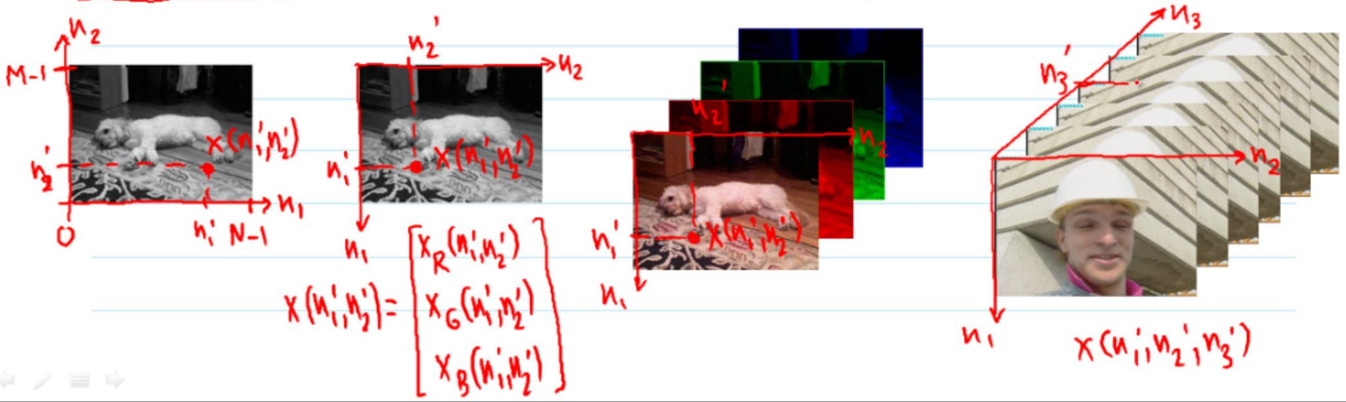
2D and 3D Discrete Signals

$$x(n_1,n_2),\quad -\infty < n_1, n_2 < \infty,\quad n_1, n_2 = 0, \pm 1, \pm 2, \ldots$$

$$x(n_1,n_2,n_3),\quad -\infty < n_1, n_2, n_3 < \infty,\quad n_1, n_2, n_3 = 0, \pm 1, \pm 2, \ldots$$

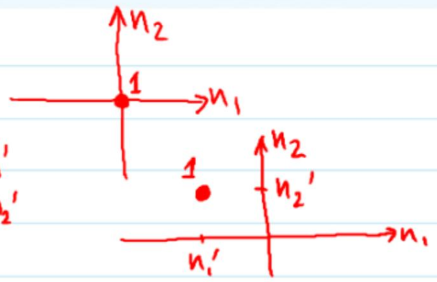


$$x(n_1',n_2') = \begin{bmatrix} x_R(n_1',n_2') \\ x_G(n_1',n_2') \\ x_B(n_1',n_2') \end{bmatrix}$$

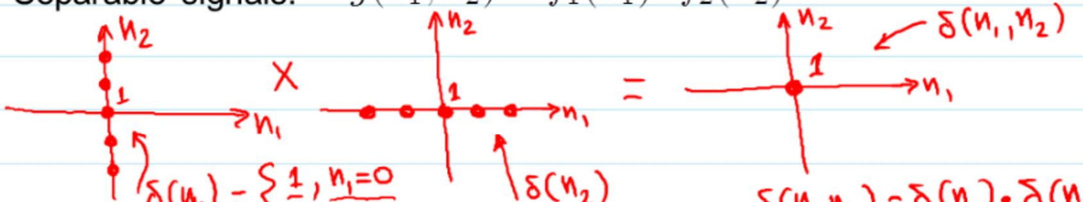
Discrete Unit Impulse

$$\delta(n_1,n_2) = \begin{cases} 1 & \text{for } n_1 = n_2 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\delta(n_1-n_1', n_2-n_2') = \begin{cases} 1, & n_1-n_1'=0 \Rightarrow n_1=n_1' \\ & n_2-n_2'=0 \Rightarrow n_2=n_2' \\ 0 & \text{otherwise} \end{cases}$$



Separable signals: $g(n_1,n_2) = f_1(n_1) \cdot f_2(n_2)$

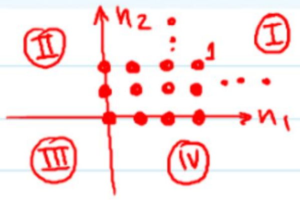


$$\delta(n_1) = \begin{cases} 1, & n_1 = 0 \\ 0, & \text{otherwise} \end{cases}$$

