A. La Rosa

Complex Variable, Addition of Waves by the Phasors Method, Light-Matter Interaction

COMPLEX NUMBERS

Addition, multiplication, reciprocal number

Euler's representation $z = a + ib = Ae^{i\theta}$

REPRESENTATION of TRAVELING HARMONIC WAVES in COMPLEX VARIABLE: PHASORS

The concept of phasors

Phasors are complex numbers (they are not vectors)

Addition of (real) waves using phasors

Waves as the real components of phasors

Graphic interpretation

Adding waves of the same frequency and wavelength

Example: Addition of two waves. Calculation of magnitude and phase

Generalization to add many waves.

LIGHT-MATTER INTERACTION

Resonant Absorption

Short-lived excited states. Return to equilibrium by re-emission of light or by thermal energy dissipation.

Line spectra from gases

The Doppler Effect and spectral line broadening

The Doppler effect and laser cooling (optical molasses)

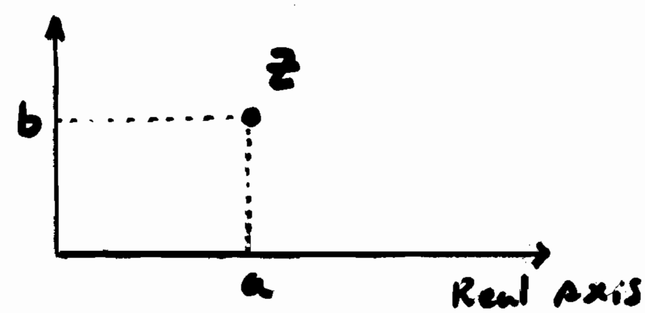
Analogy between "electronic excitations in an atom" and the motion of a mechanically forced oscillator"

COMPLEX NUMBERS (Section 2.5)

$z = a + ib$

where a, b are reals

$i^2 = -1$



Conjugate of $z = a - ib \equiv z^*$

Magnitude of $z = |z| = \sqrt{a^2 + b^2}$

OPERATIONS:

$z_1 = a_1 + ib_1$

$z_2 = a_2 + ib_2$

Addition

$z_1 + z_2 = (a_1 + a_2) + i(b_1 + b_2)$

Multiplication

$z_1 z_2 = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

Reciprocal number

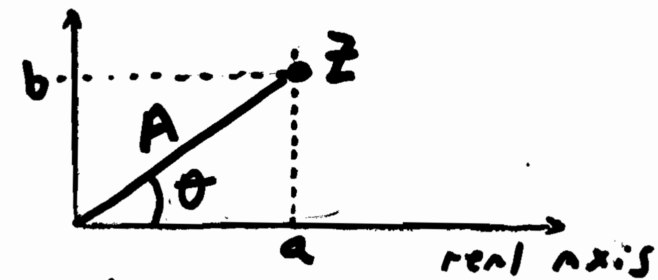
$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib}\,\frac{a-ib}{a-ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$$

$z = a + ib$

It is called the REAL part

the COMPLEX part

•



• The Euler's formula

Let $z = A(\cos\theta + i\sin\theta) = z(\theta)$

Notice

$$\frac{dz}{d\theta} = A(-\sin\theta + i\cos\theta)$$

$$= i(A\cos\theta + i\sin\theta) = iz$$

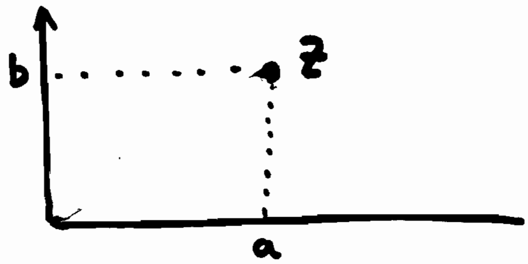
We recall that, when working with real numbers:

