Signals & Systems (BLG 354E)

Week 10

17.04.2017

G.Ü.

Frequency Response of LTI Systems

$$x[n] \longrightarrow \boxed{\text{LTI}}_{h[n]} \longrightarrow y[n] = ?$$

$$x[n] = e^{j\hat{\omega} n} \longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow \boxed{y[n] = \underbrace{e^{j\hat{\omega} n}}_{\substack{\text{sinusoidal input} \\ x(n)}} H(e^{j\hat{\omega}})}$$

Multip

$H(e^{j\hat{\omega}})$ → continuous fn. of $\hat{\omega}$.

Frequency Response

$$\hat{\omega} = \omega \cdot T$$

$$\hat{\omega} = \frac{\text{rad}}{\text{sec}} \cdot \text{sec} = \text{rad}$$

$\hat{\omega}$: Normalized (radian) frequency

angular

Frequency Response: tells us how the system responds to sinusoid inputs, at different frequencies.

Let us derive relation of impulse response $h[n]$ to $H(e^{j\hat{\omega}})$:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} h[k] \underbrace{x(n-k)}$$

: Let $x(n) = e^{+j\hat{\omega} n}$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] \underbrace{e^{+j\hat{\omega}(n-k)}}_{e^{+j\hat{\omega} n} \cdot e^{-j\hat{\omega} k}} = \underbrace{e^{+j\hat{\omega} n}}_{x[n]} \underbrace{\sum_{k=-\infty}^{\infty} h[k] e^{-j\hat{\omega} k}} \triangleq H(e^{j\hat{\omega}})$$

: frequency response of the system.

$$h[n] \xleftrightarrow{\text{DTFT}} H(e^{j\hat\omega})$$

Time-Domain Characterization

Frequency Domain characterization of the system.

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

$h[n] \Leftrightarrow H(e^{j\hat\omega}) \longrightarrow$ purely in frequency

purely in time

equivalent representation

$H(e^{j\hat\omega})$: complex fn. of $\hat\omega$:

For a specific frequency $\hat\omega_0$:

$$H(e^{j\hat\omega_0}) = \underbrace{|H(e^{j\hat\omega_0})|}_{\text{magnitude (gain) Response}} \cdot e^{j\measuredangle H(e^{j\hat\omega_0})}$$

phase Response

$\measuredangle H(e^{j\hat\omega_0})$

$$\underbrace{x[n]}_{e^{j\hat\omega n}} \longrightarrow \underset{H(e^{j\hat\omega})}{\boxed{\text{LTI}}} \longrightarrow y[n] = H(e^{j\hat\omega})\, x[n]$$

$$|H(e^{j\hat\omega})| \cdot e^{j\measuredangle H(e^{j\hat\omega})} e^{j\hat\omega n}$$

- Amplitude of $e^{j\hat\omega n}$ is Changed by magnitude response
- Phase of $e^{j\hat\omega n}$ " phase response : $e^{j(\hat\omega n + \measuredangle H(e^{j\hat\omega}))}$

