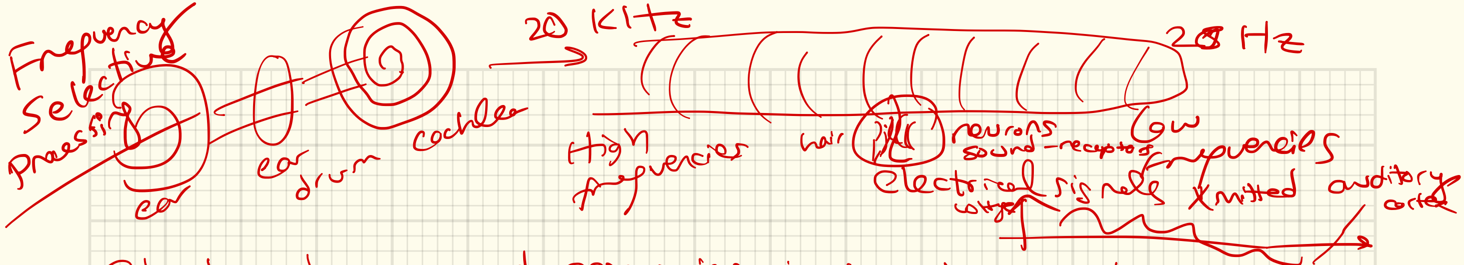
Signals & Systems

24.04.2017

BLG 354E

Week 11

Gü.



Check how sound processing is done in your brain :

Frequency Response $H(e^{j\hat{\omega}})$:

Sinusoid $\begin{cases} A e^{j\phi} e^{j\hat{\omega}_0 n} \\ A\cos(\hat{\omega}_0 n + \phi) \end{cases}$ $\rightarrow$ $\boxed{h[n] \quad H(e^{j\hat{\omega}})}$ $\rightarrow$

$$A e^{j\phi} H(e^{j\hat{\omega}_0}) e^{j\hat{\omega}_0 n}$$

$$A \underbrace{|H(e^{j\hat{\omega}_0})|}_{\text{magnitude}} \cos(\hat{\omega}_0 n + \phi + \angle H(e^{j\hat{\omega}_0}))$$

Ex: Delay System :

$x[n] \rightarrow \boxed{\text{Delay } n_0} \rightarrow y[n] = x[n-n_0]$

$$\boxed{H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\hat{\omega} n}}$$

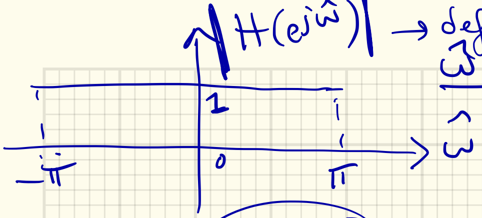
$h[n] = \delta[n-n_0]$

$$H(e^{j\hat{\omega}}) = \sum_{n=-\infty}^{\infty} \delta[n-n_0] e^{-j\hat{\omega} n} = e^{-j\hat{\omega} n_0}$$

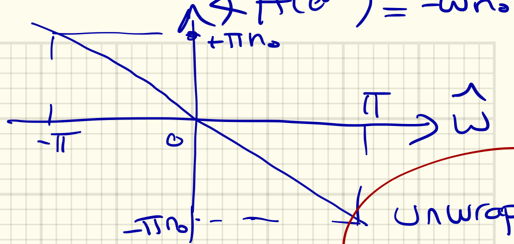
$$\underbrace{H(e^{j\hat{\omega}})}_{} = e^{-j\hat{\omega} n_0} \Rightarrow |H(e^{j\hat{\omega}})| = 1$$

$$= |H(e^{j\hat{\omega}})| \cdot e^{j\angle H(e^{j\hat{\omega}})}$$

$$\angle H(e^{j\hat{\omega}}) = -\hat{\omega} n_0$$



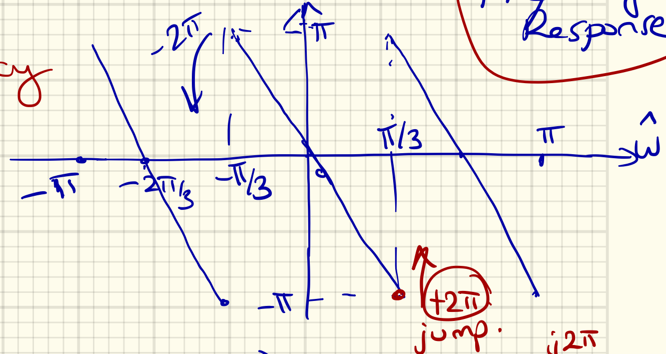
→ defined in $\hat{\omega} \in (-\pi, \pi)$ interval



