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Pyrgon Annihilation in a Five-Dimensional Model Universe

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On the basis of a five-dimensional cosmological model with the shrinking extra space, we investigate the annihilation process of pyrgons (i.e., massive modes corresponding to quantized excitations of the fifth dimension). It is shown that this annihilation goes on as a nonequilibrium process and a net entropy in a comoving volume is produced. However, no sufficient suppression of the pyrgon number density can be expected, because the contraction of the extra space stops immediately after the onset of the pyrgon annihilation. Dynamical effects of relic pyrgons in the universe are discussed.

§ 1. Introduction

Cosmological aspects of the Kaluza-Klein hypothesis, in the framework of a theory of higher-dimensional general relativity or supergravity, have received much attention recently. A basic picture is that at very early times all dimensions in the universe were of comparable size, and the smallness of the compactified extra dimensions obtained through a dynamical evolution of the universe. This cosmological dimensional-reduction process was first pointed out by Chodos and Detweiler,[1] and many authors[2),3] have tried to solve basic cosmological problems by constructing various higher-dimensional cosmological models.

The scale of the extra dimensions is at present so small as to be unobservable with any low-energy experiments. However, an effect of the existence of the compact dimensions may be observed as a remnant from the early stage when the universe was at a temperature comparable to (or higher than) their inverse size R^{-1}. This possibility was proposed by Kolb and Slansky.[4] We can expand any higher-dimensional fields in a harmonic series which describes quantized excitations of the extra dimensions. For the four-dimensional version the excited states correspond to non-zero modes of mass eigenstates. These massive states, which are called "pyrgons", will be excited during the multi-dimensional phase. Some higher-dimensional theories claim the existence of stable pyrgons which cannot decay into zero modes except through an annihilation process with antipyrgons. Then the particles with mass of order R^{-1} may survive annihilation to contribute to the present energy density of the universe as dark matter.

On the basis of the five-dimensional cosmological model with the static fifth dimension, Kolb and Slansky evaluated the annihilation cross section of pyrgons and were led to the following prediction: At a high-temperature stage of the universe the gas of pyrgons and photons (we call low-mass particles corresponding to zero modes "photons") was in equilibrium. The reaction rates decrease faster than the expansion rate of the universe, and the annihilation of pyrgons becomes ineffective before the temperature goes down below the pyrgon mass. Then the pyrgon number density today will be comparable to the photon number density. Unless the pyrgon mass is too small, the predicted energy density of relic pyrgons cannot be compatible with the present observational bound.

This difficulty may be overcome through a large amount of entropy generation at a later stage of the cosmological evolution. However we would leave aside this possibility. In this paper we reexamine the annihilation process of pyrgons in terms of a cosmological model with the shirinking fifth dimension. We do not intend here to solve the pyrgon problem. Our main aim is to clarify an influence of the contraction of the extra dimension on the time dependence of the pyrgon number density and to study why in the framework of the five-dimensional general relativity we cannot expect its sufficient suppression.

In § 2 we start from an intuitive derivation of the annihilation cross section of pyrgons in terms of the dimensional analysis, and compute their number density in the Chodos-Detweiler cosmological model[1] which is a vacuum Kasner solution of the five-dimensional Einstein equations. Because the pyrgon mass grows as the fifth dimension shrinks, their annihilation rate tends to dominate over the expansion rate of the universe when the temperature drops below their mass. Although their annihilation is not so frequent that chemical equilibrium between pyrgons and photons is established (the equilibrium number density falls too steeply), the ratio of the pyrgon energy density to the photon one decreases slowly. This is shown in § 3 by virtue of the conservation of the energy-momentum tensor. We discuss also the entropy generation due to the annihilation of pyrgons. In the final section we show that the proposed approach to the pyrgon problem is confronted with a serious constraint. If we adhere to the five-dimensional Einstein equations, the effect of matter (i.e., pyrgons or photons) becomes important at some evolutionary stage. Then the fifth dimension ceases to contract. This may happen before the sufficient annihilation of pyrgons takes place. We examine the termination of the Chodos-Detweiler era. Some implications of the results are discussed.

