Ch 11 CTFT

Forward CT FT

$$\hat{X}(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad \rightarrow \text{freq domain}$$

Inverse CT FT

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\hat{X}(j\omega)\, e^{j\omega t}}_{\text{controlling magnitude and phase of infinite sinusoids}}\, d\omega \quad \rightarrow \text{time domain}$$

Time Domain — Frequency Domain

$$x(t) \xleftrightarrow{F} \hat{X}(j\omega)$$

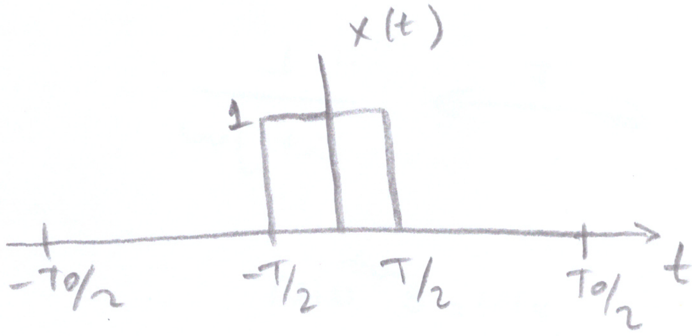
Ex 11-3: $x(t) = e^{-7t} u(t)$

$\hat{X}(j\omega) = ?$

$$\hat{X}(j\omega) = \int_{0}^{\infty} e^{-7t} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(7+j\omega)t}\, dt$$

$$= \left. \frac{e^{-7t} e^{-j\omega t}}{-(7+j\omega)} \right|_{0}^{\infty} = \frac{1}{7+j\omega}$$

Ex:



$$a_k T_0 = \int_{-T/2}^{T/2} e^{-jk\omega_0 t}\, dt$$

$$= -\frac{1}{jk\omega_0} e^{-jk\omega_0 t} \Big|_{-T/2}^{T/2}$$

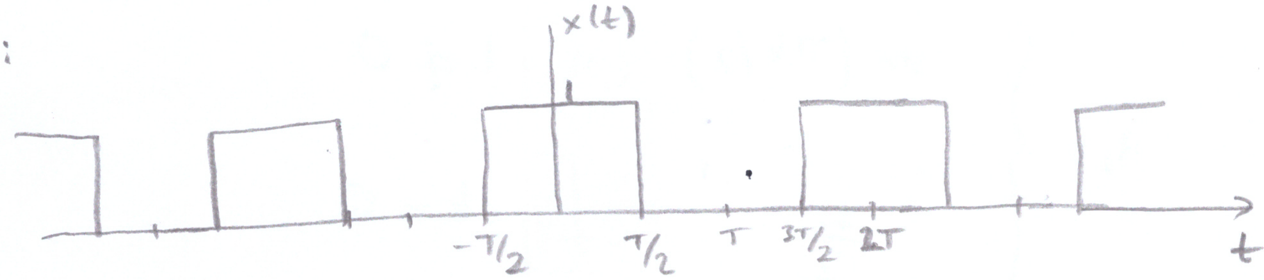
$$= -\frac{e^{-jk\omega_0 T/2} - e^{jk\omega_0 T/2}}{jk\omega_0}$$

$$= \frac{\sin(k\omega_0 T/2)}{k\omega_0 T/2}$$

$T_0 \to \infty$

$$X(j\omega) = \lim_{k\omega_0 \to \omega} \frac{\sin(k\omega_0 T/2)}{k\omega_0 T/2} = \frac{\sin(\omega T/2)}{\omega T/2}$$

Ex:



