A PVS Library for Measure and Integration

David Lester

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Contents

Part I
Local Extras

1 partitions

partitions$[T\colon$ TYPE$]\colon$ THEORY
BEGIN

a, a_1, a_2: VAR set$[T]$

A: VAR setofsets$[T]$

partition?(A, a): bool =
$\bigcup A = a \land$
$(\forall\ (x, y\colon (A))\colon x \neq y \Rightarrow$ disjoint?$(x, y))$

finite_partition?(A, a): bool =
partition?(A, a) $\land$ is_finite(A)

partition: TYPE+ = (λ A: partition?(A, fullset$[T]$)) CONTAINING singleton$[$set$[T]]$
(fullset
$[T]$)

finite_partition: TYPE+ = (λ A: finite_partition?(A, fullset$[T]$)) CONTAINING singleton
$[$set
$[T]]$
(fullset
$[T]$)

p_1, p_2: VAR finite_partition

IMPORTING finite_sets@finite_cross

join(p_1, p_2): finite_partition =
$\{a \mid \exists\ (a_1\colon (p_1), a_2\colon (p_2))\colon a = (a_1 \cap a_2)\}$

END partitions

2 pointwise_convergence

pointwise_convergence[T: TYPE]: THEORY
BEGIN

IMPORTING metric_space@convergence_aux

u, v: VAR sequence[[$T \to \mathbb{R}$]]

f, f_0, f_1: VAR [$T \to \mathbb{R}$]

g: VAR [$T \to \mathbb{R}_{\geq 0}$]

x: VAR T

c: VAR $\mathbb{R}$

n, m: VAR $\mathbb{N}$

P: VAR pred[sequence[$\mathbb{R}$]]

zero_seq(n)(x): $\mathbb{R}$ = 0

pointwise?(P)(u): bool = $\forall\ x$: $P(\lambda\ n: u(n)(x))$

pointwise_bounded_above?(u): bool =
pointwise?(bounded_above?)(u)

pointwise_bounded_below?(u): bool =
pointwise?(bounded_below?)(u)

pointwise_bounded?(u): bool = pointwise?(bounded_seq?)(u)

pointwise_bounded_def: LEMMA
pointwise_bounded?(u) $\Leftrightarrow$
(pointwise_bounded_above?(u) $\wedge$ pointwise_bounded_below?(u))

pointwise_bounded_above: TYPE+ = (pointwise_bounded_above?) CONTAINING zero_seq

pointwise_bounded_below: TYPE+ = (pointwise_bounded_below?) CONTAINING zero_seq

pointwise_bounded: TYPE+ = (pointwise_bounded?) CONTAINING zero_seq

pointwise_bounded_is_bounded_above: JUDGEMENT pointwise_bounded SUBTYPE_OF
pointwise_bounded_above

pointwise_bounded_is_bounded_below: JUDGEMENT pointwise_bounded SUBTYPE_OF
pointwise_bounded_below

$u \longrightarrow f$: bool = $\forall\ x$: $\lambda\ n$: $u(n)(x) \longrightarrow f(x)$

pointwise_convergent?(u): bool $= \exists\ f:\ u \longrightarrow f$

pointwise_convergent: TYPE+ = (pointwise_convergent?) CONTAINING zero_seq

IMPORTING reals@real_fun_ops_aux$[T]$

$u+v$: sequence$[[T \to \mathbb{R}]] =$
$\quad \lambda\ n:\ u(n)+v(n)$;

$c \times u$: sequence$[[T \to \mathbb{R}]] = \lambda\ n:\ c \times u(n)$;

$-u$: sequence$[[T \to \mathbb{R}]] = \lambda\ n:\ -(u(n))$;

$u-v$: sequence$[[T \to \mathbb{R}]] =$
$\quad \lambda\ n:\ u(n)-v(n)$;

u^+: sequence$[[T \to \mathbb{R}_{\geq 0}]] = \lambda\ n:\ u(n)^+$;

u^-: sequence$[[T \to \mathbb{R}_{\geq 0}]] = \lambda\ n:\ u(n)^-$;

pointwise_convergence_sum: LEMMA
$\quad u \longrightarrow f_0 \wedge v \longrightarrow f_1 \Rightarrow u+v \longrightarrow f_0+f_1$

pointwise_convergence_scal: LEMMA
$\quad u \longrightarrow f \Rightarrow c \times u \longrightarrow c \times f$

pointwise_convergence_opp: LEMMA $u \longrightarrow f \Rightarrow -u \longrightarrow -f$

pointwise_convergence_diff: LEMMA
$\quad u \longrightarrow f_0 \wedge v \longrightarrow f_1 \Rightarrow u-v \longrightarrow f_0-f_1$

$w,\ w_0,\ w_1$: VAR pointwise_convergent

pointwise_convergent_sum: JUDGEMENT $+(w_0,\ w_1)$ HAS_TYPE
$\quad$ pointwise_convergent

pointwise_convergent_scal: JUDGEMENT $\times(c,\ w)$ HAS_TYPE
$\quad$ pointwise_convergent

pointwise_convergent_opp: JUDGEMENT $-(w)$ HAS_TYPE
$\quad$ pointwise_convergent

pointwise_convergent_diff: JUDGEMENT $-(w_0,\ w_1)$ HAS_TYPE
$\quad$ pointwise_convergent

pointwise_convergent_is_pointwise_bounded: JUDGEMENT pointwise_convergent SUBTYPE_OF
$\quad$ pointwise_bounded

pointwise_increasing?(u): bool $=$
$\quad \forall\ x:$ increasing?$(\lambda\ n:\ u(n)(x))$

pointwise_decreasing?(u): bool $=$
  $\forall\ x$: decreasing?$(\lambda\ n:\ u(n)(x))$

$u \nearrow f$: bool $=\ u \longrightarrow f\ \wedge$ pointwise_increasing?(u)

$u \searrow f$: bool $=\ u \longrightarrow f\ \wedge$ pointwise_decreasing?(u)

plus_minus_pointwise_convergence: LEMMA
  $u \longrightarrow f \Leftrightarrow (u^+ \longrightarrow f^+ \wedge u^- \longrightarrow f^-)$

p: VAR pointwise_bounded_below

a: VAR pointwise_bounded_above

b: VAR pointwise_bounded

$\inf(p)(n)(x)$: $\mathbb{R}$ $=$
  $\inf(\text{image}[\mathbb{N}, \mathbb{R}](\lambda\ m:\ p(m)(x),\ \{m \mid m \geq n\}))$

$\text{limsup}(b)(x)$: $\mathbb{R}$ $=$
  $\sup(\text{image}[\mathbb{N}, \mathbb{R}](\lambda\ m:\ \inf(b)(m)(x),\ \text{fullset}[\mathbb{N}]))$

$\sup(a)(n)(x)$: $\mathbb{R}$ $=$
  $\sup(\text{image}[\mathbb{N}, \mathbb{R}](\lambda\ m:\ a(m)(x),\ \{m \mid m \geq n\}))$

$\text{liminf}(b)(x)$: $\mathbb{R}$ $=$
  $\inf(\text{image}[\mathbb{N}, \mathbb{R}](\lambda\ m:\ \sup(b)(m)(x),\ \text{fullset}[\mathbb{N}]))$

sup_inf_def: LEMMA $\sup(a) = -\inf(-a)$

liminf_limsup_def: LEMMA $\text{liminf}(b) = -\text{limsup}(-b)$

inf_pointwise_increasing: LEMMA pointwise_increasing?$(\inf(p))$

inf_le: LEMMA $\inf(p)(n)(x) \leq p(n)(x)$

inf_pointwise_le: LEMMA
  $p \longrightarrow f \Rightarrow (\forall\ n,\ x:\ \inf(p)(n)(x) \leq f(x))$

limsup_pointwise_convergence: LEMMA $\inf(b) \longrightarrow \text{limsup}(b)$

inf_pointwise_convergence_upto: LEMMA
  $p \longrightarrow f \Rightarrow \inf(p) \nearrow f$

pointwise_convergence_plus_minus_def: LEMMA
  $u \longrightarrow f \Rightarrow$
   $(\inf(u^+) \nearrow f^+ \wedge \inf(u^-) \nearrow f^-)$

END pointwise_convergence

3 sup_norm

sup_norm[T: TYPE]: THEORY
BEGIN

ε: VAR $\mathbb{R}_{>0}$

c: VAR $\mathbb{R}_{\geq 0}$

y: VAR $\mathbb{R}$

x: VAR T

i, n: VAR $\mathbb{N}$

IMPORTING reals@real_fun_ops_aux[T], reals@bounded_reals[$\mathbb{R}$],
structures@const_fun_def[T, $\mathbb{R}$]

bounded?(f: [$T \to \mathbb{R}$]): bool =
$\exists\ c\colon\ \forall\ x\colon\ |f(x)| \leq c$

bounded: TYPE+ = (bounded?) CONTAINING ($\lambda\ x\colon\ 0$)

f, f_1, f_2: VAR bounded

bounded_add: JUDGEMENT $+(f_1,\ f_2)$ HAS_TYPE bounded

bounded_scal: JUDGEMENT $\times(y,\ f)$ HAS_TYPE bounded

bounded_opp: JUDGEMENT $-(f)$ HAS_TYPE bounded

bounded_diff: JUDGEMENT $-(f_1,\ f_2)$ HAS_TYPE bounded

sup_norm(f): $\mathbb{R}_{\geq 0}$ =
IF $\exists\ x$: TRUE
THEN sup(extend[$\mathbb{R}$, $\mathbb{R}_{\geq 0}$, bool, FALSE]($\{c \mid \exists\ x\colon\ |f(x)| = c\}$))
ELSE 0
ENDIF

sup_norm_eq_0: LEMMA
sup_norm(f) = 0 $\Leftrightarrow$ f = const_fun[T, $\mathbb{R}$](0)

sup_norm_neg: LEMMA sup_norm($-f$) = sup_norm(f)

sup_norm_sum: LEMMA
sup_norm($f_1 + f_2$) $\leq$ sup_norm(f_1) + sup_norm(f_2)

sup_norm_prop: LEMMA
($\forall\ x\colon\ |f(x)| \leq$ sup_norm(f)) $\wedge$
($\forall\ c\colon\ (\forall\ x\colon\ |f(x)| \leq c) \Rightarrow$ sup_norm(f) $\leq c$)

u: VAR sequence[bounded]

sup_norm_converges_to?(u, f): bool =
  $\forall\ \varepsilon$:
    $\exists\ n$: $\forall\ i$: $i \geq n \Rightarrow$ sup_norm($u(i) - f$) $< \varepsilon$

sup_norm_convergent?(u): bool =
  $\exists\ f$: sup_norm_converges_to?(u, f)

sup_norm_convergent: TYPE+ = (sup_norm_convergent?) CONTAINING (λ n: λ x: 0)

IMPORTING pointwise_convergence[T]

sup_norm_convergent_is_pointwise_convergent: JUDGEMENT sup_norm_convergent SUBTYPE_OF
  pointwise_convergent

sup_norm_converges_to_pointwise_convergence: LEMMA
  sup_norm_converges_to?(u, f) $\Rightarrow$ $u \longrightarrow f$

END sup_norm

4 product_sections

product_sections$[T_1,\ T_2\colon$ TYPE$]$: THEORY
BEGIN

$X,\ Y\colon$ VAR set$[[T_1,\ T_2]]$

$a\colon$ VAR T_1

$b\colon$ VAR T_2

IMPORTING topology@cross_product$[T_1,\ T_2]$

x_section_emptyset: LEMMA x_section$(\emptyset,\ a) = \emptyset$

x_section_complement: LEMMA
x_section$(\overline{X},\ a) = \overline{\text{x_section}(X,\ a)}$

x_section_union: LEMMA
x_section$((X \cup Y),\ a) =$
(x_section$(X,\ a) \cup$ x_section$(Y,\ a)$)

x_section_intersection: LEMMA
x_section$((X \cap Y),\ a) =$
(x_section$(X,\ a) \cap$ x_section$(Y,\ a)$)

x_section_disjoint: LEMMA
disjoint?$(X,\ Y) \Rightarrow$
disjoint?(x_section$(X,\ a)$, x_section$(Y,\ a)$)

y_section_emptyset: LEMMA y_section$(\emptyset,\ b) = \emptyset$

y_section_complement: LEMMA
y_section$(\overline{X},\ b) = \overline{\text{y_section}(X,\ b)}$

y_section_union: LEMMA
y_section$((X \cup Y),\ b) =$
(y_section$(X,\ b) \cup$ y_section$(Y,\ b)$)

y_section_intersection: LEMMA
y_section$((X \cap Y),\ b) =$
(y_section$(X,\ b) \cap$ y_section$(Y,\ b)$)

y_section_disjoint: LEMMA
disjoint?$(X,\ Y) \Rightarrow$
disjoint?(y_section$(X,\ b)$, y_section$(Y,\ b)$)

END product_sections

Part II
Borel Sets and Functions

5 subset_algebra_def

subset_algebra_def[T: TYPE]: THEORY
BEGIN

IMPORTING sets_aux@countable_props,
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\leq$),
subset?[T]],
sets_aux@indexed_sets_aux[$\mathbb{N}$, T], sets_aux@countable_indexed_sets[T],
sets_aux@nat_indexed_sets[T], sets_aux@countable_image

n, i: VAR $\mathbb{N}$

a, b: VAR set[T]

S, X, Y: VAR setofsets[T]

NX: VAR (nonempty?[set[T]])

E: VAR sequence[set[T]]

subset_algebra_empty?(S): bool = ($\emptyset$[T] $\in S$)

subset_algebra_complement?(S): bool =
$\forall$ (x: (S)): ($\overline{x} \in S$)

subset_algebra_union?(S): bool =
$\forall$ (x, y: (S)): (($x \cup y$) $\in S$)

subset_algebra?(S): bool =
subset_algebra_empty?(S) $\wedge$
subset_algebra_complement?(S) $\wedge$ subset_algebra_union?(S)

sigma_algebra_union?(S): bool =
$\forall$ X:
is_countable[set[T]](X) $\wedge$ ($\forall$ (x: (X)): ($x \in S$)) $\Rightarrow$
($\bigcup X \in S$)

sigma_algebra?(S): bool =
subset_algebra_empty?(S) $\wedge$
subset_algebra_complement?(S) $\wedge$ sigma_algebra_union?(S)

sigma_union_implies_subset_union: LEMMA
sigma_algebra_union?(S) $\Rightarrow$ subset_algebra_union?(S)

sigma_algebra_implies_subset_algebra: LEMMA sigma_algebra?(S) $\Rightarrow$ subset_algebra?(S)

trivial_subset_algebra: (subset_algebra?) =
(singleton($\emptyset[T]$) $\cup$ singleton(fullset$[T]$))

subset_algebra: TYPE+ = (subset_algebra?) CONTAINING trivial_subset_algebra

sigma_algebra: TYPE+ = (sigma_algebra?) CONTAINING trivial_subset_algebra

A: VAR sigma_algebra

I: VAR set[sigma_algebra]

sigma_algebra_is_subset_algebra: JUDGEMENT sigma_algebra SUBTYPE_OF
subset_algebra

powerset_is_sigma_algebra: LEMMA
sigma_algebra?(powerset(fullset$[T]$))

$\mathcal{S}(X)$: sigma_algebra =
$\bigcap\{Y \mid$ sigma_algebra?(Y) $\wedge$ $(X \subseteq Y)\}$

generated_sigma_algebra_subset1: LEMMA $(X \subseteq \mathcal{S}(X))$

generated_sigma_algebra_subset2: LEMMA
$(X \subseteq Y)$ $\wedge$ sigma_algebra?(Y) $\Rightarrow$ $(\mathcal{S}(X) \subseteq Y)$

generated_sigma_algebra_idempotent: LEMMA $\mathcal{S}(A) = A$

intersection_sigma_algebra: LEMMA
$\forall$ (A, B: sigma_algebra): sigma_algebra?(($A \cap B$))

$\sigma(I)$: sigma_algebra =
$\mathcal{S}(\bigcup$extend [setof[setof$[T]$], sigma_algebra, bool, FALSE]$(I))$

sigma_member: LEMMA $(A \in I)$ $\Rightarrow$ $(A \subseteq \sigma(I))$

B: VAR subset_algebra

J: VAR set[subset_algebra]

powerset_is_subset_algebra: LEMMA
subset_algebra?(powerset(fullset$[T]$))