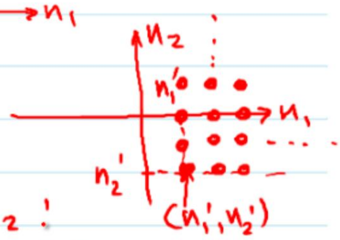
$$\delta(n_1,n_2) = \delta(n_1) \cdot \delta(n_2)$$

Discrete Unit Step

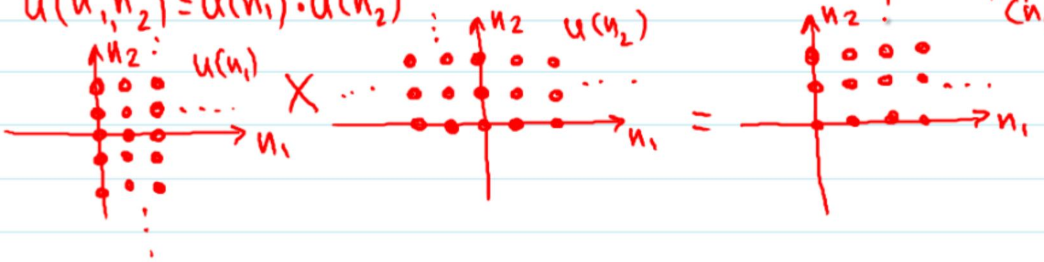
$$u(n_1, n_2) = \begin{cases} 1 & \text{for } n_1 \geq 0,\ n_2 \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$u(n_1-n_1', n_2-n_2') = \begin{cases} 1, & n_1-n_1' \geq 0 \Rightarrow n_1 \geq n_1' \\ & n_2-n_2' \geq 0 \Rightarrow n_2 \geq n_2' \\ 0 & \text{otherwise} \end{cases}$$



$$u(n_1, n_2) = u(n_1) \cdot u(n_2)$$



Exponential Sequences

$$x(n_1, n_2) = e^{j\omega_1 n_1} \cdot e^{j\omega_2 n_2}$$

Eigen-functions of LSI systems $\quad e^{j\omega_1 n_1 + j\omega_2 n_2} \longrightarrow \boxed{LSI} \longrightarrow A \cdot e^{j\omega_1 n_1 + j\omega_2 n_2}\ e^{j\phi}$

"Building Blocks" of any signal

Euler's formula

$$x(n_1, n_2) = e^{j\omega_1 n_1} \cdot e^{j\omega_2 n_2} = cos(\omega_1 n_1 + \omega_2 n_2) + jsin(\omega_1 n_1 + \omega_2 n_2)$$

$$|e^{j\omega_1 n_1}| = |e^{j\omega_2 n_2}| = 1$$

- Periodicity w.r.t. ω_1, ω_2

$$e^{j(\omega_1+2\pi)n_1}\ e^{j(\omega_2+2\pi)n_2} = e^{j\omega_1 n_1} \cdot \cancel{e^{j2\pi n_1}}^{1} \cdot e^{j\omega_2 n_2} \cdot \cancel{e^{j2\pi n_2}}^{1} = e^{j\omega_1 n_1} e^{j\omega_2 n_2}$$

Exponential Sequences

Periodicity of $e^{j\omega_1 n_1} \cdot e^{j\omega_2 n_2}$ w.r.t. n_1, n_2

If periodic w/ periods N_1, N_2

$$e^{j\omega_1(n_1+N_1)} \cdot e^{j\omega_2(n_2+N_2)} = e^{j\omega_1 n_1} \cdot e^{j\omega_2 n_2}$$

$$\Rightarrow \cancel{e^{j\omega_1 n_1}} \cdot e^{j\omega_1 N_1} \cdot \cancel{e^{j\omega_2 n_2}} \cdot e^{j\omega_2 N_2} = \cancel{e^{j\omega_1 n_1}} \cdot \cancel{e^{j\omega_2 n_2}}$$

