



PROBLEM 4.1:

$$(a) \quad x[n] = x(nT_s) = 10 \cos(880\pi n T_s + \varphi) \quad T_s = 0.0001$$

$$880 T_s = 880 \times 10^{-4} = 0.088 = 1/125$$

To find the number of samples within one period of the continuous cosine $x(t)$, find the largest integer satisfying $880\pi n T_s \leq 2\pi$

$$n \leq \frac{2}{0.088} = \frac{250}{11} = 22.73$$

There are 23 samples in one period, because samples $n=0, 1, 2, \dots, 22$ are within one period.

NOTE: the period of $x[n]$ is not 23; it is actually 250.

$$(b) \quad y[n] = 10 \cos(\omega_0 n T_s + \varphi)$$

To get the same samples for $x[n] \neq y[n]$ we solve:

$$\omega_0 n T_s = 880\pi n T_s + 2\pi l \quad l = \text{integer}$$

$$\Rightarrow \omega_0 = 880\pi + \frac{2\pi l}{T_s}$$

$$\frac{2\pi}{T_s} = 20,000\pi$$

$$\text{Take } l=1: \quad \omega_0 = 20,880\pi$$

(c) Find largest integer satisfying

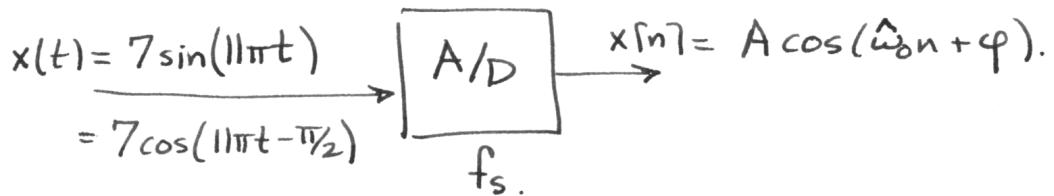
$$(20,880\pi) n T_s \leq 2\pi$$

$$n \leq \frac{2}{2.088} \quad \text{which is less than one!}$$

\therefore only one sample per period is taken



PROBLEM 4.2:



(a) $f_s = 10$ samples/sec.

$$x(t) \Big|_{t=n/f_s} = x\left(\frac{n}{f_s}\right) = 7 \cos\left(\frac{11\pi n}{10} - \pi/2\right).$$

$$= 7 \cos\left(\frac{11\pi n}{10} - 2\pi n - \pi/2\right).$$

$$= 7 \cos\left(-\frac{9\pi n}{10} - \pi/2\right) = 7 \cos(0.9\pi n + \pi/2).$$

$$\boxed{A=7, \hat{\omega}_0=0.9\pi, \varphi=\pi/2}$$

(b) $f_s = 5$ samples/sec

$$x(t) \Big|_{t=n/f_s} = x\left(\frac{n}{5}\right) = 7 \cos\left(\frac{11\pi n}{5} - \pi/2\right)$$

$$= 7 \cos\left(\frac{\pi n}{5} - \frac{\pi}{2}\right)$$

$$\boxed{A=7, \hat{\omega}_0=\frac{\pi}{5}, \varphi=-\frac{\pi}{2}}$$

(c) $f_s = 15$ samples/sec

$$x(t) \Big|_{t=n/f_s} = x\left(\frac{n}{15}\right) = 7 \cos\left(\frac{11\pi n}{15} - \frac{\pi}{2}\right)$$

$$A=7, \hat{\omega}_0 = \frac{11\pi}{15} = 2\pi\left(\frac{5.5}{15}\right) \Leftrightarrow \varphi = -\pi/2$$



PROBLEM 4.3:

$$x[n] = 2.2 \cos(0.3\pi n - \pi/3)$$

$$f_s = 6000$$

Compare to

$$x\left(\frac{n}{f_s}\right) = A \cos\left(2\pi f_0 \frac{n}{f_s} + \varphi\right) \leftarrow \boxed{\text{sampled continuous-time signal.}}$$

$$\Rightarrow \frac{2\pi f_0}{f_s} = 0.3\pi, \text{ or } 0.3\pi + 2\pi, \text{ or } 0.3\pi - 2\pi.$$

$$\text{Solve: } \frac{2\pi f_0}{f_s} = 0.3\pi \Rightarrow f_0 = f_s \left(\frac{0.3}{2}\right) = 6000 \times 0.15 \\ f_0 = 900 \text{ Hz}$$

$$\rightarrow x(t) = 2.2 \cos(1800\pi t - \pi/3)$$

NOTE:
difference
is f_s

Then $\frac{2\pi f_0}{f_s} = 2.3\pi \Rightarrow f_0 = f_s \left(\frac{2.3}{2}\right) = 6900 \text{ Hz}$

$$\rightarrow x(t) = 2.2 \cos(2\pi(6900)t - \pi/3).$$

Finally,

$$\frac{2\pi f_0}{f_s} = -1.7\pi \Rightarrow f_0 = f_s \left(-\frac{1.7}{2}\right) = -5100 \text{ Hz.}$$

$$x(t) = 2.2 \cos(2\pi(-5100)t - \pi/3)$$

$$\rightarrow x(t) = 2.2 \cos(2\pi(5100)t + \pi/3)$$



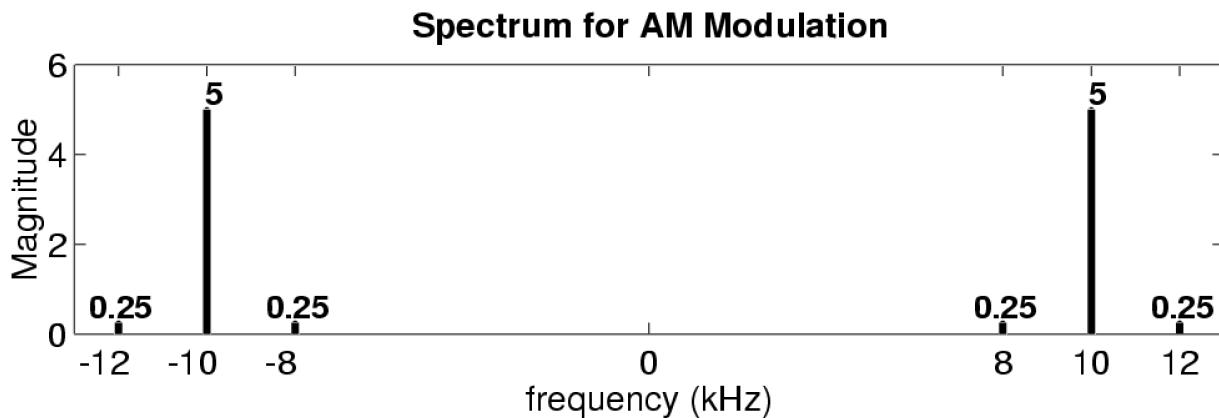
PROBLEM 4.4:

$$(a) x(t) = [10 + \frac{1}{2}e^{j2\pi(2000)t} + \frac{1}{2}e^{-j2\pi(2000)t}] \left(\frac{1}{2}e^{j2\pi \times 10^4 t} + \frac{1}{2}e^{-j2\pi \times 10^4 t} \right)$$