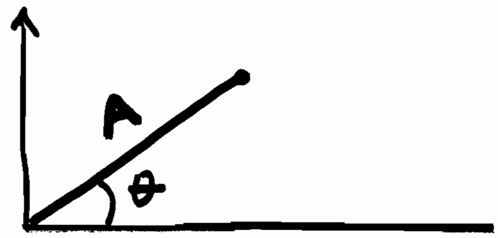
$$\frac{dAe^{\alpha\theta}}{d\theta} = \alpha(Ae^{\alpha\theta})$$

So, to the effects of derivations (and integrations) the complex variable z behaves as,

$$z = A(\cos\theta + i\sin\theta) = Ae^{i\theta}$$ EULER's formula

So, we treat these two expressions as indistinguishables





$$\boxed{z = a + i\,b}$$

$$A = (a^2 + b^2)^{1/2} \qquad \theta = \tan^{-1}\frac{b}{a}$$

$$\boxed{z = A(\cos\theta + i\,\sin\theta)}$$

To the effects of derivatives (and integrals) we have found,

$$\boxed{z = A e^{i\theta}} \qquad \text{Euler's formula}$$

Further check of Euler's formula:

↳ Is it valid for $\frac{1}{z}$?

that is, it is $\frac{1}{z} = \frac{1}{A} e^{-i\theta}$?

Indeed,

Since $\frac{1}{z} = \frac{1}{a+ib} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$

then

$$\frac{1}{z} = \frac{A\cos\theta}{A^2} - i\frac{A\sin\theta}{A^2} = \frac{1}{A}\underbrace{(\cos\theta - i\sin\theta)}_{e^{-i\theta}}$$

$$= \frac{1}{A}e^{-i\theta}$$

Thus,

$$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{Ae^{i\theta}} = \frac{e^{-i\theta}}{A}$$

As demonstrated above

↳ What about $z_1 z_2$?

If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$

Is $z_1 z_2 = A_1 A_2 e^{i(\theta_1+\theta_2)}$?

Representation of a traveling harmonic wave in complex variable: PHASORS

$$\text{wave} = \psi(x,t)$$
$$= A\cos(kx - \omega t + \alpha)$$

where

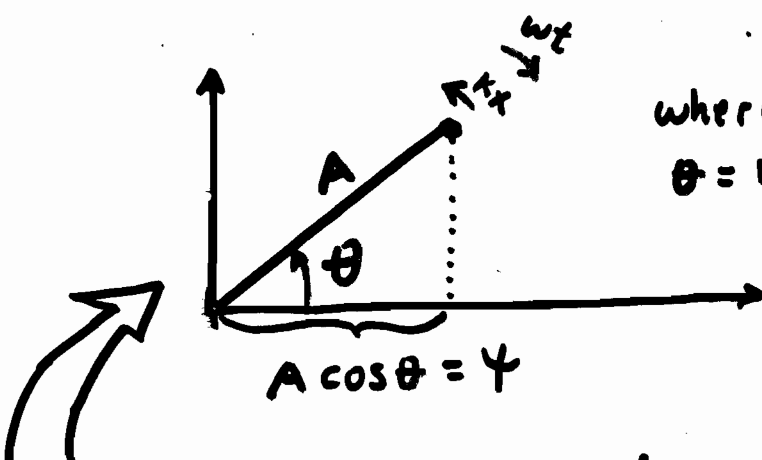
$$k = \frac{2\pi}{\lambda}$$
$$\omega = \frac{2\pi}{T}$$

Let's call $\theta = kx - \omega t + \alpha$

thus, $\boxed{\psi(x,t) = A\cos(\theta)}$ This is a REAL measurable wave

Notice, ψ can be interpreted as the REAL component of the associated complex variable

$$z = Ae^{i\theta}$$



where
$\theta = kx - \omega t + \alpha$

The rotating segment and it associated angle together constitute a PHASOR

$$\psi = \text{Real}\{z\}$$

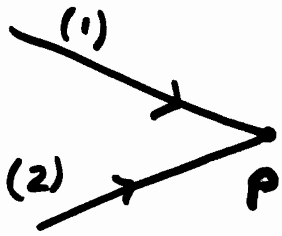
ADDITION of WAVES using the method of PHASORS

SECTION 2.6
7.1.3

Given,

wave 1 $\quad \psi_1 = A_1 \cos(k_1 x - \omega_1 t + \alpha_1) = A_1 \cos\theta_1$

wave 2 $\quad \psi_2 = A_2 \cos(k_2 x - \omega_2 t + \alpha_2) = A_2 \cos\theta_2$



At point P the waves add up.

