Ch 10 – Frequency Response : We want to know how the system response each frequency of input

– The general LTI system output

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

– If the input $x(t) = A e^{j\phi} e^{j\omega t}$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, A e^{j\phi}\, e^{j\omega(t-\tau)}\, d\tau$$

$$= \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau}_{\text{Constant value wrt } \omega}\ \underbrace{A e^{j\phi}\, e^{j\omega t}}_{x(t)}$$

→ This part can be freq. defined as function of ω

we will call $H(j\omega)$ → frequency response of a system
→ It depends on frequency of input
→ It modulates magnitude and phase of input signal

$$= H(j\omega)\, x(t) \qquad (10.3)$$

– According to (10.3), when the input is a complex exponential signal whose frequency is ω, the output is also a complex exponential signal having exactly same frequency. However, the effect of the system on a complex – exponential signal depends on the frequency of the input. This is represented by the fact that we will get different complex values for $H(j\omega)$ as ω changes.

Ex 10.1: $x(t) = 10e^{j3\pi t}$ and $H(j3\pi) = 2-j2$

Explain $y(t)$ as $Be^{j\theta}e^{j3\pi t}$

$$2-j2 \Rightarrow \tan\theta = \left(\frac{-2}{2}\right)$$

$$\theta = \operatorname{atan}\left(-\frac{2}{2}\right)$$

$$\theta = -\frac{\pi}{4}$$

$$r = 2\sqrt{2}$$

$$2\sqrt{2}\,e^{-j\frac{\pi}{4}} = H(j3\pi)$$

$$y(t) = H(j\omega)\,x(t)$$

$$y(t) = 2\sqrt{2}\,e^{-j\pi/4}\,10\,e^{j3\pi t} = 20\sqrt{2}\,e^{j(3\pi t-\pi/4)}$$

Ex 10.2: $h(t) = 2e^{-2t}u(t)$ $\qquad H(j\omega) = ?$

$$H(j\omega) = \int_{-\infty}^{\infty} 2e^{-2t}u(t)\,e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} 2e^{-2t-j\omega t}\,dt = \left.\frac{2e^{-2t-j\omega t}}{-2-j\omega}\right|_{0}^{\infty}$$

$$= 0 - \frac{2}{-2-j\omega} = \frac{2}{2+j\omega}$$

Plotting Of Frequency Response

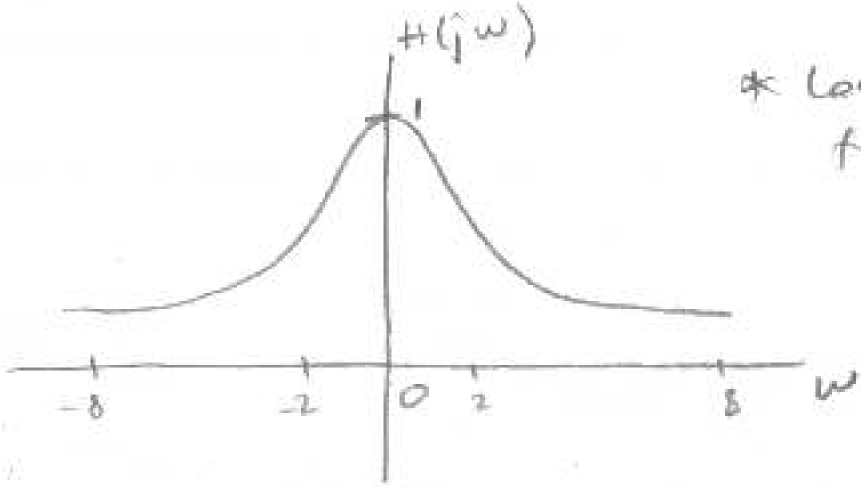
$$H(j\omega) = \frac{2}{2+j\omega}$$

$$|H(j\omega)| = \left|\frac{2}{2+j\omega}\right| = \frac{2}{\sqrt{4+\omega^2}}$$