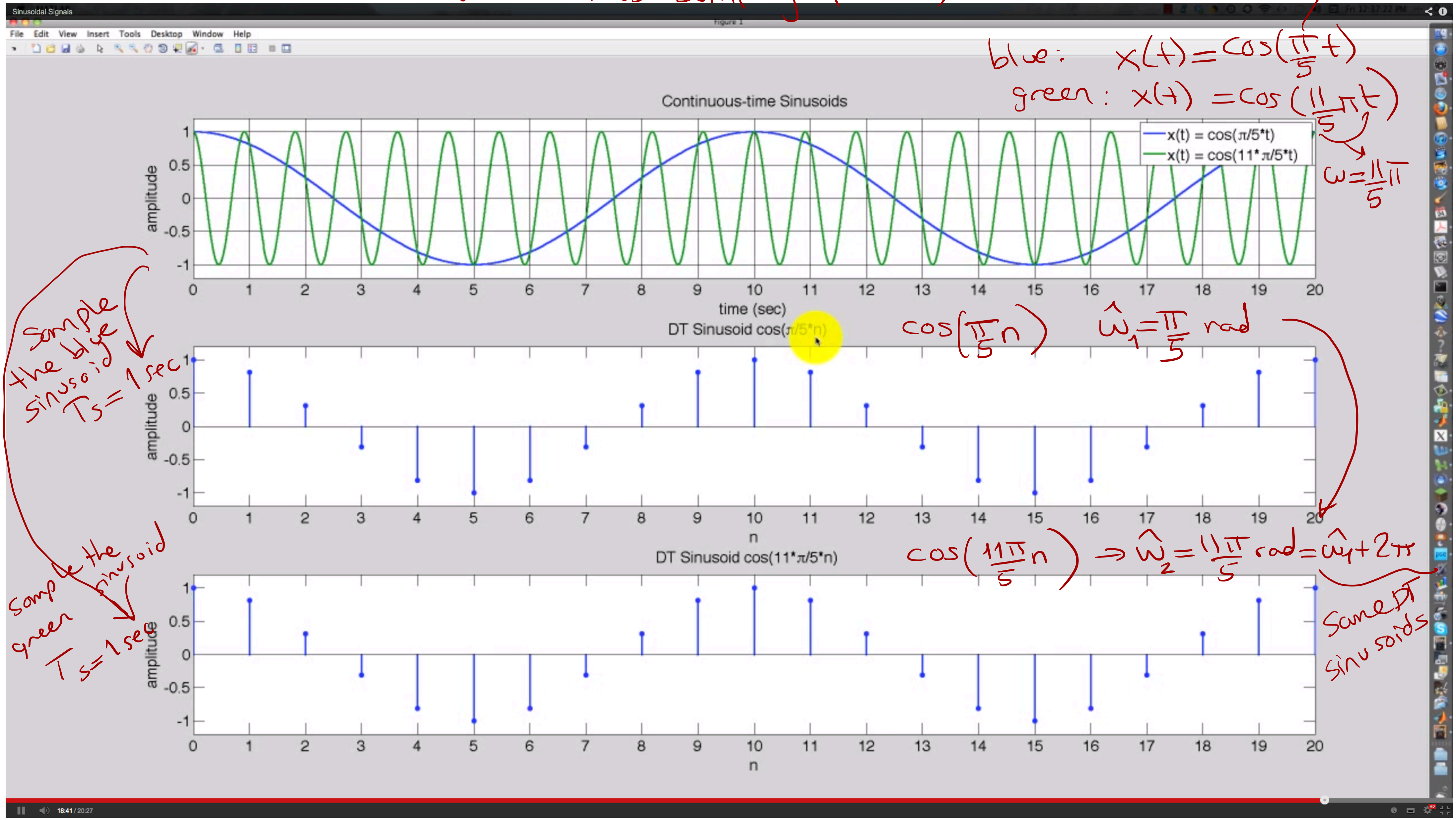
$x(t) = A\cos(\omega_0 t + \phi) \rightarrow x[n] = ?\ x(nT_s)$

Sample: $x(nT_s) = x(t)\big|_{t=nT_s} \leftarrow T_s$: sampling period)

$\omega = \frac{\pi}{5}\ \frac{rad}{s}$

blue: $x(t) = \cos(\frac{\pi}{5}t)$

green: $x(t) = \cos(\frac{11}{5}\pi t)$ $\rightarrow \omega = \frac{11}{5}\pi$



$\cos(\frac{\pi}{5}n)$ $\quad \hat{\omega}_1 = \frac{\pi}{5}$ rad

sample the blue sinusoid $T_s = 1$ sec

$\cos(\frac{11\pi}{5}n) \rightarrow \hat{\omega}_2 = \frac{11\pi}{5}$ rad $= \hat{\omega}_1 + 2\pi$

Same DT sinusoids

sample the sinusoid green $T_s = 1$ sec

Ex: I/O relation for LTI system: $y_1[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$: Averaging (smoother) FIR filter.