$\mathcal{A}(X)$: subset_algebra =
$\bigcap\{Y \mid$ subset_algebra?(Y) $\wedge$ $(X \subseteq Y)\}$

generated_subset_algebra_subset1: LEMMA $(X \subseteq \mathcal{A}(X))$

generated_subset_algebra_subset2: LEMMA
$(X \subseteq Y)$ $\wedge$ subset_algebra?(Y) $\Rightarrow$ $(\mathcal{A}(X) \subseteq Y)$

generated_subset_algebra_idempotent: LEMMA $\mathcal{A}(B) = B$

intersection_subset_algebra: LEMMA
  $\forall$ $(A, B$: subset_algebra): subset_algebra?($(A \cap B)$)

subset(J): subset_algebra =
    $\mathcal{A}(\bigcup$extend [setof[setof[T]], subset_algebra, bool, FALSE]$(J))$

subset_member: LEMMA $(B \in J) \Rightarrow (B \subseteq$ subset$(J))$

finite_disjoint_union?$(X)(a)$: bool =
    $\exists$ E, n:
      disjoint?(E) $\wedge$
       $a = \bigcup E$ $\wedge$
        ($\forall$ i:
           $(i < n \Rightarrow (E(i) \in X))$ $\wedge$
            $(i \geq n \Rightarrow$ empty?$(E(i))))$

finite_disjoint_union_of?$(X)(a)(E, n)$: bool =
    disjoint?(E) $\wedge$
     $a = \bigcup E$ $\wedge$
      ($\forall$ i:
         $(i < n \Rightarrow (E(i) \in X))$ $\wedge$
          $(i \geq n \Rightarrow$ empty?$(E(i))))$

card(X: setofsets[T], a: (finite_disjoint_union?(X))): $\mathbb{N}$ =
    min($\{n$: $\mathbb{N}$ |
              $\exists$ E: finite_disjoint_union_of?$(X)(a)(E, n)\}$)

finite_disjoint_unions(X): setofsets[T] =
    extend[setof[T], ((finite_disjoint_union?(X))), bool, FALSE]
        (fullset[(finite_disjoint_union?(X))])

disjoint_algebra_construction: LEMMA
  ($\forall$ $(a, b$: (NX)): $((a \cap b) \in$ NX)) $\wedge$
   ($\forall$ (a: (NX)): finite_disjoint_union?(NX)($\overline{a}$))
   $\Rightarrow$ $\mathcal{A}$(NX) = finite_disjoint_unions(NX)

monotone?(X): bool =
    $\forall$ E:
      ($\forall$ n: $(E(n) \in X)$) $\Rightarrow$
       ((increasing?(E) $\Rightarrow$ $(\bigcup E \in X)$) $\wedge$
          (decreasing?(E) $\Rightarrow$ $(\bigcap E \in X)$))

monotone_class: TYPE+ = (monotone?) CONTAINING trivial_subset_algebra

powerset_is_monotone: LEMMA monotone?(powerset(fullset[T]))

sigma_algebra_is_monotone_class: JUDGEMENT sigma_algebra SUBTYPE_OF
    monotone_class

monotone_algebra_is_sigma: LEMMA
  subset_algebra?(X) $\wedge$ monotone?(X) $\Rightarrow$ sigma_algebra?(X)

C: VAR monotone_class

K: VAR set[monotone_class]

monotone_class_Intersection: LEMMA
  monotone?($\bigcap$extend [setof[setof[T]], monotone_class, bool, FALSE](K))

monotone_class: THEOREM ($B \subseteq C$) $\Rightarrow$ ($\mathcal{S}(B) \subseteq C$)

END subset_algebra_def

6 subset_algebra

subset_algebra$[T$: TYPE, (IMPORTING subset_algebra_def$[T]$) S: subset_algebra$[T]]$: THEORY
BEGIN

x, y: VAR (S)

subset_algebra_emptyset: JUDGEMENT $\emptyset[T]$ HAS_TYPE (S)

subset_algebra_fullset: JUDGEMENT fullset$[T]$ HAS_TYPE (S)

subset_algebra_complement: JUDGEMENT complement(x) HAS_TYPE (S)

subset_algebra_union: JUDGEMENT union(x, y) HAS_TYPE (S)

subset_algebra_intersection: JUDGEMENT intersection(x, y) HAS_TYPE (S)

subset_algebra_difference: JUDGEMENT difference(x, y) HAS_TYPE (S)

END subset_algebra

7 sigma_algebra

sigma_algebra[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING subset_algebra[T, S], sets_aux@countable_image

x, y: VAR (S)

SS: VAR sequence[(S)]

sigma_algebra_emptyset: LEMMA ($\emptyset[T] \in S$)

sigma_algebra_fullset: LEMMA (fullset[T] $\in S$)

sigma_algebra_complement: LEMMA ($\overline{x} \in S$)

sigma_algebra_union: LEMMA ($(x \cup y) \in S$)

sigma_algebra_intersection: LEMMA ($(x \cap y) \in S$)

sigma_algebra_difference: LEMMA ($(x \setminus y) \in S$)

sigma_algebra_IUnion: LEMMA ($\bigcup \mathrm{SS} \in S$)

sigma_algebra_IIntersection: LEMMA ($\bigcap \mathrm{SS} \in S$)

sigma_algebra_emptyset_rew: JUDGEMENT $\emptyset[T]$ HAS_TYPE (S)

sigma_algebra_fullset_rew: JUDGEMENT fullset[T] HAS_TYPE (S)

sigma_algebra_complement_rew: JUDGEMENT complement(x) HAS_TYPE (S)

sigma_algebra_union_rew: JUDGEMENT union(x, y) HAS_TYPE (S)

sigma_algebra_intersection_rew: JUDGEMENT intersection(x, y) HAS_TYPE
(S)

sigma_algebra_difference_rew: JUDGEMENT difference(x, y) HAS_TYPE
(S)

sigma_algebra_IUnion_rew: JUDGEMENT IUnion(SS) HAS_TYPE (S)

sigma_algebra_IIntersection_rew: JUDGEMENT IIntersection(SS) HAS_TYPE
(S)

END sigma_algebra

8 product_sigma_def

product_sigma_def[T_1, T_2: TYPE]: THEORY
BEGIN

IMPORTING subset_algebra_def, sigma_algebra, topology@cross_product[T_1, T_2],
product_sections[T_1, T_2], sets_aux@countable_image

i, n: VAR $\mathbb{N}$

x: VAR T_1

y: VAR T_2

X: VAR set[T_1]

Y: VAR set[T_2]

NX: VAR (nonempty?[T_1])

NY: VAR (nonempty?[T_2])

Z: VAR set[[T_1, T_2]]

S_1: VAR sigma_algebra[T_1]

S_2: VAR sigma_algebra[T_2]

measurable_rectangle?(S_1, S_2)(Z): bool =
$\exists$ X, Y: $Z = X \times Y \wedge S_1(X) \wedge S_2(Y)$

measurable_rectangle(S_1, S_2): TYPE+ = (measurable_rectangle?(S_1, S_2)) CONTAINING $\emptyset$

$S_1 \times S_2$: sigma_algebra[[T_1, T_2]] =
$\mathcal{S}$(extend [setof[[T_1, T_2]], measurable_rectangle(S_1, S_2), bool, FALSE](fullset[measurable_rectangle(S_1, S_2)]

x_section_measurable: LEMMA
$(Z \in S_1 \times S_2) \Rightarrow (\text{x\_section}(Z, x) \in S_2)$

y_section_measurable: LEMMA
$(Z \in S_1 \times S_2) \Rightarrow (\text{y\_section}(Z, y) \in S_1)$

sigma_cross_projection: LEMMA
$(\text{NX} \times \text{NY} \in S_1 \times S_2) \Rightarrow ((\text{NX} \in S_1) \wedge (\text{NY} \in S_2))$

END product_sigma_def

9 product_sigma

product_sigma$[T_1,\ T_2$: TYPE, (IMPORTING subset_algebra_def) S_1: sigma_algebra$[T_1]$,
S_2: sigma_algebra$[T_2]]$: THEORY
BEGIN

IMPORTING subset_algebra_def, sigma_algebra, topology@cross_product$[T_1,\ T_2]$,
product_sigma_def$[T_1,\ T_2]$, sets_aux@countable_image

$n,\ i$: VAR $\mathbb{N}$

X: VAR (S_1)

Y: VAR (S_2)

x: VAR T_1

y: VAR T_2

Z: VAR set$[[T_1,\ T_2]]$

NX: VAR (nonempty?$[T_1]$)

NY: VAR (nonempty?$[T_2]$)

$R,\ R_1,\ R_2$: VAR set[(measurable_rectangle?$(S_1,\ S_2)$)]

$r,\ r_1,\ r_2$: VAR (measurable_rectangle?$(S_1,\ S_2)$)

cross_product_is_sigma_times: LEMMA
sigma_times$(S_1,\ S_2)(X \times Y)$

rectangle_algebra_aux: LEMMA
$\mathcal{A}$(measurable_rectangle?$(S_1,\ S_2)$) =
finite_disjoint_unions$[[T_1,\ T_2]]$
(measurable_rectangle?$(S_1,\ S_2)$)

rectangle_algebra: subset_algebra$[[T_1,\ T_2]]$ =
finite_disjoint_unions$[[T_1,\ T_2]]$
(measurable_rectangle?$(S_1,\ S_2)$)

rectangle_algebra_def: LEMMA
rectangle_algebra = $\mathcal{A}$(measurable_rectangle?$(S_1,\ S_2)$)

finite_disjoint_rectangles: LEMMA
finite_disjoint_unions$[[T_1,\ T_2]]$(measurable_rectangle?$(S_1,\ S_2)$)(Z) $\Leftrightarrow$
($\exists$ R:
$\bigcup$extend [setof$[[T_1,\ T_2]]$, ((measurable_rectangle?$[T_1,\ T_2]$$(S_1,\ S_2)$)), bool, FALSE]$(R)$
= Z
$\wedge$
is_finite(R) $\wedge$

$(\forall\ (x,\ y\colon\ (R))\colon\ x\ =\ y\ \vee\ \text{disjoint?}(x,\ y)))$

intersection_rectangle: LEMMA
finite_disjoint_union?(measurable_rectangle?(S_1, S_2))
(($r_1 \cap r_2$))

complement_rectangle: LEMMA
finite_disjoint_union?(measurable_rectangle?(S_1, S_2))($\overline{r}$)

END product_sigma

10 borel

borel[T: TYPE, (IMPORTING topology@topology_def[T]) S: topology]: THEORY
BEGIN

IMPORTING subset_algebra_def[T], topology@topology[T, S], topology@basis[T],
sets_aux@countability

x: VAR T

X: VAR open

Y: VAR closed

Z: VAR set[T]

B: VAR (base?[T](S))

borel?: sigma_algebra =
$\mathcal{S}$(extend [setof[T], open[T, S], bool, FALSE](fullset[open]))

borel: TYPE+ = (borel?) CONTAINING $\emptyset$[T]

IMPORTING sigma_algebra[T, (borel?)]

a, b: VAR borel

A: VAR countable_set[borel]

C: VAR set[borel]

emptyset_is_borel: LEMMA borel?($\emptyset$[T])

fullset_is_borel: LEMMA borel?(fullset[T])

open_is_borel: LEMMA borel?(X)

closed_is_borel: LEMMA borel?(Y)

complement_is_borel: LEMMA borel?($\overline{a}$)

union_is_borel: LEMMA borel?(($a \cup b$))

intersection_is_borel: LEMMA borel?(($a \cap b$))

difference_is_borel: LEMMA borel?(($a \setminus b$))

Union_is_borel: LEMMA
borel?($\bigcup$extend[setof[T], borel, bool, FALSE](A))

Complement_is_borel: LEMMA
every(borel?,

Complement(extend[setof[T], borel, bool, FALSE](C)))

Intersection_is_borel: LEMMA
borel?($\bigcap$ extend[setof[T], borel, bool, FALSE](A))

emptyset_is_borel_judge: JUDGEMENT $\emptyset$[T] HAS_TYPE borel

fullset_is_borel_judge: JUDGEMENT fullset[T] HAS_TYPE borel

open_is_borel_judge: JUDGEMENT open SUBTYPE_OF borel

closed_is_borel_judge: JUDGEMENT closed SUBTYPE_OF borel

complement_is_borel_judge: JUDGEMENT complement(a) HAS_TYPE borel

union_is_borel_judge: JUDGEMENT union(a, b) HAS_TYPE borel

intersection_is_borel_judge: JUDGEMENT intersection(a, b) HAS_TYPE
borel

difference_is_borel_judge: JUDGEMENT difference(a, b) HAS_TYPE borel

borel_basis: LEMMA generated_sigma_algebra(B)(Z) $\Rightarrow$ borel?(Z)

borel_countable_basis: LEMMA is_countable(B) $\Rightarrow$ borel? = $\mathcal{S}(B)$

END borel

11 hausdorff_borel

hausdorff_borel[T: TYPE, (IMPORTING topology@topology_def[T]) S: hausdorff]: THEORY
BEGIN

IMPORTING topology@hausdorff_convergence[T, S], borel[T, S]

x: VAR T

singleton_is_borel: LEMMA borel?(singleton(x))

singleton_is_borel_judge: JUDGEMENT singleton(x) HAS_TYPE borel

END hausdorff_borel

12 borel_functions

```
borel_functions[(IMPORTING topology@topology_def) T1: TYPE, S: topology[T1], T2: TYPE,
                T: topology[T2]]: THEORY
 BEGIN

  IMPORTING borel, structures@const_fun_def[T1, T2], topology@continuity_def[T1, S, T2, T],
            topology@continuity[T1, S, T2, T]

  f: VAR [T1 -> T2]

  c: VAR T2

  X: VAR open[T2, T]

  B: VAR borel[T2, T]

  borel_function?(f): bool =
      (FORALL B: borel?[T1, S](inverse_image(f, B)))

  borel_function_def: LEMMA
    borel_function?(f) <=>
     (FORALL X: borel?[T1, S](inverse_image[T1, T2](f, X)))

  borel_function: TYPE = (borel_function?)

  const_borel_function: LEMMA borel_function?(const_fun[T1, T2](c))

  continuous_is_borel: JUDGEMENT continuous SUBTYPE_OF borel_function

 END borel_functions
```

13 identity_borel

identity_borel[T: TYPE, (IMPORTING topology@topology_def[T]) S: topology]: THEORY
BEGIN

IMPORTING borel_functions[T, S, T, S]

id_borel: LEMMA borel_function?(I[T])

I_is_borel: JUDGEMENT I[T] HAS_TYPE borel_function

END identity_borel

14 composition_borel

composition_borel[(IMPORTING topology@topology_def) T_1: TYPE, S: topology[T_1], T_2: TYPE,
T: topology[T_2], T_3: TYPE, U: topology[T_3]]: THEORY

BEGIN

IMPORTING borel_functions[T_1, S, T_2, T], borel_functions[T_2, T, T_3, U],
borel_functions[T_1, S, T_3, U]

f: VAR $[T_2 \rightarrow T_3]$

g: VAR $[T_1 \rightarrow T_2]$

composition_borel: LEMMA
borel_function?(f) $\wedge$ borel_function?(g) $\Rightarrow$
borel_function?($f \circ g$)

F: VAR borel_function[T_2, T, T_3, U]

G: VAR borel_function[T_1, S, T_2, T]

composition_is_borel: JUDGEMENT $O(F, G)$ HAS_TYPE
borel_function[T_1, S, T_3, U]

END composition_borel

15 real_borel

real_borel: THEORY
BEGIN

IMPORTING metric_space@real_topology, borel[$\mathbb{R}$, metric_induced_topology],
hausdorff_borel[$\mathbb{R}$, metric_induced_topology]

borel_generated_by_rational_open_interval: LEMMA
borel? =
$\mathcal{S}$(extend [setof[$\mathbb{R}$], rational_open_interval, bool, FALSE](fullset[rational_open_interval]))

borel_generated_by_open_interval: LEMMA
borel? =
$\mathcal{S}$(extend [setof[$\mathbb{R}$], open_interval, bool, FALSE](fullset[open_interval]))

open_interval_is_borel: JUDGEMENT open_interval SUBTYPE_OF borel

closed_interval_is_borel: JUDGEMENT closed_interval SUBTYPE_OF borel

END real_borel

Part III
Measures

16 generalized_measure_def

generalized_measure_def[T: TYPE, S: setofsets[T]]: THEORY
BEGIN

ASSUMING
S_empty: ASSUMPTION $S(\emptyset)$
ENDASSUMING

IMPORTING series@series, sets_aux@indexed_sets_aux[$\mathbb{N}$, T], sets_aux@nat_indexed_sets[T],
metric_space@convergence_aux, $\overline{\mathbb{R}}_{\geq 0}$@$\overline{\mathbb{R}}_{\geq 0}$

i, j, n: VAR $\mathbb{N}$

f: VAR $[(S) \to \overline{\mathbb{R}}_{\geq 0}]$

g: VAR $[(S) \to \mathbb{R}_{\geq 0}]$

A: VAR $[\mathbb{N} \to (S)]$

a, b: VAR (S)

x: VAR set[T]

disjoint_indexed_measurable?(A): bool = disjoint?(A)

disjoint_indexed_measurable: TYPE+ = (disjoint_indexed_measurable?) CONTAINING (λ
i:
$\emptyset$
[T])

disjoint_indexed_measurable_is_disjoint_indexed_set: JUDGEMENT disjoint_indexed_measurable SUBTYPE_OF
disjoint_indexed_set[$\mathbb{N}$, T]

