} c_{n-i,k-1}$$

$$
\begin{aligned}
G(x) &= \exp(2B1(x)) \\
B(x) &= \exp(B1(x)) \\
&= \exp(\frac{\ln G(x)}{2}) \\
&= \sqrt{G}
\end{aligned}
$$

$$
\begin{aligned}
B_n^2 &= G \\
2B_n B_{(n)'} &= G'
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{E}(A(x)) &= \prod_i (1 - x^i)^{-a_i} \\
&= \exp(\sum_i \frac{A(x^i)}{i})
\end{aligned}
$$

$$nb_n = c_n + \sum_{i=1}^{n-1} c_i b_{n-i}$$

$$F(x) = x\mathcal{E}(F(x))$$

$$g_n = f_n - \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} f_i f_{n-i}$$

$$g_n = f_n - \sum_{i=\lceil \frac{n}{2} \rceil}^{n-1} f_i f_{n-i} - \binom{f_{\frac{n}{2}}}{2}$$

$$\frac{1}{\mid G \mid} \sum_{g \in G} m^{c(g)}$$

$$\frac{1}{\mid G \mid} \sum_{p \in P} w(p) m^{c(p)}$$

$$w(p) = \frac{n!}{\prod_i (p_i) \prod_i (q_{(i)!})}$$

$$n!$$

$$n$$

$$n \leq 50$$

$$g_n$$

$$n$$

$$c_n$$

$$n$$

$$\binom{n}{2}$$

$$g_n = 2^{\binom{n}{2}}$$

$$n-1$$
$$i-1$$
$$c_n$$
$$O(n^2)$$
$$n$$
$$n \leq 130000$$
$$O(n \log^2 n)$$
$$CG = H$$
$$C(x)$$
$$C(x)$$
$$G(x)$$
$$C(x)$$
$$n$$
$$n \leq 1000$$
$$g_n$$
$$c_n$$
$$b_n$$
$$g_n$$
$$n$$
$$g_n$$
$$b_n$$
$$c_{n,k}$$
$$g_n$$
$$b_n$$
$$O(n^3)$$
$$O(n^2)$$
$$b1_n$$
$$g_n$$
$$b_n$$
$$g_n$$
$$b1_n$$
$$b_n$$

$$b1_n$$

$$O(n^2)$$

$$G(x)$$

$$g_n$$

$$B1(x)$$

$$b1_n$$

$$B(x)$$

$$b_n$$

$$a_i$$

$$A(x)$$

$$A(x)$$

$$\mathcal{E}(A(x))$$

$$b_i$$

$$c_i = \sum_{d \mid n} d a_d$$

$$F(x)$$

$$n$$

$$n$$

$$\geq \left\lceil \frac{n}{2} \right\rceil$$

$$n$$

$$n \leq 200000$$

$$G$$

$$n$$

$$O(n!)$$

$$p$$

$$w(p)$$

$$c(p)$$

$$w(p)$$

$$q_i$$

$$i$$

$$p$$

$$c(p)$$

$$p$$

$$\mid p \mid$$

$$p_i$$

$$\left\lfloor \frac{p_i}{2} \right\rfloor$$

$$p_i$$

$$p_j$$

$$\operatorname{lcm}(p_i, p_j)$$

$$\frac{p_i p_j}{\operatorname{lcm}(p_i, p_j)} = \gcd(p_i, p_j)$$

$$F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$$

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...$$

$$L_0 = 2, L_1 = 1, L_n = L_{n-1} + L_{n-2}$$

$$2, 1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, ...$$

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$F_n = \left[\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n}{\sqrt{5}} \right]$$

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$\frac{L_n + F_n\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$x^2 - 5y^2 = -4$$

$$\frac{x_n + y_n\sqrt{5}}{2} = \frac{L_n + F_n\sqrt{5}}{2}$$

$$L_n^2 - 5F_n^2 = -4$$

$$[F_{n-1} \;\; F_n] = [F_{n-2} \;\; F_{n-1}][0 \;\; 11 \;\; 1]$$

$$[F_n \;\; F_{n+1}] = [F_0 \;\; F_1]P^n$$

$$F_{2k} = F_k(2F_{k+1} - F_k)$$

$$F_{2k+1} = F_{k+1}^2 + F_k^2$$

$$\frac{L_n + F_n\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$\cos nx + i \sin nx = (\cos x + i \sin x)^n$$

$$L_n^2 - 5F_n^2 = -4$$

$$\cos^2 x + \sin^2 x = 1$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^m \left(\frac{1+\sqrt{5}}{2}\right)^n = \left(\frac{1+\sqrt{5}}{2}\right)^{m+n}$$

$$2L_{m+n} = 5F_mF_n + L_mL_n$$

$$2F_{m+n} = F_mL_n + L_mF_n$$

$$L_{2n} = L_n^2 - 2(-1)^n$$

$$F_{2n} = F_nL_n$$

$$N = F_{k_1} + F_{k_2} + \dots + F_{k_r}$$

$$\begin{aligned} 1 = && 1 = && F_2 = && (11)_F \\ 2 = && 2 = && F_3 = && (011)_F \\ 6 = && 5+1 = && F_5 + F_2 = && (10011)_F \\ 8 = && 8 = && F_6 = && (000011)_F \\ 9 = && 8+1 = && F_6 + F_2 = && (100011)_F \\ 19 = && 13+5+1 = && F_7 + F_5 + F_2 = && (1001011)_F \end{aligned}$$

$$(F_1, F_2), (F_2, F_3), \dots, (F_{p^2+1}, F_{p^2+2})$$

$$F_p = \frac{2}{2^p\sqrt{5}}\left(\binom{p}{1}\sqrt{5} + \binom{p}{3}\sqrt{5}^3 + \dots + \binom{p}{p}\sqrt{5}^p\right) \equiv \sqrt{5}^{p-1} \equiv 1 \pmod p$$

$$F_{p+1} = \frac{2}{2^{p+1}\sqrt{5}}\left(\binom{p+1}{1}\sqrt{5} + \binom{p+1}{3}\sqrt{5}^3 + \dots + \binom{p+1}{p}\sqrt{5}^p\right) \equiv \frac{1}{2}\left(1 + \sqrt{5}^{p-1}\right) \equiv 1 \pmod p$$

$$F_{2p} = \frac{2}{2^{2p}\sqrt{5}}\left(\binom{2p}{1}\sqrt{5} + \binom{2p}{3}\sqrt{5}^3 + \dots + \binom{2p}{2p-1}\sqrt{5}^{2p-1}\right)$$

$$F_{2p+1} = \frac{2}{2^{2p+1}\sqrt{5}}\left(\binom{2p+1}{1}\sqrt{5} + \binom{2p+1}{3}\sqrt{5}^3 + \dots + \binom{2p+1}{2p+1}\sqrt{5}^{2p+1}\right)$$

$$F_{2p} \equiv \frac{1}{2}\binom{2p}{p}\sqrt{5}^{p-1} \equiv -1 \pmod p$$

$$F_{2p+1} \equiv \frac{1}{4}\left(\binom{2p+1}{1} + \binom{2p+1}{p}\sqrt{5}^{p-1} + \binom{2p+1}{2p+1}\sqrt{5}^{2p}\right) \equiv \frac{1}{4}(1-2+5) \equiv 1 \pmod p$$

$$F_M = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^M - \left(\frac{1-\sqrt{5}}{2}\right)^M\right) \equiv 0 \pmod{p^{k-1}}$$

$$F_{M+1} = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{M+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{M+1}\right) \equiv 1 \pmod{p^{k-1}}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^M \equiv \left(\frac{1-\sqrt{5}}{2}\right)^M \pmod{p^{k-1}}$$

$$1 \equiv \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^M \left(\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right) = \left(\frac{1+\sqrt{5}}{2}\right)^M \pmod{p^{k-1}}$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^M \equiv \left(\frac{1-\sqrt{5}}{2}\right)^M \equiv 1 \pmod{p^{k-1}}$$

$$v_p(a^t - 1) = v_p(a-1) + v_p(t)$$

$$v_2(a^t - 1) = v_2(a-1) + v_2(a+1) + v_2(t) - 1$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^{Mp} \equiv \left(\frac{1-\sqrt{5}}{2}\right)^{Mp} \equiv 1 \pmod{p^k}$$

$$n$$

$$\Theta(n)$$

$$1$$

$$L_n$$

$$F_n$$

$$\left(\frac{1+\sqrt{5}}{2}\right)^n$$

$$P = [0 \;\; 11 \;\; 1]$$

$$\Theta(\log n)$$

$$(F_n, F_{n+1})$$

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n$$

$$F_{n+k} = F_k F_{n+1} + F_{k-1} F_n$$

$$k = n$$

$$F_{2n} = F_n(F_{n+1} + F_{n-1})$$

$$\forall k \in \mathbb{N}, F_n | \; F_{nk}$$

$$\forall F_a | \ F_b, a \mid b$$

$$(F_m, F_n) = F_{(m,n)}$$

$$n$$

$$k_1 \geq k_2 + 2, \, k_2 \geq k_3 + 2, \, ..., \, k_r \geq 2$$

$$d_0 d_1 d_2 ... d_s 1$$

$$d_i = 1$$

$$F_{i+2}$$

$$n$$

$$F_i$$

$$F_i \leq n$$

$$n$$

$$F_i$$

$$i - 2$$

$$n$$

$$i$$

$$F_{i+2}$$

$$p$$

$$p^2 + 1$$

$$p$$

$$p$$

$$p^2 + 1$$

$$m$$

$$6m$$

$$m = 2 \times 5^k$$

$$n$$

$$m$$

$$n$$

$$m$$

$$2$$

$$3$$

$$5$$

$$20$$

$$a$$

$$b$$

$$ab$$

$$a$$

$$b$$

$$p \equiv 1, 4 \pmod 5$$

$$p - 1$$

$$p$$

$$p$$

$$p - 1$$

$$5^{\frac{p-1}{2}} \equiv 1 \pmod p$$

$$F_p$$

$$F_{p+1}$$

$$1$$

$$F_1$$

$$F_2$$

$$p - 1$$

$$p \equiv 2, 3 \pmod 5$$

$$2p + 2$$

$$p$$

$$p$$

$$2p + 2$$

$$5^{\frac{p-1}{2}} \equiv -1 \pmod p$$

$$p$$

$$F_{2p}$$

$$\binom{2p}{p} \equiv 2 \pmod p$$

$$F_{2p+1}$$

$$\binom{2p+1}{1} \equiv 1 \pmod p$$

$$\binom{2p+1}{p} \equiv 2 \pmod p$$

$$\binom{2p+1}{2p+1} \equiv 1 \pmod p$$

$$F_{2p}$$

$$F_{2p+1}$$

$$F_{-2}$$

$$F_{-1}$$

$$2p+2$$

$$p$$

$$M$$

$$p^{k-1}$$

$$Mp$$

$$p^k$$

$$M$$

$$p^{k-1}$$

$$Mp$$

$$p^k$$

$$\frac{1+\sqrt{5}}{2}$$

$$M$$

$$p^{k-1}$$

$$p$$

$$p=2$$

$$a$$

$$\left(\frac{1+\sqrt{5}}{2}\right)$$

$$\left(\frac{1-\sqrt{5}}{2}\right)$$

$$t$$

$$M$$

$$Mp$$

$$Mp$$

$$p^k$$

$$A(n,m) = \left\langle {n \atop m-1} \right\rangle$$

$$A(n,m) = \begin{cases} 0, & m > n \text{ or } n = 0, \\ 1, & m = 0, \\ (n-m) \cdot A(n-1,m-1) + (m+1) \cdot A(n-1,m), & \text{otherwise.} \end{cases}$$

$$\gamma$$

$$\mathrm{e}$$

$$1$$

$$n$$

$$m$$

$$m$$

$$1$$

$$3$$

$$4$$

$$A(n,m)$$

$$n$$

$$3$$

$$m$$

$$1$$

$$4$$

$$4$$

$$1$$

$$n$$

$$m$$

$$A(n,m)$$

$$A(1,0)$$

$$(1)$$

$$A(2,0)$$

$$(2,1)$$

$$A(2,1)$$

$$(1,2)$$

$$A(3,0)$$

$$(3,2,1)$$

$$A(3,1)$$

$$(1,3,2),(2,1,3),(2,3,1),(3,1,2)$$

$$A(3,2)$$

$$(1,2,3)$$

$$m \geq n$$

$$n=0$$

$$m=0$$

$$n-1$$

$$n$$

$$n$$

$$n$$

$$A(n,m)$$

$$A(n-1,m-1)$$

$$A(n-1,m)$$

$$n$$

$$p_{i-1} < p_i$$

$$n$$

$$p_i$$

$$n$$

$$n$$

$$n$$

$$A(n-1,m-1)$$

$$A(n,m)$$

$$n$$

$$n-(m-1)-1=n-m$$

$$A(n-1,m)$$

$$A(n,m)$$

$$n$$

$$m+1$$

$$E(0,0)=1$$

$$E(n,0)=0$$

$$E(n,k)=E(n,k-1)+E(n-1,n-k)$$

$$
\begin{aligned}
&E(0,0)\\
&E(1,0) \to E(1,1)\\
&E(2,2) \leftarrow E(2,1) \leftarrow E(2,0)\\
&E(3,0) \to E(3,1) \to E(3,2) \to E(3,3)\\
&E(4,4) \leftarrow E(4,3) \leftarrow E(4,2) \leftarrow E(4,1) \leftarrow E(4,0)
\end{aligned}
$$

$$
\begin{aligned}
&1\\
&0 \to 1\\
&1 \leftarrow 1 \leftarrow 0\\
&0 \to 1 \to 2 \to 2\\
&5 \leftarrow 5 \leftarrow 4 \leftarrow 2 \leftarrow 0
\end{aligned}
$$

