Signals & Systems (BLG354E)

Week 12

03.05.2017

G.Ü.

Chapter 11 CT Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \qquad \text{(Forward) F.T.}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{X(j\omega)}\, e^{j\omega t}\, d\omega \qquad \text{(Inverse) F.T.}$$

$x(t)$ Time domain $\longleftrightarrow$ $X(j\omega)$ Frequency Domain

$$x(t) \xleftrightarrow{F.T} X(j\omega) \qquad 1-1 \text{ mapping}$$

Note: F.T. exist when $\int_{-\infty}^{\infty} |x(t)|\,dt < \infty$

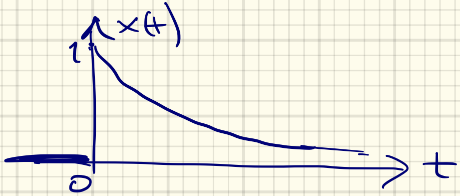
$$|X(j\omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt \right| \leq \int_{-\infty}^{\infty} |x(t)|\, \underbrace{|e^{-j\omega t}|}_{1}\, dt$$

$$= \int_{-\infty}^{\infty} |x(t)|\,dt < \infty.$$

This is a sufficient condition; not a necessary condition.
($\exists$ some signals not absolutely integrable but their F.T exists).

Now, we learned F.T. can represent non-periodic signals, let's develop a library of F.T. pairs

* $x(t) = e^{-at}u(t)$, $a>0$

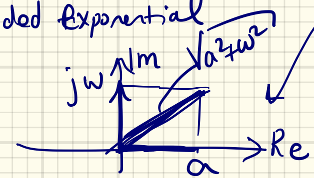


$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

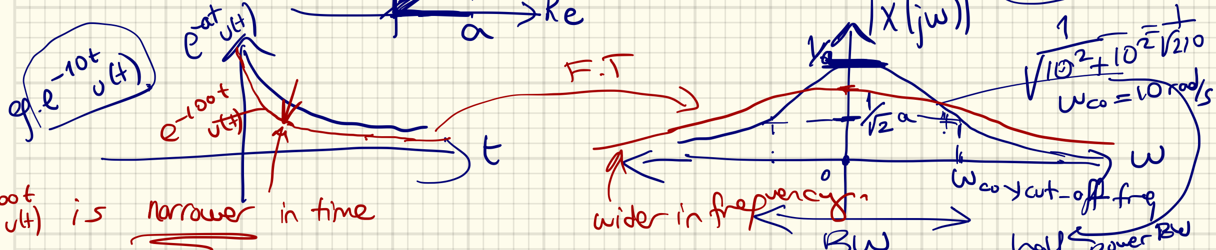
$$= \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-at} e^{-j\omega t}\, dt = \frac{1}{a+j\omega}$$

1 $\boxed{e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}}$, $a>0$

Right-sided Exponential

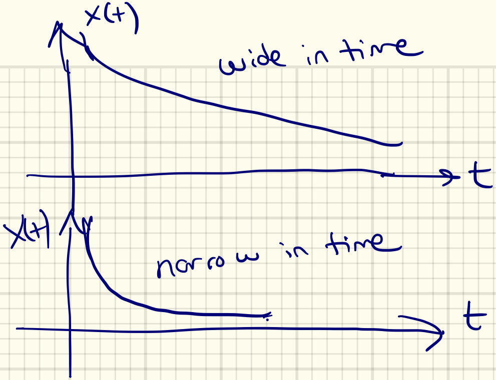


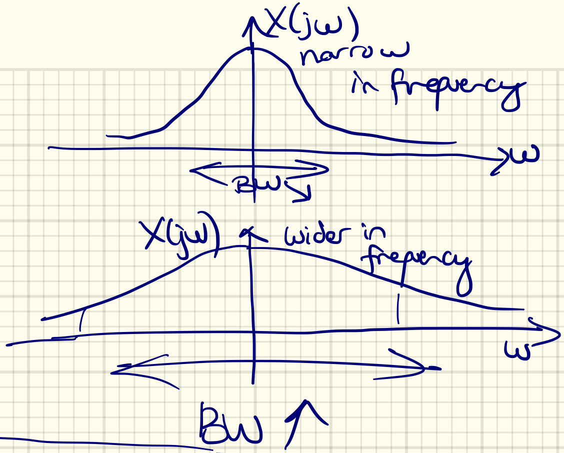
$X(j\omega) \longrightarrow |X(j\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}$



