Introduction to B&P: A Full-Fledged Example for the VRPTW

OeGOR Summer-School 2024, Krems

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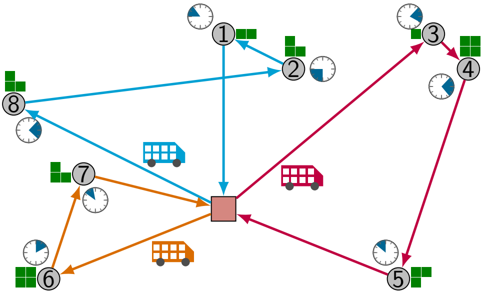
July 22 – 26, 2024

Agenda

1. Vehicle Routing Problem With Time Windows
2. From Arc-Flow to Path-Based Formulation
3. Column Generation (Solving the Root Node)
4. Solving the Integer Problem
5. Some Takeaways

VRP with Time Windows (VRPTW)



Given

- Depot
- Customers with
  - demands
  - time windows
- Vehicles (homogeneous) with capacity

Task Find a cost-minimal set of vehicle routes, such that

- each customer is visited exactly once,
- each route starts and ends at the depot,
- the vehicle capacity is respected on all routes, and
- all customers are serviced within their time window.

VRP with Time Windows (VRPTW)

Notation:

N set of customers

- $\rightarrow$ q_i demand of customer i
- $\rightarrow$ $[a_i, b_i]$ time window of customer i

o, d origin and destination depots

- $\rightarrow$ no demand $q_o = q_d = 0$
- $\rightarrow$ planning horizon $[a_o, b_o] = [a_d, b_d]$

V set of vertices, $V = N \cup \{o, d\}$

A set of arcs

- $\rightarrow$ c_{ij} cost of arc (i, j)
- $\rightarrow$ t_{ij} travel time of arc (i, j)

K set of homogeneous vehicles

- $\rightarrow$ Q capacity

Additionally:

$\delta^+(i)/\delta^-(i)$ set of all outgoing/ingoing arcs of vertex $i \in N \cup \{o, d\}$

Arc-Flow Formulation (2-Index MTZ)

Decision variables:

x_{ij} binary arc-flow variables

- $\rightarrow$ $x_{ij} = 1$ if a vehicle uses arc $(i, j) \in A$
- $\rightarrow$ $x_{ij} = 0$ otherwise

u_i load variables (continuous)

- $\rightarrow$ the load of a vehicle after visiting vertex $i \in V$

T_i time variables (continuous)

- $\rightarrow$ the start of service of a vehicle at vertex $i \in V$

Arc-Flow Formulation (2-Index MTZ)

$$
\begin{aligned}
z_{2I-MTZ}^{VRPTW} \quad = \quad & \min \sum_{(i,j)\in A} c_{ij} x_{ij} && \text{(1a)} \\
\text{s.t.} \quad & \sum_{(i,j)\in\delta^+(i)} x_{ij} = 1 \quad \forall\, i \in N && \text{(1b)} \\
& \sum_{(i,j)\in\delta^-(j)} x_{ij} = 1 \quad \forall\, j \in N && \text{(1c)} \\
& \sum_{(o,j)\in\delta^+(o)} x_{oj} = \sum_{(i,d)\in\delta^-(d)} x_{id} \le |K| && \text{(1d)} \\
& u_i - u_j + Q x_{ij} \le Q - q_j \quad \forall\, (i,j) \in A && \text{(1e)} \\
& q_i \le u_i \le Q \quad \forall\, i \in V && \text{(1f)} \\
& T_i - T_j + M x_{ij} \le M - t_{ij} \quad \forall\, (i,j) \in A && \text{(1g)} \\
& a_i \le T_i \le b_i \quad \forall\, i \in V && \text{(1h)} \\
& x_{ij} \in \{0,1\} \quad \forall\, (i,j) \in A && \text{(1i)}
\end{aligned}
$$

Arc-Flow Formulation (2-Index MTZ)

(1a) Minimize total routing costs

(1b)+(1c) Exactly one predecessor and successor for each customer

(1d) At most $|K|$ routes ($=$ vehicles)

(1e) Consistency of load variables and elimination of subtours

(1f) Capacity constraints

(1g) Consistency of time variables and elimination of subtours

(1h) Time-window constraints

(1i) Binary requirements

Arc-Flow Formulation (3-Index MTZ)

Decision variables:

x_{ij}^{k} binary arc-flow variables

- $\rightarrow$ $x_{ij}^{k} = 1$ if vehicle k uses arc $(i, j) \in A$
- $\rightarrow$ $x_{ij}^{k} = 0$ otherwise

u_{i}^{k} load variables (continuous)

- $\rightarrow$ the load of vehicle k after visiting vertex $i \in V$

T_{i}^{k} time variables (continuous)

- $\rightarrow$ the start of service of vehicle k at vertex $i \in V$

Arc-Flow Formulation (3-Index MTZ)

$$
\begin{aligned}
z_{3I-MTZ}^{VRPTW} \quad = \quad & \min \sum_{k\in K} \sum_{(i,j)\in A} c_{ij} x_{ij}^k && \text{(2a)} \\
\text{s.t.} \quad & \sum_{k\in K} \sum_{(i,j)\in \delta^+(i)} x_{ij}^k = 1 \quad \forall\, i \in N && \text{(2b)} \\
& \sum_{(o,j)\in \delta^+(o)} x_{oj}^k = \sum_{(i,d)\in \delta^-(d)} x_{id}^k \le 1 \quad \forall\, k \in K && \text{(2c)} \\
& \sum_{(i,j)\in \delta^+(i)} x_{ij}^k - \sum_{(j,i)\in \delta^-(i)} x_{ji}^k = 0 \quad \forall\, i \in N, k \in K && \text{(2d)} \\
& u_i^k - u_j^k + Q x_{ij}^k \le Q - q_j \quad \forall\, (i,j) \in A, k \in K && \text{(2e)} \\
& q_i \le u_i^k \le Q \quad \forall\, i \in N, k \in K && \text{(2f)} \\
& T_i^k - T_j^k + M_{ij} x_{ij}^k \le M_{ij} - t_{ij} \quad \forall\, (i,j) \in A, k \in K && \text{(2g)} \\
& a_i \le T_i^k \le b_i \quad \forall\, i \in N, k \in K && \text{(2h)} \\
& x_{ij}^k \in \{0,1\} \quad \forall\, (i,j) \in A, k \in K && \text{(2i)}
\end{aligned}
$$

