Signals & Systems (BLG354E)

Week 13

08.05.2017

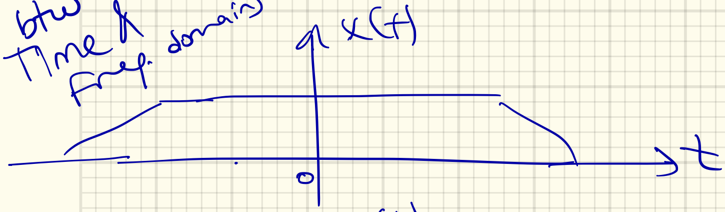
G.Ü.

Ch 11 Fourier Transform (we'll finish today & start w/ Chap 12)

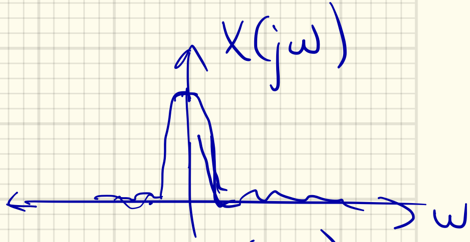
Recall: $x(t) \longleftrightarrow X(j\omega)$ 1-1 mapping

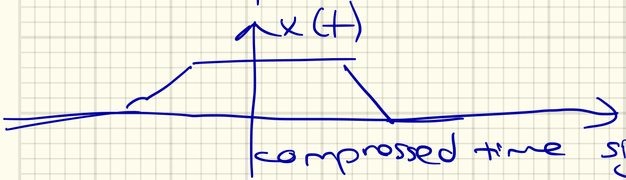
time-domain $\qquad$ frequency domain

inverse relation btw time & freq. domains



$\longleftrightarrow$





compressed time signal

$\longleftrightarrow$

wider frequency spectrum

Properties of F.T.

* Convolution property: (derived this last time)

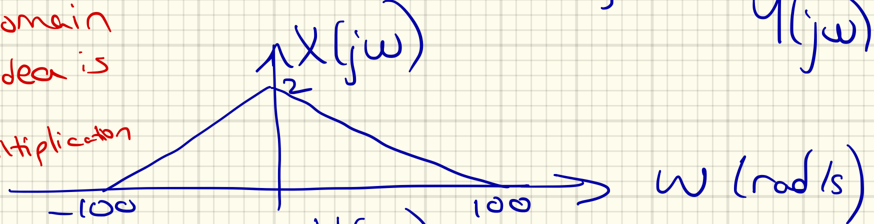
$$x(t) * y(t) \longleftrightarrow X(j\omega) \cdot Y(j\omega)$$

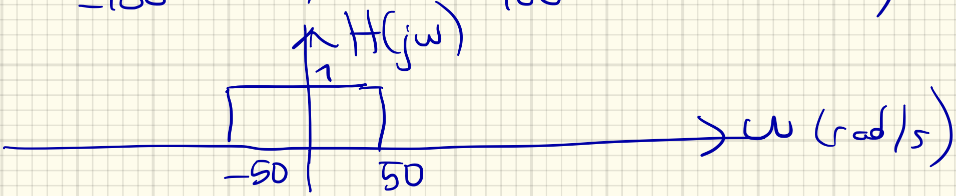
Recall in Chap 10:

LTI Systems: $x(t) \longrightarrow \boxed{\begin{matrix} h(t) \\ H(j\omega) \end{matrix}} \longrightarrow y(t) = x(t) * h(t)$

$$Y(j\omega) = X(j\omega)\,H(j\omega)$$

Frequency Domain Filtering idea is based on Convolution ⇔ multiplication property





$\longrightarrow F.T.^{-1}\{Y(j\omega)\} = y(t)$ ✓

8) Multiplication Property:

$$x(t) \cdot y(t) \longleftrightarrow \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

Exercise: derive this starting from Fourier domain.

9. Differentiation Property:

$$x(t) \longleftrightarrow X(j\omega)$$

$$\frac{d}{dt}x(t) \longleftrightarrow (j\omega)X(j\omega)$$

show:
$$\frac{d}{dt}\left(x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega \right)$$

$$\underbrace{\frac{d}{dt}x(t)}_{\triangleq\, y(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\left(\frac{d}{dt}e^{j\omega t}\right)d\omega$$

$$\int_{-\infty}^{\infty} \underbrace{\boxed{X(j\omega)\, j\omega}}_{Y(j\omega)}\, e^{j\omega t}\, d\omega$$

Generalize:

$$\Rightarrow \frac{d^n x(t)}{dt^n} \longleftrightarrow (j\omega)^n X(j\omega)$$

LTI system written as

Linear Differential Equations:

$$x(t) \longrightarrow \overset{H(j\omega)}{\boxed{LTI}} \longrightarrow y(t)$$

I/O relation

$$\frac{d^2 y}{dt^2} + 2\frac{dy(t)}{dt} = \frac{dx(t)}{dt} + 3x(t)$$

Take F.T. of both sides

$$\left[(j\omega)^2 + 2j\omega\right] Y(j\omega) = (j\omega + 3) X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega + 3}{(j\omega)^2 + 2j\omega} = \frac{N(j\omega)}{D(j\omega)}$$

In general LTI:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

System response $\Rightarrow$

$$H(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k \Big/ \left(\sum_{k=0}^{N} a_k (j\omega)^k \right)$$

Ex: $h(t) = 2e^{-2t}u(t) - e^{-t}u(t)$

$H(j\omega) = ?$ Recall $e^{-at}u(t) \longleftrightarrow \frac{1}{a+j\omega}$

$$H(j\omega) = 2\frac{1}{2+j\omega} - \frac{1}{1+j\omega}$$

$$H(j\omega) = \frac{j\omega}{(2+j\omega)(1+j\omega)} = \frac{j\omega}{(j\omega)^2+3j\omega+2}$$

s^2+3s+2

Use Partial Fraction Expansion (PFE)

Find the roots

$s_{1,2} = \begin{matrix}-1\\-2\end{matrix}$

$$\frac{j\omega}{(j\omega+2)(j\omega+1)} = \frac{A}{j\omega+2} + \frac{B}{j\omega+1}$$

$$\left.\begin{aligned} A &= 2 \\ B &= -1 \end{aligned}\right\} \text{find } A \,\&\, B.$$

Given $X(j\omega) = \dfrac{N(j\omega)}{D(j\omega)}$ rational form.