A convenient way to add waves is the method of phasor

$$\psi_1 = A_1 \cos\theta_1 = \text{Real}\{\underbrace{A_1 e^{i\theta_1}}\} \longleftarrow \text{phasor } z_1$$

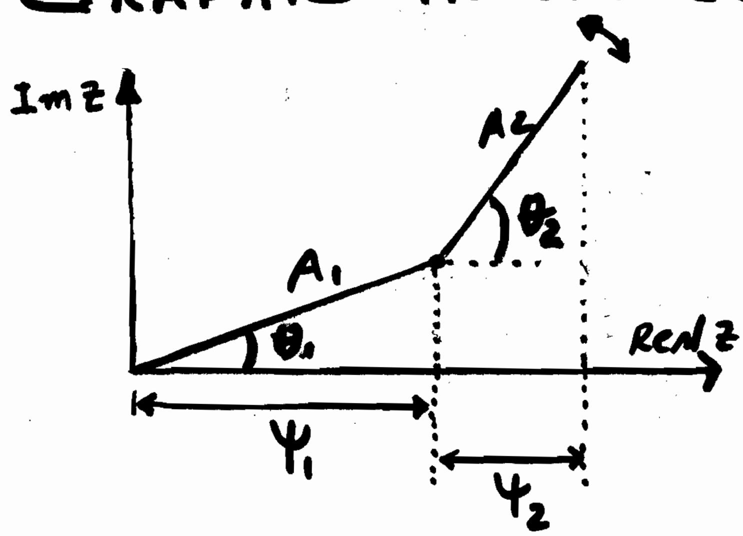
$$\psi_2 = A_2 \cos\theta_2 = \text{Real}\{\underbrace{A_2 e^{i\theta_2}}\} \longleftarrow \text{phasor } z_2$$

$$\psi_1 + \psi_2 = \text{Real}\{\underbrace{A_1 e^{i\theta_1} + A_2 e^{i\theta_2}}\}$$

working with the complex variable is sometimes easier to find a total sum.

Once the total sum (a complex number) is found, its real part will be $\psi_1 + \psi_2$

GRAPHIC INTERPRETATION

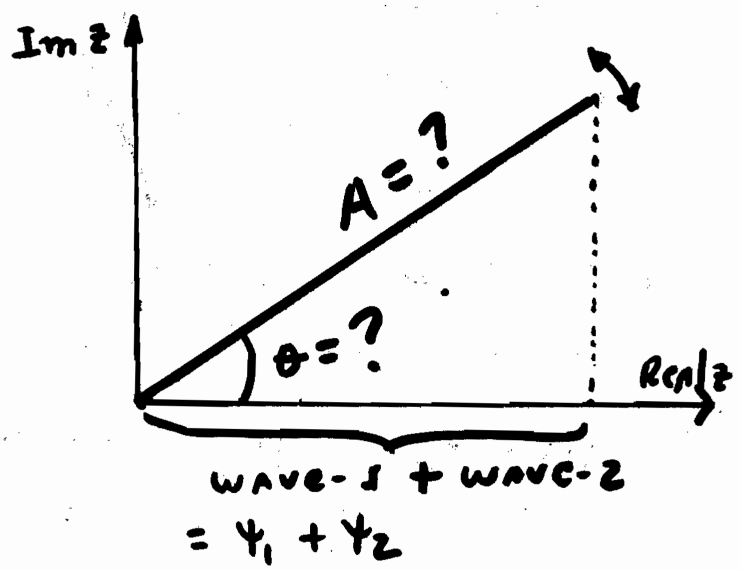


$\theta_1 = k_1 x - \omega_1 t + \alpha_1$

$\theta_2 = k_2 x - \omega_2 t + \alpha_2$

$\psi_1 = A_1 \cos\theta_1$

$\psi_2 = A_2 \cos\theta_2$



<u>STEP 1</u>: Construct the corresponding phasors

$z_1 = A_1 e^{i\theta_1}$

$z_2 = A_2 e^{i\theta_2}$

(see top graph)

