Higher Order (Poly)Spectra (HOS) Homework

Advanced Signal Processing

1st assignment

April 2022

Exercise instructions

Consider $X(k)$, given by

$$X(k) = W(k) - W(k-1), k = \pm 1, \pm 2, \ldots,$$

where $\{W(k)\}$ is a stationary stochastic process with independent, identically distributed (i.i.d) stochastic variables and $E\{W(k)\} = 0$, $E\{W^2(k)\} = 1$ and $E\{W^3(k)\} = 1$. The covariance sequence of $\{X(k)\}$ is given by:

$$\begin{aligned} c_2^x(\tau) = m_2^x(\tau) &= E\{X(k)X(k+\tau)\} \\ &= E\{(W(k) - W(k-1))(W(k+\tau) - W(k+\tau-1))\} \\ &= 2\delta(\tau) - \delta(\tau-1) - \delta(\tau+1) \end{aligned}$$

where $\delta(\tau)$ is the delta Kronecker function; hence,

$$c_2^x(\tau) = \begin{cases} 2, & \tau = 0 \\ -1, & \tau = 1,\ \tau = -1. \\ 0, & elsewhere \end{cases}$$

The corresponding Power Spectrum is given by

$$C_2^x(\omega) = \sum_{\tau=-1}^{1} c_2^x(\tau) e^{-j\omega\tau} = (2 - 2cos\omega).$$

1. Find the 3rd-order cumulants of $\{X(k)\}$, i.e., $c_3^x(\tau_1, \tau_2)$.
2. Find the skewness $\gamma_3^x = c_3^x(0,0)$. What do you observe?
3. Find the Bispectrun $C_3^x(\omega_1, \omega_2)$. Is it complex, real or imaginary?
4. How the result of 2 affects the result of 3? Can you draw a general comment?

1. Third order cumulants

We know (and it can be easily calculated) that the 3rd order cumulants of X(k) is the following:

$$c_3^x(\tau_1, \tau_2) = m_3^x(\tau_1, \tau_2) - m_1^x[m_2^x(\tau_1) + m_2^x(\tau_2) + m_2^x(\tau_2 - \tau_1)] + 2(m_1^x)^3$$

But in our case we have that:

$$m_1^x = E[X(k)] = E[W(k) - W(k-1)]E[W(k)] - E[W(k-1)] = 0$$

Therefore,

$$c_3^x(\tau_1, \tau_2) = m_3^x(\tau 1, \tau 2) \tag{1}$$

Hence,

$$
\begin{aligned}
&c_3^x(\tau_1,\tau_2) = E[X(k)X(k+\tau_1)X(k+\tau_2)) = \\
&E[(W(k)-W(k-1)(W(k+\tau_1))-W(k+\tau_1-1))(W(k+\tau_2)-W(k+\tau_2-1))] = \\
&E(W(k)W(k+\tau_1)W(k+\tau_2)-W(k)W(k+\tau_1)W(k+\tau_2-1)-W(k)W(k+\tau_1-1)W(k+\tau_2) \\
&+W(k)W(k+\tau_1-1)W(k+\tau_2-1)-W(k-1)W(k+\tau_1)W(k+\tau_2)+W(k-1)W(k+\tau_1-1)W(k+\tau_2) \\
&+W(k-1)W(k+\tau_1)W(k+\tau_2-1)-W(k-1)W(k+\tau_1-1)W(k+\tau_2-1)] = \\
&E[W(k)W(k+\tau_1)W(k+\tau_2)]-E[W(k)W(k+\tau_1)W(k+\tau_2-1)]-E[W(k)W(k+\tau_1-1)W(k+\tau_2)] \\
&+E[W(k)W(k+\tau_1-1)W(k+\tau_2-1)]-E[W(k-1)W(k+\tau_1)W(k+\tau_2)] \\
&+E[W(k-1)W(k+\tau_1-1)W(k+\tau_2)]+E[W(k-1)W(k+\tau_1)W(k+\tau_2-1)] \\
&-E[W(k-1)W(k+\tau_1-1)W(k+\tau_2-1)]
\end{aligned}
$$

Finally,

$$
\begin{aligned}
c_3^x(\tau_1,\tau_2) = {} & \delta(\tau_1)\delta(\tau_2)-\delta(\tau_1)\delta(\tau_2-1)-\delta(\tau_1-)\delta(\tau_2)+\delta(\tau_1-1)\delta(\tau_2-1)-\delta(\tau_1+1)\delta(\tau_2+1)+\delta(\tau_1)\delta(\tau_2+1) \\
& +\delta(\tau_1+1)\delta(\tau_2)-\delta(\tau_1)\delta(\tau_2)
\end{aligned}
$$

$$
c_3^x(\tau_1,\tau_2) = \begin{cases}
-1, & when \quad (\tau_1,\tau_2)=(0,1) \\
-1, & when \quad (\tau_1,\tau_2)=(1,0) \\
+1, & when \quad (\tau_1,\tau_2)=(1,1) \\
-1, & when \quad (\tau_1,\tau_2)=(-1,-1) \\
+1, & when \quad (\tau_1,\tau_2)=(-1,0) \\
+1, & when \quad (\tau_1,\tau_2)=(0,-1) \\
0, & elsewhere
\end{cases}
$$

2. Skewness

The skewness is:

$$\gamma_3^x = c_3^x(0,0) = 0$$

There is no skewness, the distribution is perfectly symmetrical around the mean.

3. Bispectrum

The Bispectrum is:

$$
\begin{aligned}
C_3^x(\omega_1,\omega_2) &= \sum_{\tau_1=-1}^{\tau_1=+1}\sum_{\tau_2=-1}^{\tau_2=+1} c_3^x(\tau_1,\tau_2)e^{-j(\omega_1\tau_1+\omega_2\tau_2)}, \text{ when } |\omega_1|\le\pi, |\omega_2|\le\pi, |\omega_1+\omega_2|\le\pi \\
&= e^{-j(\omega_1+\omega_2)} - e^{-j(\omega_1+\omega_2)} \\
&+ e^{j\omega_1} + e^{-j\omega_2} - e^{-j\omega_1} - e^{-j\omega_2} \\
&= -2j(-\sin\omega_1) - 2j(-\sin\omega_2) - 2j(\sin(\omega_1+\omega_2)) \\
&= 2j(\sin\omega_1 + \sin\omega_2 - \sin(\omega_1+\omega_2))
\end{aligned}
$$

the above indicates that the Bispectrum is imaginary.

4. General Comments

If $c_3^x(\tau_1, \tau_2) \neq 0$ there would also be a real part in the Bispectrum. As a result, we can deduce that the skewness of X(k) determines the real part of the Bispectrum.