Complex Integration

Dave Lester

March 21, 2011

cal_L_complex[(IMPORTING measure_integration@subset_algebra_def) T: TYPE,
S: sigma_algebra[T]]: THEORY
BEGIN

IMPORTING essentially_bounded, p_integrable

μ: VAR measure_type[T, S]

ν: VAR finite_measure[T, S]

h, h_1, h_2: VAR [$T \to \mathbb{C}$]

x: VAR T

c: VAR $\mathbb{C}$

p, q: VAR $\{a: \mathbb{R} \mid a \geq 1\}$

$h \in \mathcal{L}^{\infty}_{\mathbb{C}}(\mu)$: bool = essentially_bounded?[T, S, μ](h)

$\mathcal{L}^{\infty}_{\mathbb{C}}(\mu)$: TYPE+ = ($\lambda$ h: $h \in \mathcal{L}^{\infty}_{\mathbb{C}}(\mu)$) CONTAINING ($\lambda$ x: complex_ (0, 0))

$\mathcal{L}^{\infty}_{\mathbb{C}}(\nu)$: TYPE+ = cal_L_complex_infty(to_measure(ν)) CONTAINING (λ x: complex_ (0, 0))

cal_L_complex_infty_is_essentially_bounded: LEMMA
$h \in \mathcal{L}^{\infty}_{\mathbb{C}}(\mu) \Leftrightarrow$ essentially_bounded?[T, S, μ](h)

$h \in \mathcal{L}^{p}_{\mathbb{C}}(\mu)$: bool = p_integrable?[$T$, S, μ, p](h)

$\mathcal{L}^{p}_{\mathbb{C}}(\mu)$: TYPE+ = ($\lambda$ h: $h \in \mathcal{L}^{p}_{\mathbb{C}}(\mu)$) CONTAINING ($\lambda$ x: complex_ (0, 0))

$\mathcal{L}^p_{\mathbb{C}}(\nu)$: TYPE+ = cal_L_complex(to_measure(ν), p) CONTAINING (λ

x:
com-
plex_

(0,
0))

cal_L_complex_is_p_integrable: LEMMA
$h \in \mathcal{L}^p_{\mathbb{C}}(\mu) \Leftrightarrow$ p_integrable?$[T, S, \mu, p](h)$

scal_cal_L: LEMMA $h \in \mathcal{L}^p_{\mathbb{C}}(\mu) \Rightarrow c \times h \in \mathcal{L}^p_{\mathbb{C}}(\mu)$

sum_cal_L: LEMMA
$h_1 \in \mathcal{L}^p_{\mathbb{C}}(\mu) \wedge h_2 \in \mathcal{L}^p_{\mathbb{C}}(\mu) \Rightarrow h_1 + h_2 \in \mathcal{L}^p_{\mathbb{C}}(\mu)$

opp_cal_L: LEMMA $h \in \mathcal{L}^p_{\mathbb{C}}(\mu) \Rightarrow -h \in \mathcal{L}^p_{\mathbb{C}}(\mu)$

diff_cal_L: LEMMA
$h_1 \in \mathcal{L}^p_{\mathbb{C}}(\mu) \wedge h_2 \in \mathcal{L}^p_{\mathbb{C}}(\mu) \Rightarrow h_1 - h_2 \in \mathcal{L}^p_{\mathbb{C}}(\mu)$

prod_cal_L: LEMMA
$p > 1 \wedge \frac{1}{p} + \frac{1}{q} = 1 \wedge h_1 \in \mathcal{L}^p_{\mathbb{C}}(\mu) \wedge h_2 \in \mathcal{L}^q_{\mathbb{C}}(\mu) \Rightarrow$
$h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{C}}(\mu)$

scal_cal_L_infty: LEMMA $h \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \Rightarrow c \times h \in \mathcal{L}^\infty_{\mathbb{C}}(\mu)$

sum_cal_L_infty: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \Rightarrow h_1 + h_2 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu)$

opp_cal_L_infty: LEMMA $h \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \Rightarrow -h \in \mathcal{L}^\infty_{\mathbb{C}}(\mu)$

