1 complex_types

complex_types: THEORY
BEGIN

IMPORTING reals@sq, reals@sqrt

$\mathbb{C}$: DATATYPE
BEGIN
complex_(x, y: $\mathbb{R}$): complex_?
END $\mathbb{C}$

z, z_1, z_2: VAR $\mathbb{C}$

x, y: VAR $\mathbb{R}$

$\Re(z)$: $\mathbb{R}$ = CASES z OF complex_(x, y): x ENDCASES

$\Im(z)$: $\mathbb{R}$ = CASES z OF complex_(x, y): y ENDCASES

Re_rew: LEMMA $\Re$(complex_(x, y)) = x

Im_rew: LEMMA $\Im$(complex_(x, y)) = y

AUTO_REWRITE+ Re_rew

AUTO_REWRITE+ Im_rew

$\mathbb{C}_{\neq 0}$: TYPE+ = $\{z \mid \Re(z) \neq 0 \lor \Im(z) \neq 0\}$ CONTAINING complex_
(1,
0)

$\mathbb{C}_{\neq 0}$: TYPE+ = $\mathbb{C}_{\neq 0}$

$\imath$: $\mathbb{C}_{\neq 0}$ = complex_(0, 1)

Re_i: LEMMA $\Re(\imath) = 0$

Im_i: LEMMA $\Im(\imath) = 1$

AUTO_REWRITE+ Re_i

AUTO_REWRITE+ Im_i

n0z, n0z1, n0z2: VAR $\mathbb{C}_{\neq 0}$

n0x: VAR $\mathbb{R}_{\neq 0}$

$\overline{z}$: $\mathbb{C}$ = complex_($\Re(z)$, $-\Im(z)$)

Re_conjugate: LEMMA $\Re(\overline{z}) = \Re(z)$

Im_conjugate: LEMMA $\Im(\overline{z}) = -\Im(z)$

AUTO_REWRITE+ Re_conjugate

AUTO_REWRITE+ Im_conjugate

sq_abs(z): $\mathbb{R}_{\geq 0}$ = $\Re(z)^2 + \Im(z)^2$

nz_sq_abs_pos: JUDGEMENT sq_abs(n0z) HAS_TYPE $\mathbb{R}_{>0}$;

$z_1 + z_2$: $\mathbb{C}$ =
complex_($\Re(z_1) + \Re(z_2)$, $\Im(z_1) + \Im(z_2)$);

$x + z$: $\mathbb{C}$ = complex_($x + \Re(z)$, $\Im(z)$);

$z + x$: $\mathbb{C}$ = complex_($\Re(z) + x$, $\Im(z)$);

$-z$: $\mathbb{C}$ = complex_($-\Re(z)$, $-\Im(z)$);

$z_1 - z_2$: $\mathbb{C}$ =
complex_($\Re(z_1) - \Re(z_2)$, $\Im(z_1) - \Im(z_2)$);

$x - z$: $\mathbb{C}$ = complex_($x - \Re(z)$, $\Im(z)$);

$z - x$: $\mathbb{C}$ = complex_($\Re(z) - x$, $\Im(z)$);

$z_1 \times z_2$: $\mathbb{C}$ =
complex_($\Re(z_1) \times \Re(z_2) - \Im(z_1) \times \Im(z_2)$,
$\Im(z_1) \times \Re(z_2) + \Re(z_1) \times \Im(z_2)$);

$x \times z$: $\mathbb{C}$ = complex_($x \times \Re(z)$, $x \times \Im(z)$);

$z \times x$: $\mathbb{C}$ = complex_($\Re(z) \times x$, $\Im(z) \times x$);

$\frac{z}{\mathrm{n0z}}$: $\mathbb{C}$ =
complex_($\frac{(\Re(z)\times\Re(\mathrm{n0z})+\Im(z)\times\Im(\mathrm{n0z}))}{\mathrm{sq\_abs(n0z)}}$,
$\frac{(\Im(z)\times\Re(\mathrm{n0z})-\Re(z)\times\Im(\mathrm{n0z}))}{\mathrm{sq\_abs(n0z)}}$);

