Module 4

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Module 4 Study Guide and Deliverables

Topics:	**Lecture 7:** Enterprise Decision Making under Uncertainty – I **Lecture 8:** Enterprise Decision Making under Uncertainty – II
Readings:	Lectures 7 and 8 online content
Discussions:	**Module 4 Discussion**
Assignments:	**Tutorial:** Optimization libraries in R **Assignment 4**: Individual Assignment covering Lecture 7 and Lecture 8.
Assessments:	**Quiz 4**

Lecture 8

Learning Objectives

After you complete this lecture, you will be familiar with the following:

- The notion of chance constraints
- Value at Risk and Conditional Value at Risk
- Solving an optimization problem involving chance constraints
- The notion of recourse decisions
- Solving an optimization problem involving recourse decisions
- Stochastic programming and robust optimization

- Comparison of simulation optimization, stochastic programming, and robust optimization

Introduction to Chance Constraints

In Lecture 7, we covered simulation optimization as a technique to solve optimization problems involving uncertainty (these problems are generally called **stochastic optimization** problems). The area of stochastic optimization deals with optimization problems under uncertainty with the goal of finding a solution — values for the decision variables in the model — that satisfies the constraints and that maximizes or minimizes the objective in the face of uncertainty.

A feature that distinguishes stochastic optimization problems from conventional optimization problems (i.e., problems that involve no uncertainty) is the presence of **chance constraints**. "If a constraint depends on *uncertain* parameters (variables), we must specify what it means for the constraint to be satisfied. There are many possible realizations for the uncertain variables. The solver must find values for the decision variables that cause the constraint to be satisfied for all, or perhaps most realizations of the uncertainties. This is called a **chance constraint.** For example, we may specify that the constraint be satisfied 95% of the time and can be violated 5% of the time."[1]

"A chance constraint" includes:

- A left-hand side that depends on decision variables and uncertainties.
- A relation that must be either ≤ or ≥. In other words, a chance constraint cannot be an equality.
- A right-hand side that must be constant in the problem. If the right-hand side depends on decision variables or uncertainties, the usual practice is to convert "Left-hand Side ≤ Right-hand Side" into the form "Left-hand Side – Right-hand Side ≤ 0."
- A criterion that may be VaR (Value at Risk) or CVaR (Conditional Value at Risk). These criteria will be discussed shortly.
- A measure that may be a percentile 0.01 – 0.99 for VaR or CVaR."[2]

Example: The following is a simple chance-constrained problem:

Maximize $x_1 + x_2$

Subject to $a_1x_1 + a_2x_2 \leq c$ satisfied 95% of the time

$x_1, x_2 \geq 0$

x_1 and x_2 are decision variables here. a_1 and a_2 are uncertain, and c is a constant. Therefore, the left-hand side of the constraint (i.e., $a_1x_1 + a_2x_2$) depends on decision variables x_1 and x_2 and uncertainties a_1 and a_2. The relation is in the form of ≤. The right-hand side c is a constant. The measure is a percentile 0.95 for the VaR criterion which will be discussed shortly.

Value at Risk Measure

"The form of a chance constraint with the criterion that it must be satisfied for all realizations of the uncertainties up to a given percentile (say 95%) makes it a VaR (Value at Risk) measure. This constraint is represented as

VaR0.95 A1≤B1."[3]

VaR measure is a very popular measure and has been used since the early 1960s. However, it is important to be aware of its following two deficiencies:

- "A VaR constraint with probability 95% requires that the constraint be satisfied 95% of the time. However, it says nothing about the magnitude of violation that may occur 5% of the time. In other words, we do not know how much we are deviating from the constraint 5% of the time."[4]
- "As a measure of risk, VaR criterion is not subadditive. In other words, if two portfolios A and B are constrained not to lose money 95% of the time, we should not expect a combined portfolio A+B to have a 95% or better chance of not losing money since this is not guaranteed with VaR constraints."[5]

An alternative risk measure that does not suffer from these deficiencies is called CVaR (Conditional Value at Risk).