X: VAR disjoint_indexed_measurable

measure_empty?(f): bool = $f(\emptyset[T])$ = (TRUE, 0)

measure_countably_additive?(f): bool =
$\forall\ X:\ S(\bigcup X) \Rightarrow \sum f \circ X = f(\bigcup X)$

measure_complete?(f): bool =
($\forall$ x, a:
(($x \subseteq a$) $\wedge$ $f(a)$ = (TRUE, 0)) $\Rightarrow$ $S(x)$)

measure?(f): bool =
measure_empty?(f) $\wedge$ measure_countably_additive?(f)

complete_measure?(f): bool =
    measure?(f) $\wedge$ measure_complete?(f)

zero_measure(a): $\overline{\mathbb{R}}_{\geq 0}$ = (TRUE, 0)

measure_type: TYPE+ = (measure?) CONTAINING zero_measure

trivial_measure: measure_type =
    λ a: IF empty?(a) THEN (TRUE, 0) ELSE (FALSE, 0) ENDIF

complete_measure: TYPE+ = (complete_measure?) CONTAINING trivial_measure

complete_measure_is_measure: JUDGEMENT complete_measure SUBTYPE_OF
    measure_type

measure_disjoint_union: LEMMA
  measure?(f) $\wedge$ disjoint?(a, b) $\wedge$ $S((a \cup b))$ $\Rightarrow$
  $f((a \cup b)) = f(a) + f(b)$

finite_measure?(g): bool =
    $g(\emptyset[T])$ = 0 $\wedge$
     ($\forall$ X:
        $S(\bigcup X)$ $\Rightarrow$ series($g \circ X$) $\longrightarrow$ $g(\bigcup X)$)

complete_finite_measure?(g): bool =
    finite_measure?(g) $\wedge$
     ($\forall$ x, a: $(x \subseteq a)$ $\wedge$ $g(a)$ = 0 $\Rightarrow$ $S(x)$)

trivial_finite_measure(A: (S)): [$\mathbb{R}_{\geq 0}$] = 0

finite_measure: TYPE+ = (finite_measure?) CONTAINING trivial_finite_measure

complete_finite_measure: TYPE = (complete_finite_measure?)

complete_finite_measure_is_finite_measure: JUDGEMENT complete_finite_measure SUBTYPE_OF
    finite_measure

to_measure(m: finite_measure): measure_type =
    λ a: (TRUE, $m(a)$)

F: VAR sequence[measure_type]

x_sum_measure: LEMMA measure?(λ a: ($\sum \lambda$ i : $F(i)(a)$))

END generalized_measure_def

17 measure_def

measure_def[T: TYPE, (IMPORTING subset_algebra_def[T]) S: subset_algebra]: THEORY
BEGIN

IMPORTING subset_algebra[T, S], generalized_measure_def[T, S]

convergent: MACRO pred[sequence[$\mathbb{R}$]] =
convergence_sequences.convergent?;

limit: MACRO [(convergence_sequences.convergent?) $\rightarrow$ $\mathbb{R}$] =
convergence_sequences.limit;

i, j, n: VAR $\mathbb{N}$

f: VAR [(S) $\rightarrow$ $\overline{\mathbb{R}}_{\geq 0}$]

g: VAR [(S) $\rightarrow$ $\mathbb{R}_{\geq 0}$]

A: VAR [$\mathbb{N}$ $\rightarrow$ (S)]

a, b: VAR (S)

x: VAR set[T]

X: VAR disjoint_indexed_measurable

increasing_indexed_measurable?(A): bool = increasing_indexed?(A)

increasing_indexed_measurable: TYPE+ = (increasing_indexed_measurable?) CONTAINING (λ
i:
fullset
[T])

P: VAR increasing_indexed_measurable

measure_sigma_finite?(f): bool =
$\exists$ X: $\bigcup X$ = fullset[T] $\wedge$ ($\forall$ i: $f(X(i))$‘1)

sigma_finite_measure?(f): bool =
measure?(f) $\wedge$ measure_sigma_finite?(f)

complete_sigma_finite?(f): bool =
measure?(f) $\wedge$
measure_complete?(f) $\wedge$ measure_sigma_finite?(f)

sigma_finite_measure: TYPE+ = (sigma_finite_measure?) CONTAINING zero_measure

complete_sigma_finite: TYPE = (complete_sigma_finite?)

discrete_measure: measure_type =
λ a:

IF is_finite(a)
  THEN (TRUE, card$[T](a)$)
ELSE (FALSE, 0)
ENDIF

sigma_finite_measure_is_measure: JUDGEMENT sigma_finite_measure SUBTYPE_OF measure_type

complete_sigma_finite_is_complete_measure: JUDGEMENT complete_sigma_finite SUBTYPE_OF complete_measure

complete_sigma_finite_is_sigma_finite_measure: JUDGEMENT complete_sigma_finite SUBTYPE_OF sigma_finite_measure

measure_monotone: LEMMA
measure?$(f) \land (a \subseteq b) \Rightarrow f(a) \leq f(b)$

measure_union: LEMMA
measure?$(f) \Rightarrow f((a \cup b)) \leq f(a) + f(b)$

measure_def: LEMMA
(measure?$(f) \Leftrightarrow$
  (measure_empty?$(f) \land$
    $(\forall\ (a,\ b\colon (S))$:
      disjoint?$(a,\ b) \Rightarrow f((a \cup b)) = f(a) + f(b))$
    $\land$
    $(\forall\ X$:
      $S(\bigcup X) \Rightarrow$
        $f(\bigcup X) \leq \sum f \circ X)))$

finite_measure_def: LEMMA
finite_measure?$(g) \Leftrightarrow$
  $(g(\emptyset[T]) = 0 \land$
    $(\forall\ (a,\ b\colon (S))$:
      disjoint?$(a,\ b) \Rightarrow g((a \cup b)) = g(a) + g(b))$
    $\land$
    $(\forall\ X$:
      $S(\bigcup X) \land$ convergence_sequences.convergent?(series$(g \circ X)) \Rightarrow$
        $g(\bigcup X) \leq$
          convergence_sequences.limit(series$(g \circ X))))$

A_of(f: sigma_finite_measure): disjoint_indexed_measurable =
  choose($\{X$ |
    $\bigcup X$ = fullset$[T] \land$
    $(\forall\ i\colon f(X(i))$‘1)$\})$

P_of(f: sigma_finite_measure)(n): (S) =
  $\bigcup \lambda\ i$: IF $i \leq n$ THEN A_of$(f)(i)$ ELSE $\emptyset[T]$ ENDIF

μ: VAR sigma_finite_measure

A_of_def1: LEMMA $\bigcup$A_of(μ) = fullset[T]

A_of_def2: LEMMA $\forall\ n$: μ(A_of(μ)(n))‘1

P_of_def1: LEMMA $\bigcup$P_of(μ) = fullset[T]

P_of_def2: LEMMA $\forall\ n$: μ(P_of(μ)(n))‘1

P_of_def3: LEMMA
  $\forall\ i,\ j$: $i \leq j \Rightarrow$ (P_of(μ)(i) $\subseteq$ P_of(μ)(j))

sigma_finite_def1: LEMMA
  sigma_finite_measure?(f) $\Leftrightarrow$
   (measure?(f) $\wedge$
    ($\exists\ X$:
      $\bigcup X$ = fullset[T] $\wedge$ ($\forall\ i$: $f(X(i))$‘1)))

sigma_finite_def2: LEMMA
  sigma_finite_measure?(f) $\Leftrightarrow$
   (measure?(f) $\wedge$
    ($\exists\ P$:
      $\bigcup P$ = fullset[T] $\wedge$ ($\forall\ i$: $f(P(i))$‘1)))

END measure_def

18 measure_space_def

measure_space_def[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING sigma_algebra[T, S], reals@real_fun_ops_aux[T],
structures@const_fun_def[T, $\mathbb{R}$], metric_space@real_topology, topology@basis[$\mathbb{R}$],
borel[$\mathbb{R}$, metric_induced_topology], real_borel, sets_aux@countable_props,
sets_aux@inverse_image_Union, sets_aux@countable_image, sets_aux@countable_set

X: VAR set[T]

Y: VAR set[$\mathbb{R}$]

x, y, z: VAR T

f: VAR [$T \rightarrow \mathbb{R}$]

B: VAR borel

c: VAR $\mathbb{R}$

q: VAR $\mathbb{Q}$

r: VAR $\mathbb{R}_{>0}$

measurable_set?(X): bool = $S(X)$

measurable_set: TYPE+ = (measurable_set?) CONTAINING $\emptyset$[T]

a, b: VAR measurable_set

SS: VAR sequence[measurable_set]

M: VAR countable_set[(S)]

measurable_emptyset: JUDGEMENT $\emptyset$[T] HAS_TYPE measurable_set

measurable_fullset: JUDGEMENT fullset[T] HAS_TYPE measurable_set

measurable_complement: JUDGEMENT complement(a) HAS_TYPE measurable_set

measurable_union: JUDGEMENT union(a, b) HAS_TYPE measurable_set

measurable_intersection: JUDGEMENT intersection(a, b) HAS_TYPE
measurable_set

measurable_difference: JUDGEMENT difference(a, b) HAS_TYPE
measurable_set

measurable_IUnion: JUDGEMENT IUnion(SS) HAS_TYPE measurable_set

measurable_IIntersection: JUDGEMENT IIntersection(SS) HAS_TYPE
measurable_set

measurable_Union: JUDGEMENT Union(M) HAS_TYPE measurable_set

measurable_Intersection: JUDGEMENT Intersection(M) HAS_TYPE
measurable_set

measurable_function?(f): bool =
$\forall$ B: measurable_set?(inverse_image(f, B))

measurable_function: TYPE+ = (measurable_function?) CONTAINING (λ
x:
0)

g, g_1, g_2: VAR measurable_function

measurable_is_function: JUDGEMENT measurable_function SUBTYPE_OF
$[T \rightarrow \mathbb{R}]$

constant_is_measurable: JUDGEMENT (constant?$[T, \mathbb{R}]$) SUBTYPE_OF
measurable_function

U: VAR setofsets$[\mathbb{R}]$

measurable_def: LEMMA
borel? = $\mathcal{S}(U)$ $\Rightarrow$
(measurable_function?(f) $\Leftrightarrow$
($\forall$ (X: (U)): S(inverse_image(f, X))))

measurable_def2: LEMMA
measurable_function?(f) $\Leftrightarrow$
($\forall$ (i: open_interval): S(inverse_image(f, i)))

measurable_gt: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) > c\})$)

measurable_le: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) \leq c\})$)

measurable_lt: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) < c\})$)

measurable_ge: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ c: $S(\{z \mid f(z) \geq c\})$)

measurable_gt2: LEMMA
measurable_function?(f) $\Leftrightarrow$ ($\forall$ q: $S(\{z \mid f(z) > q\})$)

measurable_le2: LEMMA

measurable_function?$(f) \Leftrightarrow (\forall\ q\colon\ S(\{z \mid f(z) \le q\}))$

measurable_lt2: LEMMA
measurable_function?$(f) \Leftrightarrow (\forall\ q\colon\ S(\{z \mid f(z) < q\}))$

measurable_ge2: LEMMA
measurable_function?$(f) \Leftrightarrow (\forall\ q\colon\ S(\{z \mid f(z) \ge q\}))$

scal_measurable: JUDGEMENT $\times(c,\ g)$ HAS_TYPE measurable_function

sum_measurable: JUDGEMENT $+(g_1,\ g_2)$ HAS_TYPE measurable_function

opp_measurable: JUDGEMENT $-(g)$ HAS_TYPE measurable_function

diff_measurable: JUDGEMENT $-(g_1,\ g_2)$ HAS_TYPE measurable_function

END measure_space_def

19 measure_space

measure_space[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING measure_space_def[T, S], reals@real_fun_ops_aux[T], power@real_fun_power[T],
real_borel,
borel_functions[$\mathbb{R}$, metric_induced_topology, $\mathbb{R}$, metric_induced_topology],
topology@constant_continuity
[$\mathbb{R}$, metric_induced_topology, $\mathbb{R}$, metric_induced_topology],
metric_space@metric_continuity
[$\mathbb{R}$, (λ (x, y: $\mathbb{R}$): $|x-y|$), $\mathbb{R}$,
(λ (x, y: $\mathbb{R}$): $|x-y|$)],
metric_space@real_continuity[$\mathbb{R}$, (λ (x, y: $\mathbb{R}$): $|x-y|$)],
pointwise_convergence[T], reals@bounded_reals[$\mathbb{R}$], finite_sets@finite_cross,
finite_sets@finite_sets_minmax_props[$\mathbb{R}$, $\leq$]

f: VAR [$T \to \mathbb{R}$]

g, g_1, g_2: VAR measurable_function[T, S]

ϕ: VAR borel_function

i, j, n, m: VAR $\mathbb{N}$

s: VAR sequence[[$T \to \mathbb{R}$]]

u: VAR sequence[measurable_function[T, S]]

x: VAR T

c, c_1, c_2, y: VAR $\mathbb{R}$

X: VAR set[T]

Y: VAR set[$\mathbb{R}$]

a: VAR $\mathbb{R}_{>0}$

borel_comp_measurable_is_measurable: JUDGEMENT $O(\phi, g)$ HAS_TYPE
measurable_function[T, S]

const_measurable: LEMMA measurable_function?(λ x: c)

nn_measurable?(f): bool =
measurable_function?(f) $\wedge$ ($\forall$ x: $0 \leq f(x)$)

nn_measurable: TYPE+ = (nn_measurable?) CONTAINING (λ x: 0)

nn_measurable_is_measurable: JUDGEMENT nn_measurable SUBTYPE_OF
measurable_function

abs_measurable: JUDGEMENT abs(g) HAS_TYPE
measurable_function[T, S]

expt_nat_measurable: JUDGEMENT expt(g, n) HAS_TYPE
measurable_function[T, S]

sq_measurable: JUDGEMENT sq(g) HAS_TYPE measurable_function[T, S]

min_measurable: JUDGEMENT min(g_1, g_2) HAS_TYPE
measurable_function[T, S]

max_measurable: JUDGEMENT max(g_1, g_2) HAS_TYPE
measurable_function[T, S]

minimum_measurable: JUDGEMENT minimum(u, n) HAS_TYPE
measurable_function[T, S]

maximum_measurable: JUDGEMENT maximum(u, n) HAS_TYPE
measurable_function[T, S]

plus_measurable: JUDGEMENT plus(g) HAS_TYPE
measurable_function[T, S]

minus_measurable: JUDGEMENT minus(g) HAS_TYPE
measurable_function[T, S]

prod_measurable: JUDGEMENT $\times$(g_1, g_2) HAS_TYPE
measurable_function[T, S]

expt_measurable: JUDGEMENT ˆ(g: nn_measurable, a) HAS_TYPE
measurable_function[T, S]

measurable_plus_minus: LEMMA
measurable_function?[T, S](f) $\Leftrightarrow$
(measurable_function?[T, S](f^+) $\wedge$
measurable_function?[T, S](f^-))

measurable_bounded_above?(u): bool = pointwise_bounded_above?(u)

measurable_bounded_below?(u): bool = pointwise_bounded_below?(u)

measurable_bounded?(u): bool =
measurable_bounded_above?(u) $\wedge$ measurable_bounded_below?(u)

measurable_bounded_above: TYPE+ = (measurable_bounded_above?) CONTAINING (λ
n:
λ
x:
0)

measurable_bounded_below: TYPE+ = (measurable_bounded_below?) CONTAINING (λ
n:
λ
x:
0)

measurable_bounded: TYPE+ = (measurable_bounded?) CONTAINING (λ
n:
λ
x:
0)

measurable_bounded_above_is_bounded_above: JUDGEMENT measurable_bounded_above SUBTYPE_OF
pointwise_bounded_above

measurable_bounded_below_is_bounded_below: JUDGEMENT measurable_bounded_below SUBTYPE_OF
pointwise_bounded_below

measurable_bounded_is_measurable_bounded_above: JUDGEMENT measurable_bounded SUBTYPE_OF
measurable_bounded_above