Notice also:
As x and t change, the phasors rotate

<u>STEP 2</u> Work in the complex world

$z_1 + z_2 = A_1 e^{i\theta_1} + A_2 e^{i\theta_2}$

which will have the form

$= A e^{i\theta}$ (see graph above)

<u>STEP 3</u> $\psi_1 + \psi_2$ = Horizontal component of $\{A e^{i\theta}\}$

CASE: Adding harmonic waves of the same frequency and the same wavelength

$$\psi_1 = A_1 \cos(kx - wt + \alpha_1) = A_1 \cos(\theta_1)$$

$$\psi_2 = A_2 \cos(kx - wt + \alpha_2) = A_2 \cos(\theta_2)$$

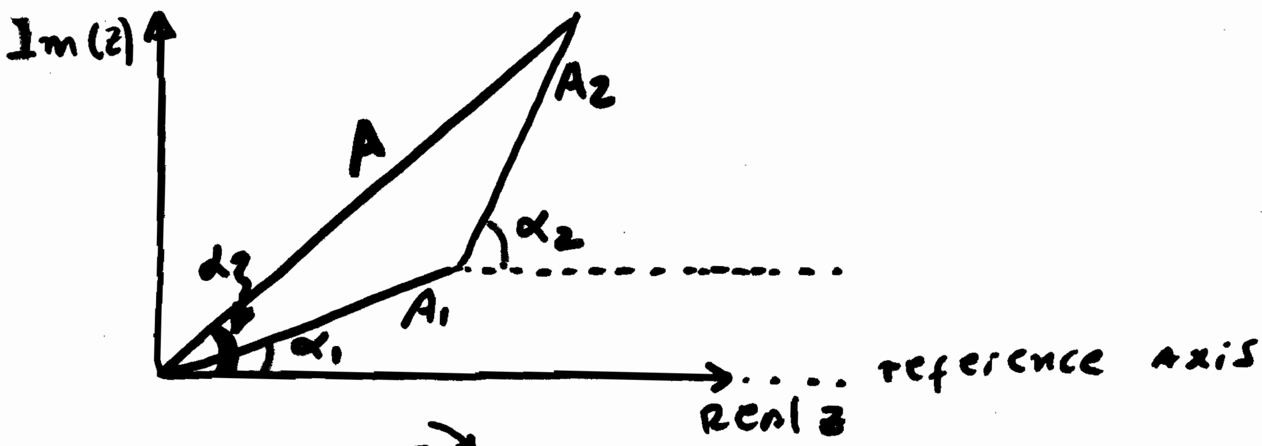
$\Rightarrow$

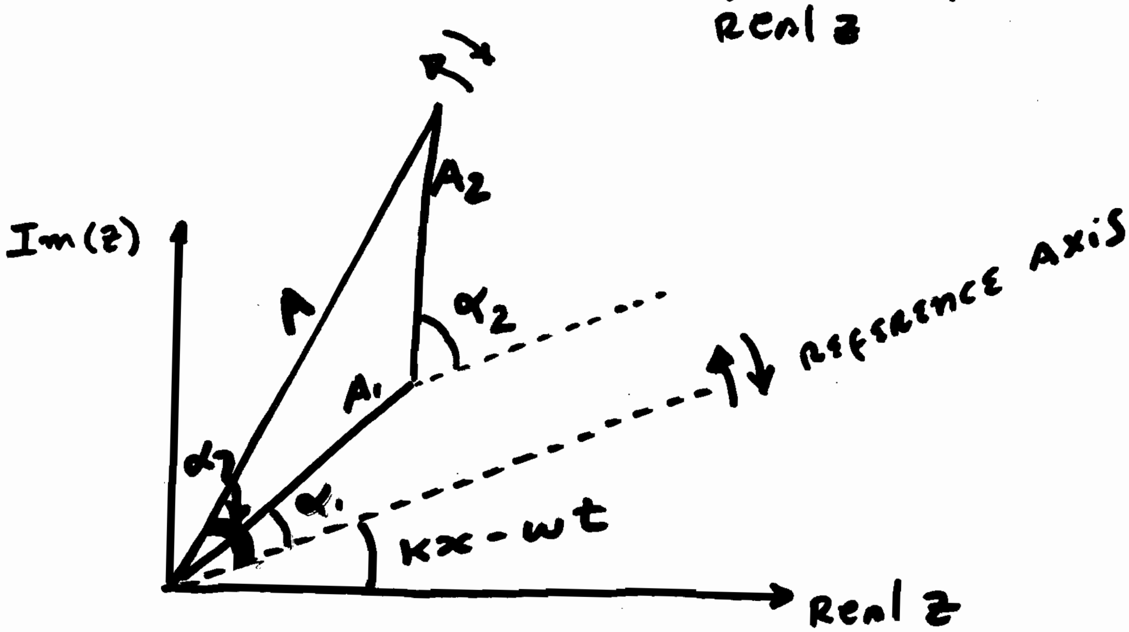
$$z_1 = A_1 e^{i\theta_1} = A_1 e^{i\alpha_1} e^{i(kx - wt)}$$

$$z_2 = A_2 e^{i\theta_2} = A_2 e^{i\alpha_2} e^{i(kx - wt)}$$

$$z_1 + z_2 = \left(A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2}\right) e^{i(kx - wt)}$$

It should be clear from here that all the phasors, z_1, z_2 and $z_1 + z_2$ rotate synchronously (graphic interpretation).





So, the problem of adding

$$\psi_1 = A_1 \cos(kx - wt + \alpha_1)$$

