A. La Rosa Lecture Notes

QUANTUM MECHANICS

CONTENT

CHAPTER-11
BOSON PARTICLES

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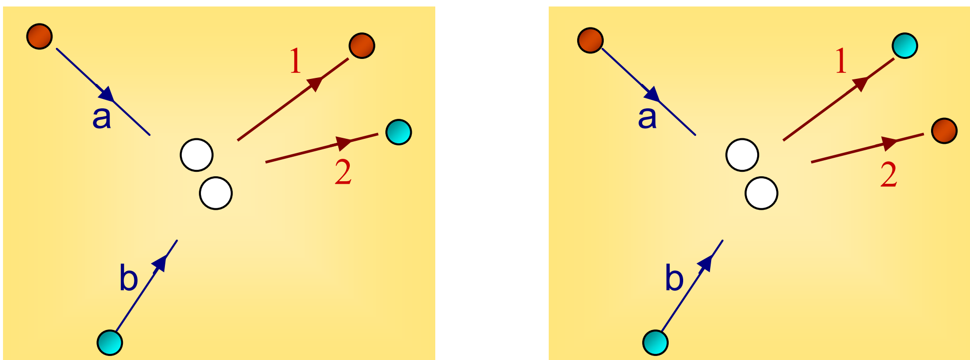
11.1 States with two particles

11.1.A Case: Two distinguishable particles

Two distinct particles are incident along the directions a and b respectively, and scattered from two different scatterers.

The finals states are defined by the outgoing directions 1 and 2. Initially, the directions 1 and 2 will be assumed to be different, but at the end we will make them become equal.



$\langle 1|a\rangle\langle 2|b\rangle$ $\qquad\qquad$ $\langle 2|a\rangle\langle 1|b\rangle$

Fig.1

If the particles were distinguishable, the final states in these two diagrams are different. So, the probability to obtain a particle in the state 1 AND the other in the state 2 is,

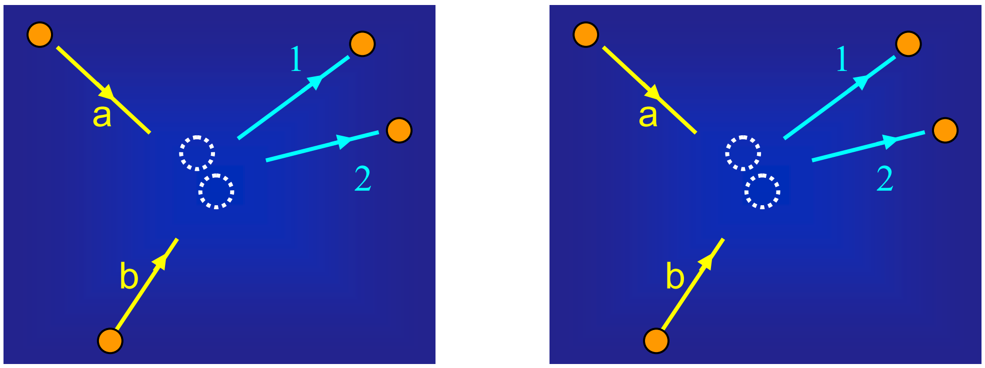
$$\left|\langle 1|a\rangle\langle 2|b\rangle\right|^2 \;+\; \left|\langle 2|a\rangle\langle 1|b\rangle\right|^2 \tag{1}$$

As we consider states 1 and 2 closer and closer to each other, we expect a smooth transition of the value $\left|\langle 1|a\rangle\right|^2$ towards the value $\left|\langle 2|a\rangle\right|^2$ (and vice versa); let's call $\left|\langle 1|a\rangle\right|^2 = \left|\langle 2|a\rangle\right|^2 \equiv |a|^2$. Similarly $\left|\langle 1|b\rangle\right|^2 = \left|\langle 2|b\rangle\right|^2 \equiv |b|^2$. Thus, the probability P_2 to pick two distinguishable particles in a given direction is,

$$P_2(distinguishables) = \ |a|^2|b|^2 \ + \ |a|^2|b|^2 \ = \ 2|a|^2|b|^2 \qquad (2)$$