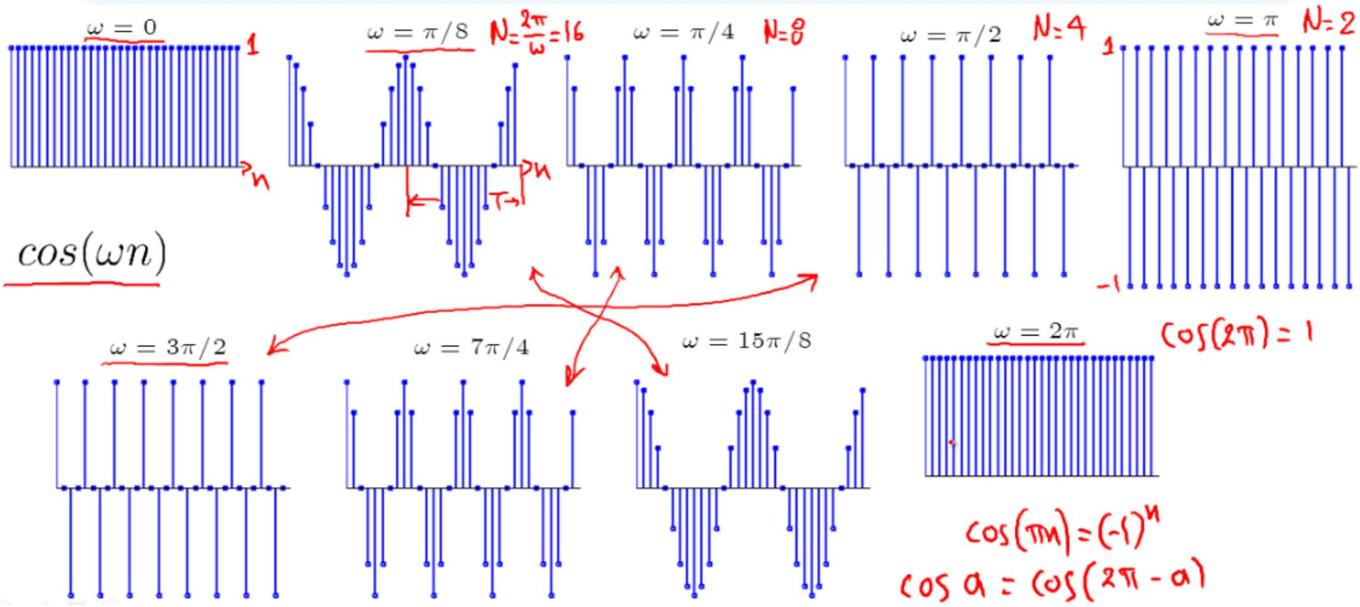
$$\Rightarrow e^{j\omega_1 N_1} \cdot e^{j\omega_2 N_2} = 1$$

$$\Rightarrow \omega_1 N_1 = k_1 \cdot 2\pi \Rightarrow \underline{N_1} = \underline{k_1} \frac{2\pi}{\omega_1} \rightarrow \text{rational number } \frac{P}{q} \rightarrow \text{integers}$$

$$\Rightarrow \omega_2 N_2 = k_2 \cdot 2\pi \Rightarrow N_2 = k_2 \cdot \frac{2\pi}{\omega_2} \rightarrow \text{" " } P/q$$

$e^{j3n_1} \cdot e^{j4n_2}$ $\quad \omega_1 = 3$, $\omega_2 = 4$ $\quad \frac{2\pi}{\omega_1} = \frac{2\pi}{3}$ Not a rational number