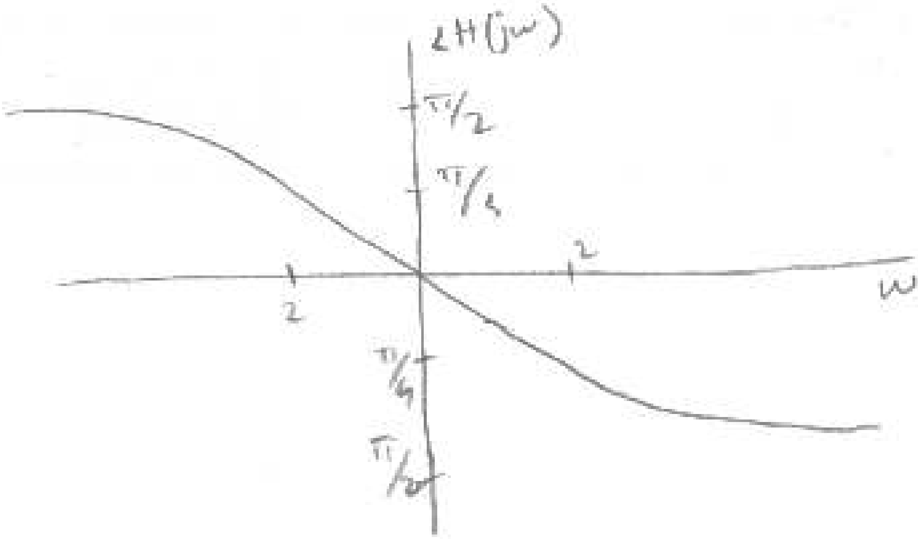
$$\angle H(j\omega) = \frac{2(2-j\omega)}{2+\omega} = \frac{4}{2+\omega} - \frac{2j\omega}{2+\omega}$$

$$= \tan^{-1}\left(-\frac{2\omega}{4}\right)$$

$$= -\tan^{-1}\left(\frac{2\omega}{4}\right)$$



* Low pass filter



Ex 10-3: Freq. response of an LTI

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

What is magnitude and phase shift of a system at $\omega = 200\pi$?

$$H(j200\pi) = \frac{j200\pi}{200\pi + j200\pi} = \frac{j}{1+j} \Rightarrow$$

$$|H(j200\pi)| = \sqrt{\frac{1^2}{1^2 + 1^2}} = \frac{1}{\sqrt{2}}$$

$$\angle H(j200\pi) = \tan^{-1}(1) = \pi/4$$

So, the system multiply the input by $1/\sqrt{2}$ and shift its phase by $\pi/4$

Ex 10-4: $x(t) = 20e^{j0.3\pi} e^{j200\pi t}$

For the system that has freq response

$$H(j\omega) = \frac{j\omega}{200\pi + j\omega}$$

$y(t) = ?$

$$y(t) = H(j\omega)\, x(t) = \frac{1}{\sqrt{2}}\, 20\, e^{j200\pi t + j0.3\pi + \pi/4}$$

$$= 10\sqrt{2}\, e^{j(200\pi t + 0.55\pi)}$$

Response to Real Sinusoidal Signal

$$A\cos(\omega t+\phi) = \underbrace{\frac{1}{2}Ae^{j\phi}e^{j\omega t}}_{x_1(t)} + \underbrace{\frac{1}{2}Ae^{-j\phi}e^{-j\omega t}}_{x_2(t)} \quad (10.17)$$

$$y_1(t) = H(j\omega)\frac{1}{2}Ae^{j\phi}e^{j\omega t}$$

$$y_2(t) = H(-j\omega)\frac{1}{2}Ae^{-j\phi}e^{-j\omega t}$$

$$y(t) = y_1(t) + y_2(t)$$

$$y(t) = H(j\omega)\frac{1}{2}Ae^{j\phi}e^{j\omega t} + H(-j\omega)\frac{1}{2}Ae^{-j\phi}e^{-j\omega t}$$