If $\deg(N(j\omega)) < \deg(D(j\omega)) \Rightarrow$ do PFE

If " $\geq$ " $\Rightarrow$ divide first to get $\deg(N) < \deg(D)$ then do PFE.

Ex: Given $Y(j\omega) = \dfrac{1-\omega^2+j\omega}{2-\omega^2+j3\omega}$ deg = 2, deg = 2

$$\begin{array}{r|l} -\omega^2 + j\omega + 1 & -\omega^2+j3\omega+2 \\ \hline -(-\omega^2+j3\omega+2) & 1 \\ \hline -2j\omega - 1 & \end{array}$$

$$Y(j\omega) = 1 + \frac{-j\omega 2 - 1}{2+j3\omega+(j\omega)^2} \Rightarrow \frac{A}{j\omega+2} + \frac{B}{j\omega+1}$$

exercise: show $y(t) = \delta(t) - 3e^{-2t}u(t) + e^{-t}u(t)$

10 Modulation Property:

$$m(t) \longrightarrow \otimes \longrightarrow y(t) = m(t)\cos\omega_0 t$$

$\uparrow$ $\cos\omega_0 t$ (into the multiplier); $\uparrow$ modulated signal (under $y(t)$)

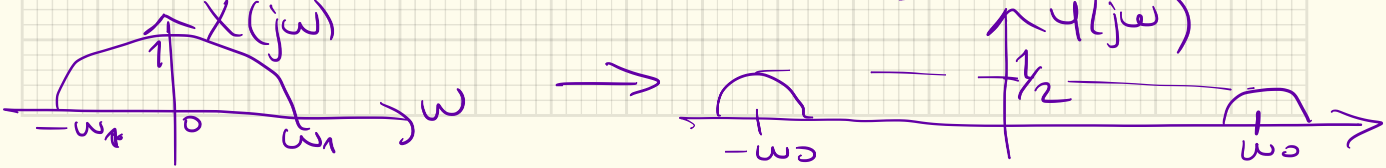
$$m(t) \longleftrightarrow M(j\omega)$$

$$Y(j\omega) = ?$$

$$Y(j\omega) = \frac{1}{2\pi}\left(M(j\omega) * \underbrace{FT\{\cos\omega_0 t\}}\right)$$

$$= \frac{1}{2\pi}\left(M(j\omega) * \left(\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)\right)\right)$$

$$Y(j\omega) = \frac{1}{2}M(j(\omega-\omega_0)) + \frac{1}{2}M(j(\omega+\omega_0))$$

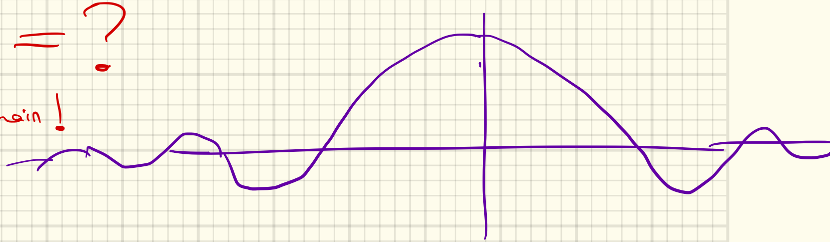


Ex: Given $x(t) = \frac{2\sin(20\pi t)}{\pi t}$, $y(t) = \frac{5\sin(10\pi t)}{\pi t}$

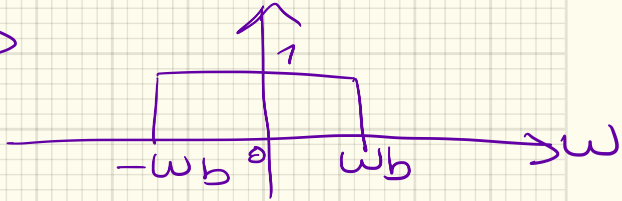
$z(t) = x(t) * y(t) = ?$

Not tractable to perform this convolution in time-domain!

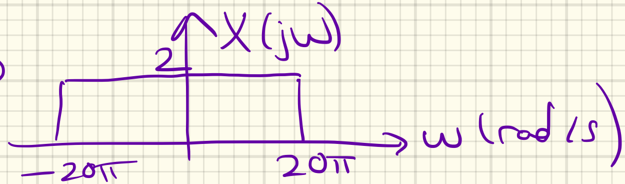
→ Use Fourier domain.



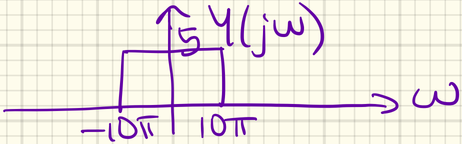
Recall $\frac{\sin(\omega_b t)}{\pi t} \Longleftrightarrow$

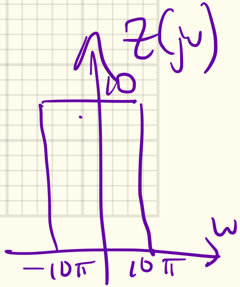


$\frac{2\sin(20\pi t)}{\pi t} \longleftrightarrow$

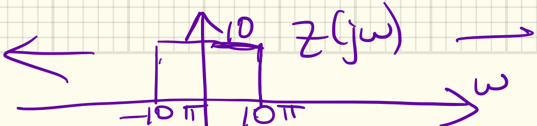


$5\frac{\sin(10\pi t)}{\pi t} \longleftrightarrow$





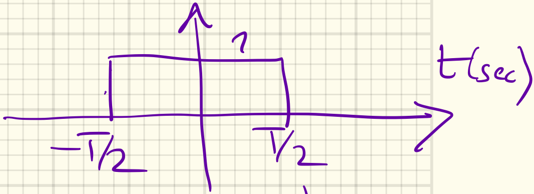
$z(t) = \frac{10\sin(10\pi t)}{\pi t} \longleftarrow$



