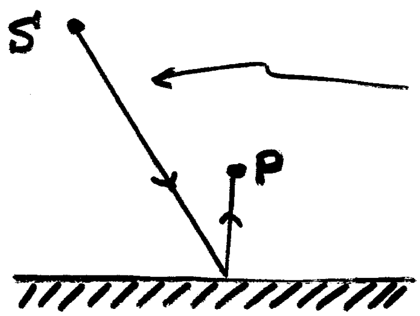
# FERMAT'S PRINCIPLE (4.5)

## EVOLUTION OF THE VARIATIONAL PRINCIPLE

① Shortest Path Principle

Example: Reflection at a surface

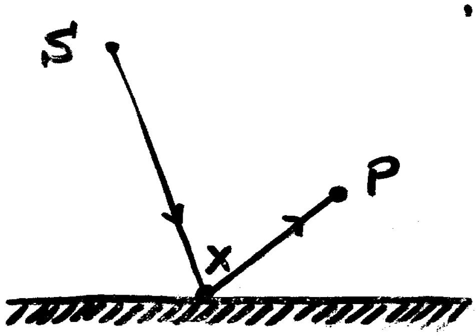



what would be the exact path the light beam follows?

Hero of Alexandria: "the path taken by a light beam in going from a point S to a point P via a reflecting surface is the shortest possible one"

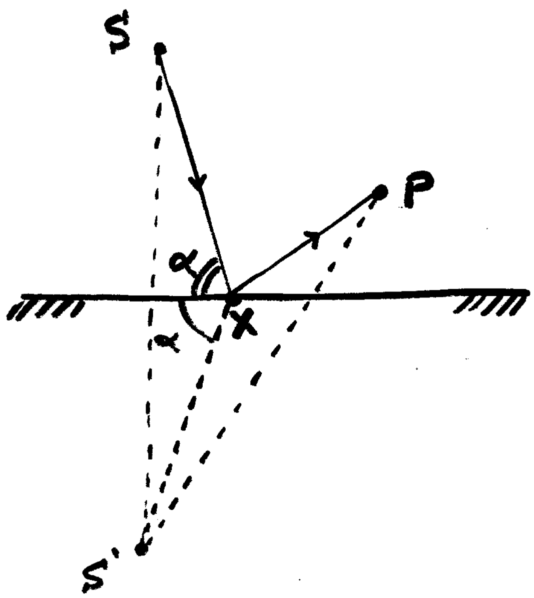
Among the infinite number of paths that join S and P via reflection, let's consider only those that follow a rectilinear path.

Where is X ?




The question, then, becomes: what is the location of the point X on the surface that makes the length of $(\overline{SX} + \overline{XP})$ minimum.

Let's draw a point S' beneath the surface and symmetric to S




Notice that, by construction: $\overline{SX} = \overline{S'X}$

Therefore:

$$\overline{SX} + \overline{XP} = \underbrace{\overline{S'X} + \overline{XP}}$$

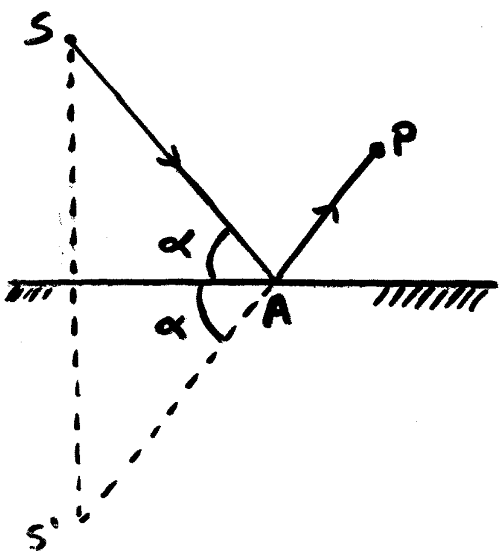
Notice in the graph this length is always greater than the straight segment $\overline{S'P}$