If $h(t)$ is real valued

$$H(-j\omega) = \left(H(j\omega)\right)^* = H^*(j\omega) \quad (10.19)$$

$$H(j\omega) = Me^{j\psi} \Rightarrow H(-j\omega) = H^*(j\omega) = Me^{-j\psi}$$

$$y(t) = H(j\omega)\frac{1}{2}Ae^{j\phi}e^{j\omega t} + H(-j\omega)\frac{1}{2}Ae^{-j\phi}e^{-j\omega t}$$

$$= Me^{j\psi}\frac{1}{2}Ae^{j\phi}e^{j\omega t} + Me^{-j\psi}\frac{1}{2}Ae^{-j\phi}e^{-j\omega t}$$

$$= \frac{1}{2}MAe^{j(\omega t+\phi+\psi)} + \frac{1}{2}MAe^{-j(\omega t+\phi+\psi)}$$

$$= \underbrace{MA}_{|H(j\omega)|}\cos(\omega t+\phi+\underbrace{\psi}_{\angle H(j\omega)})$$

Inverse Euler

Ex 10.5: Freq. response

$$H(j\omega) = \frac{40\pi}{40\pi + j\omega}$$

$$x(t) = 3\cos(40\pi t - \pi)$$

$$y(t) = ?$$

$x(t) \rightarrow$ freq of $x(t)$ 40π

so

$$|H(j40\pi)| = \frac{40\pi}{\sqrt{(40\pi)^2 + \omega^2}}$$

$$= \frac{40\pi}{\sqrt{1600\pi^2 + 1600\pi^2}} = \frac{1}{\sqrt{2}}$$

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{40\pi}\right) = -\tan^{-1}(1) = -\pi/4$$

$$y(t) = \frac{1}{\sqrt{2}} \cdot 3 \cdot \cos(40\pi t - \pi - \pi/4)$$

$$= 2.12 \cos(40\pi t - \underline{5\pi/4})$$

magnitude changed

freq. remains same

phase changed

Exercise 10.1 : Evaluate $y(t) = ?$

$$y(t) = \int_{-\infty}^{\infty} e^{-\tau} u(\tau) \underbrace{3\cos(t-\tau-\pi)}_{x(t)} d\tau$$

$\downarrow$

Find $h(t)$

$$h(t) = e^{-t} u(t)$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{-t} u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-(j\omega t + t)}\, dt$$

$$= -\frac{1}{j\omega + 1}\, e^{-(j\omega t + t)} \Big|_{0}^{\infty}$$

$$= \frac{1}{j\omega + 1}$$

ω at $1 \rightarrow |H(j\omega)| = \frac{1}{\sqrt{2}}$

$$\angle H(j\omega) = -\pi/4$$

$$y(t) = \frac{1}{\sqrt{2}}\, 3\cos(t - \pi - \pi/4) = \frac{3}{\sqrt{2}} \cos(t - 3\pi/4)$$

Response to a General Sum of Sinusoids

$$x(t) = \sum_{k=1}^{N} \underbrace{A_k}_{\text{different amplitude}} \cos(\underbrace{\omega_k}_{\text{different freq}} t + \overbrace{\phi_k}^{\text{diff. phase}}) \qquad (10.23)$$

$$y_k(t) = A_k |H(j\omega_k)| \cos\left(\omega_k t + \phi_k + \angle H(j\omega_k)\right)$$

$$y(t) = \sum_{k=1}^{N} y_k(t)$$

$$= \sum_{k=1}^{N} A_k |H(j\omega_k)| \cos\left(\omega_k t + \phi_k + \angle H(j\omega_k)\right)$$