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} E\left(m+n, \frac{1}{2}(m+n+(-1)^{m+n}(n-m))\right)\frac{x^m}{m!}\frac{x^n}{n!} = \frac{\cos x + \sin x}{\cos(x+y)}$$

$$
\begin{array}{lllll}
E(0,0) & E(1,1) & E(2,0) & E(3,3) & E(4,0)\\
E(1,0) & E(2,1) & E(3,2) & E(4,1) & \\
E(2,2) & E(3,1) & E(4,2) & & \\
E(3,0) & E(4,3) & & & \\
E(4,4) & & & &
\end{array}
$$

$$
\begin{array}{ccccc}
1 & 1 & 0 & 2 & 0\\
0 & 1 & 2 & 2 & \\
1 & 1 & 4 & & \\
0 & 5 & & & \\
5 & & & &
\end{array}
$$

$$1, 1, 2, 4, 10, 32, 122, 544, \cdots$$

$$
\begin{aligned}
&n = 1 : \{1\}\\
&n = 2 : \{1,2\}, \{2,1\}\\
&n = 3 : \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}\\
&n = 4 : \{1,3,2,4\}, \{1,4,2,3\}, \{2,1,4,3\}, \{2,3,1,4\}, \{2,4,1,3\},\\
&\qquad\quad \{3,1,4,2\}, \{3,2,4,1\}, \{3,4,1,2\}, \{4,1,3,2\}, \{4,2,3,1\}
\end{aligned}
$$

$$c_1 > c_2 < c_3 > \cdots$$

$$c_1 < c_2 > c_3 < \cdots$$

$$A_n = \frac{Z_n}{2}$$

$$A_0 = A_1 = 1$$

$$1, 1, 1, 2, 5, 16, 61, 272, \cdots$$

$$2A_{n+1} = \sum_{k=0}^{n}\binom{n}{k}A_k A_{n-k}$$

$$2(n+1)\frac{A_{n+1}}{(n+1)!} = \sum_{k=0}^{n}\frac{A_k}{k!}\frac{A_{n-k}}{(n-k)!}$$

$$2\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 + 1$$

$$y = \tan\left(\frac{1}{2}x + C\right)$$

$$y = \tan x + \sec x$$

$$A_n = E(n, n)$$

$$A_n = (-1)^{n/2} E_n$$

$$1, 1, 5, 61, 1385, \cdots$$

$$A_n = \frac{(-1)^{(n-1)/2} 2^{n+1} (2^{n+1} - 1) B_{n+1}}{n+1}$$

$$1, 2, 16, 272, 7936, \cdots$$

$$\sec x = A_0 + A_2 \frac{x^2}{2!} + A_4 \frac{x^4}{4!} + \cdots$$

$$\tan x = A_1 x + A_3 \frac{x^3}{3!} + A_5 \frac{x^5}{5!} + \cdots$$

$$\sec x + \tan x = A_0 + A_1 x + A_2 \frac{x^2}{2!} + A_3 \frac{x^3}{3!} + A_4 \frac{x^4}{4!} + A_5 \frac{x^5}{5!} + \cdots$$

$$E(n, k)$$

$$0$$

$$n$$

$$n + 1$$

$$k$$

$$E_(n, k)$$

$$E(n, k) = E(n, k-1) + E(n-1, n-k)$$

$$1$$

$$n$$

$$c_1$$

$$c_i$$

$$c_i$$

$$c_{i-1}$$

$$c_{i+1}$$

$$Z_n$$

$$n = 0$$

$$n$$

1

n

1

n

2

$n-1$

1

n

A_n

$n=0$

A_n

1

n

k

$\binom{n}{k}$

k

u

A_k

k

$n-k$

v

A_{n-k}

$n+1$

w

u

$n+1$

v

w

$n+1$

$n+1$

u

v

$n+1$

u

v

n

0

A_0

A_1

1

A_n

y

1

n

0

0

1

$E(n,k)$

k

0

n

A_n

E_n

B_n

n

S_n

n

T_n

$x=0$

$n+1$

n

1

$1\times n$

$n+1$

n

k

$\left\lceil \frac{n}{k} \right\rceil$

$$\left\lceil \frac{n}{k} \right\rceil$$

$$S \leq (\left\lceil \frac{n}{k} \right\rceil - 1) \times k = k \left\lceil \frac{n}{k} \right\rceil - k < k(\frac{n}{k} + 1) - k = n$$

$$S$$

$$S$$

$$\{A_1, A_2 ... A_k\}$$

$$\bigcup_{i=1}^{k} A_i$$

$$S$$

$$i \neq j \rightarrow A_i \cap A_j = \emptyset$$

$$S$$

$$S$$

$$\{A_1, A_2 ... A_k\}$$

$$A_i$$

$$|A_i| \geq \left\lceil \frac{|S|}{k} \right\rceil$$

$$\left| \bigcap_{i=1}^{k} S_{a_i} \right|$$

$$\left| \bigcap_{i=1}^{k} S_{a_i} \right| = (n-k)!$$

$$\begin{aligned} \left| \bigcup_{i=1}^{n} \overline{S_i} \right| &= \sum_{k=1}^{n} (-1)^{k-1} \sum_{a_{1,\cdots,k}} \left| \bigcap_{i=1}^{k} S_{a_i} \right| \\ &= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! \\ &= \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!} \\ &= n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} \end{aligned}$$

$$D_n = n! - n! \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k!} = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

$$D_n = nD_{n-1} + (-1)^n$$

$$D_n = \left\lfloor \frac{n!}{\mathrm{e}} \right\rfloor$$

$$P = \lim_{n \to \infty} \frac{D_n}{n!} = \frac{1}{\mathrm{e}}$$

$$1 \sim n$$

$$P$$

$$P_i \neq i$$

$$P$$

$$n$$

$$\{2, 3, 1\}$$

$$\{3, 1, 2\}$$

$$\{2, 1, 4, 3\}$$

$$\{2, 3, 4, 1\}$$

$$\{2, 4, 1, 3\}$$

$$\{3, 1, 4, 2\}$$

$$\{3, 4, 1, 2\}$$

$$\{3, 4, 2, 1\}$$

$$\{4, 1, 2, 3\}$$

$$\{4, 3, 1, 2\}$$

$$\{4, 3, 2, 1\}$$

$$U$$

$$1 \sim n$$

$$\mid U \mid = n!$$

$$P_i \neq i$$

$$\left| \bigcup_{i=1}^{n} \overline{S_i} \right|$$

$$\overline{S_i}$$

$$P_i = i$$

$$a_i < a_{i+1}$$

$$k$$

$$k$$

$$P_{a_i} = a_i$$

$$n-k$$

$$k$$

$$\binom{n}{k}$$

$$n$$

$$0, 1, 2, 9, 44, 265$$

$$n$$

$$1, 2, 3, 4, 5$$

$$1, 2, 3, 4, 5$$

$$n$$

$$n$$

$$n$$

$$n-1$$

$$n-1$$

$$n-1$$

$$n-1$$

$$n$$

$$D_{n-1} \times (n-1)$$

$$n-1$$

$$n-1$$

$$n$$

$$n$$

$$\mathrm{A}_n^m = n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!}$$

$$\mathrm{A}_n^m = n(n-1)(n-2)\cdots(n-m+1) = \frac{n!}{(n-m)!}$$

$$\mathrm{A}_n^n = n(n-1)(n-2)\cdots 3 \times 2 \times 1 = n!$$

$$\binom{n}{m} = \frac{\mathrm{A}_n^m}{m!} = \frac{n!}{m!(n-m)!}$$

$$\binom{n}{m} \times m! = \mathrm{A}_n^m$$

$$\binom{n}{m} = \frac{\mathrm{A}_n^m}{m!} = \frac{n!}{m!(n-m)!}$$

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

$$x_i' = x_i - a_i$$

$$\begin{aligned}(x_1' + a_1) + (x_2' + a_2) + \cdots + (x_k' + a_k) &= n \\ x_1' + x_2' + \cdots + x_k' &= n - a_1 - a_2 - \cdots - a_k \\ x_1' + x_2' + \cdots + x_k' &= n - \sum a_i\end{aligned}$$

$$x_i' \geq 0$$

$$\binom{n - \sum a_i + k - 1}{n - \sum a_i}$$

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

$$(x_1 + x_2 + \cdots + x_t)^n = \sum_{\text{□□}n_1+\cdots+n_t=n\text{□□□□□□}} \binom{n}{n_1, n_2, \cdots, n_t} x_1^{n_1} x_2^{n_2} \cdots x_t^{n_t}$$

$$\sum \binom{n}{n_1, n_2, \cdots, n_t} = t^n$$

$$\frac{n!}{\prod_{i=1}^{k} n_{(i)!}} = \frac{n!}{n_{(1)!} n_{(2)!} \cdots n_{(k)!}}$$

$$\binom{n}{n_1, n_2, \cdots, n_k} = \frac{n!}{\prod_{i=1}^{k} n_{(i)!}}$$

$$\binom{r + k - 1}{k - 1}$$

$$\forall i \in [1, k],\, x_i \leq n_i,\, \sum_{i=1}^{k} x_i = r$$

$$\left| \bigcap_{i=1}^{k} S_i \right| = \mid U \mid - \left| \bigcup_{i=1}^{k} \overline{S_i} \right|$$

$$\begin{aligned}\left| \bigcup_{i=1}^{k} \overline{S_i} \right| &= \sum_i \left| \overline{S_i} \right| - \sum_{i,j} \left| \overline{S_i} \cap \overline{S_j} \right| + \sum_{i,j,k} \left| \overline{S_i} \cap \overline{S_j} \cap \overline{S_k} \right| - \cdots \\ &\quad + (-1)^{k-1} \left| \bigcap_{i=1}^{k} \overline{S_i} \right| \\ &= \sum_i \binom{k + r - n_i - 2}{k - 1} - \sum_{i,j} \binom{k + r - n_i - n_j - 3}{k - 1} + \sum_{i,j,k} \binom{k + r - n_i - n_j - n_k - 4}{k - 1} - \cdots \\ &\quad + (-1)^{k-1} \binom{k + r - \sum_{i=1}^{k} n_i - k - 1}{k - 1}\end{aligned}$$

$$Ans = \sum_{p=0}^{k} (-1)^p \sum_A \binom{k + r - 1 - \sum_A n_{A_i} - p}{k - 1}$$

$$\mathrm{Q}_n^n \times n = \mathrm{A}_n^n \Longrightarrow \mathrm{Q}_n = \frac{\mathrm{A}_n^n}{n} = (n-1)!$$

$$\mathrm{Q}_n^r = \frac{\mathrm{A}_n^r}{r} = \frac{n!}{r \times (n-r)!}$$

$$\binom{n}{m} = \binom{n}{n-m}$$

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$$

$$\binom{n}{m} = \binom{n-1}{m} + \binom{n-1}{m-1}$$

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{i=0}^{n}\binom{n}{i} = 2^n$$

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i} = [n=0]$$

$$\sum_{i=0}^{m}\binom{n}{i}\binom{m}{m-i} = \binom{m+n}{m} \quad (n \geq m)$$

$$\sum_{i=0}^{n}\binom{n}{i}^2 = \binom{2n}{n}$$

$$\sum_{i=0}^{n} i\binom{n}{i} = n2^{n-1}$$

$$\sum_{i=0}^{n} i^2\binom{n}{i} = n(n+1)2^{n-2}$$

$$\sum_{l=0}^{n}\binom{l}{k} = \binom{n+1}{k+1}$$

$$\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$$

$$\sum_{i=0}^{n}\binom{n-i}{i} = F_{n+1}$$

$$g_n = \sum_{i=0}^{n}\binom{n}{i}f_i$$

$$f_n = \sum_{i=0}^{n}\binom{n}{i}(-1)^{n-i}g_i$$

$$
\begin{aligned}
f_n &= \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} \left[\sum_{j=0}^{i} \binom{i}{j} f_j \right] \\
&= \sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i} \binom{i}{j} (-1)^{n-i} f_j
\end{aligned}
$$

$$
\begin{aligned}
f_n &= \sum_{j=0}^{n} \sum_{i=j}^{n} \binom{n}{i} \binom{i}{j} (-1)^{n-i} f_j \\
&= \sum_{j=0}^{n} f_j \sum_{i=j}^{n} \binom{n}{i} \binom{i}{j} (-1)^{n-i}
\end{aligned}
$$

$$
\begin{aligned}
f_n &= \sum_{j=0}^{n} f_j \sum_{i=j}^{n} \binom{n}{j} \binom{n-j}{i-j} (-1)^{n-i} \\
&= \sum_{j=0}^{n} \binom{n}{j} f_j \sum_{i=j}^{n} \binom{n-j}{i-j} (-1)^{n-i}
\end{aligned}
$$

$$
f_n = \sum_{j=0}^{n} \binom{n}{j} f_j \sum_{k=0}^{n-j} \binom{n-j}{k} (-1)^{n-j-k} 1^k
$$

