习题 8

1. 解：

（1）由海伦公式知：$S(x,y,z)=\sqrt{p(p-x)(p-y)(p-z)},\ p=\dfrac{x+y+z}{2}$

（2）由已知，可以得到表面积和体积的表达式分别为：

$$S=2\pi RH+\pi R\sqrt{R^2+h^2}$$
$$V=\pi R^2H+\frac{1}{3}\pi R^2h$$

联立上述两式，消掉H，可以得到：

$$S(R,h)=\frac{2V}{R}-\frac{2\pi Rh}{3}+\pi R\sqrt{R^2+h^2}$$

2. 解：

（1）由已知:$\begin{cases}x+y>0\\x-y>0\end{cases}\Rightarrow -x<y<x$

（2）由已知:$\dfrac{x^2+y^2-x}{2x-x^2-y^2}\geq 0\Rightarrow\begin{cases}(x-\frac{1}{2})+y^2\geq\frac{1}{4}\\(x-1)^2+y^2<1\end{cases}$或$\begin{cases}(x-\frac{1}{2})+y^2\leq\frac{1}{4}\\(x-1)^2+y^2>1\end{cases}$(画图易知为空集）

$\therefore$定义域为:$\begin{cases}(x-\frac{1}{2})+y^2\geq\frac{1}{4}\\(x-1)^2+y^2<1\end{cases}$

（3）由已知：$x\sin y\geq 0\Rightarrow\begin{cases}x>0\\y\in[k\pi,\pi+k\pi],k\in Z\end{cases}$或$x=0$或$\begin{cases}x<0\\y\in[-\pi+k\pi,k\pi],k\in Z\end{cases}$

（4）由已知:$\begin{cases}-1\leq\dfrac{x}{y^2}\leq 1\\-1\leq 1-y\leq 1\end{cases}\Rightarrow\begin{cases}|x|\leq y^2\\0<y\leq 2\end{cases}$

（5）由已知:$\begin{cases}x\ln(y-x)>0\\y-x>0\end{cases}\Rightarrow\begin{cases}x>0\\y>x+1\end{cases}or\begin{cases}x<0\\x<y<x+1\end{cases}$

（6）由已知:$\begin{cases} R^2-x^2-y^2-z^2\ge 0 \\ x^2+y^2+z^2-r^2>0 \end{cases}\Rightarrow r^2<x^2+y^2+z^2\le R^2$

3. 解：

（1）$\begin{cases} 0<y<2 \\ \frac{1}{2}x-1<y<\frac{1}{2}x \end{cases}$

（2）$x^2\le y\le \sqrt{x}$

4. 解：

（1）$f(tx,ty)=(tx)^2+(ty)^2-(tx)(ty)\tan\frac{tx}{ty}=t^2 f(x,y)$

（2）$f(x+y,x-y,xy)=(x+y)^{xy}+(xy)^{(x+y)+(x-y)}=(x+y)^{xy}+(xy)^{2x}$

（3）令$\begin{cases} m=x+y \\ n=\frac{y}{x} \end{cases}\Rightarrow \begin{cases} x=\frac{m}{1+n} \\ y=\frac{mn}{1+n} \end{cases}$，从而$f(m,n)=(\frac{m}{1+n})^2-(\frac{mn}{1+n})^2=\frac{m^2(1-n)}{1+n}$

$\therefore\ f(x,y)=\frac{x^2(1-y)}{1+y}$

（4）由已知，$z(x,1)=x=1+f(\sqrt{x}-1)\Rightarrow f(\sqrt{x}-1)=x-1$

令：$m=\sqrt{x}-1\ge -1\Rightarrow x=(m+1)^2\Rightarrow f(m)=m^2+2m, m\ge -1$

$\therefore\ f(x)=x^2+2x, x\ge -1; z(x,y)=\sqrt{y}+x-1, x\ge 0$

5. 证明：

$\because\ f(tx,ty)=t^k f(x,y)$

令$t=\frac{1}{x}$,从而$f(1,\frac{y}{x})=\frac{f(x,y)}{x^k}=\frac{z}{x^k}$;

令F（$\frac{y}{x}$）$=f(1,\frac{y}{x})$,从而$z=x^kF(\frac{y}{x})$

$\therefore$原命题得证

6. 解:

（1） $\lim\limits_{(x,y)\to(0,0)}\frac{2-\sqrt{x+y+4}}{x+y}\overset{令t=x+y}{=\!=\!=}\lim\limits_{t\to0}\frac{2-\sqrt{t+4}}{t}=\lim\limits_{t\to0}\frac{-t}{t(2+\sqrt{t+4})}=-\frac{1}{4}$

（2） $\lim\limits_{(x,y)\to(0,0)}\frac{(2+x)\ln(1+xy)}{xy}\overset{\ln(1+xy)\sim xy}{=\!=\!=}\lim\limits_{(x,y)\to(0,0)}(2+x)=2$

（3） $\lim\limits_{(x,y)\to(0,1)}\frac{\sin(x^2+y^2)}{x^2+y^2}\overset{令x^2+y^2=t}{=\!=\!=}\lim\limits_{t\to1}\frac{\sin t}{t}=\sin1$

（4） $\lim\limits_{(x,y)\to(0,0)}\sqrt{x^2+y^2}\sin\frac{1}{\sqrt{x^2+y^2}}\overset{令t=\sqrt{x^2+y^2}}{=\!=\!=}\lim\limits_{t\to0}t\sin\frac{1}{t}=0$

7.解：

（1）

$\because |x\sin\frac{1}{y}+y\sin\frac{1}{x}|\le|x|+|y|$

$\lim\limits_{(x,y)\to(0,0)}|x|+|y|=0$

由夹逼定理知，原极限存在且为0.

（2）

①取路径$y=kx,k\in R$

则原极限$=\lim\limits_{x\to0}\frac{kx}{1+k}=0$

②取路径$y=x^2-x$

则原极限$=\lim\limits_{x\to0}\frac{x(x^2-x)}{x^2}=\lim\limits_{x\to0}x-1=-1$

所以极限不存在

8.解：

$\because$ 所有表达式均为初等函数及其复合形式

$\therefore$ 仅在没定义的点间断

（1）令 $\sqrt{x^2+y^2}=0\Rightarrow(x,y)=(0,0)$

（2）

$$\text{令}\sin\pi x=0\text{或}\sin\pi y=0$$
$$\Rightarrow x\in Z\text{或}y\in Z$$
$$\Rightarrow x=m\text{或}y=n(m,n\in Z)$$

（3）$\text{令}2x-y^2=0\Rightarrow y^2=2x$

（4）$\text{令}xyz=0\Rightarrow x=0\text{或}y=0\text{或}z=0$

9. 解：

（1）$f_x=1-\dfrac{2x}{2\sqrt{x^2+y^2}}=1-\dfrac{x}{\sqrt{x^2+y^2}}\Rightarrow f_x(3,4)=1-\dfrac{3}{5}=\dfrac{2}{5}$

（2）$\left.\dfrac{\partial z}{\partial x}\right|_{(1,0)}=\left.\dfrac{1-\dfrac{y}{2x^2}}{x+\dfrac{y}{2x}}\right|_{(1,0)}=1$

（3）$\left.\dfrac{\partial z}{\partial x}\right|_{(1,1)}=\left.y(1+xy)^{y-1}\cdot y\right|_{(1,1)}=1$

$$\left.\frac{\partial z}{\partial y}\right|_{(1,1)}=\left.\frac{\partial e^{\ln(1+xy)y}}{\partial y}\right|_{(1,1)}=\left.(1+xy)^y\left(\frac{xy}{1+xy}+\ln(1+xy)\right)\right|_{(1,1)}=1+2\ln 2$$

（4）$f(x,1)=x\Rightarrow f_x(x,1)=1$

10. 解：

（1）$\dfrac{\partial z}{\partial x}=\dfrac{\sqrt{x^2+y^2}-\dfrac{2x^2}{2\sqrt{x^2+y^2}}}{x^2+y^2}=\dfrac{y^2}{(x^2+y^2)^{3/2}}$

$$\frac{\partial z}{\partial y}=\frac{0-\dfrac{2xy}{2\sqrt{x^2+y^2}}}{x^2+y^2}=-\frac{xy}{(x^2+y^2)^{3/2}}$$

（2） $z = e^{\frac{y\ln 3}{x}}$

$$\frac{\partial z}{\partial x} = e^{\frac{y\ln 3}{x}} \cdot \left(-\frac{y\ln 3}{x^2}\right) = -\frac{y}{x^2} 3^{\frac{y}{x}} \ln 3$$

$$\frac{\partial z}{\partial y} = e^{\frac{y\ln 3}{x}} \cdot \left(\frac{\ln 3}{x}\right) = \frac{1}{x} 3^{\frac{y}{x}} \ln 3$$

（3） $z = (1+\frac{2y}{x-y})\sin\frac{x}{y} = (-1+\frac{2x}{x-y})\sin\frac{x}{y}$

$$\frac{\partial z}{\partial x} = -\frac{2y}{(x-y)^2}\sin\frac{x}{y} + \cos\frac{x}{y}\cdot\frac{1}{y}\cdot\frac{x+y}{x-y} = -\frac{2y}{(x-y)^2}\sin\frac{x}{y} + \frac{x+y}{y(x-y)}\cos\frac{x}{y}$$

$$\frac{\partial z}{\partial y} = \frac{2x}{(x-y)^2}\sin\frac{x}{y} + \cos\frac{x}{y}\cdot\left(-\frac{x}{y^2}\right)\cdot\frac{x+y}{x-y} = \frac{2x}{(x-y)^2}\sin\frac{x}{y} - \frac{x(x+y)}{y^2(x-y)}\cos\frac{x}{y}$$

（4） $$\frac{\partial z}{\partial x} = \frac{ye^{xy}(e^x+e^y) - e^x \cdot e^{xy}}{(e^x+e^y)^2} = \frac{e^{xy}(ye^x+ye^y-e^x)}{(e^x+e^y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{e^{xy}(xe^x+xe^y-e^y)}{(e^x+e^y)^2}$$

（5） $$\frac{\partial z}{\partial x} = \frac{\sec^2\frac{x}{y}}{\tan\frac{x}{y}}\cdot\frac{1}{y} = \frac{\frac{1}{\cos^2\frac{x}{y}}}{\frac{\sin\frac{x}{y}}{\cos\frac{x}{y}}}\cdot\frac{1}{y} = \frac{1}{y\sin\frac{x}{y}\cos\frac{x}{y}} = \frac{2}{y}\csc\frac{2x}{y}$$

$$\frac{\partial z}{\partial y} = \frac{\sec^2\frac{x}{y}}{\tan\frac{x}{y}}\cdot\left(-\frac{x}{y^2}\right) = -\frac{2x}{y^2}\csc\frac{2x}{y}$$

（6） $$\frac{\partial z}{\partial x}=\frac{-2y}{\sqrt{1-(3-2xy)^2}}+\cos(3-\frac{2x}{y})\cdot(-\frac{2}{y})=-\frac{2y}{\sqrt{1-(3-2xy)^2}}-\frac{2}{y}\cos(3-\frac{2x}{y})$$

$$\frac{\partial z}{\partial y}=\frac{-2x}{\sqrt{1-(3-2xy)^2}}+\cos(3-\frac{2x}{y})\cdot\frac{2x}{y^2}=-\frac{2x}{\sqrt{1-(3-2xy)^2}}+\frac{2x}{y^2}\cos(3-\frac{2x}{y})$$

（7） $$\sqrt{x^y}=x^{\frac{y}{2}}$$

$$\frac{\partial z}{\partial x}=\frac{1}{1+x^y}\cdot\frac{y}{2}\cdot x^{\frac{y}{2}-1}=\frac{y\sqrt{x^y}}{2x(1+x^y)}$$

$$\frac{\partial z}{\partial y}=\frac{1}{1+x^y}\cdot x^{\frac{y}{2}}\ln x\cdot\frac{1}{2}=\frac{\sqrt{x^y}\ln x}{2(1+x^y)}$$

（8） $$z=e^{(x+y)\ln(1+xy)}$$

$$\frac{\partial z}{\partial x}=e^{(x+y)\ln(1+xy)}\left[(\ln(1+xy)+\frac{y(x+y)}{1+xy})\right]=(1+xy)^{x+y}\left[\ln(1+xy)+\frac{y(x+y)}{1+xy}\right]$$

$$\frac{\partial z}{\partial y}=(1+xy)^{x+y}\left[\ln(1+xy)+\frac{x(x+y)}{1+xy}\right]$$

（9） $$\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1}$$

$$\frac{\partial u}{\partial y}=x^{\frac{y}{z}}\cdot\ln x\cdot\frac{1}{z}=\frac{x^{\frac{y}{z}}\ln x}{z}$$

$$\frac{\partial u}{\partial z}=x^{\frac{y}{z}}\cdot\ln x\cdot\frac{-y}{z^2}=-\frac{x^{\frac{y}{z}}y\ln x}{z^2}$$

（10） $$\frac{\partial u}{\partial x}=e^{x(x^2+y^2+z^2)}(x^2+y^2+z^2+2x\cdot x)=(3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$$

$$\frac{\partial u}{\partial y}=2xye^{x(x^2+y^2+z^2)}.\frac{\partial u}{\partial z}=2xze^{x(x^2+y^2+z^2)}$$

11.解：

（1）$\left.\dfrac{\partial z}{\partial x}\right|_{(2,4)}=1$

从而直线方程为：

$$\begin{cases} z-5=x-2 \\ y=4 \end{cases} \Rightarrow \frac{x-2}{1}=\frac{y-4}{0}=\frac{z-5}{1} \Rightarrow \boldsymbol{s}=(1,0,1)$$

夹角：

$$\theta=\arccos\frac{(1,0,1)\cdot(1,0,0)}{\sqrt{1+1}}=\frac{\pi}{4}$$

（2）$\left.\dfrac{\partial z}{\partial y}\right|_{(1,1)}=\left.\dfrac{2y}{2\sqrt{1+x^2+y^2}}\right|_{(1,1)}=\dfrac{\sqrt{3}}{3}$

$\therefore$ 切线方程：$\begin{cases} z-\sqrt{3}=\dfrac{\sqrt{3}}{3}(y-1) \\ x=1 \end{cases} \Rightarrow \dfrac{x-1}{0}=\dfrac{y-1}{\sqrt{3}}=\dfrac{z-\sqrt{3}}{1}$