Ex 10-6:

$$H(j\omega) = \frac{2+j\omega}{1+j\omega}$$

$$x(t) = 2\cos t + \cos(3t + 1.57)$$

$$H(j\omega)\Big|_{\omega=1} = \frac{2+j}{1+j} \Rightarrow |H(j\omega)| = \frac{\sqrt{4+1}}{\sqrt{2}} = 1.581$$

$$H(j\omega)\Big|_{\omega=1} = \frac{(2+j)(1-j)}{2} \qquad \angle H(j\omega) = \tan^{-1}(-1/3)$$

$$= \frac{3-j}{2} \qquad = -0.322$$

$$H(j\omega) = 1.581\, e^{-j0.322}$$

$$H(j\omega)\Big|_{\omega=3} = 1.14\, e^{-j0.216}$$

$$y(t) = 3.162 \cos(t - 0.322) + 1.14 \cos(3t + 1.304)$$

Periodic Input Signals

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \omega_0 = 2\pi / T_0$$

$$y_k(t) = \underbrace{H(jk\omega_0)\, a_k}_{b_k} e^{jk\omega_0 t}$$

$$y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\omega_0 t}$$

Ideal Filters

* The frequency response of delay

$$h(t) = \delta(t - t_d)$$

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_d) e^{-j\omega t} dt = \underbrace{e^{-j\omega t_d}}_{\text{magnitude is } 1}$$

it introduce phase shift that is depend on frequency ω.

Ex 10-7: $x(t) = 10e^{j\pi/4} e^{j200\pi t}$

Ideal delay system with a delay of 0.001 sec

So,
$$H(j\omega) = e^{-j\omega 0.001}$$

at $\omega = 200\pi$

$$H(j200\pi) = e^{-j200\pi\, 0.001}$$

$$y(t) = H(j200\pi)\left(10e^{j\pi/4} e^{j200\pi t}\right)$$

$$= e^{-j200\pi(0.001)}\, 10e^{j\pi/4} e^{j200\pi t}$$

$$= e^{-j0.2\pi}\, 10e^{j\pi/4} e^{j200\pi t}$$

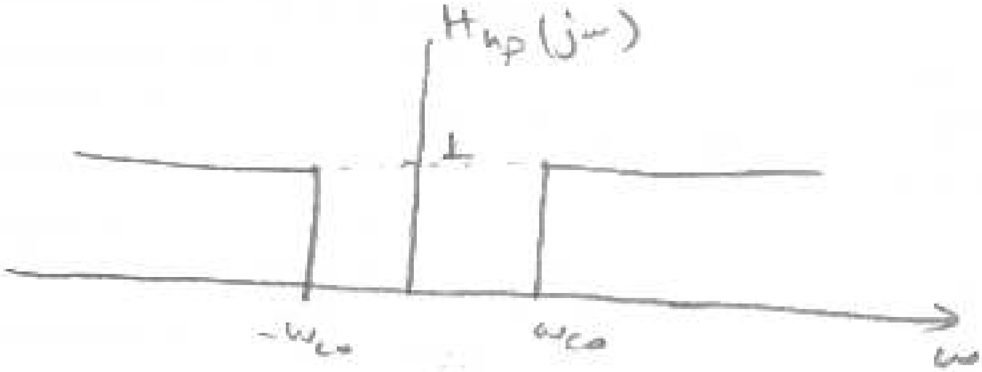
$$= 10e^{j\pi/4} e^{j200\pi\underbrace{(t-0.001)}_{\text{delay}}}$$

Ideal lowpass Filter (LPF)

$$H_{lp}(j\omega) = \begin{cases} 1 & |\omega| \le \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases} \qquad (10.32)$$