$\frac{x}{\mathrm{n0z}}$: $\mathbb{C}$ =
complex_($\frac{(x\times\Re(\mathrm{n0z}))}{\mathrm{sq\_abs(n0z)}}$,
$\frac{((-x)\times\Im(\mathrm{n0z}))}{\mathrm{sq\_abs(n0z)}}$);

$\frac{z}{\mathrm{n0x}}$: $\mathbb{C}$ =
complex_($\frac{\Re(z)}{\mathrm{n0x}}$, $\frac{\Im(z)}{\mathrm{n0x}}$);

Re_add1: LEMMA $\Re(z_1 + z_2) = \Re(z_1) + \Re(z_2)$

Re_add2: LEMMA $\Re(x + z) = x + \Re(z)$

Re_add3: LEMMA $\Re(z + x) = \Re(z) + x$

Re_neg1: LEMMA $\Re(-z) = -\Re(z)$

Re_sub1: LEMMA $\Re(z_1 - z_2) = \Re(z_1) - \Re(z_2)$

Re_sub2: LEMMA $\Re(x - z) = x - \Re(z)$

Re_sub3: LEMMA $\Re(z - x) = \Re(z) - x$

Re_mul1: LEMMA
$\Re(z_1 \times z_2) =$
$\Re(z_1) \times \Re(z_2) - \Im(z_1) \times \Im(z_2)$

Re_mul2: LEMMA $\Re(x \times z) = x \times \Re(z)$

Re_mul3: LEMMA $\Re(z \times x) = \Re(z) \times x$

Re_div1: LEMMA
$\Re(\frac{z}{\mathrm{n0z}}) =$
$\frac{(\Re(z) \times \Re(\mathrm{n0z}) + \Im(z) \times \Im(\mathrm{n0z}))}{\mathrm{sq\_abs(n0z)}}$

Re_div2: LEMMA
$\Re(\frac{x}{\mathrm{n0z}}) = \frac{(x \times \Re(\mathrm{n0z}))}{\mathrm{sq\_abs(n0z)}}$

Re_div3: LEMMA $\Re(\frac{z}{\mathrm{n0x}}) = \frac{\Re(z)}{\mathrm{n0x}}$

Im_add1: LEMMA $\Im(z_1 + z_2) = \Im(z_1) + \Im(z_2)$

Im_add2: LEMMA $\Im(x + z) = \Im(z)$

Im_add3: LEMMA $\Im(z + x) = \Im(z)$

Im_neg1: LEMMA $\Im(-z) = -\Im(z)$

Im_sub1: LEMMA $\Im(z_1 - z_2) = \Im(z_1) - \Im(z_2)$

Im_sub2: LEMMA $\Im(x - z) = \Im(z)$

Im_sub3: LEMMA $\Im(z - x) = \Im(z)$

Im_mul1: LEMMA
$\Im(z_1 \times z_2) =$
$\Im(z_1) \times \Re(z_2) + \Re(z_1) \times \Im(z_2)$

Im_mul2: LEMMA $\Im(x \times z) = x \times \Im(z)$

Im_mul3: LEMMA $\Im(z \times x) = \Im(z) \times x$

Im_div1: LEMMA
$$\Im(\frac{z}{\text{n0z}}) = \frac{(\Im(z) \times \Re(\text{n0z}) - \Re(z) \times \Im(\text{n0z}))}{\text{sq\_abs}(\text{n0z})}$$

Im_div2: LEMMA
$$\Im(\frac{x}{\text{n0z}}) = \frac{((-x) \times \Im(\text{n0z}))}{\text{sq\_abs}(\text{n0z})}$$