There are six terms:

$$x(t) = 5e^{j2\pi \times 10^4 t} + 5e^{-j2\pi \times 10^4 t} + \frac{1}{4}e^{j2\pi(12000)t} + \frac{1}{4}e^{-j2\pi(12000)t} \\ + \frac{1}{4}e^{j2\pi(8000)t} + \frac{1}{4}e^{-j2\pi(8000)t}$$

Spectrum plot was created in MATLAB:



- (b) Yes the waveform is periodic. The six frequencies $\{-12,000, -10,000, -8000, 8000, 10000, 12000\}$ are all divisible by 2000 Hz. Therefore, $f_0 = 2000$ Hz is the fundamental frequency. The period is $1/f_0 = 1/2000$ sec = $\frac{1}{2}$ msec

- (c) The sampling rate must be greater than twice the highest frequency in $x(t)$.

$$\Rightarrow f_s > 2(12,000) = 24,000 \text{ Hz}$$



PROBLEM 4.5:

(a) Let $x(t) = 10 \cos(\omega_0 t + \varphi)$

Sampling at a rate of $f_s \Rightarrow x[n] = x(t)|_{t=n/f_s} = x\left(\frac{n}{f_s}\right)$

$$x[n] = 10 \cos\left(\omega_0 \frac{n}{f_s} + \varphi\right) \quad \boxed{f_s = 1000}$$

Equate this to

$$x[n] = 10 \cos(0.2\pi n - \pi/7) \quad \varphi = -\pi/7$$

A second possible signal is the "folded alias" at $(f_s - f_0)$

$$f_s - f_0 = f_s - \frac{\omega_0}{2\pi} = 1000 - \frac{200\pi}{2\pi} = 900 \text{ Hz}$$

In this case, the phase (φ) changes.

$$\tilde{x}(t) = 10 \cos(2\pi(f_s - f_0)t + \psi)$$

$$\begin{aligned} \tilde{x}[n] &= 10 \cos(2\pi(f_s - f_0)\frac{n}{f_s} + \psi) = 10 \cos\left(2\pi n - 2\pi \frac{f_0 n}{f_s} + \psi\right) \\ &= 10 \cos\left(-2\pi \frac{f_0 n}{f_s} + \psi\right) = 10 \cos\left(2\pi \frac{f_0}{f_s} n - \psi\right). \end{aligned}$$

$$\Rightarrow \psi = +\pi/7$$

f_0 is still 100 Hz

(b) Reconstruction of $x[n]$ with $f_s = 2000$ samples/sec.

The discrete and continuous domains are related

$$\text{by: } \frac{n}{f_s} \leftrightarrow t \quad \text{or} \quad n \leftrightarrow f_s t$$

So we replace "n" in $x[n]$ with $f_s t$. This is what an ideal D-to-A would do.

$$x[n] = 10 \cos(0.2\pi n - \pi/7)$$

$$x(t) = 10 \cos(0.2\pi f_s t - \pi/7) \quad \boxed{f_s = 2000}$$

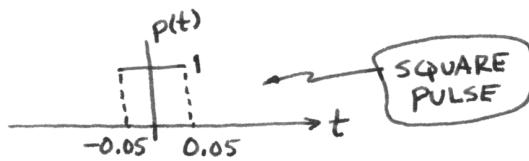
$$= 10 \cos(400\pi t - \pi/7)$$

$$\boxed{\omega_0 = 400\pi \Rightarrow f_0 = 200 \text{ Hz.}}$$



PROBLEM 4.6:

$$(a) p(t) = \begin{cases} 1 & -0.05 \leq t \leq 0.05 \\ 0 & \text{otherwise} \end{cases}$$

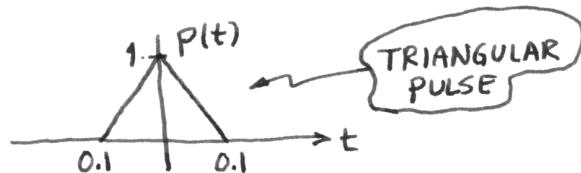


In the formula for $y(t)$

$$y(t) = \dots + y[0] p(t) + y[1] p(t-T_s) + y[2] p(t-2T_s) + \dots$$

The square pulses will not overlap, so the values of $y[n]$ will be extended over an interval of T_s .

$$(b) p(t) = \begin{cases} 1-10|t| & -0.1 \leq t \leq 0.1 \\ 0 & \text{otherwise} \end{cases}$$

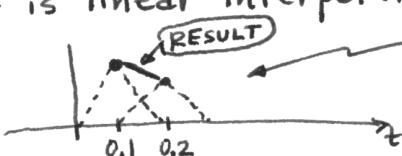


In this case, the neighboring terms do overlap

$$y(t) = \dots + y[0] p(t) + y[1] p(t-T_s) + y[2] p(t-2T_s) + \dots$$

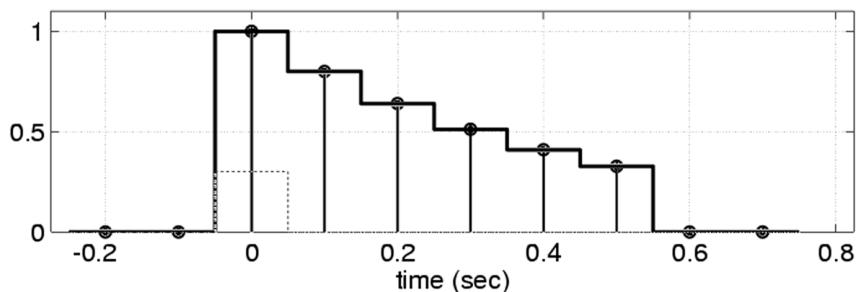
The result is linear interpolation.