$e^{-100t}u(t)$ is narrower in time

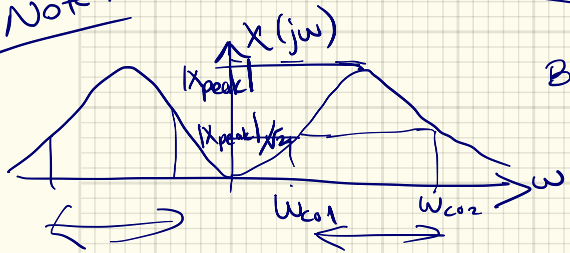
BW: Freq. range within which most part of the freq. response is contained





★ Signal is wide in time $\longleftrightarrow$ narrow in freq.
narrow " $\longleftrightarrow$ wide in freq.

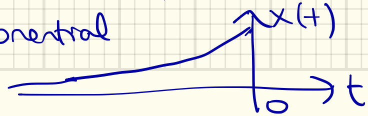
Note:



Bandpass.

Question: what if $x(t) = e^{bt}u(-t) \xleftrightarrow{F.T} X(j\omega) = \dfrac{1}{b - j\omega}$ ($b<0$)

left-handed exponential



Derive this yourself

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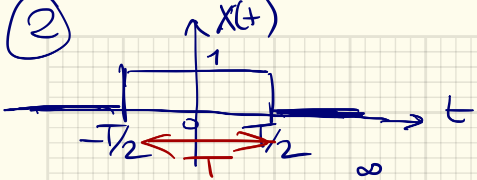


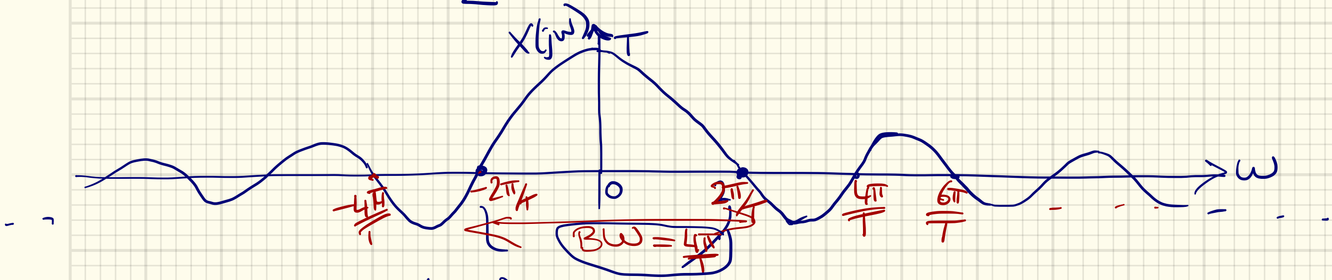
$\Longleftrightarrow \quad X(j\omega) = ?$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-T/2}^{T/2} 1 \cdot e^{-j\omega t}\, dt = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-T/2}^{T/2}$$

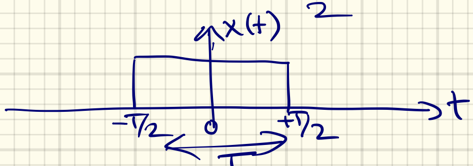
exercise fill in the blanks ...

$$= \frac{\sin(\omega T/2)}{\omega/2}$$

$\longrightarrow$ $\omega = 0$ value use l'hospital $\quad \frac{T}{2}\frac{\cos(\omega T/2)}{1/2} = T$



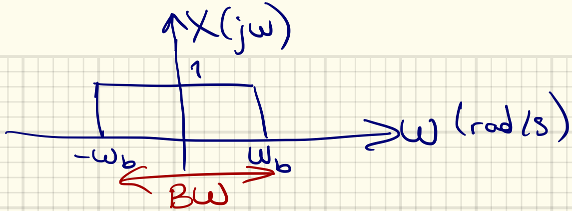
$$\sin\left(\frac{\omega T}{2}\right) = 0 \rightarrow \frac{\omega T}{2} = \pi k \Rightarrow \omega = \frac{2\pi k}{T}$$



Small T (narrow time signal) $\Longleftrightarrow$ ($BW = \frac{4\pi}{T}$) wider freq. BW $\uparrow$

Large T (wide ") $\Longleftrightarrow$ BW $\searrow$ narrow in freq.