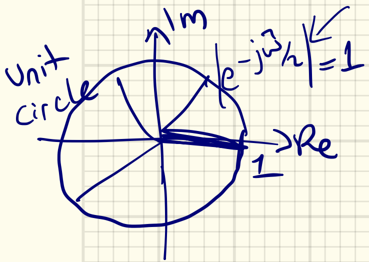
Impulse respo $\Longrightarrow$ $h_1[n] = \frac{1}{2}\delta[n] + \frac{1}{2}\delta[n-1]$; $h_1[n] = \begin{cases} \frac{1}{2}, & n=0,1 \\ 0, & o/w \end{cases}$

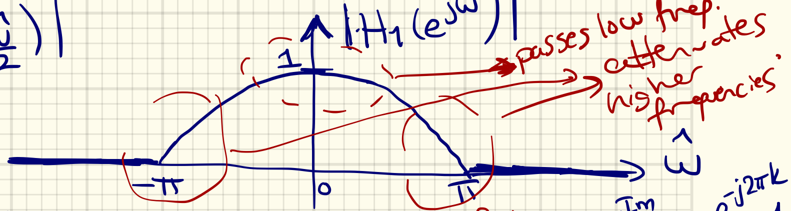
Frequency Response : $H_1(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\hat{\omega}k} = \frac{1}{2}\underbrace{e^{j0}}_{1} + \frac{1}{2}e^{-j\hat{\omega}}$

$$H_1(e^{j\hat{\omega}}) = e^{-j\hat{\omega}/2}\left(\frac{1}{2}e^{j\hat{\omega}/2} + \frac{1}{2}e^{-j\hat{\omega}/2}\right) = e^{-j\hat{\omega}/2}\cos\left(\frac{\hat{\omega}}{2}\right)$$

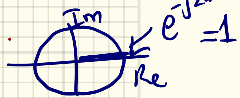
phase response

$$|H_1(e^{j\hat{\omega}})| = \underbrace{|e^{-j\hat{\omega}/2}|}_{=1}\,\left|\cos\left(\frac{\hat{\omega}}{2}\right)\right|$$





Low-Pass filter.



Note:

$-\pi \leq \angle H(e^{j\hat{\omega}}) \leq \pi$

$0 \leq \angle H(e^{j\hat{\omega}}) \leq 2\pi$

✱ Property: $H(e^{j\hat{\omega}})$ is periodic w/ 2π.

$\sum h[k]e^{-j\hat{\omega}k} \Rightarrow H(e^{j(\hat{\omega}+2\pi)}) = \sum h[k]e^{-j(\hat{\omega}+2\pi)k}$

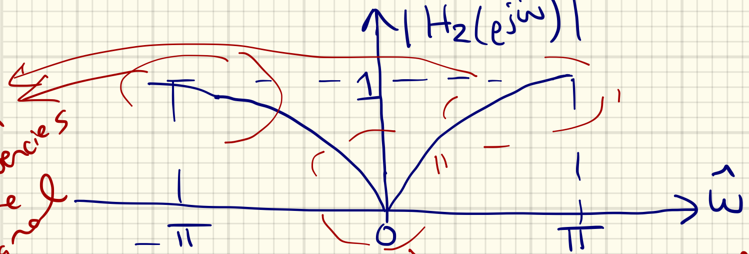
$e^{-j\hat{\omega}k}\, e^{-j2\pi k}$ $= 1 \ \forall k$