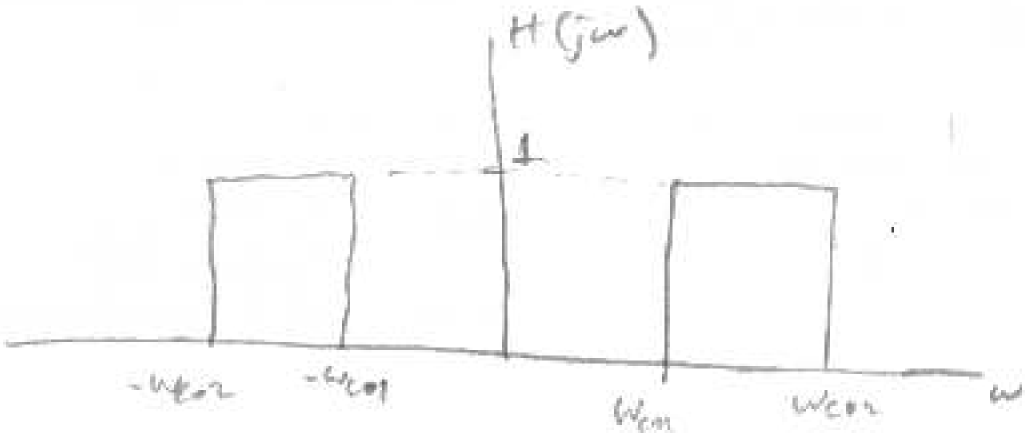
Ideal Highpass Filter

$$H_{hp}(j\omega) = \begin{cases} 0 & |\omega| \le \omega_{co} \\ 1 & |\omega| > \omega_{co} \end{cases}$$

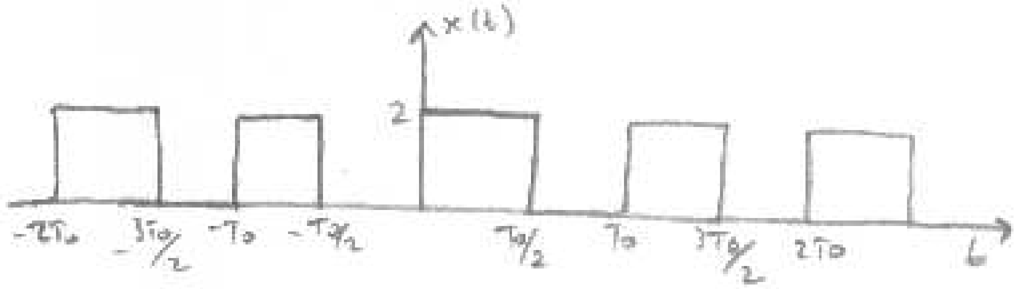


Ideal Bandpass Filter

$$H_{bp}(j\omega) = \begin{cases} 0 & |\omega| \le \omega_{co} \\ 1 & \omega_{co1} \le |\omega| \le \omega_{co2} \\ 0 & |\omega| > \omega_{co2} \end{cases} \qquad (10\text{-}36)$$



Application of Ideal Filters



$T_0 = 500\mu s$ Generate sinusoids from square signal

$$x(t) = \begin{cases} 2 & 0 \le t < T_0/2 \\ 0 & T_0/2 \le t < T_0 \end{cases}$$

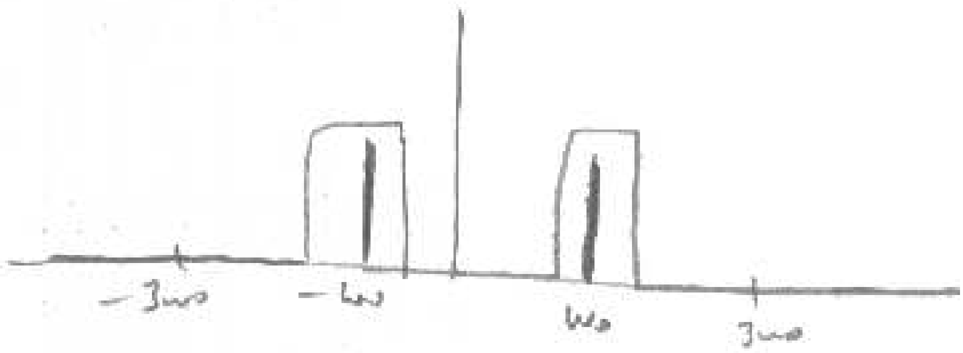
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} 2e^{-jk\omega_0 t}\, dt$$

$$a_k = \begin{cases} \frac{2}{j\pi k} & k = \pm 1, \pm 3, \pm 5 \dots \\ 0 & k = \pm 2, \pm 4, \pm 6, \dots \end{cases}$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0/2} 2\,dt = 1$$