$\Rightarrow$ 方向向量：$\vec{s}=(0,\sqrt{3},1)$

$\Rightarrow$ 法平面方程：$0(x-1)+\sqrt{3}(y-1)+1(z-\sqrt{3})=0$

$\Rightarrow$ 整理得：$\sqrt{3}y+z-2\sqrt{3}=0$

（3） 易知交点坐标为$:(\pm1,2,\frac{5}{3})$

由对称性，只需考虑$(1,2,\frac{5}{3})$处的夹角

对于$\begin{cases} z=x^2+\frac{y^2}{6} \\ y=2 \end{cases}:\left.\frac{\partial z}{\partial x}\right|_{(1,2)}=2\Rightarrow$直线方程$:\begin{cases} z-\frac{5}{3}=2(x-1) \\ y=2 \end{cases}$

$\Rightarrow$方向向量$:\overrightarrow{s_1}=(1,0,2)$

对于$\begin{cases} z=\frac{x^2+y^2}{3} \\ y=2 \end{cases}$同理可得直线方程$\begin{cases} z-\frac{5}{3}=\frac{2}{3}(x-1) \\ y=2 \end{cases}$

$\Rightarrow$方向向量$:\overrightarrow{s_2}=(3,0,2)$

夹角$\theta=\arccos\frac{\overrightarrow{s_1}\cdot\overrightarrow{s_2}}{\left|\overrightarrow{s_1}\right|\cdot\left|\overrightarrow{s_2}\right|}=\arccos\frac{7}{\sqrt{65}}=\arctan\frac{4}{7}$

12.解：

（1） $z=\frac{1-\cos(2ax+2by)}{2}$

$\frac{\partial^2 z}{\partial x^2}=2a^2\cos 2(ax+by),\frac{\partial^2 z}{\partial x\partial y}=2ab\cos 2(ax+by)$

$\frac{\partial^2 z}{\partial y^2}=2b^2\cos 2(ax+by)$

（2）$\dfrac{\partial z}{\partial x}=\dfrac{\dfrac{1-xy-(-y)(x+y)}{(1-xy)^2}}{1+(\dfrac{x+y}{1-xy})^2}=\dfrac{1}{1+x^2},\dfrac{\partial z}{\partial y}=\dfrac{1}{1+y^2}$

$\therefore\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{2x}{(1+x^2)^2},\dfrac{\partial^2 z}{\partial xy}=0,\dfrac{\partial^2 z}{\partial y^2}=-\dfrac{2y}{(1+y^2)^2}$

（3）$\dfrac{\partial z}{\partial x}=yx^{y-1}$

$\dfrac{\partial^2 z}{\partial x^2}=y(y-1)x^{y-2},\dfrac{\partial^2 z}{\partial x\partial y}=x^{y-1}(1+y\ln x),\dfrac{\partial^2 z}{\partial y^2}=x^y\ln^2 x$

（4）$\dfrac{\partial z}{\partial x}=y^{\ln x}\cdot\ln y\cdot\dfrac{1}{x}=\dfrac{y^{\ln x}\ln y}{x},\dfrac{\partial z}{\partial y}=y^{\ln x-1}\ln x$

$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{(\ln y-1)\ln y}{x^2}y^{\ln x},\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{\ln x\ln y+1}{xy}y^{\ln x}$

$\dfrac{\partial^2 z}{\partial y^2}=y^{\ln x-2}\ln x(\ln x-1),$

13.解：

（1）$z=x\ln|x|+x\ln|y|(xy>0)$

$\dfrac{\partial^3 z}{\partial x^2\partial y}=0,\dfrac{\partial^3 z}{\partial x\partial y^2}=-\dfrac{1}{y^2}$

（2）$\dfrac{\partial z}{\partial x}=\dfrac{1}{\sqrt{x^2+y^2}}\cdot\dfrac{2x}{2\sqrt{x^2+y^2}}=\dfrac{x}{x^2+y^2}$

$\dfrac{\partial^2 z}{\partial x^2}=\dfrac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\dfrac{y^2-x^2}{(x^2+y^2)^2},\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore\dfrac{\partial^2 z}{\partial x^2}+\dfrac{\partial^2 z}{\partial y^2}=0$

（3）$\frac{\partial u}{\partial x}=3x^2-3yz,\frac{\partial^2 u}{\partial x^2}=6x$

$$\therefore\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial u}{\partial z}\right)^2=9\sum(x^2-yz)^2=9\left[x^4+y^4+z^4-2xyz(x+y+z)+x^2y^2+y^2z^2+z^2x^2\right]$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=6(x+y+z)$$

14.解：

（1）$\frac{\partial u}{\partial x}=-x(x^2+y^2+z^2)^{-\frac{3}{2}}$

$$\frac{\partial^2 u}{\partial x^2}=(2x^2-y^2-z^2)(x^2+y^2+z^2)^{-\frac{5}{2}},\frac{\partial^2 u}{\partial y^2}=(2y^2-z^2-x^2)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\frac{\partial^2 u}{\partial z^2}=(2z^2-x^2-y^2)(x^2+y^2+z^2)^{-\frac{5}{2}}$$

$$\therefore\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

（2）$\frac{\partial u}{\partial x}=\frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2}+\frac{-\frac{z}{x^2}}{1+(\frac{z}{x})^2}=\frac{-y-z}{x^2+y^2},\frac{\partial^2 u}{\partial x^2}=\frac{2x(y+z)}{(x^2+y^2)^2}$

$$\frac{\partial u}{\partial y}=\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2},\frac{\partial^2 u}{\partial y^2}=\frac{-2xy}{(x^2+y^2)^2},\frac{\partial^2 u}{\partial z^2}=\frac{-2xz}{(x^2+y^2)^2}$$

$$\therefore\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$$

（3）$\frac{\partial z}{\partial x}=\frac{e^x}{e^x+e^y},\frac{\partial^2 z}{\partial x^2}=\frac{e^{2x}}{(e^x+e^y)^2},\frac{\partial^2 z}{\partial y^2}=\frac{e^{2y}}{(e^x+e^y)^2}$

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{e^{x+y}}{(e^x+e^y)^2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\partial y}\right)^2=0$$

（4）$\ln r=\frac{\ln(x^2+y^2+z^2)}{2},\frac{1}{r^2}=\frac{1}{x^2+y^2+z^2}$

$$\frac{\partial(\ln r)}{\partial x}=\frac{x}{x^2+y^2+z^2},\frac{\partial^2(\ln r)}{\partial x^2}=\frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2}=\frac{y^2+z^2-x^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2(\ln r)}{\partial y^2}=\frac{z^2+x^2-y^2}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial^2(\ln r)}{\partial z^2}=\frac{x^2+y^2-z^2}{(x^2+y^2+z^2)^2}$$

$$\therefore \frac{\partial^2(\ln r)}{\partial x^2}+\frac{\partial^2(\ln r)}{\partial y^2}+\frac{\partial^2(\ln r)}{\partial z^2}=\frac{1}{r^2}$$

15. 解：

令$z=f(x,y)$

$$\Delta z=f(2+0.1,1+0.2)-f(2,1)=\frac{1}{14}$$

$$f_x(x,y)=-\frac{y}{x^2},f_y(x,y)=\frac{1}{x},$$

$$dz\big|_{(2,1)}=f_x(2,1)\Delta x+f_y(2,1)\Delta y=\frac{3}{40}$$

16. 解：

（1）$\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2},\frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}$

$dz=\frac{2x}{x^2+y^2}dx+\frac{2y}{x^2+y^2}dy$

（2）$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}}+\frac{-x^2}{(x^2+y^2)^{\frac{3}{2}}}=\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}$

$\frac{\partial z}{\partial y}=\frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}$

$dz=\frac{y^2}{(x^2+y^2)^{\frac{3}{2}}}dx+\frac{-xy}{(x^2+y^2)^{\frac{3}{2}}}dy$

（3）$\frac{\partial z}{\partial x}=\frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2}=\frac{-y}{x^2+y^2},\frac{\partial z}{\partial y}=\frac{x}{(x^2+y^2)}$

$dz=\frac{-y}{x^2+y^2}dx+\frac{x}{(x^2+y^2)}dy$

（4）$u=e^{\frac{\ln x-\ln y}{z}}$

$\frac{\partial u}{\partial x}=\frac{u}{xz},\frac{\partial u}{\partial y}=\frac{-u}{yz},\frac{\partial u}{\partial y}=\frac{u}{z^2}(\ln y-\ln x)$

$dz=\frac{1}{z^2}(\frac{x}{y})^{\frac{1}{z}}\left[\frac{z}{x}dx-\frac{z}{y}dy-\ln(\frac{x}{y})dz\right]$

17. 解：

（1） 考察函数$z=f(x,y)=\sqrt{x^2+y^2}$

$$f_x(1,2)=\left.\frac{x}{\sqrt{x^2+y^2}}\right|_{(1,2)}=\frac{1}{\sqrt{5}},\ f_y(1,2)=\frac{2}{\sqrt{5}}$$

$$原式\approx\sqrt{1^2+2^2}+\frac{1}{\sqrt{5}}\cdot(0.02)+\frac{2}{\sqrt{5}}\cdot(-0.03)\approx 2.218179434$$

（2） 考察函数$z=f(x,y)=x^y$

$$f_x(10,2)=\left.yx^{y-1}\right|_{(10,2)}=20，f_y(10,2)=\left.x^y\ln x\right|_{(10,2)}=100\ln 10$$

$$原式\approx 10^2+20\times 0.1+100\ln 2\times 0.03=102+3\ln 10\approx 108.9077553$$

18. 解：

设水池总体积$V(a,b,c)=abc$,其中a,b,c分别为长方体的长、宽、高。

则近似值：$V_1=2V_a\Delta a+2V_b\Delta b+V_c\Delta c$

$$=2\times 4\times 3\times(0.2)+2\times 5\times 3\times(0.2)+5\times 4\times(0.2)$$

$$=14.8\mathrm{m}^3$$

准确值：$V_2=V(5,4,3)-V(5-0.4,4-0.4,3-0.2)$

$$=13.632m^2$$

19. 解：

设扇形面积为$S(r,\theta)=\frac{1}{2}r^2\theta$,其中$r$为半径，$\theta$为中心角。

由全微分可知：$\Delta S\approx S_r\Delta r+S_\theta\Delta\theta=r\theta\Delta r+\frac{1}{2}r^2\Delta\theta$

代入:$\Delta S=0$，$r=R=20m,\theta=\alpha=60°,\Delta\theta=1°$计算可得$\Delta r=-\frac{1}{6}m\approx-0.167m$

$\therefore$ 应把扇形半径减少$0.167m$

20. 证明：

（1） 设乘积$z = xy$

则相对误差为：$\frac{\partial z}{z} = \frac{y\partial x + x\partial y}{xy} = \frac{\partial x}{x} + \frac{\partial y}{y}$

故乘积的相对误差等于各因子的相对误差之和

（2） 设商$z = \frac{x}{y}$

则相对误差为：$\frac{\partial z}{z} = \frac{\frac{1}{y}\partial x - \frac{x}{y^2}\partial y}{x/y} = \frac{y\partial x - x\partial y}{xy} = \frac{\partial x}{x} - \frac{\partial y}{y}$

故商的相对误差等于被除数与除数的相对误差之差

21. 解：

（1） $\frac{dz}{dt} = e^{\sin t - 2t^3}(\cos t - 6t^2)$

（2） $\frac{dz}{dt} = -\frac{1}{\sqrt{1-(3t-4t^2)^2}} \cdot (3-8t) = \frac{8t-3}{\sqrt{1-(3t-4t^2)^2}}$

（3） $\frac{dz}{dt} = \frac{de^{ax}(a\sin x - \cos x)}{(a^2+1)dx} = \frac{ae^{ax}(a\sin x - \cos x) + e^{ax}(a\cos x + \sin x)}{a^2+1} = e^{ax}\sin x$

（4） $\frac{du}{dt} = f_x + \frac{1}{t}f_y + \sec^2 t f_z$

22. 解：

（1） $z_x = z_u u_x + z_v v_x = 2u\ln v \cdot (-\frac{y}{x^2}) + \frac{u^2}{v} \cdot (-2) = -\frac{2y^2}{x^2}\left[\frac{\ln(3y-2x)}{x} + \frac{1}{3y-2x}\right]$

$z_y = 2u\ln v \cdot \frac{1}{x} + \frac{u^2}{v} \cdot 3 = \frac{y^2}{x^2}\left[\frac{2}{y}\ln(3y-2x) + \frac{3}{3y-2x}\right]$

（2） 令$u=x^2+y^2, v=xy,$则$z=ue^{\frac{u}{v}}$

$$z_x=(e^{\frac{u}{v}}+\frac{u}{v}e^{\frac{u}{v}})2x+ue^{\frac{u}{v}}\cdot(-\frac{u}{v^2})y=e^{\frac{x^2+y^2}{xy}}(2x+\frac{x^4-y^4}{x^2y})$$

$$z_{y=}=e^{\frac{x^2+y^2}{xy}}(2y+\frac{y^4-x^4}{xy^2})$$

（3） 令$2x+y=u, z=e^{u\ln u}$

$$z_x=e^{u\ln u}(1+\ln u)2=2(2x+y)^{2x+y}\left[\ln(2x+y)+1\right]$$

$$z_y=(2x+y)^{2x+y}\left[\ln(2x+y)+1\right]$$

（4） $z=e^{e^{y\ln x}\ln x}\Rightarrow z_x=z\cdot(e^{y\ln x}\cdot\frac{y}{x}\ln x+\frac{1}{x}e^{y\ln x})=x^{x^y+y-1}(1+y\ln x), z_y=x^{x^y+y}\ln^2 x$

23. 解：

（1） $z_x=f_1+f_2, z_y=f_1-f_2$

（2） $z_x=2xf_1+ye^{xy}f_2, z_y=-2yf_1+xe^{xy}f_2$

（3） $z_x=yf'(\frac{y}{x})\cdot(-\frac{y}{x^2})=-\frac{y^2}{x^2}f'(\frac{y}{x})$

$$z_y=f(\frac{y}{x})+yf'(\frac{y}{x})\cdot\frac{1}{x}=f(\frac{y}{x})+\frac{y}{x}f'(\frac{y}{x})$$