Example:

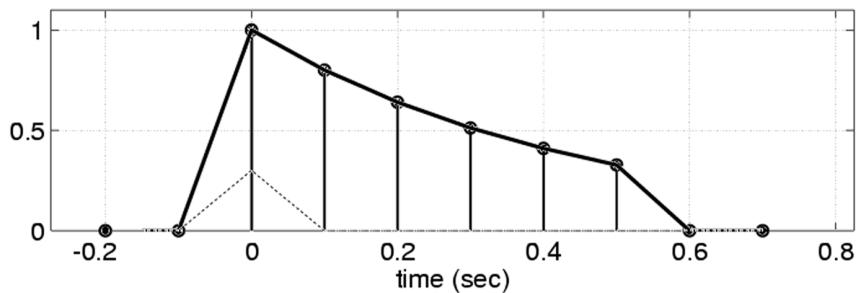


When we add these two triangles, the result between $t=0.1$ and $t=0.2$ is a straight line.

Problem 4.8(a) Square Pulse Shape



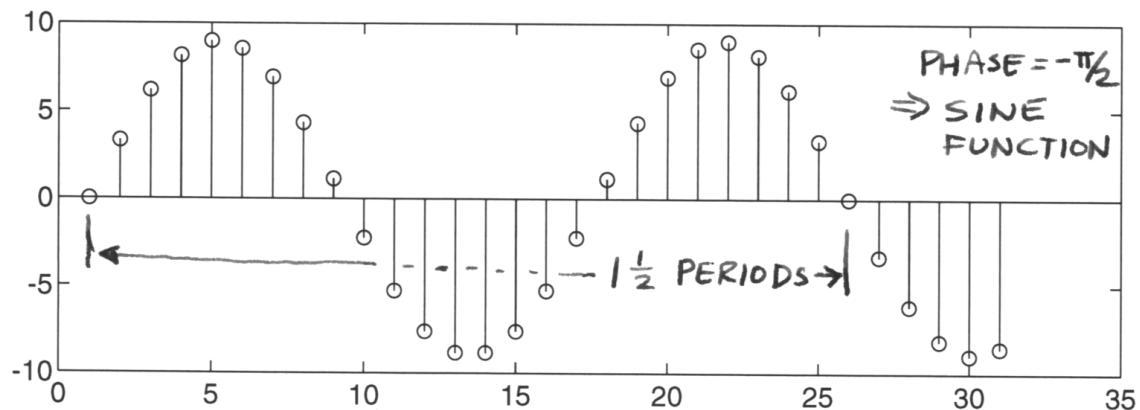
Problem 4.8(b) Triangular Reconstruction Pulse





PROBLEM 4.7:

(a)



NOTE $1\frac{1}{2}$ PERIODS = 25 samples.

$$\Rightarrow \text{PERIOD} = 50/3 \Rightarrow \hat{\omega}_0 = 2\pi(0.06) \\ \Rightarrow \frac{1}{(50/3)} = \frac{3}{50}$$

(b) The vector \mathbf{x} is actually $x[n]$

$$\begin{aligned} x[n] &= 9 \cos(2\pi(394)n(0.01) + \pi/2) \\ &= 9 \cos(2\pi(3.94)n + \pi/2) \\ &= 9 \cos(2\pi(0.94)n + \pi/2) \\ &\quad \swarrow \text{REMOVED MULTIPLE OF } 2\pi \\ &= 9 \cos(2\pi(-0.06)n + \pi/2) \end{aligned}$$

$$x[n] = 9 \cos(2\pi(0.06)n - \pi/2)$$

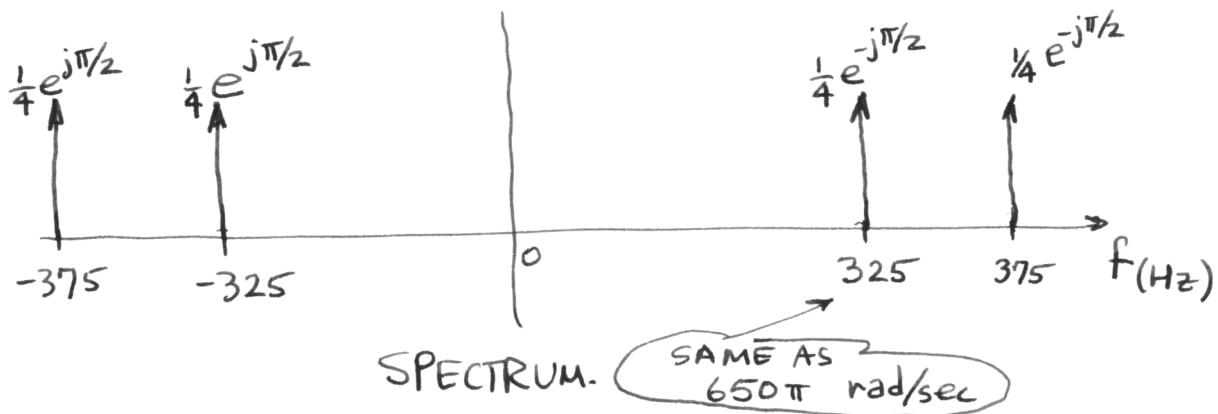
(c) Derivation in part (b) shows that $x[n]$ looks like samples of a 6 Hz sinusoid taken at $f_s = \frac{1}{0.01} = 100 \text{ Hz}$.



PROBLEM 4.8:

$$x(t) = \cos(50\pi t) \sin(700\pi t)$$

$$\begin{aligned}
 (a) \quad x(t) &= \left(\frac{1}{2} e^{j50\pi t} + \frac{1}{2} e^{-j50\pi t} \right) \left(\frac{1}{2j} e^{j700\pi t} - \frac{1}{2j} e^{-j700\pi t} \right) \\
 &= \frac{1}{4j} e^{j750\pi t} + \frac{1}{4j} e^{j650\pi t} - \frac{1}{4j} e^{-j650\pi t} - \frac{1}{4j} e^{-j750\pi t} \\
 &\quad \uparrow \qquad \qquad \qquad \uparrow \\
 &\text{SAME AS } \frac{1}{4} e^{+j\pi/2} \qquad \qquad \qquad \text{SAME AS } \frac{1}{4} e^{+j\pi/2}
 \end{aligned}$$



(b) Sampling Thm says sample at a rate greater than two times the highest freq.

