ITP20002 Discrete Mathematics

The Foundation Part 1

Propositional Logic

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TODAY'S QUESTION

- **What is proposition?** (statement, 명제) true or false
- **What is propositional logic?**
- $(p \wedge q) \rightarrow r \equiv \neg(\neg p \vee \neg q) \vee r$

- **A proposition is a declarative sentence that is either true or false.**

  - Practice
    - The Moon is made of green cheese.
    - Trenton is the capital of New Jersey.
    - Toronto is the capital of Canada.
    - 1 + 0 = 1
    - 0 + 0 = 2
    - Sit down! ✗
    - What time is it? ✗
    - x + 1 = 2
    - x + y = z

Constructing Propositions

- The proposition that is always true is denoted by T and the proposition that is always false is denoted by F.
- Compound Propositions; constructed from logical connectives and other propositions
  - Negation ¬
  - Conjunction ∧
  - Disjunction ∨
  - Implication →
  - Biconditional ↔

Negation

- Ex)
  p
  If p denotes "The earth is round.",
  then $\neg p$ denotes "It is not the case that the earth is round,"
  or more simply "The earth is not round."

$\neg p$

p	$\neg p$
T	F
F	T

❖ **Conjunction**

✓ Ex) 결합

p q

If p denotes "I am at home." and q denotes "It is raining."
then p ∧ q denotes "I am at home **and** it is raining."

and operation

p	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

- Ex)
  If p denotes "I am at home." and q denotes "It is raining."
  then $p \lor q$ denotes "I am at home or it is raining."

OR operation

p	q	$p \lor q$
T	T	T
T	F	T
F	T	T
F	F	F

❖ Exclusive Or (Xor)

✓ In p ⊕ q , one of p and q must be true, but not both

두 값이 다르면 T

p	q	$P \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

❖ Implication (or conditional statement)

- ✓ Ex)
  If p denotes "I am at home." and q denotes "It is raining."
  then $p \rightarrow q$ denotes "If I am at home then it is raining."
- ✓ In $p \rightarrow q$, p is the *hypothesis* (*antecedent* or *premise*) and q is the *conclusion* (or *consequence*).

p	q	$P \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Propositional Logic

- **Implication (or conditional statement)**
  - ✓ "If I am elected, then I will lower taxes."
    p q
    - What if the politician who is elected does not lower texes?
    - p is true and q is false
  - ✓ "If you get 100% on the final, then you will get an A."
    p q

elected promise

p	q	$P \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Implication (or conditional statement)

- Different Ways of Expressing $p \to q$

if p**, then** q

if p, q

q **unless** $\neg p$

q **if** p

q **whenever** p

q **follows from** p

p **implies** q

p **only if** q

q **when** p

p **is sufficient for** q

q **is necessary for** p

a necessary condition for p **is** q

필요

a sufficient condition for q **is** p

충분

Implication (or conditional statement)

- Converse(역), Inverse(이), and Contrapositive(대우)
  - $q \to p$ is the **converse** of $p \to q$ 역
  - $\neg p \to \neg q$ is the **inverse** of $p \to q$ 이
  - $\neg q \to \neg p$ is the **contrapositive** of $p \to q$ 대우
- Ex)

  "Raining is a sufficient condition for me not to go to town."
  - **converse**: If I do not go to town, then it is raining.
  - **inverse**: If it is not raining, then I will go to town.
  - **contrapositive**: If I go to town, then it is not raining.

    same with original proposition

Biconditional

p와 q가 같아야 True

- "p **if and only if** q ."
- Ex)
  p q
  If p denotes "I am at home." and q denotes "It is raining."
  then p ↔ q denotes "I am at home if and only if it is raining."
- Alternative expressions
  - p is necessary and sufficient for q
  - if p then q, and conversely

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

- **Truth Table**
  - ✓ Ex) $p \vee q \rightarrow \neg r$

p	q	r	$\neg r$	$p \vee q$	$p \vee q \rightarrow \neg r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

Propositional Logic

❖ **Equivalent Propositions**

- ✓ Two propositions are equivalent if they always have the same truth value.
- ✓ Ex) the conditional is equivalent to the contrapositive(대우)

=

대우

p	q	$\neg p$	$\neg q$	$p \rightarrow q$	$\neg q \rightarrow \neg p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

F	F	T
T	F	F
F	T	T
T	T	T
$\boxed{\neg q}$	$\boxed{\neg p}$	

Equivalent Propositions

- ✓ Two propositions are equivalent if they always have the same truth value.
- ✓ Ex) $p \rightarrow q \equiv \neg p \underset{\text{OR}}{\vee} q$

=

p	q	$\neg p$	$p \rightarrow q$	$\neg p \overset{\text{OR}}{\vee} q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Precedence of Logical Operators

- $p \lor q \rightarrow \neg r$ is equivalent to $(p \lor q) \rightarrow \neg r$
- If the intended meaning is $p \lor (q \rightarrow \neg r)$ then parentheses must be used.