Arc-Flow Formulation (3-Index MTZ)

(2a) Minimize total routing costs

(2b) Each customer is serviced exactly once

(2c) At most one route for each vehicle

(2d) Flow conservation

(2e) Consistency of load variables and elimination of subtours

(2f) Capacity constraints

(2g) Consistency of time variables and elimination of subtours

(2h) Time-window constraints

(2i) Binary requirements

Path-Based Formulation

Additional notation:

Ω set of feasible routes

c_r cost of route $r \in \Omega$

a_{ir} binary parameter equal to 1 if route r services customer i; 0 otherwise

Decision variables:

λ_r binary route variables

- $\rightarrow$ $\lambda_r = 1$ if route $r \in \Omega$ is selected
- $\rightarrow$ $\lambda_r = 0$ otherwise

Path-Based Formulation

$$
\begin{aligned}
z_{Path}^{VRPTW} \quad = \quad & \min \sum_{r \in \Omega} c_r \lambda_r && \text{(3a)} \\
\text{s.t.} \quad & \sum_{r \in \Omega} a_{ir} \lambda_r = 1 \quad \forall\ i \in N && \text{(3b)} \\
& \sum_{r \in \Omega} \lambda_r \leq |K| && \text{(3c)} \\
& \lambda_r \in \{0, 1\} \quad \forall\ r \in \Omega && \text{(3d)}
\end{aligned}
$$

(3a) Minimize total routing costs

(3b) Each customer is serviced exactly once

(3c) At most $|K|$ routes

(3d) Binary requirements

VRPTW Formulations – Properties

Arc-flow formulations:

- Number of variables:
  - $\rightarrow$ $\mathcal{O}\left(|V|^2\right)$ binary (2I)
  - $\rightarrow$ $\mathcal{O}\left(|V|\right)$ continuous (2I)
  - $\rightarrow$ $\mathcal{O}\left(|V|^2 \cdot |K|\right)$ binary (3I)
  - $\rightarrow$ $\mathcal{O}\left(|V| \cdot |K|\right)$ continuous (3I)
- Number of constraints:
  - $\rightarrow$ $\mathcal{O}\left(|V^2|\right)$ (2I)
  - $\rightarrow$ $\mathcal{O}\left(|V^2| \cdot |K|\right)$ (3I)
- Compact model
  - $\rightarrow$ solvable with standard software
- Very weak linear relaxations

Path-based formulation:

- Number of variables:
  - $\rightarrow$ exponential (binary)
- Number of constraints:
  - $\rightarrow$ $|N| + 1$
- Cannot be fully formulated
  - $\rightarrow$ not solvable with standard software
- Very strong linear relaxation

Agenda

Recall: 3-Index Formulation

$$
\begin{aligned}
z_{3I-MTZ}^{VRPTW} \quad = \quad & \min \sum_{k \in K} \sum_{(i,j) \in A} c_{ij} x_{ij}^k && \text{(2a)} \\
\text{s.t.} \quad & \sum_{k \in K} \sum_{(i,j) \in \delta^+(i)} x_{ij}^k = 1 \quad \forall\, i \in N && \text{(2b)} \\
& \sum_{(o,j) \in \delta^+(o)} x_{oj}^k = \sum_{(i,d) \in \delta^-(d)} x_{id}^k \leq 1 \quad \forall\, k \in K && \text{(2c)} \\
& \sum_{(i,j) \in \delta^+(i)} x_{ij}^k - \sum_{(j,i) \in \delta^-(i)} x_{ji}^k = 0 \quad \forall\, i \in N, k \in K && \text{(2d)} \\
& u_i^k - u_j^k + Q x_{ij}^k \leq Q - q_j \quad \forall\, (i,j) \in A, k \in K && \text{(2e)} \\
& q_i \leq u_i^k \leq Q \quad \forall\, i \in N, k \in K && \text{(2f)} \\
& T_i^k - T_j^k + M_{ij} x_{ij}^k \leq M_{ij} - t_{ij} \quad \forall\, (i,j) \in A, k \in K && \text{(2g)} \\
& a_i \leq T_i^k \leq b_i \quad \forall\, i \in N, k \in K && \text{(2h)} \\
& x_{ij}^k \in \{0,1\} \quad \forall\, (i,j) \in A, k \in K && \text{(2i)}
\end{aligned}
$$

Main idea:

- Constraints (2c)–(2i) can be considered separately for each vehicle
- Individual (identical) problem for each vehicle: build a feasible route
- What remains: find a combination of routes of all vehicles such that each customer is serviced exactly once and the routing costs are minimum

Problem formulation with routing variables:

$$\min \sum_{k\in K}\sum_{r\in\Omega^k} c_r\lambda_r^k$$

$$\sum_{k\in K}\sum_{r\in\Omega^k} a_{ir}\lambda_r^k = 1 \quad \forall\, i\in N$$

$$\sum_{r\in\Omega^k} \lambda_r \le 1 \quad \forall\, k\in K$$

$$\lambda_r^k \in \{0,1\} \quad \forall\, k\in K, r\in\Omega^k$$

With:

- c_r cost of route r
- a_{ir} $=1$, if route r visits customer i, $=0$ otherwise
- λ_r^k $=1$, if route r is selected, $=0$ otherwise