Unwrapped Frequency Response.

let $n_0 = 3$ $\Rightarrow$ $\measuredangle H(e^{j\hat{\omega}}) = -3\hat{\omega}$

Wrapped Frequency (Phase) Response →

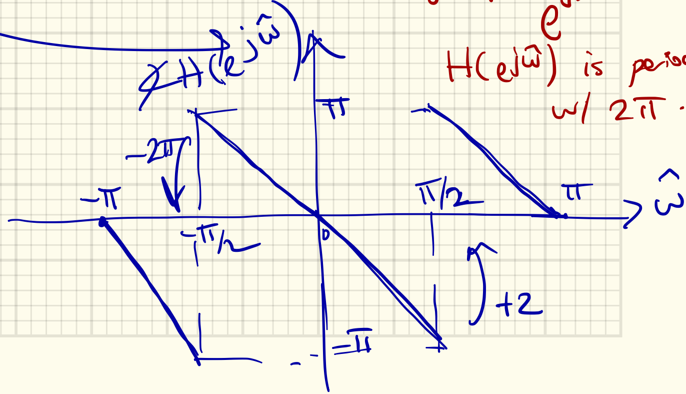


Let $n_0 = 2$

Delay system.

$e^{j2\pi}$

$H(e^{j\hat{\omega}})$ is periodic w/ 2π.



Ex: First Difference System (LTI)

I/O: $y[n] = x[n] - x[n-1]$

$h[n] = \delta[n] - \delta[n-1]$

$x[n] \longrightarrow \boxed{LTI} \longrightarrow y[n]$

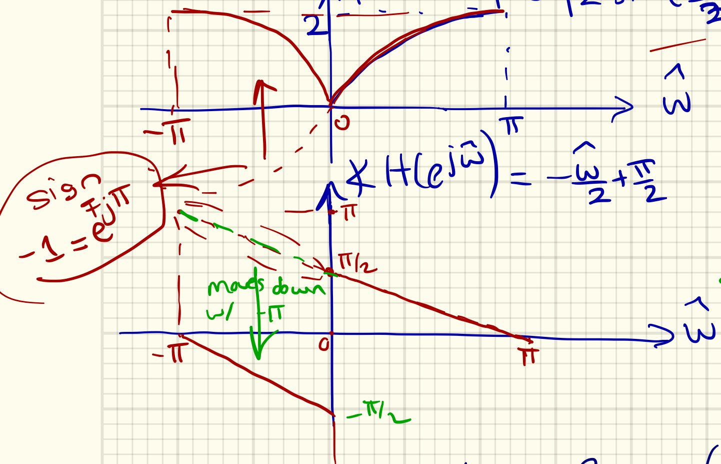
$H(e^{j\hat{\omega}})$

$$H(e^{j\hat{\omega}}) = 2\sin\left(\frac{\hat{\omega}}{2}\right) e^{-j\left(\frac{\hat{\omega}}{2} - \frac{\pi}{2}\right)}$$

$$|H(e^{j\hat{\omega}})| = \left|2\sin\left(\frac{\hat{\omega}}{2}\right)\right|$$

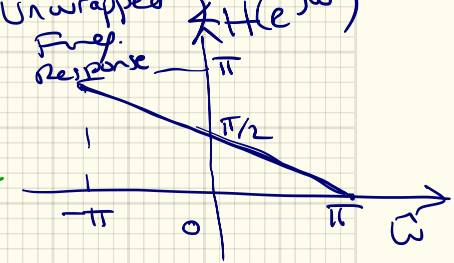
High-pass filter

$$\measuredangle H(e^{j\hat{\omega}}) = -\frac{\hat{\omega}}{2} + \frac{\pi}{2}$$



Wrapped Frequency Response

Unwrapped Freq. Response



Let $x[n] = 4 + 2\cos\left(0.3\pi n - \frac{\pi}{4}\right) + \cos(\pi n)$

$y(n) = ?$

exercise.