measurable_bounded_is_measurable_bounded_below: JUDGEMENT measurable_bounded SUBTYPE_OF
measurable_bounded_below

measurable_bounded_is_bounded: JUDGEMENT measurable_bounded SUBTYPE_OF
pointwise_bounded

inf_measurable: LEMMA
$\forall$ (u: measurable_bounded_below):
measurable_function?[T, S](inf(u)(n))

sup_measurable: LEMMA
$\forall$ (u: measurable_bounded_above):
measurable_function?[T, S](sup(u)(n))

pointwise_measurable: LEMMA
$u \longrightarrow f \Rightarrow$ measurable_function?[T, S](f)

simple?(f): bool =
measurable_function?[T, S](f) $\wedge$
is_finite(image(f, fullset[T]))

simple: TYPE+ = (simple?) CONTAINING (λ x: 0)

simple_is_measurable: JUDGEMENT simple SUBTYPE_OF measurable_function

simple_const: LEMMA simple?(λ x: c)

nn_simple?(f): bool = ($\forall$ x: $0 \le f(x)$) $\wedge$ simple?(f)

nn_simple: TYPE+ = (nn_simple?) CONTAINING (λ x: 0) nn_simple_is_simple: JUDGEMENT nn_simple SUBTYPE_OF simple

h, h_1, h_2: VAR simple

v: VAR sequence[simple]

simple_sq: JUDGEMENT sq(h) HAS_TYPE simple

simple_add: JUDGEMENT $+(h_1, h_2)$ HAS_TYPE simple

simple_scal: JUDGEMENT $\times(c, h)$ HAS_TYPE simple

simple_neg: JUDGEMENT $-(h)$ HAS_TYPE simple

simple_diff: JUDGEMENT $-(h_1, h_2)$ HAS_TYPE simple

simple_abs: JUDGEMENT abs(h) HAS_TYPE simple

simple_min: JUDGEMENT min(h_1, h_2) HAS_TYPE simple

simple_max: JUDGEMENT max(h_1, h_2) HAS_TYPE simple

simple_maximum: JUDGEMENT maximum(v, n) HAS_TYPE simple

simple_minimum: JUDGEMENT minimum(v, n) HAS_TYPE simple

simple_plus: JUDGEMENT plus(h) HAS_TYPE simple

simple_minus: JUDGEMENT minus(h) HAS_TYPE simple

simple_times: JUDGEMENT $\times(h_1, h_2)$ HAS_TYPE simple

simple_expt_nat: JUDGEMENT expt(h, n) HAS_TYPE simple

simple_expt: JUDGEMENT ˆ(h: nn_simple, a) HAS_TYPE simple

$\phi_X(x)$: $\mathbb{N}$ = IF $(x \in X)$ THEN 1 ELSE 0 ENDIF

phi_is_simple: JUDGEMENT $\phi(X: (S))$ HAS_TYPE simple

IMPORTING hausdorff_borel[$\mathbb{R}$, metric_induced_topology], partitions[T]

P: VAR finite_partition[T]

simple_def1: LEMMA
  simple?(f) $\Leftrightarrow$
    (is_finite(image(f, fullset[T])) $\wedge$
      ($\forall$ (y: (image(f, fullset[T]))):
        measurable_set?($\{x \mid y = f(x)\}$)))

constant_over?(f)(X): bool =
$\exists\ y\colon\ \forall\ (x\colon\ (X))\colon\ y\ =\ f(x)$

simple_def2: LEMMA
simple?(f) $\Leftrightarrow$
($\exists\ P$: every(S, P) $\wedge$ every(constant_over?(f), P))

simple_def3: LEMMA
simple?(f) $\Leftrightarrow$
$(\exists\ c_1,\ c_2,\ h_1,\ h_2\colon\ c_1 \times h_1 + c_2 \times h_2\ =\ f)$

IMPORTING sup_norm[T]

bounded_measurable?(f): bool =
bounded?(f) $\wedge$ measurable_function?(f)

bounded_measurable: TYPE+ = (bounded_measurable?) CONTAINING (λ
x:
0)

bounded_measurable_is_bounded: JUDGEMENT bounded_measurable SUBTYPE_OF
bounded

bounded_measurable_is_measurable: JUDGEMENT bounded_measurable SUBTYPE_OF
measurable_function

simple_is_bounded_measurable: JUDGEMENT simple SUBTYPE_OF
bounded_measurable

nn_bounded_measurable?(f): bool =
bounded_measurable?(f) $\wedge$ ($\forall\ x\colon\ 0\ \le\ f(x)$)

nn_bounded_measurable: TYPE+ = (nn_bounded_measurable?) CONTAINING (λ
x:
0)

nn_bounded_measurable_is_bounded_measurable: JUDGEMENT nn_bounded_measurable SUBTYPE_OF
bounded_measurable

increasing_nn_simple?(u): bool =
($\forall\ n$: nn_simple?($u(n)$)) $\wedge$ pointwise_increasing?(u)

increasing_nn_simple: TYPE+ = (increasing_nn_simple?) CONTAINING (λ
n:
λ
x:
0)

p: VAR nn_bounded_measurable

w: VAR increasing_nn_simple

```
sup_norm_simple: LEMMA
  ∃ h:
    (∀ x: 0 ≤ h(x) & h(x) ≤ p(x)) ∧
     sup_norm(p − h) ≤ sup_norm(p)/2

nn_simple_approx(p): nn_simple =
    choose({h |
                    (∀ x: 0 ≤ h(x) & h(x) ≤ p(x)) ∧
                     sup_norm(p − h) ≤
                      sup_norm(p)/2})

IMPORTING reals@sigma_nat

nn_simple_sequence(p)(n): RECURSIVE
      {h | ∀ x: 0 ≤ h(x) & h(x) ≤ p(x)} =
  IF n = 0
    THEN nn_simple_approx(p)
  ELSE nn_simple_sequence(p − nn_simple_approx(p))(n − 1)
  ENDIF
   MEASURE (λ p: λ n: n)

nn_bounded_measurable_as_increasing_simple_sequence: LEMMA
  ∃ w: sup_norm_converges_to?(w, p)

nn_bounded_measurable_as_sequence_prop: LEMMA
  sup_norm_converges_to?(w, p) ⇒
   (∀ n, x: w(n)(x) ≤ p(x))

bounded_measurable_as_increasing_sequence: LEMMA
  bounded_measurable?(f) ⇒
   (∃ v: sup_norm_converges_to?(v, f))

nn_measurable_def: LEMMA
  (∀ x: 0 ≤ f(x)) ⇒
   (measurable_function?(f) ⇔ (∃ w: w ↗ f))

measurable_as_limit_simple_def: LEMMA
  measurable_function?(f) ⇔ (∃ v: v ⟶ f)

END measure_space
```

20 outer_measure_def

outer_measure_def[T: TYPE]: THEORY
BEGIN

IMPORTING $\overline{\mathbb{R}}_{\geq 0}$@$\overline{\mathbb{R}}_{\geq 0}$,
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\leq$),
subset?[T]],
sets_aux@indexed_sets_aux[$\mathbb{N}$, T]

f: VAR [set[T] $\to$ $\overline{\mathbb{R}}_{\geq 0}$]

X: VAR [$\mathbb{N}$ $\to$ set[T]]

a, b: VAR set[T]

om_empty?(f): bool = $f(\emptyset[T])$ = (TRUE, 0)

om_increasing?(f): bool =
$\forall\ a,\ b:\ (a \subseteq b) \Rightarrow f(a) \leq f(b)$

om_countably_subadditive?(f): bool =
$\forall\ X:\ f(\bigcup X) \leq \sum f \circ X$

outer_measure?(f): bool =
om_empty?(f) $\wedge$
om_increasing?(f) $\wedge$ om_countably_subadditive?(f)

zero_outer_measure(a): $\overline{\mathbb{R}}_{\geq 0}$ = (TRUE, 0)

outer_measure: TYPE+ = (outer_measure?) CONTAINING zero_outer_measure

END outer_measure_def

21 ast_def

ast_def[T: TYPE, A: (nonempty?[set[T]])]: THEORY
BEGIN

ASSUMING
IMPORTING subset_algebra_def[T]

A_empty: ASSUMPTION $A(\emptyset)$

A_fullset: ASSUMPTION A(fullset)

A_intersection: ASSUMPTION $\forall$ (a, b: (A)): $A((a \cap b))$

A_difference: ASSUMPTION
$\forall$ (a, b: (A)): finite_disjoint_union?(A)($(a \setminus b)$)
ENDASSUMING

IMPORTING generalized_measure_def[T, A], outer_measure_def[T], $\overline{\mathbb{R}}_{\geq 0}$@double_index[set[$T$]]

μ: VAR measure_type

z: VAR $\overline{\mathbb{R}}_{\geq 0}$

ε: VAR $\mathbb{R}_{>0}$

X: VAR set[T]

Y: VAR (A)

I: VAR sequence[(A)]

a, b: VAR set[T]

i, n: VAR $\mathbb{N}$

A_difference_union: LEMMA
$A(a)$ $\wedge$ finite_disjoint_union?(A)(b) $\Rightarrow$
finite_disjoint_union?(A)($(a \setminus b)$)

measure_subadditive: LEMMA
$A(\bigcup I) \Rightarrow \mu(\bigcup I) \leq \sum \mu \circ I$

generalized_monotonicity: LEMMA
disjoint?(I) $\wedge$ (IUnion(n, I) $\subseteq Y$) $\wedge$ ($\forall$ i: $i \geq n \Rightarrow$ empty?($I(i)$)) $\Rightarrow$
$\sum \mu \circ I \leq \mu(Y)$

generalized_measure_monotone: LEMMA
$\forall$ (a, b: (A), μ): $(a \subseteq b) \Rightarrow \mu(a) \leq \mu(b)$

μ^*: outer_measure =

$\lambda\ X:$
$\quad \inf(\{z \mid \exists\ I:\ \sum \mu \circ I = z \ \wedge\ (X \subseteq \bigcup I)\})$

outer_measure_eq: LEMMA $\text{ast}(\mu)(Y) = \mu(Y)$

outer_measure_def: LEMMA
$\exists\ I:$
$\quad (X \subseteq \bigcup I)\ \wedge\ \sum \mu \circ I \leq \text{ast}(\mu)(X) + \varepsilon$

END ast_def

22 outer_measure

outer_measure[T: TYPE, (IMPORTING subset_algebra_def[T]) A: subset_algebra]: THEORY
BEGIN

IMPORTING subset_algebra[T, A], ast_def[T, A]

END outer_measure

23 outer_measure_props

outer_measure_props[T: TYPE, (IMPORTING outer_measure_def[T]) m: outer_measure]: THEORY
BEGIN

IMPORTING outer_measure_def[T], subset_algebra_def[T], orders@bounded_nats

i: VAR $\mathbb{N}$

x, y: VAR set[T]

A: VAR sequence[set[T]]

m_outer_empty: LEMMA $m(\emptyset[T]) = (\text{TRUE}, 0)$

m_outer_increasing: LEMMA $(x \subseteq y) \Rightarrow m(x) \leq m(y)$

m_outer_subadditive: LEMMA $m(\bigcup A) \leq \sum m \circ A$

outer_negligible?(x): bool = $m(x) = (\text{TRUE}, 0)$

outer_measurable?(x): bool =
$\forall\ y\colon\ m(y) = m((y \cap x)) + m((y \cap \overline{x}))$

outer_negligible: TYPE+ = (outer_negligible?) CONTAINING $\emptyset[T]$

outer_measurable: TYPE+ = (outer_measurable?) CONTAINING $\emptyset[T]$

pairwise_subadditive: LEMMA
$m(y) \leq m((y \cap x)) + m((y \cap \overline{x}))$

outer_measurable_def: LEMMA
outer_measurable?(x) $\Leftrightarrow$
($\forall\ y$:
$m((y \cap x)) + m((y \cap \overline{x})) \leq m(y)$)

outer_negligible_is_outer_measurable: JUDGEMENT outer_negligible SUBTYPE_OF
outer_measurable

a, b: VAR outer_measurable

X: VAR sequence[outer_measurable]

S: VAR setofsets[T]

outer_measurable_complement: JUDGEMENT complement(a) HAS_TYPE
outer_measurable

outer_measurable_emptyset: JUDGEMENT $\emptyset[T]$ HAS_TYPE outer_measurable

outer_measurable_fullset: JUDGEMENT fullset[T] HAS_TYPE
outer_measurable

outer_measurable_union: JUDGEMENT union(a, b) HAS_TYPE
outer_measurable

outer_measurable_intersection: JUDGEMENT intersection(a, b) HAS_TYPE
outer_measurable

outer_measurable_difference: JUDGEMENT difference(a, b) HAS_TYPE
outer_measurable

outer_measurable_disjoint_union: LEMMA
disjoint?(a, b) $\Rightarrow$
$m((x \cap (a \cup b))) = m((x \cap a)) + m((x \cap b))$

outer_measurable_IUnion: JUDGEMENT IUnion(X) HAS_TYPE outer_measurable

outer_measurable_IIntersection: JUDGEMENT IIntersection(X) HAS_TYPE
outer_measurable

outer_measurable_Union: LEMMA
is_countable(S) $\wedge$ every(outer_measurable?, S) $\Rightarrow$
outer_measurable?($\bigcup S$)

outer_measurable_Intersection: LEMMA
is_countable(S) $\wedge$ every(outer_measurable?, S) $\Rightarrow$
outer_measurable?($\bigcap S$)

outer_measurable_disjoint_IUnion: LEMMA
disjoint?(X) $\Rightarrow$
$m((x \cap \bigcup X)) = \sum \lambda\ i :\ m((x \cap X(i)))$

outer_measure_disjoint_IUnion: LEMMA
disjoint?(X) $\Rightarrow$ $m(\bigcup X) = \sum m \circ X$

outer_measurable_is_sigma_algebra: LEMMA
sigma_algebra?(extend[setof[T], outer_measurable, bool, FALSE]
(fullset[outer_measurable]))

induced_sigma_algebra: sigma_algebra[T] = (outer_measurable?)

IMPORTING measure_def[T, induced_sigma_algebra]

induced_measure: complete_measure =
restrict[set[T], (induced_sigma_algebra), $\overline{\mathbb{R}}_{\geq 0}$]($m$)

induced_measure_rew: LEMMA induced_measure(a) = $m(a)$

n, n_1, n_2: VAR outer_negligible

outer_negligible_emptyset: JUDGEMENT $\emptyset$[T] HAS_TYPE outer_negligible

outer_negligible_union: JUDGEMENT union(n_1, n_2) HAS_TYPE
outer_negligible

outer_negligible_subset: LEMMA $(x \subseteq n) \Rightarrow$ outer_negligible?(x)

END outer_measure_props

24 measure_props

measure_props[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space_def[T, S], sigma_algebra[T, S],
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\leq$),
subset?[T]],
measure_def[T, S], series@series_aux

n, i: VAR $\mathbb{N}$

a, b, M: VAR measurable_set

x, y: VAR $\overline{\mathbb{R}}_{\geq 0}$

X: VAR sequence[$\overline{\mathbb{R}}_{\geq 0}$]

DX: VAR disjoint_indexed_measurable

E: VAR sequence[measurable_set]

mu_fin?(M): bool $=$ $m(M)$‘1

$\mu(M$: $\{m$: (S) | mu_fin?$(m)\})$: $\mathbb{R}_{\geq 0}$ $=$ $m(M)$‘2

m_emptyset: LEMMA $m(\emptyset[T])$ $=$ (TRUE, 0)

m_countably_additive: LEMMA $\sum m \circ \mathrm{DX} = m(\bigcup \mathrm{DX})$

m_disjoint_union: LEMMA
disjoint?(a, b) $\Rightarrow$ $m((a \cup b)) = m(a) + m(b)$

m_monotone: LEMMA $(a \subseteq b)$ $\Rightarrow$ $m(a) \leq m(b)$

m_union: LEMMA $m((a \cup b)) \leq m(a) + m(b)$

m_increasing_convergence: LEMMA
increasing?(E) $\Rightarrow$ x_converges?($m \circ E$, $m(\bigcup E)$)

m_decreasing_convergence: LEMMA
decreasing?(E) $\wedge$ mu_fin?($E(0)$) $\Rightarrow$
x_converges?($m \circ E$, $m(\bigcap E)$)

END measure_props

25 measure_theory

measure_theory[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space_def[T, S], sigma_algebra[T, S], measure_props[T, S, m],
sets_aux@countable_indexed_sets

a, b: VAR measurable_set

X, Y: VAR set[T]

P: VAR set[T]

p: VAR pred[[$\mathbb{R}$, $\mathbb{R}$]]

f, g: VAR [$T \rightarrow \mathbb{R}$]