$$
f_n = \sum_{j=0}^{n} \binom{n}{j} f_j [n = j] = f_n
$$

$$n$$

$$a_i (1 \le i \le n)$$

$$i$$

$$S = a_1 + a_2 + \cdots + a_n$$

$$n$$

$$a_i (1 \le i \le n)$$

$$i$$

$$S = a_1 \times a_2 \times \cdots \times a_n$$

$$n$$

$$m$$

$$m \le n$$

$$m$$

$$n$$

$$n$$

$$m$$

$$n$$

$$m$$

$$m \le n$$

$$n$$

$$m$$
$$\mathrm{A}_n^m$$
$$\mathrm{P}_n^m$$
$$n!$$
$$n$$
$$6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6$$
$$n$$
$$m$$
$$m \leq n$$
$$n$$
$$n-1$$
$$m$$
$$n-m+1$$
$$n$$
$$n$$
$$n$$
$$n-1$$
$$n$$
$$m \leq n$$
$$n$$
$$m$$
$$n$$
$$m \leq n$$
$$n$$
$$m$$
$$\binom{n}{m}$$
$$n$$
$$m$$
$$n$$
$$m$$
$$m \leq n$$
$$\mathrm{A}_n^m$$
$$m$$
$$m!$$
$$\mathrm{C}_n^m$$

$$\mathrm{C}_n^m = \binom{n}{m}$$

$$\binom{n}{m}$$

$$\mathrm{C}_n^m$$

$$m > n$$

$$\mathrm{A}_n^m = \binom{n}{m} = 0$$

$$n$$

$$k$$

$$k-1$$

$$n$$

$$n-1$$

$$\binom{n-1}{k-1}$$

$$x_1 + x_2 + \cdots + x_k = n$$

$$k$$

$$n+k$$

$$n+k-1$$

$$k$$

$$k$$

$$k$$

$$\binom{n+k-1}{n}$$

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_i \geq 0$$

$$i$$

$$a_i, \sum a_i \leq n$$

$$x_1 + x_2 + \cdots + x_k = n$$

$$x_i \geq a_i$$

$$\sum a_i$$

$$i$$

$$a_i$$

$$1 \sim n$$

$$n$$

$$k$$

$$k$$

$$\binom{n-k+1}{k}$$

$$\binom{n}{k}+\binom{n}{k-1}=\binom{n+1}{k}$$

$$n$$

$$x_i$$

$$\binom{n}{n_1, n_2, \cdots, n_t}$$

$$S=\{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_k \cdot a_k\}$$

$$n_1$$

$$a_1$$

$$n_2$$

$$a_2$$

$$n_k$$

$$a_k$$

$$S$$

$$k$$

$$n_1, n_2, \cdots, n_k$$

$$n=n_1+n_2+\ldots+n_k$$

$$n$$

$$\binom{n}{m}$$

$$\binom{n}{m, n-m}$$

$$S=\{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_k \cdot a_k\}$$

$$n_1$$

$$a_1$$

$$n_2$$

$$a_2$$

$$n_k$$

$$a_k$$

$$r(r<n_i, \forall i \in [1, k])$$

$$S$$

$$r$$

$$x_1 + x_2 + \cdots + x_k = r$$

$$S = \{n_1 \cdot a_1, n_2 \cdot a_2, \cdots, n_k \cdot a_k, \}$$

$$n_1$$

$$a_1$$

$$n_2$$

$$a_2$$

$$n_k$$

$$a_k$$

$$r$$

$$S$$

$$r$$

$$\sum_{i=1}^{k} x_i = r$$

$$x_i \leq n_i$$

$$i$$

$$S_i$$

$$\overline{S_i}$$

$$i$$

$$x_i \geq n_i + 1$$

$$\mid U \mid = \binom{k+r-1}{k-1}$$

$$\mid A \mid = p,\, A_i < A_{i+1}$$

$$n$$

$$\mathrm{Q}_n^n$$

$$O(n^2)$$

$$a = b = 1$$

$$a = 1, b = -1$$

$$n = 0$$

$$1$$

$$(6)$$

$$n = m$$

$$(3)$$

$$S = \{a_1, a_2, \cdots, a_{n+1}\}$$

$$k+1$$

$$F$$

$$f_n$$

$$n$$

$$g_n$$

$$n$$

$$i \geq 0$$

$$f_n$$

$$g_n$$

$$g_n$$

$$f_n$$

$$g_n$$

$$f_n$$

$$g_i$$

$$j$$

$$i$$

$$k = i - j$$

$$i = k + j$$

$$H_n = \frac{\binom{2n}{n}}{n+1} (n \geq 2, n \in \mathbf{N}_+)$$

$$H_n = \begin{cases} \sum_{i=1}^{n} H_{i-1} H_{n-i} n \geq 2, n \in \mathbf{N}_+ \\ 1 \qquad\qquad\qquad n = 0, 1 \end{cases}$$

$$H_n = \frac{H_{n-1}(4n-2)}{n+1}$$

$$H_n = \binom{2n}{n} - \binom{2n}{n-1}$$

$$H_n = \sum_{i=0}^{n-1} H_i H_{n-i-1} \quad (n \geq 2)$$

$$
\begin{aligned}
H(x) &= \sum_{n\geq 0} H_n x^n \\
&= 1 + \sum_{n\geq 1}\sum_{i=0}^{n-1} H_i x^i H_{n-i-1} x^{n-i-1} x \\
&= 1 + x \sum_{i\geq 0} H_i x^i \sum_{n\geq 0} H_n x^n \\
&= 1 + xH^2(x)
\end{aligned}
$$

$$
H(x) = \frac{1 \pm \sqrt{1-4x}}{2x}
$$

$$
H(x) = \frac{2}{1 \mp \sqrt{1-4x}}
$$

$$
H(x) = \frac{1 - \sqrt{1-4x}}{2x}
$$

$$
\begin{aligned}
(1-4x)^{\frac{1}{2}} &= \sum_{n\geq 0} \binom{\frac{1}{2}}{n} (-4x)^n \\
&= 1 + \sum_{n\geq 1} \frac{\left(\frac{1}{2}\right)^{\underline{n}}}{n!} (-4x)^n
\end{aligned}
$$

$$
\begin{aligned}
\left(\frac{1}{2}\right)^{\underline{n}} &= \frac{1}{2}\frac{-1}{2}\frac{-3}{2}\cdots\frac{-(2n-3)}{2} \\
&= \frac{(-1)^{n-1}(2n-3)!!}{2^n} \\
&= \frac{(-1)^{n-1}(2n-2)!}{2^n(2n-2)!!} \\
&= \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!}
\end{aligned}
$$

$$
\begin{aligned}
(1-4x)^{\frac{1}{2}} &= 1 + \sum_{n\geq 1} \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!n!} (-4x)^n \\
&= 1 - \sum_{n\geq 1} \frac{(2n-2)!}{(n-1)!n!} 2x^n \\
&= 1 - \sum_{n\geq 1} \binom{2n-1}{n} \frac{1}{(2n-1)} 2x^n
\end{aligned}
$$

$$
\begin{aligned}
H(x) &= \frac{1-\sqrt{1-4x}}{2x} \\
&= \frac{1}{2x}\sum_{n\geq 1}\binom{2n-1}{n}\frac{1}{(2n-1)}2x^n \\
&= \sum_{n\geq 1}\binom{2n-1}{n}\frac{1}{(2n-1)}x^{n-1} \\
&= \sum_{n\geq 0}\binom{2n+1}{n+1}\frac{1}{(2n+1)}x^n \\
&= \sum_{n\geq 0}\binom{2n}{n}\frac{1}{n+1}x^n
\end{aligned}
$$

$$H_n$$

$$2n$$

$$n$$

$$n$$

$$n \times n$$

$$(0,0)$$

$$(n,n)$$

$$y = x$$

$$2n$$

$$n$$

$$1, 2, 3, \cdots, n$$

$$n$$

$$n$$

$$+1$$

$$n$$

$$-1$$

$$2n$$

$$a_1, a_2, \cdots, a_{2n}$$

$$a_1 + a_2 + \cdots + a_k \geq 0 \ (k = 1, 2, 3, \cdots, 2n)$$

$$H_0$$

$$H_1$$

$$H_2$$

$$H_3$$

$$H_4$$

$$H_5$$

$$H_6$$

$$1, 2, ..., n$$

$$H_0 = 1, H_1 = 1$$

$$H(x)$$

$$H(x)$$

$$x = 0$$

$$H(x)$$

$$H_0$$

$$H(x) = \frac{2}{1 + \sqrt{1 - 4x}}$$

$$H(0) = 1$$

$$0$$

$$\sqrt{1 - 4x}$$

$$(1)$$

$$(0, 0)$$

$$(m, n)$$

$$m$$

$$x$$

$$n$$

$$y$$

$$\binom{n + m}{m}$$

$$(0, 0)$$

$$(n, n)$$

$$y = x$$

$$y = x$$

$$(0, 0)$$

$$(1, 0)$$

$$(n, n - 1)$$

$$(n, n)$$

$$(1, 0)$$

$$(n, n - 1)$$

$$y = x$$

$$\binom{2n-2}{n-1}$$

$$y = x$$

$$(1, 0)$$

$$y = x$$

$$(0, 1)$$

$$(n, n-1)$$

$$y = x$$

$$\binom{2n-2}{n-1} - \binom{2n-2}{n}$$

$$2\binom{2n-2}{n-1} - 2\binom{2n-2}{n}$$

$$(0, 0)$$

$$(n, n)$$

$$y = x$$

$$\frac{2}{n+1}\binom{2n}{n}$$

$$1 \sim n$$

$$1 \sim n$$

$$O(n^2)$$

$$O(n \log n)$$

$$4$$

$$[2, 3, 1, 4] < [2, 3, 4, 1]$$

$$3$$

$$[2, 3, 1, 4]$$

$$[2, 3, 4, 1]$$

$$5$$

$$[2, 5, 3, 4, 1]$$

$$1$$

$$1$$

$$5$$

$$4!$$

$$5$$

$$5$$

$$5$$

$$1, 3$$

$$4$$

$$2$$

$$3 \times 3!$$

$$1 + 4! + 3 \times 3! + 2! + 1 = 46$$

$$+1$$

$$4!$$

$$3 \times 3! + 2 \times 2! + 1 \times 1!$$

$$5$$

$$x$$

$$4!$$

$$46 - 1 = 45$$

$$45$$

$$\left\lfloor \frac{45}{4!} \right\rfloor = 1$$

$$2$$

$$1 \times 4!$$

$$21$$

$$\left\lfloor \frac{21}{3!} \right\rfloor = 3$$

$$3$$

$$2$$

$$5$$

$$21 - 3 \times 3! = 3$$

$$\left\lfloor \frac{3}{2!} \right\rfloor = 1$$

$$3$$

$$3 - 1 \times 2! = 1$$

$$4$$

$$1$$

$$[2, 5, 3, 4, 1]$$

$$O(n \log n)$$

$$S_m(n) = \sum_{k=0}^{n-1} k^m = 0^m + 1^m + ... + (n-1)^m$$

$$\begin{aligned}
S_0(n) &= n \\
S_1(n) &= \frac{1}{2}n^2 - \frac{1}{2}n \\
S_2(n) &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n \\
S_3(n) &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
S_4(n) &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n
\end{aligned}$$

$$\begin{aligned}
S_m(n) &= \frac{1}{m+1}(B_0 n^{m+1} + \binom{m+1}{1} B_1 n^m + ... + \binom{m+1}{m} B_m n) \\
&= \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}
\end{aligned}$$

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0, (m > 0)$$

$$B_0 = 1$$

$$\begin{aligned}
S_{m+1}(n) + n^{m+1} &= \sum_{k=0}^{n-1} (k+1)^{m+1} \\
&= \sum_{k=0}^{n-1} \sum_{j=0}^{m+1} \binom{m+1}{j} k^j \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} S_j(n)
\end{aligned}$$

$$\begin{aligned}
S_{m+1}(n) + n^{m+1} &= \sum_{j=0}^{m+1} \binom{m+1}{j} S_j(n) \\
n^{m+1} &= \sum_{j=0}^{m} \binom{m+1}{j} S_j(n) \\
&= \sum_{j=0}^{m-1} \binom{m+1}{j} \hat{S}_j(n) + \binom{m+1}{m} S_m(n)
\end{aligned}$$

$$n^{m+1} = \sum_{j=0}^{m} \binom{m+1}{j} \hat{S}_j(n) + (m+1)(S_m(n) - \hat{S}_m(n))$$

$$
\begin{aligned}
n^{m+1} &= \sum_{j=0}^{m} \binom{m+1}{j} \hat{S}_j(n) + (m+1)\Delta \\
&= \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{j+1} \sum_{k=0}^{j} \binom{j+1}{k} B_k n^{j+1-k} + (m+1)\Delta
\end{aligned}
$$

$$
\begin{aligned}
n^{m+1} &= \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{j+1} \sum_{k=0}^{j} \binom{j+1}{j-k} B_{j-k} n^{k+1} + (m+1)\Delta \\
&= \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{j+1} \sum_{k=0}^{j} \binom{j+1}{k+1} B_{j-k} n^{k+1} + (m+1)\Delta \\
&= \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{j+1} \sum_{k=0}^{j} \frac{j+1}{k+1} \binom{j}{k} B_{j-k} n^{k+1} + (m+1)\Delta \\
&= \sum_{j=0}^{m} \binom{m+1}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{B_{j-k}}{k+1} n^{k+1} + (m+1)\Delta
\end{aligned}
$$

$$
n^{m+1} = \sum_{k=0}^{m} \frac{n^{k+1}}{k+1} \sum_{j=k}^{m} \binom{m+1}{j} \binom{j}{k} B_{j-k} + (m+1)\Delta
$$

$$
\binom{m+1}{j} \binom{j}{k} \square \binom{m+1}{k} \binom{m-k+1}{j-k}
$$

$$
\begin{aligned}
n^{m+1} &= \sum_{k=0}^{m} \frac{n^{k+1}}{k+1} \sum_{j=k}^{m} \binom{m+1}{k} \binom{m-k+1}{j-k} B_{j-k} + (m+1)\Delta \\
&= \sum_{k=0}^{m} \frac{n^{k+1}}{k+1} \binom{m+1}{k} \sum_{j=k}^{m} \binom{m-k+1}{j-k} B_{j-k} + (m+1)\Delta
\end{aligned}
$$