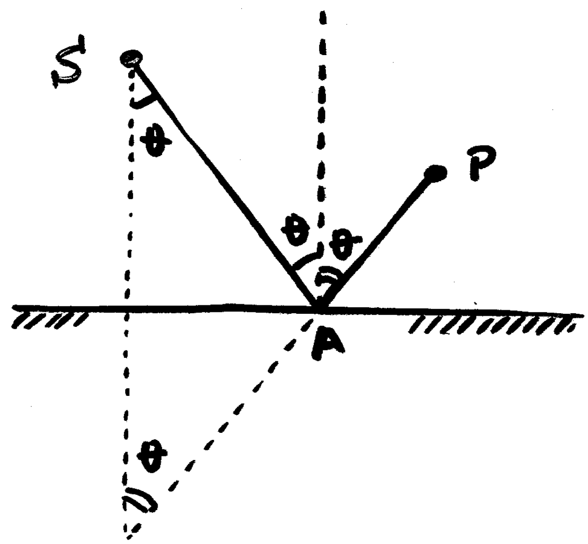
$$\overline{SX} + \overline{XP} > \overline{S'P}$$

So, we have found a minimum length

For $\overline{SX} + \overline{XP}$ to be the path of minimum length, X will have to be along the segment S'P .

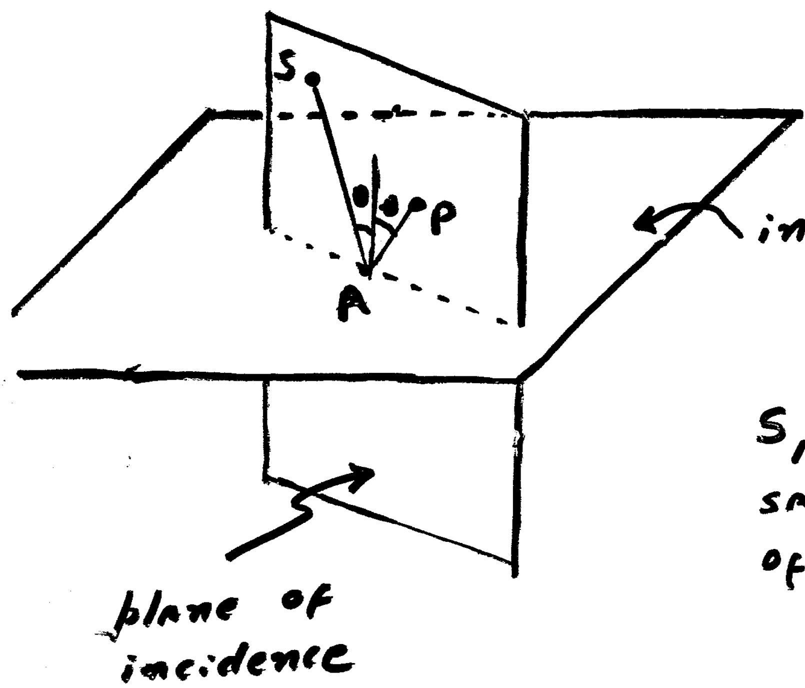
According to Hero, the light beam chooses the path SAP



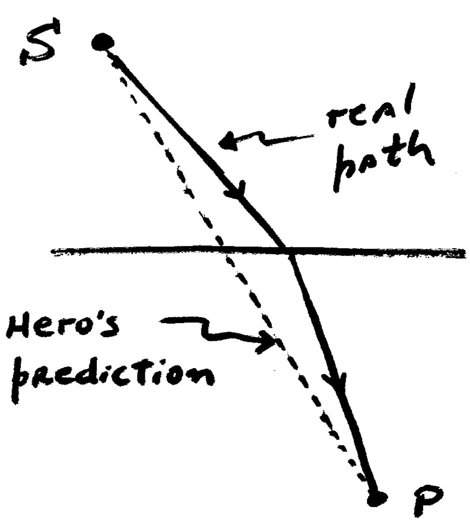

Thus, in going from S to P via reflection at a surface, the light beam "chooses" the shortest path SAP. ⊥
In doing that, it turns out that the incident angle is equal to the reflecting angle.

Following the same arguments, we will conclude that:
The shortes path SAP must lie in the same plane that is perpendicular to the reflecting surface.