Definition of the routing variables:

$$\Omega^k = \left\{ r = \left(x_{ij}^k\right) \right\}, \text{ with}$$

$$\sum_{(o,j)\in\delta^+(o)} x_{oj}^k = \sum_{(i,d)\in\delta^-(d)} x_{id}^k \le 1$$

$$\sum_{(i,j)\in\delta^+(i)} x_{ij}^k - \sum_{(j,i)\in\delta^-(i)} x_{ji}^k = 0 \quad \forall\, i\in N$$

$$u_i^k - u_j^k + Qx_{ij}^k \le Q - q_j \quad \forall\, (i,j)\in A$$

$$q_i \le u_i^k \le Q \quad \forall\, i\in N$$

$$T_i^k - T_j^k + M_{ij}x_{ij}^k \le M_{ij} - t_{ij} \quad \forall\, (i,j)\in A$$

$$a_i \le T_i^k \le b_i \quad \forall\, i\in N$$

$$x_{ij}^k \in \{0,1\} \quad \forall\, (i,j)\in A$$

All vehicles are identical: aggregation! $\rightarrow$ less variables, less symmetry

Aggregated problem formulation with routing variables:

$$\min \sum_{r \in \Omega} c_r \lambda_r$$
$$\sum_{r \in \Omega} a_{ir} \lambda_r = 1 \quad \forall\, i \in N$$
$$\sum_{r \in \Omega} \lambda_r \leq |K|$$
$$\lambda_r \in \{0,1\} \quad \forall\, r \in \Omega$$

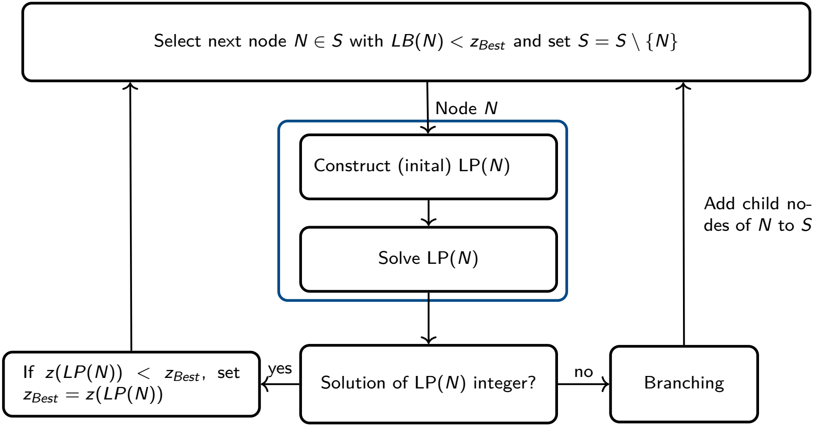
$\Rightarrow$ **path-based formulation**

Definition of the aggregated routing variables:

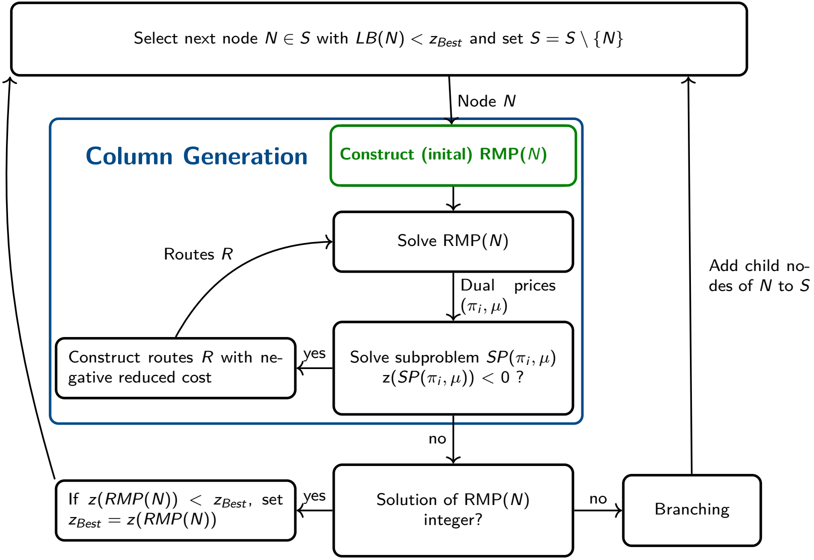
$$\Omega = \{r = (x_{ij})\}\text{, with}$$

$$\sum_{(o,j) \in \delta^+(o)} x_{oj} = \sum_{(i,d) \in \delta^-(d)} x_{id} \leq 1$$
$$\sum_{(i,j) \in \delta^+(i)} x_{ij} - \sum_{(j,i) \in \delta^-(i)} x_{ji} = 0 \quad \forall\, i \in N$$
$$u_i - u_j + Q x_{ij} \leq Q - q_j \quad \forall\, (i,j) \in A$$
$$q_i \leq u_i \leq Q \quad \forall\, i \in N$$
$$T_i - T_j + M_{ij} x_{ij} \leq M_{ij} - t_{ij} \quad \forall\, (i,j) \in A$$
$$a_i \leq T_i \leq b_i \quad \forall\, i \in N$$
$$x_{ij} \in \{0,1\} \quad \forall\, (i,j) \in A$$

Agenda


Select next node $N \in S$ with $LB(N) < z_{Best}$ and set $S = S \setminus \{N\}$
Node N
Construct (inital) LP(N)
Solve LP(N)
Add child nodes of N to S
If $z(LP(N)) < z_{Best}$, set $z_{Best} = z(LP(N))$
yes
Solution of LP(N) integer?
no
Branching

Branch-&-Price – Scheme



The **Restricted Master Problem (RMP)**:

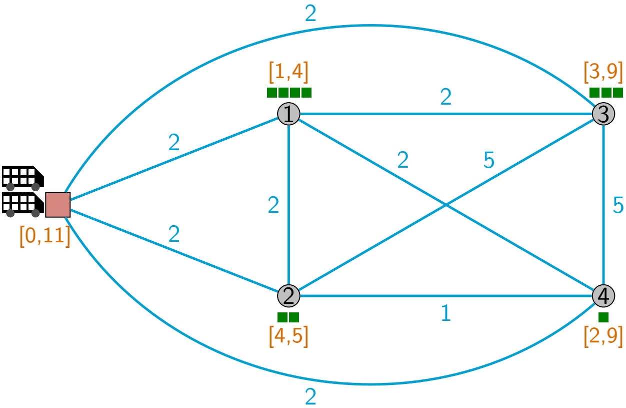
- LP-relaxation of the path-based formulation
- The set Ω of feasible routes is replaced by a (small) subset $\bar{\Omega}$

$$
\begin{aligned}
z_{RMP} \quad &= \quad \min \sum_{r \in \bar{\Omega}} c_r \lambda_r \\
\text{s.t.} \quad & \quad \sum_{r \in \bar{\Omega}} a_{ir} \lambda_r = 1 \quad \forall\, i \in N \\
& \quad \sum_{r \in \bar{\Omega}} \lambda_r \leq |K| \\
& \quad \lambda_r \geq 0 \quad \forall\, r \in \bar{\Omega}
\end{aligned}
$$