$$
n^{m+1} = \sum_{k=0}^{m} \frac{n^{k+1}}{k+1} \binom{m+1}{k} \sum_{j=0}^{m-k} \binom{m-k+1}{j} B_j + (m+1)\Delta
$$

$$
\sum_{j=0}^{m} \binom{m+1}{j} B_j = 0, (m > 0)
$$

$$
B_0 = 1
$$

$$
\sum_{j=0}^{m} \binom{m+1}{j} B_j = [m = 0]
$$

$$
\begin{aligned}
n^{m+1} &= \sum_{k=0}^{m} \frac{n^{k+1}}{k+1} \binom{m+1}{k} [m - k = 0] + (m+1)\Delta \\
&= \sum_{k=0}^{m} \frac{n^{k+1}}{k+1} \binom{m+1}{k} + (m+1)\Delta \\
&= \frac{n^{m+1}}{m+1} \binom{m+1}{m} + (m+1)\Delta \\
&= n^{m+1} + (m+1)\Delta
\end{aligned}
$$

$$\sum_{j=0}^{m+1} \binom{m+1}{j} B_j = [m=0] + B_{m+1}$$

$$\sum_{j=0}^{m} \binom{m}{j} B_j = [m=1] + B_m$$

$$\sum_{j=0}^{m} \frac{B_j}{j!} \cdot \frac{1}{(m-j)!} = [m=1] + \frac{B_m}{m!}$$

$$B(z)\mathrm{e}^z = z + B(z)$$

$$B(z) = \frac{z}{\mathrm{e}^z - 1}$$

$$F_n(z) = \sum_{m \geq 0} \frac{S_m(n)}{m!} z^m$$

$$= \sum_{m \geq 0} \sum_{i=0}^{n-1} \frac{i^m z^m}{m!}$$

$$F_n(z) = \sum_{i=0}^{n-1} \sum_{m \geq 0} \frac{i^m z^m}{m!}$$

$$= \sum_{i=0}^{n-1} \mathrm{e}^{iz}$$

$$= \frac{\mathrm{e}^{nz} - 1}{\mathrm{e}^z - 1}$$

$$= \frac{z}{\mathrm{e}^z - 1} \cdot \frac{\mathrm{e}^{nz} - 1}{z}$$

$$F_n(z) = B(z) \cdot \frac{\mathrm{e}^{nz} - 1}{z}$$

$$= \left(\sum_{i \geq 0} \frac{B_i}{i!} \right) \left(\sum_{i \geq 1} \frac{n^i z^{i-1}}{i!} \right)$$

$$= \left(\sum_{i \geq 0} \frac{B_i}{i!} \right) \left(\sum_{i \geq 0} \frac{n^{i+1} z^i}{(i+1)!} \right)$$

$$S \times m(n) = m! [z^m] F_n(z)$$

$$= m! \sum_{i=0}^{m} \frac{B \times i}{i!} \cdot \frac{n^{m-i+1}}{(m-i+1)!}$$

$$= \frac{1}{m+1} \sum_{i=0}^{m} \binom{m+1}{i} B_i n^{m-i+1}$$

$$B_n$$

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, \ldots$$

$$m$$

$$S_m(n)$$

$$n^{m+1}$$

$$\frac{1}{m+1}$$

$$n^m$$

$$-\frac{1}{2}$$

$$n^{m-1}$$

$$\frac{m}{12}$$

$$n^{m-3}$$

$$-\frac{m(m-1)(m-2)}{720}$$

$$n^{m-4}$$

$$n^{m-k}$$

$$m^{\underline{k}}$$

$$m^{\underline{k}}$$

$$\frac{m!}{(m-k)!}$$

$$\binom{2}{0}B_0 + \binom{2}{1}B_1 = 0$$

$$n$$

$$0$$

$$1$$

$$2$$

$$3$$

$$4$$

$$5$$

$$6$$

$$7$$

$$8$$

$$\cdots$$

$$B_n$$

$$1$$

$$-\frac{1}{2}$$

$$\frac{1}{6}$$

$$0$$

$$-\frac{1}{30}$$

$$0$$

$$\frac{1}{42}$$

$$0$$

$$-\frac{1}{30}$$

$$\ldots$$

$$\hat{S}_m(n) = \frac{1}{m+1} \sum_{k=0}^{m} \binom{m+1}{k} B_k n^{m+1-k}$$

$$S_m(n) = \hat{S}_m(n)$$

$$j \in [0, m)$$

$$S_j(n) = \hat{S}_j(n)$$

$$S_{m+1}(n)$$

$$\binom{m+1}{m} \hat{S}_m(n) - \binom{m+1}{m} \hat{S}_m(n)$$

$$\Delta = S_m(n) - \hat{S}_m(n)$$

$$\hat{S}_j(n)$$

$$\sum$$

$$\binom{m+1}{j} \binom{j}{k}$$

$$j - k$$

$$j$$

$$\Delta = 0$$

$$S_m(n) = \hat{S}_m(n)$$

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j = [m = 0]$$

$$B_{m+1}$$

$$B(z)=\sum_{i\geq 0}\frac{B_i}{i!}z^i$$

$$F_n(z)=\sum_{m\geq 0}\frac{S_m(n)}{m!}z^m$$

$$B(z)=\frac{z}{\mathrm{e}^z-1}$$

$$F_n(z)=\sum_{m\geq 0}\frac{S_m(n)}{m!}z^m$$

$$S_m(n)=m![z^m]F_n(z)$$

$$B_0=1,B_1=1,B_2=2,B_3=5,B_4=15,B_5=52,B_6=203,...$$

$$\begin{aligned}&\{\{a\},\{b\},\{c\}\}\\&\{\{a\},\{b,c\}\}\\&\{\{b\},\{a,c\}\}\\&\{\{c\},\{a,b\}\}\\&\{\{a,b,c\}\}\end{aligned}$$

$$B_{n+1}=\sum_{k=0}^{n}\binom{n}{k}B_k$$

$$B_n=\sum_{k=0}^{n}\left\{{n \atop k}\right\}$$

1						
1	2					
2	3	5				
5	7	10	15			
15	20	27	37	52		
52	67	87	114	151	203	
203	255	322	409	523	674	877

$$\begin{aligned}\hat{B}(x)&=\sum_{n=0}^{+\infty}\frac{B_n}{n!}x^n\\&=1+\sum_{n=0}^{+\infty}\frac{B_{n+1}}{(n+1)!}x^{n+1}\end{aligned}$$

$$\hat{B}'(x)=\sum_{n=0}^{+\infty}\frac{B_{n+1}}{n!}x^n$$

$$\frac{B_{n+1}}{n!}=\sum_{k=0}^{n}\frac{1}{(n-k)!}\frac{B_k}{k!}$$

$$\hat{B}'(x)=\mathrm{e}^x\hat{B}(x)$$

$$\hat{B}(x)=\exp(\mathrm{e}^x+C)$$

$$\hat{B}(x) = \exp(\mathrm{e}^x - 1)$$

$$B_n$$

$$B_n$$

$$n$$

$$S$$

$$S$$

$$S$$

$$B_3 = 5$$

$$a, b, c$$

$$B_0$$

$$B_{n+1}$$

$$n + 1$$

$$D_n$$

$$\{b_1, b_2, b_3, ..., b_n\}$$

$$D_{n+1}$$

$$\{b_1, b_2, b_3, ..., b_n, b_{n+1}\}$$

$$D_{n+1}$$

$$D_n$$

$$b_{n+1}$$

$$b_{n+1}$$

$$n$$

$$\binom{n}{n} B_n$$

$$n - 1$$

$$\binom{n}{n-1} B_{n-1}$$

$$n - 2$$

$$\binom{n}{n-2} B_{n-2}$$

$$n$$

$$k$$

$$a_{0,0} = 1$$

$$n \geq 1$$

$$n$$
$$a_{n,0} = a_{n-1,n-1}$$
$$m, n \geq 1$$
$$n$$
$$m$$
$$a_{n,m} = a_{n,m-1} + a_{n-1,m-1}$$
$$x = 0$$
$$\hat{B}(x) = 1$$
$$C = -1$$
$$\mathrm{e}^x - 1$$
$$n$$
$$n$$
$$O(n \log n)$$
$$O(1)$$
$$\Theta(n)$$
$$\Theta(\log(n))$$
$$\Theta(n)$$
$$\Theta(\log(n))$$
$$\Theta(n)$$
$$\Theta(\log(n))$$
$$O(1)$$
$$\Theta(n)$$
$$\Theta(\log(n))$$
$$\Theta(n)$$
$$\Theta(\log(n))$$
$$\Theta(n)$$
$$\Theta(n)$$
$$\Theta(n)$$
$$\Theta(\log(n))$$
$$O(1)$$
$$\Theta(\log(n))$$

$\Theta(\log(n))$

$\Theta(\log(n))$

$\Theta(\log(n))$

$O(1)$

$\Theta(\log(n))$

$\Theta(\log(n))$

$\Theta(\log(n))$

$\Theta(\log(n))$

$O(1)$

$\Theta(n)$

$\Theta(\log(n))$

$\Theta(\log(n))$

$O(1)$

$\Theta(n)$

$\Theta(\log(n))$

$\Theta(n)$

126271

107897

n

N

$n > N$

$2N$

$O(n)$

$O(1)$

$O(1)$

$\Theta(\text{size})$

$O(1)$

$$\sum_{i\in A}[\frac{i}{x}=1]$$
$$=\sum_{i\in A}\sum_{d\mid\frac{i}{x}}\mu(d)$$
$$=\sum_{d\in A\prime,x\mid d}\mu(\frac{d}{x})$$

$$\frac{1}{32}$$

$$\frac{1}{32}$$

$$O(n)$$

$$O(n)$$

$$O(\frac{n}{32})$$

$$\frac{1}{32}$$

$$O(\frac{n}{w})$$

$$w=32$$

$$O(\frac{n}{\log w})$$

$$w$$

$$01$$

$$0\leftrightarrow 1$$

$$1$$

$$f(i,j)$$

$$i$$

$$j$$

$$f(i,j)=\bigvee_{k=a}^{b}f(i-1,j-k^2)$$

$$O(n^5)$$

$$64$$

$$n$$

$$\gcd$$

$$A=\{\gcd(x,y)|\ x\in B,y\in C\}$$

$$10^5$$

$$10^6$$

$$7000$$

$$2$$

$$1$$

$$2$$

$$2$$

$$4$$

$$3$$

$$3$$

$$1$$

$$2$$

$$A$$

$$A'$$

$$A$$

$$x$$

$$2$$

$$-1$$

$$1$$

$$\frac{d}{x}$$

$$O(\frac{v}{w})$$

$$v = 7000, w = 32$$

$$O(v\sqrt{v})$$

$$O(v^2)$$

$$\log$$

$$O(v \log v)$$

$$5 \times 10^7$$

$$10^8$$

$$5 \times 10^8$$

$$O(n \log \log n)$$

$$O(n)$$

$$[first, last)$$

$$O(\log n)$$

$$O(n)$$

$$k$$

$$O(n)$$

$$O(\log n)$$

$$k$$

$$[first, last)$$

$$O(\log(size) + ans)$$

$$O(n)$$

$$n$$

$$x$$

$$x$$

$$O(\log n)$$

$$O(n)$$

$$O(\log n)$$

$$x$$

$$y$$

$$y[i] = x[0] + x[1] + ... + x[i]$$

$$1$$

$$9$$

$$a$$

$$x$$

$$x$$

$$x$$

$$src$$

$$dst$$

$$a$$

$$x$$

$$a$$

$$k$$

$$k$$

$$G = (V, E)$$

$$V$$

$$E$$

$$M(M \in E)$$

$$\mid M \mid$$

$$M$$

$$M$$

$$G$$

$$G$$

$$M$$

$$M$$

$$G$$

$$M$$

$$G = \langle V_1, V_2, E \rangle$$

$$\mid V_1 \mid \leq \mid V_2 \mid$$

$$M$$

$$G$$

$$\mid M \mid = \mid V_1 \mid$$

$$M$$

$$V_1$$

$$V_2$$

$$G = \langle V_1, V_2, E \rangle, \mid V_1 \mid \leq \mid V_2 \mid$$

$$G$$

$$V_1$$

$$V_2$$

$$S \subset V_1$$

$$\mid S \mid \leq \mid N(S) \mid$$

$$N(S) = \mathbb{U}_{v_i \in S} N(V_i)$$

$$S$$

$$O(\sqrt{V} E)$$

$$O(V^2 E)$$

$$O(V^2 \log V + VE)$$

$$O(V^2E)$$

$$O(V^2E)$$

$$(w(e) < 0)$$

$$G$$

$$M = \emptyset$$

$$G$$

$$w(e)$$

$$0$$

$$G$$

$$G$$

$$0$$

$$(w(e) = 0)$$

$$K = \max\{|\ w(e)| : e \in E, w(e) \leq 0\} + 1$$

$$K = 0$$

$$w(e)$$

$$K$$

$$G' = (V, E')$$

$$G$$

$$P$$

$$P$$

$$P = \sum w(e) : e \in E'$$

$$w(e)$$

$$P$$

$$G'' = (V, E'')$$

$$G$$

□$S \subseteq V$

$\gamma(S) = \{(u, v) \in E : u \in S, v \in S\}$

$O = \{B \subseteq V : |\ B\ |$ □□□□$|\ B\ | \geq 3\}$

$$\max \sum_{e \in E} w(e) x_e$$

□□□

$$x(\delta(u)) = 1 : \forall u \in V$$

$$x(\gamma(B)) \leq \left\lfloor \frac{\mid B \mid}{2} \right\rfloor : \forall B \in O$$

$$x_e \geq 0 : \forall e \in E$$

$$\min \sum_{u \in V} z_u + \sum_{B \in O} \left\lfloor \frac{\mid B \mid}{2} \right\rfloor z_B$$