Conditional Value at Risk Measure

"$VaR_{0.95}$ A1≤B1 specifies that the 95th percentile of the realizations of A1 must be less than or equal to B1. The realizations beyond the 95th percentile may be greater than B1 by any amount. On the contrary, $CVaR_{0.95}$ A1≤B1 specifies that the *expected value* of **all** the realizations of A1 up to the 95th percentile must be less than or equal to B1."[6]

"Unlike VaR, a CVaR constraint bounds the average magnitude of violations that may occur 95% of the time. Also, a CVaR constraint is subadditive; e.g., if two portfolios A and B are CVaR constrained to not lose money 95% of the time, the combined portfolio A+B has the same or better chance of not losing money."[7]

Despite the superiority of CVaR constraint over VaR constraint, it is still the VaR constraint that is widely used in practice.

A Model with Chance Constraints

In this section, we will work with the following problem.

A gas company buys gas and then sells it to its consumers. The consumer demand this year is expected to be 100 units and the price is $5 per unit. Next year's gas price and demand will depend on the weather. See the chart below for the price and demand for next year's two possible scenarios. If the weather is very cold, demand may reach 180 units and the price may reach $7.50/unit. But if the weather is warm, demand may remain at 100 units and the price may remain at $5/unit. The company must meet its demand in the first year (normal constraint) and must meet the demand in the second year with 95% probability (chance constraint), or it will encounter public relations and regulatory problems.

Weather	Price Year 2	Demand Year 2
Very Cold	$7.50	180
Warm	$5.00	100

The gas company can buy gas this year at the current price of $5/unit and store it for use next year. If gas is stored there will be an associated holding storage cost of $1/unit per year.

The company's goal is to minimize its total cost of purchased and stored gas. The company should make the following decisions:

- How much gas to purchase and resell to consumers *this year*?
- How much gas to purchase *this year* and store, at a cost of $1/unit, for use *next year*?
- How much to purchase and resell to consumers *next year*?

Source: Frontline Solvers Optimization and Simulation User Guide, Chapter "Examples: Stochastic Optimization," p. 135.

In this section, "we assume that the gas company must commit to all of its gas purchases this year, *before* next year's demand is known."[8] In the next section, we will discuss the case where the company can make decisions on a "wait-and-see" basis after the uncertainty of the weather is resolved.

- Gas Price (year 1) is given in the description of the problem.
- The demand and price are perfectly correlated since both depend on the weather.
  To model the uncertainty in the demand we will use a beta distribution and we will utilize this beta distribution in the calculation of both the demand and gas price in year 2.
- The parameters of the beta distribution are obtained from the reference material, but we encourage you to play with the distributions.

Let us parse the problem. We are asked to optimize the cost of ordering with the uncertainty of demand and pricing. Further, we are expected to meet the demand in year 2 95% of the time. The uncertainty is in how much we are going to order to cover the demand for year 2 and when. This will depend on another uncertain parameter, the price in year 2. We can order more in year 1 at a price we know is the cheapest possible price and stock the inventory. But this will incur storage costs. We could wait until year 2 and then buy, hoping that next year's price might be the same and we can avoid storage costs, but this is uncertain.

