A. La Rosa **Lecture Notes**

INTRODUCTION TO QUANTUM MECHANICS

The Variational principle:

From the Hamilton Jacobi Equation to the Schrodinger Equation

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The Variational Principle

OPTICS:

The Principle of Least Time

MECHANICS

The Variational Principle and the Lagrange Equations of Motion

The Hamilton Equations of Motion

Canonical Transformations

OPTICS

Two ways of thinking

a) The idea of **Causality**. When one thing occurs then something else happens.

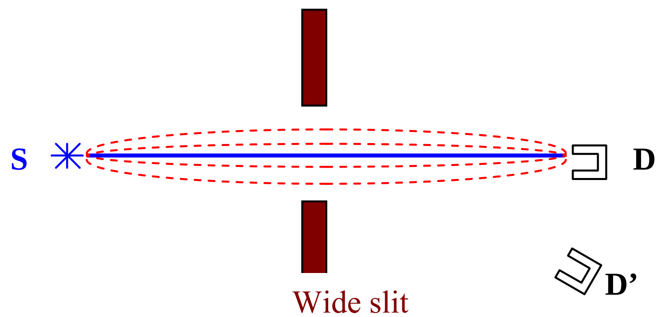
   This is the philosophy behind Snell's law. When light reaches an interface (where the index of refraction changes) its direction of propagation changes accordingly. This happens again and again as new interfaces are encountered by the light. Thus, the net path is built by piece by piece, one after another.

b) The **Principle of Least Time** is completely different. In a given situation, light "evaluates" what is the path of shortest time (or the one that makes the travel time an extreme) and takes that path.

> "*Does light smell the nearby paths and check them against each other?*"

The answer is yes, it does; in a way. …
The wavelength tells us approximately how far away the light must 'smell' the path in order to check it."

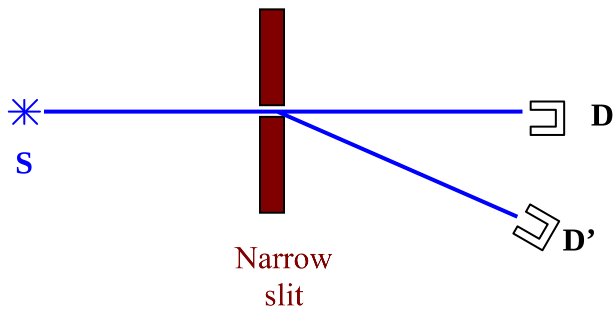
The connection between the size of the "smelling" range (*i. e.* the wavelength) and the least time principle is as follows: paths whose lengths differ from the actual one by about one wavelength constitute the only ones for which the travel time is almost the same. Paths away from the actual one by more than a wavelength have too much different travel times.)



Light goes from S to D because such straight line makes the time minimum. Nothing arrives to the detector D' because those paths around S-slit-D' make the travel time quite different (so light does not go there.)

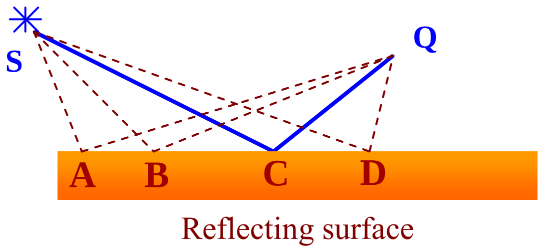
What about if we prevent the light from smelling around?

One way to do is, for example, making the aperture very narrow (smaller than the wavelength.) "Then, there is only one path available, and the light takes it!"
With a narrow slit, more radiation reaches D' than reaches it with a wide slit." This gives rise to a diffraction phenomena.



Light passing through an aperture smaller than the wavelength.

Formalization of the Quantum Dynamics View



In going from S to Q via reflection on the surface,
light does not seem to travel in the form of a wave.
Instead,
the rays seem to be made out of photons (bullets of light.)
The brightness of the light at Q is proportional to the average of photons that arrive

The photons could in principle go anywhere (that is, the exact outcome is unpredictable.) But what we can do is the following:

To calculate the probability that a photon leaving from S arrives at Q.

Such probability requires calculating first the corresponding *amplitude probability* <**Q** (final point) | **S**(initial point)>. Let's call it simply **<Q | S>**.

How to calculate the amplitude probability <Q | S>

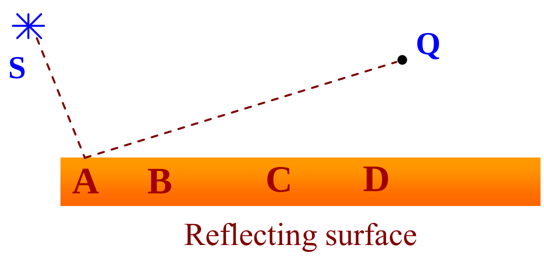
In the figure above, all the paths from S to Q are taken into account (SAQ, SBQ, etc.)