§ 2. Annihilation of pyrgons

We consider a five-dimensional field Φ, such as graviton, which represents small disturbances about the background geometry

$$ds^2=-dt^2+a^2(t)\sum_{i=1}^{3}(dx^i)^2+b^2(t)(dx^5)^2\,, \tag{1}$$

and obeys the equation

$$\Box^{(5)}\Phi=0\,, \tag{2}$$

where $\Box^{(5)}$ is the five-dimensional d'Alembertian. The topology of the extra space is S^1, and the free field Φ can be expanded in a harmonic series

$$\Phi=\sum_n \phi_n(t,\ x^i)\exp(ik_5x^5)\,, \tag{3}$$

where $0\leq x^5\leq 2\pi R$, R is a present radius of the fifth dimension, and the fifth component k_5 of wave vector is n/R. We may interpret each state ϕ_n in the harmonic expansion as a four-dimensional particle with mass $m_n=|n|/bR$. However, in the following discussions, we would treat each mode of photons ($n=0$) or pyrgons ($n\neq 0$) as a five-dimensional massless particle with wave vector k_μ ($\mu=0, 1, 2, 3, 5$ and $|k^5|=m_n$). Then the annihilation rate of pyrgons written in the five-dimensional version is

$$\Gamma_A = N\sigma_A v\,, \tag{4}$$

where N is their number density, their velocity $v=1$ and their annihilation cross section σ_A has dimension of (length)^3. In this unified theory the cross section σ_A will include the factor $\bar{G}^2$, where $\bar{G}$ is the fundamental constant that appears in the five-dimensional gravitational action. It has dimension of (length)^3 and is related to Newton's constant G by

$$\bar{G} = 2\pi RG\,. \tag{5}$$

The pyrgon-antipyrgon annihilation is a process of momentum transfer of the fifth component k^5. Hence we assume from the dimensional analysis that

$$\sigma_A = \zeta\bar{G}^2|k^5|^3\,, \tag{6}$$

where ζ is a dimensionless parameter which remains here to be free.

Now let us assume the universe to be in the Chodos-Detweiler era, i.e.,

$$a \propto t^{1/2}, \quad b \propto t^{-1/2}\,. \tag{7}$$

The expansion rate of the universe is

$$\Gamma_E \equiv \dot{V}/V = t^{-1}\,, \tag{8}$$

where $V(=a^3b)$ is the spatial volume. Whether the gas of pyrgons was in thermal equilibrium at an early stage is of crucial importance. Pyrgons as abundant as photons might be never produced, if no equilibrium was established. However we do not adhere to this problem. Our main concern is the annihilation of pyrgons in a later stage at which the temperature T is lower than $m \equiv m_{n=1} = 1/bR$ and equilibrium is no longer maintained. From a four-dimensional viewpoint, pyrgons are regarded as non-relativistic particles in non-equilibrium situation. Because all excited pyrgons ($|n|>1$) decay rapidly into stable pyrgons ($|n|=1$) and photons ($n=0$), we consider only the $|n|=1$ pyrgons in the following.

The number density N of stable pyrgons is determined by the equation

$$\dot{N} = -\Gamma_A N - \Gamma_E N\,, \tag{9}$$

since the equilibrium number density falls steeply as $\exp(-m/T)$. Taking account of the time dependence of $\sigma_A(=\zeta\bar{G}^2m^3 \propto t^{3/2})$, we obtain the solution

$$N = 3N_*(t/t_*)^{-1}\{2(t/t_*)^{3/2}+1\}^{-1}\,, \tag{10}$$

where t_* is the instant when $\Gamma_E = \Gamma_A$, and N_* is the value of N at $t=t_*$. For $t<t_*$ pyrgons suffer no annihilation, i.e.,

$$N \simeq 3N_*(t/t_*)^{-1} \propto V^{-1}\,, \tag{11}$$

while for $t>t_*$ their number density decreases more steeply, i.e.,

$$N \simeq \frac{3}{2}N_*(t/t_*)^{-5/2}\,. \tag{12}$$

The instant when their annihilation becomes effective is given by

$$t_* = (\zeta \bar{G}^2 m_*^3 N_*)^{-1}, \tag{13}$$

where m_* is the pyrgon mass at $t=t_*$. For $t>t_*$ we have

$$\Gamma_A/\Gamma_E = 3/2. \tag{14}$$

We note that the annihilation proceeds in a nonequilibrium manner, keeping a balance with the expansion rate.