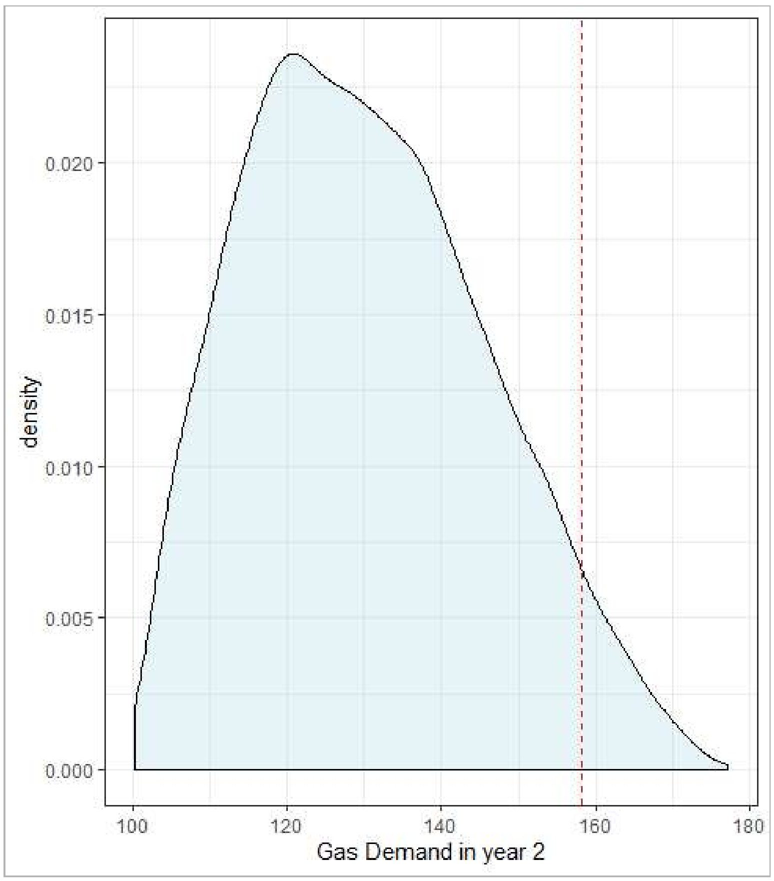
Building the optimization model

Adding Chance Constraint

We are asked to meet the demand in year 2 95% of the time. But since we won't know what the actual demand is going to be in year 2, the best approach to this problem is to simulate the demand in year 2 and take the 95th percentile of the demand as our order quantity. This is not in the strict sense of implementation a chance constraint.

```
gas_demand_2 <- c()
for (k in 1:10000){
epsilon <- rbeta(1,2,3.25)
gas_demand_2[k] <-demand_yr2_sim(epsilon)
}
quantile(gas_demand_2,0.95)
```

Figure 8.1: Simulated Demand in Year 2



Now let us simulate the optimization routine to find out what the optimal strategy is that minimizes cost.

```
opt_gas_order <- c()
opt_cost <- c()
gas_purchase <- c()
for (k in 1:100){
cost_fn <- function(gas_purchase) {
if(gas_purchase<demand_yr1) return(Inf)
gas_in_year2 <- max((quantile(gas_demand_2,0.95)[[1]]
- (gas_purchase-demand_yr1)),0)

cost <- (gas_purchase-demand_yr1)*storage_cost +
      gas_in_year2*gas_unit_price_yr2_sim(rbeta(1,2,3.2
      5))+
      gas_purchase*gas_unit_price_yr1
if (cost < 0)
      return (Inf)
return(cost)
}
temp <- DEoptim(cost_fn,lower= 100 , upper= 300)
opt_gas_order[k] <- temp$optim$bestmem['par1']
opt_cost[k] <- temp$optim$bestval
}
```

Results of the Simulation Model

Figure 8.2: Order Quantity in Yr1 (95%)