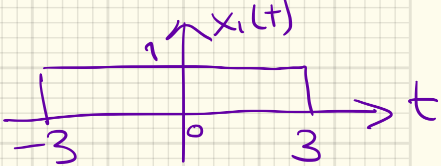
Ex: $X(j\omega) = 2\,\dfrac{\sin(3(\omega-2\pi))}{(\omega-2\pi)} \overset{F.T.^{-1}\ S.3}{\longleftrightarrow} x(t) = ?$

Use (Table 11.2 in your textbook)

$$\frac{\sin(\omega T/2)}{\omega/2} \longleftrightarrow$$



$$\frac{\sin(3\omega)}{\omega/2} \longleftrightarrow$$



Now Apply frequency shifting by 2π

$$\boxed{e^{j\omega_0 t}x_1(t) \longleftrightarrow X_1(j(\omega-\omega_0))}$$ from the table.

$$\Rightarrow x(t) = e^{j2\pi t}x_1(t) \longleftrightarrow \frac{\sin(3(\omega-2\pi))}{\frac{(\omega-2\pi)}{2}}$$

12.3 SAMPLING & Reconstruction

Sample a C.T. signal $x(t) \rightarrow \boxed{C/D} \rightarrow x[n]$

Continuous-to-Discrete Converter

$\underline{x[n]}$

$x(nT_s)$

T_s → sampling period.

$x(t) \rightarrow \otimes \rightarrow x_s(t) \rightarrow x[n]$: DT signal

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$: Impulse train

C/D Converter

D/C Converter

(Discrete-to-Continuous)

$x_s(t) \rightarrow \boxed{\text{Reconstruction Filter}} \rightarrow x_r(t)$

$H_r(j\omega)$

* Sample from a CT signal to obtain a DT signal :

C/D Converter

go back to continuous-time!

D/C Converter

$\underline{CT}$ $x(t) \rightarrow \otimes \xrightarrow{x_s(t)} x[n] \rightarrow \cdots \xrightarrow{x_s(t)} \boxed{\begin{matrix} h_r(t) \\ H_r(j\omega) \end{matrix}} \rightarrow x_r(t)$

$p(t)$

Analog

SAMPLING

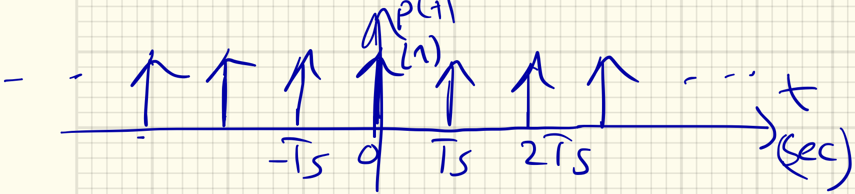
(T_s : sampling period)

Reconstruction

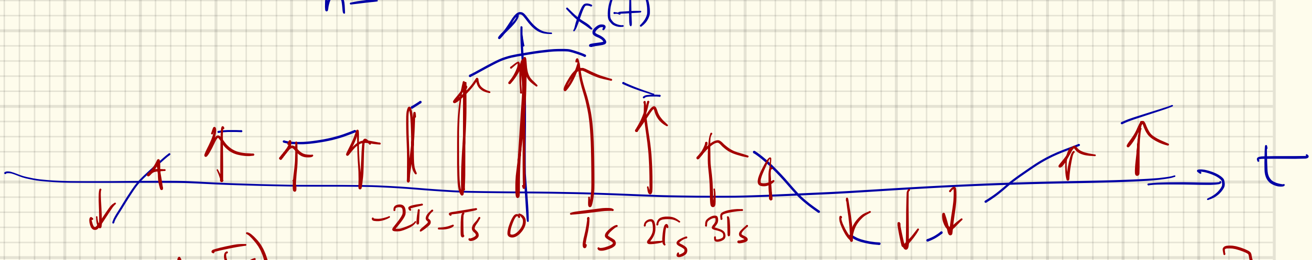
Fourier-Domain Analysis of Sampling:

$$x_s(t) = x(t)\,p(t) = x(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT_s)$$

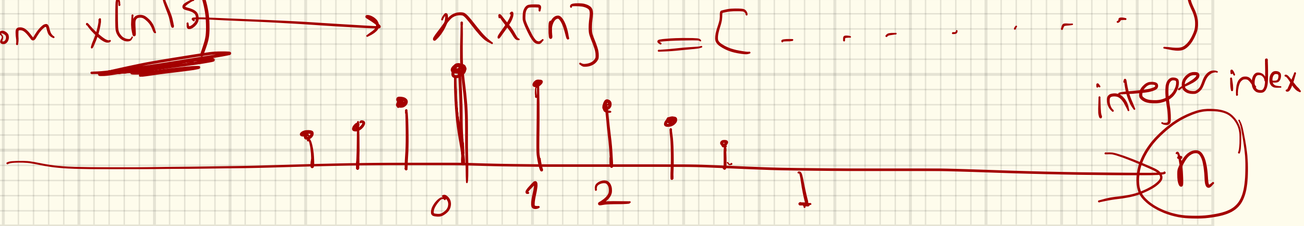
sampling period



$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\,\delta(t-nT_s)$$



from $x(nT_s)$ $\longrightarrow$ $x[n] = [\,\dots\,]$



Use multiplication prop of F.T.

$$x_s(t) = x(t)\,p(t) \longleftrightarrow X_s(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$X_s(j\omega) = \frac{1}{\cancel{2\pi}} X(j\omega) * \left(\sum_{k=-\infty}^{\infty} \left(\frac{\cancel{2\pi}}{T_s}\right) \delta\left(\omega - k\left(\frac{2\pi}{T_s}\right)\right)\right)$$