$$\psi_2 = A_2 \cos(kx - wt + \alpha_2)$$

reduces to adding

$$A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2}$$

This sum will have the form

$A e^{i\alpha}$ A and α to be determined in terms of A_1, A_2, α_1 and α_2

key recipe

Finding A and α

$$A e^{i\alpha} = A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2}$$

- Taking the magnitude square (Remember $|z|^2 = z z^*$, $z = Ae^{i\alpha} \Rightarrow z^* = A e^{-i\alpha}$)

$$|A e^{i\alpha}|^2 = (A_1 e^{i\alpha_1} + A_2 e^{i\alpha_2})(A_1 e^{-i\alpha_1} + A_2 e^{-i\alpha_2})$$

$$= A_1^2 + A_2^2 + A_1 A_2 \left[e^{i(\alpha_2 - \alpha_1)} + e^{-i(\alpha_2 - \alpha_1)}\right]$$

$$A^2 = A_1^2 + A_2^2 + 2A_1 A_2 \cos(\alpha_2 - \alpha_1)$$ ← solution

- Using Euler's identity

$$A \sin\alpha = A_1 \sin\alpha_1 + A_2 \sin\alpha_2$$

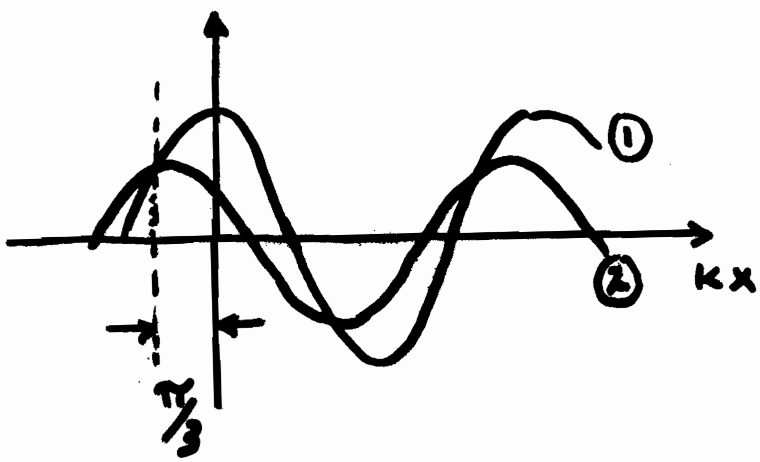
$$A \cos\alpha = A_1 \cos\alpha_1 + A_2 \cos\alpha_2$$

$$\tan\alpha = \frac{A_1 \sin\alpha_1 + A_2 \sin\alpha_2}{A_1 \cos\alpha_1 + A_2 \cos\alpha_2}$$ ← solution