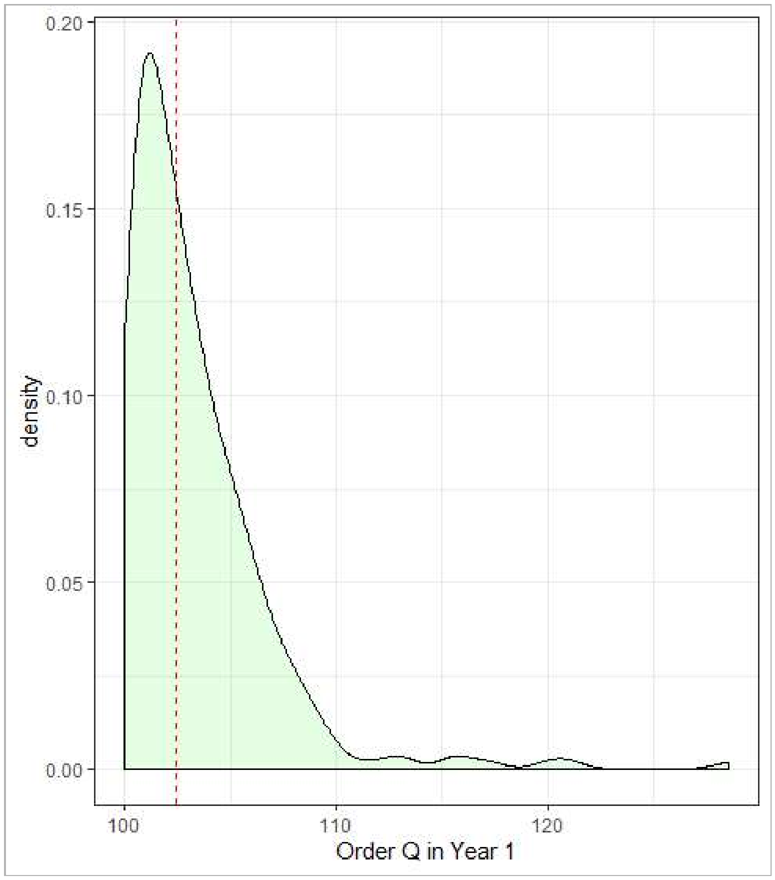
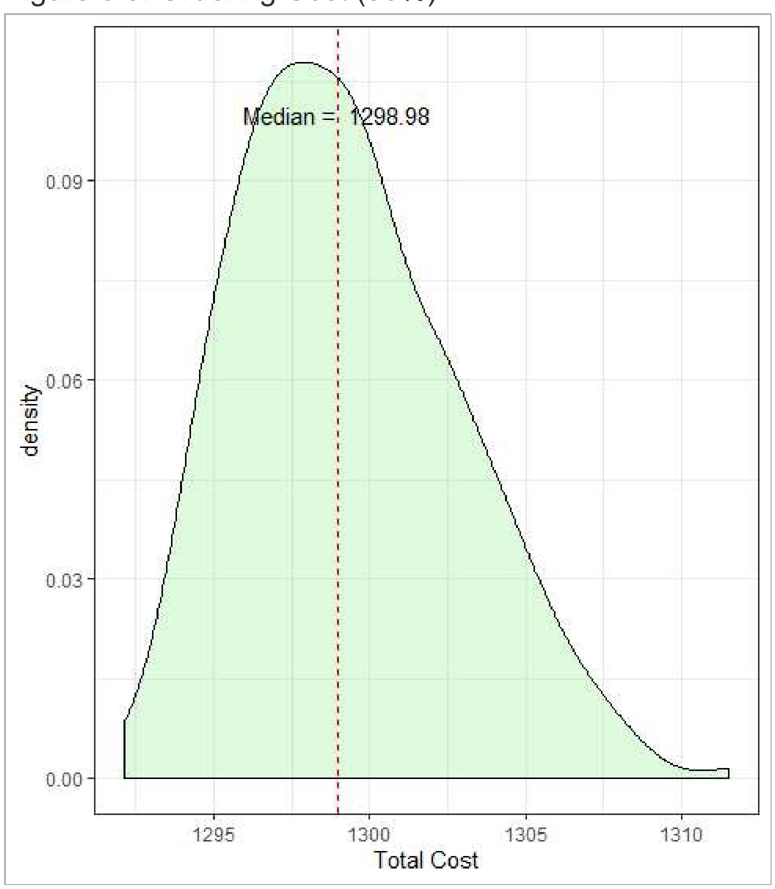
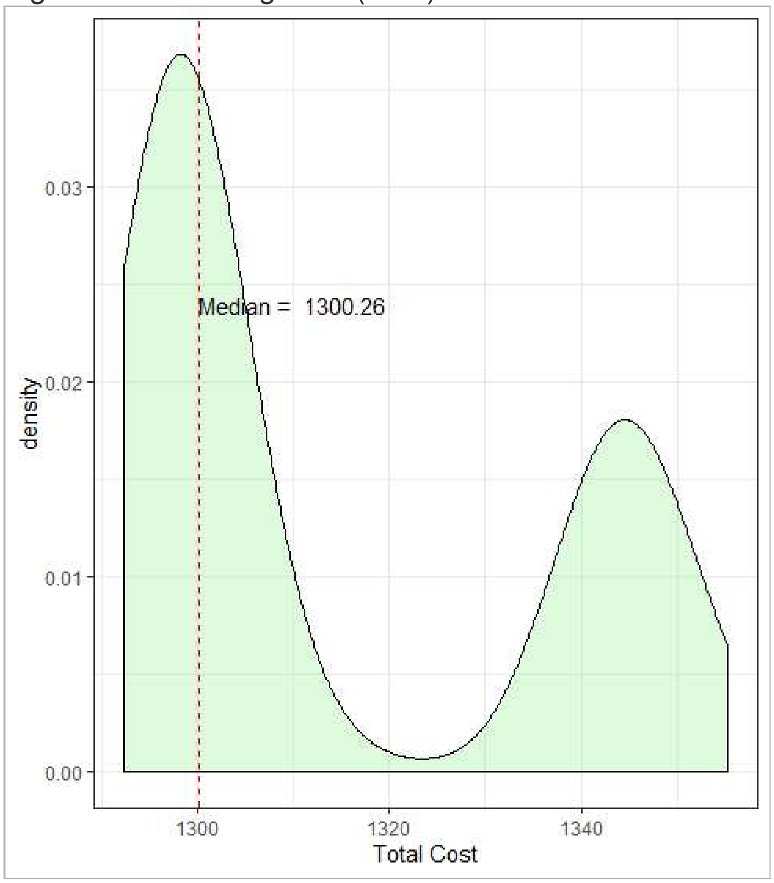