11.1.B Two indistinguishable Bose particles

If we replicate the diagram drawn above but this time emphasize that the particles are identical, we obtain



$\langle 1|a\rangle\langle 2|b\rangle$ $\qquad$ $\langle 2|a\rangle\langle 1|b\rangle$

Fig.2

We notice these two possible configurations define the same final state. Accordingly, the amplitudes must be added. So the probability to get a boson scattered in the direction 1 AND the other in the direction 2 is,

$$\left| \ \langle 1|a\rangle\langle 2|b\rangle \ + \ \langle 2|a\rangle\langle 1|b\rangle \ \right|^2 \qquad (3)$$

Bye requesting the two outgoing directions to coincide ($1 \leftrightarrow 2$), we obtain $\langle 1|a\rangle = \langle 2|a\rangle \equiv a$ and $\langle 1|b\rangle = \langle 2|b\rangle \equiv b$. Accordingly the probability P_2 to pick two bosons in a given direction is,

$$P_2(bosons) = \left| \ ab \ + \ ab \ \right|^2 = \left| \ 2ab \ \right|^2$$

$$= 4\left| \ a \ \right|^2 \left| \ b \ \right|^2 \qquad (4)$$

From (2) and (4) we are led to the conclusion that, It is twice as likely to find two Bose particles scattered into the same state compared to the case of distinguishable particles

11.2 Density of probability

Making reference to a specific (narrow) scattering direction may not be convenient. No only because the Heisenberg's principle forces to use uncertainty in the physics variables that describe the state of a system, but also because, in the context of the specific experiment being described here, the detection system always have a finite entrance window. Accordingly, let's call

$\langle 1|a\rangle(dS_1)$ is the amplitude probability that the particle incident from a direction a is scattered into of cross section area dS_1 around the direction 1. (5)

Since the particles are being detected by a detector of cross section area ΔS, the probability that a particle incident in the direction a will be scattered into the detector will be,

$$\int_{\Delta S} |\langle 1|a\rangle|^2 \, dS_1 \tag{6}$$

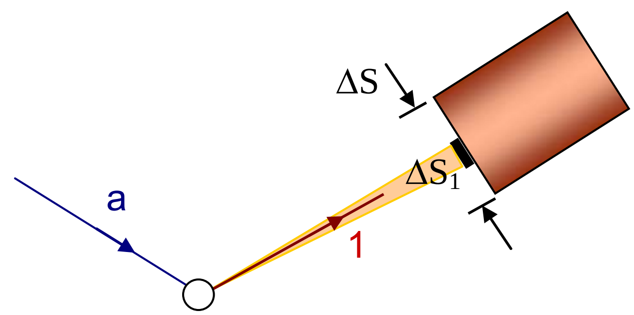


Fig.3

If the area ΔS area is small enough as to consider that the corresponding values of $|\langle 1|a\rangle|^2$ are the same for all the directions pointing into the detector, then (calling the common value $|\langle 1|a\rangle|^2 \equiv |a|^2$) we obtain,

$|a|^2 \Delta S$ probability that a particle incident from the direction a makes it into the detector.

11.2.A Case: Two distinguishable particles

The probability that the particle incident in the direction a is scattered into dS_1 AND the particle incident in the direction b into dS_2 is,

$$\left|\langle 1|a\rangle\right|^2 (dS_1) \left|\langle 2|b\rangle\right|^2 (dS_2) = |a|^2 dS_1 \, |b|^2 dS_2 \tag{7}$$