diff_cal_L_infty: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \Rightarrow h_1 - h_2 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu)$

prod_cal_L_infty: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \Rightarrow h_1 \times h_2 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu)$

prod_cal_L_1_infty: LEMMA
$h_1 \in \mathcal{L}^1_{\mathbb{C}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{C}}(\mu) \Rightarrow h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{C}}(\mu)$

prod_cal_L_infty_1: LEMMA

$$h_1 \in \mathcal{L}^{\infty}_{\mathbb{C}}(\mu) \ \wedge \ h_2 \in \mathcal{L}^{1}_{\mathbb{C}}(\mu) \ \Rightarrow \ h_1 \times h_2 \in \mathcal{L}^{1}_{\mathbb{C}}(\mu)$$

END cal_L_complex

cal_L_real[(IMPORTING measure_integration@subset_algebra_def) T: TYPE,
S: sigma_algebra[T]]: THEORY
BEGIN

IMPORTING cal_L_complex[T, S]

μ: VAR measure_type[T, S]

ν: VAR finite_measure[T, S]

h, h_1, h_2: VAR [$T \to \mathbb{R}$]

x: VAR T

c: VAR $\mathbb{R}$

p, q: VAR $\{a\colon \mathbb{R} \mid a \geq 1\}$

$h \in \mathcal{L}^{\infty}_{\mathbb{R}}(\mu)$: bool $=$ $\lambda\ x:$ complex_($h(x)$, 0) $\in \mathcal{L}^{\infty}_{\mathbb{C}}(\mu)$

$\mathcal{L}^{\infty}_{\mathbb{R}}(\mu)$: TYPE+ = ($\lambda\ h$: $h \in \mathcal{L}^{\infty}_{\mathbb{R}}(\mu)$) CONTAINING ($\lambda\ x$: 0)

$h \in \mathcal{L}^{\infty}_{\mathbb{R}}(\nu)$: bool $=$ $h \in \mathcal{L}^{\infty}_{\mathbb{R}}$(to_measure($\nu$))

$\mathcal{L}^{\infty}_{\mathbb{R}}(\nu)$: TYPE+ = cal_L_real_infty(to_measure(ν)) CONTAINING (λ
x:
0)

$h \in \mathcal{L}^{p}_{\mathbb{R}}(\mu)$: bool $=$
$\lambda\ x:$ complex_($h(x)$, 0) $\in \mathcal{L}^{p}_{\mathbb{C}}(\mu)$

$\mathcal{L}^{p}_{\mathbb{R}}(\mu)$: TYPE+ = ($\lambda\ h$: $h \in \mathcal{L}^{p}_{\mathbb{R}}(\mu)$) CONTAINING ($\lambda$
x:
0)

$h \in \mathcal{L}^{p}_{\mathbb{R}}(\nu)$: bool $=$ $h \in \mathcal{L}^{p}_{\mathbb{R}}$(to_measure($\nu$))

$\mathcal{L}^{p}_{\mathbb{R}}(\nu)$: TYPE+ = cal_L_real(to_measure(ν), p) CONTAINING (λ
x:
0)

cal_L_real_1_def: LEMMA

$h \in \mathcal{L}^1_{\mathbb{R}}(\mu) \Leftrightarrow$ integrable?$[T, S, \mu](h)$

scal_cal_LR: LEMMA $h \in \mathcal{L}^p_{\mathbb{R}}(\mu) \Rightarrow c \times h \in \mathcal{L}^p_{\mathbb{R}}(\mu)$

sum_cal_LR: LEMMA
$h_1 \in \mathcal{L}^p_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^p_{\mathbb{R}}(\mu) \Rightarrow h_1 + h_2 \in \mathcal{L}^p_{\mathbb{R}}(\mu)$

opp_cal_LR: LEMMA $h \in \mathcal{L}^p_{\mathbb{R}}(\mu) \Rightarrow -h \in \mathcal{L}^p_{\mathbb{R}}(\mu)$

diff_cal_LR: LEMMA
$h_1 \in \mathcal{L}^p_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^p_{\mathbb{R}}(\mu) \Rightarrow h_1 - h_2 \in \mathcal{L}^p_{\mathbb{R}}(\mu)$