Im_div3: LEMMA $\Im(\frac{z}{\text{n0x}}) = \frac{\Im(z)}{\text{n0x}}$

AUTO_REWRITE+ Re_add1

AUTO_REWRITE+ Re_add2

AUTO_REWRITE+ Re_add3

AUTO_REWRITE+ Re_neg1

AUTO_REWRITE+ Re_sub1

AUTO_REWRITE+ Re_sub2

AUTO_REWRITE+ Re_sub3

AUTO_REWRITE+ Re_mul1

AUTO_REWRITE+ Re_mul2

AUTO_REWRITE+ Re_mul3

AUTO_REWRITE+ Re_div1

AUTO_REWRITE+ Re_div2

AUTO_REWRITE+ Re_div3

AUTO_REWRITE+ Im_add1

AUTO_REWRITE+ Im_add2

AUTO_REWRITE+ Im_add3

AUTO_REWRITE+ Im_neg1

AUTO_REWRITE+ Im_sub1

AUTO_REWRITE+ Im_sub2

AUTO_REWRITE+ Im_sub3

AUTO_REWRITE+ Im_mul1

AUTO_REWRITE+ Im_mul2

AUTO_REWRITE+ Im_mul3

AUTO_REWRITE+ Im_div1

AUTO_REWRITE+ Im_div2

AUTO_REWRITE+ Im_div3

c_eq(z_1, z_2): bool =
    $\Re(z_1) = \Re(z_2) \wedge \Im(z_1) = \Im(z_2)$;

c_eq(x, z): bool = $x = \Re(z) \wedge \Im(z) = 0$;

c_eq(z, x): bool = $\Re(z) = x \wedge \Im(z) = 0$;

=(z_1, z_2): MACRO bool = c_eq(z_1, z_2);

=(x, z): MACRO bool = c_eq(x, z);

$=(z,\ x)$: MACRO bool $=$ c_eq$(z,\ x)$;

c_ne$(z_1,\ z_2)$: bool $= \neg$ c_eq$(z_1,\ z_2)$;

c_ne$(x,\ z)$: bool $= x \neq \Re(z) \vee \Im(z) \neq 0$;

c_ne$(z,\ x)$: bool $= \Re(z) \neq x \vee \Im(z) \neq 0$;

$\neq(z_1,\ z_2)$: MACRO bool $=$ c_ne$(z_1,\ z_2)$;

$\neq(x,\ z)$: MACRO bool $=$ c_ne$(x,\ z)$;

$\neq(z,\ x)$: MACRO bool $=$ c_ne$(z,\ x)$;

c_eq1: LEMMA
  c_eq$(z_1,\ z_2) \Leftrightarrow$
    $\Re(z_1) = \Re(z_2) \wedge \Im(z_1) = \Im(z_2)$

c_eq2: LEMMA c_eq$(x,\ z) \Leftrightarrow x = \Re(z) \wedge \Im(z) = 0$

c_eq3: LEMMA c_eq$(z,\ x) \Leftrightarrow \Re(z) = x \wedge \Im(z) = 0$

c_ne1: LEMMA c_ne$(z_1,\ z_2) \Leftrightarrow \neg$ c_eq$(z_1,\ z_2)$

c_ne2: LEMMA c_ne$(x,\ z) \Leftrightarrow x \neq \Re(z) \vee \Im(z) \neq 0$

c_ne3: LEMMA c_ne$(z,\ x) \Leftrightarrow \Re(z) \neq x \vee \Im(z) \neq 0$

AUTO_REWRITE+ c_eq1

AUTO_REWRITE+ c_eq2

AUTO_REWRITE+ c_eq3

AUTO_REWRITE+ c_ne1

AUTO_REWRITE+ c_ne2

AUTO_REWRITE+ c_ne3

plus_conjugate: LEMMA c_eq$(z + \overline{z},\ 2 \times \Re(z))$

minus_conjugate: LEMMA
  c_eq$(z - \overline{z},\ 2 \times \Im(z) \times \imath)$

conjugate_plus: LEMMA
  c_eq($\overline{z_1 + z_2}$, $\overline{z_1} + \overline{z_2}$)

conjugate_neg: LEMMA c_eq($\overline{-z}$, $-\overline{z}$)

conjugate_minus: LEMMA
  c_eq($\overline{z_1 - z_2}$, $\overline{z_1} - \overline{z_2}$)

conjugate_times: LEMMA
  c_eq($\overline{z_1 \times z_2}$, $\overline{z_1} \times \overline{z_2}$)