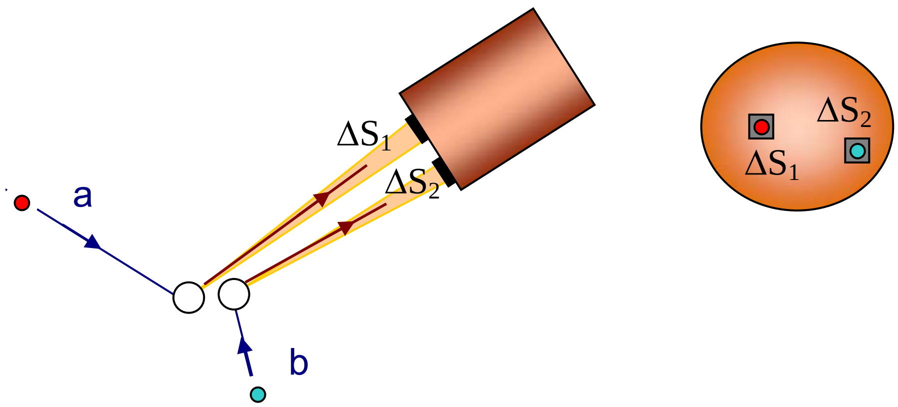


Fig. 4

The probability that the particle incident in the direction a is scattered into dS_2 AND the particle incident in the direction b into dS_1 is,

$$\left|\langle 2|a\rangle\right|^2 (dS_2) \left|\langle 1|b\rangle\right|^2 (dS_1) = |a|^2 dS_2 \, |b|^2 dS_1 \tag{8}$$

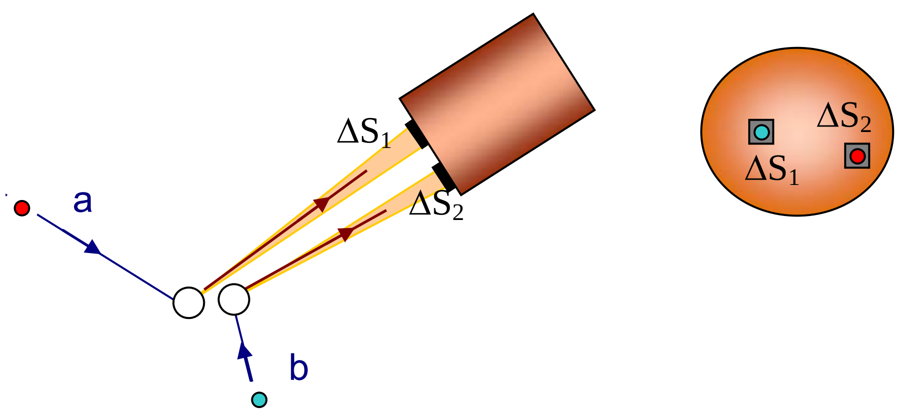


Fig. 5

(The calculation reduces to the addition of areas.)

The probability that a is scattered into dS_1 and b anywhere into the detector is,

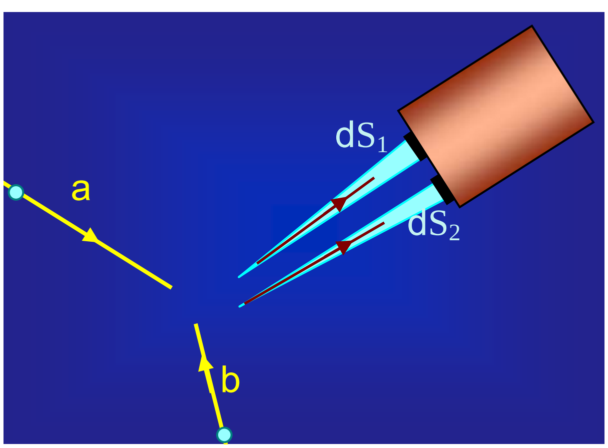
$$|a|^2 dS_1 \left[\int_{\Delta S} |b|^2 dS_2 \right] = |a|^2 dS_1 \left[\ |b|^2 \Delta S \right] \qquad (9)$$

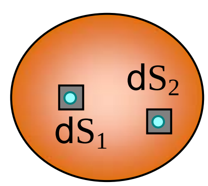
The probability that the two particles are scattered anywhere into the detector is, $\left[\int_{\Delta S} |a|^2 dS_1 \right]\left[\ |b|^2 \Delta S \right]$. Thus,

$$P_2 = \left[\ |a|^2 \Delta S \right] \left[\ |b|^2 \Delta S \right] \qquad \text{(\textit{for distinguishable particles})} \qquad (10)$$

11.2.B Case: Two indistinguishable particles

Notice in figures 4 and 5 above that if the particles were identical, both situations could not be distinguished from each other; their initial and final states are the same). Accordingly, to calculate probabilities we first have to add the amplitude probabilities and only then square the resultant quantity.