3) Ideal LP



$\longrightarrow x(t) = ?$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(jw)\, e^{jwt}\, dw = \frac{1}{2\pi}\int_{-w_b}^{w_b} 1 \cdot e^{jwt}\, dw = \frac{1}{2\pi}\frac{e^{jwt}}{jt}\Bigg|_{-w_b}^{w_b} = \frac{1}{\pi t}\,\frac{e^{jw_bt} - e^{-jw_bt}}{2j}$$

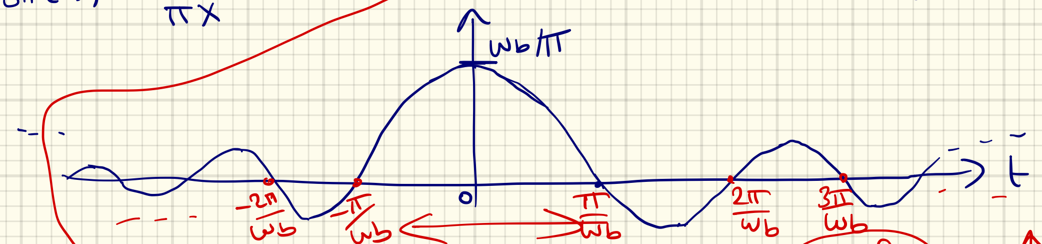
$$\Rightarrow \boxed{x(t) = \frac{\sin(w_bt)}{\pi t}}$$

Note: $\mathrm{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

$\rightarrow \infty$ length signal.

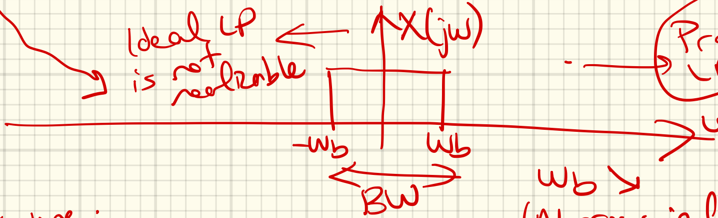
zeros of $x(t)$: $w_bt = \pi k$

$$\boxed{t = \frac{\pi}{w_b}k}$$

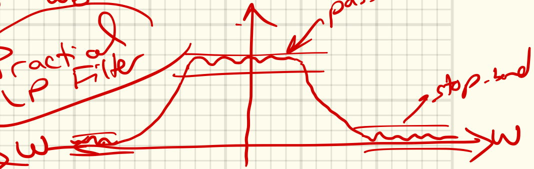


Recall Uncertainty Principle:
Time resol^n ↑
Freq resol^n
cannot be improved at the same time.

Ideal LP is not realizable



Practical LP Filter



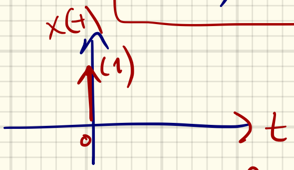
$w_b \searrow$ (Narrow in freq) $\longleftrightarrow$ Wide in time $\left(\frac{2\pi}{w_b}\right)$ ↗

wide in freq $\longleftrightarrow$ narrow in time

④ $F.T\{\delta(t)\} = ?$ $x(t)=\delta(t) \longleftrightarrow X(j\omega) = ?$

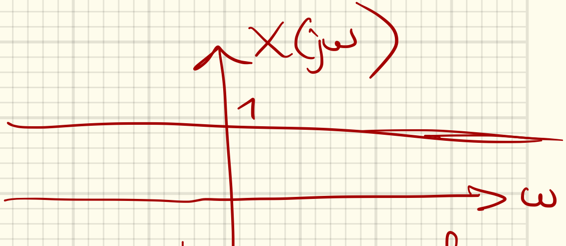
$$X(j\omega)=\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{j0}\, dt = 1$$