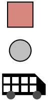
Initialization of $\bar{\Omega}$, e.g., with:

- Heuristic solution
- Dummy variables

Heuristic solution:



Given


Depot

Customers with demands and time windows [ ]

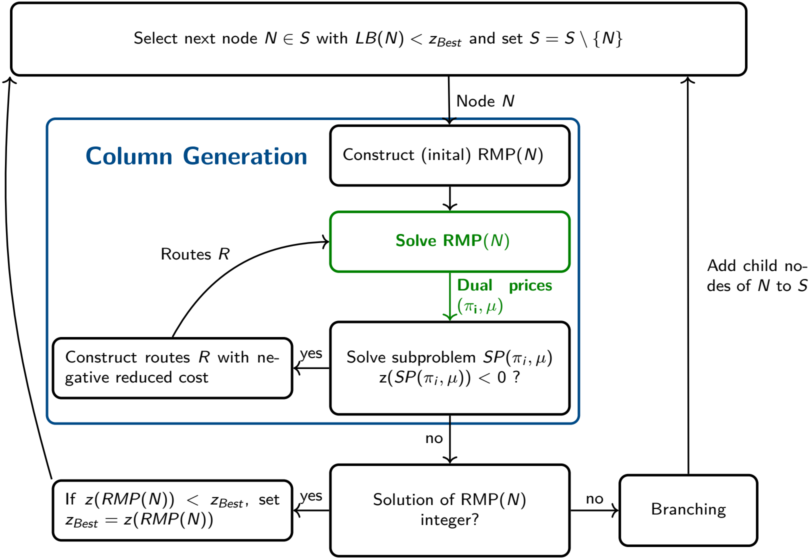
Vehicles with capacity $Q = 6$

Restricted Master Problem (RMP) – Example

Initial RMP with routes of heuristic solution:

$$
\begin{array}{llll}
\min & 6\lambda_1 & +9\lambda_2 & \\
& \lambda_1 & & = 1 \\
& \lambda_1 & & = 1 \\
& & \lambda_2 & = 1 \\
& & \lambda_2 & = 1 \\
& \lambda_1 & +\lambda_2 & \leq 2 \\
& \lambda_1 & \geq 0 & \\
& \lambda_2 & \geq 0 &
\end{array}
$$

Branch-&-Price – Scheme



RMP Solution and Dual Prices – Example

RMP with heuristic solution in standard form:

$$
\begin{array}{lcccccccl}
\min & 6\lambda_1 & 9\lambda_2 & +Ms_1 & +Ms_2 & +Ms_3 & +Ms_4 & & \\
 & \lambda_1 & & +s_1 & & & & & = 1 \\
 & \lambda_1 & & & +s_2 & & & & = 1 \\
 & & \lambda_2 & & & +s_3 & & & = 1 \\
 & & \lambda_2 & & & & +s_4 & & = 1 \\
 & \lambda_1 & +\lambda_2 & & & & & +s_5 & = 2 \\
\end{array}
$$

$$\lambda_1, \lambda_2, s_1, s_2, s_3, s_4, s_5 \geq 0$$

1. Simplex tableau:

	c_j	6	9	M	M	M	M		
c_B	x_B	λ_1	λ_2	s_1	s_2	s_3	s_4	s_5	b'
M	s_1	1	0	1					1
M	s_2	1	0		1				1
M	s_3	0	1			1			1
M	s_4	0	1				1		1
0	s_5	1	1					1	2
	$\tilde{c}_j$	$6-2M$	$9-2M$						$4M$

2. Simplex tableau:

	c_j	6	9	M	M	M	M		
c_B	x_B	λ_1	λ_2	s_1	s_2	s_3	s_4	s_5	b'
6	λ_1	1	0	1					1
M	s_2		0	-1	1				0
M	s_3		1	0		1			1
M	s_4		1	0			1		1
0	s_5		1	-1				1	1
	$\tilde{c}_j$		$9-2M$	$2M-6$					$2M+6$

3. Simplex tableau (final tableau):

	c_j	6	9	M	M	M	M		
c_B	x_B	λ_1	λ_2	s_1	s_2	s_3	s_4	s_5	b'
6	λ_1	1		1					1
M	s_2			-1	1				0
9	λ_2		1	0		1			1
M	s_4			0		-1	1		0
0	s_5			-1		-1		1	0
	$\tilde{c}_j$			$2M-6$		$2M-9$			15

RMP Solution and Dual Prices

Dual variables and dual prices:

- Each constraint is associated with a dual variable
- In the optimum: the value of the dual variable (also: *shadow price*, *dual price*) of a constraint indicates (within certain limits) how the objective function value changes when the right hand side is increased by one unit
- Alternative interpretation (in CG/VRP-context): Attractiveness to service the customer in a missing route, w.r.t. the current RMP solution
- Dual prices $c_B^\top B^{-1}$

$$
\begin{aligned}
z_{RMP} \quad = \quad & \min \sum_{r \in \bar{\Omega}} c_r x_r & & \text{(dual prices)} \\
\text{s.t.} \quad & \sum_{r \in \bar{\Omega}} a_{ir} x_r = 1 \quad \forall\ i \in N & & (\pi_i) \\
& \sum_{r \in \bar{\Omega}} x_r \leq |K| & & (\mu) \\
& x_r \geq 0 \quad \forall\ r \in \bar{\Omega}
\end{aligned}
$$