conjugate_inv: LEMMA c_eq($\overline{\frac{1}{\text{n0z}}}$, $\frac{1}{\overline{\text{n0z}}}$)

conjugate_div: LEMMA
  c_eq($\overline{\frac{z}{\text{n0z}}}$, $\frac{\overline{z}}{\overline{\text{n0z}}}$)

zero_times: LEMMA
  c_eq($z_1 \times z_2$, 0) $\Leftrightarrow$ c_eq(z_1, 0) $\vee$ c_eq(z_2, 0)

neg_nzcomplex: JUDGEMENT $-$(n0z) HAS_TYPE $\mathbb{C}_{\neq 0}$

mul_nzcomplex1: JUDGEMENT $\times$(n0z1, n0z2) HAS_TYPE $\mathbb{C}_{\neq 0}$

mul_nzcomplex2: JUDGEMENT $\times$(n0x, n0z) HAS_TYPE $\mathbb{C}_{\neq 0}$

mul_nzcomplex3: JUDGEMENT $\times$(n0z, n0x) HAS_TYPE $\mathbb{C}_{\neq 0}$

div_nzcomplex1: JUDGEMENT /(n0z1, n0z2) HAS_TYPE $\mathbb{C}_{\neq 0}$

div_nzcomplex2: JUDGEMENT /(n0x, n0z) HAS_TYPE $\mathbb{C}_{\neq 0}$

div_nzcomplex3: JUDGEMENT /(n0z, n0x) HAS_TYPE $\mathbb{C}_{\neq 0}$

z^2: $\mathbb{C}$ = $z \times z$

sq_def: LEMMA c_eq(z^2, $z \times z$)

AUTO_REWRITE+ sq_def

END complex_types

2 polar

polar: THEORY
BEGIN

IMPORTING complex_types, reals@sqrt, trig@atan2, trig@atan2_props

argrng: TYPE+ = $\{x: \mathbb{R} \mid -\pi < x \;\&\; x \leq \pi\}$ CONTAINING 0

z, z_1, z_2: VAR $\mathbb{C}$

n0x, n0y, n0z: VAR $\mathbb{C}_{\neq 0}$

nzx: VAR $\mathbb{R}_{\neq 0}$

r, x, y: VAR $\mathbb{R}$

j: VAR $\mathbb{Z}$

θ: VAR argrng;

$=(z_1, z_2)$: MACRO bool = c_eq(z_1, z_2);

$=(x, z)$: MACRO bool = c_eq(x, z);

$=(z, x)$: MACRO bool = c_eq(z, x);

$\neq(z_1, z_2)$: MACRO bool = c_ne(z_1, z_2);

$\neq(x, z)$: MACRO bool = c_ne(x, z);

$\neq(z, x)$: MACRO bool = c_ne(z, x);

$|z|$: $\mathbb{R}_{\geq 0}$ = $\sqrt{\text{sq\_abs}(z)}$

abs_def: LEMMA $|z| = \sqrt{\Re(z)^2 + \Im(z)^2}$

abs_nzcomplex: JUDGEMENT abs(n0z) HAS_TYPE $\mathbb{R}_{>0}$

abs_nz_iff_nz: LEMMA $|z| > 0 \Leftrightarrow$ c_ne(z, 0)

abs_is_0: LEMMA $|z| = 0 \Leftrightarrow$ c_eq(z, 0)