$$\text{HIGHEST FREQ} = 375 \text{ Hz}$$

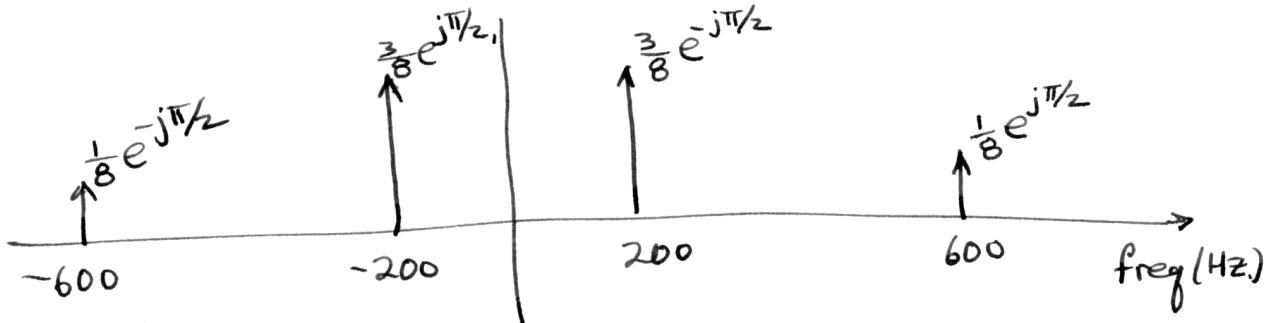
$$\Rightarrow f_s \geq 750 \text{ Hz.}$$



PROBLEM 4.9:

- (a) Draw a sketch of the spectrum of $x(t)$ which is "sine-cubed" $x(t) = \sin^3(400\pi t)$

$$\begin{aligned} x(t) &= \left(\frac{e^{j400\pi t} - e^{-j400\pi t}}{2j} \right)^3 \\ &= \frac{1}{-8j} \left\{ e^{j1200\pi t} - 3e^{j400\pi t} + 3e^{-j400\pi t} - e^{-j1200\pi t} \right\}. \end{aligned}$$



$$\frac{1}{-8j} = \frac{1}{8} e^{j\pi/2}$$

- (b) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

$$f_s \geq 2f_{\text{HIGH}}$$

$$\Rightarrow f_s \geq 1200 \text{ Hz}$$



PROBLEM 4.10:

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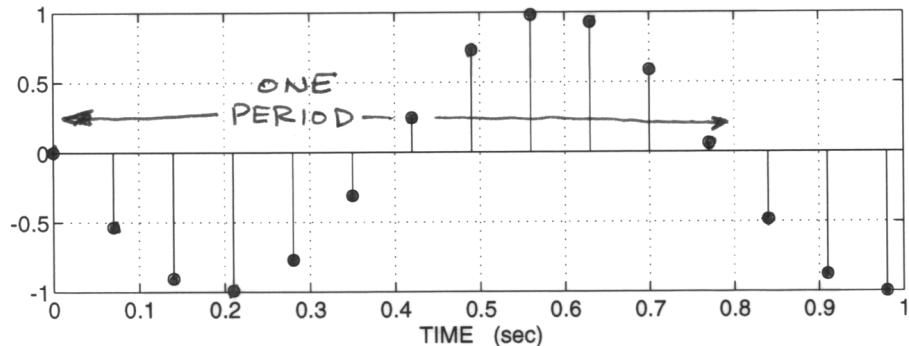
Fo = 13;
Period = 1/Fo;
Ts = 0.07;
tt = 0 : Ts : (13*Period);
j = sqrt(-1);
xx = real( exp( j*(2*pi*Fo*tt - pi/2) ) );
%
stem( tt, xx ), xlabel('TIME (sec)'), grid

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$$F_{SAMP} = \frac{1}{Ts} = \frac{1}{0.07} = 14.28 \text{ Hz.}$$

↑ NOT GREATER THAN $2F_o$

(a)



we observe
1 sec of
the signal
which is
1.28 periods

$$\begin{aligned}
 X[n] &= \cos(2\pi(13)(0.07n) - \pi/2) \\
 &= \cos(2\pi(0.91)n - \pi/2) \quad \text{FOLDING} \\
 &= \cos(2\pi(0.09)n + \pi/2)
 \end{aligned}$$

in continuous-time, the
folded frequency is
 $14.28 - 13 = 1.28 \text{ Hz}$
 $\Rightarrow \text{period} \approx \frac{1}{1.28} \approx 0.8 \text{ sec}$

(b) SAMPLING THM $\Rightarrow F_{SAMP} \geq 2F_o = 2(13) = 26$

$$F_{SAMP} = \frac{1}{Ts}$$

To get [SMOOTH] plot need about 20 samples
per period, which is a sampling rate of

$$20F_o \quad \therefore \quad Ts \leq \frac{1}{20F_o} = \frac{1}{20(13)} = \frac{1}{260}.$$



PROBLEM 4.11:

$$x(t) = [3 + \sin(\pi t)] \cos(13\pi t + \pi/2)$$

$\underbrace{\hspace{10em}}$ $\cos(\pi t - \pi/2)$

(a) Use *phasors* to show that $x(t)$ can be expressed in the form:

$$x(t) = A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(\omega_2 t + \phi_2) + A_3 \cos(\omega_3 t + \phi_3)$$

where $\omega_1 < \omega_2 < \omega_3$; i.e., find A_1 , A_2 , A_3 , ϕ_1 , ϕ_2 , ϕ_3 , ω_1 , ω_2 , ω_3 in terms of A , ω_0 , and ω_c .

$$\begin{aligned} x(t) &= \left[3 + \frac{1}{2} e^{j(\pi t - \pi/2)} + \frac{1}{2} e^{-j(\pi t - \pi/2)} \right] \left(\frac{1}{2} e^{j(13\pi t + \pi/2)} + \frac{1}{2} e^{-j(13\pi t + \pi/2)} \right) \\ &= \frac{3}{2} e^{j\pi/2} e^{j13\pi t} + \frac{3}{2} e^{-j\pi/2} e^{-j13\pi t} + \frac{1}{4} e^{j14\pi t} + \frac{1}{4} e^{-j14\pi t} \\ &\quad + \frac{1}{4} e^{j\pi} e^{j12\pi t} + \frac{1}{4} e^{-j\pi} e^{-j12\pi t} \\ &= 3 \cos(13\pi t + \pi/2) + \frac{1}{2} \cos 14\pi t + \frac{1}{2} \cos(12\pi t + \pi) \end{aligned}$$