Figure 8.3: Ordering Cost (95%)



The graphs in Figures 8.2 and 8.3 show the ordering quantity in year 1 and the resulting total cost. The cost to meet the demand at 95% is about $1284 and we order close to 100 units in year 1 and the rest in year 2. In Figure 8.4 we did the same optimization but now at 99% guarantee, and the cost is now $1300, but we also notice a second peak on the right centered around $1350. Which implies that there is a likelihood that we might end up spending ~$1350 in order to serve the 99% level.

Figure 8.4: Ordering Cost (99%)



A Model with Recourse Decisions

In the gas company problem discussed above, we assumed that the gas company must commit to all of its gas purchases this year, *before* next year's demand is known. This means that we have to make a decision in the face of uncertainty. "On the other hand, if the business situation allows us to make decisions on a 'wait and see' basis after the uncertainty in the weather is resolved, we can improve the solution by reducing the total expected cost significantly. This type of decision, which can be made on a 'wait and see' basis, after the uncertainty is resolved is called a 'recourse decision.'"[9]

In some cases, the decisions need to be made on a "here and now" basis, i.e., we have the make decisions for the future in the face of uncertainty today. In other cases, we cannot make decisions on a "wait and see" basis. "If the business situation allows you to make decisions on a wait and see basis, then it is critical to incorporate this into your model. Otherwise, you may find an optimal solution to the wrong problem."[10]

In our gas company example, let us assume that the "gas supplier is willing to accept our purchase order next year and not require us to commit to a specific amount of gas this year for delivery next year."[11]

We are now trying to accomplish this without using any mathematical notation, but only because this is a trivial case of the recourse problem, mathematically known as a multistage (two, in this case) stochastic programming problem. This is just to give you an overview of how you might approach such a problem.

The recourse decision variable allows us to do the following: Once the uncertainty in weather is resolved and the consumer demand is known, the company will purchase just the amount of gas needed to meet the demand.

Source: Frontline Solvers Optimization and Simulation User Guide, Chapter "Examples: Stochastic Optimization," p. 140.