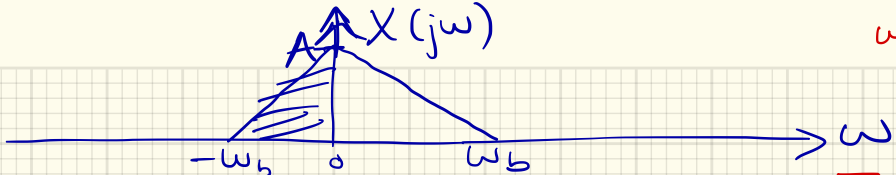
ω_s sampling frequency

$$\boxed{\omega_s = \frac{2\pi}{T_s}}$$

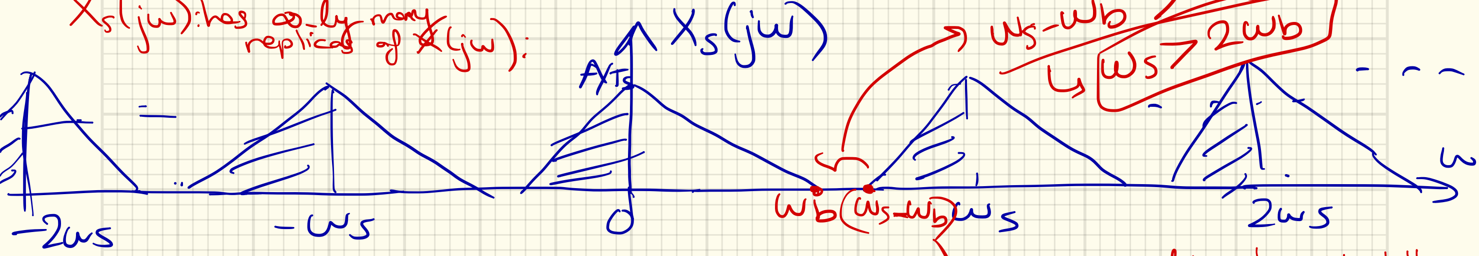
$$\boxed{X_s(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T_s} X(j(\omega - k\omega_s))}$$

Repeated replicas of the original spectrum, whose amplitude is scaled by $\frac{1}{T_s}$ & shifted by $\mp k\,\omega_s$, $\forall k$ integers

ω_s : sampling freq. should be "carefully" chosen.



$X_s(j\omega)$: has ∞-ly many replicas of $X(j\omega)$:



$\rightarrow \omega_s - \omega_b > \omega_b$

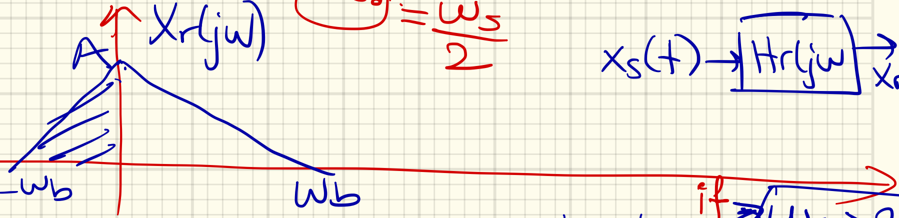
$\hookrightarrow \boxed{\omega_s > 2\omega_b}$

$H_r(j\omega)$: reconstruction filter to get back the original CT signal

$\omega_c = \dfrac{\omega_s}{2}$

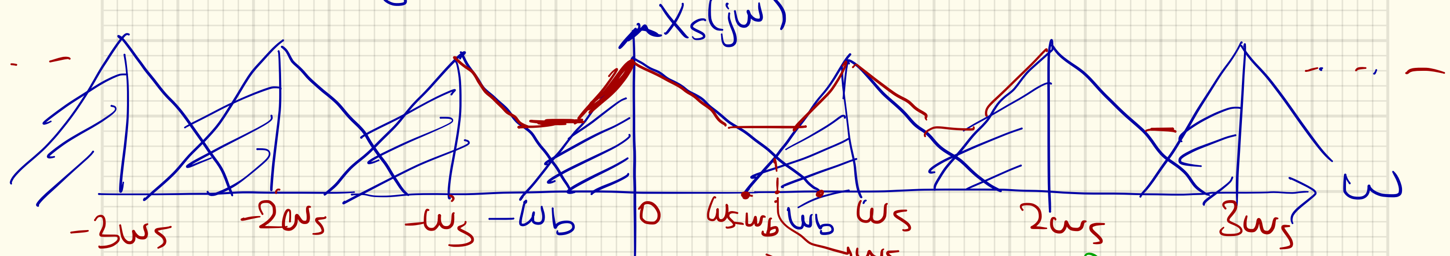
$$X_r(j\omega) = X_s(j\omega)\,H_r(j\omega)$$

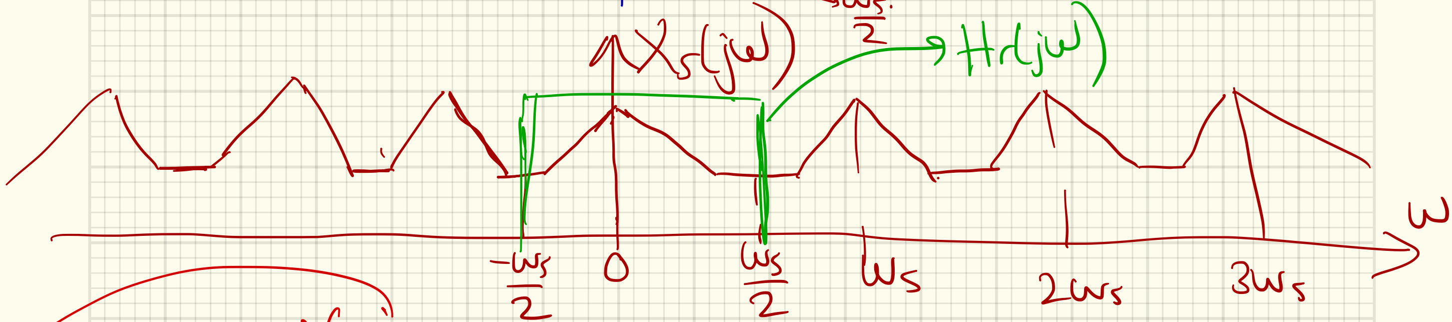
$x_s(t) \rightarrow \boxed{H_r(j\omega)} \rightarrow x_r(t)$

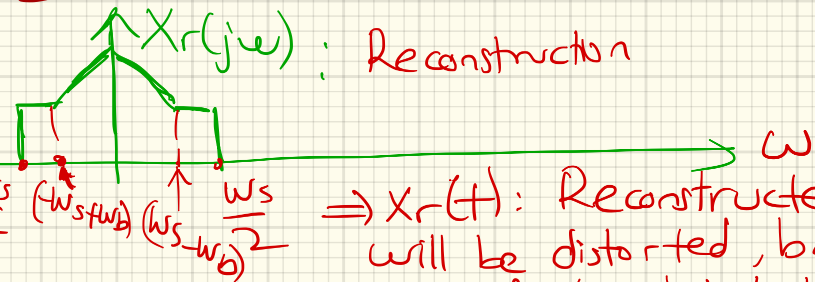


$\Rightarrow x_r(t) = x(t)$: Perfect Reconstruction if $\boxed{\omega_s > 2\omega_b}$

→ What if $\omega_s < 2\omega_b$?