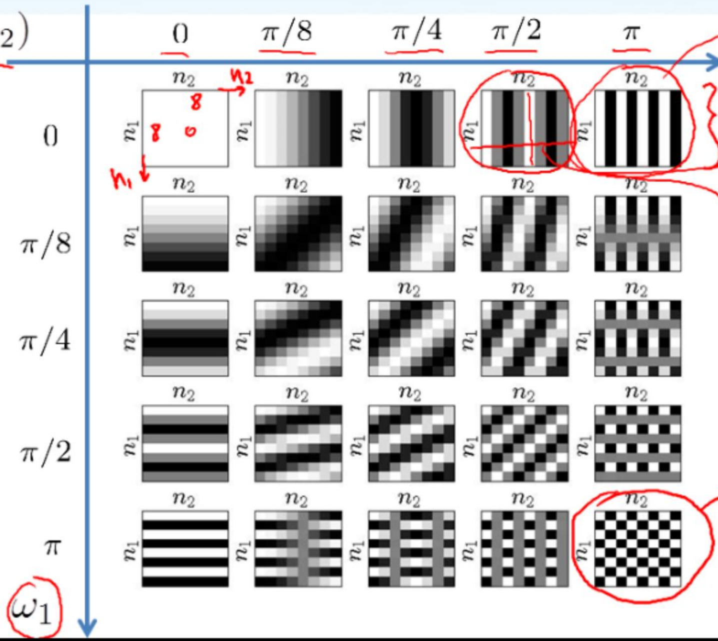
1D Discrete Cosine



2D Discrete Cosine

$\cos(\omega_1 n_1 + \omega_2 n_2)$

white → 1
black → -1
grey → 0



$\cos(\pi n_2) = (-1)^{n_2}$

$\cos(0 n_1 + \omega_2 n_2) = \cos(\omega_2 n_2)$

$\cos(\pi/2\, n_2)$

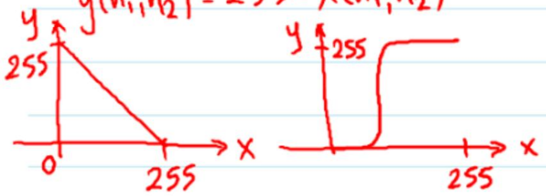
$N_2 = \frac{2\pi}{\pi/2} = 4$

$\cos(\pi n_1 + \pi n_2) = (-1)^{n_1}(-1)^{n_2}$

2D Systems

$$x(n_1, n_2) \rightarrow \boxed{\mathbf{T}[\cdot]} \rightarrow y(n_1, n_2) = \mathbf{T}[x(n_1, n_2)]$$

$y(n_1, n_2) = 255 - x(n_1, n_2)$



$y(n_1, n_2) = \text{average}(N(x(n_1, n_2)))$ ← Neighborhood

$y(n_1, n_2) = \text{median}(N(x(n_1, n_2)))$

System Properties

- Stability
- With/out Memory
- Causality
- Linearity
- Spatial Invariance

Linear Systems

If

$$\mathbf{T}[\alpha_1 x_1(n_1,n_2) + \alpha_2 x_2(n_1,n_2)] = \alpha_1 \mathbf{T}[x_1(n_1,n_2)] + \alpha_2 \mathbf{T}[x_2(n_1,n_2)]$$

then $\mathbf{T}[\cdot]$ *is linear*

- $\alpha_2 = 0 \quad T[\alpha_1 x_1(n_1,n_2)] = \alpha_1 T[x_1(n_1,n_2)]$
- $\alpha_1 = -\alpha_2 \;,\; x_1(n_1,n_2) = x_2(n_1,n_2)$

$$T[\alpha_1 x_1(n_1,n_2) - \alpha_1 x_1(n_1,n_2)] = T[0] = \alpha_1 T[x_1(n_1,n_2)] - \alpha_1 T[x_1(n_1,n_2)] = 0$$

$$T[0] = 0$$

- $y(n_1,n_2) = T[x(n_1,n_2)] = 255 - x(n_1,n_2)$

$T[0] = 255 \neq 0 \Rightarrow T[\cdot]$ is non-linear

Spatially Invariant Systems

$$\mathbf{T}[x(n_1,n_2)] = y(n_1,n_2)$$

If $\mathbf{T}[x(n_1-k_1, n_2-k_2)] = y(n_1-k_1, n_2-k_2)$

then $\mathbf{T}[\cdot]$ *is spatially invariant*

- $y(n_1,n_2) = T[x(n_1,n_2)] = 255 - x(n_1,n_2)$ $\left\{ \begin{array}{l} \text{non-linear} \\ \text{SI} \end{array} \right.$

$$T[x(n_1-k_1, n_2-k_2)] = 255 - x(n_1-k_1, n_2-k_2) = y(n_1-k_1, n_2-k_2).$$

- $y(n_1,n_2) = \underbrace{c(n_1,n_2)}_{\text{gain}} \cdot x(n_1,n_2)$ → Linear; → Not spatially-invariant or spatially-varying