$$\omega_1 = 12\pi$$

$$A_1 = \frac{1}{2}$$

$$\phi_1 = \pi$$

$$\omega_2 = 13\pi$$

$$A_2 = 3$$

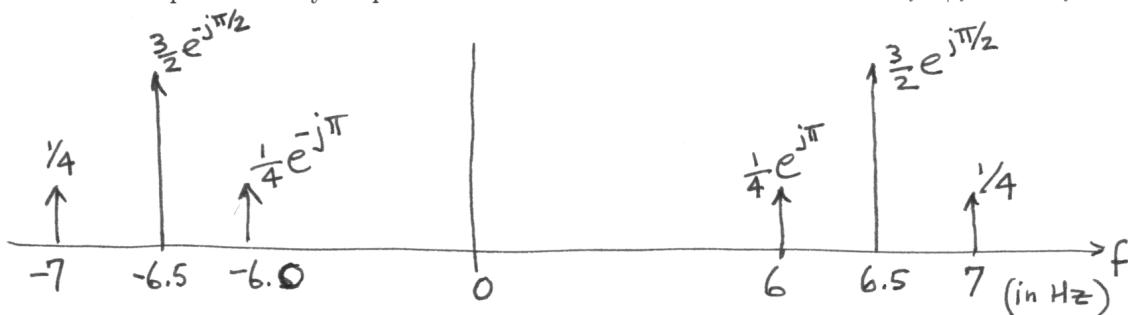
$$\phi_2 = \pi/2$$

$$\omega_3 = 14\pi$$

$$A_3 = \frac{1}{2}$$

$$\phi_3 = 0$$

(b) Sketch the two-sided spectrum of this signal on a frequency axis. Be sure to label important features of the plot. Label your plot in terms of the numerical values of the A_i 's ϕ_i 's and ω_i 's.



(c) Determine the minimum sampling rate that can be used to sample $x(t)$ without any aliasing.

$$\text{HIGHEST FREQ} = 7 \text{ Hz.}$$

$$\Rightarrow F_{\text{samp}} \geq 2(7) = 14 \text{ Hz}$$



PROBLEM 4.12:

(a) $x[n] = 10 \cos(0.13\pi n + \pi/13)$ the sampling rate is $f_s = 1000$ samples/second

$$0.13\pi n = 2\pi(0.065)n = 2\pi(65)\frac{n}{1000} \Rightarrow 65 \text{ Hz is one Freq.}$$

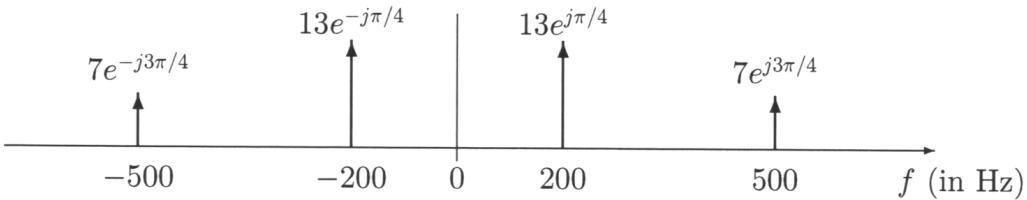
$$x_1(t) = 10 \cos(2\pi(65)t + \pi/13)$$

Also, it could be "folded" case: $1000 - 65 = 935 \text{ Hz}$

$$x_2(t) = 10 \cos(2\pi(935)t - \pi/13)$$

\leftarrow NOTE phase reversal

- (b) If the input $x(t)$ is given by the two-sided spectrum representation shown below, determine a simple formula for $y(t)$ when $f_s = 700$ samples/sec. (for both the C/D and D/C converters).



$$x(t) = 26 \cos(2\pi(200)t + \pi/4) + 14 \cos(2\pi(500)t + 3\pi/4)$$

$$x[n] = 26 \cos\left(2\pi\left(\frac{2}{7}\right)n + \pi/4\right) + 14 \cos\left(2\pi\left(\frac{5}{7}\right)n + 3\pi/4\right)$$

\curvearrowleft THIS TERM FOLDS

$$x[n] = 26 \cos\left(2\pi\left(\frac{2}{7}\right)n + \pi/4\right) + 14 \cos\left(2\pi\left(-\frac{2}{7}\right)n + 3\pi/4\right).$$

$$\Rightarrow y(t) = 26 \cos(2\pi(200)t + \pi/4) + 14 \cos(2\pi(200)t - 3\pi/4).$$

$$y(t) = 12 \cos(2\pi(200)t + \pi/4)$$



PROBLEM 4.13:

Assume that the sampling rates of a C-to-D and D-to-C conversion system are equal, and the input to the Ideal C-to-D converter is

$$x(t) = 2 \cos(2\pi(50)t + \pi/2) + \cos(2\pi(150)t)$$

- (a) If the output of the ideal D-to-C Converter is equal to the input $x(t)$, i.e.,

$$y(t) = 2 \cos(2\pi(50)t + \pi/2) + \cos(2\pi(150)t)$$

what general statement can you make about the sampling frequency f_s in this case?

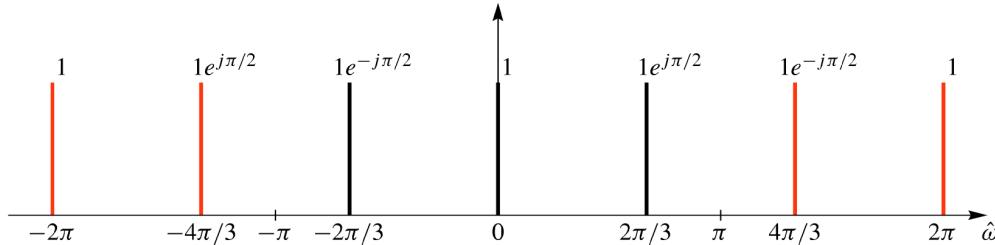
Solution: The sampling frequency must be greater than twice the highest frequency, because there was no aliasing. Thus, we can say that

$$F_s > 2 \times 150 = 300 \text{ Hz}$$

- (b) If the sampling rate is $f_s = 250$ samples/sec., determine the discrete-time signal $x[n]$, and give an expression for $x[n]$ as a sum of cosines. *Make sure that all frequencies in your answer are positive and less than π radians.* *Solution:* Replace t with $n/f_s = n/250$ to get

$$\begin{aligned} x[n] &= x(n/250) = 2 \cos(2\pi(50)(n/250) + \pi/2) + \cos(2\pi(150)(n/250)) \\ &= 2 \cos(2\pi(0.2)n + \pi/2) + \cos(2\pi(0.6)n) \\ &= 2 \cos(2\pi(0.2)n + \pi/2) + \cos(2\pi(0.4)n) \end{aligned}$$

- (c) Plot the spectrum of the signal in part (b) over the range of frequencies $-\pi \leq \hat{\omega} \leq \pi$. The plot below shows the periodicity of the DT spectrum.



- (d) If the output of the Ideal D-to-C Converter is

$$y(t) = 2 \cos(2\pi(50)t + \pi/2) + 1$$

determine the value of the sampling frequency f_s . (Remember that the input signal is $x(t)$ defined above.)