: Reconstruction

ALIASING:

Negative frequencies of repeated copies overlapped w/ positive frequencies of $X(j\omega)$.

$\Rightarrow x_r(t)$: Reconstructed signal will be distorted, b/c freq content of $X(j\omega)$ is lost for some frequencies: $(\omega_s-\omega_b, \frac{\omega_s}{2})$

SHANNON _ NYQUIST SAMPLING THEOREM:

Two conditions for avoiding aliasing distortion & perfect reconstruction when sampling a signal $x(t)$:

① The signal is band limited; $X(j\omega)=0$ for $|\omega| \geq \omega_b$.

② The sampling frequency; $\omega_s \geq 2\omega_b$

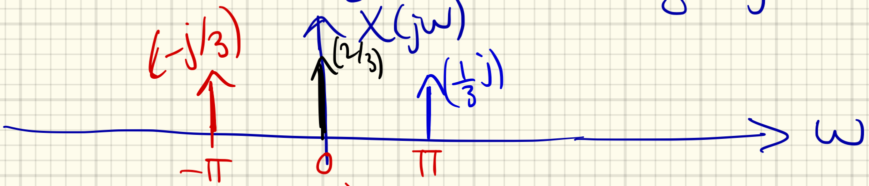
We should follow the above theorem in choosing a sampling frequency (or $\equiv$ sampling period) for sampling a CT signal.

Ex: $x(t) = \frac{1}{3\pi} + \frac{1}{3\pi}\cos\left(\pi t + \frac{\pi}{2}\right)$: Sample this signal w/ (i) $\omega_s = 3\pi$ rad/s

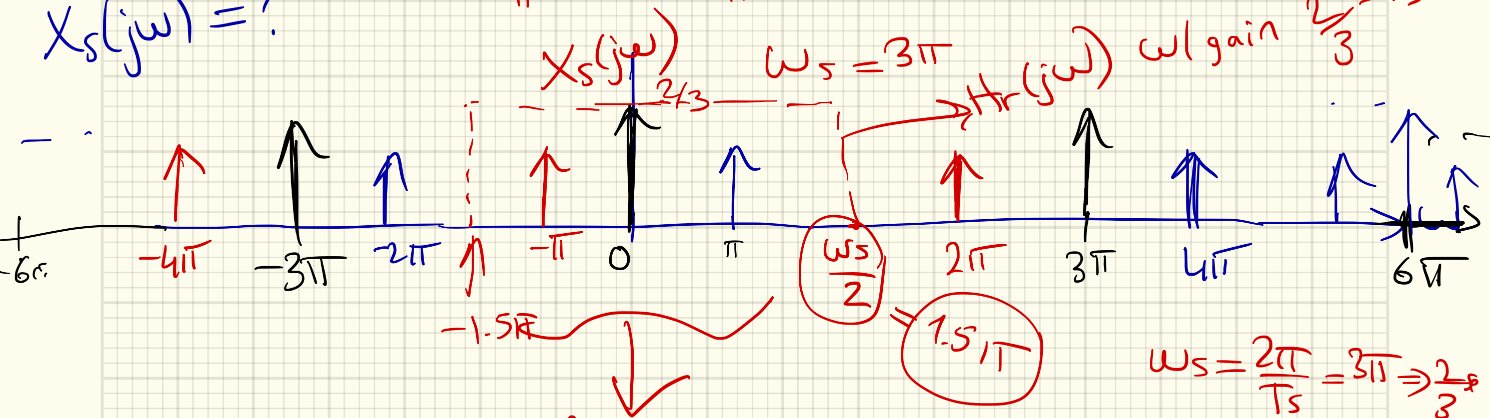
(ii) $\omega_s = 1.5\pi$ "

derive this

$$X(j\omega) = \frac{2}{3}\delta(\omega) + \frac{1}{3}(j)\,\delta(\omega-\pi) + \frac{1}{3}(-j)\,\delta(\omega+\pi)$$



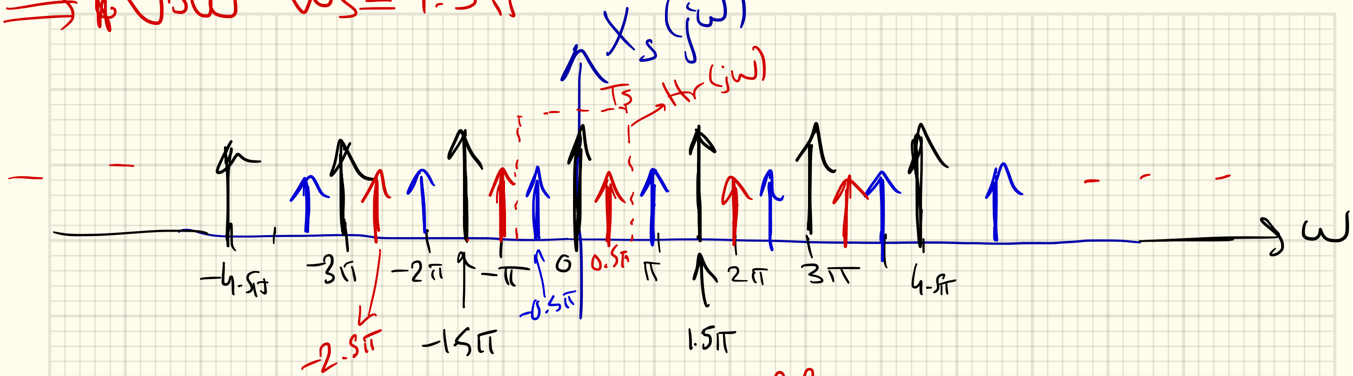
$X_s(j\omega) = ?$



$\omega_s = \frac{2\pi}{T_s} = 3\pi \Rightarrow \frac{2}{3}$

$x_r(t) = x(t)$ perfectly reconstructed.