$X(j\omega) = ?$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

FT of $e^{j\omega_0 t}$ is

$$FT\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

coeffs for square wave

$$a_k = \frac{1}{T_0}\int_{-T/2}^{T/2} e^{-jk\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\frac{e^{-jk\omega_0 t}}{(-jk\omega_0)}\Bigg|_{-T/2}^{T/2} = \frac{\sin(k\omega_0 T/2)}{k\omega_0 T_0/2}$$

$$a_0 = \frac{1}{T_0}\int_{-T/2}^{T/2} dt = \frac{T}{T_0} = \frac{1}{2}$$

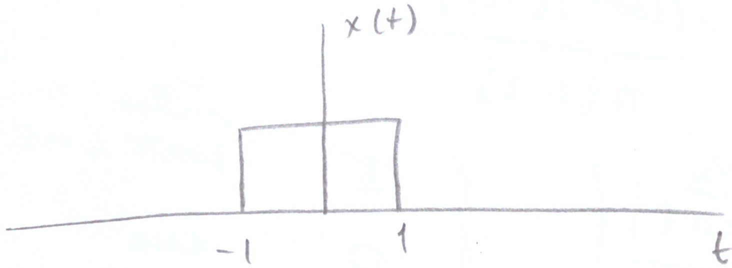
Fourier transform of a periodic signals

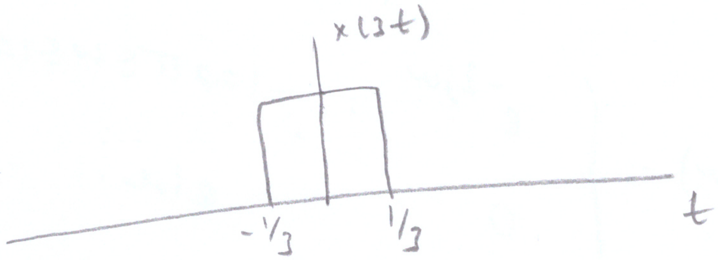
$$P(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_s)$$

$$P(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s} \, \delta(\omega - k\omega_s)$$

→ FT of a periodic impulse is also a periodic impulse train

<u>P-11.1</u>: $x(t) = u(t+1) - u(t-1)$ Draw $y(t) = x(3t)$





Duration of $y(t) = 2/3$ sec

c)

$$x(t) = e^{-t}u(t) - e^{-t}u(t-4)$$

$$FT\{e^{-t}u(t)\} = \frac{1}{1+j\omega}$$

$$e^{-t}u(t-4) = e^{-4}e^{-(t-4)}u(t-4)$$

$$FT\{e^{-4}e^{-(t-4)}u(t-4)\} = e^{-4}e^{-4j\omega}\frac{1}{1+j\omega}$$

$$X(j\omega) = \frac{1}{1+j\omega} - e^{-4}e^{-4j\omega}\frac{1}{1+j\omega}$$

$$= \frac{1}{1+j\omega}\left(1 - e^{-4(j\omega+1)}\right)$$

P 11-3: Find $x(t)$ using FT table

a) $$X(j\omega) = \frac{j\omega}{0.1+j\omega}e^{-j\omega 0.2} = X_1(j\omega)\underbrace{e^{-j0.2\omega}}_{\text{delay}}$$

$$X_1(j\omega) = \underbrace{j\omega}_{\text{derivative}} X_2(j\omega)$$

$$X_2(j\omega) = \frac{1}{0.1+j\omega} \Rightarrow x_2(t) = e^{-0.1t}u(t)$$

$$x_1(t) = \frac{d}{dt}x_2(t) = \frac{d\left(e^{-0.1t}u(t)\right)}{dt} = \underbrace{e^{-0.1t}\delta(t)}_{\delta(t)} - 0.1e^{-0.1t}u(t)$$

$$x(t) = x_1(t-0.2) = \delta(t-0.2) - 0.1e^{-0.1(t-0.2)}u(t-0.2)$$

P 11-4:

a) $X(j\omega) = ?$ when $x(t) = \begin{cases} 1 & 0 \le t < 4 \\ 0 & \text{else} \end{cases}$

$x(t) = u(t) - u(t-4) = \delta(t-2) * [u(t+2) - u(t-2)]$

$FT\left\{u(t+\frac{1}{2}T) - u(t-\frac{1}{2}T)\right\} = \frac{\sin(\omega T/2)}{\omega/2}$