□□□

$$z_B \geq 0 : \forall B \in O$$

$$z_e \geq 0 : \forall e \in E$$

□$e = (u, v)$□□□

z_e
$=$
$z_u + z_v - w(e) + \sum_{\substack{B \in O \\ u, v \in \gamma(B)}} z_B$

$$x_e > 0 \longrightarrow z_e = 0, \quad \forall e \in E$$

$$z_B > 0 \longrightarrow x(\gamma(B)) = \left\lfloor \frac{\mid B \mid}{2} \right\rfloor, x(\delta(B)) = 1 \quad \forall B \in O$$

$$d1 = \min(\{z_e : e = (u^+, v^\emptyset)\})$$

$$d2 = \min(\{z_e : e = (u^+, v^+),\ u^+ \in T_i,\ v^+ \in T_j,\ i \neq j\})/2$$

$$d3 = \min(\{z_{B^-} : B^- \in O\})/2$$

$$z_{u^+} - = d$$

$$z_{v^-} + = d$$

$$z_{B^+} + = 2d$$

$$z_{B^-} - = 2d$$

$$z_u$$

$$u$$

$$KM$$

$$e(u, v)$$

$$u$$

$$v$$

$$e$$

$$z_u + z_v = w(e)$$

$$z_e = z_u + z_v - w(e) = 0$$

$$O$$

$$\geq 3$$

$$\gamma(S)$$

$$S$$

$$x_e = 1$$

$$x_e = 0$$

$$x_e \in \{0, 1\} : \forall e \in E$$

$$z_B$$

$$z_B$$

$$0$$

$$z_B = 0$$

$$z_B > 0$$

$$x(\gamma(B)) = \left\lfloor \frac{\mid B \mid}{2} \right\rfloor \square x(\delta(B)) = 1$$

$$z_B > 0$$

$$z_B > 0$$

$$e$$

$$z_e = 0$$

$$z_B > 0 \longrightarrow x(\gamma(B)) = \left\lfloor \frac{\mid B \mid}{2} \right\rfloor$$

$$z_B > 0$$

$$B$$

$$B$$

$$x(\delta(B)) = 1$$

$$B$$

$$z_B > 0$$

$$z_B = 0$$

$$z_B$$

$$u^-$$

$$u$$

$$u^+$$

$$u$$

$$u^\emptyset$$

$$u$$

$$B$$

$$B^{+}$$

$$B^{-}$$

$$B^{\emptyset}$$

$$T_i = (U_{t_i}, V_{t_i}) : 1 \leq i \leq r$$

$$d = min(d1, d2, d3)$$

$$z_B = 0(d = d3)$$

$$z_B < 0$$

$$\emptyset$$

$$z_e \geq 0 : \forall e \in E$$

$$z_B$$

$$u$$

$$z_u > 0$$

$$z_u = max(\{w(e) : e \in E\})/2$$

$$0$$

$$2$$

$$z_e$$

$$\{6, 5, 8\} \in b1, \{b1, 4, 3, 2, 11, 10, 9\} \in b2$$

$$O(\mid V \mid)$$

$$O(\mid V \mid)$$

$$O(\mid V \mid^2)$$

$$O(\mid V \mid + \mid E \mid)$$

$$O(\mid V \mid + \mid E \mid) + O(\mid V \mid^2) = O(\mid V \mid^2)$$

$$\mid V \mid$$

$$O(\mid V \mid^3)$$

$$\tilde{A}(G)_{i,j} = \begin{cases} x_{i,j}, & i < j,\, (v_i, v_j) \in E \\ -x_{i,j}, & i > j,\, (v_i, v_j) \in E \\ 0, & \text{otherwise} \end{cases}$$

$$\det A = \sum_{\pi} (-1)^{\pi} \prod_{i} A_{i,\pi_i}$$

$$A = \begin{bmatrix} a_{1,1} & v^T \\ u & B \end{bmatrix} \quad A^{-1} = \begin{bmatrix} \hat{a}^{1,1} & \hat{v}^T \\ \hat{u} & \hat{B} \end{bmatrix}$$

$$B^{-1} = \hat{B} - \frac{\hat{u}\hat{v}^T}{\hat{a}_{1,1}}$$

$$M$$

$$v$$

$$u$$

$$u$$

$$u$$

$$u$$

$$u$$

$$G$$

$$G'$$

$$G$$

$$G'$$

$$G'$$

$$G$$

$$(u, v)$$

$$h = LCA(u, v)$$

$$h$$

$$h$$

$$h$$

$$O(\mid E \mid^2)$$

$$O(\mid V \mid \mid E \mid^2)$$

$$O(n^3)$$

$$n$$

$$G = (V, E)$$

$$\tilde{A}(G)$$

$$n \times n$$

$$x_{i,j}$$

$$\tilde{A}(G)$$

$$\mid E \mid$$

$$\tilde{A}(G)$$

$$\tilde{A}$$

G

$\det \tilde{A} \neq 0$

G

G

G

G

G

G

$\tilde{A} \neq 0$

π

$(-1)^{\pi}$

π

-1

1

G

0

0

$\operatorname{rank} \tilde{A}$

G

$\operatorname{rank} \tilde{A}$

$\mid E \mid$

p

$\mathcal{Z}_p$

$\mathcal{Z}_p$

$\tilde{A}$

$\operatorname{rank} \tilde{A}$

G

$1 - \frac{n}{p}$

n

10^3

p

$$10^9$$

$$\operatorname{rank} \tilde{A}$$

$$G$$

$$\tilde{A}$$

$$G$$

$$i$$

$$v_i$$

$$\tilde{A}^{-1}_{j,i} \neq 0 \Longleftrightarrow G - \{v_i, v_j\}$$

$$n$$

$$A$$

$$A^*_{i,j} = (-1)^{i+j} M_{j,i}$$

$$M_{j,i}$$

$$j$$

$$i$$

$$A$$

$$M$$

$$A^* = M^T$$

$$A$$

$$A^{-1} = \frac{1}{\det A} A^*$$

$$A^{-1}_{j,i} \neq 0 \Longleftrightarrow M_{i,j} \neq 0$$

$$A$$

$$i$$

$$j$$

$$(v_i, v_j) \in E$$

$$\tilde{A}^{-1}_{j,i} \neq 0$$

$$(v_i, v_j)$$

$$G$$

$$i, j$$

$$(v_i, v_j)$$

$$\tilde{A}^{-1}_{j,i} \neq 0$$

$$(v_i, v_j)$$

$$G$$

$$\tilde{A}^{-1}$$

$$\frac{n}{2}$$

$$O(n^3)$$

$$O(n^4)$$

$$\tilde{A}^{-1}$$

$$\hat{a}_{1,1} \neq 0$$

$$\tilde{A}^{-1}$$

$$O(n^2)$$

$$\frac{n}{2}$$

$$O(n^2)$$

$$O(n^3)$$

$$G$$

$$\operatorname{rank} \tilde{A}$$

$$\tilde{A}$$

$$G$$

$$G$$

$$\tilde{A}$$

$$\tilde{A}$$

$$\tilde{A}$$

$$\tilde{A}$$

$$\tilde{A}$$

$$O(n^3)$$

$$0$$

$$i$$

$$l(i)$$

$$(u, v)$$

$$w(u, v) \leq l(u) + l(v)$$

$$w(u,v) = l(u) + l(v)$$

$$(u,v)$$

$$M$$

$$val(M) = \sum_{(u,v)\in M} w(u,v) \le \sum_{(u,v)\in M} l(u) + l(v) \le \sum_{i=1}^{n} l(i)$$

$$M'$$

$$val(M') = \sum_{(u,v)\in M} l(u) + l(v) = \sum_{i=1}^{n} l(i)$$

$$val(M')$$

$$M'$$

$$n$$

$$lx(i)$$

$$i$$

$$ly(i)$$

$$i$$

$$w(u,v)$$

$$u$$

$$v$$

$$lx(i) = \max_{1\le j\le n}\{w(i,j)\}, ly(i) = 0$$

$$S$$

$$T$$

$$S'$$

$$T'$$

$$S - T'$$

$$S' - T$$

$$S$$

$$S$$

$$-a$$

$$T$$

$$+a$$

$$S - T$$

$$S' - T'$$

$$S - T'$$

$$lx + ly$$

$$S' - T$$

$$lx + ly$$

$$a$$

$$S - T'$$

$$a = \min\{lx(u) + ly(v) - w(u, v) \mid u \in S, v \in T'\}$$

$$(u, v)$$

$$v$$

$$v$$

$$S'$$

$$n$$

$$T$$

$$v$$

$$slack(v) = \min\{lx(u) + ly(v) - w(u, v) \mid u \in S\}$$

$$O(n)$$

$$a$$

$$a = \min\{slack(v) \mid v \in T'\}$$

$$S$$

$$O(n)$$

$$slack(v)$$

$$O(n)$$

$$slack(v)$$

$$a$$

$$n$$

$$n$$

$$n$$

$$O(n^3)$$

$$u_i$$

$$v_j$$

$$lx(u_i) = max(w_{ij} - v_j), \forall j$$

$$u_i$$

$$u_i$$

$$v_j$$

$$ly(v_j) = max(w_{ij} - u_i), \forall i$$

$$v_j$$

$$u_i$$

$$v_j$$

$$u_i$$

$$v_j$$

$$\alpha^i$$

$$\beta^j$$

$$G_l$$

$$w_{ij} = alpha_i + beta_j$$

$$M^*$$

$$\alpha^{i^*}$$

$$\beta^{j^*}$$

$$i^*$$

$$v_{i^*}$$

$$v_{i^*}$$

$$w_{i^*j} \leq alpha_{i^*} + beta_j$$

$$u_j$$

$$i^*$$

$$v_{i^*}$$

$$x$$

$$y$$

$$1$$

$$0$$

$$1$$

$$0$$

$$u$$

$$v$$

$$w$$

$$u$$

$$v$$

1

w

n

m

G

s

t

$O(m)$

n

$O(nm)$

1

1

$O(\sqrt{n}m)$

$O(m)$

$O(nm)$

1

$O(m)$

$O(\sqrt{n})$

$O(\sqrt{n}m)$

$O(\sqrt{n})$

$\sqrt{n}$

$\sqrt{n}$

$\sqrt{n}$

$O(\sqrt{n}m)$

$=$

$= n -$

$a - b$

x

P_x

P_x

$a - b$

P_x

$$P_x$$
$$a - b$$
$$a - b$$
$$a - b$$
$$x$$
$$a - b$$
$$x$$
$$r$$
$$r$$
$$T = (V_t, E_t)$$
$$1$$
$$1$$
$$G = (V, E)$$
$$S$$
$$T = V - S$$
$$s \in S$$
$$t \in T$$
$$(S, T)$$
$$c(S, T)$$
$$S$$
$$T$$
$$c(S, T) = \sum_{u \in S, v \in T} c(u, v)$$
$$c(s, t)$$
$$c(S, T)$$
$$(S, T)$$
$$c(S, T)$$
$$s$$
$$0$$
$$S$$
$$\infty$$
$$1$$
$$1$$

n

A, B

A

a_i

B

b_i

u_i, v_i, w_i

u_i

v_i

w_i

s

t

i

s

a_i

t

b_i

u, v, w

u, v

w

s

t

A

B

0

s

t

u

s

u

u

u

t

∞

s

$$s$$

$$t$$

$$\infty$$

$$=$$

$$-$$

$$+$$

$$s$$

$$t$$

$$=$$

$$-$$

$$=$$

$$-$$

$$=$$

$$-$$

$$G = (V, E)$$

$$c(u, v)$$

$$w(u, v)$$

$$(u, v)$$

$$f(u, v)$$

$$f(u, v) \times w(u, v)$$

$$w$$

$$w(u, v) = -w(v, u)$$

$$\sum_{(s,v) \in E} f(s, v)$$

$$\sum_{(u,v) \in E} f(u, v) \times w(u, v)$$

$$i$$

$$f_i$$

$$f_0 = 0$$

$$f_i$$

$$f_i$$

$$f_{i+1}$$

$$f_{i+1} - f_i$$

$$f_{i+1}$$

$$f'_{i+1}$$

$$f_{i+1} - f_i$$

$$f'_{i+1} - f_i$$

$$f_i$$

$$s$$

$$f_i$$

$$O(nm)$$

$$f$$

$$O(nmf)$$

$$O(nmf)$$

$$m = n^2, f = 2^{n/2}$$

$$O(n^3 2^{n/2})$$

$$O(nm)$$

$$h_i$$

$$u$$

$$v$$

$$w$$

$$w + h_u - h_v$$

$$i$$

$$d'_i$$

$$h_i$$

$$d'_i$$

$$(i, j)$$

$$(j, i)$$

$$d'_i + (w(i, j) + h_i - h_j) = d'_j$$

$$(i, j)$$

$$w(j, i) + (h_j + d'_j) - (h_i + d'_i) = 0$$

$$d'_i + (w(i, j) + h_i - h_j) - d'_j \geq 0$$

$$w(i, j) + (d'_i + h_i) - (d'_j + h_j) \geq 0$$

$$h_i + d'_i$$

$$(i, j)$$

$$O(m \log n)$$

$$O(n^2)$$

$$\begin{aligned}
\mid f \mid &= f(s) \\
&= \sum_{u \in S} f(u) \\
&= \sum_{u \in S} \left(\sum_{v \in V} f(u,v) - \sum_{v \in V} f(v,u) \right) \\
&= \sum_{u \in S} \left(\sum_{v \in T} f(u,v) + \sum_{v \in S} f(u,v) - \sum_{v \in T} f(v,u) - \sum_{v \in S} f(v,u) \right) \\
&= \sum_{u \in S} \left(\sum_{v \in T} f(u,v) - \sum_{v \in T} f(v,u) \right) + \sum_{u \in S} \sum_{v \in S} f(u,v) - \sum_{u \in S} \sum_{v \in S} f(v,u) \\
&= \sum_{u \in S} \left(\sum_{v \in T} f(u,v) - \sum_{v \in T} f(v,u) \right) \\
&\leq \sum_{u \in S} \sum_{v \in T} f(u,v) \\
&\leq \sum_{u \in S} \sum_{v \in T} c(u,v) \\
&= \mid \mid S, T \mid \mid
\end{aligned}$$