Linear and Spatially Invariant (LSI) Systems

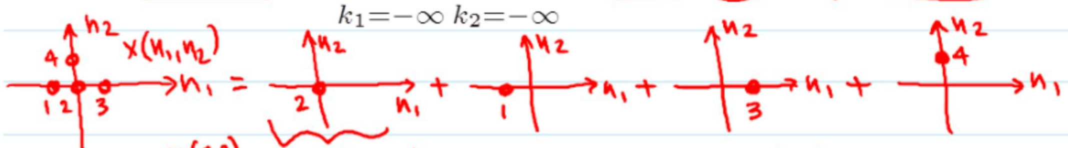
$$\delta(n_1,n_2) \longrightarrow \boxed{\textbf{LSI}} \longrightarrow h(n_1,n_2) \triangleq \text{impulse response of the LSI system}$$

$$x(n_1,n_2) \longrightarrow \boxed{\mathbf{h(n_1,n_2)}} \longrightarrow y(n_1,n_2) = x(n_1,n_2) \underbrace{**}_{\text{2D discrete convolution}} h(n_1,n_2)$$

$$y(n_1,n_2) = x(n_1,n_2) ** h(n_1,n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\, h(n_1-k_1, n_2-k_2)$$
$$= h(n_1,n_2) ** x(n_1,n_2)$$

Derivation of Convolution

$$\underline{x(n_1,n_2)} = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \underline{x(k_1,k_2)}\underbrace{\delta(n_1-k_1,n_2-k_2)}$$



$$\underbrace{x(0,0)}\rightarrow 2\cdot\delta(n_1,n_2) + x(-1,0)\,\delta(n_1+1,n_2) + x(1,0)\,\delta(n_1-1,n_2) + x(0,1)\,\delta(n_1,n_2-1)$$

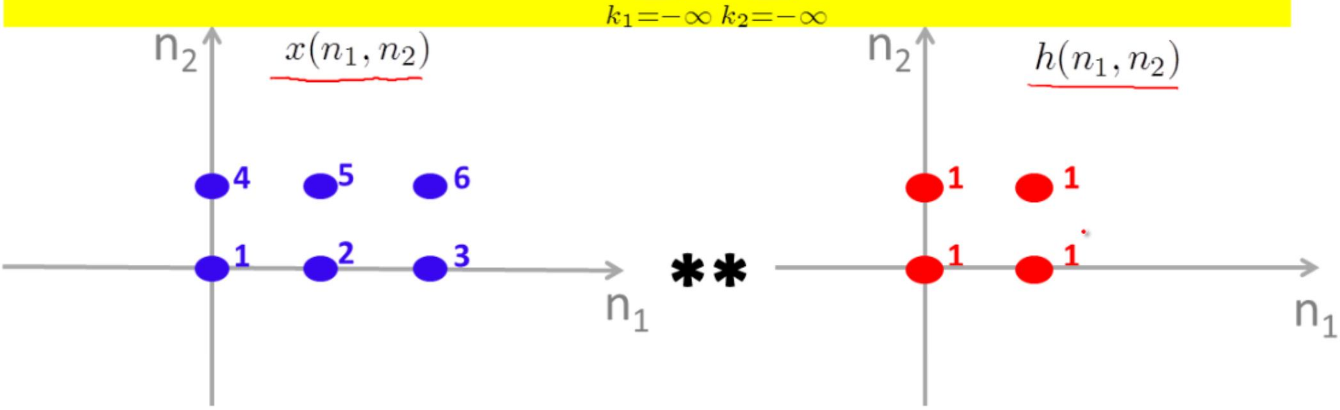
$$y(n_1,n_2) = T[x(n_1,n_2)] = T\Big[\sum_{k_1}\sum_{k_2} \overbrace{x(k_1,k_2)}^{\text{weight}}\underbrace{\delta(n_1-k_1,n_2-k_2)}\Big]$$

linearity
$$\overset{\sim}{=} \sum_{k_1}\sum_{k_2} x(k_1,k_2)\, T[\delta(n_1-k_1,n_2-k_2)]$$