The probability that one of the incident bosons goes to dS_1 while the other goes to dS_2 is given by,

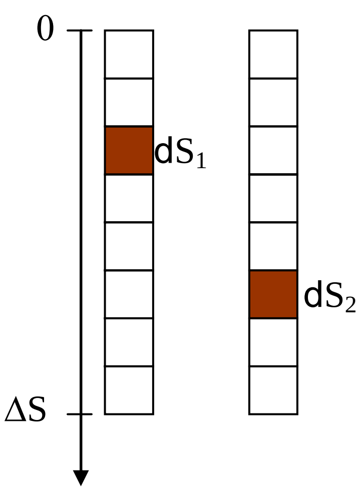
$$\left| \langle 1|a\rangle\langle 2|b\rangle + \langle 2|a\rangle\langle 1|b\rangle \right|^2 dS_1 dS_2 =$$

$$= \left| ab + ab \right|^2 dS_1 dS_2 = 4\left| ab \right|^2 dS_1 dS_2$$

The probability that the boson incident from the direction a is scattered into dS_1, while the other goes anywhere in the detector is given by,

$$4 \left| a \right|^2 dS_1 \left[\int_{\Delta S} \left| b \right|^2 dS_2 \right] = 4 \left| a \right|^2 dS_1 \left[\left| b \right|^2 \Delta S \right]$$

To calculate the probability that the two bosons make it to the detector we can integrate the variable dS_1 in the expression above, but we need to take into account that an interchange in the ($dS_1 ; dS_2$) configuration does not lead to a new final state. If we blindly integrate the variable dS_1 we would be counting each final state twice.



Thus, we can still go ahead and evaluate the integral in dS_1 from zero to ΔS but we need to divide the final expression by a factor of 2:

$$4 \left[\frac{1}{2} \int_{\Delta S} \left| a \right|^2 dS_1 \right] \left[\left| b \right|^2 \Delta S \right]$$

$$P_2 = 2 \left| a \right|^2 \left| b \right|^2 (\Delta S)^2 \quad \textit{(for indistinguishable particles)} \qquad (11)$$

Compare expressions (10) and (11). The latter is twice the value obtained in the former. That is,

> *the probability to find two indistinguishable particles in the same state is twice greater than what it would be if the particles were distinguishable.*

11.3 States with *n* particles

Lets extend the above results to n particles , which are incident along the directions a, b, c, … towards scattering centers. We want to calculate the probability that all of them are scattered into a detector of cross section are ΔS .

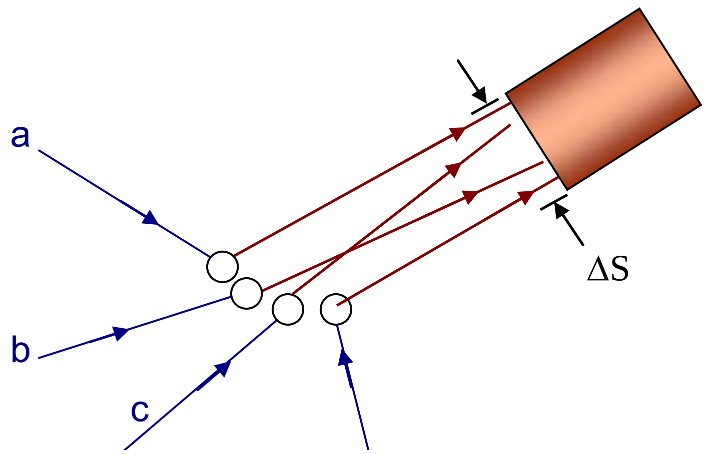
11.3.A The case of distinguishable particles