Frequency Response in CT : (Chapter 10)

$\omega : \frac{rad}{s} = 2\pi f = \omega$ (f: 1/s)

LTI

$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$

$\delta(t) \longrightarrow \boxed{} \longrightarrow h(t)$

Recall DT systems $H(e^{j\hat{\omega}})$

If $x(t)$: sinusoidal signal

$x(t) = A e^{j\phi} e^{j\omega t} \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t) = ? = A e^{j\phi} e^{j\omega t} \cdot H(j\omega)$

Given $h(t)$ & this specific $x(t)$ $\rightarrow y(t) = h(t) * x(t)$:

CT convolution:

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau) A e^{j\phi} e^{j\omega(t-\tau)} d\tau$$

$$y(t) = A e^{j\phi} \cdot e^{j\omega t} \boxed{\int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau} \triangleq H(j\omega)$$

$$H(e^{j\hat{\omega}}) = \sum_{k=-\infty}^{\infty} h[k] e^{-j\hat{\omega}k}$$

$$\boxed{H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} dt}$$

Freq. Response of LTI system given by $h(t)$.

$$H(j\omega) = |H(j\omega)|\, e^{j\angle H(j\omega)}$$

$h(t) \longleftrightarrow H(j\omega)$

Time-domain — Freq. domain. representation.

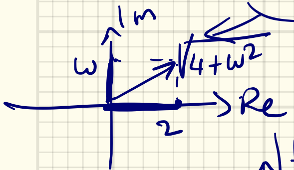
Ex: LTI system: $h(t) = 2e^{-2t}u(t)$: filter : what type?

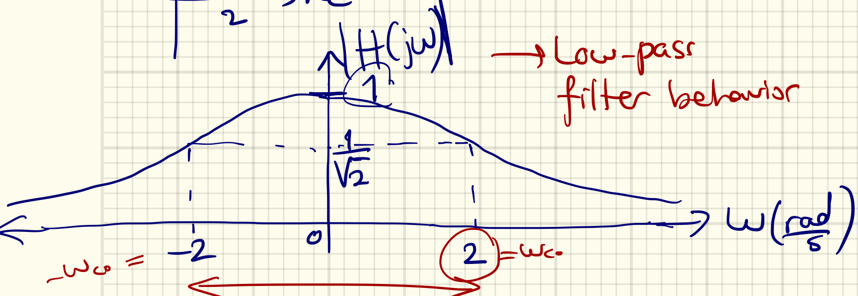
$$H(j\omega) = ? = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau = \int_{-\infty}^{\infty} 2e^{-2\tau}u(\tau)e^{-j\omega\tau}\,d\tau$$

$$H(j\omega) = \int_0^{\infty} \underbrace{2e^{-2\tau}\cdot 1\cdot e^{-j\omega\tau}}_{2\cdot e^{-(j\omega+2)\tau}}\,d\tau = \frac{2}{(-1)(2+j\omega)}\,e^{-(2+j\omega)\tau}\Big|_0^{\infty} = \frac{2}{2+j\omega}$$

$$H(j\omega) = \frac{2}{2+j\omega} \quad ; \quad |H(j\omega)| = \frac{2}{\sqrt{4+\omega^2}}$$

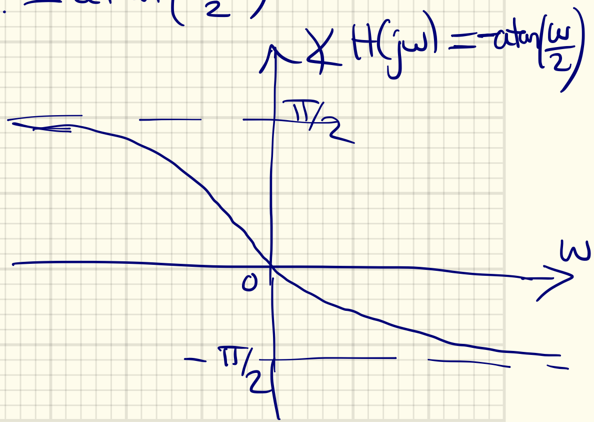
$$\measuredangle H(j\omega) = 0 - \text{atan}\left(\frac{\omega}{2}\right)$$