$$\boxed{\delta(t) \xleftrightarrow{F.T} 1}$$



narrowest possible signal in time

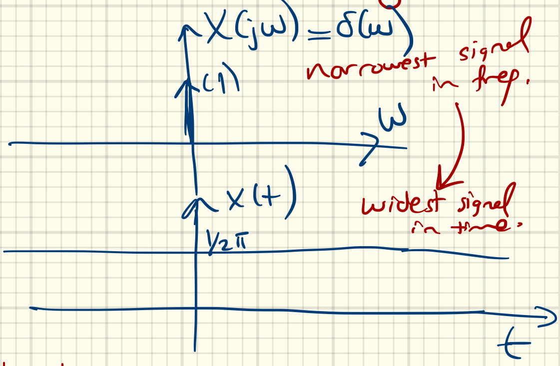
$\longleftrightarrow$



widest possible waveform (signal) in frequency.

⑤ $F.T^{-1}\{\delta(\omega)\} = ?$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi} = x(t)$$



narrowest signal in freq.

widest signal in time.

$$\boxed{1 \xleftrightarrow{F.T} 2\pi\,\delta(\omega)}$$

signal has 0 frequency

Frequency: impulse at $\omega=0$.

(6) $\delta(t-t_0) \longleftrightarrow\ ?\quad e^{-j\omega t_0}$

(7) $?\ \longleftrightarrow \delta(\omega-\omega_0)$

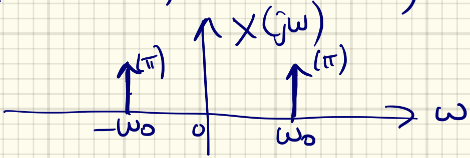
$\rightarrow \mathcal{F.T.}^{-1}\{\delta(\omega-\omega_0)\}$

$$= \frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega t}d\omega}_{e^{j\omega_0 t}}$$

$$= \frac{1}{2\pi}e^{j\omega_0 t}$$

(7) $e^{j\omega_0 t} \xleftrightarrow{F.T.} 2\pi\delta(\omega-\omega_0)$

(8) $x(t) = \underbrace{\cos(\omega_0 t)}_{\frac{1}{2}\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)} \longleftrightarrow \pi\left(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right)$



Recall old spectrum repres.

9 F.T. of a periodic signal w/ period T_0: $\rightarrow \omega_0$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longleftarrow \text{F.S. expansion}$$

Using $e^{jk\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega - k\omega_0)$

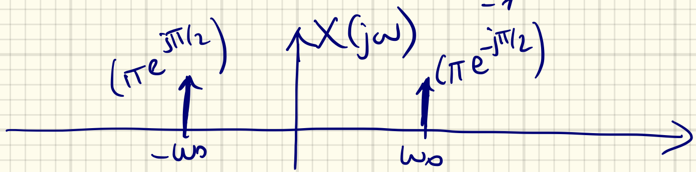
$$\hookrightarrow \boxed{X(j\omega) = \sum_{k=-\infty}^{\infty} a_k\, 2\pi\,\delta(\omega - k\omega_0)}$$

ex: $x(t) = \sin(\omega_0 t) = \frac{1}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right) = \sum_{k=-1,1} \underbrace{\left(\frac{1}{2jk}\right)}_{a_k} e^{jk\omega_0 t}$

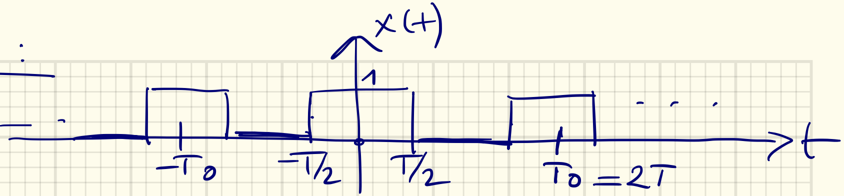
$$\Rightarrow \text{F.T.: } X(j\omega) = \sum_{k=-1,1} \frac{2\pi}{2jk}\,\delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{2j}\,\delta(\omega - \omega_0) - \frac{2\pi}{2j}\,\delta(\omega + \omega_0)$$

$$= \pi e^{-j\pi/2}\,\delta(\omega - \omega_0) + \underbrace{e^{j\pi}}_{-1}\,\pi e^{-j\pi/2}\,\delta(\omega + \omega_0)$$