Ex: $y[n] = \frac{1}{2}x[n] - \frac{1}{2}x[n-1]$: forming a difference btw consecutive values of the signal

$$h_2[n] = \begin{cases} \frac{1}{2}, & n=0 \\ -\frac{1}{2}, & n=1 \\ 0, & o/w \end{cases} \Rightarrow H_2(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} h[k]\, e^{-j\hat{\omega}k}$$

$$H_2(e^{j\hat{\omega}}) = \frac{1}{2} - \frac{1}{2}e^{-j\hat{\omega}}$$

$$H_2(e^{j\hat{\omega}}) = je^{-j\hat{\omega}/2}\left(\frac{1}{2j}e^{+j\hat{\omega}/2} - \frac{1}{2j}e^{-j\hat{\omega}/2}\right) = e^{j\pi/2}e^{-j\hat{\omega}/2}\sin\left(\frac{\hat{\omega}}{2}\right)$$



$$x[n] \longrightarrow \boxed{H_2(e^{j\omega})} \longrightarrow y[n]$$

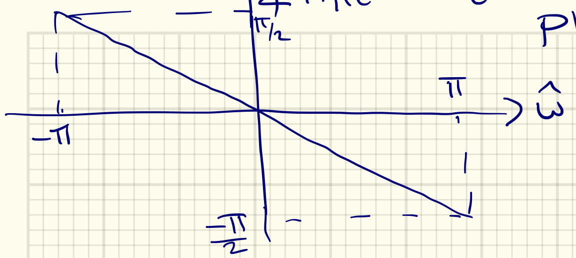
High frequencies in the signal less affected

attenuates low frequencies in the input

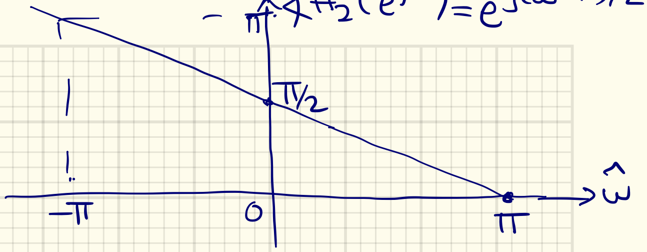
eg. At 0 frequency in the input $\rightarrow y[n]=0$ b/c: $\underbrace{H(e^{j0})}_{\text{Gain}=0 \text{ at } \hat{\omega}=0} \cdot x(n)$

High-Pass Filter:

$\measuredangle H_1(e^{j\hat{\omega}}) = e^{-j\hat{\omega}/2}$



$\measuredangle H_2(e^{j\hat{\omega}}) = e^{-j(\hat{\omega}-\pi)/2}$



Phase Response

$\downarrow$

Linear Phase

CASCADED LTI SYSTEMS :

$$x(n) \longrightarrow \boxed{H_1} \xrightarrow{w(n)} \boxed{H_2(e^{j\hat{\omega}})} \longrightarrow y[n] = \underbrace{e^{j\hat{\omega}n}}_{x[n]} H_1(e^{j\hat{\omega}}) H_2(e^{j\hat{\omega}})$$

$= e^{j\hat{\omega}n}$

$$w[n] = \left(e^{j\hat{\omega}n} H_1(e^{j\hat{\omega}})\right)$$

$$y[n] = x(n) * (h_1[n] * h_2[n])$$

Next time

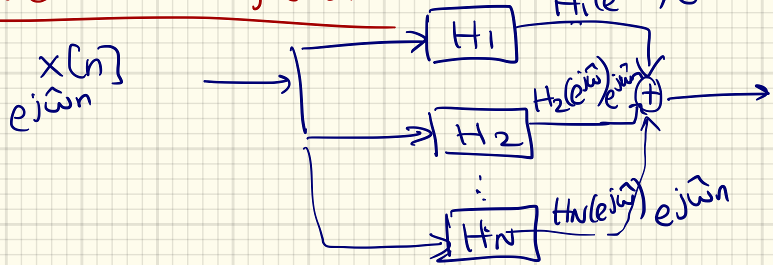
$$\left[\, y(n) = h[n] * x[n] \quad \overset{\text{DTFT}}{\longleftrightarrow} \quad Y(e^{j\hat{\omega}}) = H(e^{j\hat{\omega}})\, X(e^{j\hat{\omega}}) \,\right]$$

$$e^{j\hat{\omega}n} \longrightarrow \underbrace{\boxed{H_1} \longrightarrow \boxed{H_2} \longrightarrow \cdots \longrightarrow \boxed{H_N}}_{H_{overall}(e^{j\hat{\omega}})} \longrightarrow y[n] = \underbrace{\left(\prod_{i=1}^{N} H_i(e^{j\hat{\omega}})\right)}_{H_{overall}(e^{j\hat{\omega}})} e^{j\hat{\omega}n}$$

$H_{overall}(e^{j\hat{\omega}})$: overall system's freq. response

$H_{overall}(e^{j\hat{\omega}})$: multiplic. of freq. responses.