$T = 2$

$= \frac{\sin(\omega)}{\omega/2}$

$FT\{\delta(t-2)\} = e^{-j\omega 2}$

$X(j\omega) = e^{-j\omega 2} \frac{\sin(\omega)}{\omega/2}$

b) $s(t) = ?$

$S(j\omega) = 4\pi\delta(\omega) + 2\pi\delta(\omega - 10\pi) + 2\pi\delta(\omega + 10\pi)$

$2\pi\delta(\omega) \xleftrightarrow{F} 1$

$2\pi\delta(\omega - 10\pi) \longleftrightarrow e^{j10\pi t}$

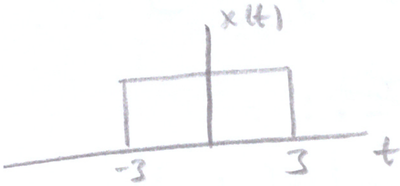
$2\pi\delta(\omega + 10\pi) \longleftrightarrow e^{-j10\pi t}$

$s(t) = 2 + e^{j10\pi t} + e^{-j10\pi t} = 2 + 2\cos(10\pi t)$

P-11.6:

a) $x(t) = u(t+3)\,u(3-t) \xleftrightarrow{F} X(jw) = ?$

$u(t+3)\,u(3-t) \Rightarrow$



$$X(jw) = \frac{\sin(w3)}{w/2}$$

b) $x(t) = \sin(4\pi t)\overset{\sin}{\wedge}(50\pi t) \xleftrightarrow{F} X(jw)$

$\sin(4\pi t)\sin(50\pi t) \xleftrightarrow{F} \left(FT\{\sin(4\pi t)\} * FT\{\sin(50\pi t)\}\right)\frac{1}{2\pi}$

$FT\{\sin(4\pi t)\} = -j\pi\,\delta(w-4\pi) + j\pi\,\delta(w+4\pi)$

$FT\{\sin(50\pi t)\} = -j\pi\,\delta(w-50\pi) + j\pi\,\delta(w+50\pi)$

$-j^2\pi^2\,\delta(w-46\pi) + j^2\pi^2\,\delta(w+54\pi)$

$+j^2\pi^2\,\delta(w-54\pi) - j^2\pi^2\,\delta(w+46\pi) - j^2\pi^2\,\delta(w-46\pi) + j^2\pi^2\,\delta(w+54\pi)$

$\frac{1}{2\pi}\left(-\pi^2\,\delta(w-54\pi) + \pi^2\,\delta(w+46\pi) + \pi^2\,\delta(w-46\pi) - \pi^2\,\delta(w+54\pi)\right)$

$-\frac{\pi}{2}\,\delta(w-54\pi) + \frac{\pi}{2}\,\delta(w+46\pi) + \frac{\pi}{2}\,\delta(w-46\pi) + \frac{\pi}{2}\,\delta(w+54\pi)$

Limit of Fourier Series

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{-jk\omega_0 t}\, dt \qquad (11.9)$$

$$x_{T_0}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad (11.10)$$

$$\lim_{T_0 \to \infty} x_{T_0}(t) = x(t)$$

$$\lim_{T_0 \to \infty} \omega_0 = \lim_{T_0 \to \infty} 2\pi / T_0 = 0$$

$$(a_k T_0) = \int_{-T_0/2}^{T_0/2} x_{T_0}(t)\, e^{\overbrace{-jk\omega_0 t}^{T_0 \to \infty \Rightarrow k\omega_0 = \omega}}\, dt$$

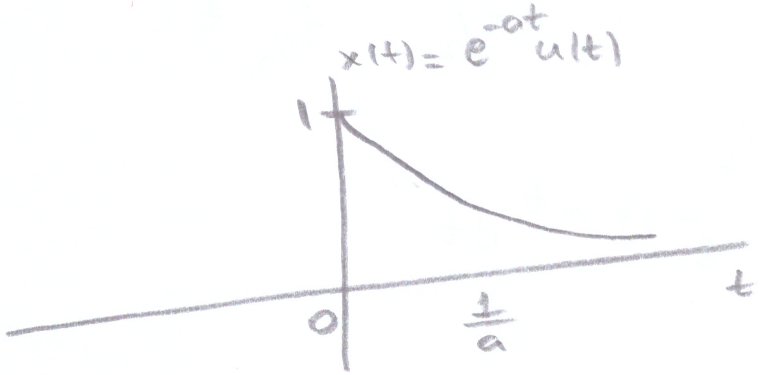
$$x_{T_0}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} (a_k T_0)\, e^{jk\omega_0 t} \underbrace{\left(\frac{2\pi}{T_0}\right)}_{d\omega \Rightarrow T_0 \to \infty}$$