The probability that the particle incident along the direction a will be scattered into the area dS_1 , AND b into dS_2 , AND ... etc., is given by,

$$\left|\langle 1|a\rangle\right|^2 (dS_1)\left|\langle 2|b\rangle\right|^2 (dS_2)\left|\langle 3|c\rangle\right|^2 (dS_3)...$$

Assuming again that the amplitudes depend only on where the particle come from and not on where dS_j is located in the detector (i.e. ΔS is small enough for this approximation to take place,) we obtain,

$$|a|^2 dS_1\ |b|^2 dS_2 |c|^2 dS_3 ...$$

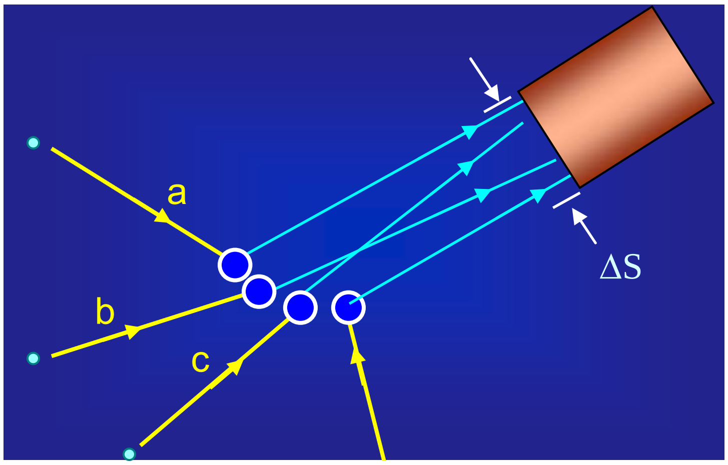


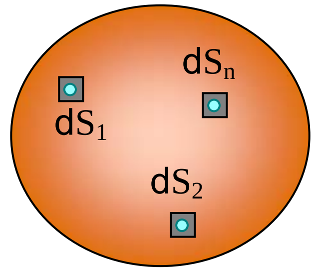
The probability that all the n particles make it to the detector is,

$P_n = \left[\int_{\Delta S} |a|^2 dS_2\right]\left[\int_{\Delta S} |b|^2 dS_2\right]\left[\int_{\Delta S} |c|^2 dS_2\right]\cdots$, which leads to,

$$P_n(distinguishable\ \ particles) = |a|^2 |b|^2 |c|^2 ...(\Delta S.)^n \tag{12}$$

11.3.B Bose particles





Upon the arrival at n sites on the detector, such final state could not distinguish which particle came from which direction, whether from a, or b, or c, etc.; therefore, there are $n!$ possible paths with the same final state. Accordingly, all these paths will interfere:

- Probability to receive one boson at dS_1, another at dS_2, another at dS_3, … etc., is given by:

$$\Big| \ \langle 1|a\rangle\langle 2|b\rangle\langle 3|c\rangle ... + \ \langle 1|a\rangle\langle 2|c\rangle\langle 3|b\rangle ... + \ \langle 1|c\rangle\langle 2|a\rangle\langle 3|b\rangle ... +$$

$$+ \langle 1|b\rangle\langle 2|a\rangle\langle 3|c\rangle ... + \langle 1|b\rangle\langle 2|c\rangle\langle 3|a\rangle ... + \langle 1|c\rangle\langle 2|b\rangle\langle 3|a\rangle ... + ... \ \Big|^2 dS_1 dS_2 dS_2 ...$$

Since the amplitudes do not depend on where the dS_j is located in the detector, we have,