RMP Solution and Dual Prices

3. Simplex tableau (final tableau):

	c_j	6	9	M	M	M	M		
c_B	x_B	λ_1	λ_2	s_1	s_2	s_3	s_4	s_5	b'
6	λ_1	1		1					1
M	s_2			-1	1				0
9	λ_2		1	0		1			1
M	s_4			0		-1	1		0
	s_5			-1		-1		1	0
	$\tilde{c}_j$			$2M-6$		$2M-9$			15

Dual price of a constraint: difference of cost and reduced cost of the associated slack/auxiliary variable

$\pi_1 := c_{s_1} - \tilde{c}_{s_1} = -M + 6$

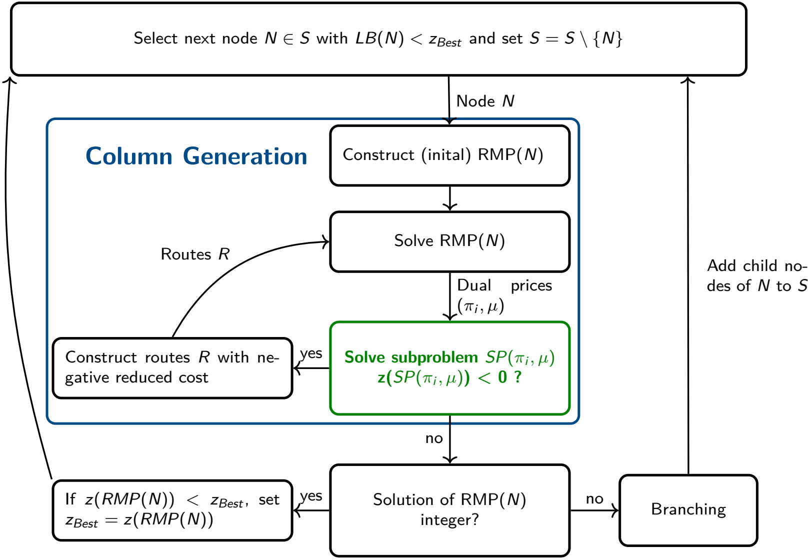
$\pi_2 := c_{s_2} - \tilde{c}_{s_2} = M$

$\pi_3 := c_{s_3} - \tilde{c}_{s_3} = -M + 9$

$\pi_4 := c_{s_4} - \tilde{c}_{s_4} = M$

$\mu := c_{s_5} - \tilde{c}_{s_5} = 0$

Branch-&-Price – Scheme



Subproblem task: identify missing routes or prove that none exists

- Missing routes r are those that improve the objective function value
  - $\rightarrow$ route $r \in \Omega$ is attractive, if its reduced cost $\tilde{c}_r$ is negative
  - $\rightarrow$ inclusion in the basis may(!) improve the objective function value
- Reduced cost of a route is defined as:

$$\tilde{c}_r := c_r - c_B^\top B^{-1} a^r = c_r - \sum_{i \in N} \pi_i a_{ir} - \mu$$

  - $\rightarrow$ problem: route r is unknown
  - $\rightarrow$ cost c_r and visited customers a_{ir} are unknown
- Routes r need to be feasible, i.e., $r \in \Omega$

$\Rightarrow$ **Optimization problem:** Find a feasible route r (i.e., determine c_r and a_{ir}) with minimum reduced cost given π_i and μ

$$\min_{r \in \Omega} \tilde{c}_r$$

Subproblem (= Pricing Problem)

$$
\begin{aligned}
z_{Sub} \quad = \quad & \min \sum_{(i,j)\in A} c_{ij}x_{ij} - \sum_{i\in N} \pi_i \Big(\sum_{(i,j)\in\delta^+(i)} x_{ij} \Big) - \mu && \text{(4a)} \\
& \sum_{(o,j)\in\delta^+(o)} x_{oj} = \sum_{(i,d)\in\delta^-(d)} x_{id} \le 1 && \text{(4b)} \\
& \sum_{(i,j)\in\delta^+(i)} x_{ij} - \sum_{(j,i)\in\delta^-(i)} x_{ji} = 0 \quad \forall\, i \in N && \text{(4c)} \\
& u_i - u_j + Qx_{ij} \le Q - q_j \quad \forall\, (i,j) \in A && \text{(4d)} \\
& q_i \le u_i \le Q \quad \forall\, i \in N && \text{(4e)} \\
& T_i - T_j + M_{ij}x_{ij} \le M_{ij} - t_{ij} \quad \forall\, (i,j) \in A && \text{(4f)} \\
& a_i \le T_i \le b_i \quad \forall\, i \in N && \text{(4g)} \\
& x_{ij} \in \{0,1\} \quad \forall\, (i,j) \in A && \text{(4h)}
\end{aligned}
$$

- Constraints (4b)–(4h) correspond with Constraints (2c)–(2i) of the 3-index formulation (removing index k)
- Elementary shortest path problem with resource constraint (ESPPRC)
- *NP*-hard

Subproblem (= Pricing Problem)

Our example:

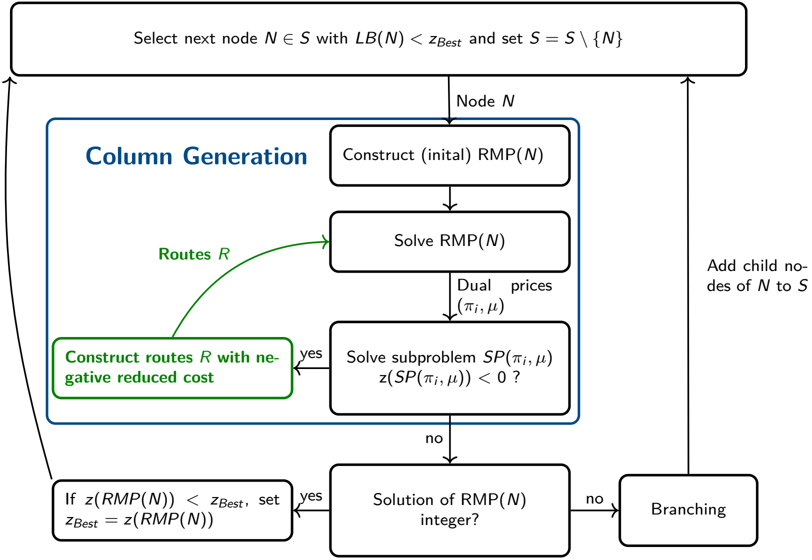
- Recall: $\pi_1 = -M + 6, \pi_2 = M, \pi_3 = -M + 9, \pi_4 = M, \mu = 0$
- Optimal solution (e.g., via MIP solver):