Operator	Precedence
$\neg$	1
$\land$ and	2
$\lor$ or	3
$\rightarrow$	4
$\leftrightarrow$	5

Translating English Sentences

- "If I go to Harry's or to the country, I will not go shopping."
  p q ¬r
  - p: I go to Harry's
  - q: I go to the country.
  - r: I will go shopping.
  - If *p* or *q* then not *r*.

$$\left(p \vee q\right) \rightarrow \neg r$$

OR then

p → q

- Some more
  a c ∨ ¬f
  - "You can access the Internet from campus only if you are a computer science major or you are not a freshman."
  - What are a, c, and f?

p→q
p only if q

f: you are a freshman

$$a \rightarrow \left(c \vee \neg f\right)$$

Consistent System Specifications

- "The automated reply cannot be sent when the file system is full"
  - Let p denote "The automated reply can be sent" and q denote "The file system is full."

$\neg p$ q p

$$q \rightarrow \neg p$$

Conditional statement

Propositional Equivalences

- Tautologies(동어반복), Contradictions(모순), and Contingencies(우연)
  - A tautology is a proposition which is always true.
    - Example: $p \vee \neg p$ TRUE
  - A contradiction is a proposition which is always false.
    - Example: $p \wedge \neg p$ FALSE
  - A contingency is a proposition which is neither a tautology nor a contradiction, such as p

P	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

- **Propositional Equivalences**
  - ✓ De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Augustus De Morgan
1806-1871

p	q	$\neg p$	$\neg q$	$(p \vee q)$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Propositional Equivalences

- Identity Laws:

$$p \underset{\text{and}}{\wedge} T \equiv p, \qquad p \underset{\text{or}}{\vee} F \equiv p$$

- Domination Laws:

$$p \underset{\text{or}}{\vee} T \equiv T, \qquad p \underset{\text{and}}{\wedge} F \equiv F$$

- Idempotent laws:

$$p \vee p \equiv p, \qquad p \wedge p \equiv p$$

- Double Negation Law:

$$\neg(\neg p) \equiv p$$

- Negation Laws:

$$p \vee \neg p \equiv T, \qquad p \wedge \neg p \equiv F$$

❖ Propositional Equivalences

- ✓ Commutative Laws:

$$p \vee q \equiv q \vee p, \quad p \wedge q \equiv q \wedge p$$

OR AND

- ✓ Associative Laws:

$$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$(p \vee q) \vee r \equiv p \vee (q \vee r)$$

- ✓ Distributive Laws:

$$(p \vee (q \wedge r)) \equiv (p \vee q) \wedge (p \vee r)$$

$$(p \wedge (q \vee r)) \equiv (p \wedge q) \vee (p \wedge r)$$

- ✓ Absorption Laws:

$$p \vee (p \wedge q) \equiv p \quad p \wedge (p \vee q) \equiv p$$

Propositional Equivalences

- Involving Conditional Statements
- Involving Biconditional Statements

$$p \rightarrow q \equiv \neg p \vee q$$
$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$
$$p \vee q \equiv \neg p \rightarrow q$$
$$p \wedge q \equiv \neg(p \rightarrow \neg q)$$
$$\neg(p \rightarrow q) \equiv p \wedge \neg q$$
$$(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$$
$$(p \rightarrow r) \wedge (q \rightarrow r) \equiv (p \vee q) \rightarrow r$$
$$(p \rightarrow q) \vee (p \rightarrow r) \equiv p \rightarrow (q \vee r)$$
$$(p \rightarrow r) \vee (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$
$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$
$$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$$
$$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Propositional Equivalences

- Equivalence Proofs
- Ex) Show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$

  – Solution

$\neg(p \vee (\neg p \wedge q)) \equiv \neg p \wedge \neg(\neg p \wedge q)$	by the second De Morgan law
$\equiv \neg p \wedge [\neg(\neg p) \vee \neg q]$	by the first De Morgan law
$\equiv \neg p \wedge (p \vee \neg q)$	by the double negation law
$\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	by the second distributive law
$\equiv F \vee (\neg p \wedge \neg q)$	because $\neg p \wedge p \equiv F$
$\equiv (\neg p \wedge \neg q) \vee F$	by the commutative law for disjunction
$\equiv (\neg p \wedge \neg q)$	By the identity law for **F**

Propositional Equivalences

- Equivalence Proofs
- Ex) Show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.
  always true
  - Solution