F, G: VAR sequence[[$T \rightarrow \mathbb{R}$]]

x: VAR T

i, j, n: VAR $\mathbb{N}$

Z: VAR setofsets[T]

null_set?(X): bool =
measurable_set?(X) $\wedge$ mu_fin?(X) $\wedge$ $\mu(X) = 0$

negligible_set?(Y): bool = $\exists$ X: null_set?(X) $\wedge$ $(Y \subseteq X)$

null_set: TYPE+ = (null_set?) CONTAINING $\emptyset$[T]

negligible: TYPE+ = (negligible_set?) CONTAINING $\emptyset$[T]

N, N_1, N_2: VAR null_set

NS: VAR sequence[null_set]

E, E_1, E_2: VAR negligible

ES: VAR sequence[negligible]

negligible_iff_measurable_null: LEMMA
(negligible_set?(X) $\wedge$ measurable_set?(X)) $\Leftrightarrow$ null_set?(X)

null_set_is_measurable: JUDGEMENT null_set SUBTYPE_OF measurable_set

null_is_negligible: JUDGEMENT null_set SUBTYPE_OF negligible null_emptyset: JUDGEMENT $\emptyset[T]$ HAS_TYPE null_set

null_union: JUDGEMENT union(N_1, N_2) HAS_TYPE null_set

null_intersection: JUDGEMENT intersection(N_1, N_2) HAS_TYPE null_set

null_difference: JUDGEMENT difference(N_1, N_2) HAS_TYPE null_set

null_IUnion: JUDGEMENT IUnion(NS) HAS_TYPE null_set

null_Union: LEMMA
every(null_set?, Z) $\wedge$ is_countable(Z) $\Rightarrow$ null_set?($\bigcup Z$)

negligible_emptyset: JUDGEMENT $\emptyset[T]$ HAS_TYPE negligible

negligible_union: JUDGEMENT union(E_1, E_2) HAS_TYPE negligible

negligible_intersection: JUDGEMENT intersection(E_1, E_2) HAS_TYPE negligible

negligible_IUnion: JUDGEMENT IUnion(ES) HAS_TYPE negligible

negligible_Union: LEMMA
every(negligible_set?, Z) $\wedge$ is_countable(Z) $\Rightarrow$ negligible_set?($\bigcup Z$)

negligible_subset: LEMMA $(X \subseteq E) \Rightarrow$ negligible_set?(X)

ae_in?(P)(X): bool =
$\exists\ E$: $\forall\ (x$: $(X))$: $(\neg\ (x \in E)) \Rightarrow (x \in P)$

ae?(P): bool = ae_in?(P)(fullset$[T]$)

pointwise_ae?(p)(f, g): bool =
ae?($\lambda\ x$: $p(f(x),\ g(x))$)

ae?(p)(f, g): bool = pointwise_ae?(p)(f, g)

$f = 0$ *a.e.*: bool =
pointwise_ae?(restrict$[[$number, number$]$, $[\mathbb{R},\ \mathbb{R}]$, boolean$]$(=))
(f, $\lambda\ x$: 0)

$f \geq 0$ *a.e.*: bool = pointwise_ae?($\leq$)(($\lambda\ x$: 0), f)

$f > 0$ *a.e.*: bool = pointwise_ae?($<$)(($\lambda\ x$: 0), f)

$f \leq g$ *a.e.*: bool = pointwise_ae?($\leq$)(f, g)

$f = g$ *a.e.*: bool =
pointwise_ae?(restrict$[[$number, number$]$, $[\mathbb{R},\ \mathbb{R}]$, boolean$]$(=))
(f, g)

ae_eq_equivalence: LEMMA equivalence?(ae_eq?)

ae_le_reflexive: LEMMA reflexive?(ae_le?)

ae_le_antisymmetric: LEMMA
  $f \leq g$ *a.e.* $\wedge$ $g \leq f$ *a.e.* $\Rightarrow$ $f = g$ *a.e.*

ae_le_transitive: LEMMA transitive?(ae_le?)

ae_convergence_in?(X)(F, f): bool =
    ae_in?(λ x: λ n: $F(n)(x) \longrightarrow f(x)$)($X$)

ae_cauchy_in?(X)(F): bool =
    ae_in?(λ x: cauchy?(λ n: $F(n)(x)$))(X)

$F \longrightarrow f$ *a.e.*: bool = ae_convergence_in?(fullset[T])(F, f)

ae_cauchy?(F): bool = ae_cauchy_in?(fullset[T])(F)

ae_convergence_cauchy: LEMMA $F \longrightarrow f$ *a.e.* $\Rightarrow$ ae_cauchy?(F)

ae_convergence_eq: LEMMA
  $F \longrightarrow f$ *a.e.* $\Rightarrow$ ($F \longrightarrow g$ *a.e.* $\Leftrightarrow$ $f = g$ *a.e.*)

ae_eq_convergence: LEMMA
  $F \longrightarrow f$ *a.e.* $\wedge$ ($\forall$ n: $F(n) = G(n)$ *a.e.*) $\Rightarrow$
   $G \longrightarrow f$ *a.e.*

increasing?(F) *a.e.*: bool =
    $\exists$ E:
      $\forall$ x:
        $\neg$ ($x \in E$) $\Rightarrow$
         ($\forall$ i, j: $i \leq j$ $\Rightarrow$ $F(i)(x) \leq F(j)(x)$)

decreasing?(F) *a.e.*: bool =
    $\exists$ E:
      $\forall$ x:
        $\neg$ ($x \in E$) $\Rightarrow$
         ($\forall$ i, j: $i \leq j$ $\Rightarrow$ $F(j)(x) \leq F(i)(x)$)

ae_monotonic_converges?(F, f): bool =
    $F \longrightarrow f$ *a.e.* $\wedge$ (increasing?(F) *a.e.* $\vee$ decreasing?(F) *a.e.*)

ae_convergent?(F): bool = $\exists$ f: $F \longrightarrow f$ *a.e.*

END measure_theory

26 monotone_classes

```
monotone_classes[T: TYPE, C: (nonempty?[set[T]])]: THEORY
 BEGIN

  IMPORTING subset_algebra_def[T]

  a, b: VAR (C)

  x: VAR (S(C))

  IMPORTING measure_def[T, (S(C))], sigma_algebra, measure_props

  monotone_finite_measures: COROLLARY
    ∀ (ν, μ: finite_measure):
      (∀ a, b: ((a ∩ b) ∈ C)) ∧
       (∀ a: finite_disjoint_union?(C)(ā)) ∧ (∀ a: μ(a) = ν(a))
       ⇒ (∀ x: μ(x) = ν(x))

 END monotone_classes
```

27 hahn_kolmogorov

hahn_kolmogorov[T: TYPE, (IMPORTING subset_algebra_def[T]) A: subset_algebra,
(IMPORTING measure_def[T, A]) μ: measure_type]: THEORY

BEGIN

IMPORTING outer_measure[T, A], outer_measure_props[T, μ^*]

x: VAR (A)

algebra_in_induced_sigma_algebra: LEMMA ($A \subseteq$ induced_sigma_algebra)

IMPORTING measure_theory[T, induced_sigma_algebra, induced_measure]

induced_measure_measure: LEMMA induced_measure(x) $= \mu(x)$

END hahn_kolmogorov

Part IV
Finite Measures

28 finite_measure

finite_measure[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: finite_measure]: THEORY
BEGIN

IMPORTING sets_aux@sets_lemmas_aux, sets_aux@indexed_sets_aux[$\mathbb{N}$, T], sigma_algebra[T, S],
series@series_aux,
structures@fun_preds_partial
[$\mathbb{N}$, set[T], restrict[[$\mathbb{R}$, $\mathbb{R}$], [$\mathbb{N}$, $\mathbb{N}$], boolean](reals.$\le$),
subset?[T]]

X: VAR $[\mathbb{N} \to (S)]$

A, B: VAR (S)

fm_emptyset: LEMMA $\mu(\emptyset) = 0$

fm_convergence: LEMMA
$\text{disjoint?}(X) \Rightarrow \text{series}(\mu \circ X) \longrightarrow \mu(\bigcup X)$

fm_disjointunion: LEMMA
$\text{disjoint?}(A, B) \Rightarrow$
$\mu((A \cup B)) = \mu(A) + \mu(B)$

fm_complement: LEMMA $\mu(\overline{A}) = \mu(\text{fullset}) - \mu(A)$

fm_union: LEMMA
$\mu((A \cup B)) =$
$\mu(A) + \mu(B) - \mu((A \cap B))$

fm_intersection: LEMMA
$\mu((A \cap B)) =$
$\mu(A) + \mu(B) - \mu((A \cup B))$

fm_difference: LEMMA
$\mu((A \setminus B)) =$
$\mu(A) - \mu(B) + \mu((B \setminus A))$

fm_subset: LEMMA
$(A \subseteq B) \Rightarrow \mu(B) = \mu(A) + \mu((B \setminus A))$

fm_subset_le: LEMMA $(A \subseteq B) \Rightarrow \mu(A) \le \mu(B)$

fm_monotone: LEMMA $(A \subseteq B) \Rightarrow \mu(A) \le \mu(B)$

fm_IUnion: LEMMA $\text{increasing?}(X) \Rightarrow \mu \circ X \longrightarrow \mu(\bigcup X)$

fm_IIntersection: LEMMA
  decreasing?$(X) \Rightarrow \mu \circ X \longrightarrow \mu(\bigcap X)$

IMPORTING measure_def$[T, S]$

measure_from: measure_type = λ A: (TRUE, $\mu(A)$)

END finite_measure

Part V

Complete Measures

29 complete_measure_theory

complete_measure_theory$[T$: TYPE, (IMPORTING subset_algebra_def$[T]$) S: sigma_algebra,
(IMPORTING measure_def$[T, S]$) μ: complete_measure$]$: THEORY

BEGIN

IMPORTING measure_space_def$[T, S]$, sigma_algebra$[T, S]$, measure_theory$[T, S, \mu]$,
measure_props$[T, S, \mu]$

N: VAR null_set

X: VAR set$[T]$

E: VAR negligible

f: VAR $[T \to \mathbb{R}]$

g: VAR measurable_function

null_subset: LEMMA $(X \subseteq N) \Rightarrow$ null_set?(X)

null_is_negligible: LEMMA null_set?$(X) \Leftrightarrow$ negligible_set?(X)

ae_eq_measurable: LEMMA $f = g$ *a.e.* $\Rightarrow$ measurable_function?(f)

END complete_measure_theory

30 measure_completion_aux

measure_completion_aux[T: TYPE]: THEORY
BEGIN

IMPORTING subset_algebra_def[T], measure_def, measure_theory, measure_props

XS: VAR setofsets[T]

A, B, X: VAR set[T]

z: VAR $\overline{\mathbb{R}}_{\geq 0}$

almost_measurable?(S: sigma_algebra[T], m: measure_type[T, S])(X): bool =
$\exists$ (Y: (S), N_1, N_2: negligible[T, S, m]):
$X = ((Y \cup N_1) \setminus N_2)$

empty_almost_measurable: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
almost_measurable?(S, m)($\emptyset$[T])

complement_almost_measurable: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
almost_measurable?(S, m)(X) $\Leftrightarrow$
almost_measurable?(S, m)($\overline{X}$)

Union_almost_measurable: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
every(almost_measurable?(S, m), XS) $\wedge$ is_countable(XS) $\Rightarrow$
almost_measurable?(S, m)($\bigcup$XS)

completion(S: sigma_algebra[T], m: measure_type[T, S]): sigma_algebra[T] =
{X | almost_measurable?(S, m)(X)}

generated_completion: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
$\mathcal{S}$(($S \cup$ extend [setof[T], negligible[T, S, m], bool, FALSE](fullset[negligible[T, S, m]])))
= completion(S, m)

completion_extends: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
$S(X) \Rightarrow$ completion(S, m)(X)

negligible_completion: LEMMA
$\forall$ (S: sigma_algebra[T], m: measure_type[T, S]):
negligible_set?[T, S, m](X) $\Rightarrow$ completion(S, m)(X)

is_completion(S: sigma_algebra[T], m: measure_type[T, S])(A, B): bool =
completion(S, m)(A) $\wedge$ $S(B)$ $\Rightarrow$
($\exists$ (N_1, N_2: negligible[T, S, m]):
$A = ((B \cup N_1) \setminus N_2)$)

m_completions: LEMMA
  $\forall$ (S: sigma_algebra[T], m: measure_type[T, S], X, A, B):
    completion(S, m)(X) $\wedge$
     $S(A)$ $\wedge$
      $S(B)$ $\wedge$ is_completion(S, m)(X, A) $\wedge$ is_completion(S, m)(X, B)
     $\Rightarrow$ $m(A) = m(B)$

choose_completion: LEMMA
  $\forall$ (S: sigma_algebra[T], m: measure_type[T, S], X):
    completion(S, m)(X) $\Rightarrow$
     is_completion(S, m)
       (X,
        choose({Y: (S) |
          $\exists$ (N_1, N_2: negligible[T, S, m]):
           $X = ((Y \cup N_1) \setminus N_2)$}))

completion(S: sigma_algebra[T], m: measure_type[T, S]):
      complete_measure[T, completion(S, m)] =
    λ (X: (completion(S, m))):
      m(choose({Y: (S) |
        $\exists$ (N_1, N_2: negligible[T, S, m]):
         $X = ((Y \cup N_1) \setminus N_2)$}))

completion_measurable: LEMMA
  $\forall$ (S: sigma_algebra[T], m: measure_type[T, S], X: (S)):
    completion(S, m)(X) = $m(X)$

completion_negligible: LEMMA
  $\forall$ (S: sigma_algebra[T], m: measure_type[T, S], N: negligible[T, S, m]):
    completion(S, m)(N) = (TRUE, 0)

END measure_completion_aux

31 measure_completion

measure_completion[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_completion_aux[T], measure_theory[T, S, m]

X: VAR (S)

N: VAR negligible[T, S, m]

sigma_algebra_completion: sigma_algebra[T] = completion(S, m)

generated_completion: LEMMA
$\mathcal{S}((S \cup$ extend [setof[T], negligible[T, S, m], bool, FALSE](fullset[negligible[T, S, m]])))
= sigma_algebra_completion

completion: complete_measure[T, completion(S, m)] =
completion(S, m)

completion_measurable: LEMMA completion(X) = $m(X)$

completion_negligible: LEMMA completion(N) = (TRUE, 0)

END measure_completion

Part VI

Integration

32 isf

isf[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space[T, S], measure_theory[T, S, m], measure_props[T, S, m]

x: VAR T

f: VAR $[T \to \mathbb{R}]$

g: VAR measurable_function

X: VAR (S)

Y: VAR set[T]

nonzero_set?(f): set[T] = $\{x \mid f(x) \neq 0\}$

nonzero_measurable: LEMMA measurable_set?(nonzero_set?(g))

nonzero_set_phi: LEMMA nonzero_set?(ϕ_X) = X

isf?(f): bool = simple?(f) $\wedge$ mu_fin?(nonzero_set?(f))

isf_zero: LEMMA isf?(λ x: 0)

isf: TYPE+ = (isf?) CONTAINING (λ x: 0)

isf_is_simple: JUDGEMENT isf SUBTYPE_OF simple

i, i_1, i_2: VAR isf

w: VAR sequence[isf]

c: VAR $\mathbb{R}$

n: VAR $\mathbb{N}$

pn: VAR $\mathbb{N}_{>0}$

E: VAR (mu_fin?)

h: VAR simple

nnx: VAR $\mathbb{R}_{\geq 0}$

isf_add: JUDGEMENT $+(i_1,\ i_2)$ HAS_TYPE isf

isf_scal: JUDGEMENT $\times(c,\ i)$ HAS_TYPE isf

isf_opp: JUDGEMENT $-(i)$ HAS_TYPE isf

isf_diff: JUDGEMENT $-(i_1,\ i_2)$ HAS_TYPE isf

isf_abs: JUDGEMENT abs(i) HAS_TYPE isf

isf_min: JUDGEMENT min$(i_1,\ i_2)$ HAS_TYPE isf

isf_max: JUDGEMENT max$(i_1,\ i_2)$ HAS_TYPE isf

isf_minimum: JUDGEMENT minimum$(w,\ n)$ HAS_TYPE isf

isf_maximum: JUDGEMENT maximum$(w,\ n)$ HAS_TYPE isf

isf_plus: JUDGEMENT plus(i) HAS_TYPE isf

isf_minus: JUDGEMENT minus(i) HAS_TYPE isf

isf_sq: JUDGEMENT sq(i) HAS_TYPE isf

isf_prod: JUDGEMENT $\times(i_1,\ i_2)$ HAS_TYPE isf

isf_phi: JUDGEMENT $\phi(E)$ HAS_TYPE isf

isf_expt: JUDGEMENT expt$(i,$ pn$)$ HAS_TYPE isf

isf_times_simple_is_isf: JUDGEMENT $\times(i,\ h)$ HAS_TYPE isf

P: VAR pred[isf]

isf_induction: LEMMA
  $(P(\lambda\ x\colon\ 0)\ \wedge\ (\forall\ c,\ E,\ i\colon\ P(i)\ \Rightarrow\ P(c \times \phi_E + i)))\ \Rightarrow$
    $P(i)$