（4） $z_t=f_1+sf_2+srf_3, z_s=tf_2+trf_3, z_r=tsf_3$

24. 解：

（1）

$$\frac{\partial z}{\partial x}=yf'(x^2-y^2)\cdot 2x=2xyf'(x^2-y^2)$$

$$\frac{\partial z}{\partial y}=f(x^2-y^2)+yf'(x^2-y^2)\cdot(-2y)=f(x^2-y^2)-2y^2f'(x^2-y^2)$$

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x}+\frac{1}{y}\cdot\frac{\partial z}{\partial y}=2yf'(x^2-y^2)+\frac{f(x^2-y^2)}{y}-2yf'(x^2-y^2)=\frac{z}{y^2}$$

（2）

$$\frac{\partial u}{\partial x}=kx^{k-1}f(\frac{z}{x},\frac{y}{x})+\left[f_1\cdot(-\frac{z}{x^2})+(f_2\cdot(-\frac{y}{x^2})\right]x^k=x^{k-2}\left[kxf(\frac{z}{x},\frac{y}{x})-zf_1-yf_2\right]$$

$$\frac{\partial u}{\partial y}=x^k\cdot\frac{1}{x}f_2=x^{k-1}f_2,\frac{\partial u}{\partial z}=x^{k-1}f_1$$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=kx^k f(\frac{z}{x},\frac{y}{x})=ku$$

25. 证明：

$$x=r\cos\theta, y=r\sin\theta$$

$$\because\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta,\frac{\partial u}{\partial\theta}=-\frac{\partial u}{\partial x}r\sin\theta+\frac{\partial u}{\partial y}r\cos\theta$$

$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial x}\cos\theta++\frac{\partial v}{\partial y}\sin\theta,\frac{\partial v}{\partial\theta}=-\frac{\partial v}{\partial x}r\sin\theta+\frac{\partial v}{\partial y}r\cos\theta$$

$$\therefore\frac{\partial u}{\partial r}=\frac{1}{r}(\frac{\partial u}{\partial x}r\cos\theta+\frac{\partial u}{\partial y}r\sin\theta)=\frac{1}{r}(\frac{\partial v}{\partial y}r\cos\theta-\frac{\partial v}{\partial x}r\sin\theta)=\frac{1}{r}\frac{\partial v}{\partial\theta}$$

$$\frac{\partial v}{\partial r}=\frac{1}{r}(\frac{\partial v}{\partial x}r\cos\theta++\frac{\partial v}{\partial y}r\sin\theta)=-\frac{1}{r}(\frac{\partial u}{\partial y}r\cos\theta-\frac{\partial u}{\partial x}r\sin\theta)=-\frac{1}{r}\frac{\partial u}{\partial\theta}$$

$$\therefore\frac{\partial u}{\partial r}=\frac{1}{r}\cdot\frac{\partial v}{\partial\theta},\frac{\partial v}{\partial r}=-\frac{1}{r}\cdot\frac{\partial u}{\partial\theta}$$

26. 解：

（1）

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{1}{2}\cdot\frac{2x}{x^2+y^2}+\frac{\partial z}{\partial v}\cdot\frac{-\frac{y}{x^2}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}\frac{\partial z}{\partial u}-\frac{y}{x^2+y^2}\frac{\partial z}{\partial v}$$

$$\frac{\partial z}{\partial y}=\frac{y}{x^2+y^2}\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\cdot\frac{\frac{1}{x}}{1+(\frac{y}{x})^2}=\frac{x}{x^2+y^2}\frac{\partial z}{\partial u}+\frac{x}{x^2+y^2}\frac{\partial z}{\partial v}$$

$$（x+y)\frac{\partial z}{\partial x}-（x-y)\frac{\partial z}{\partial y}=\frac{x^2+xy}{x^2+y^2}\frac{\partial z}{\partial u}-\frac{xy+y^2}{x^2+y^2}\frac{\partial z}{\partial v}-\frac{x^2-xy}{x^2+y^2}\frac{\partial z}{\partial u}+\frac{x^2-xy}{x^2+y^2}\frac{\partial z}{\partial v}$$

$$=\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial u}-\frac{2xy}{x^2+y^2}\frac{\partial z}{\partial v}$$

$$=0$$

$\therefore$变换结果为：$\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=0$

（2）

$$z=e^{x+y+\omega}$$

$$\frac{\partial z}{\partial x}=e^{x+y+\omega}(1+2x\omega_u-\frac{1}{x^2}\omega_v),\frac{\partial z}{\partial y}=e^{x+y+\omega}(1+2y\omega_u-\frac{1}{y^2}\omega_v)$$

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}-(y-x)z=e^{x+y+\omega}(y+2xy\omega_u-\frac{y}{x^2}\omega_v-x-2xy\omega_u+\frac{x}{y^2}\omega_v-y+x)$$

$$=e^{x+y+\omega}(\frac{x}{y^2}--\frac{y}{x^2})\omega_v$$

$$=0$$

$\therefore$变换结果为：$\omega_v=0$

27. 解：

（1）

$$\frac{\partial z}{\partial x}=y^2 f_1+2xyf_2,\frac{\partial z}{\partial y}=2xyf_1+x^2 f_2$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=(y^2 f_{11}+2xyf_{12})y^2+2yf_2+(y^2 f_{21}+2xyf_{22})2xy\\&=y^4 f_{11}+4xy^3 f_{12}+4x^2y^2 f_{22}+2yf_2\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y}&=2yf_1+(2xyf_{11}+x^2 f_{12})y^2+2xf_2+(2xyf_{21}+x^2 f_{22})2xy\\&=2xy^3 f_{11}+5x^2y^2 f_{12}+2x^3yf_{22}+2yf_1+2xf_2\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2}&=2xf_1+(2xyf_{11}+x^2 f_{12})2xy+(2xyf_{21}+x^2 f_{22})x^2\\&=4x^2y^2 f_{11}+4x^3yf_{12}+x^4 f_{22}+2xf_1\end{aligned}$$

（2）

$$\frac{\partial z}{\partial x}=f_1+\frac{1}{y}f_2,\frac{\partial z}{\partial y}=-\frac{x}{y^2}f_2$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=f_{11}+\frac{1}{y}f_{12}+\frac{1}{y}(f_{21}+\frac{1}{y}f_{22})\\&=f_{11}+\frac{2}{y}f_{12}+\frac{1}{y^2}f_{22}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial x\partial y}&=-\frac{x}{y^2}f_{12}-\frac{1}{y^2}f_2-\frac{x}{y^3}f_{22}\\&=-\frac{x}{y^2}f_{12}-\frac{x}{y^3}f_{22}-\frac{1}{y^2}f_2\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2}&=\frac{2x}{y^3}f_2-\frac{x}{y^2}f_{22}\cdot(-\frac{x}{y^2})\\&=\frac{2x}{y^3}f_2+\frac{x^2}{y^4}f_{22}\end{aligned}$$

（3）

$$\frac{\partial z}{\partial x}=2xf',\frac{\partial z}{\partial y}=2yf'$$

$$\frac{\partial^2 z}{\partial x^2}=2f'+2xf''(2x)$$
$$=2f'+4x^2f''$$

$$\frac{\partial^2 z}{\partial x\partial y}=4xyf''$$

$$\frac{\partial^2 z}{\partial y^2}=2f'+4y^2f''$$

（4）

$$\frac{\partial z}{\partial x}=f_1+yf_2+\frac{1}{y}f_3,\frac{\partial z}{\partial y}=f_1+xf_2-\frac{x}{y^2}f_3$$

$$\frac{\partial^2 z}{\partial x^2}=f_{11}+yf_{12}+\frac{1}{y}f_{13}+(f_{21}+yf_{22}+\frac{1}{y}f_{23})y+(f_{31}+yf_{32}+\frac{1}{y}f_{33})\frac{1}{y}$$
$$=f_{11}+y^2f_{22}+\frac{1}{y^2}f_{33}+2yf_{12}+2f_{23}+\frac{2}{y}f_{13}$$

$$\frac{\partial^2 z}{\partial x\partial y}=f_{11}+xf_{12}-\frac{x}{y^2}f_{13}+f_2+(f_{21}+xf_{22}-\frac{x}{y^2}f_{23})y-\frac{1}{y^2}f_3+(f_{31}+xf_{32}-\frac{x}{y^2}f_{33})\frac{1}{y}$$
$$=f_{11}+xyf_{22}-\frac{x}{y^3}f_{33}+(x+y)f_{12}+\frac{y-x}{y^2}f_{13}+f_2-\frac{1}{y^2}f_3$$

$$\frac{\partial^2 z}{\partial y^2}=f_{11}+xf_{12}-\frac{x}{y^2}f_{13}+(f_{21}+xf_{22}-\frac{x}{y^2}f_{23})x+\frac{2x}{y^3}f_3+(f_{31}+xf_{32}-\frac{x}{y^2}f_{33})\cdot(-\frac{x}{y^2})$$
$$=f_{11}+x^2f_{22}+\frac{x^2}{y^4}f_{33}+2xf_{12}-\frac{2x^2}{y^2}f_{23}-\frac{2x}{y^2}f_{13}+\frac{2x}{y^3}f_3$$

28. 解：

（1）

$$F(x,y)=\int_1^x\left[f(u)\int_0^{yu}g(\frac{t}{u})dt\right]du$$

$$\frac{\partial F}{\partial x}=f(x)\int_0^{yx}g(\frac{t}{x})dt$$

$$\frac{\partial^2 F}{\partial x\partial y}=xf(x)g(y)$$

（2）

$$\frac{\partial F}{\partial x}=2xyf(x^2y,e^{x^2y})$$

$$\frac{\partial^2 F}{\partial x\partial y}=2xf+(x^2f_1+x^2e^{x^2y}f_2)\cdot(2xy)$$

$$=2xf+2x^3y(f_1+e^{x^2y}f_2)$$

29. 证明：

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta,\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}(-r\sin\theta)+\frac{\partial u}{\partial y}r\cos\theta$$

$$\frac{\partial^2 u}{\partial r^2}=\frac{\partial^2 u}{\partial x^2}\cos^2\theta+\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial u}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\theta$$

$$\frac{\partial^2 u}{\partial \theta^2}=(\frac{\partial^2 u}{\partial x^2}(-r\sin\theta)+\frac{\partial^2 u}{\partial x\partial y}r\cos\theta)(-r\sin\theta)-\frac{\partial u}{\partial x}r\cos\theta$$

$$+\frac{\partial u}{\partial y\partial x}(-r\sin\theta)+\frac{\partial^2 u}{\partial y^2}r\cos\theta)r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta$$

$$=\frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta+\frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta-\frac{\partial^2 u}{\partial x\partial y}r^2\sin\theta\cos\theta-\frac{\partial u}{\partial y\partial x}r^2\sin\theta\cos\theta-\frac{\partial u}{\partial x}r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta$$

$$\text{右边}=\frac{\partial^2 u}{\partial x^2}\cos^2\theta+\frac{\partial^2 u}{\partial x\partial y}\sin\theta\cos\theta+\frac{\partial u}{\partial y\partial x}\sin\theta\cos\theta+\frac{\partial^2 u}{\partial y^2}\sin^2\theta+\frac{1}{r}(\frac{\partial u}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta)$$

$$+\frac{1}{r^2}(\frac{\partial^2 u}{\partial x^2}r^2\sin^2\theta+\frac{\partial^2 u}{\partial y^2}r^2\cos^2\theta-\frac{\partial^2 u}{\partial x\partial y}r^2\sin\theta\cos\theta-\frac{\partial u}{\partial y\partial x}r^2\sin\theta\cos\theta-\frac{\partial u}{\partial x}r\cos\theta-\frac{\partial u}{\partial y}r\sin\theta)$$

$$=\frac{\partial^2 u}{\partial x^2}(\cos^2\theta+\sin^2\theta)+\frac{\partial^2 u}{\partial y^2}(\sin^2\theta+\cos^2\theta)$$

$$+\frac{\partial^2 u}{\partial x\partial y}(\sin\theta\cos\theta-\sin\theta\cos\theta)+\frac{\partial u}{\partial y\partial x}(\sin\theta\cos\theta-\sin\theta\cos\theta)$$

$$\frac{\partial u}{\partial x}(\frac{\cos\theta}{r}-\frac{\cos\theta}{r})+\frac{\partial u}{\partial y}(\frac{\sin\theta}{r}-\frac{\sin\theta}{r})$$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\text{左边}$$

$\therefore$ 原等式得以验证

30. 解：

$$\frac{\partial z}{\partial x}=e^x\sin yf',\frac{\partial^2 z}{\partial x^2}=e^x\sin yf'+e^{2x}\sin^2 yf''$$

$$\frac{\partial z}{\partial y}=e^x\cos yf',\frac{\partial^2 z}{\partial y^2}=-e^x\sin yf'+e^{2x}\cos^2 yf''$$

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=e^{2x}f''=e^{2x}f\Rightarrow f''=f$$

特征方程：$r^2=1\Rightarrow r=\pm 1$

$$f(u)=C_1e^u+C_2e^{-u}$$

31. 解：

（1）解法 1：

$$\frac{\partial z}{\partial u}=e^u\cos vz_x+e^u\sin vz_y$$
$$\frac{\partial z}{\partial v}=-e^u\sin vz_x+e^u\cos vz_y$$

$$\begin{aligned}\frac{\partial^2 z}{\partial u^2}&=e^u\cos vz_x+(z_{xx}e^u\cos v+z_{xy}e^u\sin v)e^u\cos v\\&\quad+e^u\sin vz_y+(z_{yx}e^u\cos v+z_{yy}e^u\sin v)e^u\sin v\\&=e^{2u}\cos^2 vz_{xx}+e^{2u}\sin v\cos vz_{xy}+e^{2u}\sin^2 vz_{yy}\\&\quad+e^u\cos vz_x+e^u\sin vz_y\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial v^2}&=-e^u\cos vz_x+(-e^u\sin vz_{xx}+e^u\cos vz_{xy})\cdot(-e^u\sin v)\\&\quad-e^u\sin vz_y+(-e^u\sin vz_{yx}+e^u\cos vz_{yy})\cdot e^u\cos v\\&=e^{2u}\sin^2 vz_{xx}-e^{2u}\sin v\cos vz_{xy}+e^{2u}\cos^2 vz_{yy}\\&\quad-e^u\cos vz_x-e^u\sin vz_y\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}&=e^{2u}\sin^2 vz_{xx}-e^{2u}\sin v\cos vz_{xy}+e^{2u}\cos^2 vz_{yy}\\&\quad-e^u\cos vz_x-e^u\sin vz_y\\&\quad+e^{2u}\cos^2 vz_{xx}+e^{2u}\sin v\cos vz_{xy}+e^{2u}\sin^2 vz_{yy}\\&\quad+e^u\cos vz_x+e^u\sin vz_y\\&=e^{2u}z_{xx}+e^{2u}z_{yy}\\&=e^{2u}(z_{xx}+z_{yy})\\&=-m^2ze^{2u}\end{aligned}$$