  $\rightarrow$ $z_{Sub} = 5 - 2M < 0$
  $\rightarrow$ $x^*_{o2} = x^*_{24} = x^*_{4d} = 1$ $\quad u^*_2 = 2; u^*_4 = 3$ $\quad T^*_2 = 4; T^*_4 = 5$
  $\rightarrow$ all other variables: value of zero

Remarks:

- Route with negative reduced cost found
  $\rightarrow$ add to RMP ($\rightarrow$ following slides)
- No route with $\tilde{c}_r < 0$ exists
  $\rightarrow$ RMP is solved to optimality
- It is not necessary to identify a route with minimum $\tilde{c}_r$
- Instead of adding only one new route to the RMP, multiple (or all) routes with negative reduced costs can be added
- ESPPRC subproblem can be solved more effectively with a labeling algorithm

Branch-&-Price – Scheme



Route Construction and RMP

The following properties of a route $r \in \Omega$ are relevant for the RMP:

c_r cost of the route

a_{ir} the number of services to customers i in the route

Can be easily extracted from the solution of the subproblem (let (x^*, u^*, T^*) be the optimal solution):

- We have

$$c_r = \sum_{(i,j)\in A} c_{ij} x^*_{ij}$$

  and

$$a_{ir} = \sum_{(i,j)\in\delta^+(i)} x_{ij} \quad \forall\ i \in N \quad \text{and} \quad a_{|N|+1,r} = 1$$

- The new column $a^{r'}$ in the current Simplex tableau is given by:

$$a^{r'} := (B^{-1} \cdot (a_{1r}, \ldots, a_{|N|,r}, a_{|N|+1,r})^\top)$$

Route Construction and RMP

Our example:

- Recall: optimal subproblem solution:

  $\rightarrow$ $z_{Sub} = 5 - 2M < 0$
  $\rightarrow$ $x^*_{o2} = x^*_{24} = x^*_{4d} = 1 \qquad u^*_2 = 2; u^*_4 = 3 \qquad T^*_2 = 4; T^*_4 = 5$
  $\rightarrow$ all other variables: value of zero

- The corresponding route variable λ_3 is given by:

  $\rightarrow$ $c_3 = 2x^*_{o2} + 1x^*_{24} + 2x^*_{4d} = 5$
  $\rightarrow$ $a_{23} = a_{43} = 1 \qquad a_{13} = a_{33} = 0 \qquad a_{53} = 1$

- The column in the current Simplex tableau:

$$a^{3'} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

Our example: Simplex tableau extended by new column

1. Simplex tableau of 2. RMP iteration:

	c_j	6	9	5	M	M	M	M		
c_B	x_B	λ_1	λ_2	λ_3	s_1	s_2	s_3	s_4	s_5	b'
6	λ_1	1		0	1		0			1
M	s_2			1	-1	1	0			0
9	λ_2		1	0	0		1			1
M	s_4			1	0		-1	1		0
0	s_5			1	-1		-1		1	0
	$\tilde{c}_j$			$5-2M$	$2M-6$		$2M-9$			15

2. Simplex tableau of 2. RMP iteration:

	c_j	6	9	5	M	M	M	M		
c_B	x_B	λ_1	λ_2	λ_3	s_1	s_2	s_3	s_4	s_5	b'
6	λ_1	1			1	0	0			1
5	λ_3			1	-1	1	0			0
9	λ_2		1		0	0	1			1
M	s_4				1	-1	-1	1		0
0	s_5				0	-1	-1		1	0
	$\tilde{c}_j$				-1	$2M-5$	$2M-9$			15

Route Construction and RMP

3. Simplex tableau of 2. RMP iteration:

	c_j	6	9	5	M	M	M	M		
c_B	x_B	λ_1	λ_2	λ_3	s_1	s_2	s_3	s_4	s_5	b'
6	λ_1	1				1	1	-1		1
5	λ_3			1		0	-1	1		0
9	λ_2		1			0	1	0		1
M	s_1				1	-1	-1	1		0
0	s_5					-1	-1	0	1	0
	$\tilde{c}_j$					$2M-6$	$2M-10$	1		15

Dual prices:

$\pi_1 := c_{s_1} - \tilde{c}_{s_1} = M$

$\pi_2 := c_{s_2} - \tilde{c}_{s_2} = -M + 6$

$\pi_3 := c_{s_3} - \tilde{c}_{s_3} = -M + 10$

$\pi_4 := c_{s_4} - \tilde{c}_{s_4} = M - 1$

$\mu := c_{s_5} - \tilde{c}_{s_5} = 0$

(An) optimal solution of the subproblem:

$z_{Sub} = 7 - 2M < 0$ $\quad x^*_{o4} = x^*_{41} = x^*_{1d} = 1$

$u^*_1 = 5; u^*_4 = 1$ $\quad T^*_1 = 4; T^*_4 = 2$

all other variables: value of zero

corresponding route λ_4: $\quad c_4 = 6$

$a_{14} = a_{44} = a_{54} = 1$ $\quad a_{24} = a_{34} = 0$

Example – Continued

Iter.	Dual prices π_1	π_2	π_3	π_4	μ	new column #	c_r	a_{1r}	a_{2r}	a_{3r}	a_{4r}	a_{5r}
3	M-3	9-M	M	9-M	0	5	4	0	0	1	0	1
4	6	5	9	5	-5	6	10	0	1	1	1	1
5	4	3	5	3	-1	$z_{Sub} \geq 0$						