⇒ Now Choose $\omega_s = 1.5\pi$ rad/s



Now place $H_r(j\omega)$ w/ cut-off at $\frac{\omega_s}{2} = 0.75\pi$ rad/s

We ended up w/ another signal due to aliasing: now a cosine w/ 0.5π rad/s : a different signal !!!

Properties of F.T. (One last property)

* If $x(t)$ is real $\Longrightarrow X(-j\omega) = X^*(j\omega)$

Conjugate Symmetry

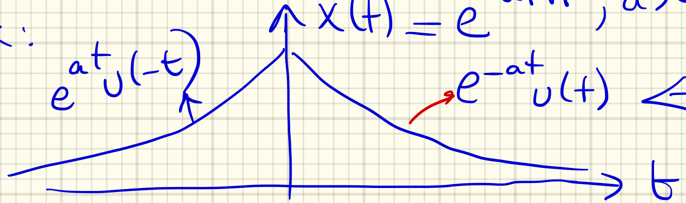
(Recall, we derived these in Freq. Response chapters)

Also $|X(-j\omega)| = |X(j\omega)|$: even symmetry

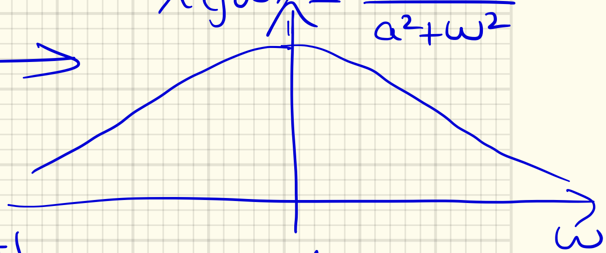
$\measuredangle X(-j\omega) = -\measuredangle X(j\omega)$: odd symmetry

* If $x(t)$ is real & even symmetric $\Longleftrightarrow X(j\omega)$ is real & even symmetric

ex: $x(t) = e^{-a|t|}$; $a>0$ $\qquad X(j\omega) = \dfrac{2a}{a^2+\omega^2}$



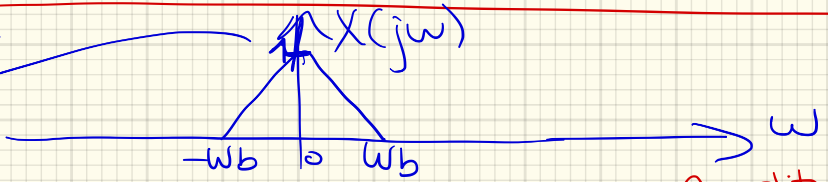
$\longleftrightarrow$



exercise derive this pair: $e^{-a|t|} \longleftrightarrow X(j\omega)$ ↑.

12.2 AMPLITUDE MODULATION:

Signal to be x-mitted:

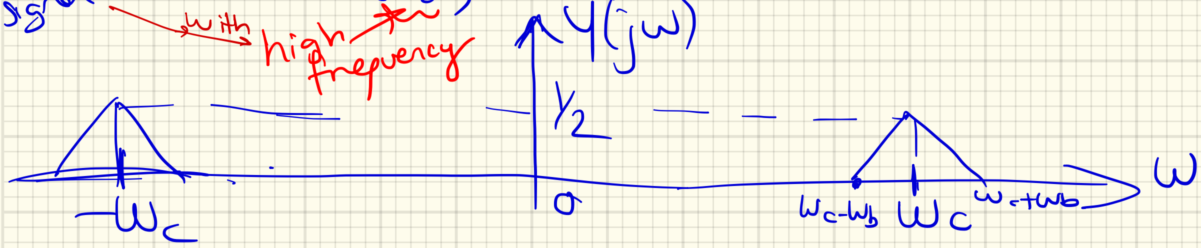


Amplitude Modulation:

$$x(t) \xrightarrow{\text{Modulation}} \otimes \rightarrow y(t) = x(t).p(t)$$

Recall: $Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_c)) + \frac{1}{2} X(j(\omega + \omega_c))$

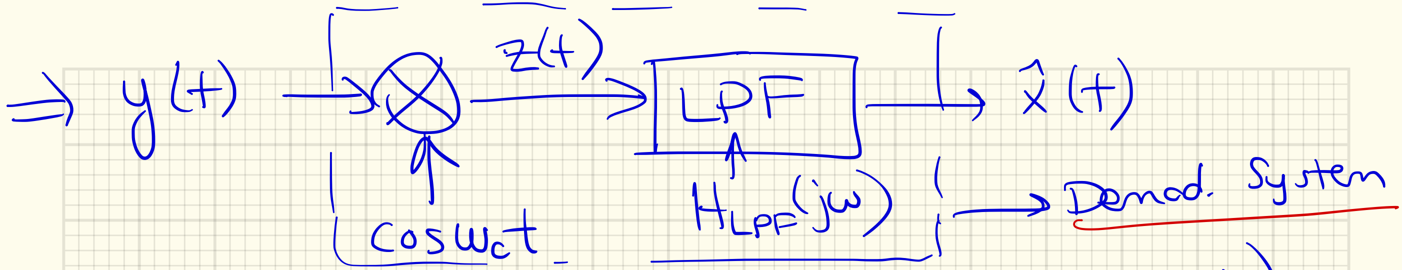
carrier signal: $p(t) = \cos(\omega_c t)$ with high frequency



Demodulation:

Received Signal: $y(t) \longrightarrow$ Demod ? $\longrightarrow x(t)$

Goal: recover xmitted signal.