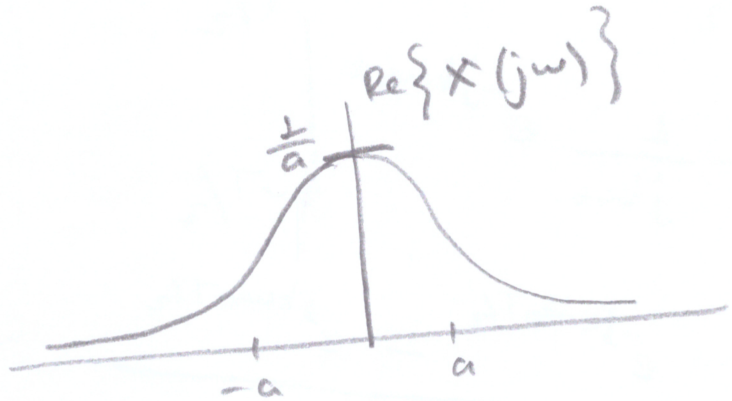
$T_0 \to \infty$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

Ex: $e^{-at}u(t) \xleftrightarrow{F} \dfrac{1}{a+j\omega}$





- If a increase, the exponential dies out more quickly making x(t) more concenterated near t=0. In frequency domain, on the other hand, as a increases the FT spread out

$$a_k = \begin{cases} \sin(\pi k/2) & k \neq 0 \\ \frac{1}{2} & k = 0 \end{cases}$$

$$X(j\omega) = \pi\delta(\omega) + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2\sin(\pi k/2)}{k} \delta(\omega - k\omega_0)$$

<u>Ex 11-6:</u> Transform of Impulse Train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$p(t)$ can be expressed as a Fourier series

$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t} \qquad \omega_s = 2\pi/T_s$$

$$a_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, e^{-jk\omega_s t}\, dt$$

$$= \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\, dt = \frac{1}{T_s}$$

P 11.2 : $x(t-t_d) \xleftrightarrow{F} e^{-j\omega t_d} X(j\omega)$

Find FT of followings

a) $x(t) = \delta(t+1) + 2\delta(t) + \delta(t-1)$

$$F.T\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = 1$$

$$F.T.\{\delta(t+1)\} = e^{j\omega t_d}$$

$$F.T.\{\delta(t-1)\} = e^{-j\omega t_d}$$

$$X(j\omega) = e^{j\omega} + 2 + e^{-j\omega} = 2 + 2\cos\omega$$

b) $$x(t) = \frac{\sin(100\pi(t-2))}{\pi(t-2)}$$

$$F.T.\left\{\frac{\sin(\overbrace{100\pi}^{w_b} t)}{\pi t}\right\} = \begin{cases} 1 & \overbrace{-100\pi}^{w_b} \le \omega \le \overbrace{100\pi}^{w_b} \\ 0 & \text{else} \end{cases}$$

$$X(j\omega) = \begin{cases} e^{-2j\omega} & -100\pi \le \omega \le 100\pi \\ 0 & \text{else} \end{cases}$$

c)

$$X(j\omega) = \frac{1}{1+j\omega} - \frac{1}{2+j\omega}$$

$$FT\left\{\frac{1}{1+j\omega}\right\} = e^{-t}u(t)$$

$$FT\left\{\frac{1}{2+j\omega}\right\} = e^{-2t}u(t)$$

$$x(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

d)