S, A, and P lie in the same plane, the "plane of incidence"

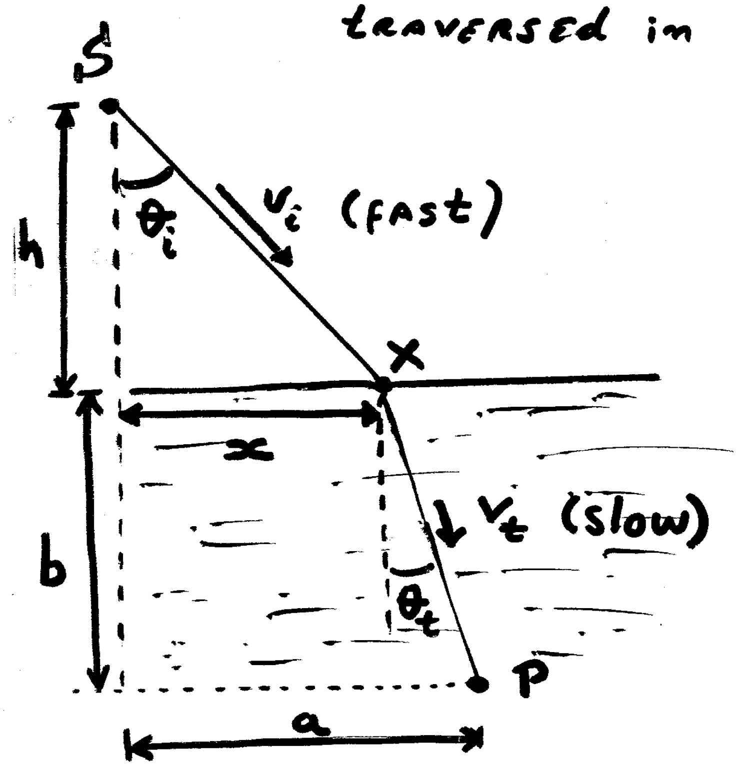
(2) Hero's Principle is not applicable to describe phenomena of refraction




In going from S to P the light beam does not choose the path of shortest length

## ③ PRINCIPLE of LEAST TIME

FERMAT (1657) "The actual path taken by a light beam in going from a point S to a point P is the one traversed in the least time"




To get from S to P in the minimum time, the light beam may want to maximize $\overline{SX}$ (where it travels faster) and minimize $\overline{XP}$ (where it travels slower)

CASE $n_i < n_t$

$t(x) =$ time to travel along SXP

$$v_i = \frac{c}{n_i}, \quad v_t = \frac{c}{n_t}$$

$$= \frac{\overline{SX}}{v_i} + \frac{\overline{XP}}{v_t}$$

$$t(x) = \frac{(h^2 + x^2)^{1/2}}{v_i} + \frac{\left[b^2 + (a-x)^2\right]^{1/2}}{v_t} ,$$

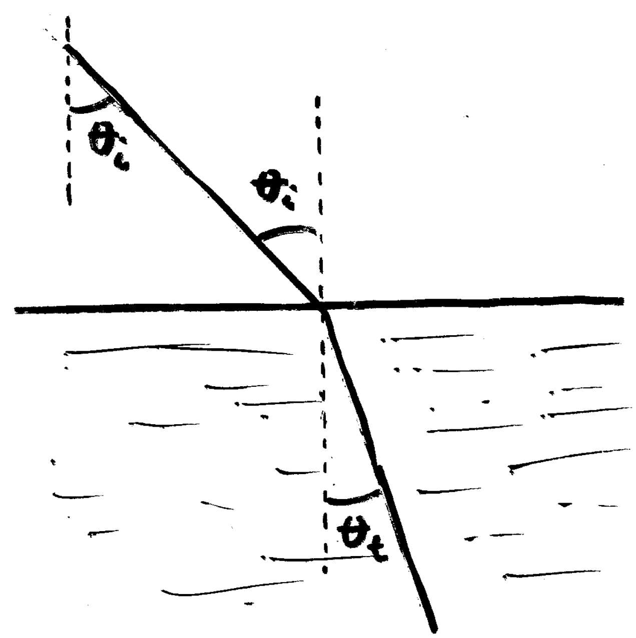
$$\frac{dt}{dx} = \frac{2x}{2(h^2+x^2)^{1/2}} \cdot \frac{1}{v_i} + \frac{2(a-x)\cdot(-1)}{2\left[b^2+(a-x)^2\right]^{1/2}} \cdot \frac{1}{v_t}$$