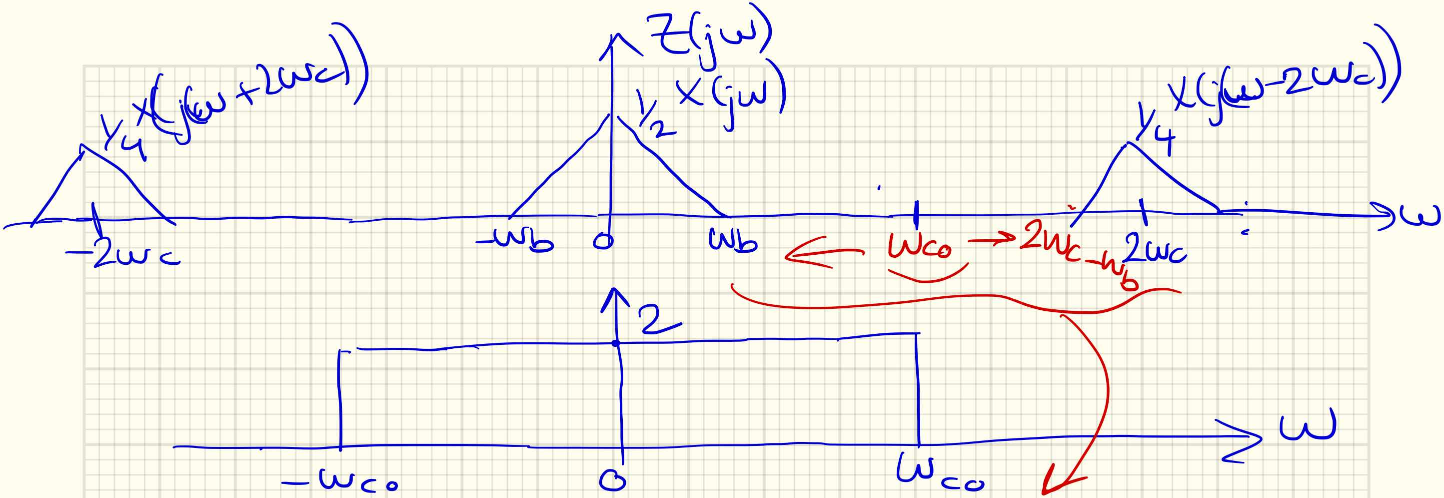


$$Z(jw) = \frac{1}{2}\left(Y(j(w-w_c)) + Y(j(w+w_c))\right)$$

$$= \frac{1}{4}X(j(w-2w_c)) + \frac{1}{4}X(j(w)) + \frac{1}{4}X(jw) + \frac{1}{4}X(j(w+2w_c))$$

$$Z(jw) = \frac{1}{2}X(jw) + \frac{1}{4}X(j(w-2w_c)) + \frac{1}{4}X(j(w+2w_c))$$

$\Rightarrow$



$$\boxed{\omega_b < \omega_{co} < 2\omega_c - \omega_b}$$

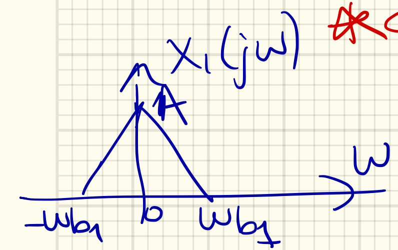
cut-off frequency conditions

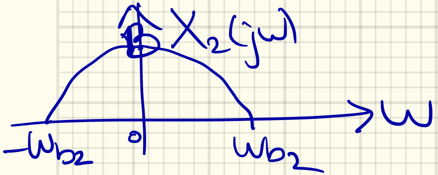
$$H_{LPF}(j\omega) = \begin{cases} 2, & \omega_b < |\omega| < 2\omega_c - \omega_b \\ 0, & o/w \end{cases}$$

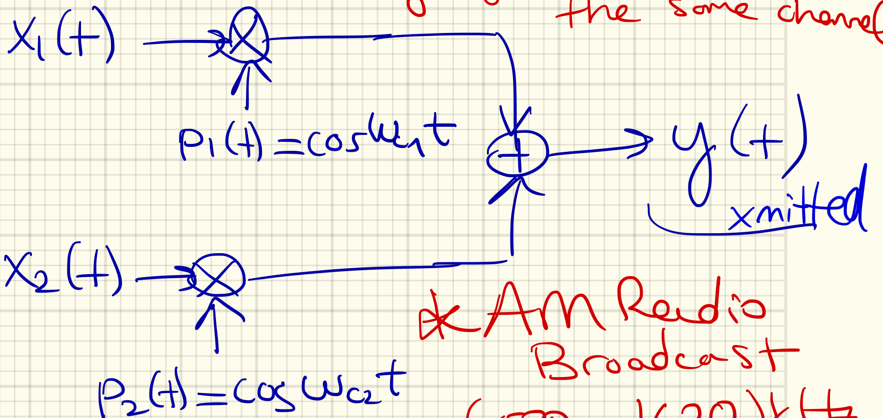
$$\hat{X}(j\omega) = H_{LPF}(j\omega)\, Z(j\omega) = X(j\omega)$$

FREQUENCY DIVISION MULTIPLEXING:

* Can xmit a multiplicity of signals over the same channel

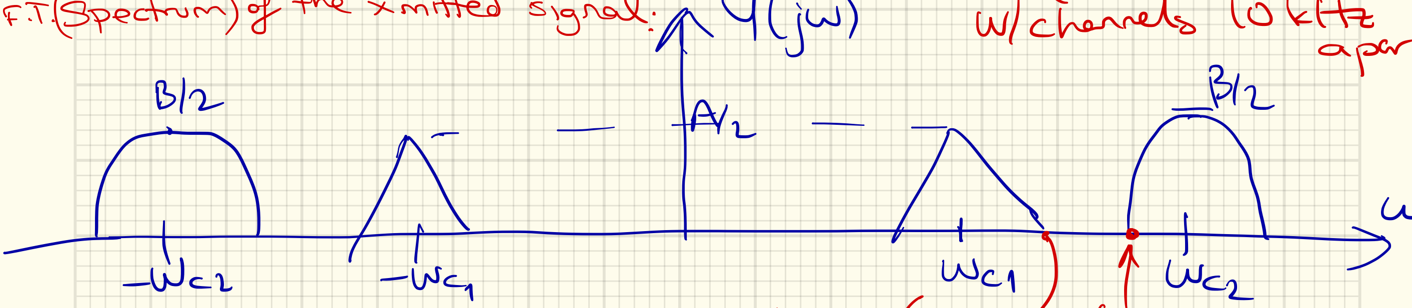






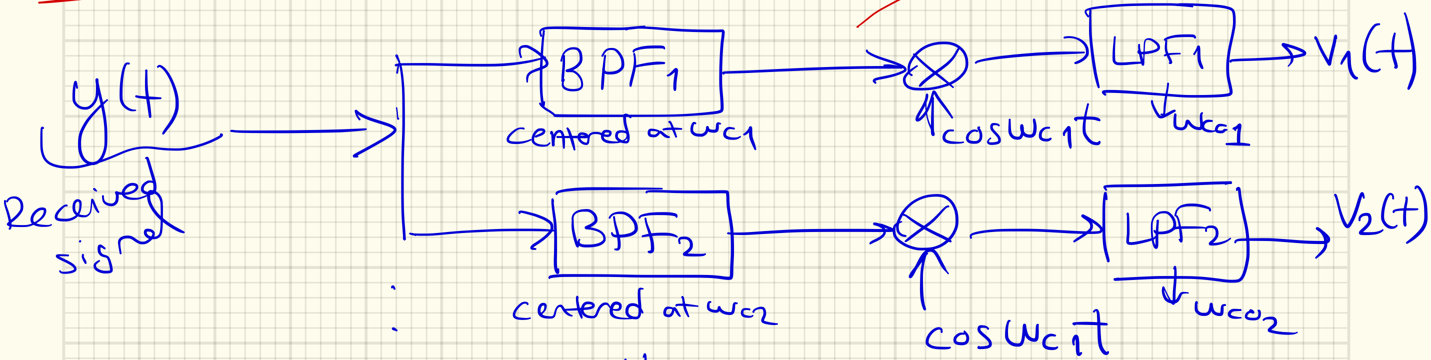
* AM Radio Broadcast (530 – 1620) kHz w/ channels 10 kHz apart

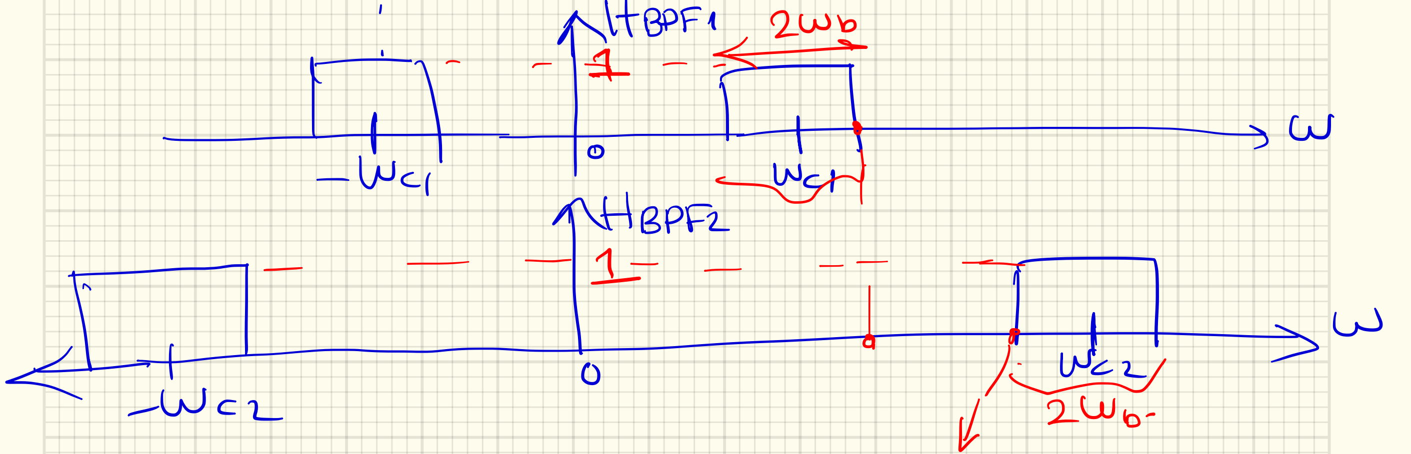
F.T. (Spectrum) of the xmitted signal:



Constraint $(\omega_{c_1} + \omega_{b_1}) < (\omega_{c_2} - \omega_{b_2})$

Receiver-Side : DEMUX (Demultiplexer), Demodulator:





$\omega_{c2} - \omega_b > \omega_{c1} + \omega_b$

$\therefore$ Voice & music signals must be band-limited before modulation w/ $\boxed{5\,kHz} = \omega_b$

CTFT: $x(t) \longleftrightarrow X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$

DTFT: $x[n] \longleftrightarrow X(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\hat{\omega} n}$

sampled signal

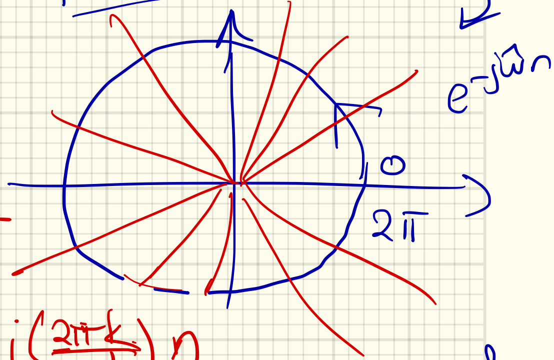
Recall: periodic w/ 2π

Recall chap6: $\boxed{\hat{\omega} = \omega T_s}$

Normalized radian frequency: rad $=$ $\frac{\text{rad}}{s} \cdot s$

$\hat{\omega} \in [0, 2\pi]$

or $\hat{\omega} \in [-\pi, \pi]$.



DFT (Discrete Fourier Transform): discretize $\hat{\omega}$ into N values

$\hat{\omega} = \frac{2\pi k}{N}$, $k = 0, 1, \dots, N-1$

This is numerically implementable

$X[k] = \sum_{n=\cancel{-\infty}\ 0}^{\cancel{\infty}\ M} x[n]\, e^{-j\left(\frac{2\pi k}{N}\right) n}$

typically finite range (length) signal $x[n]$

This is the basis of famous FFT algorithm

Freq ↓ / Time →	n (Discrete)	t (continuous)
k (Discrete)	DFT	Fourier Series.
w (continuous)	DTFT	CTFT

→ Check yourself against the Learning Objectives of our course.

THE END.