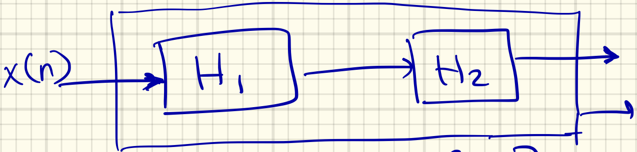
Parallel LTI Systems:



$$y(n) = \boxed{\sum_{k=1}^{N} H_k(e^{j\hat{\omega}})} e^{j\hat{\omega}n}$$

$H_{overall}(e^{j\hat{\omega}})$: sum of freq. responses

Ex:



$y(n)$

$h_o(n)$ or $H_o(e^{j\hat{\omega}})$

Given $h_1[n] = 2\delta[n] + 4\delta[n-1] + 4\delta[n-2] + 2\delta[n-3] \rightarrow h_1 = \begin{cases} 2, & n=0 \\ 4, & n=1 \\ 4, & n=2 \\ 2, & n=3 \end{cases}$

$h_2(n) = \delta(n) - 2\delta[n-1] + \delta[n-2]$

Find $h_o[n]$ using Freq response $H_o(e^{j\hat{\omega}}) = H_1(e^{j\hat{\omega}})H_2(e^{j\hat{\omega}})$

$$H_1(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} \underbrace{h(k)} e^{-j\hat{\omega}k} = 2 \cdot e^{j0} + 4e^{-j\hat{\omega}} + 4e^{-j2\hat{\omega}} + 2e^{-j3\hat{\omega}}$$

$$H_2(e^{j\hat{\omega}}) = 1 - 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

$$\hookrightarrow H_o(e^{j\hat{\omega}}) = (2 + 4e^{-j\hat{\omega}} + 4e^{-j2\hat{\omega}} + 2e^{-j3\hat{\omega}})(1 - 2e^{j\hat{\omega}} + e^{-j2\hat{\omega}})$$

$$H_o(e^{j\hat{\omega}}) = 2 - 2e^{-j\hat{\omega}2} - 2e^{-j\hat{\omega}3} + 2e^{-j\hat{\omega}5} \Rightarrow h_o(n) = ? \Rightarrow$$

$$h_o[n] = 2\delta[n] - 2\delta[n-2] - 2\delta[n-3] + 2\delta[n-5]$$

Frequency Response $H(e^{j\hat{\omega}})$ Properties :

1) $H(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\hat{\omega}k}$: complex fn. of $\underline{\hat{\omega}}$ only (no time variable)

2) $H(e^{j\hat{\omega}})$ is periodic w/ 2π ; $\hat{\omega}$: normalized angular freq.

3) When $h[n]$ is $\underline{\text{real}}$; $\boxed{H^*(e^{j\hat{\omega}}) = H(e^{-j\hat{\omega}})}$. Conjugate Symmetry

— Show :

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\hat{\omega}n}$$

$$H^*(e^{j\hat{\omega}}) = (\quad)^* = \sum \underbrace{h^*(n)}_{=h[n]} e^{j\hat{\omega}n}$$

$=h[n]$ b/c it is a real fn.