Solution: Since the frequency of 50 Hz is preserved, the other frequency of 150 Hz must have been aliased to 0 Hz. This can happen if the sampling frequency is $f_s = 150$ Hz, in which case the discrete-time signal is

$$\begin{aligned} x[n] &= x(n/150) = 2 \cos(2\pi(50)(n/150) + \pi/2) + \cos(2\pi(150)(n/150)) \\ &= 2 \cos(2\pi n/3 + \pi/2) + \cos(2\pi n) \\ &= 2 \cos(2\pi n/3 + \pi/2) + 1 \end{aligned}$$

When $x[n]$ is reconstructed by the D/A converter running at $f_s = 150$ Hz, the final output will be

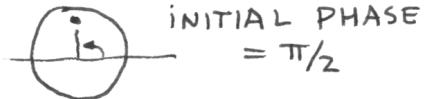
$$y(t) = x[n]|_{n \rightarrow f_s t} = 2 \cos(2\pi(150t)/3 + \pi/2) + 1 = 2 \cos(2\pi(50)t + \pi/2) + 1$$



PROBLEM 4.14:

- (a) Assume that the disk is rotating clockwise at a constant speed of 13 revolutions per second. If the flashing rate is 15 times per second, express the movement of the spot on the disk as a complex phasor, $p[n]$, that gives the position of the spot at the n -th flash. Assume that the spot is at the top when $n = 0$ (the first flash).

$$p(t) = r e^{j\pi/2} e^{-j2\pi(13)t}$$



$$\begin{aligned} p[n] &= r e^{j\pi/2} e^{-j2\pi(\frac{13}{15})n} \\ &= r e^{j\pi/2} e^{+j2\pi(\frac{2}{15})n} \end{aligned}$$

ADD $2\pi n$ in EXPONENT.

- (b) For the conditions in part (a), determine the apparent speed (in revolutions per second) and direction of movement of the "strobed" spot.

Convert $e^{j2\pi(\frac{2}{15})n}$ back to "continuous-time"

Replace n with $15t$

$$\Rightarrow e^{j2\pi(2)t}$$

$$\therefore \text{speed} = 2 \text{ rev/sec}$$

direction = COUNTER-CLOCKWISE

because sign of exponent is positive

- (c) Now assume that the rotation speed of the disk is unknown. If the flashing rate is 13 times per second, and the spot on the disk moves counter-clockwise by 15 degrees with each flash, determine the rotation speed of the disk (in rev/sec). If the answer is not unique give all possible rotation speeds.

15° per flash $\Rightarrow 1 \text{ rev per } 24 \text{ flashes}$

$$\Rightarrow p[n] = e^{j2\pi n/24} \quad (\text{i.e., period}=24)$$

$$\hat{p}(t) = e^{j2\pi f_0 t} \quad \text{where } f_0 = \text{unknown speed.}$$

sample at 13 flashes/sec.

$$e^{j2\pi f_0 n/13} = e^{j2\pi n/24} \cdot e^{j2\pi ln}$$

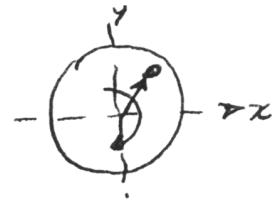
$$\frac{f_0}{13} = \frac{1}{24} + l \quad \Rightarrow \quad \boxed{f_0 = \frac{13}{24} + 13l} \quad l = \text{integer}$$



PROBLEM 4.15:

(a) $720 \text{ rpm} = 12 \text{ rotations/sec.}$

If (x, y) = the co-ordinates of the spot we can also use polar co-ords: $r\angle\theta$.



The radius of the spot is constant: r
The angle of the vector from the origin to the spot changes LINEARLY

$$\theta = \varphi_0 + \omega_0 t$$

where φ_0 is the initial phase
 ω_0 = freq of rotation in rad/sec.

$$\therefore \omega_0 = 2\pi(12) = 24\pi.$$

So the position of the spot = $r e^{-j24\pi t + j\varphi_0}$
The minus sign is for clockwise rotation

(b) Disk spot will stand still if flash rate is once per rotation, once every 2 rotations etc. \Rightarrow possible flash rate = $\frac{12}{l}$, $l=1,2,3,\dots$ to hold spot still

So we get an answer of [rate = 1, 2, 3, 4, 6, & 12 per sec]
The rate must be a factor of 12

(c) $f_s = 13 \text{ per sec.} \Rightarrow \text{sample at } \frac{n}{13}.$

$$x[n] = r e^{j\varphi_0} e^{-j24\pi n/13} \Rightarrow \omega_0 = -\frac{24\pi}{13}$$

But ω_0 is same as: $-\frac{24\pi}{13} + 2\pi = \frac{2\pi}{13}$

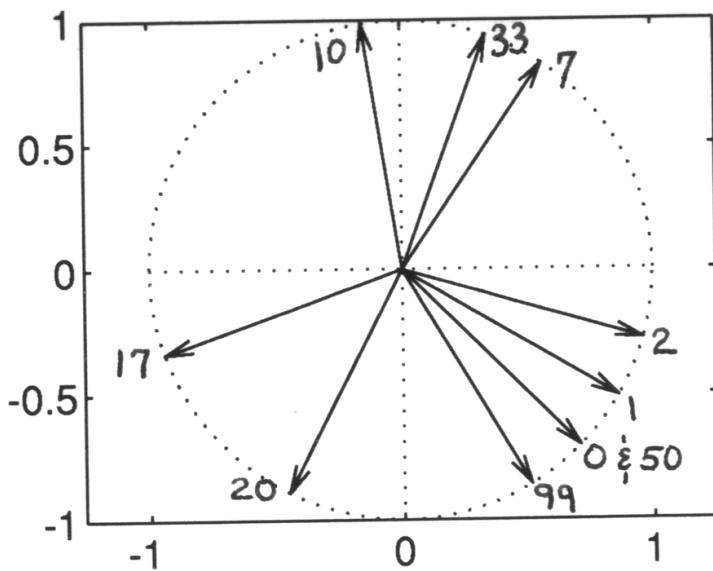
$$x[n] = r e^{j\varphi_0} e^{+j2\pi n/13} \leftarrow (\text{will take 13 flashes per revolution})$$

Spot will move counter-clockwise (due to + sign)