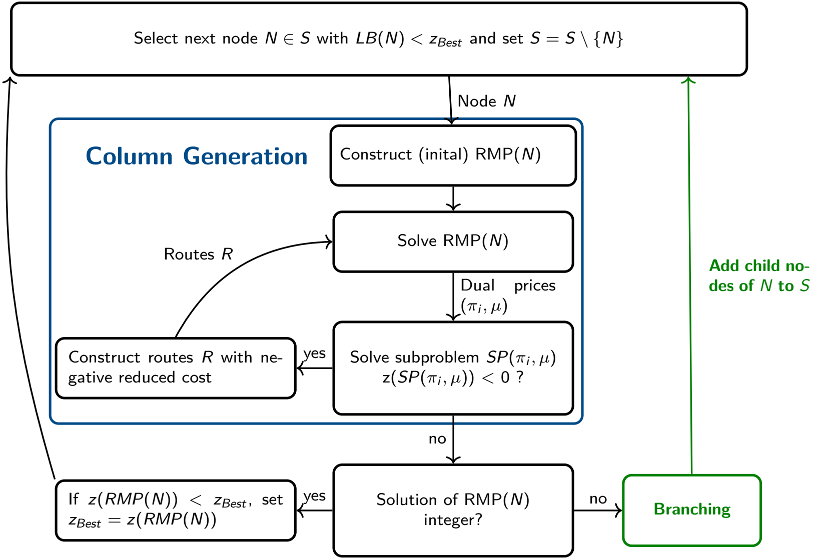
Final Simplex tableau of 5. RMP iteration:

	c_j	6	9	5	6	4	10	M	M	M	M		
c_B	x_B	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	s_1	s_2	s_3	s_4	s_5	b'
6	λ_1	1	$-\frac{1}{2}$	0	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$
6	λ_4	0	$\frac{1}{2}$	0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	0	$\frac{1}{2}$
4	λ_5	0	$\frac{1}{2}$	0	0	1	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{1}{2}$
10	λ_6	0	$\frac{1}{2}$	0	0	0	1	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	-1	$\frac{1}{2}$
5	λ_3	0	0	1	0	0	0	-1	0	-1	0	1	0
	$\tilde{c}_j$		2					$M-4$	$M-3$	$M-5$	$M-3$	1	13

Optimal RMP solution:

$\lambda_1^* = \lambda_4^* = \lambda_5^* = \lambda_6^* = \frac{1}{2}$ all other variables: value of zero

Agenda

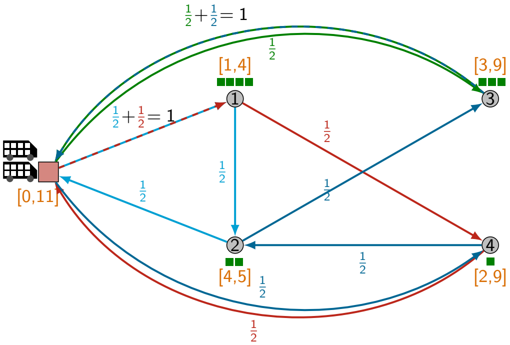

Select next node $N \in S$ with $LB(N) < z_{Best}$ and set $S = S \setminus \{N\}$
Node N
Column Generation
Construct (inital) RMP(N)
Solve RMP(N)
Routes R
Dual prices (π_i, μ)
Construct routes R with negative reduced cost
yes
Solve subproblem $SP(\pi_i, \mu)$ z($SP(\pi_i, \mu)) < 0$?
no
If $z(RMP(N)) < z_{Best}$, set $z_{Best} = z(RMP(N))$
yes
Solution of RMP(N) integer?
no
Branching
Add child nodes of N to S

If the optimal RMP solution λ_r^* is fractional, then branch-&-bound is required to obtain an integer solution:

- Branching directly on the λ_r variables is not advisable
  1. decision $\lambda_r = 1$ is very strong, $\lambda_r = 0$ is very weak
     $\rightarrow$ unbalanced branch-&-bound tree
  2. difficult to enforce in the subproblem
- Branch on the (aggregated) arc variables $x_{ij} = \sum_{k \in K} x_{ij}^k$ of the original 3-index formulation:
  - $\rightarrow$ *support graph* of solution λ_r^*: route variables λ_r are translated back into the aggregated arc variables x_{ij}
  - $\rightarrow$ branch on a variable x_{ij} with fractional solution value
    - > e.g., value closest to 0.5
    - > two child nodes: $x_{ij} = 0$ and $x_{ij} = 1$

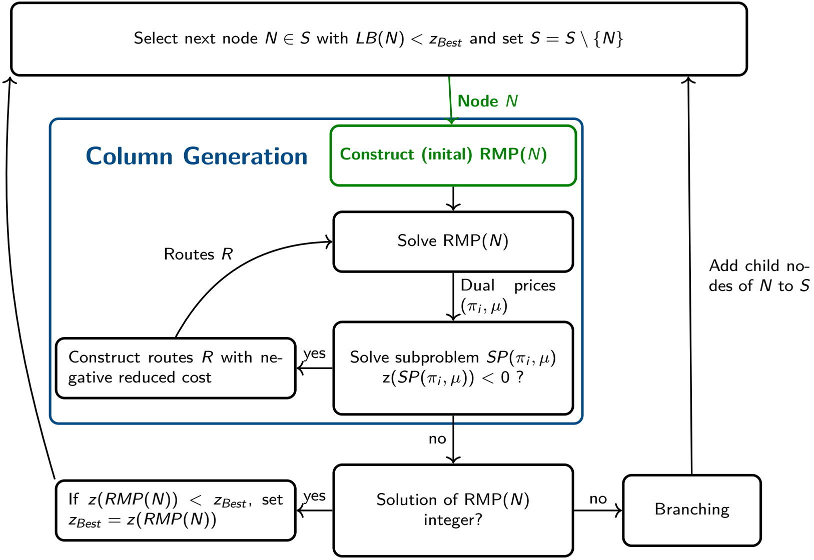
Our example:

- Recall: optimal solution of LP relaxation is $\lambda_1^* = \lambda_4^* = \lambda_5^* = \lambda_6^* = \frac{1}{2}$
- Corresponding routes:
  $\lambda_1 \hat{=} (o\text{-}1\text{-}2\text{-}d)$ $\lambda_4 \hat{=} (o\text{-}1\text{-}4\text{-}d)$ $\lambda_5 \hat{=} (o\text{-}3\text{-}d)$ $\lambda_6 \hat{=} (o\text{-}4\text{-}2\text{-}3\text{-}d)$
- Support graph:



$\Rightarrow$ **branching, e.g., on** x_{12}

Branch-&-Price – Scheme



RMP Construction and Subproblem after Branching

Active branching decisions must be respected by RMP and subproblem:

- **Here:** Branching decisions enforce or forbid arcs
  - $\rightarrow$ can be realized using a reduced arc set $A' \subset A$ when solving the node
    - $x_{ij}=0$ remove arc (i, j) from A'
    - $x_{ij}=1$ remove arcs $(i, k), k \neq j$ and $(h, j), h \neq i$ from A'
  - $\rightarrow$ the second implication follows because each customer is serviced exactly once
  - $\rightarrow$ (be careful if $i = o$ or $j = d$)
- Implementation of branching decision:
  - $\rightarrow$ Remove all routes from the RMP that use excluded arcs
  - $\rightarrow$ Solve the subproblem over the reduced arc set A'

RMP Construction and Subproblem after Branching

Our example: (node N_1)

- Routes in the RMP: $\lambda_1 \mathrel{\hat{=}} (o\text{-}1\text{-}2\text{-}d)$ $\lambda_2 \mathrel{\hat{=}} (o\text{-}3\text{-}4\text{-}d)$
  $\lambda_3 \mathrel{\hat{=}} (o\text{-}2\text{-}4\text{-}d)$ $\lambda_4 \mathrel{\hat{=}} (o\text{-}1\text{-}4\text{-}d)$ $\lambda_5 \mathrel{\hat{=}} (o\text{-}3\text{-}d)$ $\lambda_6 \mathrel{\hat{=}} (o\text{-}4\text{-}2\text{-}3\text{-}d)$
- Branching decision $x_{12} = 0 \Rightarrow$ remove route λ_1
- 1. Simplex tableau:

	c_j	9	5	6	4	10	M	M	M	M		
c_B	x_B	λ_2	λ_3	λ_4	λ_5	λ_6	s_1	s_2	s_3	s_4	s_5	b'
M	s_1	0	0	1	0	0	1					1
M	s_2	0	1	0	0	1		1				1
M	s_3	1	0	0	1	1			1			1
M	s_4	1	1	1	0	1				1		1
0	s_5	1	1	1	1	1					1	2
	$\tilde{c}_j$	$9-2M$	$5-2M$	$6-2M$	$4-M$	$10-3M$						$4M$

- Newly generated route: $\lambda_7 \mathrel{\hat{=}} (o\text{-}1\text{-}d)$ with $c_7 = 4$
- Optimal solution of LP relaxation: $\lambda_6^* = \lambda_7^* = 1$ (integer!)
  with objective function value 14

RMP Construction and Subproblem after Branching

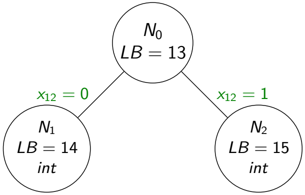
Our example: (node N_2)

- Routes in the RMP: $\lambda_1 \mathrel{\hat{=}} (o\text{-}1\text{-}2\text{-}d)$ $\lambda_2 \mathrel{\hat{=}} (o\text{-}3\text{-}4\text{-}d)$
  $\lambda_3 \mathrel{\hat{=}} (o\text{-}2\text{-}4\text{-}d)$ $\lambda_4 \mathrel{\hat{=}} (o\text{-}1\text{-}4\text{-}d)$ $\lambda_5 \mathrel{\hat{=}} (o\text{-}3\text{-}d)$ $\lambda_6 \mathrel{\hat{=}} (o\text{-}4\text{-}2\text{-}3\text{-}d)$
- Branching decision $x_{12} = 1 \Rightarrow$ remove routes λ_3, λ_4, and λ_6
- 1. Simplex tableau:

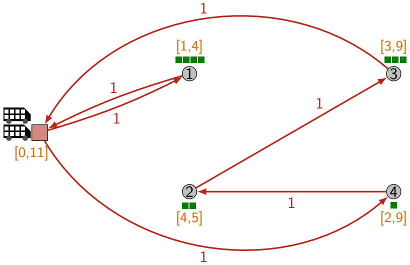
	c_j	6	9	4	M	M	M	M		
c_B	x_B	λ_1	λ_2	λ_5	s_1	s_2	s_3	s_4	s_5	b'
M	s_1	1	0	0	1					1
M	s_2	1	0	0		1				1
M	s_3	0	1	1			1			1
M	s_4	0	1	0				1		1
0	s_5	1	1	1					1	2
	$\tilde{c}_j$	$6-2M$	$9-2M$	$4-M$						$4M$

- Newly generated route: $\lambda_8 \mathrel{\hat{=}} (\text{o}-4-\text{d})$ with $c_8 = 4$
- Optimal solution of LP relaxation: $\lambda_1^* = \lambda_2^* = 1$ (integer!)
  with objective function value 15

Branch-&-Bound Tree – Example



- The branch-&-price algorithm terminates after processing the two child nodes N_1 and N_2
- Optimal solution: solution of node N_1 with total routing costs of 14.



Agenda

Some Takeaways

- Path-based formulation was natural for the VRPTW
  - $\rightarrow$ extended set-partitioning formulation that selects the best routes to service all customers
- Structured way to obtain path-based formulation (= extensive or CG formulation) from 3-index formulation (= original formulation)
  - $\rightarrow$ (knowledge of) original formulation also exploited in the B&P algorithm
- Branching is somewhat tricky
  - $\rightarrow$ major impact on implementation effort
  - $\rightarrow$ cannot rely on branch-&-bound tree of standard solver (as opposed to branch-&-cut with user cuts)