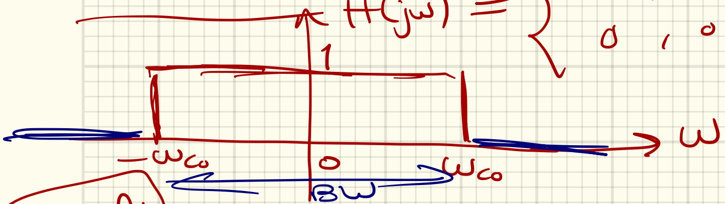


→ Low-pass filter behavior

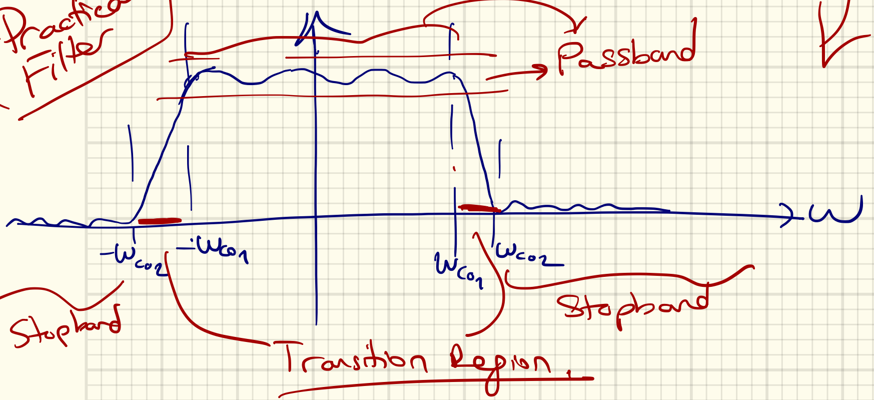
~ BW: Bandwidth of the filter.



Ideal LP Filter $H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_{co} \\ 0, & o/w \end{cases}$ → Not realizable.



Practical Filter:



Properties of $H(j\omega)$

Almost Same as $H(e^{j\hat{\omega}})$

Difference is:

$$H(j(\omega+2\pi)) \neq H(j\omega)$$

* $H(j\omega)$: frequency response is not periodic w/ 2π.

$$H(j(\omega+2\pi)) = \int_{-\infty}^{\infty} h(t)\, \underbrace{e^{-j(\omega+2\pi)t}}_{e^{-j\omega t}\, \underbrace{e^{-j2\pi t}}_{\neq 1\ \forall t}}\, dt \neq \int h(t) e^{-j\omega t} dt$$

; b/c t is a continuous variable

* $h(t)$ is real $\Rightarrow H(-j\omega) = H^*(j\omega)$

* $|H(j\omega)| = |H(-j\omega)|$; $\measuredangle H(j\omega) = -\measuredangle H(-j\omega)$.

* For a real sinusoid

$$x(t) = A\cos(\omega t + \phi) \longrightarrow \boxed{H(j\omega)} \longrightarrow y(t) = A|H(j\omega)|\cos(\omega t + \phi + \measuredangle H(j\omega))$$

* Superposition applies

$$\sum_k A_k \cos(\omega_k t + \phi_k) \longrightarrow \boxed{H(j\omega)} \longrightarrow \sum A_k |H(j\omega_k)| \cdot \cos(\omega_k + \phi_k + \measuredangle H(j\omega_k))$$

Ex: Given $h(t) = e^{-2t}u(t) \Rightarrow H(j\omega) = \frac{1}{2+j\omega}$ ✓

Let $x(t) = \underbrace{5}_{x_1(t)} + \underbrace{8\cos\left(2t + \frac{\pi}{3}\right)}_{x_2(t)} + \underbrace{3\,\delta(t-0.1)}_{x_3(t)}$

$y(t) = ?$

$$y_3(t) = x_3(t) * h(t) = 3e^{-2(t-0.1)}u(t-0.1)$$

$x_2(t) \to \omega_2 = 2\text{ rad/s}$ $\quad y_2(t) = 8 \cdot \frac{1}{2\sqrt{2}} \cdot \cos\left(2t + \frac{\pi}{3} - \frac{\pi}{4}\right)$

$$H(j\omega)\Big|_{\omega = 2\text{ rad/s}} = \frac{1}{2+j2} \Rightarrow \underbrace{|H(j\omega)|\Big|_{\omega=2}}_{= |H(j2)|} = \frac{1}{\sqrt{4+4}} = \frac{1}{2\sqrt{2}}, \quad \underbrace{\measuredangle H(j\omega)\Big|_{\omega=2}}_{= \measuredangle H(j2)} = -\frac{\pi}{4}$$