$$= (\sin\theta_i)\cdot\frac{1}{v_i} - (\sin\theta_r)\frac{1}{v_t}$$

minimum time occurs when $\dfrac{\sin\theta_i}{v_i} = \dfrac{\sin\theta_r}{v_t}$

or

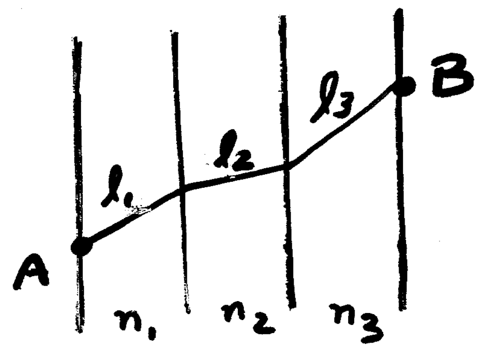
$$n_i \sin\theta_i = n_t \sin\theta_t$$



# ④ OPTICAL PATH LENGTH (OPL)

$$t_{AB} = \sum_i \frac{l_i}{v_i} = \sum_i \frac{n_i l_i}{c}$$

$$= \frac{1}{c} \underbrace{\sum_i n_i l_i}_{\text{optical path length}}$$




Definition

$$\boxed{OPL \equiv \sum_i n_i l_i}$$

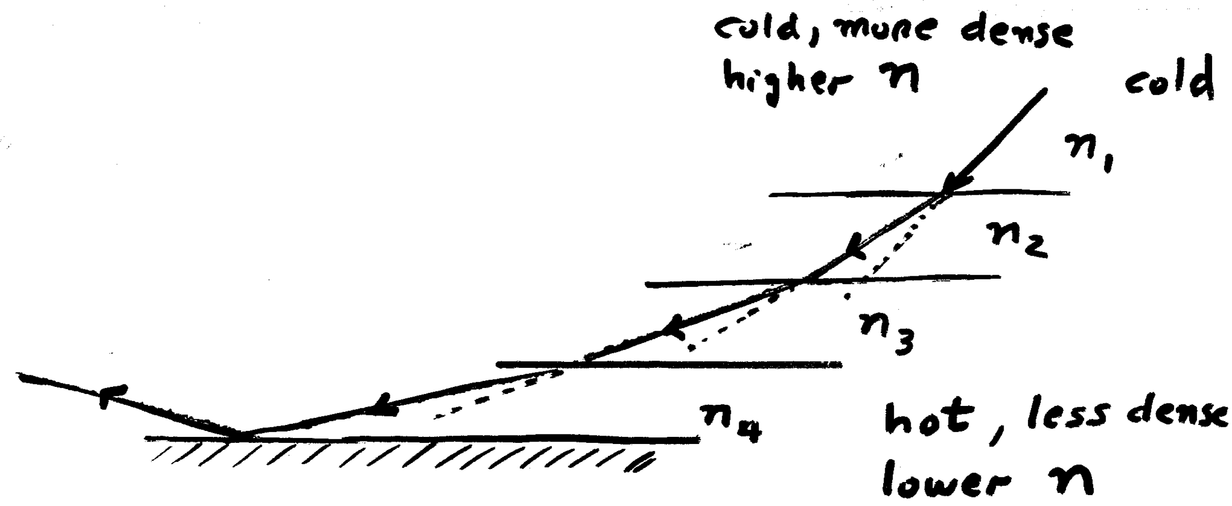
Principle of least time $\Longleftrightarrow$ (equivalent) Principle of least OPL