$\therefore$ 原方程变换为：$\dfrac{\partial^2 z}{\partial u^2}+\dfrac{\partial^2 z}{\partial v^2}+m^2ze^{2u}=0$

解法 2：

$$\begin{cases} x = e^u \cos v \\ y = e^u \sin v \end{cases} \xrightarrow{\text{两边同时取微分}} \begin{cases} dx = e^u \cos v du - e^u \sin v dv \\ dy = e^u \sin v du + e^u \cos v dv \end{cases}$$

$$\xrightarrow{\text{解方程组可得}} \begin{cases} du = -e^{-u} \cos v dx - e^{-u} \sin v dy \\ dv = -e^{-u} \sin v dx + e^{-u} \cos v dy \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x} = -e^{-u} \cos v, \frac{\partial u}{\partial y} = -e^{-u} \sin v, \frac{\partial v}{\partial x} = -e^{-u} \sin v, \frac{\partial v}{\partial y} = e^{-u} \cos v$$

根据$\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial x} \\ \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial y} \end{cases}$，从而逐阶求导即可

（2）

$$\frac{\partial z}{\partial x} = yz_u + \frac{1}{y} z_v$$

$$\frac{\partial z}{\partial y} = xz_u - \frac{x}{y^2} z_v$$

$$\begin{aligned} \frac{\partial^2 z}{\partial x^2} &= (yz_{uu} + \frac{1}{y} z_{uv})y + (yz_{vu} + \frac{1}{y} z_{vv})\frac{1}{y} \\ &= y^2 z_{uu} + 2z_{uv} + \frac{1}{y^2} z_{vv} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 z}{\partial y^2} &= (xz_{uu} - \frac{x}{y^2} z_{uv})x + \frac{2x}{y^3} z_v + (xz_{vu} - \frac{x}{y^2} z_{vv})(-\frac{x}{y^2}) \\ &= x^2 z_{uu} - \frac{2x^2}{y^2} z_{uv} + \frac{x^2}{y^4} z_{vv} + \frac{2x}{y^3} z_v \end{aligned}$$

$$\begin{aligned} x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} &= x^2 (y^2 z_{uu} + 2z_{uv} + \frac{1}{y^2} z_{vv}) \\ &\quad - y^2 (x^2 z_{uu} - \frac{2x^2}{y^2} z_{uv} + \frac{x^2}{y^3} z_{vv} + \frac{2x}{y^3} z_v) \\ &= 4x^2 z_{uv} - \frac{2x}{y} z_v \\ &= 4uvz_{uv} - 2vz_v = 0 \end{aligned}$$

$$\therefore \text{原方程变换为}: 2u \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} = 0$$

32. 解：

（1）

$$z_x = z_u + z_v, z_y = az_u + bz_v$$

$$z_{xx} = z_{uu} + 2z_{uv} + z_{vv}$$

$$\begin{aligned} z_{xy} &= az_{uu} + bz_{uv} + az_{vu} + bz_{vv} \\ &= az_{uu} + (a+b)z_{uv} + bz_{vv} \end{aligned}$$

$$\begin{aligned} z_{yy} &= (az_{uu} + bz_{uv})a + (az_{vu} + bz_{vv})b \\ &= a^2 z_{uu} + 2abz_{uv} + b^2 z_{vv} \end{aligned}$$

$$\begin{aligned} z_{xx} + 4z_{xy} + 3z_{yy} &= z_{uu} + 2z_{uv} + z_{vv} + 4\left[az_{uu} + (a+b)z_{uv} + bz_{vv}\right] + 3(a^2 z_{uu} + 2abz_{uv} + b^2 z_{vv}) \\ &= (1+4a+3a^2)z_{uu} + \left[2+4(a+b)+6ab\right]z_{uv} + (1+4b+3b^2)z_{vv} \\ &= 0 \end{aligned}$$

$$\Rightarrow 1+4a+3a^2 = 0, 2+4(a+b)+6ab \neq 0, 1+4b+3b^2 = 0$$

$$\Rightarrow (a,b) = (-\frac{1}{3},-1),(-1,-\frac{1}{3})$$

（2）

$$z_x = z_u + z_v, z_y = -2z_u + az_v$$

$$z_{xx} = z_{uu} + 2z_{uv} + z_{vv}$$

$$\begin{aligned} z_{xy} &= -2z_{uu} + az_{uv} - 2z_{vu} + az_{vv} \\ &= -2z_{uu} + (a-2)z_{uv} + az_{vv} \end{aligned}$$

$$\begin{aligned} z_{yy} &= -2(-2z_{uu} + az_{uv}) + a(-2z_{vu} + az_{vv}) \\ &= 4z_{uu} - 4az_{uv} + a^2 z_{vv} \end{aligned}$$

$$\begin{aligned} 6z_{xx} + z_{xy} - z_{yy} &= 6(z_{uu} + 2z_{uv} + z_{vv}) \\ &\quad + (-2z_{uu} + (a-2)z_{uv} + az_{vv}) \\ &\quad - (4z_{uu} - 4az_{uv} + a^2 z_{vv}) \\ &= (5a+10)z_{uv} + (6 + a - a^2)z_{vv} \\ &= 0 \end{aligned}$$

$$\Rightarrow 5a + 10 \neq 0, 6 + a - a^2 = 0$$

$$\Rightarrow a = 3$$

33. 解：

（1）

两边同时取微分得：

$$(xdy + ydx)\cos xy - e^{xy}(xdy + ydx) - 2xydx - x^2 dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y(\cos xy - e^{xy} - 2x)}{x(\cos xy - e^{xy} - x)}$$

（2）

两边同时取微分得:

$$\frac{1}{1+\frac{y^2}{x^2}}(\frac{dy}{x}-\frac{ydx}{x^2})=\frac{1}{\sqrt{x^2+y^2}}\cdot(\frac{2xdx+2ydy}{2\sqrt{x^2+y^2}})$$

$$\Rightarrow\frac{1}{x^2+y^2}(xdy-ydx)=\frac{xdx+ydy}{x^2+y^2}$$

$$\Rightarrow(x-y)dy=(x+y)dx$$

$$\Rightarrow\frac{dy}{dx}=\frac{x+y}{x-y}$$

（3）

两边同时取对数得：

$x\ln y=y\ln x$

两边同时取微分得：

$$\ln ydx+\frac{x}{y}dy=\frac{y}{x}dx+\ln xdy$$

$$\Rightarrow(\ln y-\frac{y}{x})dx=(\ln x-\frac{x}{y})dy$$

$$\Rightarrow\frac{dy}{dx}=\frac{\ln y-\frac{y}{x}}{\ln x-\frac{x}{y}}=\frac{xy\ln y-y^2}{xy\ln x-x^2}=\frac{y(y-x\ln y)}{x(x-y\ln x)}$$

（4）

原等式变形为：

$\sin xy=\ln|x+1|-\ln|y|+1,$令$x=0,$得$y(0)=e$

两边同时取微分得：

$$(ydx+xdy)\cos xy=\frac{dx}{x+1}-\frac{dy}{y}$$

$$\Rightarrow(y\cos xy-\frac{1}{x+1})dx=-(\frac{1}{y}+x\cos xy)dy$$

$$\Rightarrow y'=\frac{y\cos xy-\frac{1}{x+1}}{-(\frac{1}{y}+x\cos xy)}$$

代入$x=0,y(0)=e,$计算得：$y'(0)=\dfrac{e-1}{-(\frac{1}{e}+0)}=e-e^2$

34. 解：

（1）

两边同时对x求偏导：

$$\frac{z-xz_x}{z^2}=\frac{\cos\frac{z}{y}}{\sin\frac{z}{y}}\cdot\frac{z_x}{y}$$

$$\Rightarrow z_x=\frac{yz}{xy+z^2\cot\frac{z}{y}}$$

同理$z_y=\dfrac{z^3\cot\frac{z}{y}}{y(xy+z^2\cot\frac{z}{y})}$

（2）

两边同时对x求偏导：

$e^z z_x-yz-xyz_x=0$

$$\Rightarrow z_x=\frac{yz}{e^z-xy}=\frac{yz}{xyz-xy}=\frac{z}{x(z-1)}$$

由对称轮换性知：$z_y=\dfrac{z}{y(z-1)}$

（3）

原式中，令$x=y=0$，则$z(0,0)=1$

两边同时对x求偏导：

$e^z z_x+yz+xyz_x=0$

令$x=y=0,z=1$

$\Rightarrow z_x(0,0)=0$

（4）

两边同时对x求偏导：

$$2x+2zz_x=2z_x\Rightarrow z_x=\frac{x}{1-z}$$

同理$z_y=\dfrac{y}{1-z}$

$$dz=\frac{xdx+ydy}{1-z}$$

（5）

两边同时对x求偏导:

$$(1+y)F_1+(yz+xyz_x)F_2=0\Rightarrow z_x=-\frac{(1+y)F_1+yzF_2}{xyF_2}$$

同理可得$z_y=-\dfrac{F_1+zF_2}{yF_2}$

$$\Rightarrow dz=-\frac{(1+y)F_1+yzF_2}{xyF_2}dx-\frac{F_1+zF_2}{yF_2}dy=-\frac{\left[(1+y)F_1+yzF_2\right]dx+(xF_1+xzF_2)dy}{xyF_2}$$

35. 解：

（1）

$z=xf(x+y)$,两边同时对x求导：

$$\frac{dz}{dx}=f(x+y)+(1+y')xf'(x+y)$$

$F(x,y,z)=0$，两边同时对x求导：

$$F_x+y'F_y+z'F_z=0$$

上述两式联立，消去y'：

$$\frac{dz}{dx}=\frac{(f+xf')F_y-xf'F_x}{F_y+xf'F_z}$$

（2）

$u=f\left(x,y,z\right)$,两边同时对x求偏导：

$$u_x=f_x+\cos x\cdot f_y+z_x f_z$$

$\varphi\left(x^2,e^y,z\right)=0$,两边同时对$x$求偏导：

$$2x\varphi_1+\cos x\cdot e^y\varphi_2+z_x\varphi_3=0$$

两式联立，消去z_x得：

$$u_x=f_x+\cos x\cdot f_y-\frac{2x\varphi_1+\cos x\cdot e^y\varphi_2}{\varphi_3}f_z$$

（3）

$e^{xy}-xy=2$,两边同时对x求偏导：
$(y+xy')e^{xy}-(y+xy')=0$
$e^{x}=\int_0^{x-z}\frac{\sin t}{t}dt$,两边同时对$x$求偏导：
$e^{x}=\frac{\sin(x-z)}{x-z}(1-z')$
$\frac{du}{dx}=f_x+y'f_y+z'f_z=f_x-\frac{y}{x}f_y+\left[1-\frac{e^{x}(x-z)}{\sin(x-z)}\right]f_z$

36. 解：

（1）

$\varphi(cx-az,cy-bz)=0$,两边同时对x,y分别求偏导：
$(c-az_x)\varphi_1+(-bz_x)\varphi_2=0 \qquad (-az_y)\varphi_1+(c-bz_y)\varphi_2=0$
则有$:z_x=\frac{c\varphi_1}{a\varphi_1+b\varphi_2},z_y=\frac{c\varphi_2}{a\varphi_1+b\varphi_2}$
$az_x+bz_y=\frac{ac\varphi_1+bc\varphi_2}{a\varphi_1+b\varphi_2}=c$

（2）

$u=f(z)$,两边同时对x,y分别求偏导：
$u_x=z_xf' \qquad u_y=z_yf'$
$z=x+y\varphi(z)$,两边同时对x,y分别求偏导：
$z_x=1+y(z_x\varphi') \qquad z_y=\varphi+z_y\varphi'=\frac{z-x}{y}+z_y\varphi'$
$u_y=\frac{\varphi}{1-\varphi'}f'==\frac{\varphi}{1-\frac{z_x-1}{z_x}}f'=\varphi z_xf'=\varphi u_x$

（3）

$\varphi(\frac{x}{z},\frac{y}{z})=0,$两边同时对$x,y$分别求偏导：

$(\frac{z-xz_x}{z^2})\varphi_1+(-\frac{y}{z^2}z_x)\varphi_2=0 \qquad (-\frac{x}{z^2}z_y)\varphi_1+(\frac{z-yz_y}{z^2})\varphi_2=0$

$\Rightarrow z_x=\frac{z\varphi_1}{x\varphi_1+y\varphi_2} \qquad z_y=\frac{z\varphi_2}{x\varphi_1+y\varphi_2}$

$xz_x+yz_y=\frac{xz\varphi_1+yz\varphi_2}{x\varphi_1+y\varphi_2}=\frac{z(x\varphi_1+y\varphi_2)}{x\varphi_1+y\varphi_2}=z$

（4）

$ax+by+cz=\varphi(x^2+y^2+z^2),$两边同时对$x,y$分别求偏导：

$a+cz_x=(2x+2zz_x)\varphi' \qquad b+cz_y=(2y+2zz_y)\varphi'$

$\Rightarrow z_x=\frac{a-2x\varphi'}{2z\varphi'-c} \qquad z_y=\frac{b-2y\varphi'}{2z\varphi'-c}$

$$
\begin{aligned}
(cy-bz)z_x+(az-cx)z_y &= \frac{(cy-bz)(a-2x\varphi')+(az-cx)(b-2y\varphi')}{2z\varphi'-c} \\
&= \frac{(-2cxy+2bxz-2ayz+2cxy)\varphi'-bcx+acy+(ab-ab)z}{2z\varphi'-c} \\
&= \frac{2z\varphi'(bx-ay)-c(bx-ay)}{2z\varphi'-c} \\
&= \frac{(2z\varphi'-c)(bx-ay)}{2z\varphi'-c} \\
&= bx-ay
\end{aligned}
$$