$p,\ p_1,\ p_2$: VAR finite_partition$[T]$

finite_partition_of?$(f)(p)$: bool =
  $\forall\ (E\colon\ (p))$:
    $S(E)\ \wedge$
     constant_over?$(f)(E)\ \wedge$
      (empty?$(E)\ \vee\ f($choose$(E))\ =\ 0\ \vee$ mu_fin?$(E))$

isf_def: LEMMA
  isf?$(f)\ \Leftrightarrow\ (\exists\ (p\colon$ (finite_partition_of?$(f)))$: TRUE$)$

IMPORTING sigma_set@sigma_countable

isf_integral(i): $\mathbb{R}$ =
$\sum_{\text{image}[T,\ \mathbb{R}](i,\ \text{fullset}[T])} \lambda\ c:$ IF $c = 0$ THEN 0 ELSE $c \times \mu(\text{inverse\_image}\ [T,\ \mathbb{R}](i,\ \text{singleton}\ [\mathbb{R}](c))$

isf_integral_phi: LEMMA isf_integral(ϕ_E) = $\mu(E)$

isf_integral_zero: LEMMA isf_integral($\lambda\ x$: 0) = 0

isf_integral_def: LEMMA
finite_partition_of?(i)(p) $\Rightarrow$
isf_integral(i) =
LET f =
$\lambda\ Y$:
IF $(\neg\ p(Y)) \vee$ empty?(Y) $\vee\ i(\text{choose}[T](Y)) = 0$
THEN 0
ELSE $i(\text{choose}[T](Y)) \times \mu(Y)$
ENDIF
IN $\sum_p f$

isf_integral_scal: LEMMA
isf_integral($c \times i$) = $c \times$ isf_integral(i)

isf_integral_opp: LEMMA
isf_integral($-i$) = $-$isf_integral(i)

isf_integral_add: LEMMA
isf_integral($i_1 + i_2$) =
isf_integral(i_1) + isf_integral(i_2)

isf_integral_diff: LEMMA
isf_integral($i_1 - i_2$) =
isf_integral(i_1) $-$ isf_integral(i_2)

isf_integral_pos: LEMMA
$(\forall\ x:\ i(x) \geq 0) \Rightarrow$ isf_integral(i) ≥ 0

isf_integral_le: LEMMA
$(\forall\ x:\ i_1(x) \leq i_2(x)) \Rightarrow$
isf_integral(i_1) $\leq$ isf_integral(i_2)

isf_integral_abs: LEMMA
|isf_integral(i)| $\leq$ isf_integral($|i|$)

isf_bounded: LEMMA
$\exists$ nnx: $\forall\ x:\ -\text{nnx} \leq i(x) \wedge i(x) \leq \text{nnx}$

isf_integral_bound: LEMMA
$(\forall\ x:\ |i(x)| \leq \text{nnx}) \Rightarrow$
isf_integral($|i|$) $\leq$ nnx $\times\ \mu$(nonzero_set?(i))

isf_ae_0: LEMMA

$(\mathrm{simple?}(f)\ \wedge\ f = 0\ \ a.e.) \Leftrightarrow$
$(\mathrm{isf?}(f)\ \wedge\ \mathrm{isf\_integral}(|f|)\ =\ 0)$

isf_ae_eq: LEMMA
$i_1 = i_2\ \ a.e.\ \Rightarrow\ \mathrm{isf\_integral}(i_1)\ =\ \mathrm{isf\_integral}(i_2)$

isf_ae_0_le: LEMMA $i \geq 0\ \ a.e.\ \Rightarrow\ 0\ \leq\ \mathrm{isf\_integral}(i)$

isf_ae_le: LEMMA
$i_1 \leq i_2\ \ a.e.\ \Rightarrow\ \mathrm{isf\_integral}(i_1)\ \leq\ \mathrm{isf\_integral}(i_2)$

isf_ae_ge_0: LEMMA $i \geq 0\ \ a.e.\ \wedge\ \mathrm{isf\_integral}(i)\ =\ 0\ \Rightarrow\ i = 0\ \ a.e.$

u: VAR increasing_nn_simple

isf_convergence: LEMMA
$u \nearrow i\ \Rightarrow\ (\mathrm{isf\_integral} \circ u) \nearrow \mathrm{isf\_integral}(i)$

END isf

33 nn_integral

nn_integral[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
    (IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space[T, S], measure_props[T, S, m], measure_theory[T, S, m],
    isf[T, S, m]

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
    topological_convergence.convergent?

limit: MACRO [convergent → $\mathbb{R}$] = topological_convergence.limit

n: VAR $\mathbb{N}$

pn: VAR $\mathbb{N}_{>0}$

x: VAR T

c: VAR $\mathbb{R}_{\geq 0}$

E: VAR measurable_set

F: VAR (mu_fin?)

g: VAR measurable_function

h: VAR nn_bounded_measurable

nn_isf?(i: isf): bool = $\forall$ x: $i(x) \geq 0$

nn_isf: TYPE+ = (nn_isf?) CONTAINING (λ x: 0)

i: VAR nn_isf

w: VAR sequence[nn_isf]

increasing_nn_isf?(u: sequence[nn_isf]): bool =
    pointwise_increasing?(u)

increasing_nn_isf: TYPE+ = (increasing_nn_isf?) CONTAINING (λ
    n:
    λ
    x:
    0)

u, u_1, u_2: VAR increasing_nn_isf

nn_integrable?(g: [T → $\mathbb{R}_{\geq 0}$]): bool =
    $\exists$ u:

$u \longrightarrow g \ \wedge$
topological_convergence.convergent?(isf_integral $\circ$ u)

nn_integrable_zero: LEMMA nn_integrable?(λ x: 0)

nn_integrable: TYPE+ = (nn_integrable?) CONTAINING (λ x: 0)

f, f_1, f_2: VAR nn_integrable

nn_integrable_is_nonneg: LEMMA $f(x) \ \geq \ 0$

nn_integrable_is_measurable: JUDGEMENT nn_integrable SUBTYPE_OF
measurable_function

nn_convergence: LEMMA
$u_1 \longrightarrow f \ \wedge$
$u_2 \longrightarrow f \ \wedge$ topological_convergence.convergent?(isf_integral $\circ$ u_1)
$\Rightarrow$
(topological_convergence.convergent?(isf_integral $\circ$ u_2) $\wedge$
topological_convergence.limit(isf_integral $\circ$ u_1) =
topological_convergence.limit(isf_integral $\circ$ u_2))

nn_integral(f): $\mathbb{R}_{\geq 0}$ =
topological_convergence.limit
(isf_integral $\circ$ choose($\{u \ | \ u \longrightarrow f\}$))

nn_integrable_add: JUDGEMENT +(f_1, f_2) HAS_TYPE nn_integrable

nn_integrable_scal: JUDGEMENT $\times$(c, f) HAS_TYPE nn_integrable

nn_isf_is_nn_integrable: JUDGEMENT nn_isf SUBTYPE_OF nn_integrable

nn_integral_isf: LEMMA nn_integral(i) = isf_integral(i)

nn_integrable_le: LEMMA
($\forall$ x: $0 \ \leq \ g(x) \ \wedge \ g(x) \ \leq \ f(x)$) $\Rightarrow$
(nn_integrable?(g) $\wedge$ nn_integral(g) $\leq$ nn_integral(f))

nn_integral_zero: LEMMA nn_integral(λ x: 0) = 0

nn_integral_phi: LEMMA nn_integral(ϕ_F) = $\mu(F)$

nn_integral_add: LEMMA
nn_integral($f_1 + f_2$) =
nn_integral(f_1) + nn_integral(f_2)

nn_integral_scal: LEMMA
nn_integral($c \times f$) = $c \times$ nn_integral(f)

nn_integrable_prod: JUDGEMENT $\times$(f, h) HAS_TYPE nn_integrable nn_indefinite_integrable: LEMMA nn_integrable?($\phi_E \times f$)

nn_0_le: LEMMA $0 \leq$ nn_integral(f)

nn_integral_def: LEMMA
$\exists\ u$:
$u \longrightarrow f \ \wedge$ isf_integral $\circ\, u \longrightarrow$ nn_integral(f)

END nn_integral

34 integral

$\int[T\colon$ TYPE, (IMPORTING subset_algebra_def$[T]$) $S\colon$ sigma_algebra,
(IMPORTING measure_def$[T,\ S]$) $m\colon$ measure_type$]\colon$ THEORY
BEGIN

IMPORTING measure_space$[T,\ S]$, measure_theory$[T,\ S,\ m]$, nn_integral$[T,\ S,\ m]$

$g,\ g_1,\ g_2,\ g_3,\ g_4\colon$ VAR nn_integrable

$x\colon$ VAR T

integrable?($f\colon [T \to \mathbb{R}]$): bool $=$
$\exists\ (g,\ h\colon$ nn_integrable$)\colon\ f\ =\ g - h$

integrable: TYPE+ = (integrable?) CONTAINING ($\lambda\ x\colon\ 0$)

nn_integrable_is_integrable: JUDGEMENT nn_integrable SUBTYPE_OF
integrable

isf_is_integrable: JUDGEMENT isf SUBTYPE_OF integrable

integrable_is_measurable: JUDGEMENT integrable SUBTYPE_OF
measurable_function

$f,\ f_1,\ f_2\colon$ VAR integrable

$w\colon$ VAR sequence[integrable]

$f_0\colon$ VAR $[T \to \mathbb{R}]$

$h\colon$ VAR measurable_function

$\varepsilon\colon$ VAR $\mathbb{R}_{>0}$

$c\colon$ VAR $\mathbb{R}$

nnc: VAR $\mathbb{R}_{\geq 0}$

$E\colon$ VAR measurable_set

$F\colon$ VAR (mu_fin?)

$i\colon$ VAR isf

$n\colon$ VAR $\mathbb{N}$

integrable_equiv: LEMMA
$g_1 - g_3\ =\ g_2 - g_4\ \Rightarrow$
nn_integral(g_1) $-$ nn_integral(g_3) $=$
nn_integral(g_2) $-$ nn_integral(g_4)

integrable_add: JUDGEMENT $+(f_1, f_2)$ HAS_TYPE integrable

integrable_scal: JUDGEMENT $\times(c, f)$ HAS_TYPE integrable

integrable_opp: JUDGEMENT $-(f)$ HAS_TYPE integrable

integrable_diff: JUDGEMENT $-(f_1, f_2)$ HAS_TYPE integrable

integrable_zero: LEMMA integrable?($\lambda\ x$: 0)

integrals(f): set$[\mathbb{R}]$ =
  $\{c$ |
    $\exists$ (g, h: nn_integrable):
      $f = g - h \wedge$
      c = nn_integral(g) $-$ nn_integral(h)$\}$

nonempty_integrals: LEMMA nonempty?$[\mathbb{R}]$(integrals(f))

singleton_integrals: LEMMA singleton?$[\mathbb{R}]$(integrals(f))

$\int f$: $\mathbb{R}$ = choose$[\mathbb{R}]$(integrals(f))

nn_integrable_is_nn_integrable: LEMMA
  ($\forall\ x$: $f(x) \geq 0$) $\Rightarrow$ nn_integrable?(f)

integral_nn: LEMMA $\int g$ = nn_integral(g)

integral_zero: LEMMA $\int \lambda\ x:\ 0 = 0$

integral_phi: LEMMA $\int \phi_F = \mu(F)$

integral_add: LEMMA $\int f_1 + f_2 = \int f_1 + \int f_2$

integral_scal: LEMMA $\int c \times f = c \times \int f$

integral_opp: LEMMA $\int -f = -\int f$

integral_diff: LEMMA $\int f_1 - f_2 = \int f_1 - \int f_2$

integral_nonneg: LEMMA ($\forall\ x$: $f(x) \geq 0$) $\Rightarrow \int f \geq 0$

integrable_abs: JUDGEMENT abs(f) HAS_TYPE integrable

integrable_max: JUDGEMENT max(f_1, f_2) HAS_TYPE integrable

integrable_min: JUDGEMENT min(f_1, f_2) HAS_TYPE integrable

integrable_plus: JUDGEMENT plus(f) HAS_TYPE integrable

integrable_minus: JUDGEMENT minus(f) HAS_TYPE integrable integral_abs: LEMMA $\left|\int f\right| \ \leq\ \int |f|$

integrable_pm_def: LEMMA
integrable?(f_0) $\Leftrightarrow$
(integrable?(${f_0}^{+}$) $\wedge$ integrable?(${f_0}^{-}$))

integral_pm: LEMMA $\int f \ = \ \int f^{+} - \int f^{-}$

integrable_abs_def: LEMMA integrable?($|h|$) $\Leftrightarrow$ integrable?(h)

integrable_nz_finite: LEMMA
measurable_set?($\{x \ \mid \ |f(x)| \ \geq \ \varepsilon\}$) $\wedge$
mu_fin?($\{x \ \mid \ |f(x)| \ \geq \ \varepsilon\}$)

isf_integral: LEMMA $\int i \ = $ isf_integral(i)

integral_ae_eq: LEMMA
$f = h$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int f \ = \ \int h$)

integral_prod: LEMMA
$|h| \leq \lambda \ x :$ nnc *a.e.* $\Rightarrow$
(integrable?($f \times h$) $\wedge$
$\int |f \times h| \ \leq$ nnc $\times \int |f|$)

indefinite_integrable: LEMMA integrable?($\phi_E \times f$)

integral_ae_le: LEMMA $f_1 \leq f_2$ *a.e.* $\Rightarrow \int f_1 \ \leq \ \int f_2$

integral_ae_abs: LEMMA
$|h| \leq |f|$ *a.e.* $\Rightarrow$
(integrable?(h) $\wedge$ $\left|\int h\right| \ \leq \ \int |f|$)

bounded_is_indefinite_integrable: LEMMA
bounded?($\phi_F \times h$) $\Rightarrow$
(integrable?($\phi_F \times h$) $\wedge$
$\left|\int \phi_F \times h\right| \ \leq$
$\mu(F) \times$ sup_norm($\phi_F \times h$))

integral_abs_0: LEMMA $\int |f| \ = \ 0 \ \Rightarrow \ f = 0$ *a.e.*

measurable_ae_0: LEMMA
$h = 0$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int h \ = \ 0$)

integral_ae_ge_0: LEMMA $f \geq 0$ *a.e.* $\wedge$ $\int f \ = \ 0 \ \Rightarrow \ f = 0$ *a.e.*

integrable_maximum: JUDGEMENT maximum(w, n) HAS_TYPE integrable

integrable_minimum: JUDGEMENT minimum(w, n) HAS_TYPE integrable

integrable_split: LEMMA

$\forall \ (h\colon \ [T \ \rightarrow \ \mathbb{R}])\colon$
integrable?$(h) \ \Leftrightarrow$
integrable?$(\phi_E \times h) \ \wedge$
integrable?$(\phi_{\overline{E}} \times h)$

integral_split: LEMMA
$\int f \ =$
$\int \phi_E \times f + \int \phi_{\overline{E}} \times f$

END $\int$

35 finite_integral

finite_integral[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: finite_measure]: THEORY

BEGIN

IMPORTING $\int$[T, S, to_measure(μ)]

bounded_measurable_is_integrable: JUDGEMENT bounded_measurable SUBTYPE_OF
integrable

END finite_integral

36 integral_convergence_scaf

integral_convergence_scaf[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_space[T, S], measure_theory[T, S, m], $\int$[T, S, m]

f: VAR $[T \to \mathbb{R}]$

F: VAR sequence[integrable]

monotone_convergence_scaf: LEMMA
$F \nearrow f \ \wedge$ bounded?$(\int \circ F) \Rightarrow$
(integrable?$(f) \ \wedge \ (\int \circ F) \nearrow \int f$)

END integral_convergence_scaf

37 integral_convergence

integral_convergence[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY

BEGIN

IMPORTING integral_convergence_scaf[T, S, m]

i, j, n: VAR $\mathbb{N}$

f, g: VAR integrable

F: VAR sequence[integrable]

E: VAR negligible

x: VAR T

monotone_convergence: THEOREM
increasing?(F) *a.e.* $\Rightarrow$
$(((\exists\ f:\ F \longrightarrow f\ \ a.e.) \Leftrightarrow$ bounded?$(\int \circ F)) \wedge$
$(\forall\ f:\ F \longrightarrow f\ \ a.e. \Rightarrow (\int \circ F) \nearrow \int f))$

dominated_convergence: THEOREM
$(\forall\ n:\ |F(n)| \leq f\ \ a.e.) \wedge$ ae_convergent?(F) $\Rightarrow$
$(\exists\ g:\ F \longrightarrow g\ \ a.e. \wedge \int \circ F \longrightarrow \int g)$

END integral_convergence

38 complete_integral

complete_integral$[T$: TYPE, (IMPORTING subset_algebra_def$[T]$) S: sigma_algebra,
(IMPORTING measure_def$[T, S]$) μ: complete_measure$]$: THEORY

BEGIN

IMPORTING complete_measure_theory$[T, S, \mu]$, $\int[T, S, \mu]$,
integral_convergence$[T, S, \mu]$

f: VAR integrable

h: VAR $[T \rightarrow \mathbb{R}]$

F: VAR sequence[integrable]

n: VAR $\mathbb{N}$

x: VAR T

complete_integral_ae_eq: LEMMA
$f = h$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int f = \int h$)

complete_measurable_ae_0: LEMMA
$h = 0$ *a.e.* $\Rightarrow$ (integrable?(h) $\wedge$ $\int h = 0$)

monotone_convergence_complete: THEOREM
ae_monotonic_converges?(F, h) $\wedge$ bounded?($\int \circ F$) $\Rightarrow$
(integrable?(h) $\wedge$
monotonic_converges?(($\int \circ F$), $\int h$))

dominated_convergence_complete: THEOREM
($\forall$ n: $|F(n)| \leq f$ *a.e.*) $\wedge$ $F \longrightarrow h$ *a.e.* $\Rightarrow$
(integrable?(h) $\wedge$ $\int \circ F \longrightarrow \int h$)

END complete_integral

39 indefinite_integral

indefinite_integral[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) m: measure_type]: THEORY
BEGIN

IMPORTING measure_props[T, S, m], $\int$[T, S, m], integral_convergence[T, S, m]

f, f_1, f_2: VAR integrable

g: VAR $[T \to \mathbb{R}]$

h: VAR measurable_function[T, S]

DX: VAR disjoint_indexed_measurable

E, E_1, E_2: VAR measurable_set

F: VAR (mu_fin?)