EXAMPLE Two waves travel along a string in the same direction

$$\psi_1 = 4\,mm \cos(kx - \omega t)$$

$$\psi_2 = 3\,mm \cos(kx - \omega t + \tfrac{\pi}{3})$$



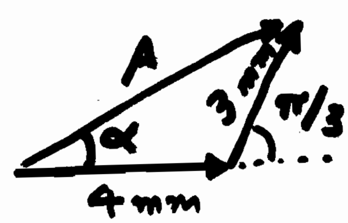
Find $\psi_1 + \psi_2$

SOLUTION

$\psi_1 \longrightarrow 4\,mm$

$\psi_2 \longrightarrow 3\,mm\, e^{i\pi/3}$

REAL   COMPLEX VARIABLE



$$4\,mm + 3\,mm\, e^{i\pi/3} = A e^{i\alpha}$$

From the phasors graph

$$A^2 = 16 + 9 + 2 \times 4 \times 3 \cos \pi/3 = 37$$

$$\Rightarrow A = 6.1\,mm$$

$$\tan\alpha = \frac{3 \sin \pi/3}{4 + 3\cos\pi/3} = 0.47$$

$$\Rightarrow \alpha = 0.44\ rad$$

Thus, we have found

$$4\,\text{mm} + 3\,\text{mm}\, e^{i\pi/3} = 6.1\,\text{mm}\, e^{i\,0.44\,\text{rad}}$$

Accordingly

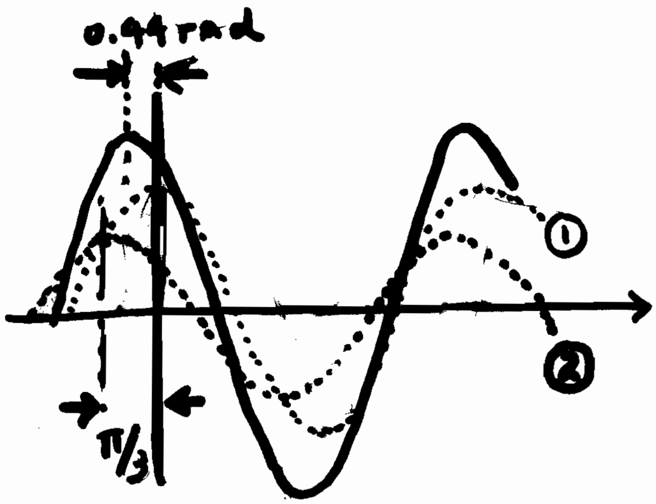
$$z_1 + z_2 = 6.1\,\text{mm}\, e^{i\,0.44\,\text{rad}}\; e^{i(kx - wt)}$$