37 解：

（1）

$z^3-3xyz=a^3,$两边同时对x求偏导：

$3z^2z_x-3yz-3xyz_x=0\Rightarrow z_x=\frac{yz}{z^2-xy}$

且：$z^2z_x-yz-xyz_x=0$，两边同时对x求导：

$2z(z_x)^2+z^2z_{xx}-yz_x-yz_x-xyz_{xx}=0$

代入z_x解得：$z_{xx}=-\frac{2xy^3z}{(z^3-xy)^3}$

（2）

$e^z - xyz = 0$，两边同时对x，y分别求偏导：

$$z_x e^z - yz - xyz_x = 0 \Rightarrow z_x = \frac{yz}{e^z - xy} = \frac{yz}{xyz - xy} = \frac{z}{x(z-1)}$$

由对称轮换性知：$z_y = \dfrac{z}{y(z-1)}$

$z_x = \dfrac{z}{x(z-1)}$,再对y求偏导:

$$z_{xy} = \frac{z_y\left[x(z-1)\right] - xz_y z}{x^2(z-1)^2} = \frac{xz_y(z-1-z)}{x^2(z-1)^2} = -\frac{z_y}{x(z-1)^2} = -\frac{z}{xy(z-1)^3}$$

（3）

$x^2 + y^2 + z^2 = 4z$，两边同时对y求偏导：

$$2y + 2zz_y = 4z_y \Rightarrow z_y = \frac{y}{2-z}$$

再对y求偏导：

$$z_{yy} = \frac{(2-z) - (-z_y)y}{(2-z)^2} = \frac{(2-z) + yz_y}{(2-z)^2} = \frac{(2-z) + \dfrac{y^2}{2-z}}{(2-z)^2} = \frac{(2-z)^2 + y^2}{(2-z)^3}$$

（4）

$z^5 - xz^4 + yz^3 = 1$,两边同时分别对x, y求偏导:

$$5z^4 z_x - z^4 - 4xz^3 z_x + 3yz^2 z_x = 0 \qquad 5z^4 z_y - 4xz^3 z_y + z^3 + 3yz^2 z_y = 0$$

代入$(x, y) = 0$得：$z_x(0,0) = \dfrac{1}{5}$　　　$z_y(0,0) = -\dfrac{1}{5}$

$5z^4 z_x - z^4 - 4xz^3 z_x + 3yz^2 z_x = 0$,令$x = 0$;

$5z^4 z_x - z^4 + 3yz^2 z_x = 0$（$x = 0$)，两边同时对$y$求导:

$$20z^3 z_y z_x + 5z^4 z_{xy} - 4z^3 z_y + 3z^2 z_x + 3y(z^2 z_x)' = 0$$

令$y = 0$,解得：$z_{xy}(0,0) = -\dfrac{3}{25}$

38.证明：

由于 $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{F_x}{F_y}$，于是有

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-\frac{(F_{xx}+F_{xy}\frac{\mathrm{d}y}{\mathrm{d}x})F_y-F_x(F_{yx}+F_{yy}\frac{\mathrm{d}y}{\mathrm{d}x})}{(F_y)^2}$$

$$=-\frac{\left(F_{xx}+F_{xy}\cdot\left(-\frac{F_x}{F_y}\right)\right)\cdot F_y-F_x\cdot\left(F_{yx}+F_{yy}\cdot\left(-\frac{F_x}{F_y}\right)\right)}{(F_y)^2}$$

$$=-\frac{F_{xx}(F_y)^2-2F_xF_{yx}F_y+F_{yy}(F_x)^2}{(F_y)^3}$$

39.解：

（1）

$$\begin{cases}z-x^2-y^2=0\\x^2+2y^2+3z^2=4\end{cases}\xrightarrow{\text{两边同时对}x\text{求导}}\begin{cases}z'-2x-2yy'=0\\2x+4yy'+6zz'=0\end{cases}$$

$$\xrightarrow{\text{解方程组可得}}\begin{cases}y'=-\dfrac{x(1+6z)}{y(2+6z)}\\z'=\dfrac{x}{1+3z}\end{cases}$$

（2）

$$\begin{cases}xu-yu=0\\yu+xv=1\end{cases}\xrightarrow{\text{两边同时取微分}}\begin{cases}xdu+udx-ydu-udy=0\\ydu+udy+xdv+vdx=0\end{cases}$$

$$\xrightarrow{\text{解方程组可得}}\begin{cases}du=-\dfrac{xu+yv}{x^2+y^2}dx+\dfrac{xv-yu}{x^2+y^2}dy\\dv=\dfrac{-xv+yu}{x^2+y^2}dx-\dfrac{xu+yv}{x^2+y^2}dy\end{cases}$$

$$\Rightarrow\frac{\partial u}{\partial x}=-\frac{xu+yv}{x^2+y^2},\frac{\partial u}{\partial y}=\frac{xv-yu}{x^2+y^2},\frac{\partial v}{\partial x}=\frac{-xv+yu}{x^2+y^2},\frac{\partial v}{\partial y}=-\frac{xu+yv}{x^2+y^2}$$

（3）

$$\begin{cases} u^3+xv=y \\ v^3+yu=x \end{cases} \xrightarrow{\text{两边同时取微分}} \begin{cases} 3u^2du+xdv+vdx=dy \\ 3v^2dv+ydu+udy=dx \end{cases}$$

$$\xrightarrow{\text{解方程组可得}} \begin{cases} du=-\dfrac{x+3v^2}{9x^2v^2-xy}dx+\dfrac{3v^2+xu}{9u^2v^2-xy}dy \\ dv=\dfrac{3u^2+yu}{9u^2v^2-xy}dx-\dfrac{3u^2+y}{9u^2v^2-xy}dy \end{cases}$$

$$\Rightarrow \frac{\partial u}{\partial x}=-\frac{x+3v^2}{9x^2v^2-xy},\frac{\partial u}{\partial y}=\frac{3v^2+xu}{9u^2v^2-xy},\frac{\partial v}{\partial x}=\frac{3u^2+yu}{9u^2v^2-xy},\frac{\partial v}{\partial y}=-\frac{3u^2+y}{9u^2v^2-xy}$$

（4）

$$\begin{cases} x=e^u\cos v \\ y=e^u\sin v \\ z=uv \end{cases} \xrightarrow{\text{整理得}} \begin{cases} x^2+y^2=e^{2u} \\ \tan v=\dfrac{y}{x} \\ z=uv \end{cases} \xrightarrow{\text{整理得}} \begin{cases} u=\dfrac{\ln(x^2+y^2)}{2} \\ v=\arctan\dfrac{y}{x} \\ z=uv \end{cases} \xrightarrow{\text{两边同时取微分}} \begin{cases} du=\dfrac{xdx+ydy}{x^2+y^2} \\ dv=\dfrac{xdy-ydx}{x^2+y^2} \\ dz=udv+vdu \end{cases}$$

$$\xrightarrow{\text{解方程组可得}} dz=\frac{x\arctan\frac{y}{x}-\frac{y}{2}\ln(x^2+y^2)}{x^2+y^2}dx+\frac{y\arctan\frac{y}{x}+\frac{x}{2}\ln(x^2+y^2)}{x^2+y^2}dy$$

$$\Rightarrow \frac{\partial z}{\partial x}=\frac{x\arctan\frac{y}{x}-\frac{y}{2}\ln(x^2+y^2)}{x^2+y^2},\frac{\partial z}{\partial y}=\frac{y\arctan\frac{y}{x}+\frac{x}{2}\ln(x^2+y^2)}{x^2+y^2}$$

40. 解：

（1）

$$\boldsymbol{l}^0=\frac{1}{2}\boldsymbol{i}+\frac{\sqrt{3}}{2}\boldsymbol{j}, z_x\big|_{M_0}=2\times1=2, z_y\big|_{M_0}=2\times2=4$$

$$\frac{dz}{d\boldsymbol{l}}=2\times\frac{1}{2}+4\times\frac{\sqrt{3}}{2}=1+2\sqrt{3}$$

（2）

$$\boldsymbol{l}^0=\frac{1}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{2}{3}\boldsymbol{k}, z_x\big|_{M_0}=\frac{x}{\sqrt{x^2+y^2+z^2}}\Bigg|_{M_0}=\frac{1}{\sqrt{2}}, z_y\big|_{M_0}=0, z_z\big|_{M_0}=\frac{1}{\sqrt{2}}$$

$$\frac{dz}{d\boldsymbol{l}}=\frac{1}{\sqrt{2}}\times\frac{1}{3}+0\times\frac{2}{3}+\frac{1}{\sqrt{2}}\times\frac{2}{3}=\frac{\sqrt{2}}{2}$$

（3）

$$\boldsymbol{l}^0=\frac{1}{\sqrt{3}}\boldsymbol{i}+\frac{1}{\sqrt{3}}\boldsymbol{j}-\frac{1}{\sqrt{3}}\boldsymbol{k}, z_x\big|_{M_0}=\arctan\frac{y}{z}\Big|_{M_0}=-\frac{\pi}{4}, z_y\big|_{M_0}=\frac{x\cdot\frac{1}{z}}{1+(\frac{y}{z})^2}\Bigg|_{M_0}=-\frac{1}{4}, z_z\big|_{M_0}=\frac{x\cdot(-\frac{y}{z^2})}{1+(\frac{y}{z})^2}\Bigg|_{M_0}=-\frac{1}{4}$$

$$\frac{dz}{d\boldsymbol{l}}=-\frac{\pi}{4}\times\frac{1}{\sqrt{3}}-\frac{1}{4}\times\frac{1}{\sqrt{3}}-\frac{1}{4}\times(-\frac{1}{\sqrt{3}})=-\frac{\pi}{4\sqrt{3}}$$

（4）

$$\boldsymbol{l}=(5-2,5-1,15-3)=(3,4,12)$$

$$\boldsymbol{l}^0=\frac{3}{13}\boldsymbol{i}+\frac{4}{13}\boldsymbol{j}+\frac{12}{13}\boldsymbol{k}, z_x\big|_{M_0}=y+z\big|_{M_0}=4, z_y\big|_{M_0}=x+z\big|_{M_0}=5, z_z\big|_{M_0}=x+y\big|_{M_0}=3$$

$$\frac{dz}{d\boldsymbol{l}}=4\times\frac{3}{13}+5\times\frac{4}{13}+3\times\frac{12}{13}=\frac{68}{13}$$

41.解：

令 $u=u(x,y,z)$

（1）

$$u_x=2xy^3z^4, u_y=3x^2y^2z^4, u_z=4x^2y^3z^3$$

$$\nabla u=(2xy^3z^4,3x^2y^2z^4,4x^2y^3z^3)=2xy^3z^4\boldsymbol{i}+3x^2y^2z^4\boldsymbol{j}+4x^2y^3z^3\boldsymbol{k}$$

（2）

$$u_x=6x, u_y=-4y, u_z=6z$$

$$\nabla u=(6x,-4y,6z)=6x\boldsymbol{i}-4y\boldsymbol{j}+6z\boldsymbol{k}$$

（3）

$$u_x=\frac{xz^2}{\sqrt{x^2+2y^2}}, u_y=\frac{2yz^2}{\sqrt{x^2+2y^2}}, u_z=2z\sqrt{x^2+2y^2}$$

$$\nabla u=(\frac{xz^2}{\sqrt{x^2+2y^2}},\frac{2yz^2}{\sqrt{x^2+2y^2}},2z\sqrt{x^2+2y^2})=\frac{xz^2}{\sqrt{x^2+2y^2}}\boldsymbol{i}+\frac{2yz^2}{\sqrt{x^2+2y^2}}\boldsymbol{j}+2z\sqrt{x^2+2y^2}\boldsymbol{k}$$

$$\therefore\nabla u(1,\frac{\sqrt{2}}{2},1)=\frac{\sqrt{2}}{2}\boldsymbol{i}+\boldsymbol{j}+2\sqrt{2}\boldsymbol{k}$$

42.解：

$$f_x=2x,\ f_y=-2z+2y,\ f_z=-2y$$
$$\nabla f=(2x,-2z+2y,-2y)=2x\boldsymbol{i}+(-2z+2y)\boldsymbol{j}-2y\boldsymbol{k}$$
$$\nabla f(-1,2,1)=-2\boldsymbol{i}+2\boldsymbol{j}-4\boldsymbol{k}$$
$$\left|\nabla f(-1,2,1)\right|=2\sqrt{6},\frac{\nabla f(-1,2,1)}{\left|\nabla f(-1,2,1)\right|}=-\frac{1}{\sqrt{6}}\boldsymbol{i}+\frac{1}{\sqrt{6}}\boldsymbol{j}-\frac{2}{\sqrt{6}}\boldsymbol{k}$$

（1）

$\boldsymbol{l}=\dfrac{\nabla f(-1,2,1)}{\left|\nabla f(-1,2,1)\right|}=-\dfrac{1}{\sqrt{6}}\boldsymbol{i}+\dfrac{1}{\sqrt{6}}\boldsymbol{j}-\dfrac{2}{\sqrt{6}}\boldsymbol{k}=（-\dfrac{1}{\sqrt{6}},\dfrac{1}{\sqrt{6}},-\dfrac{2}{\sqrt{6}})$时最大，最大值为$\left|\nabla f(-1,2,1)\right|=2\sqrt{6}$

（2）

$\boldsymbol{l}=-\dfrac{\nabla f(-1,2,1)}{\left|\nabla f(-1,2,1)\right|}=\dfrac{1}{\sqrt{6}}\boldsymbol{i}-\dfrac{1}{\sqrt{6}}\boldsymbol{j}+\dfrac{2}{\sqrt{6}}\boldsymbol{k}=（\dfrac{1}{\sqrt{6}},-\dfrac{1}{\sqrt{6}},+\dfrac{2}{\sqrt{6}})$时最小，最大值为$-\left|\nabla f(-1,2,1)\right|=-2\sqrt{6}$

43.证明：

“充分性”：由 $u=ax+by+cz+d$ 知 $grad\ u=(a,b,c)$.