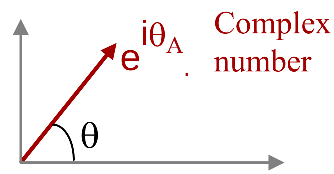
- Each path, SAQ for example, makes a contribution to the amplitude probability **<Q | S>** the following way:

  i) Calculate the corresponding time t_{SAQ}

  ii) Draw a complex number $e^{i(\omega t_{SAQ})}$.

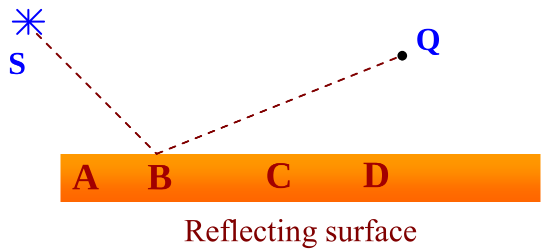
  where ω is the frequency of the light

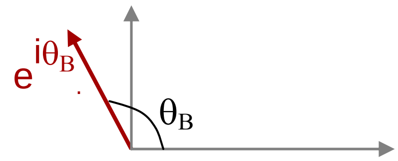




$\theta_A = \omega\, t_{PAQ}$

- Take another path, SBQ for example



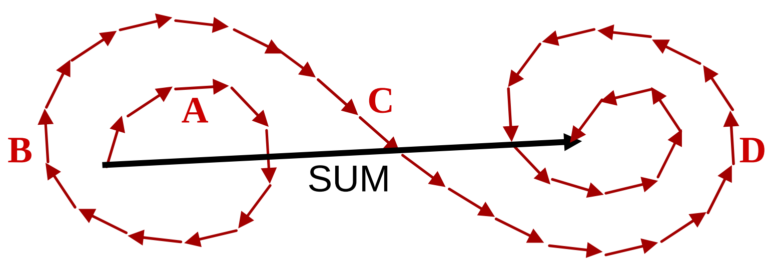


$\theta_B = \omega\, t_{PBQ}$

Since the time t_{SBQ} is in general quite different than t_{SAQ} then the corresponding angle θ will be different.

- Take ALL the available paths and add up their corresponding individual complex numbers, thus obtaining a total SUM

$$\text{SUM} = \sum_{\theta} e^{i\theta} \qquad (1)$$



Addition of amplitude probabilities contributed by all possible paths that gp from S to Q via reflection on the mirror.

The SUM is a complex number and defines the amplitude probability <**Q**|**S**>.

How to calculate the probability

The probability that a photon leaving from S arrives at Q., via reflection on the surface, is given by (2)

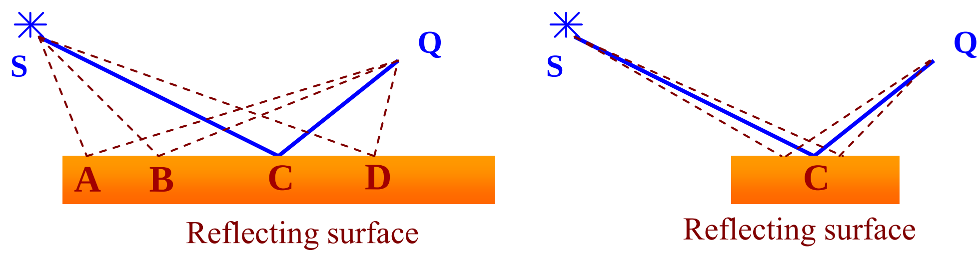
$|\langle \mathbf{Q}|\mathbf{S}\rangle|^2 \equiv |\text{SUM}|^2.$

Amplitude probability and the least time principle

In the drawing above (used to obtain the amplitude probability) notice that,

- the net contribution to the amplitude probability from paths away from SCQ (the one that makes the time minimum) is practically equal to zero. What happens is that their corresponding travel times are quite different and so it is their phase, which makes their complex numbers to add up to practically zero value.

- However, when adding the complex number $e^{i\theta}$ associated to paths neighbor to the SCQ (the one that makes the time minimum), their travelling time does not change too much, so their phases θ are about the same. That makes the corresponding complex numbers to constructively add up. Hence their notable contribution to the SUM observed in the graph above.
- The two points above suggest that is we decreased the size of the reflecting mirror, cutting the side extremes, the amplitude probability will not change too much, as the contribution from those regions add up to zero.



Cutting the side extremes of the mirror will not affect the amplitude probability.

This observation also reinforces the notion that light "smells" the paths neighborhood of SCQ (the one that makes the time minimum) within a range of λ. For the phase of $e^{i(\omega t_{SCQ})}$ would change dramatically only if t_{SCQ} changed by a period. The latter would occur if the path changed by λ.