abs_neg: LEMMA $|-z| = |z|$

abs_mult: LEMMA $|z_1 \times z_2| = |z_1| \times |z_2|$

abs_inv: LEMMA $\left|\frac{1}{\text{n0z}}\right| = \frac{1}{|\text{n0z}|}$

abs_div: LEMMA $\left|\frac{z}{\text{n0z}}\right| = \frac{|z|}{|\text{n0z}|}$

abs_triangle: LEMMA $|z_1 + z_2| \leq |z_1| + |z_2|$

abs_abs: LEMMA $||z|| = |z|$

abs_i: LEMMA $|\imath| = 1$

abs_div2: LEMMA $\left|\frac{z}{\text{nzx}}\right| = \frac{|z|}{|\text{nzx}|}$

abs_div3: LEMMA $\left|\frac{x}{\text{n0z}}\right| = \frac{|x|}{|\text{n0z}|}$

AUTO_REWRITE+ abs_neg

AUTO_REWRITE+ abs_mult

AUTO_REWRITE+ abs_inv

AUTO_REWRITE+ abs_div

AUTO_REWRITE+ abs_abs

AUTO_REWRITE+ abs_i

AUTO_REWRITE+ abs_div2

AUTO_REWRITE+ abs_div3

arg(z): argrng =
IF c_eq(z, 0)
THEN 0
ELSIF $\Im(z) < 0$ THEN atan2($\Re(z)$, $\Im(z)$) $- 2 \times \pi$
ELSE atan2($\Re(z)$, $\Im(z)$)
ENDIF

arg_is_0_nz: LEMMA
arg(n0z) = 0 $\Leftrightarrow$ ($\Re$(n0z) $> 0 \wedge \Im$(n0z) $= 0$)

arg_is_0: LEMMA
$\arg(z) = 0 \Leftrightarrow (\Re(z) \geq 0 \wedge \Im(z) = 0)$

arg_is_pi2: LEMMA
$\arg(z) = \frac{\pi}{2} \Leftrightarrow (\Re(z) = 0 \wedge \Im(z) > 0)$

arg_is_pi: LEMMA

$\arg(z) = \pi \Leftrightarrow (\Re(z) < 0 \wedge \Im(z) = 0)$

arg_is_mpi2: LEMMA
$\arg(z) = \frac{-\pi}{2} \Leftrightarrow$
$(\Re(z) = 0 \wedge \Im(z) < 0)$

arg_lt_0: LEMMA $\arg(z) < 0 \Leftrightarrow \Im(z) < 0$

arg_p_lt_pi: LEMMA
$(0 < \arg(z) \wedge \arg(z) < \pi) \Leftrightarrow \Im(z) > 0$

arg_gt_0: LEMMA
$\arg(z) > 0 \Leftrightarrow$
$(\Im(z) > 0 \vee$
$(\Im(z) = 0 \wedge \Re(z) < 0))$

arg_div_abs: LEMMA $\arg(\mathrm{n0x}) = \arg(\frac{\mathrm{n0x}}{|\mathrm{n0x}|})$

Re_cos_abs1: LEMMA
$|\mathrm{n0x}| = 1 \Rightarrow \Re(\mathrm{n0x}) = \cos(\arg(\mathrm{n0x}))$

Im_sin_abs1: LEMMA
$|\mathrm{n0x}| = 1 \Rightarrow \Im(\mathrm{n0x}) = \sin(\arg(\mathrm{n0x}))$

abs_cos_arg: LEMMA $|z| \times \cos(\arg(z)) = \Re(z)$

abs_sin_arg: LEMMA $|z| \times \sin(\arg(z)) = \Im(z)$

arg_nnreal: LEMMA
$\Im(z) = 0 \wedge \Re(z) \geq 0 \Rightarrow \arg(z) = 0$

arg_nreal: LEMMA
$\Im(z) = 0 \wedge \Re(z) < 0 \Rightarrow \arg(z) = \pi$

arg_i: LEMMA $\arg(\imath) = \frac{\pi}{2}$

arg_neg: LEMMA
$\arg(-\mathrm{n0x}) =$
IF $0 < \arg(\mathrm{n0x})$
THEN $\arg(\mathrm{n0x}) - \pi$
ELSE $\arg(\mathrm{n0x}) + \pi$
ENDIF

arg_conjugate: LEMMA
$\arg(\overline{z}) =$
IF $\arg(z) = 0 \vee \arg(z) = \pi$
THEN $\arg(z)$
ELSE $-\arg(z)$
ENDIF

arg_mult: LEMMA

$\arg(\text{n0x} \times \text{n0y}) =$
LET $r = \arg(\text{n0x}) + \arg(\text{n0y})$ IN
IF $r > \pi$
THEN $r - 2 \times \pi$
ELSIF $r \leq -\pi$ THEN $r + 2 \times \pi$
ELSE r
ENDIF