$$p_{in}(v) = \sum_{(u,v) \in L} (c(u,v) - f(u,v))$$

$$p_{out}(v) = \sum_{(v,u) \in L} (c(v,u) - f(v,u))$$

$$p(v) = \min(p_{in}(v), p_{out}(v))$$

$$e(u) = \sum_{(x,u) \in E} f(x,u) - \sum_{(u,y) \in E} f(u,y)$$

$$\forall (u,v) \in E, \quad f(u,v) = \begin{cases} c(u,v), u = s \\ 0, \qquad u \neq s \end{cases}$$

$$\forall u \in V, \quad h(u) = \begin{cases} \mid V \mid, u = s \\ 0, \qquad u \neq s \end{cases}$$

$$e(u) = \sum_{(x,u) \in E} f(x,u) - \sum_{(u,y) \in E} f(u,y)$$

$$G = (V, E)$$

$$G$$

$$f$$

$$\mid f \mid$$

$$\sum_{x\in V} f(s,x) - \sum_{x\in V} f(x,s)$$

$$G$$

$$G$$

$$f$$

$$(u,v)$$

$$c_f(u,v)$$

$$c_f(u,v) = c(u,v) - f(u,v)$$

$$G$$

$$0$$

$$G_f$$

$$G_f = (V, E_f)$$

$$E_f = \{(u,v) \mid c_f(u,v) > 0\}$$

$$E_f$$

$$E$$

$$G_f$$

$$s$$

$$t$$

$$(u,v)$$

$$(u,v)$$

$$(v,u)$$

$$f(u,v) = -f(v,u)$$

$$f(u,v)$$

$$f(v,u)$$

$$g_{u,v} \leftrightarrow g_{v,u}$$

$$0$$

$$i$$

$$i \oplus 1$$

$$f(v,u)$$

$$c_f$$

$$f(v,u)$$

$$c_f(v,u)$$

$$G$$

$$G$$

$$u,v$$

$$1$$

$$1$$

$$2$$

$$u$$

$$v$$

$$c_f(v,u)$$

$$1$$

$$(v,u)$$

$$p,v,u,q$$

$$(u,v)$$

$$G_f$$

$$f$$

$$G=(V,E)$$

$$f$$

$$\{S,T\}$$

$$\mid f \mid = \mid \mid S,T \mid \mid$$

$$G=(V,E)$$

$$f$$

$$\{S,T\}$$

$$\mid f \mid \leq \mid \mid S,T \mid \mid$$

$$\{(u,v) \mid u \in S, v \in T\}$$

$$\{(u,v) \mid u \in T, v \in S\}$$

$$f$$

$$G_f$$

$$G_f$$

$$s$$

$$t$$

$$s$$

$$S$$

$$T = V \setminus S$$

$$\{S, T\}$$

$$G_f$$

$$\mid \mid S, T \mid \mid = \sum_{u \in S} \sum_{v \in T} c_f(u, v) = 0$$

$$u \in S, v \in T, (u, v) \in E_f$$

$$c_f(u, v) = 0$$

$$(u, v) \in E$$

$$c_f(u, v) = c(u, v) - f(u, v) = 0$$

$$c(u, v) = f(u, v)$$

$$\{(u, v) \mid u \in S, v \in T\}$$

$$(v, u) \in E$$

$$c_f(u, v) = c(u, v) - f(u, v) = 0 - f(u, v) = f(v, u) = 0$$

$$\{(v, u) \mid u \in S, v \in T\}$$

$$f$$

$$f$$

$$G$$

$$\{S, T\}$$

$$G$$

$$G = (V, E)$$

$$O(\mid E \mid \mid f \mid)$$

$$f$$

$$G$$

$$O(\mid E \mid)$$

$$\mid f \mid$$

$$G_f$$

$$G_f$$

$$s$$

$$t$$

$$p$$

$$p$$

$$\Delta = \min_{(u,v)\in p} c_f(u,v)$$

$$p$$

$$\Delta$$

$$\Delta$$

$$\Delta$$

$$G_f$$

$$G_f$$

$$O(\mid E \mid)$$

$$O(\mid V \mid \mid E \mid)$$

$$d_f(u)$$

$$G_f$$

$$u$$

$$s$$

$$f$$

$$f'$$

$$u$$

$$d_{f'}(u) \geq d_f(u)$$

$$(u,v)$$

$$(u,v) \not\in E_{f'}$$

$$(v,u) \in E_{f'}$$

$$(u,v)$$

$$G_f$$

$$(u,v)$$

$$d_f(u) + 1 = d_f(v)$$

$$G_{f'}$$

$$G_{f'}$$

$$(v,u)$$

$$d_{f'}(v) + 1 = d_{f'}(u)$$

$$d_{f'}(v) \geq d_f(v)$$

$$d_{f'}(u) \geq d_f(u) + 2$$

$$(u, v)$$

$$u$$

$$s$$

$$2$$

$$s$$

$$| V |$$

$$O(| V |)$$

$$O(| V | | E |)$$

$$d_{f'}(u) \geq d_f(u)$$

$$s$$

$$v$$

$$s$$

$$v = \arg \min_{x \in V, d_{f'}(x) < d_f(x)} d_{f'}(x)$$

$$d_{f'}(v) < d_f(v)$$

$$O(| V | | E |^2)$$

$$G_f$$

$$u$$

$$s$$

$$d(u)$$

$$u$$

$$v$$

$$u$$

$$G_f$$

$$G_L = (V, E_L)$$

$$G_f = (V, E_f)$$

$$E_L = \{(u, v) \mid (u, v) \in E_f, d(u) + 1 = d(v)\}$$

$$G_L$$

$$f_b$$

G_L

f_b

G_L

G_f

G_L

G_L

f_b

f_b

f

$f \leftarrow f + f_b$

s

t

f

G_L

f_b

G_L

u

u

u

$O(\mid E \mid^2)$

(u, v)

(u, v)

v

u

(u, v)

u

u

s

t

p

s

p

$O(|V||E|)$

f_b

G_L

$|V|$

f_b

E_1

G_L

E_1

E_L

E_2

E_2

E_1

E_2

$E_1 \cup E_2$

E_L

$E_1 \cup E_2$

$|V|$

$|V||E_L|$

$|E|$

$O(|V||E|)$

$|E|$

$|E|$

u

$|V|$

$O(|V|)$

$d_f(u)$

G_f

u

t

f

$$f'$$

$$G_L = (V, E_L)$$

$$G'_L = (V, E'_L)$$

$$G = (V, E)$$

$$h$$

$$V$$

$$N$$

$$h$$

$$G$$

$$h(u) \leq h(v) + 1$$

$$(u, v) \in E$$

$$E_{f'}$$

$$(u, v)$$

$$(u, v) \in E_{f'}$$

$$(u, v) \in E_f$$

$$d_f(u) \leq d_f(v) + 1$$

$$(u, v) \not\in E_f$$

$$(v, u)$$

$$d_f(u) + 1 = d_f(v)$$

$$d_f$$

$$G_{f'}$$

$$G_{f'}$$

$$G'_L$$

$$G'_L$$

$$p = (s, ..., u, v, ..., t)$$

$$p$$

$$t$$

$$s$$

$$v$$

$$u$$

$$u$$

$d_{f'}(u)$

$d_{f'}(v)$

1

d_f

G'_L

$d_f(u)$

$d_f(v)$

1

$d_{f'}(s) \geq d_f(s)$

$d_f(u) = d_f(v) + 1$

$(u, v) \in p$

$d_{f'}(s) > d_f(s)$

$d_{f'}(s) = d_f(s)$

G'_L

p

(u, v)

(u, v)

G_L

$d_f(s) = d_{f'}(s)$

G_L

(u, v)

$(u, v) \in E_f$

$d_f(u) \leq d_f(v) + 1$

$(u, v) \not\in E_L$

E'_L

$(u, v) \not\in E_f$

(u, v)

(v, u)