$$= 6.1\,\text{mm}\, e^{i(kx - wt + 0.44)}$$

Back to the real world

$$\psi_1 + \psi_2 = \text{Real}\{z_1 + z_2\}$$

$$= 6.1\,\text{mm} \cos(kx - wt + 0.44\,\text{rad})$$

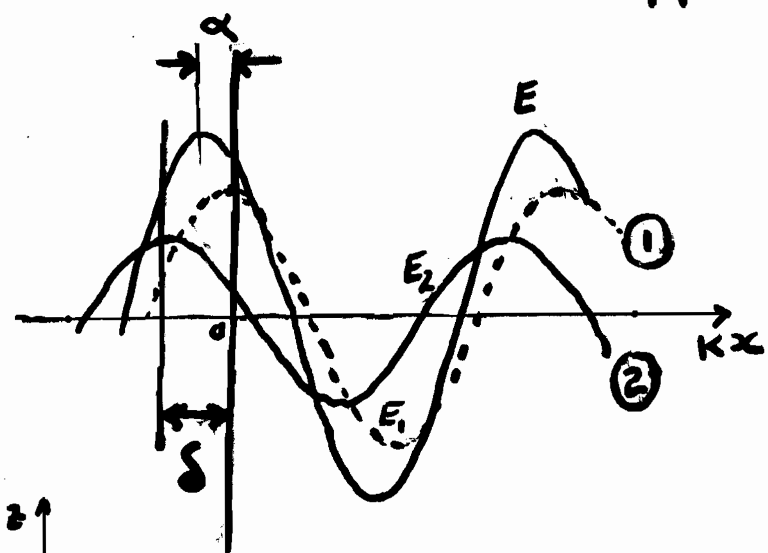


Notice:

The phase 0.44 rad of the resulting wave is between the phase values of the component waves.

$$0 < 0.44\,\text{rad} < \pi/3$$

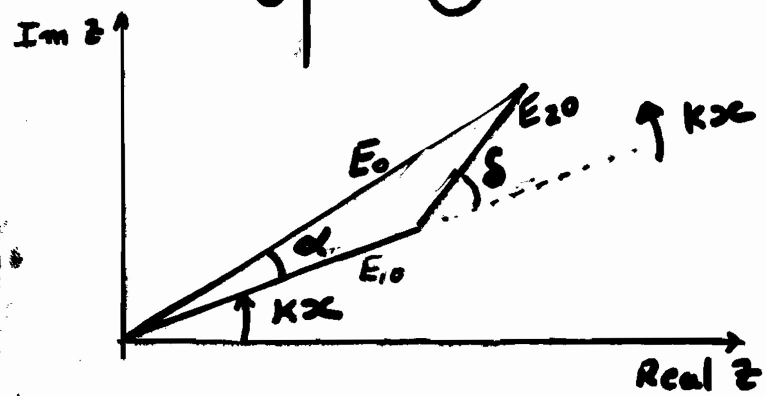
Example: Addition of two waves having a phase difference δ



$E_1 = E_{10} \cos kx \qquad (\alpha_1 = 0)$

$E_2 = E_{20} \cos(kx + \delta)$

δ is given



- while
  $z_1 = E_{10}\, e^{ikx}$
  $z_2 = E_{20}\, e^{i(kx+\delta)}$,
  it is enough to consider the addition of
  $E_{10} + E_{20} e^{i\delta}$ ←
- The sum will be written as $E_0 e^{i\alpha}$ ←

SOLUTION:

$$E_0^2 = E_{10}^2 + E_{20}^2 + 2E_{10}E_{20}\cos\delta$$

$$\operatorname{Tan}\alpha = \frac{E_{20}\sin\delta}{E_{10} + E_{20}\cos\delta}$$

$$z_1 + z_2 = \left[E_0 e^{i\alpha}\right] e^{i(kx-\omega t)} = E_0\, e^{i(kx-\omega t+\alpha)}$$

then

$$E_{10}\cos kx + E_{20}\cos(kx+\delta) = E_0 \cos(kx - \omega t + \alpha)$$

Phasors Method

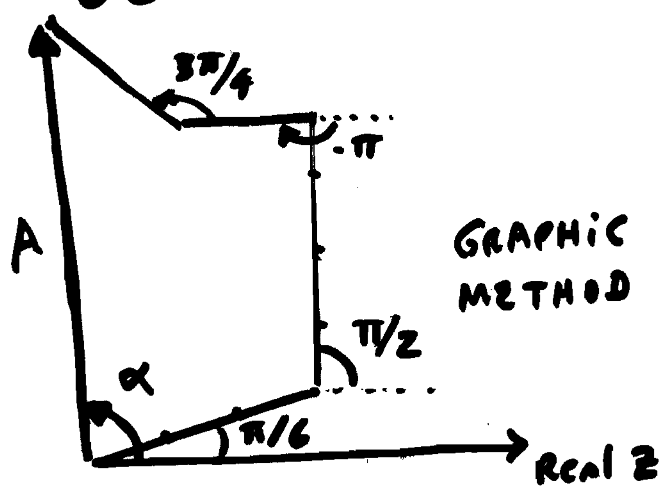
Generalization to add many waves

$$\psi = 3\cos(\omega t + \pi/6) + 4\cos(\omega t + \pi/2) + 2\cos(\omega t - \pi) + 2.5\cos(\omega t + \tfrac{3}{4}\pi)$$