$$= \underbrace{\sum_n h[n] e^{-j(-\hat{\omega})n}}$$

$$H^*(e^{j\hat{\omega}}) = H(e^{-j\hat{\omega}}) \checkmark$$

— Also, we can show

$$H(e^{-j\hat{\omega}}) = |H(e^{-j\hat{\omega}})| e^{j\measuredangle H(e^{-j\hat{\omega}})}$$

take conjugate

$$(H(e^{-j\hat{\omega}}))^* = |H(e^{-j\hat{\omega}})| e^{-j\measuredangle H(e^{-j\hat{\omega}})}$$

$$H(e^{j\hat{\omega}}) = |H(e^{j\hat{\omega}})| e^{j\measuredangle H(e^{j\hat{\omega}})} = |H(e^{-j\hat{\omega}})| e^{-j\measuredangle H(e^{-j\hat{\omega}})}$$

This shows $\Longrightarrow$

Gain or magnitude Response (4) $\ast\, |H(e^{j\hat{\omega}})|$ is even symmetric : $|H(e^{-j\hat{\omega}})| = |H(e^{+j\hat{\omega}})|$

Phase Response (5) $\ast\, \measuredangle H(e^{j\hat{\omega}})$ is odd-symmetric : $\measuredangle H(e^{-j\hat{\omega}}) = -\measuredangle H(e^{j\hat{\omega}})$

General Even symm: $f(x) = f(-x)$; $f(-x) = -f(x) \Longleftrightarrow f(x) = -f(-x)$

Superposition property of LTI systems :

$$x[n] = \underbrace{\alpha_1 e^{j\hat{\omega}_1 n}} + \underbrace{\alpha_2 e^{j\hat{\omega}_2 n}}$$

$$x[n] \longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow y[n] = y_1[n] + y_2[n]$$

$$\alpha_1 e^{j\hat{\omega}_1 n} \longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow y_1[n] = \alpha_1 H(e^{j\hat{\omega}_1}) e^{j\hat{\omega}_1 n}$$

$$\alpha_2 e^{j\hat{\omega}_2 n} \longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow y_2[n] = \alpha_2 H(e^{j\hat{\omega}_2}) e^{j\hat{\omega}_2 n}$$

Sum of Sinusoids $\sum \alpha_i e^{j\hat{\omega}_i n}$

$$\longrightarrow \boxed{H(e^{j\hat{\omega}})} \longrightarrow \underbrace{\alpha_i \sum_i H(e^{j\hat{\omega}_i}) \cdot e^{j\hat{\omega}_i n}}_{\text{also Sum of sinusoids}}$$

$$x[n] = A_0 + A_1 \cos(\hat{\omega}_1 n + \phi) + \cdots$$

$$x[n] \longrightarrow \boxed{H(\ )} \longrightarrow y[n]$$

$$x[n] = A_0 + \frac{A_1}{2} e^{j(\hat{\omega}_1 n + \phi)} + \frac{A_1}{2} e^{-j(\hat{\omega}_1 n + \phi)}$$

use superposition

LTI

$$A_0 \longrightarrow \boxed{H} \longrightarrow \underbrace{H(e^{j0})}_{H(e^{j\hat{\omega}})\big|_{\hat{\omega}=0}} A_0$$

$$\frac{A_1}{2} e^{j\phi} \cdot e^{j\hat{\omega}_1 n} \longrightarrow \boxed{H} \longrightarrow \frac{A_1}{2} e^{j\phi} H(e^{j\hat{\omega}_1}) e^{j\hat{\omega}_1 n}$$

$$\frac{A_1}{2} e^{-j\phi} e^{-j\hat{\omega}_1 n} \longrightarrow \boxed{H} \longrightarrow \frac{A_1}{2} e^{-j\phi} H(e^{-j\hat{\omega}_1}) e^{-j\hat{\omega}_1 n}$$

$$y[n] = H(e^{j0})A_0 + \underbrace{H(e^{j\hat{\omega}_1})}_{|H(e^{j\hat{\omega}_1})| e^{j\angle H(e^{j\hat{\omega}_1})}} \frac{A_1}{2} e^{j\phi} e^{j\hat{\omega}_1 n} + \underbrace{H(e^{-j\hat{\omega}_1})}_{|H(e^{-j\hat{\omega}_1})| e^{j\angle H(e^{-j\hat{\omega}_1})}} \frac{A_1}{2} e^{-j\phi} e^{-j\hat{\omega}_1 n}$$