N: VAR null_set

c: VAR $\mathbb{R}$

x: VAR T

n: VAR $\mathbb{N}$

integrable?(E)(g): bool = integrable?($\phi_E \times g$)

$\int_{E:\ \text{measurable\_set}} f$: (integrable?(E)): $\mathbb{R}$ =
$\int \phi_E \times f$

indefinite_emptyset: LEMMA $\int_\emptyset g = 0$

indefinite_fullset: LEMMA $\int_{\text{fullset}} f = \int f$

indefinite_eq_0: LEMMA
∀ (E: measurable_set, f: (integrable?(E))):
ae_in?($\lambda\ x$: $f(x) > 0$)(E) ∧ $\int \phi_E \times f = 0 \Rightarrow$
(mu_fin?(E) ∧ $\mu(E) = 0$)

indefinite_eq: LEMMA
(∀ E: $\int_E f_1 = \int_E f_2$) ⇒ $f_1 = f_2$ *a.e.*

indefinite_phi: LEMMA $\int_E \phi_F = \mu((E \cap F))$

indefinite_add: LEMMA
∀ (E: measurable_set, f_1, f_2: (integrable?(E))):
$\int_E f_1 + f_2 = \int_E f_1 + \int_E f_2$

indefinite_scal: LEMMA
$\forall$ (E: measurable_set, f: (integrable?(E))):
$\int_E (c \times f) = c \times \int_E f$

indefinite_opp: LEMMA
$\forall$ (E: measurable_set, f: (integrable?(E))):
$\int_E -f = -\int_E f$

indefinite_diff: LEMMA
$\forall$ (E: measurable_set, f_1, f_2: (integrable?(E))):
$\int_E f_1 - f_2 = \int_E f_1 - \int_E f_2$

indefinite_ae_eq: LEMMA
$f_1 = f_2$ *a.e.* $\Leftrightarrow$ ($\forall$ E: $\int_E f_1 = \int_E f_2$)

indefinite_0_le: LEMMA $f \geq 0$ *a.e.* $\Leftrightarrow$ ($\forall$ E: $0 \leq \int_E f$)

indefinite_le: LEMMA
$f_1 \leq f_2$ *a.e.* $\Leftrightarrow$ ($\forall$ E: $\int_E f_1 \leq \int_E f_2$)

indefinite_pm: LEMMA
$\forall$ (E: measurable_set, f: (integrable?(E))):
$\int_E f = \int_E f^+ - \int_E f^-$

indefinite_union: LEMMA
$\forall$ (E_1, E_2: measurable_set, f: (integrable?(($E_1 \cup E_2$)))):
disjoint?(E_1, E_2) $\Rightarrow$
$\int_{(E_1 \cup E_2)} f = \int_{E_1} f + \int_{E_2} f$

indefinite_subset: LEMMA
$\forall$ (E_1, E_2: measurable_set, f: (integrable?(E_2))):
($E_1 \subseteq E_2$) $\wedge$ $f \geq 0$ *a.e.* $\Rightarrow$ $\int_{E_1} f \leq \int_{E_2} f$

indefinite_null: LEMMA $\int_N h = 0$

indefinite_countably_additive: LEMMA
series(λ n : $\int_{\mathrm{DX}(n)} f$) $\longrightarrow \int_{\bigcup \mathrm{DX}} f$

END indefinite_integral

40 measure_equality

measure_equality[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING measure_def[T, S]

μ, ν: VAR measure_type

x: VAR T

f: VAR $[T \to \mathbb{R}]$

g: VAR $[T \to \mathbb{R}_{\geq 0}]$

E: VAR (S)

IMPORTING $\int$

measure_eq_isf?: LEMMA
$(\forall\ E:\ \mu(E) = \nu(E)) \Rightarrow$
(isf?[T, S, μ](f) $\Leftrightarrow$ isf?[T, S, ν](f))

measure_eq_isf: LEMMA
$(\forall\ E:\ \mu(E) = \nu(E)) \wedge$
(isf?[T, S, μ](f) $\vee$ isf?[T, S, ν](f))
$\Rightarrow$
(isf_integral[T, S, μ](f) $=$
isf_integral[T, S, ν](f))

measure_eq_nn_integrable?: LEMMA
$(\forall\ E:\ \mu(E) = \nu(E)) \Rightarrow$
(nn_integrable?[T, S, μ](g) $\Leftrightarrow$
nn_integrable?[T, S, ν](g))

measure_eq_nn_integral: LEMMA
$(\forall\ E:\ \mu(E) = \nu(E)) \wedge$
(nn_integrable?[T, S, μ](g) $\vee$ nn_integrable?[T, S, ν](g))
$\Rightarrow$
(nn_integral[T, S, μ](g) $=$ nn_integral[T, S, ν](g))

measure_eq_integrable?: LEMMA
$(\forall\ E:\ \mu(E) = \nu(E)) \Rightarrow$
(integrable?[T, S, μ](f) $\Leftrightarrow$ integrable?[T, S, ν](f))

measure_eq_integral: LEMMA
$(\forall\ E:\ \mu(E) = \nu(E)) \wedge$
(integrable?[T, S, μ](f) $\vee$ integrable?[T, S, ν](f))
$\Rightarrow (\int f\ =\ \int f)$

END measure_equality

41 measure_contraction

measure_contraction[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra]: THEORY
BEGIN

IMPORTING measure_def[T, S], measure_space[T, S]

$f \times g$: $[T \to \mathbb{R}]$: MACRO $[T \to \mathbb{R}] = f \times g$

μ: VAR measure_type

ν: VAR sigma_finite_measure

A, E: VAR measurable_set

f: VAR measurable_function

i: VAR $\mathbb{N}$

contraction(μ, A): measure_type = λ E: $\mu((A \cap E))$

fm_contraction(μ: measure_type, A: $\{E \mid \mu(E)`1\}$): finite_measure =
λ E: $\mu((A \cap E))`2$

sigma_finite_contraction_def: LEMMA
$\nu(E) = \sum \lambda\ i$: (TRUE, fm_contraction(ν, A_of(ν)(i))(E))

IMPORTING isf, nn_integral, $\int$, indefinite_integral, integral_convergence

contraction_is_sigma_finite: JUDGEMENT contraction(ν, A) HAS_TYPE
sigma_finite_measure

contraction_isf: LEMMA
$\forall$ (f: simple):
isf?[T, S, contraction(μ, A)](f) $\Leftrightarrow$
isf?[T, S, μ](($\phi_A \times f$))

contraction_isf_integral: LEMMA
$\forall$ (f: isf[T, S, contraction(μ, A)]):
isf_integral[T, S, contraction(μ, A)](f) =
isf_integral[T, S, μ](($\phi_A \times f$))

contraction_nn_integrable: LEMMA
$\forall$ (f: nn_measurable):
nn_integrable?[T, S, contraction(μ, A)](f) $\Leftrightarrow$
nn_integrable?[T, S, μ](($\phi_A \times f$))

contraction_nn_integral: LEMMA
$\forall$ (f: nn_integrable[T, S, contraction(μ, A)]):
nn_integral[T, S, contraction(μ, A)](f) =
nn_integral[T, S, μ](($\phi_A \times f$))

contraction_integrable: LEMMA
  integrable?$[T,\ S,\ \text{contraction}(\mu,\ A)](f) \Leftrightarrow$
   integrable?$[T,\ S,\ \mu]((\phi_A \times f))$

contraction_integral: LEMMA
  $\forall\ (f\colon \text{integrable}[T,\ S,\ \text{contraction}(\mu,\ A)])$:
    $\int f\ =\ \int (\phi_A \times f)$

END measure_contraction

42 measure_contraction_props

measure_contraction_props[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: measure_type]: THEORY
BEGIN

IMPORTING measure_props[T, S, μ], measure_contraction[T, S], integral_convergence_scaf

A: VAR disjoint_indexed_measurable

h: VAR measurable_function

x: VAR T

n: VAR $\mathbb{N}$

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
topological_convergence.convergent?

contraction_integrable_def: LEMMA
$\bigcup A$ = fullset[T] $\wedge$ ($\forall$ x: $h(x) \geq 0$) $\Rightarrow$
(integrable?[T, S, μ](h) $\Leftrightarrow$
(($\forall$ n: integrable?[T, S, contraction(μ, $A(n)$)](h)) $\wedge$
topological_convergence.convergent?
(series(λ n: $\int h$))))

END measure_contraction_props

43 sigma_finite_measure_props

sigma_finite_measure_props[T: TYPE, (IMPORTING subset_algebra_def[T]) S: sigma_algebra,
(IMPORTING measure_def[T, S]) μ: sigma_finite_measure]: THEORY

BEGIN

IMPORTING measure_contraction_props[T, S, μ], measure_equality[T, S]

f: VAR nn_integrable[T, S, μ]

g: VAR integrable[T, S, μ]

h: VAR nn_measurable[T, S]

A: VAR (S)

x: VAR T

n: VAR $\mathbb{N}$

F: VAR sequence[[$T \to \mathbb{R}$]]

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
topological_convergence.convergent?

sfm_integrable: LEMMA
(($\forall$ n: integrable?[T, S, contraction(μ, A_of(μ)(n))](h)) $\wedge$
topological_convergence.convergent?(series(λ n: $\int h$)))
$\Leftrightarrow$ integrable?[T, S, μ](h)

sfm_integral: LEMMA series(λ n : $\int f$) $\longrightarrow \int f$

sfm_component_eq: LEMMA
to_measure(fm_contraction(μ, A_of(μ)(n)))(A) = contraction(μ, A_of(μ)(n))(A)

IMPORTING integral_convergence[T, S, μ]

sfm_monotone_convergence: LEMMA
increasing?(F) *a.e.* $\wedge$
($\forall$ n, x: $F(n)(x) \geq 0$) $\wedge$
($\forall$ n: integrable?[T, S, contraction(μ, P_of(μ)(n))]($F(n)$))
$\Rightarrow$
((($\exists$ g: $F \longrightarrow g$ *a.e.*) $\Leftrightarrow$ bounded?(λ n: $\int F(n)$)) $\wedge$
($\forall$ g: $F \longrightarrow g$ *a.e.* $\Rightarrow$ λ n : $\int F(n) \nearrow \int g$))

END sigma_finite_measure_props

Part VII

Product Measures

44 product_finite_measure

product_finite_measure[(IMPORTING subset_algebra_def) T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2]]: THEORY
BEGIN

IMPORTING product_sigma_def[T_1, T_2], product_sigma[T_1, T_2, S_1, S_2], measure_def[T_1, S_1],
measure_def[T_2, S_2], measure_def[[T_1, T_2], $S_1 \times S_2$], $\int$, finite_measure,
monotone_classes, integral_convergence

M: VAR $(S_1 \times S_2)$

x: VAR T_1

y: VAR T_2

X: VAR (S_1)

Y: VAR (S_2)

E: VAR sequence[$(S_1 \times S_2)$]

μ: VAR finite_measure[T_1, S_1]

ν: VAR finite_measure[T_2, S_2]

x_section_bounded: LEMMA
$0 \leq (\nu \circ \text{x\_section}(M))(x) \wedge$
$(\nu \circ \text{x\_section}(M))(x) \leq \nu(\text{fullset}[T_2])$

y_section_bounded: LEMMA
$0 \leq (\mu \circ \text{y\_section}(M))(y) \wedge$
$(\mu \circ \text{y\_section}(M))(y) \leq \mu(\text{fullset}[T_1])$

x_section_measurable: LEMMA
measurable_function?[T_1, S_1]($\nu \circ$ x_section(M))

y_section_measurable: LEMMA
measurable_function?[T_2, S_2]($\mu \circ$ y_section(M))

x_section_integrable: LEMMA
integrable?[T_1, S_1, to_measure(μ)]($\nu \circ$ x_section(M))

y_section_integrable: LEMMA
integrable?[T_2, S_2, to_measure(ν)]($\mu \circ$ y_section(M))

rectangle_measure1: LEMMA

$M \ = \ X \times Y \ \Rightarrow$
$\int \nu \circ \text{x\_section}(M) \ = \ \mu(X) \times \nu(Y)$

rectangle_measure2: LEMMA
$M \ = \ X \times Y \ \Rightarrow$
$\int \mu \circ \text{y\_section}(M) \ = \ \mu(X) \times \nu(Y)$

$\mu \times \nu$: finite_measure$\left[\left[T_1, \ T_2\right], \ S_1 \times S_2\right] \ =$
$\lambda \ M: \ \int \nu \circ \text{x\_section}(M)$

fm_times_alt: LEMMA
finite_measure?$\left[\left[T_1, \ T_2\right], \ S_1 \times S_2\right]$
$(\lambda \ M: \ \int \mu \circ \text{y\_section}(M))$

finite_product_alt: THEOREM
fm_times$(\mu, \ \nu)(M) \ = \ \int \mu \circ \text{y_section}(M)$

END product_finite_measure

45 product_measure

product_measure[(IMPORTING subset_algebra_def) T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2]]: THEORY

BEGIN

IMPORTING product_sigma[T_1, T_2, S_1, S_2], measure_contraction[T_1, S_1],
measure_contraction[T_2, S_2], measure_contraction[[T_1, T_2], $S_1 \times S_2$],
product_finite_measure[T_1, T_2, S_1, S_2], $\overline{\mathbb{R}}_{\geq 0}$@double_index[set[[$T_1$, T_2]]]

μ: VAR sigma_finite_measure[T_1, S_1]

ν: VAR sigma_finite_measure[T_2, S_2]

X: VAR (S_1)

Y: VAR (S_2)

M: VAR ($S_1 \times S_2$)

z: VAR [T_1, T_2]

i, j, n: VAR $\mathbb{N}$

product_measure_approx(μ, ν)(i, j): finite_measure[[T_1, T_2], $S_1 \times S_2$] =
fm_contraction[T_1, S_1](μ, A_of(μ)(i)) $\times$ fm_contraction [T_2, S_2](ν, A_of(ν)(j))

