The Friedmann Universe and Compact Internal Spaces in Higher-Dimensional Gravity Theories

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Abstract

We consider gravity theories in $4+N$ dimensions which are governed by the Lagrangian written as an extended Gauss-Bonnet density. We can find a naturally generalized Einstein gravity where the maximal symmetric compactification leads to vanishing four-dimensional cosmological constant in the static limit. A later stage in the generalized Kaluza-Klein cosmology is also examined.

In the last decade, the unification of the gauge interactions and the gravity through higher dimensions has much interest.[1] This renewed Kaluza-Klein idea is originated from the study of supergravity theories[2] and superstring theories,[3] which have a simple or unique structure in higher dimensions than four.

Most attractive feature of Kaluza-Klein theories is that the isometry group of compact extra spaces will be viewed as the gauge symmetry group in the four-dimensional effective theory.[1] Various gauge groups can be obtained as a consequence of the corresponding compactification in the Kaluza-Klein (super) gravity.[2]

Another aspect of the Kaluza-Kein model is the cosmological evolution of the scale factors. In the early universe, it is natural to expect the "dimensional reduction transition"[4] which explains the reason why the length scale of extra spaces is too small to be obervable. And also, we can ask whether the cosmological inflation associated with this phase transition takes place or not.[5] However, before such investigations, we have to explain why the four-dimensional cosmological constant is so small in our universe.[6] From the cosmological observation it follows that the cosmological constant in the universe at present cannot be greater than the critical density $\sim 10^{-120} m_{\rm pl}^4$. In usual Kaluza-Klein supergravity theories, though the higher-dimensional cosmological term is forbidden by supersymmetry, the compactification of extra spaces brings about a large four-dimensional cosmological constant $\sim m_{\rm pl}^4$. On the other hand, in non-super-theories, we can adjust the higher-dimensional cosmological constant in order to lead a vanishing four-dimensional cosmological constant. However, this fine-tuning seems to be unnatural while we have no principle to decide any fixed values for higher-dimensional cosmological constant.

There are many attempts to solve this "cosmological constant problem". Wetterich and his coworkers adopted non-compact spaces as extra spaces, and showed that no fine-tuning is needed to obtain a vanishing four-dimensional cosmological constant.[7] On the other hand, in the low energy limit of the superstring theories, it is favoured that extra six dimensions are compactified on the Calabi-Yau manifolds, Ricci curvature of which is zero.[8] In both cases, the compactification yields few or zero numbers of Kaluza-Klein gauge fields, because the symmetries of the internal spaces are far from maximal. Thus the "beautiful concept" of Kaluza-Klein theories is badly spoiled.

Another approach to the cosmological constant problem is given by Gasperini.[9] He demonstrated that an induced gravity model of Zee's type[10] with another matter field in higher dimensions solves the problem. But in this case, the choice of the matter fields is very crucial.

In this paper, we respect simplicity. We abandon the Einstein-Hilbert action in higher dimensions and, instead, we consider a generalized pure gravity action. First of all, we look for symmetric tensors with the following properties:[11]

(1) It is a concomitant of the metric tensor and its first two derivatives.

(2) It is divergence free.

When we set such a tensor equals zero, it can be regarded as the gravitational fIeld equation in vacua.

In four dimensions, the tensor which satisfies the above condition is only a linear combination of the Einstein tensor and the metric tensor. But in higher dimensions, other tensors with these properties have been found.[11] Recently, it is well mentioned that the Lagrangian which leads to such a field equation is expressed in a linear combination of extended Gauss-Bonnet densities.[12, 13] We have partly motivated the generalized gravity theory based on such a Lagrangian with the recent results on the low energy limit of string theories.[14] However, according to more recent investigations, it seems that there is no exact Gauss-Bonnet form of pure gravity in the effective field theory of strings.[15, 16]

In the present paper, we consider gravity theories in $(4+N)$ dimensions, Lagrangian of which is a monomial of the Gauss-Bonnet density, and investigate the maximal symmetric compactification of N dimensions. We also investigate the later stage of evolutions of scale factors in the model where a vanishing four dimensional cosmological constant appears naively after the compactification.