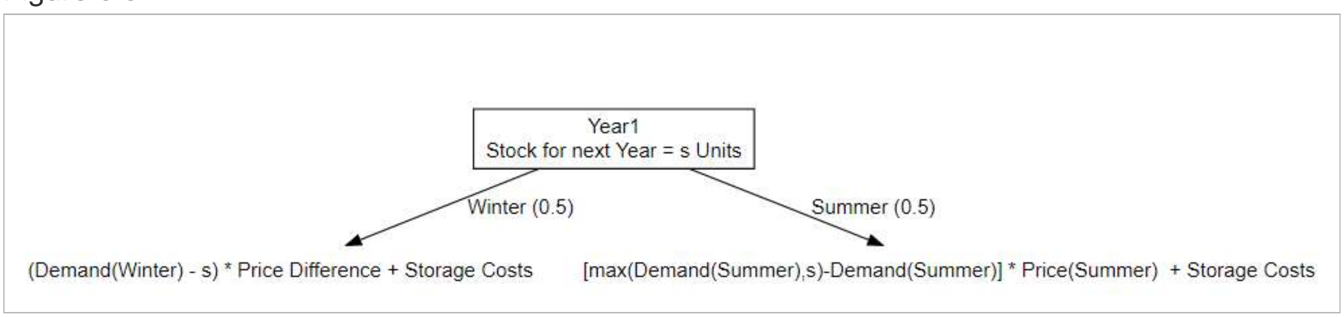
Now, how do we solve this problem?

Solving the problem with recourse constraints

So far, we have used simulation optimization to solve an optimization problem involving uncertainty. Simulation optimization is a very popular method; however, one drawback of the method is that ***it cannot solve problems with recourse decisions.***

Assume for the time being both summer or winter are equally likely, our decision chart is going to look like this:

Figure 8.5



There are two types of costs associated with this decision. The underage cost, or the penalty of not buying when the stock is cheap and the demand turns out to be higher than our stock quantity (note: this also includes the scenario where we stock 0), and overage costs where we stock too much. If both the problems are equally probable we can calculate the expected demand

0.5*(Winter Scenario) + 0.5(Summer Scenario)

However, in this case the scenarios are a continuous set of probabilities resulting in a large set of outcomes. Luckily for us we can rephrase this as a linear optimization problem wherein we just have to minimize the cost, considering we are going to make an optimal decision once we reach the end of year 1 (since we will know the demand exactly). To solve this problem, we will just make a few changes to the solution we arrived at above. First of all, if we waited the until the end of year 1 we will know the demand in the year 2 exactly. So there won’t be any wastage. This is typical of a recourse problem.

Let us model our objective function with a parameter ‘stock’ that tells us how much to stock for next year. We will now optimize for the penalty of understocking and overstocking with simulation optimization.

```
costs <- c()
opt_gas_order <- c()
opt_cost <- c()
for (k in 1:100){
costs_yr2 <- function(stock){
epsilon <- rbeta(1,2,3.25)
return(stock*gas_unit_price_yr1 + stock*storage_costs + max(demand_yr2_sim(epsilon)-stock,0)*gas_unit_price_yr2_sim(epsilon))
}
temp <- DEoptim(costs_yr2,lower=0,upper=190)
opt_gas_order[k] <- list(temp$optim$bestmem)[[1]]
opt_cost[k] <- temp$optim$bestval + 500 #Adding the 1st year costs
}
```