In going from A to B the light beam chooses the path that has the lowest OPL

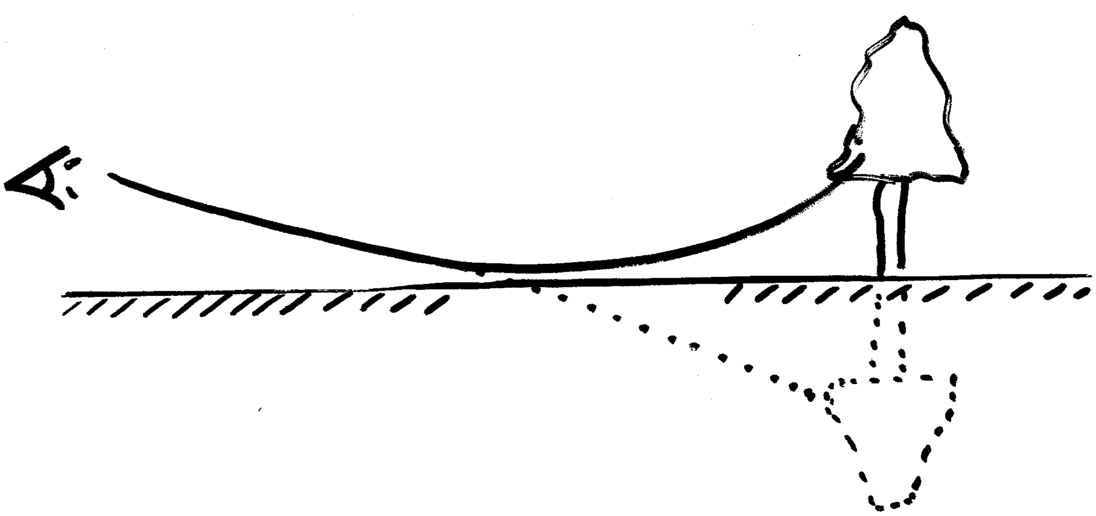
when the index of refraction change almost continuously:

$$OPL = \int_A^B n \, dl$$

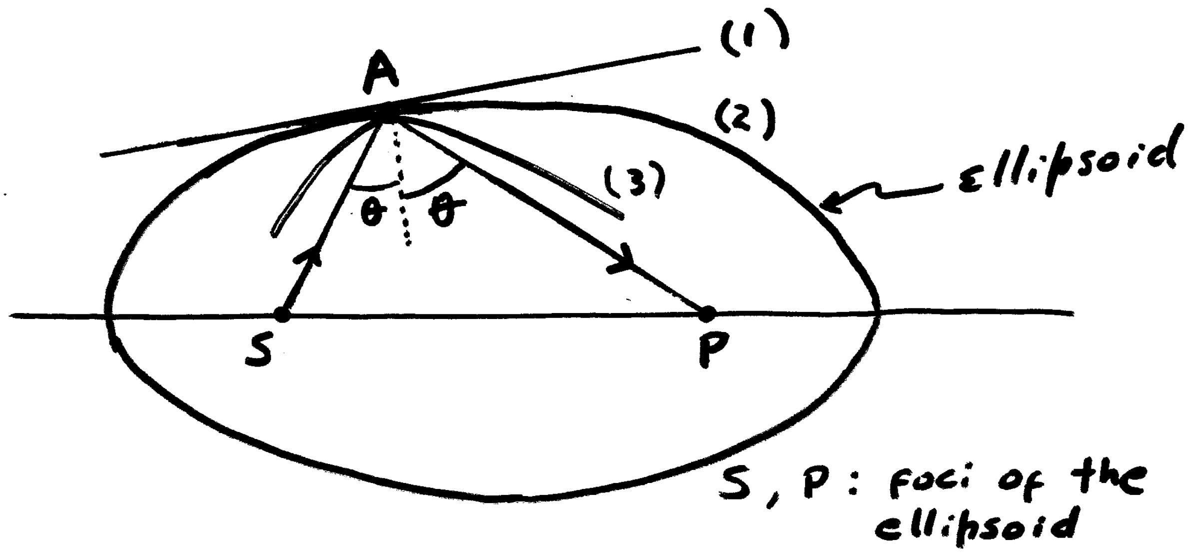
"



cold, more dense
higher $n$
cold
$n_1$
$n_2$
$n_3$
$n_4$
hot, less dense
lower $n$

## (5) Least time Principle may not work all the time




The three surfaces (1), (2) and (3) are tangent at point A. But physically we have to consider one at a time.

Plane mirror:

the path SAP is the one with the least OPL among the many other path that reflect from the mirror

Ellipsoid:

the path SAP is not the one with the least OPL among the many others that reflect from the ellipsoid.