$x_1(t) = 5 \rightarrow \omega_1 = 0 \text{ rad/s} \rightarrow H(j0) = \frac{1}{2}$

$$\Rightarrow y_1(t) = x_1(t) \cdot H(j0) = \frac{5}{2}.$$

$\Rightarrow y(t) = y_1(t) + y_2(t) + y_3(t)$: Superposition rule. ✓

Exercise: $x(t) = 2\cos t + \cos(3t + 1.57)$

Given $H(j\omega) = \frac{2+j\omega}{1-j\omega}$: Use superposition to find the output $y(t)$.

Soln

$y_1(t) = 2|H(j1)|\cos(t + \measuredangle H(j1)) = 2(1.58)\cos(t - 0.32)$

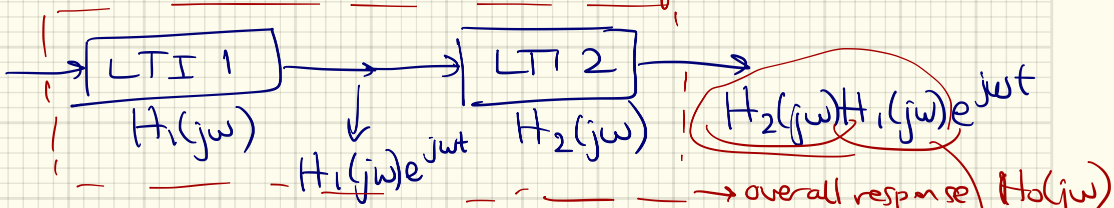
$y_2(t) = |H(j3)|\cos(3t + 1.57 + \measuredangle H(j3)) = 1.74\cos(3t + 1.57 - 0.25)$

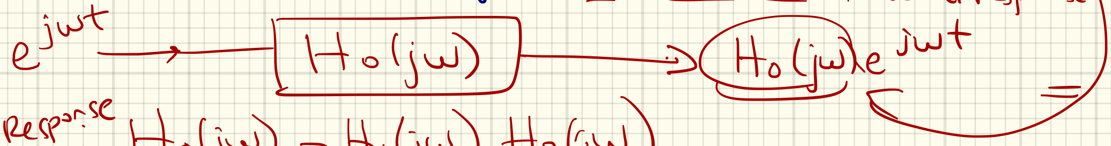
$y(t) = y_1(t) + y_2(t)$.

check yourself.

CASCADE & PARALLEL CONNECTIONS of LTI Systems

Cascade:

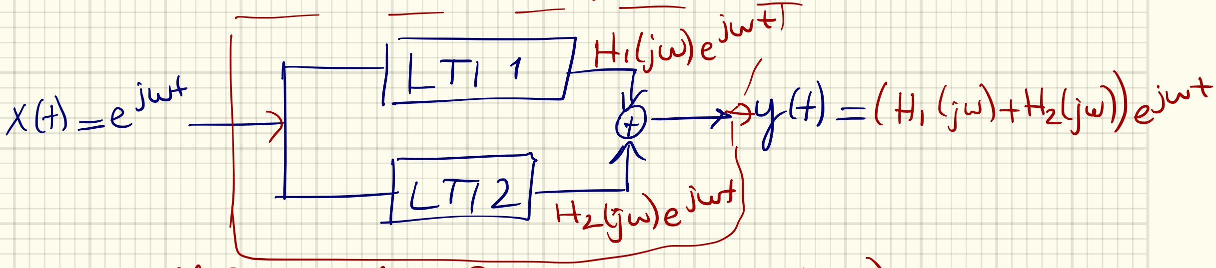




Overall Freq. Response: $H_0(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$