At what rate? once per sec = 60 rpm



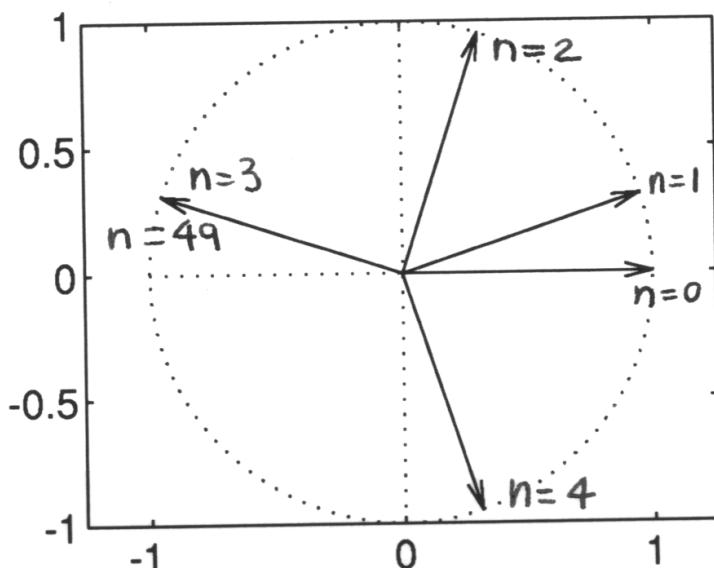
PROBLEM 4.16:



$$\text{Period} = \frac{2\pi}{0.08\pi} = 25$$

$$\Rightarrow z[50] = z[0]$$

$$z[33] = z[8]$$



When $n=7$

$$e[7] = e^{j0.1\pi(49)}$$

$$= e^{j4.9\pi}$$

$$= e^{j0.9\pi}$$

NOT PERIODIC



PROBLEM 4.17:

(a) $\Theta[n] = \pi(0.7 \times 10^{-3})n^2$ ← This is $\Theta[n]$ in $\operatorname{Re}\{e^{j\Theta[n]}\}$

For $n=10$:

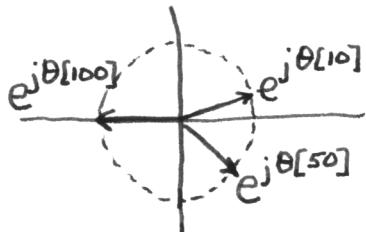
$$\Theta[10] = \pi(0.7 \times 10^{-3})10^2 = 0.07\pi = 12.6^\circ$$

For $n=50$:

$$\Theta[50] = \pi(0.7 \times 10^{-3})(50)^2 = \pi(0.7 \times 10^{-3} \times 25 \times 10^2) = 1.75\pi \quad \text{at } 315^\circ$$

For $n=100$:

$$\Theta[100] = \pi(0.7 \times 10^{-3})10^4 = 7\pi = \pi \text{ rads, or } 180^\circ$$



(c) Work part (c) before part (b)

$$v[n] = \cos(0.7\pi n) \quad f_s = 8000 \text{ Hz}$$

↑ Ideal D/A ⇒ replace n with $f_s t$

$$v(t) = v[n] \Big|_{n=8000t} = \cos(0.7\pi \cdot 8000t) = \cos(2\pi(2800)t)$$

Freq is 2800 Hz

(b) $x[n] = \cos(\pi(0.7 \times 10^{-3})n^2)$ Replace n with $8000t$

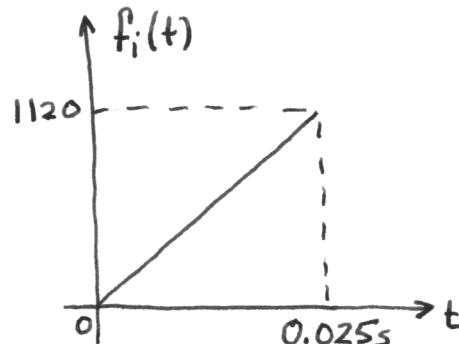
$$\begin{aligned} x(t) &= \cos(\pi(0.7 \times 10^{-3}) \times 64 \times 10^6 t^2) \\ &= \cos(\pi(44.8 \times 10^3) t^2) \end{aligned} \quad \underbrace{n=0, 1, \dots, 200}_{0 \leq t \leq \frac{200}{f_s}} = 0.025 \text{ sec.}$$

$$\Psi(t) = \pi(44.8 \times 10^3) t^2$$

$$f_i(t) = \frac{1}{2\pi} \frac{d}{dt} \Psi(t)$$

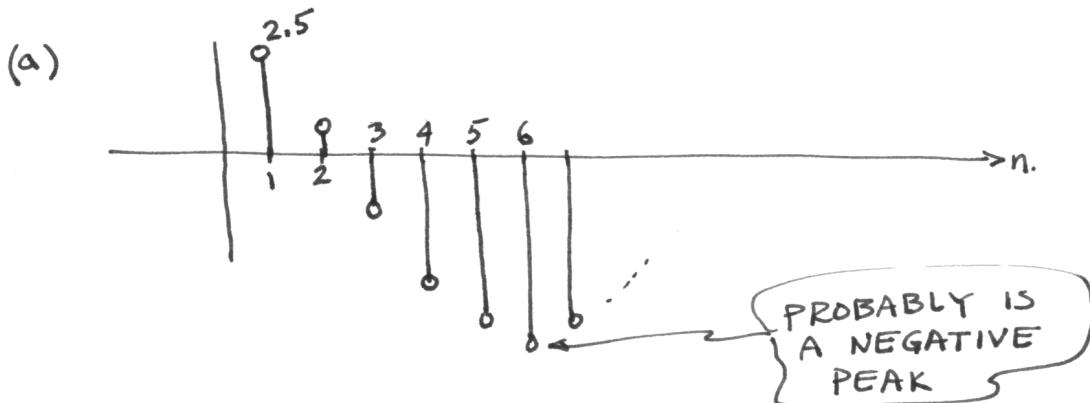
$$= \frac{1}{2\pi} (2\pi)(44.8 \times 10^3) t$$

$$= 44800 t \text{ Hz}$$





PROBLEM 4.18:



$$(b) \beta \cos n\omega_0 = \cos(n+1)\omega_0 + \cos(n-1)\omega_0$$

$$\beta \left(\frac{e^{jn\omega_0} + e^{-jn\omega_0}}{2} \right) = \frac{e^{j(n+1)\omega_0} + e^{-j(n+1)\omega_0}}{2} + \frac{e^{j(n-1)\omega_0} + e^{-j(n-1)\omega_0}}{2}$$

$$\Rightarrow \beta(e^{jn\omega_0} + e^{-jn\omega_0}) = (e^{j\omega_0} + e^{-j\omega_0})e^{jn\omega_0} + (e^{j\omega_0} + e^{-j\omega_0})e^{-jn\omega_0}$$

$$\Rightarrow \beta = e^{j\omega_0} + e^{-j\omega_0} = 2\cos\omega_0$$

This result can be generalized to cosines with any phase so we get a method for extracting the frequency directly from $x[n]$ when we know that $x[n]$ is $x[n] = A \cos(\omega_0 n + \varphi)$.