Ex: Periodic square wave:



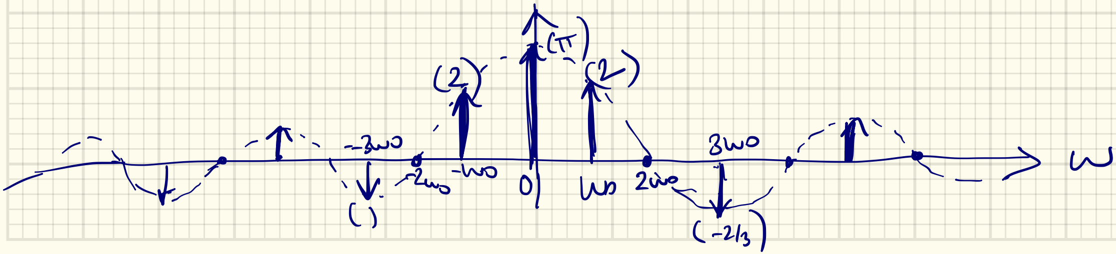
Fund. freq. $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{2T} = \frac{\pi}{T}$

Derive F.S. coeff. (recall or exercise)

$$a_k = \begin{cases} \frac{\sin(\pi k/2)}{\pi k}, & k \neq 0, \ k \text{ odd} \\ 1/2, & k = 0 \\ 0, & k \text{ even} \end{cases}$$

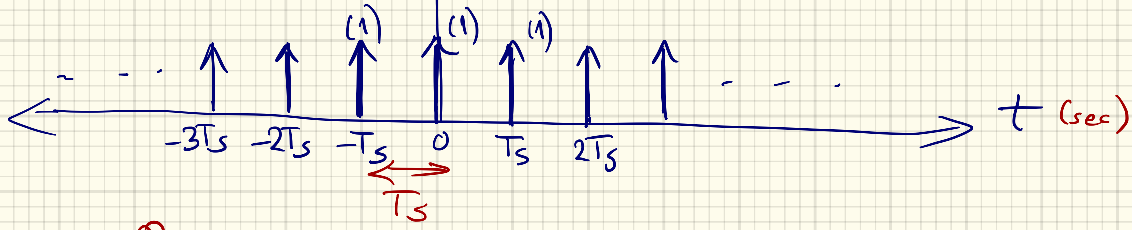
$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \, \delta(\omega - k\omega_0)$$

$$= \frac{1}{\cancel{2}} \cancel{2}\pi \, \delta(\omega) + \sum_{\substack{k=-\infty \\ (k \neq 0, \, k \text{ odd})}}^{\infty} 2 \cancel{\pi} \, \frac{\sin(\pi k/2)}{\cancel{\pi} k} \, \delta(\omega - k\omega_0)$$



(10) (Periodic) Impulse train Signal

F.T. $\{x(t)\} = ?$



$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s)$$ is periodic w/ $\boxed{T_s = T_0}$ → $f_0 = \frac{1}{T_s}$

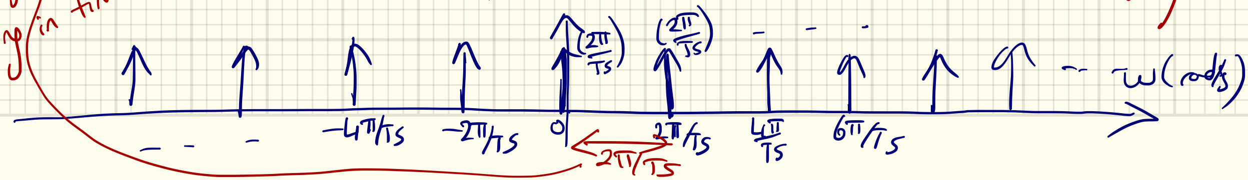
Find F.S. coeff a_k to find its F.T.

$$a_k = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \underbrace{x(t)}_{\delta(t)} e^{-jk\omega_0 t}\,dt = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)\underbrace{e^{j0}}_{1}\,dt = \frac{1}{T_s}$$

$$\boxed{a_k = \frac{1}{T_s}} \Rightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi\, a_k\, \delta(\omega - k\omega_0) \rightarrow \boxed{\omega_0 = \frac{2\pi}{T_s}}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \left(\frac{2\pi}{T_s}\right) \delta\left(\omega - k\frac{2\pi}{T_s}\right) \rightarrow$$ Another periodic impulse train in freq.