“必要性”：若 $grad\ u=(a,b,c)$ ，即 $\dfrac{\partial u}{\partial x}=a,\dfrac{\partial u}{\partial y}=b,\dfrac{\partial u}{\partial z}=c$ ，则有 $u=ax+\varphi(y,z)$ ，

从而 $\varphi_y(y,z)=b$ ，因此 $\varphi(y,z)=by+\psi(z)$ ，进而有 $\psi'(z)=c$ ，导出 $\psi(z)=cz+d$. 依

次代入得 $u=ax+by+cz+d$.

44. 解：

（1）

$$x=\frac{a(1-\cos 2t)}{2},y=\frac{b\sin 2t}{2},z=\frac{c(1+\cos 2t)}{2}$$
$$(dx,dy,dz)=(a\sin 2t,b\cos 2t,-c\sin 2t)dt$$
$$\Rightarrow \boldsymbol{n}=(a\sin 2t,b\cos 2t,-c\sin 2t)$$

令$t=\dfrac{\pi}{4}$,则$\boldsymbol{n}=(a,0,-c)$,点$(\dfrac{a}{2},\dfrac{b}{2},\dfrac{c}{2})$

切线为:$\dfrac{x-\frac{a}{2}}{a}=\dfrac{y-\frac{b}{2}}{0}=\dfrac{z-\frac{c}{2}}{-c}$,法平面为：$a(x-\dfrac{a}{2})-c(z-\dfrac{c}{2})=0$

（2）

$$\begin{cases} x^2+z^z=10 \\ y^2+z^2=10 \end{cases} \xrightarrow{\text{两边同时取微分}} \begin{cases} 2xdx+2zdz=0 \\ 2ydy+2zdz=0 \end{cases} \xrightarrow{\text{令（}x,y,z)=(1,1,3\text{）}} \begin{cases} dx+3dz=0 \\ dy+3dz=0 \end{cases}$$

取$dz=-1$，则$dx=dy=3 \Rightarrow \boldsymbol{n}=(3,3,-1)$

$\Rightarrow$ 切线: $\dfrac{x-1}{3}=\dfrac{y-1}{3}=\dfrac{z-3}{-1}$

法平面: $3(x-1)+3(y-1)-(z-3)=0 \Rightarrow 3x+3y-z-3$

（3）

$$\begin{cases} x^2+y^2+z^2-3x=0 \\ 2x-3y+5z-4=0 \end{cases} \xrightarrow{\text{两边同时取微分}} \begin{cases} 2xdx+2ydy+2zdz-3dx=0 \\ 2dx-3dy+5dz=0 \end{cases}$$

$$\xrightarrow{\text{令}x=y=z=1} \begin{cases} -dx+2dy+2dz=0 \\ 2dx-3dy+5dz=0 \end{cases}$$

$\boldsymbol{n}=(-1,2,2)\times(2,-3,5)=(16,9,-1)$

切线: $\dfrac{x-1}{16}=\dfrac{y-1}{9}=\dfrac{z-1}{-1}$

法平面: $16(x-1)+9(y-1)-(z-1)=0 \Rightarrow 16x+9y-z-24=0$

45. 证明：

曲线上参数为t点处，其切向量$\vec{\tau}=\mathrm{e}^t\big((\cos t-\sin t),(\sin t+\cos t),1\big)$，圆锥面上过该点的母线方向向量可取为$\vec{s}=\mathrm{e}^t(\cos t,\sin t,1)$，于是

$$\cos(\widehat{\vec{s},\vec{\tau}})=\frac{\vec{s}\cdot\vec{\tau}}{|\vec{s}||\vec{\tau}|}=\sqrt{\frac{2}{3}}.$$

46.解：

（1）

$$F(x,y,z)=\arctan\frac{y}{x}-z,$$

$$\boldsymbol{n}=(F_x,F_y,F_z)=(\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}},\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}},-1)=(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2},-1)$$

令$x=y=1,z=\frac{\pi}{4},\boldsymbol{n}=(-\frac{1}{2},\frac{1}{2},-1)$

切平面$:(-\frac{1}{2})(x-1)+\frac{1}{2}(y-1)-(z-\frac{\pi}{4})=0\Rightarrow x-y+2z-\frac{\pi}{2}=0$

法线$:\frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-\frac{\pi}{4}}{2}$

（2）

$$F(x,y,z)=ax^2+by^2+cz^2-1=0$$

$\boldsymbol{n}=(ax,by,cz),$令$x=x_0,y=y_0,z=z_0\Rightarrow\boldsymbol{n}=(ax_0,by_0,cz_0)$

切平面$:ax_0(x-x_0)+by_0(y-y_0)+cz_0(z-z_0)=0$

$\Rightarrow ax_0x+by_0y+cz_0z=1$

法线$:\frac{x-x_0}{ax_0}=\frac{y-y_0}{by_0}=\frac{z-z_0}{cz_0}$

（3）

$$F(x,y,z)=e^{\frac{x}{z}}+e^{\frac{y}{z}}-4$$

$\boldsymbol{n}=(\frac{e^{\frac{x}{z}}}{z},\frac{e^{\frac{y}{z}}}{z},-\frac{xe^{\frac{x}{z}}+ye^{\frac{y}{z}}}{z^2})$，令$x=y=\ln 2,z=1\Rightarrow\boldsymbol{n}=(2,2,-4\ln 2)$

切平面$:2(x-\ln 2)+2(y-\ln 2)-(4\ln 2)(z-1)=0$

$\Rightarrow x+y-(2\ln 2)z=0,$

法线$:\frac{x-\ln 2}{1}=\frac{y-\ln 2}{1}=\frac{z-1}{-2\ln 2}$

47. 解：

检验知道： $4x+y-z-3=0$ 和 $x+y-z=0$ 均不是切平面

设切点为 (x_0,y_0,z_0) ， 曲面在该处的法向量为 $(3x_0,y_0,-z_0)$. 作平面束

$$(\lambda+4)x+(\lambda+1)y-(\lambda+1)z=3,$$

依题意有

$$\begin{cases}\dfrac{\lambda+4}{3x_0}=\dfrac{\lambda+1}{y_0}=\dfrac{-(\lambda+1)}{-z_0}\\ 3x_0^2+y_0^2-z_0^2=3\\ (\lambda+4)x_0+(\lambda+1)y_0-(\lambda+1)z_0=3\end{cases}\Rightarrow y_0=z_0,\ x_0=\pm1,\ \lambda=-1\text{或}-7.$$

故所求切平面为 $x=1$ 或 $x+2y-2z+1=0$.

48. 解：

（1）

先分别求两个曲面的切平面的法向量，法向量夹角即为平面夹角:

对于球面 $:x^2+y^2+z^2=14,\boldsymbol{n}_1=(-1,-2,3)$

对于椭球面 $:3x^2+y^2+z^2=16,\boldsymbol{n}_2=(-3,-2,3)$

$$\theta=\arccos\frac{|\boldsymbol{n}_1\cdot\boldsymbol{n}_2|}{|\boldsymbol{n}_1||\boldsymbol{n}_2|}=\arccos\frac{3+4+9}{\sqrt{1+4+9}\times\sqrt{9+4+9}}=\arccos\frac{8}{\sqrt{77}}$$

（2）

证明：对于 $x^2+y^2+z^2=ax,\boldsymbol{n}_1=(2x_0-a,2y_0,2z_0)$

对于 $x^2+y^2+z^2=by,\boldsymbol{n}_2=(2x_0,2y_0-b,2z_0)$

$$\boldsymbol{n}_1\cdot\boldsymbol{n}_2=4x_0^2-2ax_0+4y_0^2-2by_0+4z_0^2=2(x_0^2+y_0^2+z_0^2-ax_0)+2(x_0^2+y_0^2+z_0^2-by_0)=0$$

故曲面 $x^2+y^2+z^2=ax$ 与 $x^2+y^2+z^2=by$ 相互正交

49.证明：

（1）曲面在点(x_0,y_0,z_0)处的切平面方程为

$$y_0z_0(x-x_0)+x_0z_0(y-y_0)+x_0y_0(z-z_0)=0,$$

即$\dfrac{x}{\frac{3a^3}{y_0z_0}}+\dfrac{y}{\frac{3a^3}{x_0z_0}}+\dfrac{z}{\frac{3a^3}{x_0y_0}}=1$，其与坐标轴围成的四面体体积$V=\dfrac{1}{3}\cdot\dfrac{1}{2}\cdot\dfrac{27a^9}{|x_0y_0z_0|^2}=\dfrac{9a^3}{2}$.

（2）曲面在点(x_0,y_0,z_0)处的切平面方程为

$$x_0^{-\frac{1}{3}}(x-x_0)+y_0^{-\frac{1}{3}}(y-y_0)+z_0^{-\frac{1}{3}}(z-z_0)=0,$$

即$\dfrac{x}{\sqrt[3]{x_0}}+\dfrac{y}{\sqrt[3]{y_0}}+\dfrac{z}{\sqrt[3]{z_0}}=a^{\frac{2}{3}}$，它在各坐标轴上的截距平方和为

$$\left(a^{\frac{2}{3}}x_0^{\frac{1}{3}}\right)^2+\left(a^{\frac{2}{3}}y_0^{\frac{1}{3}}\right)^2+\left(a^{\frac{2}{3}}z_0^{\frac{1}{3}}\right)^2=a^2.$$

（3）曲面在点(x_0,y_0,z_0)处的切平面方程为

$$\left[f\left(\frac{y_0}{x_0}\right)-\frac{y_0}{x_0}f'\left(\frac{y_0}{x_0}\right)\right](x-x_0)+f'\left(\frac{y_0}{x_0}\right)(y-y_0)-(z-z_0)=0,$$

即$\left[f\left(\dfrac{y_0}{x_0}\right)-\dfrac{y_0}{x_0}f'\left(\dfrac{y_0}{x_0}\right)\right]x+f'\left(\dfrac{y_0}{x_0}\right)y-z=0$，该平面总过原点.

50.解：

（1）

$f(x,y)=-(x-2)^2-(y+2)^2+8,\ f(2,-2)$为极大值（也是最大值）

（2）

令$f_x=2x+y+1=0,\ f_y=x+2y-1=0$

$\Rightarrow x=-1,y=1$

令$A=f_{xx}\big|_{(-1,1)}=2,B=f_{xy}\big|_{(-1,1)}=1,C=f_{yy}\big|_{(-1,1)}=2$

$H=AC-B^2=3>0,A>0\ \therefore\ f(-1,1)=0$为极小值

另解：$f(x,y)=\dfrac{1}{2}\left[(x+y)^2+(x+1)^2+(y-1)^2\right]$显然$f(-1,1)=0$为极小值

（3）

$f(x,y)=x^6-(x^4+x^2)y+y^2$
令$f_x=6x^5-(4x^3+2x)y=0,\ f_y=-(x^4+x^2)+2y=0$
$\Rightarrow x=\pm\frac{\sqrt{2}}{2},y=\frac{3}{8}$,由于$g(x)=f$为偶函数，所以只需要考虑$x=\frac{\sqrt{2}}{2}$即可
令$A=f_{xx}\big|_{(\frac{\sqrt{2}}{2},\frac{3}{8})}=\frac{9}{2}$，$B=f_{xy}\big|_{(\frac{\sqrt{2}}{2},\frac{3}{8})}=-2\sqrt{2},C=f_{yy}\big|_{(\frac{\sqrt{2}}{2},\frac{3}{8})}=2$
$H=AC-B^2=1>0,A>0,\ f(\pm\frac{\sqrt{2}}{2},\frac{3}{8})=-\frac{1}{64}$为极小值
Tip:多出的解$(\pm1,1),(0,0)$无法判断或非极值点，已舍去。

（4）

令$f_x=y-\frac{50}{x^2}=0$，$f_y=x-\frac{20}{y^2}=0$
$\Rightarrow x=5,y=2$
令$A=f_{xx}\big|_{(5,2)}=\frac{4}{5},B=f_{xy}\big|_{(5,2)}=1,C=f_{yy}\big|_{(5,2)}=5$
$H=AC-B^2=3>0,A>0,\ f(5,2)=30$为极小值

51. 解：

（1）

令$f_x=2x=0,\ f_y=-2y=0$
$\Rightarrow x=y=0$
令$A=f_{xx}\big|_{(0,0)}=2,B=f_{xy}\big|_{(0,0)}=0,C=f_{yy}\big|_{(0,0)}=-2$
$H=AC-B^2<0\Rightarrow$最值点在边界取
令$x=\cos\theta,y=2\sin\theta,\theta\in[0,2\pi)$
$f(x,y)=\cos^2\theta-4\sin^2\theta+2=3-5\sin^2\theta\in[-2,3]$
$\therefore$最小值为-2,最大值为3

（2）

令$f_x=\cos x-\cos(x+y)=f_y=\cos y-\cos(x+y)=0$

$\begin{cases}\cos x=\cos(x+y)\\ \cos y=\cos(x+y)\end{cases}$在$x\ge 0,y\ge 0,x+y\le 2\pi$中解该方程组

$\because \cos x=\cos(x+y)\Rightarrow x+(x+y)=2\pi\ or\ x=y=0\ or\ y=2\pi$

$\because \cos y=\cos(x+y)\Rightarrow y+(x+y)=2\pi\ or\ x=y=0\ or\ x=2\pi$

$\Rightarrow (x,y)=(0,0),(0,2\pi),(2\pi,0),(\frac{2\pi}{3},\frac{2\pi}{3})$,忽略掉边界点，则仅有$(\frac{2\pi}{3},\frac{2\pi}{3})$

$A=f_{xx}=-\sin x+\sin(x+y),B=f_{xy}=\sin(x+y),C=f_{yy}=-\sin y+\sin(x+y)$

$$\begin{aligned}H&=AC-B^2\\&=[-\sin x+\sin(x+y)][-\sin y+\sin(x+y)]-\sin^2(x+y)\\&=\sin x\sin y-\sin(x+y)[\sin x+\sin y]\end{aligned}$$