We consider $D = 4 + N$-dimensional space-time. Let e^A be an orthonormal basis for the metric ds^2:

$$ds^2 = e^A \otimes e^B \eta_{AB}\,, \quad \eta = \mathrm{diag}(-1, 1, \cdots, 1)\,. \tag{1}$$

It is convenient to introduce the differential forms,

$$\varepsilon_{A_1 \cdots A_m} = \frac{1}{(D-m)!} \varepsilon_{A_1 \cdots A_m A_{m+1} \cdots A_D} e^{A_{m+1}} \wedge \cdots \wedge e^{A_D} \tag{2}$$

where the Levi-Civita symbol is totally antisymmetric with $\varepsilon_{1 \cdots D} = 1$. The connection one-form ω_{AB} is defined by

$$de^A + \omega^A{}_B \wedge e^B = 0\,, \quad \omega_{AB} = -\omega_{BA}\,. \tag{3}$$

(Here we only consider torsion-free theories.) The curvature two-form Θ_{AB} related to the Riemann tensor R_{ABCD} by

$$\Theta_{AB} = \frac{1}{2} R_{ABCD} e^C \wedge e^D\,, \tag{4}$$

is given by $\Theta^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B$.

The Lagrangian to be considered here is

$$\mathcal{L} = K\mathcal{L}_m\,, \tag{5}$$

where K is a constant and

$$\mathcal{L}_m = \Theta^{A_1B_1} \wedge \cdots \wedge \Theta^{A_mB_m} \wedge \varepsilon_{A_1B_1\cdots A_mB_m}\,, \quad (2m \leq D) \tag{6}$$

This is the so-called "dimensionally-extended Gauss-Bonnet density."[12, 13, 14] In particular, we can see

$$\begin{aligned}
\mathcal{L}_0 &= e\,, & (7)\\
\mathcal{L}_1 &= \Theta^{AB} \wedge \varepsilon_{AB} = eR\,, & (8)\\
\mathcal{L}_2 &= \Theta^{AB} \wedge \Theta^{CD} \wedge \varepsilon_{ABCD}\,, & \\
&= e(R^2_{ABCD} - 4R^2_{AB} + R^2)\,, \quad \text{etc.}\,, & (9)
\end{aligned}$$

where R_{AB} is the Ricci tensor, $R_{AB} = R^C{}_{ACB}$ and R is the scalar curvature, $R = R^A_A$. For later convenience, we define L_m as $eL_m = \mathcal{L}_m$.

Now we consider the compactification. First of all, we consider the case

$$e^\alpha = e^\alpha(x)\,, \quad e^a = e^a(y)\,, \tag{10}$$

where e^α is a vierbein in four-dimensional spacetime, e^a is a vierbein in the N-dimensional internal space, and x^μ and y^m represent four-dimensional and N-dimensional coordinates respectively. As we will see later, this case includes the static compactification.

The curvature splits according to

$$\Theta^{AB} = \begin{cases} \Theta^{\alpha\beta}(x)\,, \\ \Theta^{ab}(y)\,, \end{cases} \tag{11}$$

then it can be shown that [12]

$$L_m = \sum_{\nu=0}^{m} \begin{pmatrix} m \\ \nu \end{pmatrix} \hat{L}_{m-\nu} \tilde{L}_\nu\,, \tag{12}$$

where a roof refers to four-dimensional spacetime whereas a tilde refers to the interal space.

For simplicity we take a maximal symmetric space as the internal space. In short:

$$\Theta^{ab} = \kappa e^a \wedge e^b \quad \text{with} \quad \kappa = \text{constant}. \tag{13}$$

Hereafter, for concreteness, we make use of the hyperspheres as maximal symmetric spaces, and set $\kappa = I/r^2$, where r is the radius of the sphere S^N.