Spacing in freq $(2\pi/T_s)$ inversely related to spacing in time (T_s)



→ In your textbook, Table 11.2 F.T. pairs
Table 11.3 F.T. properties. } will be given in the exam.

F.T. properties :

(1) Linearity :
$$x(t) \longrightarrow X(j\omega)$$
$$y(t) \longleftrightarrow Y(j\omega)$$
$$ax(t)+by(t) \longleftrightarrow aX(j\omega)+bY(j\omega)$$

(2) Time-Shifting :
$$x(t) \longleftrightarrow X(j\omega)$$
$$x(t-t_0) \longleftrightarrow e^{-j\omega t_0}X(j\omega)$$

Show
$$\int_{-\infty}^{\infty} x(\underbrace{t-t_0}_{\tau=t-t_0})\, e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau+t_0)}\,d\tau = \underbrace{\left(\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\,d\tau\right)}_{X(j\omega)}\cdot e^{-j\omega t_0}$$

Do a change of variables $\tau = t-t_0 \Rightarrow t = \tau+t_0$

$$= X(j\omega)\,e^{-j\omega t_0}$$

(3) Frequency shifting :
$$x(t) \longleftrightarrow X(j\omega)$$
$$e^{j\omega_0 t}x(t) \longleftrightarrow X(j(\omega-\omega_0))$$

pf:
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\underbrace{(\omega-\omega_0)}_{\sigma=\omega-\omega_0})\,e^{j\omega t}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\sigma)\,e^{j\sigma t}\,e^{+j\omega_0 t}\,d\sigma$$
$$= e^{j\omega_0 t}\,x(t)$$

(4) Time Scaling

$x(t) \longleftrightarrow X(j\omega)$

$x(at) \longleftrightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right) \quad = ?$

pf: $\displaystyle\int_{-\infty}^{\infty} \underbrace{x(at)}_{\substack{\tau = at \\ d\tau = |a|\, dt}} e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau/a} \frac{d\tau}{|a|}$

$\displaystyle = \frac{1}{|a|} \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau}$

$$\boxed{F.T\{x(at)\} \Rightarrow \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)}$$

Again narrow in time $x(2t) \longleftrightarrow \underbrace{\frac{1}{2} X\left(j\frac{\omega}{2}\right)}_{\text{wider in freq.}}$

Recall the inverse relation btw time & frequency domains.

(5) Time-Flip : $x(-t) \longleftrightarrow X(-j\omega)$

(6) $\underset{\text{Freq}}{X(j0)} = \displaystyle\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \underbrace{\int_{-\infty}^{\infty} x(t)\, dt}_{\text{area under } x(t)}$

$\omega = 0$

$\underset{\text{time}}{x(0)} = \displaystyle\frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, d\omega$: area under F.T. signal.

7 Convolution Property:

$x(t) \longleftrightarrow X(j\omega)$

$y(t) \longleftrightarrow Y(j\omega)$

$z(t) = x(t) * y(t) \longleftrightarrow Z(j\omega) = ?$

pf: $$Z(j\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty}_{(t)} \Big(\underbrace{\int_{-\infty}^{\infty}_{(\tau)} x(\tau)\, y(t-\tau)\, d\tau}_{x(t) * y(t)} \Big) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty}_{(\tau)} x(\tau) \underbrace{\int_{-\infty}^{\infty}_{(t)} y(t-\tau)\, e^{-j\omega t}\, dt}_{e^{-j\omega\tau}\, \underline{Y(j\omega)}}\, d\tau$$

$\rightarrow$ from time-shifting property

$$= Y(j\omega) \underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau}_{X(j\omega)}$$

$$\boxed{Z(j\omega) = X(j\omega) \cdot Y(j\omega)}$$

* Instead of $\underline{x(t) * y(t)} \longleftrightarrow F.T^{-1}\{X(j\omega)\, Y(j\omega)\}$