prod_cal_LR: LEMMA
$p > 1 \wedge \frac{1}{p} + \frac{1}{q} = 1 \wedge h_1 \in \mathcal{L}^p_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^q_{\mathbb{R}}(\mu) \Rightarrow$
$h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$

scal_cal_L_inftyR: LEMMA $h \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \Rightarrow c \times h \in \mathcal{L}^\infty_{\mathbb{R}}(\mu)$

sum_cal_L_inftyR: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \Rightarrow h_1 + h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu)$

opp_cal_L_inftyR: LEMMA $h \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \Rightarrow -h \in \mathcal{L}^\infty_{\mathbb{R}}(\mu)$

diff_cal_L_inftyR: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \Rightarrow h_1 - h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu)$

prod_cal_L_inftyR: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \Rightarrow h_1 \times h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu)$

prod_cal_L_1_inftyR: LEMMA
$h_1 \in \mathcal{L}^1_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \Rightarrow h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$

prod_cal_L_infty_1R: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\mu) \wedge h_2 \in \mathcal{L}^1_{\mathbb{R}}(\mu) \Rightarrow h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{R}}(\mu)$

scal_cal_L_fmR: LEMMA $h \in \mathcal{L}^p_{\mathbb{R}}(\nu) \Rightarrow c \times h \in \mathcal{L}^p_{\mathbb{R}}(\nu)$

sum_cal_L_fmR: LEMMA
$h_1 \in \mathcal{L}^p_{\mathbb{R}}(\nu) \wedge h_2 \in \mathcal{L}^p_{\mathbb{R}}(\nu) \Rightarrow h_1 + h_2 \in \mathcal{L}^p_{\mathbb{R}}(\nu)$

opp_cal_L_fmR: LEMMA $h \in \mathcal{L}^p_{\mathbb{R}}(\nu) \Rightarrow -h \in \mathcal{L}^p_{\mathbb{R}}(\nu)$

diff_cal_L_fmR: LEMMA
$h_1 \in \mathcal{L}^p_{\mathbb{R}}(\nu) \ \wedge \ h_2 \in \mathcal{L}^p_{\mathbb{R}}(\nu) \ \Rightarrow \ h_1 - h_2 \in \mathcal{L}^p_{\mathbb{R}}(\nu)$

prod_cal_L_fmR: LEMMA
$p \ > \ 1 \ \wedge \ \frac{1}{p} + \frac{1}{q} \ = \ 1 \ \wedge \ h_1 \in \mathcal{L}^p_{\mathbb{R}}(\nu) \ \wedge \ h_2 \in \mathcal{L}^q_{\mathbb{R}}(\nu) \ \Rightarrow$
$h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{R}}(\nu)$

scal_cal_L_infty_fmR: LEMMA $h \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \Rightarrow \ c \times h \in \mathcal{L}^\infty_{\mathbb{R}}(\nu)$

sum_cal_L_infty_fmR: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \wedge \ h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \Rightarrow \ h_1 + h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu)$

opp_cal_L_infty_fmR: LEMMA $h \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \Rightarrow \ -h \in \mathcal{L}^\infty_{\mathbb{R}}(\nu)$

diff_cal_L_infty_fmR: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \wedge \ h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \Rightarrow \ h_1 - h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu)$

prod_cal_L_infty_fmR: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \wedge \ h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \Rightarrow \ h_1 \times h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu)$

prod_cal_L_1_infty_fmR: LEMMA
$h_1 \in \mathcal{L}^1_{\mathbb{R}}(\nu) \ \wedge \ h_2 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \Rightarrow \ h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{R}}(\nu)$

prod_cal_L_infty_1_fmR: LEMMA
$h_1 \in \mathcal{L}^\infty_{\mathbb{R}}(\nu) \ \wedge \ h_2 \in \mathcal{L}^1_{\mathbb{R}}(\nu) \ \Rightarrow \ h_1 \times h_2 \in \mathcal{L}^1_{\mathbb{R}}(\nu)$

END cal_L_real