$$X(j\omega) = j\,\delta(\omega - 100\pi) - j\,\delta(\omega + 100\pi)$$

$$2\pi\,\delta(\omega - \omega_0) \rightarrow e^{j\omega_0 t}$$

$$X(j\omega) = j\frac{2\pi}{2\pi}\delta(\omega - 100\pi) - \frac{j2\pi}{2\pi}\delta(\omega + 100\pi)$$

$$= \frac{j}{2\pi}e^{+j100\pi t} - \frac{j}{2\pi}e^{-j100\pi t}$$

$$= -\frac{1}{\pi}\left(\frac{1}{2j}e^{j100\pi t} - \frac{1}{2j}e^{-j100\pi t}\right) \rightarrow \text{use inverse euler}$$

$$= -\frac{1}{\pi}\sin(100\pi t)$$

c) $r(t) = ?$

$$R(j\omega) = \frac{j\omega}{4+j2\omega} = \underbrace{\frac{1}{2}}_{\frac{\delta(t)}{2}} - \frac{2}{4+2j\omega}$$

$$= \frac{1}{2} - \underbrace{\frac{1}{2+j\omega}}_{e^{-2t}u(t)}$$

$$r(t) = \frac{\delta(t)}{2} - e^{-2t}u(t)$$

d) Same as P-11.2 a

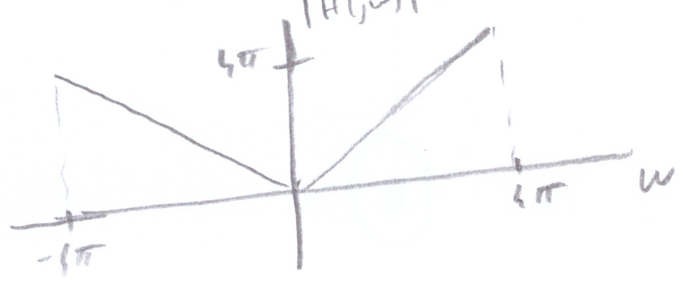
P-11.5:
$$h(t) = \frac{d}{dt}\left\{ \underbrace{\frac{\sin(4\pi t)}{\pi t}}_{\text{sinc}} \right\}$$

$H(j\omega) = ?$

$$FT\left\{ \frac{\sin(4\pi t)}{\pi t} \right\} = [u(\omega + 4\pi) - u(\omega - 4\pi)]$$

FT of FT is multiplication by $j\omega$

$$H(j\omega) = j\omega\,[u(\omega+4\pi) - u(\omega-4\pi)]$$



12

P. 11-9 : Prove that for a real $h(t)$, the magnitude $|H(j\omega)|$ has even symmetry and the phase $\angle H(j\omega)$ has odd symmetry

$$|H(j\omega)| = |H(-j\omega)|$$
$$\angle H(j\omega) = -\angle H(-j\omega)$$

If $h(t)$ is purely imaginary, describe how the symmetries of the magnitude and phase change

$h(t)$ is real $\Rightarrow h^*(t) = h(t)$

If $h(t) \rightarrow H(j\omega)$, then $h^*(t) \rightarrow H^*(j\omega)$

$$\Rightarrow \quad H^*(-j\omega) = H(j\omega)$$

$$H(j\omega) = A(\omega) + jB(\omega)$$
$$H^*(-j\omega) = A(-\omega) - jB(-\omega)$$
$$A(\omega) = A(-\omega) \quad \text{and} \; -B(\omega) = B(-\omega)$$

The magnitude

$$|H(j\omega)| = \sqrt{A^2(-\omega) + B^2(-\omega)}$$
$$= \sqrt{A^2(\omega) + B(\omega)} = |H(j\omega)| \quad \underline{\text{even}}$$

The phase

$$\angle H(-j\omega) = \tan^{-1}\left(\frac{B(-\omega)}{A(-\omega)}\right) = \tan^{-1}\left(\frac{-B(\omega)}{A(\omega)}\right) = -\tan^{-1}\left(\frac{B(\omega)}{A(\omega)}\right)$$
$$= -\angle H(j\omega)$$