$\mu \times \nu$: sigma_finite_measure[[T_1, T_2], $S_1 \times S_2$] =
λ M:
$\sum \lambda\ i$: $\sum \lambda\ j$: to_measure(product_measure_approx(μ, ν)(i, j))(M)

m_times_alt: LEMMA
m_times(μ, ν)(M) = $\sum \lambda\ j$: $\sum \lambda\ i$: to_measure(product_measure_approx(μ, ν)(i, j))(M)

rectangle_measure: LEMMA
m_times(μ, ν)($X \times Y$) = $\mu(X) \times \nu(Y)$

phi1(X): simple[[T_1, T_2], $S_1 \times S_2$] =
$\phi_{X \times \mathrm{fullset}[T_2]}$

phi2(Y): simple[[T_1, T_2], $S_1 \times S_2$] =
$\phi_{\mathrm{fullset}[T_1] \times Y}$

END product_measure

Part VIII
Product Integrals

46 product_integral_def

product_integral_def[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2], μ: measure_type[T_1, S_1],
ν: measure_type[T_2, S_2]]: THEORY

BEGIN

IMPORTING $\int[T_1, S_1, \mu]$, $\int[T_2, S_2, \nu]$, reals@real_fun_ops[[T_1, T_2]]

h: VAR $[[T_1, T_2] \rightarrow \mathbb{R}]$

f: VAR integrable[T_1, S_1, μ]

g: VAR integrable[T_2, S_2, ν]

N_1: VAR null_set[T_1, S_1, μ]

N_2: VAR null_set[T_2, S_2, ν]

x: VAR T_1

y: VAR T_2

c: VAR $\mathbb{R}$

integrable1?(h): bool =
$\exists\ N_1, f:$
$\forall\ x:$
$\neg\ (x \in N_1) \Rightarrow$
integrable?($\lambda\ y: h(x, y)$) $\wedge$
$\int \lambda\ y:\ h(x, y) = f(x)$

integrable2?(h): bool =
$\exists\ N_2, g:$
$\forall\ y:$
$\neg\ (y \in N_2) \Rightarrow$
integrable?($\lambda\ x: h(x, y)$) $\wedge$
$\int \lambda\ x:\ h(x, y) = g(y)$

integrable1: TYPE+ = (integrable1?) CONTAINING ($\lambda\ x, y: 0$)

integrable2: TYPE+ = (integrable2?) CONTAINING ($\lambda\ x, y: 0$)

g_1, h_1: VAR integrable1

g_2, h_2: VAR integrable2

integrable1_zero: LEMMA integrable1?(λ x, y: 0)

integrable1_add: JUDGEMENT $+(g_1, h_1)$ HAS_TYPE integrable1

integrable1_scal: JUDGEMENT $\times(c, h_1)$ HAS_TYPE integrable1

integrable1_opp: JUDGEMENT $-(h_1)$ HAS_TYPE integrable1

integrable1_diff: JUDGEMENT $-(g_1, h_1)$ HAS_TYPE integrable1

integrable2_zero: LEMMA integrable2?(λ x, y: 0)

integrable2_add: JUDGEMENT $+(g_2, h_2)$ HAS_TYPE integrable2

integrable2_scal: JUDGEMENT $\times(c, h_2)$ HAS_TYPE integrable2

integrable2_opp: JUDGEMENT $-(h_2)$ HAS_TYPE integrable2

integrable2_diff: JUDGEMENT $-(g_2, h_2)$ HAS_TYPE integrable2

null_integrable1(h_1): [null_set[T_1, S_1, μ], integrable[T_1, S_1, μ]] =
choose({N_1, f |
$\forall$ x:
$\neg$ $(x \in N_1)$ $\Rightarrow$
integrable?(λ y: $h_1(x, y)$) $\wedge$
$\int \lambda$ y: $h_1(x, y) = f(x)$})

null_integrable2(h_2): [null_set[T_2, S_2, ν], integrable[T_2, S_2, ν]] =
choose({N_2, g |
$\forall$ y:
$\neg$ $(y \in N_2)$ $\Rightarrow$
integrable?(λ x: $h_2(x, y)$) $\wedge$
$\int \lambda$ x: $h_2(x, y) = g(y)$})

null_integral1_def: LEMMA
(N_1, f) = null_integrable1(h_1) $\wedge$ $(\neg\ (x \in N_1))$ $\Rightarrow$
(integrable?(λ y: $h_1(x, y)$) $\wedge$
$\int \lambda$ y: $h_1(x, y) = f(x)$)

null_integral2_def: LEMMA
(N_2, g) = null_integrable2(h_2) $\wedge$ $(\neg\ (y \in N_2))$ $\Rightarrow$
(integrable?(λ x: $h_2(x, y)$) $\wedge$
$\int \lambda$ x: $h_2(x, y) = g(y)$)

integral1(h_1): integrable[T_1, S_1, μ] =
λ x:
LET (N_1, f) = null_integrable1(h_1) IN
IF $(x \in N_1)$ THEN 0 ELSE $f(x)$ ENDIF

integral2(h_2): integrable[T_2, S_2, ν] =
λ y:

LET $(N_2,\ g) = \text{null\_integrable2}(h_2)$ IN
IF $(y \in N_2)$ THEN 0 ELSE $g(y)$ ENDIF

integral1_zero: LEMMA integral1($\lambda\ x,\ y\colon 0) = (\lambda\ x\colon 0)$

integral1_add: LEMMA
$\text{integral1}(g_1 + h_1) = \text{integral1}(g_1) + \text{integral1}(h_1)$ *a.e.*

integral1_scal: LEMMA
$\text{integral1}(c \times h_1) = c \times \text{integral1}(h_1)$ *a.e.*

integral1_opp: LEMMA
$\text{integral1}(-(h_1)) = -\text{integral1}(h_1)$ *a.e.*

integral1_diff: LEMMA
$\text{integral1}(g_1 - h_1) = \text{integral1}(g_1) - \text{integral1}(h_1)$ *a.e.*

integral2_zero: LEMMA integral2($\lambda\ x,\ y\colon 0) = (\lambda\ y\colon 0)$

integral2_add: LEMMA
$\text{integral2}(g_2 + h_2) = \text{integral2}(g_2) + \text{integral2}(h_2)$ *a.e.*

integral2_scal: LEMMA
$\text{integral2}(c \times h_2) = c \times \text{integral2}(h_2)$ *a.e.*

integral2_opp: LEMMA
$\text{integral2}(-(h_2)) = -\text{integral2}(h_2)$ *a.e.*

integral2_diff: LEMMA
$\text{integral2}(g_2 - h_2) = \text{integral2}(g_2) - \text{integral2}(h_2)$ *a.e.*

END product_integral_def

47 finite_fubini_scaf

finite_fubini_scaf$\big[$(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra$[T_1]$, S_2: sigma_algebra$[T_2]$, μ: finite_measure$[T_1, S_1]$,
ν: finite_measure$[T_2, S_2]\big]$: THEORY

BEGIN

IMPORTING product_sigma$[T_1, T_2, S_1, S_2]$, measure_def$[T_1, S_1]$, measure_def$[T_2, S_2]$,
measure_def$\big[[T_1, T_2], S_1 \times S_2\big]$, product_finite_measure$[T_1, T_2, S_1, S_2]$

IMPORTING nn_integral$\big[[T_1, T_2], S_1 \times S_2, \text{to_measure}(\mu \times \nu)\big]$

g: VAR nn_integrable

i: VAR isf

n: VAR nn_isf

E: VAR $(S_1 \times S_2)$

IMPORTING $\int\big[[T_1, T_2], S_1 \times S_2, \text{to\_measure}(\mu \times \nu)\big]$

f: VAR integrable

h: VAR nn_measurable$\big[[T_1, T_2], S_1 \times S_2\big]$

m: VAR measurable_function$\big[[T_1, T_2], S_1 \times S_2\big]$

x: VAR T_1

y: VAR T_2

IMPORTING $\int[T_1, S_1, \text{to\_measure}(\mu)]$, $\int[T_2, S_2, \text{to\_measure}(\nu)]$

measurable_x_section: LEMMA
measurable_function?$[T_2, S_2](\lambda\ y: m(x, y))$

measurable_y_section: LEMMA
measurable_function?$[T_1, S_1](\lambda\ x: m(x, y))$

isf_x_section: LEMMA isf?$(\lambda\ y: i(x, y))$

isf_y_section: LEMMA isf?$(\lambda\ x: i(x, y))$

integral_phi1: LEMMA
$(\lambda\ x: \text{isf\_integral}[T_2, S_2, \text{to\_measure}(\nu)](\lambda\ y: \phi(E)(x, y))) =$
$\nu \circ \text{x\_section}(E)$

integral_phi2: LEMMA
$(\lambda\ y: \text{isf\_integral}[T_1, S_1, \text{to\_measure}(\mu)](\lambda\ x: \phi(E)(x, y))) =$
$\mu \circ \text{y\_section}(E)$

integral_phi3: LEMMA
isf_integral(ϕ_E) =
$\int \lambda\ x:$ isf_integral($\lambda\ y:\ \phi(E)(x,\ y)$)

integral_phi4: LEMMA
isf_integral(ϕ_E) =
$\int \lambda\ y:$ isf_integral($\lambda\ x:\ \phi(E)(x,\ y)$)

isf_integral_x: LEMMA
integrable?($\lambda\ x$: isf_integral($\lambda\ y$: $i(x,\ y)$))

isf_integral_y: LEMMA
integrable?($\lambda\ y$: isf_integral($\lambda\ x$: $i(x,\ y)$))

isf_fubini_tonelli_3: LEMMA
isf_integral(i) =
$\int \lambda\ x:$ isf_integral($\lambda\ y:\ i(x,\ y)$)

isf_fubini_tonelli_4: LEMMA
isf_integral(i) =
$\int \lambda\ y:$ isf_integral($\lambda\ x:\ i(x,\ y)$)

END finite_fubini_scaf

48 finite_fubini_tonelli

finite_fubini_tonelli[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2],
μ: finite_measure[T_1, S_1], ν: finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING finite_fubini_scaf[T_1, T_2, S_1, S_2, μ, ν],
product_integral_def[T_1, T_2, S_1, S_2, to_measure(μ), to_measure(ν)],
integral_convergence, indefinite_integral

g: VAR
nn_integrable
[[T_1, T_2], $S_1 \times S_2$, to_measure($\mu \times \nu$)]

h: VAR nn_measurable[[T_1, T_2], $S_1 \times S_2$]

finite_fubini_tonelli_1: LEMMA integrable?(h) $\Leftrightarrow$ integrable1?(h)

finite_fubini_tonelli_2: LEMMA integrable?(h) $\Leftrightarrow$ integrable2?(h)

finite_fubini_tonelli_3: LEMMA $\int g = \int$ integral1(g)

finite_fubini_tonelli_4: LEMMA $\int g = \int$ integral2(g)

END finite_fubini_tonelli

49 finite_fubini

finite_fubini[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2], μ: finite_measure[T_1, S_1],
ν: finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING sigma_algebra[T_1, S_1], sigma_algebra[T_2, S_2],
finite_fubini_tonelli[T_1, T_2, S_1, S_2, μ, ν],
finite_integral[[T_1, T_2], $S_1 \times S_2$, $\mu \times \nu$],
finite_integral[T_1, S_1, μ], finite_integral[T_2, S_2, ν]

f: VAR
integrable
[[T_1, T_2], $S_1 \times S_2$, to_measure($\mu \times \nu$)]

finite_integrable_is_integrable1: LEMMA integrable1?(f)

finite_integrable_is_integrable2: LEMMA integrable2?(f)

finite_fubini1: COROLLARY $\int f = \int$ integral1(f)

finite_fubini2: COROLLARY $\int f = \int$ integral2(f)

h: VAR bounded_measurable[[T_1, T_2], $S_1 \times S_2$]

x: VAR T_1

y: VAR T_2

integrable_x_section: LEMMA integrable?($\lambda\ y$: $h(x,\ y)$)

integrable_y_section: LEMMA integrable?($\lambda\ x$: $h(x,\ y)$)

integrable_integral_x_section: LEMMA
integrable?($\lambda\ x$: $\int \lambda\ y:\ h(x,\ y)$)

integrable_integral_y_section: LEMMA
integrable?($\lambda\ y$: $\int \lambda\ x:\ h(x,\ y)$)

integral_integral_x_section: LEMMA
$\int \lambda\ x:\ \int \lambda\ y:\ h(x,\ y) = \int h$

integral_integral_y_section: LEMMA
$\int \lambda\ y:\ \int \lambda\ x:\ h(x,\ y) = \int h$

END finite_fubini

50 fubini_tonelli_scaf

fubini_tonelli_scaf[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2],
μ: sigma_finite_measure[T_1, S_1],
ν: sigma_finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING product_measure[T_1, T_2, S_1, S_2], $\int$[[T_1, T_2], $S_1 \times S_2$, $\mu \times \nu$]

E: VAR ($S_1 \times S_2$)

X: VAR (S_1)

Y: VAR (S_2)

x: VAR T_1

y: VAR T_2

i, j, n: VAR $\mathbb{N}$

IMPORTING product_integral_def[T_1, T_2, S_1, S_2, μ, ν], measure_contraction_props,
measure_equality, finite_fubini_tonelli, finite_fubini, indefinite_integral,
$\overline{\mathbb{R}}_{\geq 0}$@double_nn_sequence, sigma_finite_measure_props

h: VAR nn_measurable[[T_1, T_2], $S_1 \times S_2$]

IMPORTING product_integral_def, sigma_finite_measure_props

convergent?: MACRO pred[sequence[$\mathbb{R}$]] =
topological_convergence.convergent?

product_measure_contraction: LEMMA
contraction($\mu \times \nu$, $X \times Y$)(E) = m_times(contraction(μ, X), contraction(ν, Y))(E)

product_sfm_contraction: LEMMA
contraction($\mu \times \nu$, A_of(μ)(i) $\times$ A_of(ν)(j))(E) = product_measure_approx(μ, ν)(i, j)(E)

product_measure_contraction_n: LEMMA
m_times(contraction(μ, P_of(μ)(n)), contraction(ν, P_of(ν)(n)))(E) = to_measure(fm_contraction(μ, P_of(μ)(n))

fubini_tonelli_scaf1: LEMMA
(integrable?(h) $\Leftrightarrow$ integrable1?(h)) $\wedge$
(integrable?(h) $\Rightarrow$
$\int h$ = $\int$ integral1[T_1, T_2, S_1, S_2, μ, ν](h))

fubini_tonelli_scaf2: LEMMA
(integrable?(h) $\Leftrightarrow$ integrable2?(h)) $\wedge$
(integrable?(h) $\Rightarrow$
$\int h$ = $\int$ integral2[T_1, T_2, S_1, S_2, μ, ν](h))

END fubini_tonelli_scaf

51 fubini_tonelli

fubini_tonelli[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE,
S_1: sigma_algebra[T_1], S_2: sigma_algebra[T_2],
μ: sigma_finite_measure[T_1, S_1], ν: sigma_finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING product_measure[T_1, T_2, S_1, S_2], $\int$[[T_1, T_2], $S_1 \times S_2$, $\mu \times \nu$]

g: VAR nn_integrable

h: VAR nn_measurable[[T_1, T_2], $S_1 \times S_2$]

x: VAR T_1

y: VAR T_2

IMPORTING product_integral_def[T_1, T_2, S_1, S_2, μ, ν],
fubini_tonelli_scaf[T_1, T_2, S_1, S_2, μ, ν]

fubini_tonelli_1: THEOREM integrable?(h) $\Leftrightarrow$ integrable1?(h)

fubini_tonelli_2: THEOREM integrable?(h) $\Leftrightarrow$ integrable2?(h)

fubini_tonelli_3: THEOREM
$\int g$ = $\int$ integral1[T_1, T_2, S_1, S_2, μ, ν](g)

fubini_tonelli_4: THEOREM
$\int g$ = $\int$ integral2[T_1, T_2, S_1, S_2, μ, ν](g)

END fubini_tonelli

52 fubini

fubini[(IMPORTING subset_algebra_def) measure_def, T_1, T_2: TYPE, S_1: sigma_algebra[T_1],
S_2: sigma_algebra[T_2], μ: sigma_finite_measure[T_1, S_1],
ν: sigma_finite_measure[T_2, S_2]]: THEORY

BEGIN

IMPORTING fubini_tonelli[T_1, T_2, S_1, S_2, μ, ν]

f: VAR integrable[[T_1, T_2], $S_1 \times S_2$, $\mu \times \nu$]

x: VAR T_1

y: VAR T_2

integrable_is_integrable1: LEMMA integrable1?(f)

integrable_is_integrable2: LEMMA integrable2?(f)

fubini1: LEMMA
$\int f = \int$ integral1[T_1, T_2, S_1, S_2, μ, ν](f)

fubini2: LEMMA
$\int f = \int$ integral2[T_1, T_2, S_1, S_2, μ, ν](f)

END fubini