The above results lead to the following picture: If the equilibrium was destroyed at an early stage, after the decay of excited pyrgons there exists a period of freeze-out of stable pyrgons. Their annihilation begins at $t=t_*$ again, and for $t>t_*$ their energy density

$$\rho_P \equiv mN \simeq \frac{3}{2} m_* N_* (t/t_*)^{-2} \tag{15}$$

is fully suppressed during the Chodos-Detweiler era, in comparison with the free streaming case.

§ 3. Energy density of photons

The annihilation of pyrgons heats the photon gas. This process can be calculated from the conservation of the energy-momentum tensor of photons and pyrgons. We adopt the kinetic-theory definition[5] for the five-dimensional energy-momentum tensor, taking account of topology of the fifth dimension, i.e.,

$$T_{\mu\nu} = (a^3 bR)^{-1} \sum_{|n|=0,1} \int d^3k_i f(k_i, k_5) k_\mu k_\nu (k^0)^{-1}, \tag{16}$$

where

$$k^0 = \left(\sum_{i=1}^{3} k_i^2/a^2 + k_5^2/b^2\right)^{1/2}, \quad k_5 = n/R, \tag{17}$$

and $f(k_i, k_5)$ is a distribution function of the five-dimensional particles. For the later stage of evolution considered here we have approximately $k^0(|n|=1) \simeq m$. Then we find the approximate form satisfying traceless condition,

$$\begin{aligned} &T_{00} = \rho_\gamma + \rho_P, \\ &\sum_{i=1}^{3} T_i^{\ i} = \rho_\gamma, \quad T^5{}_5 = \rho_P, \end{aligned} \tag{18}$$

where ρ_γ is the energy density of photons $(n=0)$. We further assume that the stress tensor is diagonal, and is isotropic for the ordinary spatial dimensions $(T^i{}_i = \rho_\gamma/3)$.

The equation for conservation of $T_{\mu\nu}$ in the Chodos-Detweiler geometry is reduced to

$$\dot{\rho}_\gamma + \dot{\rho}_P = -(3\rho_\gamma + \rho_P)/2t. \tag{19}$$

This gives the solution

$$\rho_\gamma = (t/t_*)^{-3/2} \left\{ \rho_0 + 9\rho_{P*} \int_0^{t/t_*} x^{3/2} (2x^{3/2}+1)^{-2} dx \right\}, \tag{20}$$

where $\rho_{P*} = m_* N_*$. During the period of freeze-out of pyrgons the energy density depends on time as

$$\rho_\gamma=\rho_0(t/t_*)^{-3/2}\propto(a^4b)^{-1}, \tag{21}$$

and for $t\gg t_*$ the annihilation of pyrgons replaces the factor ρ_0 by $\rho_0+2^{4/3}\pi\rho_{P*}/\sqrt{3}$. The constant ρ_0 becomes the energy density of photons at $t=t_*$ if no annihilation occurs. Even if for $t<t_*$ the pyrgon energy density was larger than the photon one ($\rho_0<\rho_{P*}$), for $t>t_*$ the ratio ρ_P/ρ_γ decreases as

$$\rho_P/\rho_\gamma\simeq(t/t_*)^{-1/2}<1. \tag{22}$$

We observe the period of freeze-out of photons (i.e., $\rho_\gamma\propto(a^4b)^{-1}$) both for $t<t_*$ and for $t>t_*$. In this period the distribution function of photons, $f(k_i, k_5=0)$, will preserve a thermal distribution established previously. That is, we assume

$$f(k_i, k_5=0)=f\left(a'\sqrt{\sum_{i=1}^{3}k_i^2}\Big/T'\right), \tag{23}$$

where a' and T' are the values of the scale factor of 3-dimensional space and the temperature at the instant of freeze-out, respectively. Then, except a numerical factor, ρ_γ has the form

$$\rho_\gamma=mT^4 \tag{24}$$

with the temperature T defined by

$$Ta=T'a'=\text{const}. \tag{25}$$

The entropy S_γ of photons in a comoving volume defined by

$$S_\gamma=\rho_\gamma V/T=mT^3V \tag{26}$$

is conserved during the period. We find for $t\ll t_*$

$$S_\gamma=S_1=\rho_0^{3/4}m_*^{1/4}V_* \tag{27}$$

and for $t>t_*$

$$S_\gamma=S_2=(\rho_0+2^{4/3}\pi\rho_{P*}/\sqrt{3})^{3/4}m_*^{1/4}V_*, \tag{28}$$