<u>odd</u>

Ex: $H(j\omega) = \dfrac{\sin^2(3\omega)\cos(\omega)}{\omega^2} \longleftrightarrow h(t) = ?$

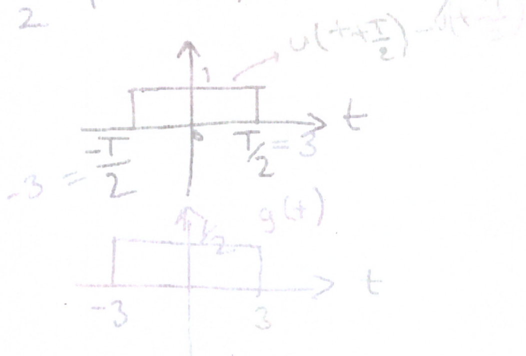
$$= \underbrace{\cos\omega}_{H_1(j\omega)} \underbrace{\left(\frac{\sin(3\omega)}{\omega}\right)^2}_{H_2(j\omega)} \longleftrightarrow h_1(t) * h_2(t)$$

$$\cos(\omega) = \frac{1}{2}e^{j\omega} + \frac{1}{2}e^{-j\omega} \longleftrightarrow \frac{1}{2}\delta(t-1) + \frac{1}{2}\delta(t+1) = h_1(t)$$

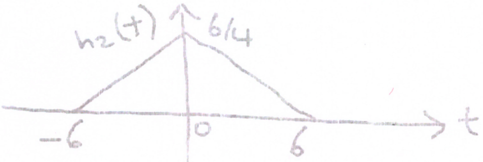
$$H_2(j\omega) = \underbrace{\frac{\sin 3\omega}{\omega}}_{G(j\omega)} \cdot \underbrace{\frac{\sin 3\omega}{\omega}}_{G(j\omega)} \longleftrightarrow g(t) * g(t) = h_2(t)$$

$$\frac{\sin 3\omega}{\omega} \longleftrightarrow \frac{1}{2}\left(u(t+3) - u(t-3)\right)$$

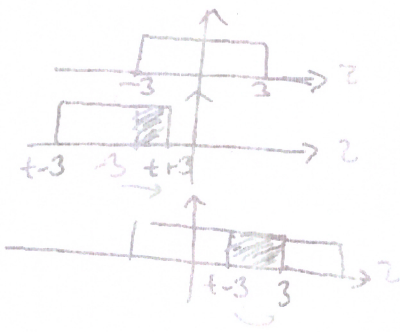
Recall $\dfrac{\sin(\omega T/2)}{\omega/2} \longleftrightarrow$



$g(t) * g(t)$

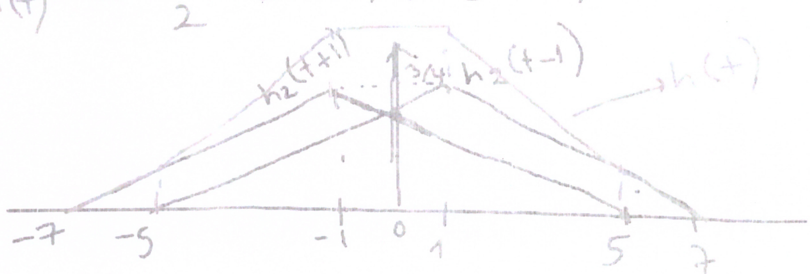


Do the Convolution



$$h_2(t) = \begin{cases} 0, & t < -6 \\ \dfrac{t+6}{4}, & -6 \le t < 0 \\ \dfrac{6-t}{4}, & 0 \le t < 6 \\ 0, & 6 < t \end{cases}$$

$$h(t) = h_1(t) * h_2(t)$$
$$= \frac{1}{2}\left(\delta(t+1) + \delta(t-1)\right) * h_2(t)$$
$$h(t) = \frac{1}{2}\, h_2(t+1) + h_2(t-1)$$