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) && \text{by truth table for} \rightarrow \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by associative and commutative laws laws for disjunction} \\
&\equiv T \vee T && \text{by truth tables} \\
&\equiv T && \text{by the domination law}
\end{aligned}
$$

❖ **Satisfiability**

- ✓ A compound proposition is **satisfiable**
  if there is an assignment of truth values to its variables that **make the compound proposition true**.
- ✓ When no such assignments exist, the compound proposition is unsatisfiable.
- ✓ A compound proposition is unsatisfiable if and only if its negation is a tautology.
- ✓ Ex)

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p)$$

p, q, r에 T 대입하고
prove 되면 → satisfiable
(T이면)

$$(p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

$$(p \vee \neg q) \wedge (q \vee \neg r) \wedge (r \vee \neg p) \wedge (p \vee q \vee r) \wedge (\neg p \vee \neg q \vee \neg r)$$

(T ∨ F) ∧ (T ∨ F) ∧ (T ∨ F) ∧ (T) ∧ (F) ?

PROPOSITIONAL LOGIC

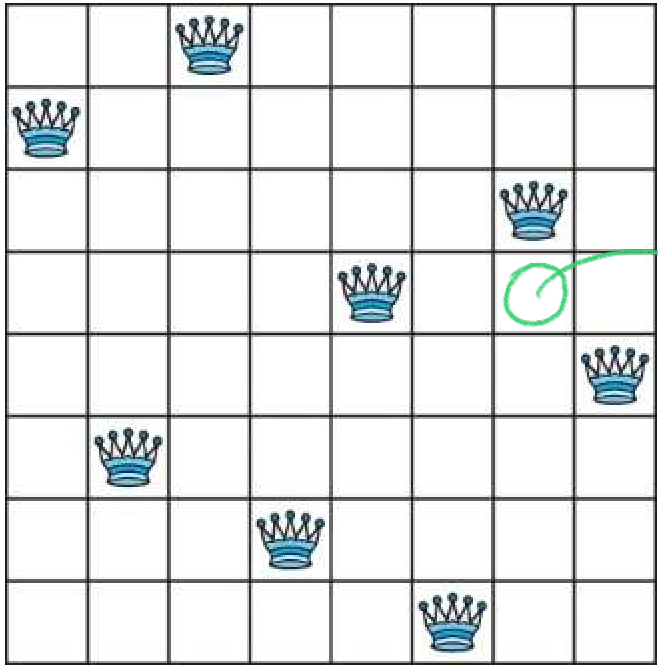
❖ **Some more Notation**

$$\bigvee_{j=1}^{n} p_j \text{ is used for } \overbrace{p_1 \vee p_2 \vee \ldots \vee p_n}^{1 \sim n}$$

$$\bigwedge_{j=1}^{n} p_j \text{ is used for } \underbrace{p_1 \wedge p_2 \wedge \ldots \wedge p_n}_{1 \sim n}$$

❖ N-Queen Problem

- ✓ Problem 8
  - – Place N Queens on a NxN grid, while not placing two Queens on the same vertical, horizontal or diagonal line 8x8
- ✓ Modeling
  - – Proposition $p_{i,j}$ indicates whether a Queen is placed at the i-row and at the j-th column P_{ij}



P_{4n} is F

N-Queen Problem

- ✓ $Q = Q_1 \land Q_2 \land Q_3 \land Q_4 \land Q_5$
- ✓ Solution is the truth values to $p(i, j)$ that satisfies Q.

Q_1	Every row contains at least one queen	$Q_1 = \bigwedge_{i=1..n} \bigvee_{j=1..n} p_{i,j}$
Q_2	At most one queen in each row	$Q_2 = \bigwedge_{i=1..n} \bigwedge_{j=1..n-1} \bigwedge_{k=j+1..n} \neg(p_{i,j} \land p_{i,k})$
Q_3	At most one queen in each column	$Q_3 = \bigwedge_{i=1..n} \bigwedge_{j=1..n-1} \bigwedge_{k=j+1..n} \neg(p_{j,i} \land p_{k,i})$
Q_4	No diagonal contains two queens (toward the top-right direction) *i-1: limit toward top, n-j: limit toward right*	$Q_4 = \bigwedge_{i=2..n} \bigwedge_{j=1..n-1} \bigwedge_{k=1..\min(i-1,n-j)} \neg(p_{i,j} \land p_{i-k,j+k})$
Q_5	No diagonal contains two queens (toward the bottom-right direction) *n-1: limit toward bottom, n-j: limit toward right*	$Q_5 = \bigwedge_{i=1..n-1} \bigwedge_{j=1..n-1} \bigwedge_{k=1..\min(n-i,n-j)} \neg(p_{i,j} \land p_{i+k,j+k})$

IGU

Thank you!