The method is

$$\cos\omega_0 = \frac{x[n+1] + x[n-1]}{2x[n]}$$

where we use any 3 consecutive values in $x[n]$.



PROBLEM 4.18 (more):

$$\cos \omega_0 = \frac{x[5] + x[7]}{2x[6]} = \frac{-4.5677 - 4.5677}{2(-5.0)} = .9135$$

$$\Rightarrow \omega_0 = \cos^{-1}(0.9135) = 0.4190 = 2\pi/15$$

Now set up simultaneous eqns for $A \& \varphi$

$$\text{Let } Z = \frac{1}{2}A e^{j\varphi}$$

$$x[n] = Z e^{jn\omega_0} + Z^* e^{-jn\omega_0}$$

$$\begin{bmatrix} e^{j\omega_0} & e^{-j\omega_0} \\ e^{j2\omega_0} & e^{-j2\omega_0} \end{bmatrix} \begin{bmatrix} Z \\ Z^* \end{bmatrix} = \begin{bmatrix} x[1] \\ x[2] \end{bmatrix}.$$

Invert 2×2 matrix:

$$\begin{bmatrix} Z \\ Z^* \end{bmatrix} = \frac{1}{e^{-j\omega_0} - e^{j\omega_0}} \begin{bmatrix} e^{-j2\omega_0} & -e^{-j\omega_0} \\ -e^{j2\omega_0} & e^{j\omega_0} \end{bmatrix} \begin{bmatrix} 2.50 \\ .5226 \end{bmatrix}.$$

$$\therefore Z = \frac{e^{-j2\omega_0} x[1] - e^{-j\omega_0} x[2]}{e^{-j\omega_0} - e^{j\omega_0}} = \boxed{\frac{1}{2} A e^{j\varphi}}$$

Now, plug in the numbers & CRUNCH:

$$\frac{1}{2} A e^{j\varphi} = 2.023 + j1.47 = 2.5 e^{j\pi/5}$$

$$\Rightarrow A = 5$$

$$\varphi = \pi/5$$

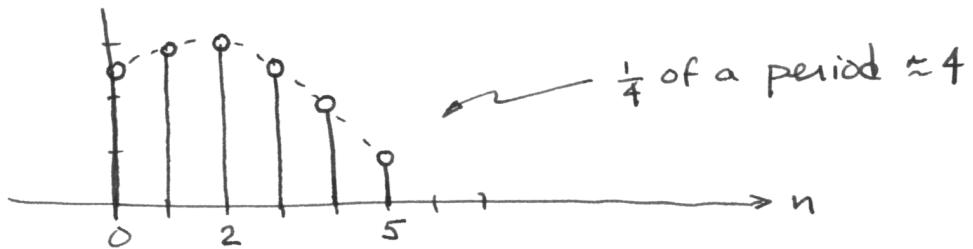
$$5 \cos\left(\frac{2\pi}{15}n + \pi/5\right)$$

$$\omega_0 = 2\pi/15$$



PROBLEM 4.19:

You could estimate the values from a plot.



$$\text{Looks like } A \approx 3 \quad \omega_0 \approx 2\pi \left(\frac{1}{\text{period}} \right) = 2\pi \frac{1}{16} = \frac{\pi}{8}$$

$$\varphi = -2\pi \left(\frac{t_1}{T} \right) \approx -2\pi \left(\frac{2}{16} \right) = -\pi/4$$

EXACT:

Write 3 consecutive values of $x[n]$.

$$x[n-1] = \frac{A}{2} e^{j\varphi} e^{j\omega_0 n} e^{-j\omega_0} + \frac{A}{2} e^{-j\varphi} e^{-j\omega_0 n} e^{j\omega_0}$$

$$x[n] = \frac{A}{2} e^{j\varphi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\varphi} e^{-j\omega_0 n}$$

$$x[n+1] = \frac{A}{2} e^{j\varphi} e^{j\omega_0 n} e^{j\omega_0} + \frac{A}{2} e^{-j\varphi} e^{-j\omega_0 n} e^{-j\omega_0}$$

$$\begin{aligned} \Rightarrow x[n-1] + x[n+1] &= \frac{A}{2} e^{j\varphi} e^{j\omega_0 n} (2 \cos \omega_0) + \frac{A}{2} e^{-j\varphi} e^{-j\omega_0 n} (2 \cos \omega_0) \\ &= (2 \cos \omega_0) x[n]. \end{aligned}$$

$$\begin{aligned} \Rightarrow \cos \omega_0 &= \frac{x[n-1] + x[n+1]}{2x[n]} = \frac{2.4271 + 2.9816}{2(2.9002)} = 0.9325 \\ \Rightarrow \boxed{\omega_0 = 2\pi/17} \end{aligned}$$

$$\text{Let } Z = \frac{A}{2} e^{j\varphi}$$

$$x[0] = Z + Z^* = 2.4271$$

$$x[1] = e^{j2\pi/17} Z + e^{-j2\pi/17} Z^* = 2.9002$$

2 EQUATIONS in 2 UNKNOWNs

$$Z = 1.5 e^{-j\pi/5}$$

$$\boxed{A = 3 \quad \varphi = -\pi/5}$$



PROBLEM 4.19 (more):

SOLVE SIMULTANEOUS EQNS FOR $A \pm \varphi$

$$\text{Let } Z = \frac{1}{2} A e^{j\varphi}$$

$$\begin{bmatrix} 1 & 1 \\ e^{j2\pi/17} & e^{-j2\pi/17} \end{bmatrix} \begin{bmatrix} Z \\ Z^* \end{bmatrix} = \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

Invert the 2×2 matrix: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

$$\Rightarrow \begin{bmatrix} Z \\ Z^* \end{bmatrix} = \frac{1}{e^{-j2\pi/17} - e^{j2\pi/17}} \begin{bmatrix} e^{-j2\pi/17} & -1 \\ -e^{+j2\pi/17} & 1 \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \end{bmatrix}$$

$$Z = \frac{x[0]e^{-j2\pi/17} - x[1]}{-2j \sin 2\pi/17} = \frac{1}{2} A e^{j\varphi}$$

PLUG IN $\begin{cases} x[0] = 2.4271 \\ x[1] = 2.9002 \end{cases}$ AND SIMPLIFY

$$\Rightarrow Z = 1.2135 - j0.8817 = 1.5 e^{-j\pi/5}$$

$$\Rightarrow A = 3 \quad \varphi = -\pi/5$$