Recall $h_0(t) = h_1(t) * h_2(t)$

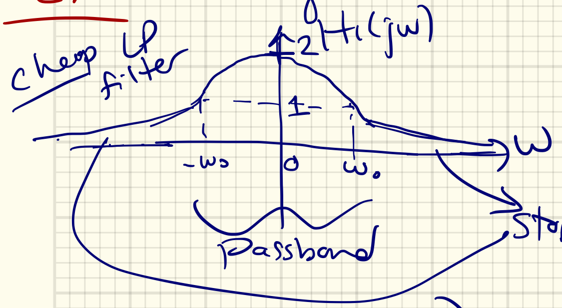
Parallel

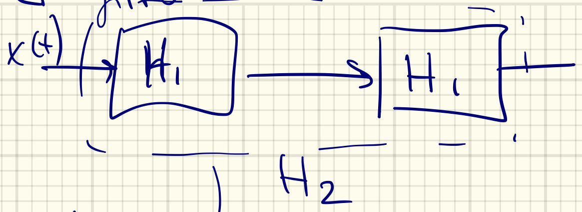


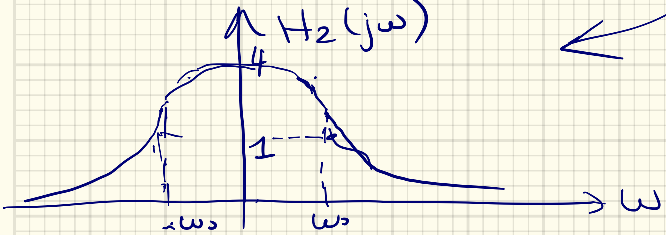
Overall Freq Resp: $H_0(j\omega) = H_1(j\omega) + H_2(j\omega)$

$h_0(t) = h_1(t) + h_2(t)$

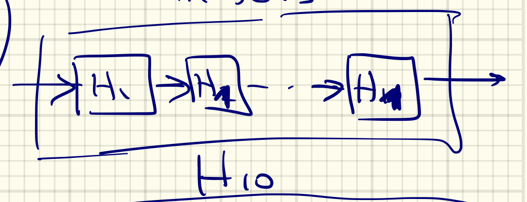
Ex: Let's say we have a LP filter

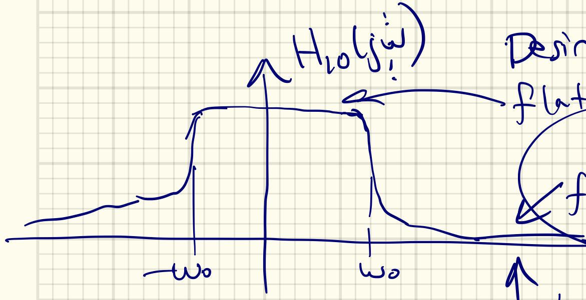






Connect 10 of these cheap filters in series





Desired effect

flatter in passband

flatter & closer to zero in stopband.

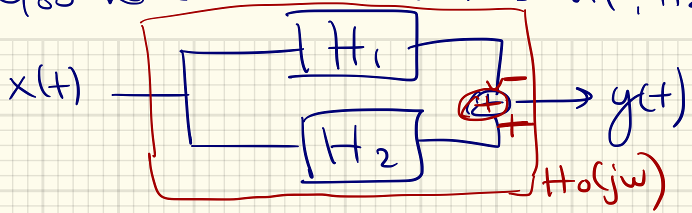
Check in matlab

You have 2 cheap filters H_1, H_2 : LP filters w/ cut-off frequencies at

$f_1 = 10\ \text{kHz}$

$f_2 = 20\ \text{kHz}$.

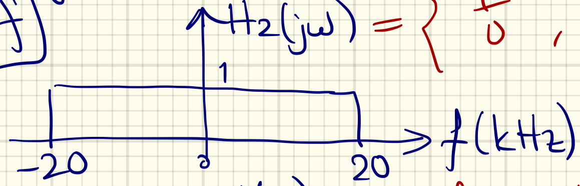
Ex:



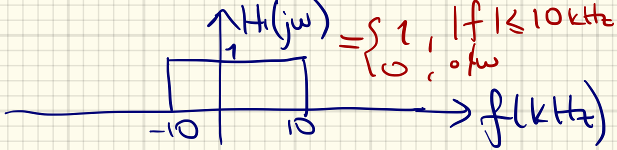
Goal: Create a Band pass filter by parallel connection of H_1 & H_2.