代入$x=y=\frac{2\pi}{3}$，$H=\frac{9}{4}>0,A=-\sqrt{3},\Rightarrow f(\frac{2\pi}{3},\frac{2\pi}{3})=\frac{3\sqrt{3}}{2}$为最大值

$\therefore$最小值在边界取$x=0,y=0,x+y=2\pi$,发现$f\equiv 0$;

$\therefore f|_{\partial D}=0$为最小值

52. 证明：

设三角形三边a,b,c，所对角为A,B,C，圆半径为R

由正弦定理知：$\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=2R$

$$
\begin{aligned}
S&=\frac{1}{2}ab\sin C\\
&=2R^2\sin A\sin B\sin C\\
&\le 2R^2\left(\frac{\sin A+\sin B+\sin C}{3}\right)^3 \cdots\cdots \text{利用了三元基本不等式}\\
&\le 2R^2\left(\frac{3\sin\frac{A+B+C}{3}}{3}\right)^3 \cdots\cdots \text{利用了Jensen不等式}\\
&=2R^2(\sin\frac{\pi}{3})^3=\frac{3\sqrt{3}}{4}R^2
\end{aligned}
$$

当且仅当$\sin A=\sin B=\sin C, A=B=C, A+B+C=\pi$

即$A=B=C=\dfrac{\pi}{3}$时取等号

等号能取到，且此时三角形恰好为正三角形。

综上，命题得证

53. 解：

设点的坐标$(x,y,0)$,距离$d=d(x,y)$

则$d(x,y)=y^2+x^2+\dfrac{(2x+y-16)^2}{2^2+1^2}$

令$d_x=d_y=0\Rightarrow(x,y)=(\dfrac{16}{5},\dfrac{8}{5})$

Ps:由图像容易知道，当点$(x,y,0)$不在三条直线所围成区域时，$d(x,y)$会越来越大

所以只需要在三角直线所围成的区域考虑即可。

54. 解：

（1）

两边同时对x求导: $2x+2y+4yy'=0$

再对两边求导得: $2+2y'+4(y')^2+4yy''=0$

$$\begin{cases} x^2+2xy+2y^2=1 \\ 2x+2y+4yy'=0 \\ 2+2y'+4(y')^2+4yy''=0 \end{cases}$$
中，令$y'=0$，解方程组有: $(x,y,y'')=(-1,1,-\frac{1}{2}),(1,-1,\frac{1}{2})$

$\therefore y(-1)=1$为极大值，$y(1)=-1$为极小值

（2）

两边分别对x,y求偏导:

$4x+2zz_x+8z+8xz_x-z_x=0\cdots①$　　$4y+2zz_y+8xz_y-z_y=0\cdots②$

再对①式两边对x,y分别求偏导, ②式两边对y求偏导

$4+2(z_x)^2+2zz_{xx}+8z_x+8z_x+8xz_{xx}-z_{xx}=0\cdots③$

$2z_yz_x+2zz_{xy}+8z_y+8xz_{xy}-z_{xy}=0\cdots④$

$4+2(z_y)^2+2zz_{yy}+8xz_{yy}-z_{yy}=0\cdots⑤$

令$A=z_{xx},B=z_{xy},C=z_{yy},H=AC-B^2$

令$z_x=z_y=0$, 并让①～⑤式与$2x^2+2y^2+z^2+8xz-z+8=0$联立可得:

$$(x,y,z,H,A)=(\frac{16}{7},0,-\frac{8}{7},\frac{16}{225},-\frac{4}{15}),(-2,0,1,\frac{16}{225},\frac{4}{15})$$

$\therefore z(\frac{16}{7},0)=-\frac{8}{7}$为极大值，$z(-2,0)=1$为极小值，

55. 解：

（1）

构造$Lagrange$函数：$L(x,y,\lambda)=xy+\lambda(x+y-1)$

$$\Rightarrow \begin{cases} L_x=y+\lambda=0 \\ L_y=x+\lambda=0 \\ L_\lambda=x+y-1=0 \end{cases} \Rightarrow (x,y,\lambda)=(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$$

$\Rightarrow z(\frac{1}{2},\frac{1}{2})=\frac{1}{4}$为极大值

另解：$z=xy=x(1-x)=-x^2+x=-(x-\frac{1}{2})^2+\frac{1}{4}$

（2）

构造$Lagrange$函数：$L(x,y,\lambda)=x^2+y^2+\lambda(\frac{x}{a}+\frac{y}{b}-1)$

$$\Rightarrow\begin{cases}L_x=2x+\dfrac{\lambda}{a}=0\\ L_y=2y+\dfrac{\lambda}{b}=0\\ L_\lambda=\dfrac{x}{a}+\dfrac{y}{b}-1=0\end{cases}\Rightarrow(x,y,\lambda)=(\frac{ab^2}{a^2+b^2},\frac{a^2b}{a^2+b^2},-\frac{2a^2b^2}{a^2+b^2})$$

$\Rightarrow z(\frac{ab^2}{a^2+b^2},\frac{a^2b}{a^2+b^2})=\frac{a^2b^2}{a^2+b^2}$为极小值

另解:z等价于直线上一点到原点距离的平方，显然存在极小值，无极大值

且极小值为原点到直线的距离的平方$d^2=\left(\dfrac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}\right)^2=\dfrac{a^2b^2}{a^2+b^2}$

（3）

构造$Lagrange$函数：$L(x,y,z,\lambda)=x-2y+2z+\lambda(x^2+y^2+z^2-1)$

$$\Rightarrow\begin{cases}L_x=1+2\lambda x=0\\ L_y=-2+2\lambda y=0\\ L_z=2+2\lambda z=0\\ L_\lambda=x^2+y^2+z^2-1=0\end{cases}\Rightarrow(x,y,z,\lambda)=(\frac{1}{3},-\frac{2}{3},\frac{2}{3},-\frac{3}{2}),(-\frac{1}{3},\frac{2}{3},-\frac{2}{3},\frac{3}{2})$$

$\Rightarrow z(\frac{1}{3},-\frac{2}{3},\frac{2}{3})=3$为极大值，$z(-\frac{1}{3},\frac{2}{3},-\frac{2}{3})=-3$为极小值

另解：构造向量$\boldsymbol{n}_1=(1,-2,2),\boldsymbol{n}_2=(x,y,z)$，其中$x^2+y^2+z^2=1$

$z=\boldsymbol{n}_1\cdot\boldsymbol{n}_2\Rightarrow|z|\le|\boldsymbol{n}_1|\cdot|\boldsymbol{n}_2|=3\Rightarrow 3$为极大值$,-3$为极小值(由几何意义显然可得）

56. 解：

距离函数$d=\sqrt{x^2+y^2+z^2} \Rightarrow d^2=x^2+y^2+z^2$

构造Lagrange函数：$L(x,y,z,\lambda,\mu)=x^2+y^2+z^2+\lambda(x^2+y^2-z)+\mu(x+y+z-1)$

$$\Rightarrow \begin{cases} L_x=2x+2\lambda x+\mu=0 \\ L_y=2y+2\lambda y+\mu=0 \\ L_z=2z-\lambda+\mu=0 \\ L_\lambda=x^2+y^2-z=0 \\ L_\mu=x+y+z-1=0 \end{cases} \Rightarrow (x,y,z)=(\frac{-1+\sqrt{3}}{2},\frac{-1+\sqrt{3}}{2}.2-\sqrt{3}),(\frac{-1-\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2},2+\sqrt{3})$$

$$\Rightarrow \begin{cases} d_{\max}=d(\frac{-1-\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2},2+\sqrt{3})=\sqrt{9+5\sqrt{3}} \\ d_{\min}=d(\frac{-1-\sqrt{3}}{2},\frac{-1-\sqrt{3}}{2},2+\sqrt{3})=\sqrt{9-5\sqrt{3}} \end{cases}$$

57.解：

（1）

设点坐标为(x,y,z)

$$\begin{aligned} f(x,y,z)&=(x-1)^2+(y-1)^2+(z-1)^2+(x-2)^2+(y-3)^2+(z-4)^2 \\ &=2x^2+2y^2+2z^2-6x-8y-10z+32 \end{aligned}$$

构造Lagrange函数：$L(x,y,z,\lambda)=2x^2+2y^2+2z^2-6x-8y-10z+32+\lambda(3x-2z)$

$$\Rightarrow \begin{cases} L_x=4x-6+3\lambda=0 \\ L_y=4y-8=0 \\ L_z=4z-10-2\lambda=0 \\ L_\lambda=3x-2z=0 \end{cases} \Rightarrow (x,y,z)=(\frac{21}{13},2,\frac{63}{26})$$

（2）

设点坐标为$(u,v,\omega),u,v,\omega>0$

$F(x,y,z)=x^2+y^2+z-2,\boldsymbol{n}=(F_x,F_y,F_z)=(2x,2y,1)\Rightarrow \boldsymbol{n}=(2u,2v,1)$

切平面:$2u(x-u)+2v(y-v)+(z-\omega)=0$

$\Rightarrow 2ux+2vy+z=2u^2+2v^2+\omega \Rightarrow 2ux+2vy+z=4-\omega$

$$\Rightarrow \frac{x}{{4-\omega}/{2u}}+\frac{y}{{4-\omega}/{2v}}+\frac{z}{4-\omega}=1$$

$$\Rightarrow V=\frac{1}{6}\times\frac{(4-\omega)^3}{4uv}=\frac{(4-\omega)^3}{24uv}$$

构造$Lagrange$函数：L（$u,v,\omega,\lambda)=\frac{(4-\omega)^3}{24uv}+\lambda(u^2+v^2+\omega-2)$

$$\Rightarrow\begin{cases}L_u=-\frac{(4-\omega)^3}{24u^2v}+2\lambda u=0\\ L_v=-\frac{(4-\omega)^3}{24uv^2}+2\lambda v=0\\ L_\omega=-\frac{(4-\omega)^2}{8uv}+\lambda=0\\ L_\lambda=u^2+v^2+\omega-2=0\end{cases}\Rightarrow (u,v,\omega)=(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2},1)$$

（3）

构造$Lagrange$函数：$L(x,y,z,\lambda)=xyz^3+\lambda(x^2+y^2+z^2-5R^2)$

$$\Rightarrow\begin{cases}L_x=yz^3+2\lambda x=0\\ L_y=xz^3+2\lambda y=0\\ L_z=3xyz^2+2\lambda z=0\\ L_\lambda=x^2+y^2+z^2-5R^2=0\end{cases}\quad\Rightarrow (x,y,z)=(R,R,\sqrt{3}R)$$

另解：

$$\because x^2+y^2+z^2=x^2+y^2+\frac{z^2}{3}+\frac{z^2}{3}+\frac{z^2}{3}\geq 5\sqrt[5]{\frac{x^2y^2z^6}{27}}$$

$$\therefore 5R^2\geq 5\sqrt[5]{\frac{x^2y^2z^6}{27}}\Rightarrow xyz^3\leq 3\sqrt{3}R^5$$

当且仅当$x^2=y^2=\frac{z^2}{3},x^2+y^2+z^2=5R^2$,即$(x,y,z)=(R,R,\sqrt{3}R)$时取等号

58.解：

（1）

设抛物线上一点P（x,y)，则P到直线$x-y-2=0$的距离为：

$$d=\frac{|x-y-2|}{\sqrt{1^2+(-1)^2}}=\frac{|x-x^2-2|}{\sqrt{2}}=\frac{\left|(x-\frac{1}{2})^2+\frac{7}{4}\right|}{\sqrt{2}}\geq\frac{\frac{7}{4}}{\sqrt{2}}=\frac{7\sqrt{2}}{8}$$

$\therefore$最短距离为$\frac{7\sqrt{2}}{8}$,此时$x=\frac{1}{2},y=\frac{1}{4}$

（2）

设曲面上一点$P(x,y,z)$，则P到平面$x+y-4z=1$的距离为：

$$d=\frac{|x+y-4z-1|}{\sqrt{1^2+1^2+(-4)^2}}=\frac{|x+y-(3x^2-2xy+3y^2)-1|}{\sqrt{18}}=\frac{\left|(x-y)^2+2(x-\frac{1}{4})^2+2(y-\frac{1}{4})^2+\frac{3}{4}\right|}{\sqrt{18}}\geq\frac{\sqrt{2}}{8}$$

$\therefore$最短距离为$\frac{\sqrt{2}}{8}$,此时$x=y=\frac{1}{4},z=\frac{1}{16}$

Ps:也可构造二元函数$f(x,y)=3x^2-2xy+3y^2-x-y+1$，通过$f_x=f_y=0$，进而求最值

59.解：

设长宽高分别为x,y,z(单位：m)。则：$xyz=V$

用料为$S=xy+2yz+2zx=xy+\frac{2V}{x}+\frac{2V}{y}\geq3\sqrt[3]{xy\cdot\frac{2V}{x}\cdot\frac{2V}{y}}=3\sqrt[3]{4V^2}$

当且仅当$xy=\frac{2V}{x}=\frac{2V}{y}$时即$x=y=\sqrt[3]{2V},z=\frac{\sqrt[3]{2V}}{2}$时取最小值。

$\therefore$当水箱的长宽高分别为$\sqrt[3]{2V}$米、$\sqrt[3]{2V}$米、$\frac{\sqrt[3]{2V}}{2}$米时，用料最少

60. 解：

由已知：$\alpha+\beta=1$,产出量$Q=2x_1^{\alpha}x_2^{\beta}=12(\alpha,\beta>0)$,成本$C=p_1x_1+p_2x_2$

构造$Lagrange$函数：L（$x_1,x_2,\lambda)=p_1x_1+p_2x_2+\lambda(2x_1^{\alpha}x_2^{\beta}-12)$

$$\Rightarrow\begin{cases}L_{x_1}=p_1+2\alpha\lambda x_1^{\alpha-1}x_2^{\beta}=0\\ L_{x_2}=p_2+2\beta\lambda x_1^{\alpha}x_2^{\beta-1}=0\\ L_{\lambda}=2x_1^{\alpha}x_2^{\beta}-12=0\end{cases}\Rightarrow x_1=6\left(\frac{p_2\alpha}{p_1\beta}\right)^{\beta},x_2=6\left(\frac{p_1\beta}{p_2\alpha}\right)^{\alpha}$$

Tip:取对数，再解方程组$(\lambda<0)$

补充题：

1.解

不一定. 所给例子中$\lim\limits_{\substack{x\to 0\\ y=kx}} f(x,y)=\lim\limits_{x\to 0}\dfrac{x^2\cdot kx}{x^4+(kx)^2}=0$，但是

$$\lim_{\substack{x\to 0\\ y=x^2}} f(x,y)=\lim_{x\to 0}\frac{x^2\cdot x^2}{x^4+(x^2)^2}=\frac{1}{2}.$$

因此$\lim\limits_{\substack{x\to x_0\\ y\to y_0}} f(x,y)$不存在.