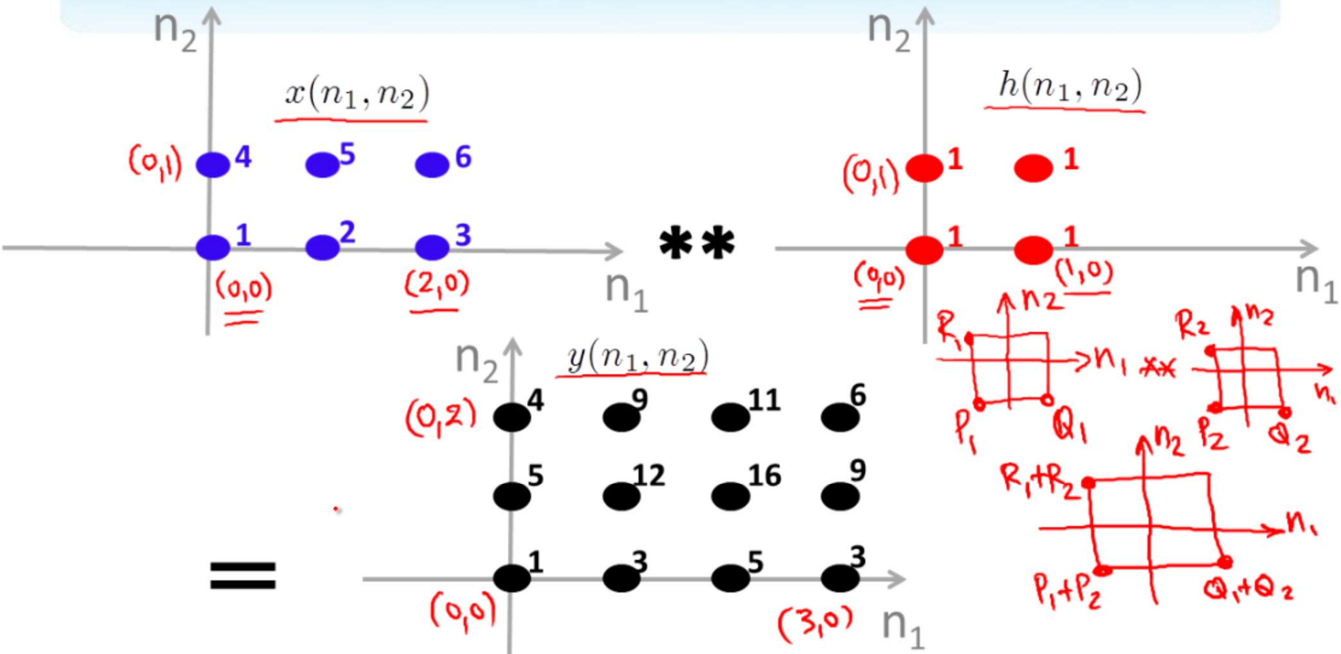
SI
$$\overset{\sim}{=} \sum_{k_1}\sum_{k_2} x(k_1,k_2)\cdot h(n_1-k_1,n_2-k_2) = x(n_1,n_2) ** h(n_1,n_2)$$
$$= h(n_1,n_2) ** x(n_1,n_2)$$

2D Convolution Example

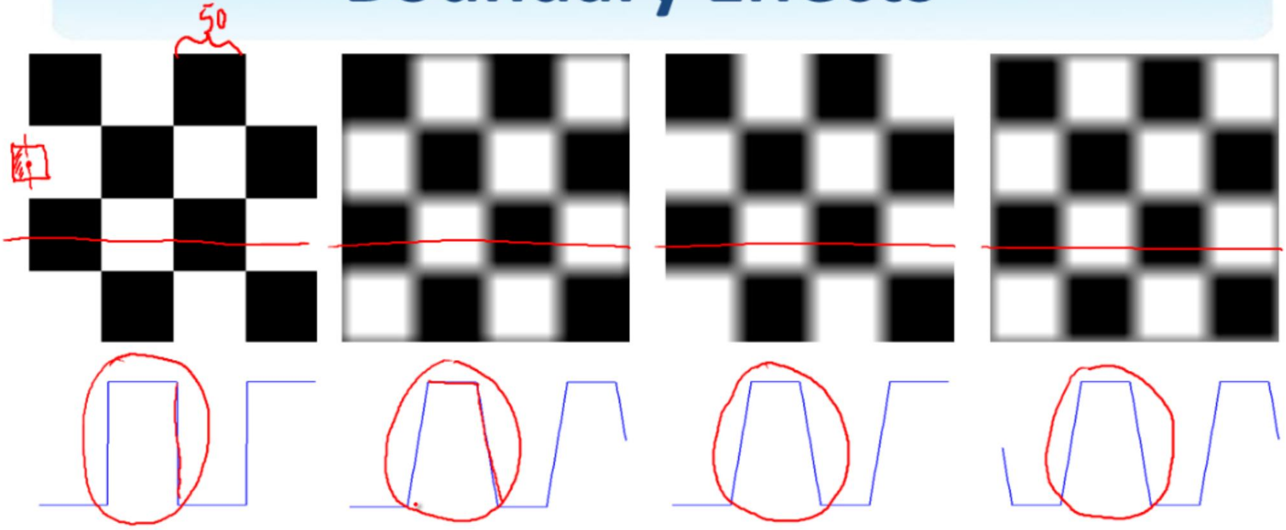
$$y(n_1,n_2) = x(n_1,n_2) ** h(n_1,n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1,k_2)h(n_1-k_1, n_2-k_2)$$