$d_f(u) = d_f(v) - 1$

$$d_f(u) \neq d_f(v) + 1$$

$$d_{f'}(s) \geq d_f(s)$$

$$d_{f'}(s) = d_f(s)$$

$$s$$

$$t$$

$$1$$

$$s$$

$$t$$

$$s$$

$$t$$

$$O(\mid V \mid \mid E \mid)$$

$$O(\mid V \mid)$$

$$O(\mid V \mid^2 \mid E \mid)$$

$$\mid V \mid, \mid E \mid$$

$$\mid V \mid^2 \mid E \mid$$

$$G = (V, E)$$

$$1$$

$$c(u, v) \in \{0, 1\}$$

$$(u, v) \in E$$

$$G$$

$$O(\mid E \mid)$$

$$O(\mid E \mid^{\frac{1}{2}})$$

$$s$$

$$d_f(u)$$

$$G_f$$

$$u$$

$$s$$

$$\left\{u \mid u \in V, d_f(u) = k\right\}$$

$$k$$

$$D_k$$

$$S_k = \cup_{i \leq k} D_i$$

$$| E |^{\frac{1}{2}}$$

$$k$$

$$\{(u,v) \mid u \in D_k, v \in D_{k+1}, (u,v) \in E_f\}$$

$$\frac{| E |}{| E |^{\frac{1}{2}}} \approx | E |^{\frac{1}{2}}$$

$$\{S_k, V - S_k\}$$

$$G_f$$

$$s$$

$$t$$

$$| E |^{\frac{1}{2}}$$

$$G_f$$

$$| E |^{\frac{1}{2}}$$

$$G_f$$

$$| E |^{\frac{1}{2}}$$

$$O(| E |^{\frac{1}{2}})$$

$$O(| V |^{\frac{2}{3}})$$

$$2 | V |^{\frac{2}{3}}$$

$$| V |^{\frac{2}{3}}$$

$$| V |^{\frac{1}{3}}$$

$$k$$

$$| V |^{\frac{1}{3}}$$

$$| D_k | \leq | V |^{\frac{1}{3}}$$

$$| D_{k+1} | \leq | V |^{\frac{1}{3}}$$

$$D_k$$

$$D_{k+1}$$

$$\{(u,v) \mid u \in D_k, v \in D_{k+1}, (u,v) \in E_f\}$$

$$| V |^{\frac{2}{3}}$$

$$\{S_k, V - S_k\}$$

$$G_f$$

$$s$$

$$t$$

$$| V |^{\frac{2}{3}}$$

$$G_f$$

$$| V |^{\frac{2}{3}}$$

$$G_f$$

$$| V |^{\frac{2}{3}}$$

$$O(| V |^{\frac{2}{3}})$$

$$u$$

$$deg_{in}(u) = 1$$

$$deg_{out}(u) = 1$$

$$O(| V |^{\frac{1}{2}})$$

$$deg_{in}(u)$$

$$deg_{out}(u)$$

$$u$$

$$| V |^{\frac{1}{2}}$$

$$| V |^{\frac{1}{2}}$$

$$G_f$$

$$\frac{| V |}{| V |^{\frac{1}{2}}} \approx | V |^{\frac{1}{2}}$$

$$G_f$$

$$| V |^{\frac{1}{2}}$$

$$O(| V |^{\frac{1}{2}})$$

$$O(| E | \min(| E |^{\frac{1}{2}}, | V |^{\frac{2}{3}}))$$

$$u$$

$$deg_{in}(u) = 1$$

$$deg_{out}(u) = 1$$

$$O(| E | \, | V |^{\frac{1}{2}})$$

$$O(n^3 \log n)$$

$$O(n^3)$$

$$O(n^3)$$

$$G$$

$$O(n^2)$$

$$L$$

$$v$$

$$p(v)$$

$$r$$

$$p(r) = \min p(v)$$

$$r$$

$$r$$

$$p(r)$$

$$0$$

$$L$$

$$L$$

$$p(r)$$

$$s$$

$$r$$

$$t$$

$$p(r)$$

$$L$$

$$s$$

$$t$$

$$O(V^2)$$

$$V$$

$$V$$

$$O(V^2 + E) = O(V^2)$$

$$V$$

$$O(V^3)$$

$$V$$

$$s$$

$$level_{i+1}[v] \geq level_i[v]$$

$$i$$

$$v$$

$$G_i^R$$

$$s$$

$$v$$

$$P$$

$$P$$

$$level_i[v]$$

$$G_i^R$$

$$G_i^R$$

$$P$$

$$G_i^R$$

$$level_{i+1}[v] \geq level_i[v]$$

$$P$$

$$G_i^R$$

$$P$$

$$(u, w)$$

$$level_{i+1}[u] \geq level_i[u]$$

$$(u, w)$$

$$G_i^R$$

$$(u, w)$$

$$level_i[u] = level_i[w] + 1$$

$$level_{i+1}[w] = level_{i+1}[u] + 1$$

$$level_{i+1}[u] \geq level_i[u]$$

$$level_{i+1}[w] \geq level_i[w] + 2$$

$$level_{i+1}[t] > level_i[t]$$

$$level_{i+1}[t] \geq level_i[t]$$

$$level_{i+1}[t] = level_i[t]$$

$$G_i^R$$

G_i^R

G_i^R

t

s

1

i

d_i

i

i

j

$d_i \leftarrow d_j + 1$

i

$d_i \leftarrow n$

$d_s \geq n$

i

num_i

x

y

num

$num_x = 0$

d_s

n

$u(u \in V - \{s, t\})$

$e(u)$

$e(u) > 0$

u

s

t

$h(u)$

u

u

u

u

$$u$$

$$h : V \to \mathbf{N}$$

$$h(s) = \mid V \mid, h(t) = 0$$

$$\forall (u, v) \in E_f, h(u) \leq h(v) + 1$$

$$h$$

$$G_f = (V_f, E_f)$$

$$G_f$$

$$h$$

$$u, v \in V$$

$$h(u) > h(v) + 1$$

$$(u, v)$$

$$G_f$$

$$h(u) = h(v) + 1$$

$$u$$

$$v((u, v) \in E_f, c(u, v) - f(u, v) > 0, h(u) = h(v) + 1)$$

$$(u, v)$$

$$u$$

$$v$$

$$c(u, v) - f(u, v)$$

$$v$$

$$(u, v)$$

$$u$$

$$\forall (u, v) \in E_f, h(u) \leq h(v)$$

$$u$$

$$h(u)$$

$$\min_{(u,v) \in E_f} h(v) + 1$$

$$(s, v) \in E$$

$$h(s)$$

$$(s, v) \notin E_f$$

$$h(s) > h(v)$$

$$(s, v)$$

$$e(s)$$

$$\sum_{(s,v)\in E} f(s, v)$$

$$u$$

$$h(u)$$

$$e(u)$$

$$c(u, v) - f(u, v)$$

$$h(u) = h(v) + 1$$

$$(u, v)$$

$$E_f$$

$$s$$

$$f$$

$$e(t)$$

$$n$$

$$O(n^2\sqrt{m})$$

$$u$$

$$u$$

$$O(n^2\sqrt{m})$$

$$h(u)$$

$$u$$

$$t$$

$$h(s) = n$$

$$h(u) = h(v) + 1$$

$$k$$

$$h(u) = k$$

$$0$$

$$h(u) > k$$

$$t$$

$$s$$

$$n + 1$$

$$s$$

$$N*2-1$$
$$i$$
$$n$$
$$c(u,v)$$
$$b(u,v)$$
$$b(u,v) \le f(u,v) \le c(u,v)$$
$$G$$
$$b(u,v)$$
$$u$$
$$v$$
$$c(u,v)-b(u,v)$$
$$0$$
$$M$$
$$M=0$$
$$M>0$$
$$S'$$
$$S'$$
$$M$$
$$M<0$$
$$T'$$
$$T'$$
$$-M$$
$$S'$$
$$T'$$
$$S'$$
$$G$$
$$S$$
$$T$$
$$T$$
$$S$$
$$\infty$$
$$0$$
$$S$$

T

T

S

G

S

T

S

T

G

T

S

x

y

v

∞

1

T

c

∞

1

S

1

$$f_{i,j} = \begin{cases} \max(f_{i-1,j}, f_{i,j-1}), A_i \neq B_j \\ f_{i-1,j-1} + 1 \qquad\qquad , A_i = B_j \end{cases}$$

$$f_{s,i} = \bigvee_{j \in s, j \neq i} f_{s \setminus \{i\},j} \wedge g_{j,i} (i \in s)$$

n, m

A, B

A, B

$(n \leq 10^6, m \leq 10^3)$

$f_{i,j}$

A

i

B

$$j$$

$$O(nm)$$

$$m$$

$$f_{i,j}$$

$$B$$

$$i$$

$$j$$

$$A$$

$$A$$

$$a, b, \cdots, z$$

$$O(1)$$

$$O(m^2 + 26n)$$

$$n$$

$$(2 \le n \le 20)$$

$$f_{s,i}$$

$$1$$

$$s$$

$$i$$

$$g$$

$$O(n^2 \times 2^n)$$

$$dp$$

$$s$$

$$f_{s,1}, f_{s,2}, \ldots, f_{s,n}$$

$$O(1)$$

$$O(n^2/w \times 2^n)$$

$$w$$

$$f_i - (s_i - L')^2 = \min_{j<i}\{f_j + s_j^2 + 2s_j(L' - s_i)\}$$

$$x_j = s_j$$

$$y_j = f_j + s_j^2$$

$$k_i = -2(L' - s_i)$$

$$b_i = f_i - (s_i - L')^2$$

$$f_i - (s_i - L')^2 = \min_{j<i}\{f_j + s_j^2 + 2s_j(L' - s_i)\}$$

$$n$$

$$i$$

$$c_i$$

$$n$$

$$[l, r]$$

$$(r - l + \sum_{i=l}^{r} c_i - L)^2$$

$$L$$

$$1 \leq n \leq 5 \times 10^4, 1 \leq L, c_i \leq 10^7$$

$$f_i$$

$$i$$

$$f_i = \min_{j<i}\{f_j + (i - (j+1) + pre_i - pre_j - L)^2\} = \min_{j<i}\{f_j + (pre_i - pre_j + i - j - 1 - L)^2\}$$

$$pre_i$$

$$i$$

$$\sum_{j=1}^{i} c_j$$

$$O(n^2)$$

$$s_i = pre_i + i, L' = L + 1$$

$$f_i = \min_{j<i}\{f_j + (s_i - s_j - L')^2\}$$

$$j$$

$$y = kx + b$$

$$b = y - kx$$

$$j$$

$$y$$

$$i, j$$

$$kx$$

$$i$$

$$b$$

$$b_i = \min_{j<i}\{y_j - k_i x_j\}$$

$$(x_j, y_j)$$

$$k_i$$

$$b_i$$

$$(x_j, y_j)$$

$$k_i$$

$$j$$

$$1 \leq j < i$$

$$k_i$$

$$(x_p, y_p)$$

$$b_i = y_p - k_i x_p$$

$$b_i$$

$$f_i$$

$$(x_i, y_i)$$

$$b_i$$

$$p$$

$$i - 1$$

$$k_i$$

$$i$$

$$K(a, b)$$

$$(x_a, y_a)$$

$$(x_b, y_b)$$

$$q_l, q_{l+1}, ..., q_r$$

$$l < i < r$$

$$K(q_{i-1}, q_i) < K(q_i, q_{i+1})$$

$$e$$

$$b_i$$

$$K(q_{e-1}, q_e) \leq k_i < K(q_e, q_{e+1})$$

$$e$$

$$e = l$$

$$e = r$$

$$p = q_e$$

$$q_e$$

$$i$$

$$k_i$$

$$e$$

$$O(1)$$

$$(x_i, y_i)$$

$$K(q_{r-1}, q_r) < K(q_r, i)$$

$$q_r$$

$$i$$

$$q$$

$$O(n)$$

$$i$$

$$f(i)$$

$$dp_i$$

$$dp_i$$

$$dp_i$$

$$dp_i$$

$$i$$

$$i$$

$$f(i)$$

$$i$$

$$n$$

$$i$$

$$c_i$$

$$n$$

$$[l, r]$$

$$(r - l + \sum_{i=l}^{r} c_i - L)^2$$

$$L$$

$$1 \leq n \leq 5 \times 10^4, 1 \leq L \leq 10^7, -10^7 \leq c_i \leq 10^7$$

$$f_i$$

$$i$$

$$f_i = \min_{j<i}\{f_j + (pre_i - pre_j + i - j - 1 - L)^2\}$$

$$pre_i = \sum_{j=1}^{i} c_j$$

$$\text{CDQ}(l, r)$$

$$f_i, i \in [l, r]$$

$$\text{CDQ}(1, n)$$

$$\text{CDQ}(1, mid)$$

$$f_i, i \in [1, mid]$$

$$[1, mid]$$

$$f_i, i \in [mid+1, n]$$

$$i \in [mid+1, n]$$

$$f_i$$

$$k_i$$

$$[mid+1, n]$$

$$[1, mid]$$

$$[1, mid]$$

$$\text{CDQ}(mid+1, n)$$

$$O(n \log^2 n)$$

$$f(i) = \min_{1\le j\le i} w(j, i) \qquad (1 \le i \le n)$$

$$w(a, c) + w(b, d) \le w(a, d) + w(b, c),$$

$$\forall j \in [1, n] : a_j \le a_i + f_i - \sqrt{\mid i - j \mid}.$$

$$f_i = \min_{j\le i}\{-a_j - \sqrt{i-j} + a_i\}.$$

$$f(i) = \min_{1\le j\le i} f(j-1) + w(j, i) \qquad (1 \le i \le n)$$

$$f_i = \min_{j=1}^{i} f_{j-1} + \left(i - j + \sum_{k=j}^{i} C_k\right)^2.$$

$$f(k, i) = \min_{1\le j\le i} f(k-1, j-1) + w(j, i) \qquad (1 \le k \le m,\, 1 \le i \le n)$$

$$\begin{aligned}
&w(b_{k+1}, c_{k+1}) + \cdots + w(b_{j+1}, c_{j+1}) + w(b_j, c_j) + w(b_{j-1}, c_{j-1}) + \cdots + w(b_1, c_1) \\
&\quad \le w(b_{k+1}, c_{k+1}) + \cdots + w(b_{j+1}, c_{j+1}) + w(b_j, d_j) + w(a_{j-1}, d_{j-1}) + \cdots + w(a_1, d_1).
\end{aligned}$$

$$
\begin{aligned}
&w(a_k, d_k) + \cdots + w(a_{j+1}, d_{j+1}) + w(a_j, d_j) + w(a_{j-1}, d_{j-1}) + \cdots + w(a_1, d_1) \\
&\quad < w(a_k, d_k) + \cdots + w(a_{j+1}, d_{j+1}) + w(a_j, c_j) + w(b_{j-1}, c_{j-1}) + \cdots + w(b_1, c_1).
\end{aligned}
$$

$$
\begin{aligned}
&[a_1, d_1], \cdots, [a_{j-1}, d_{j-1}], [a_j, c_{j+1}], [b_{j+2}, c_{j+2}], \cdots, [b_{k+1}, c_{k+1}], \\
&[b_1, c_1], \cdots, [b_j, c_j], [b_{j+1}, d_j], [a_{j+1}, d_{j+1}], \cdots, [a_{k-1}, d_{k-1}].
\end{aligned}
$$

$$
\begin{aligned}
2g(k) &\le w(a_1, d_1) + \cdots + w(a_{j-1}, d_{j-1}) + w(a_j, c_{j+1}) + w(b_{j+2}, c_{j+2}) + \cdots + w(b_{k+1}, c_{k+1}) \\
&\quad + w(b_1, c_1) + \cdots + w(b_j, c_j) + w(b_{j+1}, d_j) + w(a_{j+1}, d_{j+1}) + \cdots + w(a_{k-1}, d_{k-1}) \\
&\le w(a_1, d_1) + \cdots + w(a_{j-1}, d_{j-1}) + w(a_j, d_j) + w(a_{j+1}, d_{j+1}) + \cdots + w(a_{k-1}, d_{k-1}) \\
&\quad + w(b_1, c_1) + \cdots + w(b_j, c_j) + w(b_{j+1}, c_{j+1}) + w(b_{j+2}, c_{j+2}) + \cdots + w(b_{k+1}, c_{k+1}) \\
&= g(k-1) + g(k+1).
\end{aligned}
$$

$$
f(j, i) = \min_{j \le k < i} f(j, k) + f(k+1, i) + w(j, i) \qquad (1 \le j < i \le n)
$$

$$
w(b, c) \le w(a, d),
$$

$$
\begin{aligned}
f(a, d) + f(b, c) &= f(a, d') + f(d'+1, d) + w(a, d) + f(b, c) \\
&\ge f(a, c) + f(b, d') + f(d'+1, d) + w(a, d) \\
&\ge f(a, c) + f(b, d') + f(d'+1, d) + w(b, d) \\
&\ge f(a, c) + f(b, d).
\end{aligned}
$$

$$
\begin{aligned}
f(a, d) + f(b, c) &= f(a, d') + f(d'+1, d) + w(a, d) + f(b, c') + f(c'+1, c) + w(b, c) \\
&\ge f(a, c') + f(c'+1, c) + w(b, c) + f(b, d') + f(d'+1, d) + w(a, d) \\
&\ge f(a, c') + f(c'+1, c) + w(a, c) + f(b, d') + f(d'+1, d) + w(b, d) \\
&\ge f(a, c) + f(b, d).
\end{aligned}
$$

$$
\operatorname{opt}(j, i-1) \le \operatorname{opt}(j, i) \le \operatorname{opt}(j+1, i). \qquad (j+1 < i)
$$

$$
\begin{aligned}
\Delta_i \Delta_j h(w(j, i)) &= h(w(b, d)) - h\big(w(a, c) + \Delta_j w(j, c) + \Delta_i w(a, i)\big) \\
&\quad + h\big(w(a, c) + \Delta_j w(j, c) + \Delta_i w(a, i)\big) - h\big(w(a, c) + \Delta_j w(j, c)\big) \\
&\quad - h(w(a, c) + \Delta_i w(a, i)) + h(w(a, c)).
\end{aligned}
$$