To extract sinusoidal signal

$$0 < \omega_{c1} < 2\pi(2000)$$

$$2\pi(2000) < \omega_{c2} < 2\pi(6000)$$

The output signal coefficients

$$b_k = H(jk\omega_0)\, a_k = \begin{cases} \frac{2}{j\pi k} & k = \pm 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$b_k = H(j\, 2\pi(2000)\, k)\, a_k = \begin{cases} \frac{2}{j\pi k} & k = \pm 1 \\ 0 & \text{else} \end{cases}$$

$$y(t) = \frac{2}{j\pi} e^{j\pi(2000)t} + \frac{2}{j\pi(-1)} e^{-j2\pi(2000)t}$$

$$= \frac{4}{\pi}\left(\frac{1}{2j} e^{j2\pi(2000)t} - \frac{1}{2j} e^{-j2\pi t(2000)}\right)$$

$$= \frac{4}{\pi} \sin(2\pi\, 2000 t) = \frac{4}{\pi} \cos\left(2\pi\, 2000\, t - \pi/2\right)$$

Time Domain or Freq. Domain?

- When the input signal can be represented as a sum of sinusoids or complex exponentials that are nonzero for all time $-\infty < t < \infty$, then the frequency response method usually provides simplest solution.
- If a signal consists of impulses or step functions or other non sinusoidal signals, convolution of that signal with the impulse response of the system may be the simplest approach.

Ex 10-8: Suppose LTI system

$$h(t) = \delta(t) - 200\pi e^{-200\pi t} u(t)$$

$$x(t) = \underbrace{10}_{x_1(t)} + \underbrace{20\delta(t-0.1)}_{x_2(t)} + \underbrace{40\cos(200\pi t + 0.3\pi)}_{x_3(t)}$$

Freq response of the system

$$H(j\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[\delta(t) - 200\pi e^{-200\pi t} u(t)\right] e^{-j\omega t}\, dt$$

$$= \underbrace{\int_{-\infty}^{\infty} \delta(t) e^{-j\omega t}\, dt}_{1} - \underbrace{\int_{-\infty}^{\infty} 200\pi e^{-200\pi t} u(t) e^{-j\omega t}\, dt}_{200\pi \int_0^{\infty} e^{-200\pi t} e^{-j\omega t}\, dt}$$

$$= \frac{200\pi}{200\pi + j\omega}$$

$$H(j\omega) = 1 - \frac{200\pi}{200\pi + j\omega} = \frac{j\omega}{200\pi + j\omega}$$

$x_1(t) \rightarrow \omega = 0$

$$y_1(t) = H(j0)\, x_1(t) = \frac{j0}{200\pi + j\omega}\, 10 = 0$$

$x_2(t)$ is a δ function, so we can use impulse response

$$y_2(t) = x_2(t) * h(t)$$

$$= 20\left[\delta(t-0.1) - 200\pi\, e^{-200\pi(t-0.1)} u(t-0.1)\right]$$

$$= 20\delta(t-0.1) - 4000\pi\, e^{-200\pi(t-0.1)} u(t-0.1)$$

$$x_3(t) = 40\cos(200\pi t + 0.3\pi)$$

$$y_3(t) = H(j200\pi)\, x_3(t)$$

$$= |H(j200\pi)|\, 40\cos(200\pi t + 0.3\pi + \angle H(j200\pi))$$

$$|H(j200\pi)| = \left|\frac{200\pi j}{200\pi + 200\pi j}\right| = \left|\frac{200\pi j(200\pi - 200\pi j)}{40000\pi^2 + 40000\pi^2}\right|$$