$$\Big| \ abc... + \ acb... + \ cab... +$$

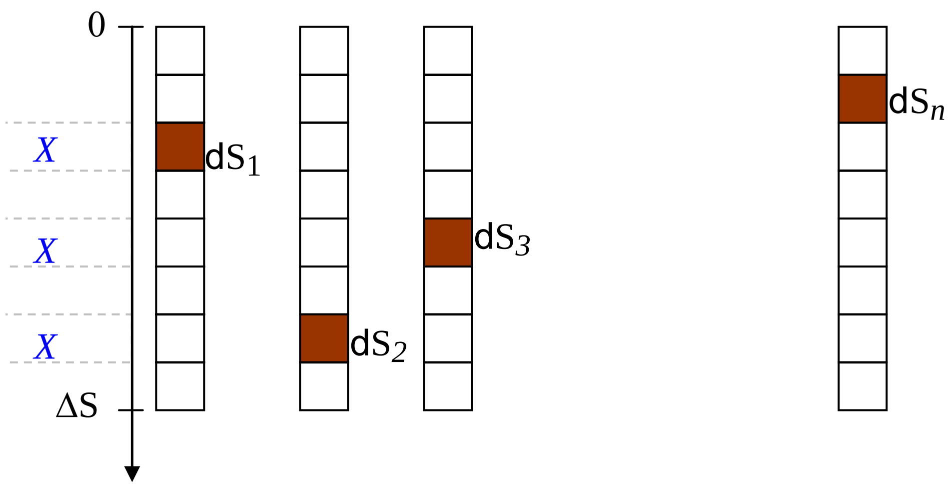
$$+ bac... + bca... + cba... + ... \ \Big|^2 dS_1 dS_2 dS_3 ...$$

There are $n!$ Identical terms inside the bars, which leads to,

$$\left| n!\ \ abc... \ \right|^2 dS_1 dS_2 dS_3 ... \quad \text{Probability that } n \text{ boson arrive at specific } n \text{ sites } dS_1, dS_2, dS_3, \ ... \text{ on the detector} \tag{13}$$

- Probability that the n bosons make it to the detector

We can integrate each variable $dS_1, dS_2, dS_3,\ ...$ from 0 to ΔS. However we must realized that in doing so a given configuration $(dS_1;\ dS_2;\ dS_3;\ ...)$ we would be counted $n!$ times redundantly. Thus, we can still go ahead and evaluate the integral in every dS_j from zero to ΔS but we need to divide the final expression by a factor of $n!$.



$$\left| n!\ \ abc... \ \right|^2 \left[\frac{1}{n!} \int_0^{\Delta S} dS_1 \int_0^{\Delta S} dS_2 \int_0^{\Delta S} dS_3 ... \right] = \left| n!\ \ abc... \ \right|^2 \left[\frac{1}{n!} (\Delta S)^n \right],$$

which reduces to,

$$P_n(Bose) = n! \left| \ abc... \ \right|^2 (\Delta S)^n \tag{14}$$

Comparing with (12),

$$P_n(Bose) = n!\,P_n(distinguishable) \tag{15}$$

The probability in the Bose case is greater by a factor of $n!$ compared to the calculation with distinguishable particles.

Comparison

If we had started with one more particle, the latter incident from a direction ω, the result would be

$$\begin{aligned} P_{n+1}(Bose) &= (n+1)!\left|abc...\omega\right|^2 \left(\Delta S\right)^{n+1} \\ &= (n+1)\left|\omega\right|^2 n!\left(\Delta S\right)\left|abc...\right|^2 \left(\Delta S\right)^n \\ &= (n+1)\left|\omega\right|^2 \left(\Delta S\right)P_n(Bose) \end{aligned} \tag{16}$$

That is, the participation of one more particle into a set of n particles, enhances the probability that *all* the particles make it to the detector in a factor $n+1$ times the probability of the particle participating alone. (It pays off to collaborate.)

In other words:

If a particle goes alone, its probability to make it to the detector is $\left|\omega\right|^2\left(\Delta S\right)$

In joining a group of n bosons, the particle (identical to the other bosons) enhances its probability to make it to the detector in a factor $(n+1)P_n(Bose)$ (17)

Appendix

How many permutations can be done with n different particles

For n=2: ab

ba

Total=2

For n=3: With each of the two (2) cases above, we insert c in alternating places

cab acb abc

That is, the extra element c gives 3 new cases for each of the (2) alternatives in the previous case

Total=3x(2)

For n=4: With each of the two (6) cases above, we insert d in alternating places

dcab cdab cadb cabd

That is, the extra element d gives 4 new cases for each of the (3x2) alternatives in the previous case

Total=4x(3)(2)

Generalizing, there are $n!$ different ways to permute n distinguishable particles.