$\boxed{w = 2\pi f}$

$$H_2(jw) = \begin{cases} 1, & |f| \leq 20\text{kHz} \\ 0, & \text{o/w} \end{cases}$$

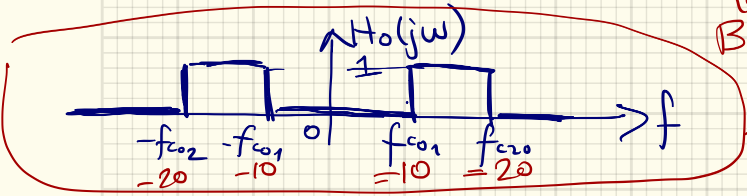


$$H_1(jw) = \begin{cases} 1, & |f| \leq 10\text{kHz} \\ 0, & \text{o/w} \end{cases}$$

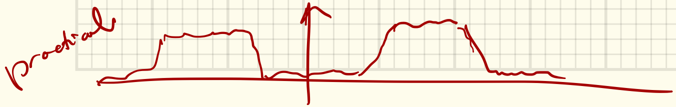


$$H_0(jw) = H_2(jw) - H_1(jw)$$

$$H_0(jw) = \begin{cases} 1, & 10\text{kHz} \leq |f| \leq 20\text{kHz} \\ 0, & \text{o/w}. \end{cases}$$



Ideal Band-pass filter.



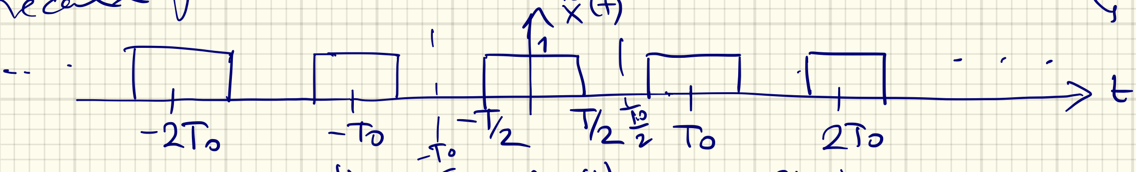
CHAPTER 11 CT Fourier Transform:

"Frequency content" = Spectrum of non-periodic signals

For any signal $x(t)$!

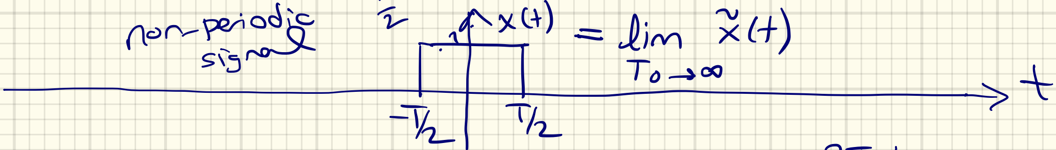
Recall for a periodic signal: $\tilde{x}(t)$ fund. period $= T_0$ sec.

$\hookrightarrow f_0 = \frac{1}{T_0}$ Hz.



$\underbrace{\omega_0 = 2\pi f_0}_{\text{fundamental freq.}}$

non-periodic signal



$x(t) = \lim_{T_0 \to \infty} \tilde{x}(t)$

F.S. expansion $\tilde{x}(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longleftarrow \omega_0 = 2\pi/T_0 .$$

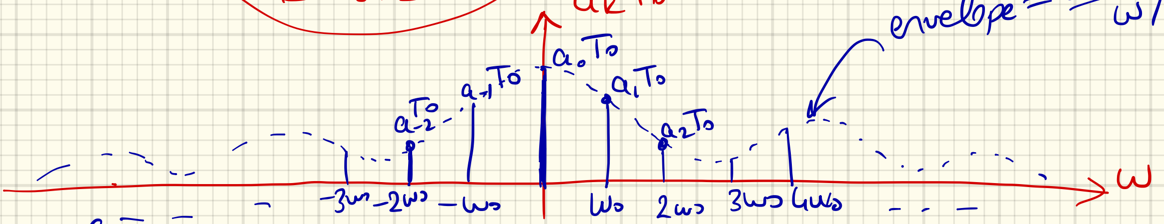
$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\omega_0 t} = \underbrace{\frac{1}{T_0}\int_{-\cancel{T_0/2}\ -T/2}^{\cancel{T_0/2}\ T/2} 1 \cdot e^{-jk\omega_0 t} dt}\ \ldots$$