where V_* is the spatial volume V at $t=t_*$. The definition of the entropy S_P of pyrgons in a comoving volume is obscure. Here we regard the number of pyrgons in V as S_P. For $t<t_*$ it is conserved as

$$S_P\equiv NV=3N_*V_*, \tag{29}$$

while for $t>t_*$ it becomes negligible. The annihilation of pyrgons proceeds without maintaining a thermal equilibrium between photons and pyrgons. This nonequilibrium process should generate entropy in a comoving volume. In fact we find the increase of total entropy as

$$S_2/(S_1+3N_*V_*)\simeq(\rho_{P*}/\rho_0)^{3/4}. \tag{30}$$

The equality results from the assumptions that for $t<t_*$ there are pyrgons as abundant as photons, i.e.,

$$\rho_0^{3/4} m_*^{1/4} \simeq N_* , \tag{31}$$

and the temperature $T=(\rho_0 t_*^2/m_* t^2)^{1/4}$ is lower than the pyrgon mass m, i.e.,

$$\rho_0 < m_*^5 . \tag{32}$$

These relations assure that $\rho_{P^*} > \rho_0$.

In Ref. 3) it was found that entropy is pumped from the extra dimensions into the effective four-dimensional universe via the annihilation of pyrgon modes. However, the whole process was assumed to be adiabatic. It will be necessary to examine that the condition for equilibrium is fulfilled in the cosmological models considered there. The destruction of equilibrium will lead to an enhancement of entropy generation from the extra dimensions.

§ 4. Termination of the Chodos-Detweiler era

In the previous sections we studied the time variation of the pyrgon abundance in the Chodos-Detweiler model universe. We found interesting aspects of the annihilation process of pyrgons due to the shrinking extra dimension: There is a period of freeze-out of pyrgons at the early stage. Their subsequent annihilation proceeds without maintaining thermal equilibrium. This process generates a net entropy in a comoving volume in addition to the entropy flow from the extra dimension into the ordinary three-dimension,[3] and gives rise to the suppression of the pyrgon energy density below the photon one.

The picture proposed in the previous sections is based on the assumption that the extra dimension shrinks at all times. However, the Chodos-Detweiler solution satisfies only the vacuum Einstein equations. As the evolution of the universe goes on, we must consider the effect of matter as follows:

$$\begin{aligned}
&(\dot{a}/a)^2 + \dot{a}\dot{b}/ab = 8\pi\bar{G}(\rho_P+\rho_\gamma)/3 , \\
&\ddot{a}/a + 2(\dot{a}/a)^2 + \dot{a}\dot{b}/ab = 8\pi\bar{G}\rho_\gamma/3 , \\
&\ddot{b}/b + 3\dot{a}\dot{b}/ab = 8\pi\bar{G}\rho_P .
\end{aligned} \tag{33}$$

The Chodos-Detweiler era continues during the period when the anisotropic motion between the extra and ordinary dimensions governs the dynamical evolution, that is,

$$\Gamma_E^2 \gg \bar{G}(\rho_P+\rho_\gamma) . \tag{34}$$

If the annihilation of pyrgons remains inactive, the photon and pyrgon components of the energy-momentum tensor (18) are respectively conserved in the following way:

$$\rho_\gamma \propto (a^4 b)^{-1}, \quad \rho_P \propto (a^3 b^2)^{-1} . \tag{35}$$

Hence, during the Chodos-Detweiler era, ρ_P decreases slower than ρ_γ and Γ_E^2 towards the breakdown of relation (34). The solution of Eq. (33) under the condition $\rho_P \gg \rho_\gamma$ is given by

$$a \propto \exp(\Lambda\tau), \quad b \propto \exp\{-\Lambda\tau + c\exp(3\Lambda\tau)\} , \tag{36}$$

where Λ and c are arbitrary positive constants and the new time-coordinate τ is introduced via the equation

$$d\tau = V^{-1} dt\,. \tag{37}$$

The scale of the fifth dimension increases rapidly after its contraction stops. This pyrgon-dominated era is followed by the isotropic-expansion era, i.e., $\rho_P \simeq \rho_\gamma/3$ and $a \propto b \propto t^{2/5}$.