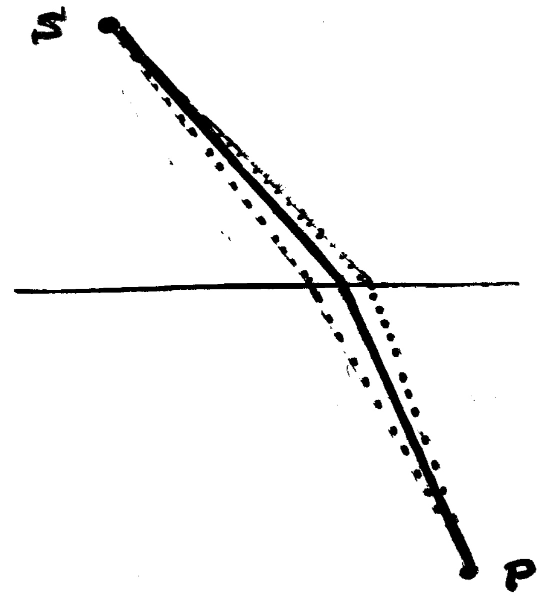
In fact, it is a property of the ellipse that any ray from S will be reflected by the ellipsoid toward the point P (regardlees which point on the surface the reflection occurs) AND all of them have THE SAME OPL !

Surface (3)

It appeas from the figure that path SAP is the one with MAX OPL among the other hypothetical path that start at S and reflect toward P.

That is, path SA'P where A' is on the surface (3) very close to A will have a smaller OPL (which makes the OPL of SAP a maximum.)

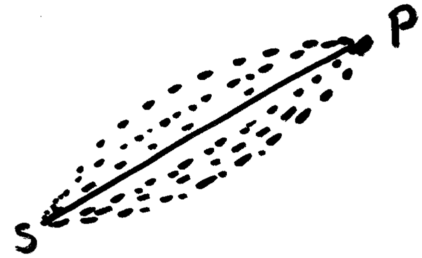
## ⑥ MODERN FORMULATION OF FERMAT'S PRINCIPLE



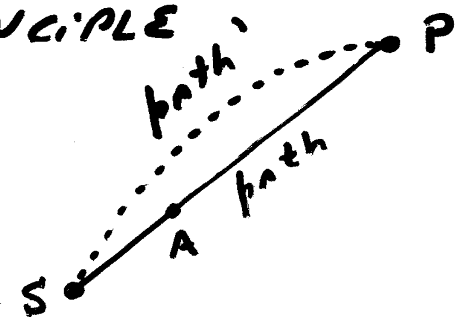

"A ray going in a certain particular path has the property that
IF we make a small change (say 1% shift) in the ray in any manner whatever,
say in the location at which it comes to the mirror
or the shape of the curve
or anything"
THEN
there will be NO first order change in the time.

$$\delta(OPL) = 0$$

$$\delta(t_{SP}) = 0$$



# THE VARIATIONAL PRINCIPLE




- the time the light beam takes to go from $S$ to $P$ is a function of the particular path it takes.

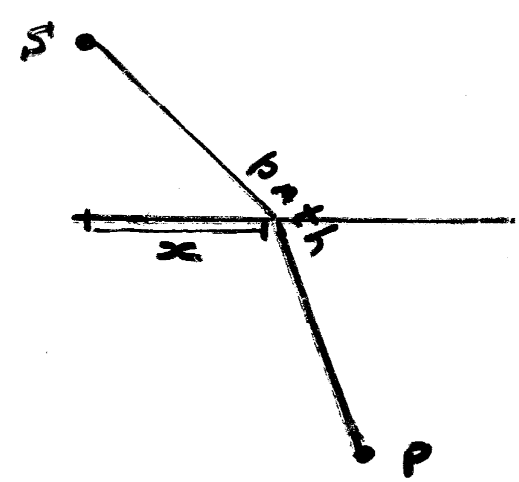
$$t_{SP} = f(\text{path } S \to P)$$

that is to say that $t_{SP}$ depends on some parameters that specify the path from $S$ to $P$