$$
\frac{\partial^2}{\partial x \partial y} h(w(x, y)) = h''(w(x, y)) \frac{\partial}{\partial x} w(x, y) \frac{\partial}{\partial y} w(x, y) + h'(w(x, y)) \frac{\partial^2}{\partial x \partial y} w(x, y) \le 0.
$$

$$w(j, i)$$

$$O(1)$$

$$i$$

$$i$$

$$i$$

$$j$$

$$i$$

$$i$$

$$j$$
$$j$$
$$f(i)$$
$$i$$
$$\operatorname{opt}(i)$$
$$O(n^2)$$
$$i$$
$$j$$
$$i_1 < i_2$$
$$\operatorname{opt}(i_1) \leq \operatorname{opt}(i_2)$$
$$i$$
$$A$$
$$B$$
$$A \leq B$$
$$a \in A$$
$$b \in B$$
$$\min\{a, b\} \in A$$
$$\max\{a, b\} \in B$$
$$i_1 < i_2$$
$$\operatorname{optmax}(i_1) > \operatorname{optmin}(i_2)$$
$$j_1 = \operatorname{opt}(i_1)$$
$$j_2 = \operatorname{opt}(i_2)$$
$$j_1 < j_2$$
$$i_1 < i_2$$
$$j_1 < j_2$$
$$[l_{j_1}, r_{j_1}]$$
$$[l_{j_2}, r_{j_2}]$$
$$r_{j_1} < l_{j_2}$$
$$a \leq b \leq c \leq d$$
$$w$$
$$w$$

$$a \leq b \leq c \leq d$$

$$w$$

$$c < d$$

$$a = \text{opt}(d) < \text{opt}(c) = b$$

$$a < b \leq c < d$$

$$w(a, d) \leq w(b, d)$$

$$w(b, c) < w(a, c)$$

$$w(a, d) - w(b, d) \leq 0 < w(a, c) - w(b, c)$$

$$w$$

$$\Delta_i \Delta_j w(j, i)$$

$$O(n \log n)$$

$$1 \leq i \leq n$$

$$\text{opt}(i)$$

$$\text{opt}(n/2)$$

$$1 \leq i < n/2$$

$$n/2 < i \leq n$$

$$\text{opt}(i)$$

$$\text{opt}(i)$$

$$1$$

$$\text{opt}(n/2)$$

$$\text{opt}(i)$$

$$\text{opt}(n/2)$$

$$\text{opt}(n)$$

$$O(n \log n)$$

$$O(\log n)$$

$$O(n \log n)$$

$$j$$

$$i$$

$$j$$

$$l_j$$

r_j

$[l_j, r_j]$

j

j

$j \leq i$

i

$l_j \leq i \leq r_j$

j

$r_j < i$

j

j'

$l_{j'}$

j

j'

$w(j, l_{j'}) < w(j', l_{j'})$

j'

j'

j

$l_{j'}$

$(j, j+1, n)$

j

j'

$r_{j'}$

j

$r_{j'} < n$

$(j, r_{j'}+1, n)$

j

$[r_{j'}+1, n]$

j

j'

$$j$$

$$l_{j'}$$

$$r_{j'}$$

$$i \in [l_{j'}, r_{j'}]$$

$$w(j,i) < w(j',i)$$

$$i-1$$

$$(j,i,n)$$

$$O(1)$$

$$O(\log n)$$

$$O(n \log n)$$

$$n$$

$$a_1, a_2, \cdots, a_n$$

$$1 \le i \le n$$

$$f_i$$

$$f_i = \max_j \{a_j + \sqrt{|\ i-j\ |} - a_i\}$$

$$j \le i$$

$$-\sqrt{x}$$

$$w(l,r) = -a_l - \sqrt{r-l} + a_r$$

$$O(n \log n)$$

$$[1,n]$$

$$[a_1, b_1], \cdots, [a_k, b_k]$$

$$b_1 = 1$$

$$a_k = n$$

$$b_i + 1 = a_{i+1}$$

$$i < k$$

$$\sum_{i=1}^{k} w(a_i, b_i)$$

$$f(0) = 0$$

$$w(j,i)$$

$$f(j-1) + w(j,i)$$

$$j$$

$$i$$

$$O(n \log n)$$

$$n$$

$$C_i$$

$$[j, i]$$

$$i - j + \sum_{k=j}^{i} C_k$$

$$K$$

$$f_i$$

$$i$$

$$i$$

$$s(i) = i + \sum_{k=1}^{i} C_k$$

$$w(j, i) = (s(i) - s(j-1) - 1 - K)^2$$

$$s(i)$$

$$s(i) - s(j-1) - 1 - K$$

$$x^2$$

$$w(j, i)$$

$$m$$

$$f(0, 0) = 0$$

$$f(0, i) = f(k, 0) = \infty$$

$$1 \leq k \leq m$$

$$1 \leq i \leq n$$

$$f(k-1, j-1) + w(j, i)$$

$$i$$

$$O(mn \log n)$$

$$w$$

$$\operatorname{opt}(k-1, i) \leq \operatorname{opt}(k, i) \leq \operatorname{opt}(k, i+1)$$

$$k$$

$$\operatorname{opt}(k, i) \leq \operatorname{opt}(k+1, i)$$

$$[1, i]$$

$$[a_k, d_k], \cdots, [a_1, d_1]$$

$$[b_{k+1}, c_{k+1}], \cdots, [b_1, c_1]$$

$$d_j$$

$$c_j$$

$$[a_j, i]$$

$$[b_j, i]$$

$$j$$

$$a_{j-1} > b_{j-1}$$

$$d_j > c_j$$

$$a_j > c_j$$

$$a_1 > b_1$$

$$a_k > b_k$$

$$a_1 > b_1$$

$$a_k < b_k$$

$$j > 1$$

$$a_j \leq b_j$$

$$a_{j-1} > b_{j-1}$$

$$d_j > c_j$$

$$a_j \leq b_j \leq c_j < d_j$$

$$[b_{k+1}, c_{k+1}], \cdots, [b_{j+1}, c_{j+1}], [b_j, d_j], [a_{j-1}, d_{j-1}], \cdots, [a_1, d_1]$$

$$(k+1)$$

$$[a_k, d_k], \cdots, [a_{j+1}, d_{j+1}], [a_j, c_j], [b_{j-1}, c_{j-1}], \cdots, [b_1, c_1]$$

$$k$$

$$a_1 > b_1$$

$$a_1$$

$$k$$

$$w(b_j, c_j) + w(a_j, d_j) < w(b_j, d_j) + w(a_j, c_j)$$

$$j$$

$$k$$
$$i$$
$$j$$
$$O(n(n+m))$$
$$O(nm)$$
$$n \times m$$
$$(i,k)$$
$$\mathrm{opt}(k-1,i) \le j \le \mathrm{opt}(k,i+1)$$
$$i-k$$
$$O(n)$$
$$(n+m)$$
$$O(n(n+m))$$
$$w$$
$$g(k) := f(n,k)$$
$$k$$
$$g(k-1)+g(k+1) \ge 2g(k)$$
$$(k-1)$$
$$(k+1)$$
$$[a_1,d_1],\cdots,[a_{k-1},d_{k-1}]$$
$$[b_1,c_1],\cdots,[b_{k+1},c_{k+1}]$$
$$1 \le j \le k-1$$
$$c_{j+1} \le d_j$$
$$c_k < n = d_{k-1}$$
$$b_{j+1} > a_j$$
$$a_j < b_{j+1} \le c_{j+1} \le d_j$$
$$k$$
$$w_c(j,i) := w(j,i)+c$$
$$f_c(n)$$
$$c$$
$$m$$
$$c$$

$$f(n,m) = f_c(n) - cm$$

$$c$$

$$O(n \log n \log C)$$

$$C$$

$$n$$

$$[i,i]$$

$$[1,n]$$

$$[j,k]$$

$$[k+1,i]$$

$$w(j,i)$$

$$f(i,i) = w(i,i)$$

$$O(n^3)$$

$$O(n^2)$$

$$a \leq b \leq c \leq d$$

$$w$$

$$w(j,i)$$

$$j$$

$$i$$

$$w$$

$$f(j,i)$$

$$a \leq b \leq c \leq d$$

$$f(a,d) + f(b,c) \geq f(a,d) + f(b,c)$$

$$d - a$$

$$a = b$$

$$c = d$$

$$d' = \mathrm{opt}(a,d)$$

$$c \leq d'$$

$$d' < b$$

$$[b,c]$$

$$[a,d']$$

$$[d'+1,d]$$

$$c \leq d'$$

$$f(a,c) + f(b,d') \leq f(a,d') + f(b,c)$$

$$w(b,d) \leq w(a,d)$$

$$f(b,d) \leq f(b,d') + f(d'+1,d) + w(b,d)$$

$$b \leq d' < c$$

$$d'$$

$$[b,c]$$

$$c' = \mathrm{opt}(b,c)$$

$$c' \leq d'$$

$$[b,c']$$

$$[a,d']$$

$$f(a,c') + f(b,d') \leq f(a,d') + f(b,c')$$

$$w(a,c) + w(b,d) \leq w(a,d) + w(b,c)$$

$$f(a,c)$$

$$f(b,d)$$

$$w$$

$$\mathrm{opt}(j,i)$$

$$f(j,i)$$

$$f(j,k) + f(k+1,i) + w(j,i)$$

$$j$$

$$(k,i)$$

$$\mathrm{opt}(j,i-1) \leq \mathrm{opt}(j,i)$$

$$(k,i)$$

$$i$$

$$(k,j)$$

$$\mathrm{opt}(j,i) \leq \mathrm{opt}(j+1,i)$$

$$k$$

$$i - j + 1$$

$$[j,i]$$

$$\mathrm{opt}(j,i-1)$$

$$\mathrm{opt}(j+1,i)$$

$$k$$

$$f(j,i)$$

$$\mathrm{opt}(j,i)$$

$$O(n)$$

$$O(n)$$

$$O(n^2)$$

$$w_1(j,i)$$

$$w_2(j,i)$$

$$c_1, c_2 \geq 0$$

$$c_1 w_1 + c_2 w_2$$

$$f(x)$$

$$g(x)$$

$$w(j,i) = f(j) - g(i)$$

$$w$$

$$f$$

$$g$$

$$w$$

$$h(x)$$

$$w(j,i)$$

$$h(w(j,i))$$

$$h(x)$$

$$w(j,i)$$

$$h(w(j,i))$$

$$h(x)$$

$$h(w(j,i))$$

$$a \leq j \leq b \leq c \leq i \leq d$$

$$\Delta_i w(a,i) := w(a,d) - w(a,c) \geq 0$$

$$\Delta_j w(j,c) := w(b,c) - w(a,c) \leq 0$$

$$h(x)$$

$$t_1, t_2 \geq 0$$

$$h(x+t_1-t_2)-h(x+t_1) \leq h(x-t_2)-h(x)$$

$$w(b,d) \leq w(a,c)+\Delta_j w(j,c)+\Delta_i w(a,i) = w(b,c)+w(a,d)-w(a,c)$$

$$h(x)$$

$$h' \geq 0$$

$$h'' \geq 0$$

$$w_x \leq 0$$

$$w_y \geq 0$$

$$w_{xy} \leq 0$$

$$w$$

$$f_{i,j} = \max_{k=0}^{k_i}(f_{i-1,j-k\times w_i}+v_i \times k)$$

$$g_{x,y} = \max_{k=0}^{k_i}({g'}_{x-k,y}+v_i \times k)$$

$$g_{x,y} = \max_{k=0}^{k_i}(G_{x-k,y})+v_i \times x$$

$$n$$

$$m$$

$$i$$

$$t_i$$

$$a_i$$

$$x$$

$$b_i - \mid a_i - x \mid$$

$$d$$

$$f_{i,j}$$

$$i$$

$$j$$

$$f_{i,j} = \max\{f_{i-1,k}+b_i - \mid a_i - j \mid\}$$

$$k$$

$$j-(t_i-t_{i-1}) \times d \leq k \leq j+(t_i-t_{i-1}) \times d$$

$$\max$$

$$b_i$$

$$f_{i,j} = \max\{f_{i-1,k}+b_i - \mid a_i - j \mid\} = \max\{f_{i-1,k} - \mid a_i - j \mid\} + b_i$$

$$i$$

$$j$$

$$\mid a_i - j \mid$$

$$f_{i,j} = \max\{f_{i-1,k} - \mid a_i - j \mid\} + b_i = \max\{f_{i-1,k}\} - \mid a_i - j \mid + b_i$$

$$\max$$

$$i$$

$$f_{i-1}$$

$$O(1)$$

$$\max\{f_{i-1,k}\}$$

$$f_{i,j}$$

$$O(nm)$$

$$n$$

$$w_i$$

$$v_i$$

$$k_i$$

$$W$$

$$f_{i,j}$$

$$i$$

$$j$$

$$O(W \sum k_i)$$

$$f_i$$

$$g_{x,y} = f_{i,x \times w_i + y}, g'_{x,y} = f_{i-1,x \times w_i + y}$$

$$G_{x,y} = g'_{x,y} - v_i \times x$$

$$G_{x,y}$$

$$O(1)$$

$$y$$

$$O\left(\left\lfloor \frac{W}{w_i} \right\rfloor\right)$$

$$g_{x,y}$$

$$g_{x,y}$$

$$O\left(\left\lfloor \frac{W}{w_i} \right\rfloor\right) \times O(w_i) = O(W)$$

$$O(nW)$$