The compactification on S^N gives the four-dimensional action as follows:

$$\begin{aligned} I &= KV_N \int d^4x\, \hat{e} \Big(\tilde{L}_m + m\tilde{L}_{m-1}\hat{R} \\ &\quad + \frac{m(m-1)}{2}\tilde{L}_{m-2}(\hat{R}^2_{\alpha\beta\gamma\delta} - 4\hat{R}^2_{\alpha\beta} + \hat{R}^2) \Big) \,, \end{aligned} \tag{14}$$

where V_N is the volume of S^N. After a simple combinatorial counting, we get

$$\begin{aligned} \tilde{L}_m &= \frac{N!}{(N-2m)!}(1/r^2)^m \,, \quad (N \geq 2m) \\ &= 0 \,, \quad (N < 2m) \end{aligned} \tag{15}$$

in the case of S^N. The first and second terms of the action (14) correspond to the cosmological term and the Einstein action respectively. We require here $\tilde{L}_m = 0$ and $\tilde{L}_{m-1} \neq 0$, which are fulfilled when $D = 4 + N = 2m + 2$ or $2m + 3$. We adopt here the case $D = 4 + N = 2m + 2$, because this case can be regarded as a natural generalization from the Einstein gravity in four dimensions, since a constant K has dimension (mass)^2 and is the same as the inverse of the Newton constant in the four-dimensional Einstein gravity.

We write the action with an appropriate normalization,

$$I = \int d^{4+N}z\, e \left[\frac{1}{2N!m\kappa^2} L_m + L_{\text{matter}} \right] \,, \tag{16}$$

where $4+N = 2m+2$, L_{matter}, is the Lagrangian of matter and κ^2 is a generalized Newton constant. For convenience, we still use both N and m through this paper.

Next, we investigate cosmological solutions. We assume the following metric in $4 + N$ dimensions:

$$ds^2 = -dt^2 + R^2(t)(d\Omega_3)^2 + r^2(t)(d\Omega_N)^2 \,, \tag{17}$$

where $(d\Omega_{3(N)})^2$ is the line element of $3(N)$-dimensional maximal symmetric space with a unit radius. As the previous case, We take S^N as extra spaces. Under these assumptions, the action is written as

$$\begin{aligned} I \simeq & \int dt\, (R^3 r^N) \left[\frac{1}{2\kappa^2} \left\{ \left(\frac{1+\dot{r}^2}{r^2} \right)^{m-1} \left(-6\frac{\dot{R}^2}{R^2} + \frac{k}{R^2} \right) \right. \right. \\ & \qquad -4(m-1)\frac{\dot{r}}{r}\left(\frac{\dot{R}}{R}\right)^3 \left(\frac{1+\dot{r}^2}{r^2} \right)^{m-2} \\ & \left. \left. -12(m-1)\ \frac{(rR)^{\cdot}}{R^3 r^N} k \sum_{n=0}^{m-2} \binom{m-2}{n} \frac{1}{2n+1}\dot{r}^{2n+1} \right\} - \rho(R,r) \right] , \end{aligned} \tag{18}$$

where $k = 1, 0, -1$, corresponds to closed, flat and open three-dimensional space, respectively. We have performed the partial integration for the action to involve no second derivatives of scale factors. Then this action leads to equations of motions with at most second derivatives for R or r. Taking variations with the metric, we obtain the following equations of motions:

$$\begin{aligned}
& 3\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-1}\left(\frac{\dot{R}}{R}\right)^2 \\
+ \quad & 6(m-1)\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-2}\left\{\left(\frac{\dot{r}}{r}\right)^2\left(\frac{\dot{R}}{R}\right)^2+\frac{\dot{r}}{r}\left(\frac{\dot{R}}{R}\right)^3\right\} \\
+ \quad & 4(m-1)(m-2)\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-3}\left(\frac{\dot{r}}{r}\right)^3\left(\frac{\dot{R}}{R}\right)^3 = \kappa^2\rho\,,
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \left(\frac{1+\dot{r}^2}{r^2}\right)^{m-1}\left\{2\frac{\ddot{R}}{R}+\left(\frac{\dot{R}}{R}\right)^2\right\} \\
+ \quad & 2(m-1)\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-2}\left\{\left(\frac{\dot{r}}{r}\right)^2\left(\frac{\dot{R}}{R}\right)^2+\frac{\ddot{r}}{r}\left(\frac{\dot{R}}{R}\right)^2+2\left(\frac{\ddot{r}}{r}+\frac{\ddot{R}}{R}\right)\frac{\dot{r}}{r}\frac{\dot{R}}{R}\right\} \\
+ \quad & 4(m-1)(m-2)\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-3}\frac{\ddot{r}}{r}\left(\frac{\dot{r}}{r}\right)^2\left(\frac{\dot{R}}{R}\right)^2 = -\kappa^2 p\,,
\end{aligned} \tag{20}$$