exercise: derive this a_k

$$a_k = \frac{\sin(\omega_0 k T/2)}{T_0 \cdot k\frac{\omega_0}{2}}$$

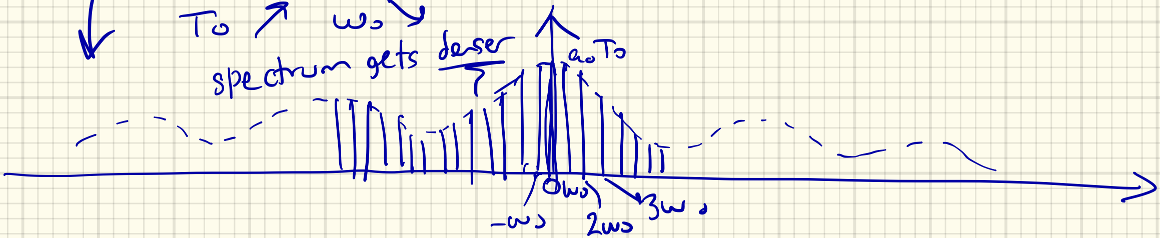
$$\Rightarrow a_k T_0 = \frac{\sin(k\omega_0 T/2)}{k\omega_0/2}$$

envelope $= \dfrac{\sin(\omega T/2)}{\omega/2}$



$T_0 \nearrow$ $\omega_0 \searrow$

spectrum gets denser



as $T_0 \to \infty$: $\lim_{T_0 \to \infty} \tilde{x}(t) = x(t)$

$\omega_0 \to d\omega$

$k\omega_0 \to \omega$

cont freq variable

$$a_k T_0 = \int_{-T_0/2}^{T_0/2} \tilde{x}(t) e^{-jk\omega_0 t} dt$$

envelope

$$\boxed{X(j\omega) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt}$$

Forward Fourier transform.

Fourier Transform

F.T. of $\cancel{x}(t)$

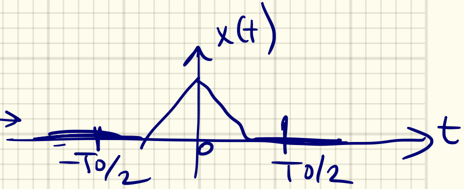
non-periodic.

Note that (i) $x(t)$ is ~~not~~ periodic $\rightarrow$ it has a Fourier Transform (F.T) <u>not</u> a Fourier series

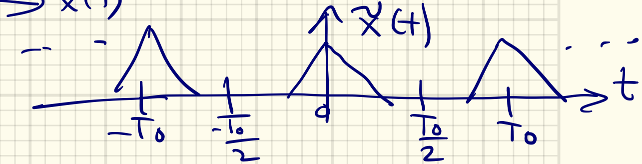
(ii) $X(j\omega)$: shows the "frequency content" of $x(t)$.
spectrum representation.

Now, let's derive the inverse F.T.

Take a non-periodic signal:



Create a periodic extension of $x(t) \Rightarrow \tilde{x}(t)$



$X(j\omega)$ is the F.T. of $x(t)$.

F.S. coeff of $\tilde{x}(t)$

$$a_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \tilde{x}(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-jk\omega_0 t}\, dt}_{X(jk\omega_0)}$$

$$\Rightarrow \boxed{a_k = \frac{X(jk\omega_0)}{T_0}}$$

Relation btw.
F.S. coef of periodic signal corresp. to $x(t) \rightarrow$ non-periodic. part.

$\rightarrow$ Can we obtain $x(t)$ from $X(j\omega)$? $\longrightarrow$

F.S. expansion $\Rightarrow$ $\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ $\longleftarrow$ insert $a_k = \dfrac{X(jk\omega_0)}{T_0}$

$= \sum_{k=-\infty}^{\infty} \dfrac{X(jk\omega_0)}{T_0} e^{jk\omega_0 t}$ $\longleftarrow$ $T_0 = \dfrac{2\pi}{\omega_0}$

$= \dfrac{1}{2\pi} \sum_{k=-\infty}^{\infty} X(jk\omega_0) e^{jk\omega_0 t} \cdot \omega_0$

$x(t) = \lim_{T_0 \to \infty} \tilde{x}(t)$

$T_0 \to \infty$

$\omega_0 \to d\omega$

$k\omega_0 \to \omega$

$\lim_{T_0 \to \infty} \tilde{x}(t) = \boxed{x(t) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} \, d\omega}$ Inverse F.T.

$x(t) = \int \cancel{b}\, a_k \, \phi_k(t) \, dt.$

$x(t) \xleftrightarrow{F.T.} X(j\omega)$