= due even symmetry

$$= e^{-j\angle H(e^{j\hat{\omega}_1})}$$ due odd sym

$$= H(e^{j0})A_0 + |H(e^{j\hat{\omega}_1})| \frac{A_1}{2} \left(e^{j(\phi + \angle H(e^{j\hat{\omega}_1}))} \cdot e^{j\hat{\omega}_1 n} + e^{-j(\phi + \angle H(e^{j\hat{\omega}_1}))} \cdot e^{-j\hat{\omega}_1 n} \right)$$

$$y[n] = H(e^{j0})A_0 + \underbrace{|H(e^{j\hat{\omega}_1})|}_{\text{magnitude changed}} A_1 \cos(\hat{\omega}_1 n + \underbrace{\angle H(e^{j\hat{\omega}_1})}_{\text{new phase affected by the system.}})$$

Frequency Response and Difference Equations:

General Linear Difference Equation: $\Rightarrow \displaystyle\sum_{k=0}^{N} a_k\, y[n-k] = \sum_{k=0}^{M} b_k\, x[n-k]$

Let $\underbrace{x(n) = e^{j\hat{\omega} n}}$ $\longrightarrow \boxed{H(\;)} \longrightarrow \underline{y[n] = H(e^{j\hat{\omega}})\, e^{j\hat{\omega} n}}$

$\Rightarrow$ Substitute into diff. eqn

$$\sum_{k=0}^{N} a_k\, H(e^{j\hat{\omega}})\, e^{j\hat{\omega}(n-k)} = \sum_{k=0}^{M} b_k\, e^{j\hat{\omega}(n-k)}$$

$$\Rightarrow \boxed{H(e^{j\hat{\omega}}) = \frac{\sum_{k=0}^{M} b_k e^{-j\hat{\omega}k}}{\sum_{k=0}^{N} a_k e^{-j\hat{\omega}k}}}$$

General form of frequency response for LTI systems

Recall, we set $a_0 = 1$, $k \neq 0;\ a_k = 0$ } FIR system

General case: IIR system

Recall eg. $\underline{h(n) = (\tfrac{1}{2})^n u(n)}$ → infinite-impulse response

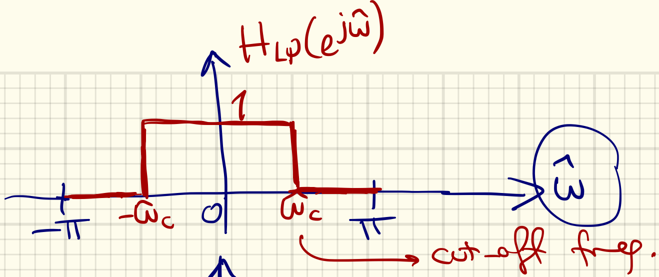
$\Rightarrow$ FIR system: $H(e^{j\hat{\omega}}) = \displaystyle\sum_{\cancel{k=0}\ -\infty}^{\cancel{M}\ \infty} b_k\, e^{-j\hat{\omega}k}$

$a_0 = 1$

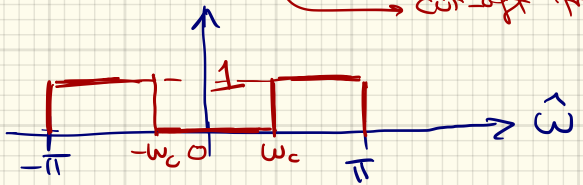
other a_k's are 0

Ideal Filters:

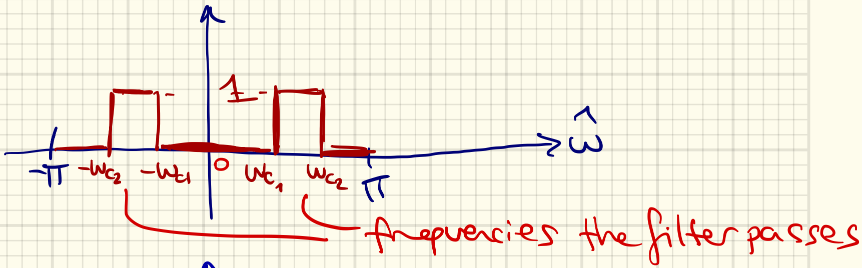
Low-Pass Filter:



High-Pass Filter



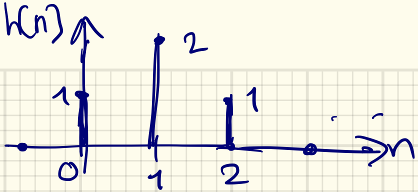
Band-Pass Filter:



Band-stop Filter:

Ex: Given FIR filter

$b_k = \{1, 2, 1\}$, $\downarrow n=0$

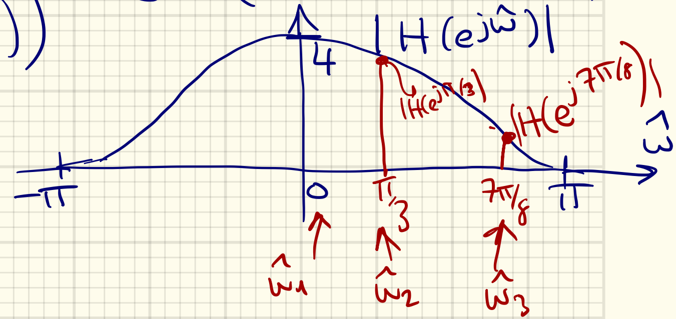


$$H(e^{j\hat{\omega}}) = ? = \sum_{k=0}^{2} h(k) e^{-j\hat{\omega}k} = 1 + 2e^{-j\hat{\omega}} + e^{-j2\hat{\omega}}$$

$$= e^{-j\hat{\omega}}\left(e^{j\hat{\omega}} + 2 + e^{-j\hat{\omega}}\right)$$

$$H(e^{j\hat{\omega}}) = e^{-j\hat{\omega}}\left(2 + 2\cos(\hat{\omega})\right)$$

$$x(n) \longrightarrow \boxed{H} \longrightarrow y[n]$$



Given $x[n] = \underbrace{4}_{x_1(n)} + \underbrace{3\cos\left(\frac{\pi}{3}n - \frac{\pi}{2}\right)}_{+\ x_2(n)} + \underbrace{3\cos\left(\frac{7\pi}{8}n\right)}_{+\ x_3(n)}$

$y_1[n] = ?\quad H(e^{j0}) \cdot 4 = \underline{16} = y_1[n]$

$y_2(n) = ? \Rightarrow \hat{\omega}_2 = \frac{\pi}{3} \rightarrow H(e^{j\hat{\omega}_2}) = H(e^{j\pi/3}) = 3e^{-j\pi/3}$

$y_3[n] = ? \Rightarrow \hat{\omega}_3 = 7\frac{\pi}{8} \Rightarrow \underbrace{H(e^{j\hat{\omega}_3})}_{} = H(e^{j7\pi/8}) = \left(2 + \underbrace{2\cos 7\frac{\pi}{8}}_{}\right) e^{-j7\pi/8} = 0.152e^{-j\frac{7\pi}{8}}$

$\Longrightarrow$

$$\Rightarrow y[n] = 16 + 9\cos\Big(\frac{\pi}{3}n - \underbrace{\frac{\pi}{2} - \frac{\pi}{3}}_{\measuredangle H(e^{j\hat{\omega}_2})}\Big) + 3(0.152)\cos\Big(\frac{7\pi}{8}n - \underbrace{\frac{7\pi}{8}}_{\measuredangle H(e^{j\hat{\omega}_3})}\Big)$$

Q: What if $x_0[n] = x[n] + \delta[n-4]$?

Next time.

Reading Assignment: Chapter 6 from your textbook