Results of Optimization

Figure 8.6: Order Quantity in Year 2

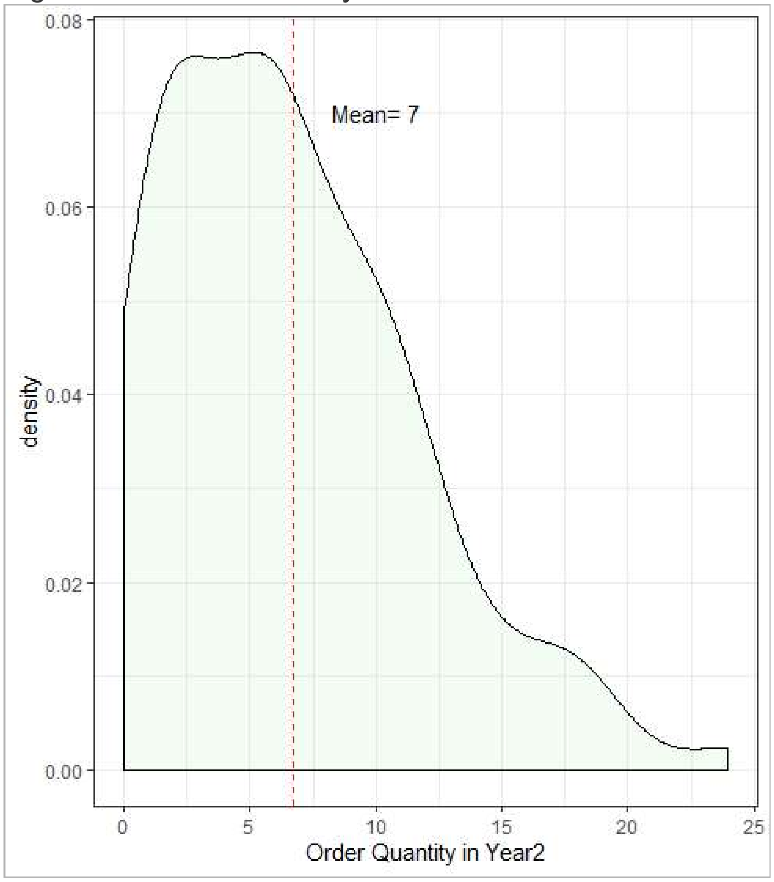
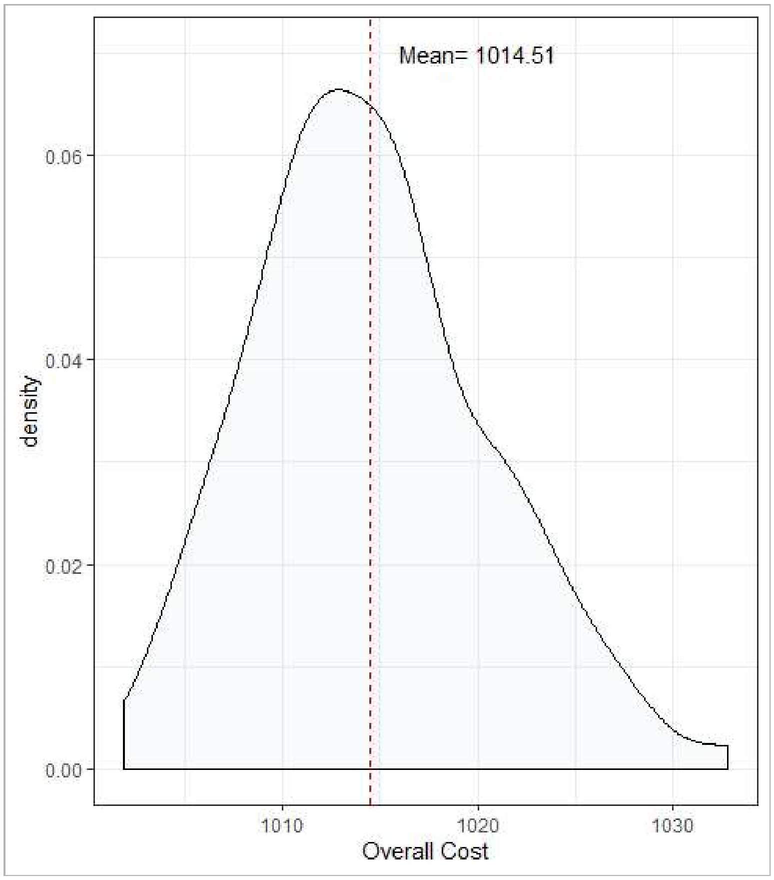


Figure 8.7: Overall Cost after optimization



From the results we note that the overall expected cost is about $1014.5, which is lower than our earlier estimate without the recourse.