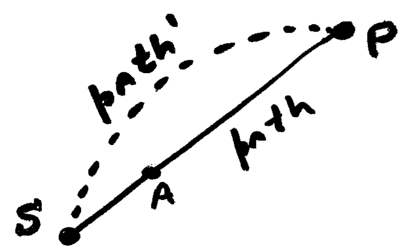
[ We have already seen an example of that before.
With $S$ and $P$ being fixed points, the rectilinear path is specified by the parameter $x$.
thus

$$t_{SP} = f(x)$$

]



. When considering two different paths, there will be, in general, a difference in the time the light take to go from S to P along those paths




$$\Delta t_{SP} = f(\text{path}') - f(\text{path})$$

for which we use the notation

$$= \delta f$$

In general $\Delta t_{SP} = a\,\Delta x + b(\Delta x)^2 + \dots$

where, for simplicity, we are assuming that $x$ is one of the parameters that specify an arbitrary path.

But, IF SAP (see figure above) were the actual path followed by the light beam,

THEN

there would be no first order change in the time $t_{SP}$;

there would be only a second order change in the time.

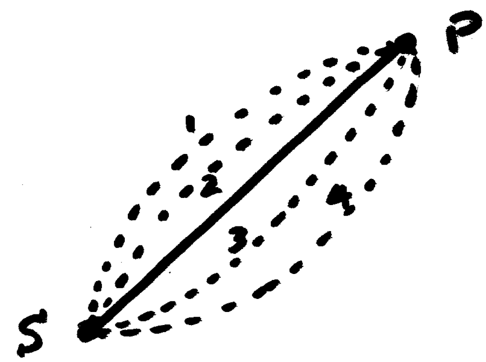
In symbolic form

$$\delta f \left( \begin{smallmatrix} \text{actual} \\ \text{path} \end{smallmatrix} \right) = 0$$

VARIATIONAL PRINCIPLE

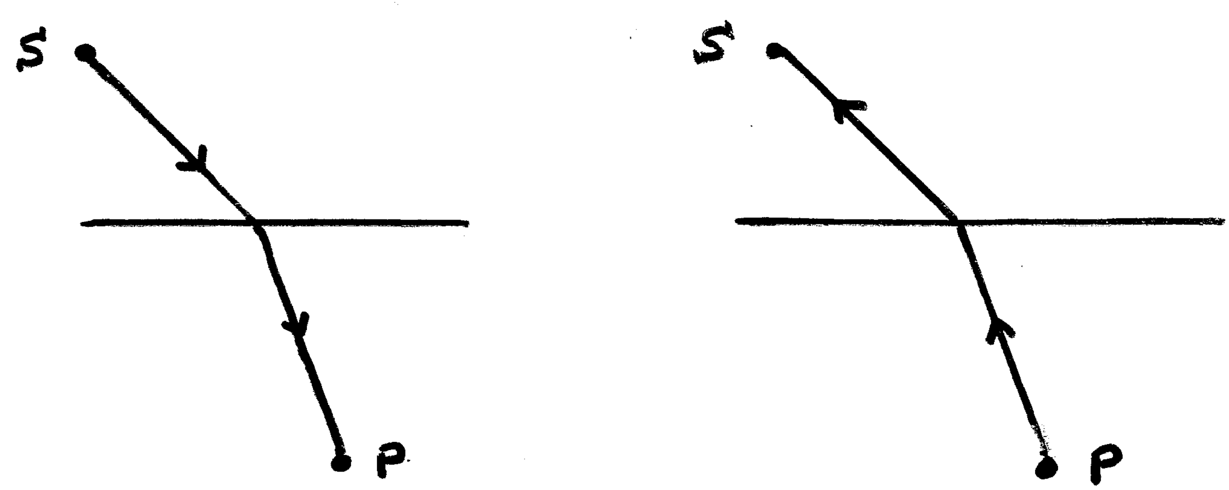
## STATIONARY PATH

If $t_{SP} = f(\text{path})$ we say: the actual path followed by the light beam is the one that makes the function f stationary



# PRINCIPLE OF REVERSIBILITY

- Notice that the variational principle speaks only about the stationary path (without specification of directions along with it).
- If the roles of points S and P are interchanged, so that P is the source of light, the variational principle will predict the path path as determined for the original direction of light propagation




A ray going from S to P will trace the same path as one from P to S