2D Convolution Example



Boundary Effects



A 15x15 symmetric flat filter is used (amplitude of each sample = 1/255)

Spatial Filtering (LPF)

$$\begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

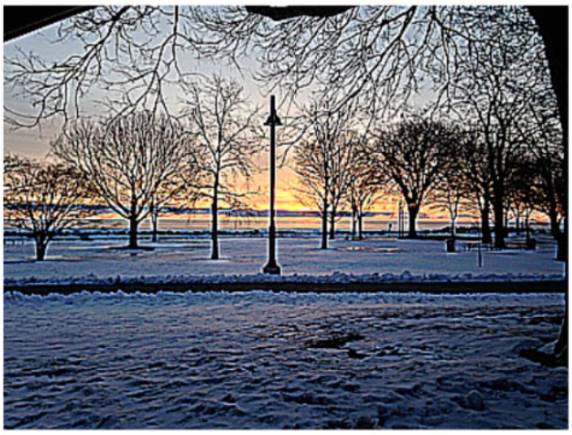
h(0,0)

Spatial Filtering (LPF)

Original

Flat 5x5 LPF

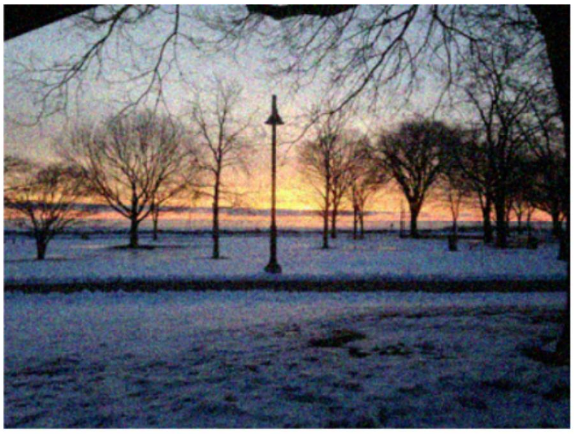
Spatial Filtering (HPF)



$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & 9 & -1 \\ -1 & -1 & -1 \end{bmatrix}$$

h(0,0)

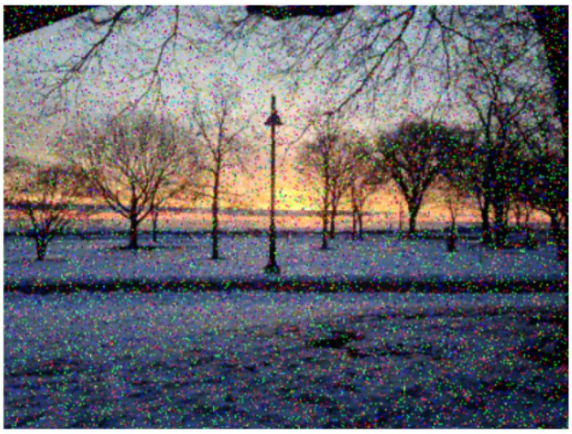
Noise Reduction



$$\begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

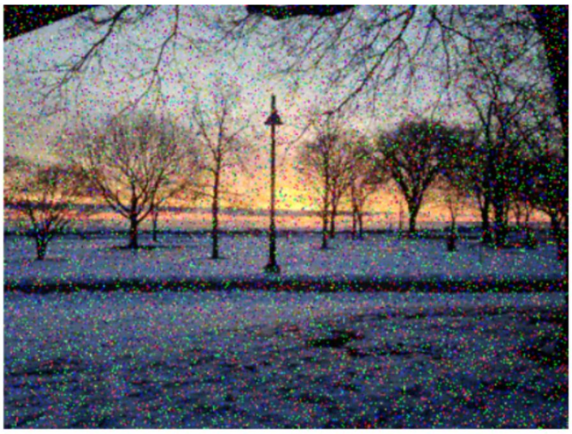
h(0,0)

Noise Reduction



$$\begin{bmatrix} 0.075 & 0.124 & 0.075 \\ 0.124 & 0.204 & 0.124 \\ 0.075 & 0.124 & 0.075 \end{bmatrix}$$

Noise Reduction



Median Filtering