Comparison Among Stochastic Optimization Techniques

Simulation optimization is the most widely used method in practice mostly because of its availability in all software products such as Crystal Ball Professional and @RISK Industrial. Furthermore, it is the only method that can deal with the problems where uncertainties may be dependent on the decisions. However, it is important to note that as discussed in the previous section, simulation optimization does not have the concept of recourse decisions. Furthermore,

- "It takes time and it is not scalable to large models.
- It yields only "improved" solutions, not optimal solutions."[12]

The methods of stochastic programming and robust optimization, on the other hand, can be applied only to linear and quadratic programming models with uncertainty and they cannot represent decision-dependent uncertainties. However, they are scalable to larger models, yield optimal solutions and can solve models involving recourse decisions and recourse constraints.

- If your problem involves recourse decisions, then use stochastic programming or robust optimization.
- If the uncertainty you are modeling is dependent on the decision variables (decision-dependent uncertainties), i.e., they change if the decisions change, then use simulation optimization.
- If your model is linear (i.e., "the objective function and the constraints can be written as a sum of terms, where each term consists of one decision variable multiplied by a positive or negative constant"[13]) or quadratic (i.e., the "objective function and/or the constraints is a sum of terms, where each term is a positive or negative constant multiplied by a variable or the product of two variables"[14]) and it is a large model, use robust optimization or stochastic programming.
- If your problem includes chance constraints, you can use robust optimization or simulation optimization to solve this model.

In deciding what method to use to solve a particular stochastic programming problem, one needs to look at several factors together. For instance, if your model has decision dependent uncertainties and chance constraints, despite the fact that we can use any method to solve a chance-constrained problem, the presence of decision-dependent uncertainties forces us to use simulation optimization to solve this model.

Lecture 8 Footnotes

[1] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 342.

[2] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 345.

[3] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 345.

[4] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 345.

[5] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," pp. 345–346.

[6] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 346.

[7] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 346.

[8] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 342.

[9] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 342.

[10] Frontline Solvers Optimization and Simulation User Guide, Chapter "Examples: Stochastic Optimization," p. 139.

[11] Frontline Solvers Optimization and Simulation User Guide, chapter "Examples: Stochastic Optimization", p. 139.

[12] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p.

348.

[13] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Conventional Optimization Concepts," p. 287.

[14] Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Conventional Optimization Concepts," p. 287.

Lecture 8 References

Adapted from Frontline Solvers Optimization and Simulation User Guide, Chapter "Examples: Stochastic Optimization," p. 136.
Adapted from Frontline Help, Examples, Stochastic Optimization Examples, Gas Company Example with Chance Constraints.
Frontline Solvers Optimization and Simulation User Guide, Chapter "Mastering Stochastic Optimization Concepts," p. 342.
Adapted from Frontline Help, Examples, Stochastic Optimization Examples, Gas Company Example with Chance Constraints.
Frontline Solvers Optimization and Simulation User Guide, chapter "Mastering Stochastic Optimization Concepts."
Adapted from Frontline Help, Examples, Stochastic Optimization Examples, Gas Company Example with Chance Constraints.
Frontline Solvers Optimization and Simulation User Guide, chapter "Mastering Conventional Optimization Concepts."

Lecture 8 Resources

The spreadsheet for the gas company problem with chance constraints.
The spreadsheet for the gas company problem with recourse constraints.

Lecture 8 Summary Questions

1. What is a chance constraint? What is the importance of it in a business setting?
2. What is a recourse decision? Why is it important to consider recourse variables (if any) in the decision-making process?
3. What are the techniques used to solve optimization problems involving uncertainty?
4. What are the pros and cons of “simulation optimization” as a technique used to solve optimization problems involving uncertainty?

5. Please comment on the following statement: *"One can use simulation optimization to solve a problem involving a recourse constraint."*

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