arg_inv: LEMMA
$\arg(\frac{1}{\text{n0z}}) =$
IF $\arg(\text{n0z}) = 0$
THEN 0
ELSIF $\arg(\text{n0z}) = \pi$ THEN π
ELSE $-\arg(\text{n0z})$
ENDIF

arg_div: LEMMA
$\arg(\frac{\text{n0x}}{\text{n0y}}) =$
LET $r = \arg(\text{n0x}) - \arg(\text{n0y})$ IN
IF $r > \pi$
THEN $r - 2 \times \pi$
ELSIF $r \leq -\pi$ THEN $r + 2 \times \pi$
ELSE r
ENDIF

$\text{polar}(z)\colon\ [\mathbb{R}_{\geq 0},\ \text{argrng}] = (|z|,\ \arg(z))$

$\text{rectangular}(z)\colon\ [\mathbb{R},\ \mathbb{R}] = (\Re(z),\ \Im(z))$

$\text{from\_polar}(r,\ \theta)\colon\ \mathbb{C} =$
$r \times \cos(\theta) + r \times \sin(\theta) \times \imath$

$\text{from\_rectangular}(x,\ y)\colon\ \mathbb{C} = x + y \times \imath$

idempotent_rectangular: LEMMA
$\text{c\_eq}(z,\ \text{from\_rectangular}(\text{rectangular}(z)))$

idempotent_polar: LEMMA $\text{c\_eq}(\text{n0z},\ \text{from\_polar}(\text{polar}(\text{n0z})))$

END polar

3 complex_lnexp

complex_lnexp: THEORY
BEGIN

IMPORTING polar, trig@trig_ineq, trig_aux, lnexp@ln_exp

r: VAR $\mathbb{R}$

x, y, z: VAR $\mathbb{C}$

n0x, n0y, n0z: VAR $\mathbb{C}_{\neq 0}$

θ: VAR argrng

j: VAR $\mathbb{Z}$;

$=(x, y)$: MACRO bool $=$ c_eq(x, y);

$=(r, z)$: MACRO bool $=$ c_eq(r, z);

$=(z, r)$: MACRO bool $=$ c_eq(z, r);

$\neq(x, y)$: MACRO bool $=$ c_ne(x, y);

$\neq(r, z)$: MACRO bool $=$ c_ne(r, z);

$\neq(z, r)$: MACRO bool $=$ c_ne(z, r);

exp(z): $\mathbb{C}_{\neq 0}$ $=$
complex_$(\exp(\Re(z)) \times \cos(\Im(z)),$
$\exp(\Re(z)) \times \sin(\Im(z)))$

Re_exp: LEMMA $\Re(\exp(z)) = \exp(\Re(z)) \times \cos(\Im(z))$

Im_exp: LEMMA $\Im(\exp(z)) = \exp(\Re(z)) \times \sin(\Im(z))$

AUTO_REWRITE+ Re_exp

AUTO_REWRITE+ Im_exp

exp_i_pi: LEMMA c_eq$(\exp(\imath \times \pi), -1)$

exp_plus: LEMMA
c_eq$(\exp(x + y), \exp(x) \times \exp(y))$

exp_negate: LEMMA c_eq$(\exp(-x), \frac{1}{\exp(x)})$

exp_minus: LEMMA

c_eq(exp($x-y$), $\frac{\exp(x)}{\exp(y)}$)

exp_periodicity: LEMMA
c_eq(exp($x+(2\times j\times\pi)\times\imath$), exp($x$))

abs_exp: LEMMA $|\exp(z)| = \exp(\Re(z))$

arg_exp: LEMMA
$-\pi < \Im(z) \wedge \Im(z) \leq \pi \Rightarrow$
arg(exp(z)) $= \Im(z)$

ln(n0z): $\mathbb{C}$ = complex_(ln(|n0z|), arg(n0z))

Re_ln: LEMMA $\Re$(ln(n0z)) = ln(|n0z|)

Im_ln: LEMMA $\Im$(ln(n0z)) = arg(n0z)