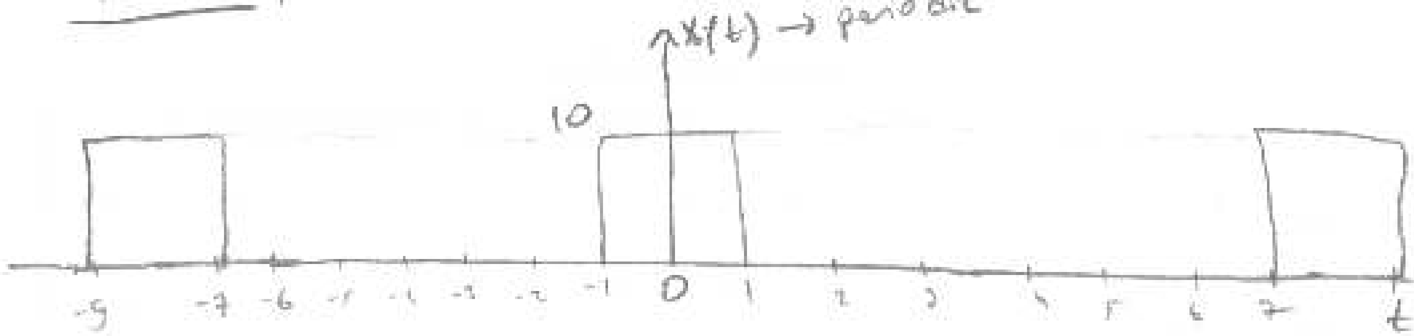
$$= \left|\frac{40000 j + 40000}{40000 + 40000}\right| = \left|\frac{j+1}{2}\right| = \frac{1}{\sqrt{2}}$$

$$\angle H(j200\pi) = \tan^{-1}(1) = \pi/4$$

$$y_3(t) = \frac{40}{\sqrt{2}}\cos(200\pi t + 0.55\pi)$$

$$y(t) = y_1(t) + y_2(t) + y_3(t) = 0 + 20\delta(t-0.1) - 4000\pi e^{-200\pi(t-0.1)} u(t-0.1) + \frac{40}{\sqrt{2}}\cos(200\pi t + 0.55\pi)$$

P10.5:



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{where} \quad a_k = \frac{10\sin(\pi k/4)}{\pi k}$$

a) $\omega_0 = ?$, $\quad \omega_0 = 2\pi f_0 \qquad T_0 = 8s$

$$\omega_0 = \frac{2\pi}{8s} \qquad f_0 = \frac{1}{8s}$$

$$= \frac{\pi}{4s}$$

Write down integral to obtain a_k?

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\, e^{-jk\omega_0 t}\, dt$$

So, $x(t) = \begin{cases} 10, & |t| \leq 1 \\ 0, & \text{else} \end{cases}$

$$a_k = \frac{1}{8}\int_{-1}^{1} 10\, e^{-jk\omega_0 t}\, dt = \frac{5}{4}\int_{-1}^{1} e^{-jk\omega_0 t}\, dt$$

$$= -\frac{5}{4jk\omega_0}\, e^{-jk\omega_0 t}\Big|_{-1}^{1}$$

$$a_k = -\frac{5}{4jk\omega_0} e^{-jk\omega_0} + \frac{5}{4jk\omega_0} e^{jk\omega_0}$$

$$a_k = \frac{5}{2k\omega_0}\left(-\frac{1}{2j} e^{-jk\omega_0} + \frac{1}{2j} e^{jk\omega_0}\right)$$