2.解

设$\overrightarrow{l^0}=(\cos\alpha,\cos\beta)$是任一方向. 当$\cos\beta\neq 0$时，

$$\lim_{t\to 0}\frac{f(t\cos\alpha,t\cos\beta)-f(0,0)}{t}=\lim_{t\to 0}\frac{t^4\cos^5\alpha}{(t\cos\beta-t^2\cos^2\alpha)^2+t^6\cos^6\alpha}$$

$$=\lim_{t\to 0}\frac{t^2\cos^5\alpha}{(\cos\beta-t\cos^2\alpha)^2+t^4\cos^6\alpha}=0$$

.

当$\cos\beta=0$时，则$\cos\alpha=1$，此时

$$\lim_{t\to 0}\frac{f(t,0)-f(0,0)}{t}=\lim_{t\to 0}\frac{t^4}{(-t^2)^2+t^6}=1.$$

故$f(x,y)$在点(0,0)沿任意方向的方向导数都存在.

另一方面，由于当$x\neq 0$时，$f(x,x^2)=\dfrac{1}{x}$，从而$f(x,y)$在点(0,0)的任意邻域内无界，因此在点(0,0)处不连续.

3.

证 首先 $$f_x(0,0)=\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x}=\lim_{x\to 0}\frac{x^2\sin\frac{1}{x^2}}{x}=0,$$

类似有 $f_y(0,0)=0$. 当 $x^2+y^2\neq 0$ 时,

$$f_x(x,y)=2x\sin\frac{1}{x^2+y^2}-\frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2},$$

$$f_y(x,y)=2y\sin\frac{1}{x^2+y^2}-\frac{2y}{x^2+y^2}\cos\frac{1}{x^2+y^2},$$

故在点 $(0,0)$ 的邻域内 $f_x(x,y)$, $f_y(x,y)$ 存在.

令 $x_n=\dfrac{1}{\sqrt{2n\pi}}, y_n=0$，则 $f_x(x_n,y_n)=-2\sqrt{2n\pi}$，故 $f_x(x,y)$ 在 $(0,0)$ 点的任何邻域内无界，从而在 $(0,0)$ 点不连续, $f_x(x,y)$ 的情形完全类似.

另一方面，在 $(0,0)$ 点有

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\Delta f-(0\cdot\Delta x+0\cdot\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}}=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\sqrt{\Delta x^2+\Delta y^2}\sin\frac{1}{\Delta x^2+\Delta y^2}=0,$$

因此 $f(x,y)$ 在 $(0,0)$ 点可微，且 $\mathrm{d}f\big|_{(0,0)}=0$.

4.

解 由于

$$z(x,y)=\int_0^1 f(t)|xy-t|\mathrm{d}t=\int_0^{xy} f(t)(xy-t)\mathrm{d}t+\int_{xy}^1 f(t)(t-xy)\mathrm{d}t$$

$$=xy\int_0^{xy} f(t)\mathrm{d}t-\int_0^{xy} tf(t)\mathrm{d}t+\int_{xy}^1 tf(t)\mathrm{d}t-xy\int_{xy}^1 f(t)\mathrm{d}t$$

因此 $\dfrac{\partial z}{\partial x}=y\left(\int_0^{xy} f(t)\mathrm{d}t-\int_{xy}^1 f(t)\mathrm{d}t\right)$，从而

$$\frac{\partial^2 z}{\partial x^2}=y\left(yf(xy)+yf(xy)\right)=2y^2 f(xy).$$

5.

解

$$\frac{\partial f}{\partial l^0}=(\cos\alpha,\sin\alpha)\cdot(1+y,2+x)=(\cos\alpha,\sin\alpha)\cdot(1+\cos\beta,2+\sin\beta)$$
$$\le 1\cdot\sqrt{6+2\cos\beta+4\sin\beta}$$
$$\le\sqrt{6+\sqrt{2^2+4^2}}$$
$$=\sqrt{5}+1$$

6.

解 记 $J=\begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix}$. 对方程组取全微分得

$$\begin{cases} f_u\mathrm{d}u+f_v\mathrm{d}v=\mathrm{d}x \\ g_u\mathrm{d}u+g_v\mathrm{d}v=\mathrm{d}y \\ \mathrm{d}z=h_u\mathrm{d}u+h_v\mathrm{d}v \end{cases},$$

由前两式得

$$\mathrm{d}u=\frac{g_v\mathrm{d}x-f_v\mathrm{d}y}{J},\ \mathrm{d}v=\frac{-g_u\mathrm{d}x+f_u\mathrm{d}y}{J},$$

代入最后一式得 $$\mathrm{d}z=\frac{g_vh_u-g_uh_v}{J}\mathrm{d}x+\frac{f_uh_v-f_vh_u}{J}\mathrm{d}x.$$

从而 $$\frac{\partial z}{\partial x}=-\frac{\frac{\partial(g,h)}{\partial(u,v)}}{J},\quad \frac{\partial z}{\partial y}=-\frac{\frac{\partial(h,f)}{\partial(u,v)}}{J}.$$

7.

解 对方程组取全微分得

$$\begin{cases} \mathrm{d}z = f\mathrm{d}x + xf' \cdot (\mathrm{d}x + \mathrm{d}y) \\ F_x\mathrm{d}x + F_y\mathrm{d}y + F_z\mathrm{d}z = 0 \end{cases},$$

导出

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{(f + xf')F_y - xf'F_x}{F_y + xf'F_z}.$$

8.

解 曲面在点 (x_0, y_0, z_0) 处的法向量

$$\vec{n} = \left(-\left[f_u \frac{y_0 - b}{(x_0 - a)^2} + f_v \frac{z_0 - c}{(x_0 - a)^2} \right], \frac{f_u}{x_0 - a}, \frac{f_v}{x_0 - a} \right)$$

$$// \left(-\left[f_u \frac{y_0 - b}{x_0 - a} + f_v \frac{z_0 - c}{x_0 - a} \right], f_u, f_v \right).$$

切平面方程为

$$-\left[f_u \frac{y_0 - b}{x_0 - a} + f_v \frac{z_0 - c}{x_0 - a} \right](x - x_0) + f_u \cdot (y - y_0) + f_v \cdot (z - z_0) = 0,$$

该平面过定点 (a, b, c).

9.

解 曲面在$(1,-2,5)$处的法向量$\vec{n}=(2,-4,-1)$. 作平面束

$$(\lambda+1)x+(\lambda+a)y-z+\lambda b-3=0.$$

依题意有$\dfrac{\lambda+1}{2}=\dfrac{\lambda+a}{-4}=\dfrac{-1}{-1}$，从而$\lambda=1, a=-5$，故切平面为$2x-4y-z+b-3=0$.

将$(1,-2,5)$代入得$b=-2$.

10.

证 (1) 令$F(x,y,\lambda)=\ln x+3\ln y+\lambda(x^2+y^2-4r^2)$. 由

$$\begin{cases} F_x=\dfrac{1}{x}+2\lambda x=0 \\ F_y=\dfrac{3}{y}+2\lambda y=0 \\ F_\lambda=x^2+y^2-4r^2=0 \end{cases} \quad 解得\begin{cases} x=r \\ y=\sqrt{3}r \end{cases}.$$

故所求最大值为$\ln 3\sqrt{3}r^4$. 于是$\ln x+3\ln y\le \ln 3\sqrt{3}r^4$，从而

$$xy^3\le 3\sqrt{3}r^4=3\sqrt{3}\left(\frac{x^2+y^2}{4}\right)^2,$$

于是$x^2y^6\le 27\left(\dfrac{x^2+y^2}{4}\right)^4$. 令$x^2=a, y^2=b$得所证不等式.

(2) 令 $F(x,y,z,\lambda)=\ln x+2\ln y+3\ln z+\lambda(x^2+y^2+z^2-6r^2)$. 由

$$\begin{cases} F_x=\dfrac{1}{x}+2\lambda x=0 \\ F_y=\dfrac{2}{y}+2\lambda y=0 \\ F_z=\dfrac{3}{z}+2\lambda z=0 \\ F_\lambda=x^2+y^2+z^2-6r^2=0 \end{cases} \quad 解得 \begin{cases} x=r \\ y=\sqrt{2}r \\ z=\sqrt{3}r \end{cases}.$$

故所求最大值为 $\ln 6\sqrt{3}r^6$. 所以有 $\ln x+2\ln y+3\ln z\le \ln 6\sqrt{3}r^6$，从而

$$xy^2z^3\le 6\sqrt{3}r^6=6\sqrt{3}\left(\frac{x^2+y^2+z^2}{6}\right)^3 \text{导出}\ x^2y^4z^6\le 108\left(\frac{x^2+y^2+z^2}{6}\right)^6.$$

令 $x^2=a,\ y^2=b,\ z^2=c$ 得所证不等式.

11.

解 设长方体位于平面 $z=h$ 上的一个顶点为 $(r\cos\theta, r\sin\theta, h)$ $(0<r<R,\ 0<\theta<\frac{\pi}{2})$，则其高为 $h\left(1-\frac{r}{R}\right)$，体积

$$V=4r^2h\sin\theta\cos\theta\left(1-\frac{r}{R}\right)=2r^2h\sin 2\theta\left(1-\frac{r}{R}\right).$$

由

$$\begin{cases} V_\theta=4hr^2\cos 2\theta\left(1-\dfrac{r}{R}\right)=0 \\ V_r=2h\sin 2\theta\left(2r-\dfrac{3r^2}{R}\right)=0 \end{cases} 解得 \begin{cases} \theta=\dfrac{\pi}{4} \\ r=\dfrac{2R}{3} \end{cases}.$$

此时，底面是边长为 $\dfrac{2\sqrt{2}R}{3}$ 的正方形，高为 $\dfrac{h}{3}$，最大体积为 $\dfrac{8R^2h}{27}$.

12.

证 对 $x=x(u,v),\ y=y(u,v)$ 取全微分得 $\begin{cases} x_u\mathrm{d}u+x_v\mathrm{d}v=\mathrm{d}x \\ y_u\mathrm{d}u+y_v\mathrm{d}v=\mathrm{d}y \end{cases}$,

导出 $$\mathrm{d}u=\frac{y_v}{\frac{\partial(x,y)}{\partial(u,v)}}\mathrm{d}x-\frac{x_v}{\frac{\partial(x,y)}{\partial(u,v)}}\mathrm{d}y,\quad \mathrm{d}v=-\frac{y_u}{\frac{\partial(x,y)}{\partial(u,v)}}\mathrm{d}x+\frac{x_u}{\frac{\partial(x,y)}{\partial(u,v)}}\mathrm{d}y.$$

于是有 $$\frac{\partial u}{\partial x}=\frac{y_v}{\frac{\partial(x,y)}{\partial(u,v)}},\ \frac{\partial u}{\partial y}=-\frac{x_v}{\frac{\partial(x,y)}{\partial(u,v)}},\ \frac{\partial v}{\partial x}=-\frac{y_u}{\frac{\partial(x,y)}{\partial(u,v)}},\ \frac{\partial v}{\partial y}=\frac{x_u}{\frac{\partial(x,y)}{\partial(u,v)}}.$$

从而 $$\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=\frac{1}{\frac{\partial(x,y)}{\partial(u,v)}}\begin{vmatrix} y_v & -x_v \\ -y_u & x_u \end{vmatrix}=1.$$

13.

证 “必要性”：设 $u(x,y)=f(x)g(y)$，则

$$\frac{\partial u}{\partial x}=f'(x)g(y),\ \frac{\partial u}{\partial y}=f(x)g'(y),\ \frac{\partial^2 u}{\partial x\partial y}=f'(x)g'(y),$$

从而 $$u\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}.$$

“充分性”证法 1：由 $u\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}$ 有

$$\frac{u\frac{\partial^2 u}{\partial x\partial y}-\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}}{u^2}=0,\ 即\ \left(\frac{\frac{\partial u}{\partial x}}{u}\right)'_y=0,\ 从而\ \frac{\frac{\partial u}{\partial x}}{u}=\varphi(x).$$

导出 $\ln|u|=\int\varphi(x)\mathrm{d}x+C(y)$，因此

$$u=\mathrm{e}^{\int\varphi(x)\mathrm{d}x}\cdot\mathrm{e}^{C(y)}\overset{def}{=}f(x)g(y).$$

“充分性”证法 2：

$$uu_{xy}=u_xu_y\Rightarrow\frac{u_{xy}}{u_x}=\frac{u_y}{u}\Rightarrow\int\frac{u_{xy}}{u_x}dy=\int\frac{u_y}{u}dy\Rightarrow\ln|u_x|=\ln|C_1(x)u|\Rightarrow u_x=C_1(x)u$$

$$\Rightarrow\frac{u_x}{u}=C_1(x)\Rightarrow\int\frac{u_x}{u}dx=\int C_1(x)dx\Rightarrow\ln|u|=\int C_1(x)dx+C_2(y)$$

$$\Rightarrow u=e^{\int C_1(x)dx}\cdot e^{C_2(y)}\overset{def}{=\!=}f(x)g(y)$$

14.解：

$$u_{xy}(x,y)=2xy+2x$$

两边同时对y取不定积分：

$$u_x(x,y)=xy^2+2xy+C_1(x)\Rightarrow u_x(x,0)=C_1(x)=xe^x$$

$$\Rightarrow u_x(x,y)=xy^2+2xy+xe^x$$

两边同时对x取不定积分：

$$u(x,y)=\frac{x^2y^2}{2}+x^2y+(x-1)e^x+C_2(y)\Rightarrow u(0,y)=-1+C_2(y)=\cos y$$

$$\Rightarrow u(x,y)=\frac{x^2y^2}{2}+x^2y+(x-1)e^x+\cos y+1$$