$$\begin{aligned}
& 3\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-2}\left\{\left(\frac{\ddot{R}}{R}+\frac{\ddot{r}}{r}\right)\left(\frac{\dot{R}}{R}\right)^2+\frac{\dot{r}}{r}\left(\frac{\dot{R}}{R}\right)^3+2\frac{\ddot{R}}{R}\frac{\dot{r}}{r}\frac{\dot{R}}{R}\right\} \\
+ \quad & 2(m-2)\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-3}\left\{\left(\frac{\dot{r}}{r}\right)^3\left(\frac{\dot{R}}{R}\right)^3+3\frac{\ddot{r}}{r}\frac{\dot{r}}{r}\left(\frac{\dot{R}}{R}\right)^3\right. \\
& \qquad\qquad \left.+3\left(\frac{\ddot{r}}{r}+\frac{\ddot{R}}{R}\right)\left(\frac{\dot{r}}{r}\right)^2\left(\frac{\dot{R}}{R}\right)^2\right\} \\
+ \quad & 4(m-2)(m-3)\left(\frac{1+\dot{r}^2}{r^2}\right)^{m-4}\frac{\ddot{r}}{r}\left(\frac{\dot{r}}{r}\right)^3\left(\frac{\dot{R}}{R}\right)^3 = -\kappa^2 q\,,
\end{aligned} \tag{21}$$

where $p = -(1/3R^2)(\partial/\partial R)(R^3\rho)$, $q = -(1/Nr^{N-1})(\partial/\partial r)(r^N\rho)$.[17] We wrote down them only in the case $k = 0$ for later use. We will examine here the later stage of evolutions of scale factors. In such a case, we can regard the matter as "four-dimensional radiation".[6, 18] In other words, the equation of state is expressed as $\rho = 3p$ ($q = 0$). This ansatz should be valid in the region $1/R < T < 1/r$, where T is the temperature.[18]

There is an approximate Friedmann (Tolemann) solution with the relatively slow time variation of r, we find that in the case $k = 0$,

$$R \sim R_0 t^{1/2}(1 + O((m-1)r_0^2 t^{2(\beta-1)}))\,, \tag{22}$$
$$r \sim r_0 t^{\beta}(1 + O((m-1)r_0^2 t^{2(\beta-1)}))\,, \tag{23}$$

where $\beta = (3 - \sqrt{13})/4 \sim -0.151$. These results may be compared with the solution in the usual five-dimensional Kaluza-Klein cosmology with "four-dimensional radiation".[19] Equation (23) shows that the scale of the compact space shrinks asymptotically more slowly in our model.

However, our model has a distinct property. The third equation of motion (21) shows that $\dot{r} = 0$ is forbidden even if we intended to set it by hand in the case $q = 0$, it means that the particle creation by quantum effects[20] plays an important role because the leading term on the left-hand side of the equation involves the fourth-order of time denvatives. This problem will be examined in the near future.

If we want to investigate whether the Kaluza-Klein inflation takes place or not, we should take into consideration highly nonlinear effects of the differential equations and need to perform numerical calculations carefully. The Regge calculus[21] may be suitable for the calculation.

Next, we should remark on the static solutions including dimensional reduction.[22] In our model, the metric of (Schwarzschild solution)$\times$(constant S^N) is not a solution. The study of properties of the static solutions in our theory is currently in progress.

Finally, we will comment on quantum nature of our theory. In flat space-time, it is known that the extended Gauss-Bonnet terms contain only the graviton interaction vertex.[14] However, contents of these terms is still unknown in curved space-time. While we need the principle that forbids the existence of the cosmological term, the Einstein-Hilbert term and all others unrequired, we should consider the supersymmetric extension of our model. Therefore, it is important to investigate quantum nature of the theory, as well as contributions of other matters. Particularly, to supersymmetrize the extended Gauss-Bonnet term, we need to introduce antisymetric tensors,[23] which also affects cosmological solutions.

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