In order to avoid the rapid expansion of the extra dimension, the annihilation of pyrgons should become active before the pyrgon-dominated era sets in. From Eq. (15), indeed, ρ_P turns out to remain negligibly small in comparison to $\Gamma_E{}^2$ if it is so at the instant $t=t_*$. Then after $t=t_*$ the photon energy density ρ_γ tends to be dominant on the dynamical evolution. For the period $\rho_P \ll \rho_\gamma$ the increase of ρ_γ due to the energy flow ρ_P becomes so small that ρ_γ evolves adiabatically ($\rho_\gamma \propto a^{-4} b^{-1}$). We obtain the solution of Eq. (33) for the photon-dominated era as

$$a = a_0 t^{1/2}, \quad b = b_0 t^{-1/2} + c_0\,, \tag{38}$$

where a_0, b_0 and c_0 are arbitrary positive constants. Equation (38) shows that the Chodos-Detweiler universe evolves into the Friedmann one with a static extra dimension ($b \simeq c_0$). We denote the instant of the transition by t_s, which will be estimated by the equation

$$\Gamma_E{}^2(t=t_s) \simeq \bar{G}\rho_\gamma(t=t_s > t_*)\,, \tag{39}$$

that is,

$$t_s/t_* \simeq (\bar{G} t_*{}^2 \rho_{P*})^{-2}\,. \tag{40}$$

For the Friedmann era the annihilation of pyrgons stops, since the fifth dimension does not shrink. The contribution of relic pyrgons to the dynamical evolution is gradually recovered. If we expect a sufficiently long Friedmann era, their sufficient annihilation during the Chodos-Detweiler era is necessary. This leads to the condition

$$\bar{G} t_*{}^2 \rho_{P*} \ll 1\,, \tag{41}$$

which is identical with the requirement that at $t=t_*$ the pyrgon-dominated era is not established yet, that is,

$$\Gamma_E{}^2(t=t_*) \gg \bar{G}\rho_{P*}\,. \tag{42}$$

Unless condition (41) is satisfied, the Chodos-Detweiler era will terminate before the annihilation of pyrgons becomes remarkable. Unfortunately, Eq. (41) implies an unacceptable situation. Using Eq. (13), we can reduce Eq. (41) to the form

$$\zeta \bar{G} m_*{}^2 \gg t_*\,. \tag{43}$$

The pyrgon mass m_* at $t=t_*$ will be smaller than R^{-1}. (R is the present radius of the fifth dimension.) Hence, from Eq. (5), we have a constraint for the parameter ζ of the annihilation cross section,

$$\zeta \gg \frac{R}{t_{\mathrm{Pl}}} \cdot \frac{t_*}{t_{\mathrm{Pl}}} > 1\,, \tag{44}$$

where t_{Pl} is the Planck time defined by $t_{\mathrm{Pl}} = G^{1/2}$. Our approach based on the Chodos-Detweiler model universe also needs a too large value of ζ.

We could not give any resolution of the pyrgon problem pointed out by Kolb and Slansky. The main reason for the failure of our approach is that in the framework of the five-dimensional general relativity (i.e., Eq. (33)) the pyrgon energy density can govern the dynamical behaviour of the extra dimension. If pyrgons are so abundant that their annihilation becomes effective, then their energy density can stop the contraction of the extra dimension to prevent the further progress of the annihilation process. This situation is represented by the incompatibility between Eqs. (13) and (42). Some higher-dimensional theories of supergravity may claim that classical bosonic fields which induce spontaneous compactification can drive a contraction of the extra dimensions irrespective of the pyrgon abundance. If so, we need not worry about condition (42), and in the framework of the higher-dimensional theories our picture for the annihilation process of pyrgons may survive. The kinetic treatment of pyrgons also should be reconsidered. If we introduce a field-theoretical treatment for the pyrgon modes of quantum disturbances, their contribution to the energy-momentum tensor may be different from the one given by Eq. (18). This may delay the time when the extra dimension begins to expand. Further development of the analysis is needed.

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