AUTO_REWRITE+ Re_ln

AUTO_REWRITE+ Im_ln

AUTO_REWRITE+ abs_exp

AUTO_REWRITE+ arg_exp

ln_exp: LEMMA
$(2\times j-1)\times\pi < \Im(z) \wedge$
$\Im(z) \leq (2\times j+1)\times\pi$
$\Rightarrow$ c_eq(ln(exp(z)), $z-(2\times j\times\pi)\times\imath$)

exp_ln: LEMMA c_eq(exp(ln(n0z)), n0z)

ln_mult: LEMMA
c_eq(ln(n0x $\times$ n0y),
ln(n0x) + ln(n0y) − IF arg(n0x) + arg(n0y) > π THEN $2\times\pi$ ELSIF arg(n0x) + arg(n0y) $\leq$ $-\pi$ THEN -

ln_inv: LEMMA
c_eq(ln($\frac{1}{\text{n0x}}$),
IF arg(n0x) = π THEN $2\times\pi$ ELSE 0 ENDIF $\times\,\imath$ − ln(n0x))

ln_div: LEMMA
c_eq(ln($\frac{\text{n0x}}{\text{n0y}}$),
ln(n0x) − ln(n0y) − IF arg(n0x) − arg(n0y) > π THEN $2\times\pi$ ELSIF arg(n0x) − arg(n0y) $\leq$ $-\pi$ THEN -

END complex_lnexp

4 complex_sqrt

complex_sqrt: THEORY
 BEGIN

  IMPORTING polar, trig_aux

  r: VAR $\mathbb{R}$

  nnx: VAR $\mathbb{R}_{\geq 0}$

  npx: VAR $\mathbb{R}_{\leq 0}$

  x, y, z: VAR $\mathbb{C}$

  n0x, n0y, n0z: VAR $\mathbb{C}_{\neq 0}$;

  $=(x, y)$: MACRO bool = c_eq(x, y);

  $=(r, z)$: MACRO bool = c_eq(r, z);

  $=(z, r)$: MACRO bool = c_eq(z, r);

  $\neq(x, y)$: MACRO bool = c_ne(x, y);

  $\neq(r, z)$: MACRO bool = c_ne(r, z);

  $\neq(z, r)$: MACRO bool = c_ne(z, r);

  $\sqrt{z}$: $\mathbb{C}$ = from_polar($\sqrt{|z|}$, $\frac{\arg(z)}{2}$)

  sqrt_nz_is_nz: JUDGEMENT sqrt(n0z) HAS_TYPE $\mathbb{C}_{\neq 0}$

  sqrt_eq_0: LEMMA c_eq($\sqrt{z}$, 0) $\equiv$ c_eq(z, 0)

  sqrt_sq: LEMMA
    c_eq($\sqrt{z^2}$,
        IF $\frac{-\pi}{2} < \arg(z) \land \arg(z) \leq \frac{\pi}{2}$
          THEN z
        ELSE $-z$
        ENDIF)

  sq_sqrt: LEMMA c_eq($\sqrt{z}^2$, z)

  sqrt_times: LEMMA
    c_eq($\sqrt{x \times y}$,
        IF $-\pi < \arg(x) + \arg(y) \land \arg(x) + \arg(y) \leq \pi$
          THEN $\sqrt{x} \times \sqrt{y}$
        ELSE $-\sqrt{x} \times \sqrt{y}$
        ENDIF)

  sqrt_neg: LEMMA

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  c_eq(√(−z),
          IF arg(z) ≤ 0
             THEN ı × √z
          ELSE −ı × √z
          ENDIF)

sqrt_inv: LEMMA
  c_eq(√(1/n0z),
          IF arg(n0z) = π
             THEN −1/√n0z
          ELSE 1/√n0z
          ENDIF)

sqrt_div: LEMMA
  c_eq(√(x/n0y),
          IF (arg(n0y) = π & arg(x) > 0) ∨
                arg(n0y) = 0 ∨
                 (−π < arg(x) − arg(n0y) &
                      arg(x) − arg(n0y) ≤ π)
             THEN √x/√n0y
          ELSE −√x/√n0y
          ENDIF)

END complex_sqrt
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