$$k\omega_0 = \frac{\pi}{4}$$

$$a_k = \frac{10}{k\pi} \sin\left(\pi k/4\right)$$

b) $a_0 = \frac{5}{2}$

$$a_1 = \frac{10}{\pi} \sin(\pi/4) = \frac{5\sqrt{2}}{\pi}$$

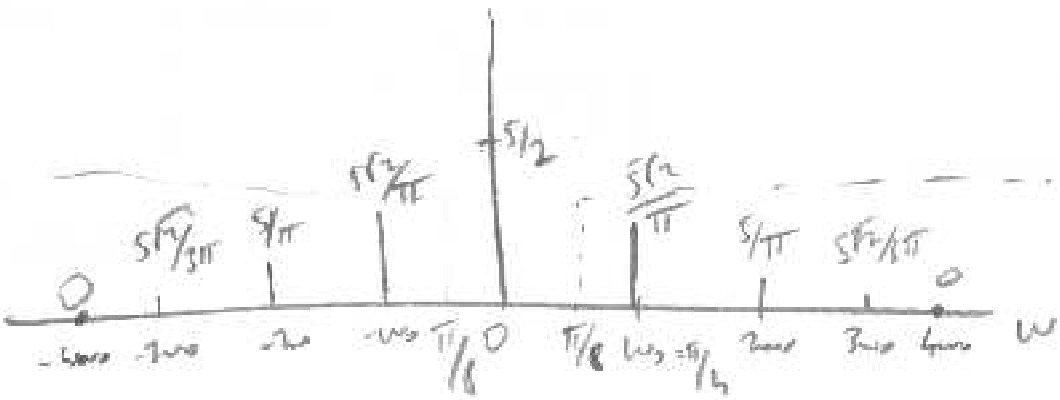
$$a_{-1} = +\frac{5\sqrt{2}}{\pi}$$

$$a_2 = \frac{5}{\pi} \sin(\pi/2) = \frac{5}{\pi}$$

$$a_{-2} = \frac{5}{\pi} \sin(-\pi/2) = 5/\pi$$

$$a_{\pm 3} = \frac{10}{3\pi} \frac{\sqrt{2}}{2} = \frac{5\sqrt{2}}{3\pi}$$

$$a_{\pm 4} = 0$$

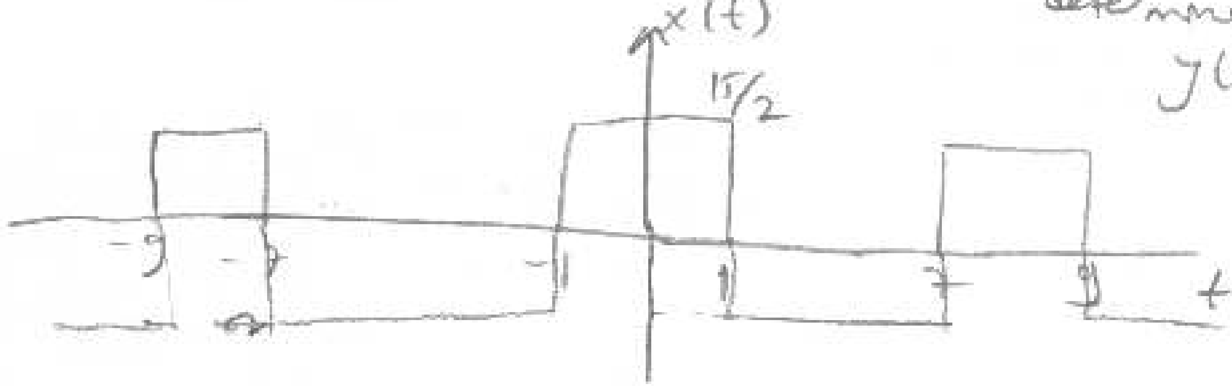


c) If the freq. response of the system is the ideal highpass

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{kj\omega_0 t} - a_0$$

$$H(j\omega) = \begin{cases} 0 & |\omega| < \pi/8 \\ 1 & |\omega| > \pi/8 \end{cases}$$

determine $y(t)$



d) If the freq response low pass

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

for what values of ω_{co} will the output of the system have the form

$$y(t) = A + B\cos(\omega_0 t + \phi)$$

$$A \neq 0, \; B \neq 0$$

$$\pi/4 < \omega_{co} < \pi/2$$

e) If the freq. response of the LTI system is $H(j\omega) = 1 - e^{-j2\omega}$, plot the output of the system $y(t)$ when the input is $x(t)$ as plotted above

$$H(j\omega) = 1 - e^{-j2\omega}$$

$$h(t) = \delta(t) - \delta(t-2)$$

$$y(t) = h(t) * x(t)$$
$$= x(t) - x(t-2)$$