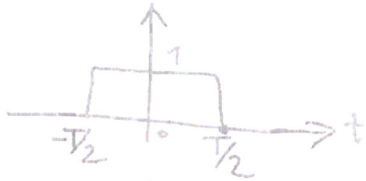


Ex: $X(j\omega) = 2\,\dfrac{\sin(3(\omega-2\pi))}{(\omega-2\pi)} \Longleftrightarrow \mathcal{F}^{-1}\{.\}$

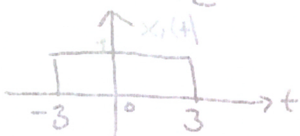
$x(t) = ?$

Recall (Table 11.2 in your textbook)

$\dfrac{\sin(\omega T/2)}{\omega/2} \Longleftrightarrow$



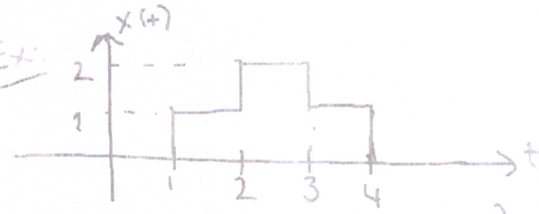
$\dfrac{\sin(3\omega)}{\omega/2} \Longleftrightarrow$



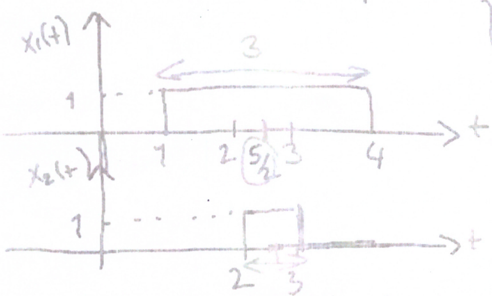
note the Frequency shifting by 2π : $e^{j\omega_0 t}x(t) \Longleftrightarrow X(j(\omega-\omega_0))$

$\Longrightarrow \quad x(t) = e^{j2\pi t}x_1(t) \Longleftrightarrow \dfrac{\sin(3(\omega-2\pi))}{(\omega-2\pi)/2}$

Ex:



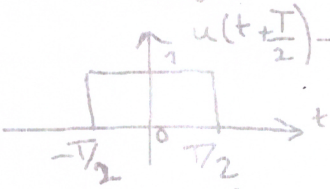
$X(j\omega) = ?$



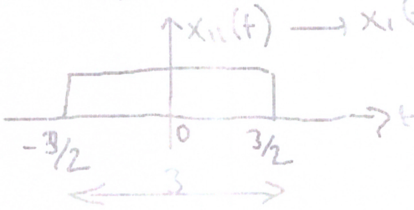
$x(t) = x_1(t) + x_2(t)$

$X(j\omega) = X_1(j\omega) + X_2(j\omega)$

Recall

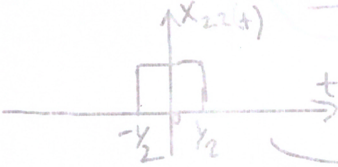


$u(t+\frac{T}{2}) - u(t-\frac{T}{2}) \Longleftrightarrow \dfrac{\sin(\omega T/2)}{\omega/2}$



$\longrightarrow x_1(t) = x_{11}(t-\frac{5}{2}) \Longleftrightarrow X_{11}(j\omega) = \dfrac{\sin(3\omega/2)}{\omega/2}$

$\Longrightarrow X_1(j\omega) = e^{-j\omega\frac{5}{2}}X_{11}(j\omega)$



$x_2(t) = x_{22}(t-\frac{5}{2})$

$\xrightarrow{F.T} \dfrac{\sin(\omega/2)}{\omega/2} = X_{22}(j\omega)$

$X_2(j\omega) = e^{-j\omega\frac{5}{2}}X_{22}(j\omega)$

$\Longrightarrow X(j\omega) = X_1(j\omega) + X_2(j\omega)$

$= e^{-j\omega 5/2}\left(\dfrac{\sin(3\omega/2)}{\omega/2} + \dfrac{\sin(\omega/2)}{\omega/2}\right)$