Since all the waves have the same frequency, we get to add

$$3e^{i\pi/6} + 4e^{i\pi/2} + 2e^{-i\pi} + 2.5\,e^{i\frac{3}{4}\pi} =$$

$$= Ae^{i\alpha}$$



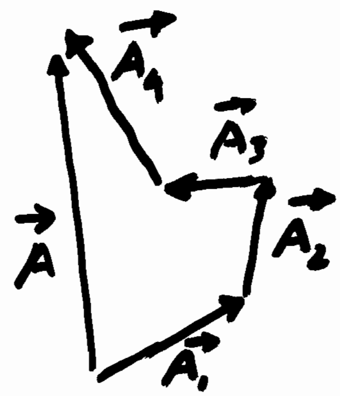
GRAPHIC METHOD

Thus

$$\psi = A\cos(\omega t + \alpha)$$

An analytical expression for A, in terms of the amplitudes A_i of the component waves, can be obtained if we consired the phasors $A_i e^{i\alpha_i}$ as if they were vectors.

$$|\vec{A}|^2 = \vec{A} \cdot \vec{A}$$
$$= \left(\sum_i \vec{A}_i\right) \cdot \left(\sum_i \vec{A}_i\right)$$



$$A^2 = \sum_i \vec{A}_i^2 + \sum_i \sum_{j \neq i} \vec{A}_i \cdot \vec{A}_j$$

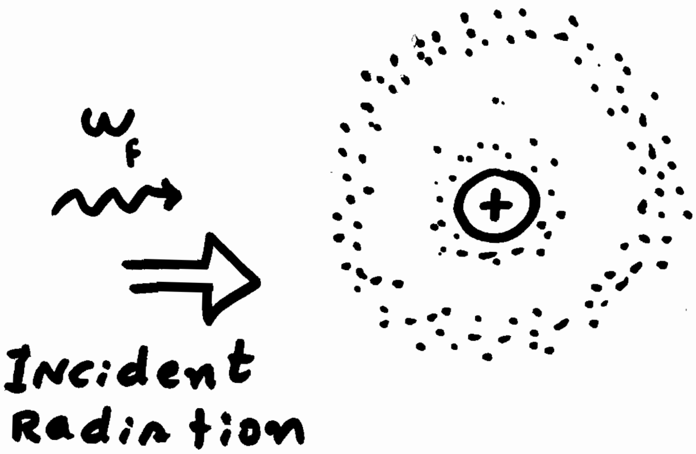
$$A^2 = \sum A_i^2 + \sum_i \sum_{j \neq i} A_i A_j \cos(\alpha_i - \alpha_j) \quad \leftarrow$$

and

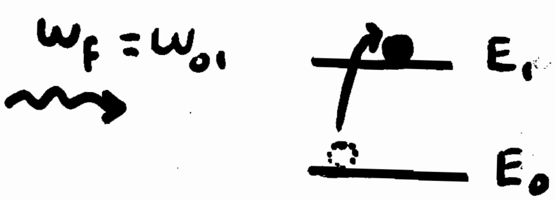
$$\tan \alpha = \frac{\sum_i A_i \sin \alpha_i}{\sum_i A_i \cos \alpha_i} \quad \leftarrow$$

Light-Matter Interaction

Section 3.4.4



If ω_f matches anyone of the atom's resonant frequencies (ω_{01}, ω_{02}, ω_{03}, ...), then there will be a strong absorption (resonant absorption)

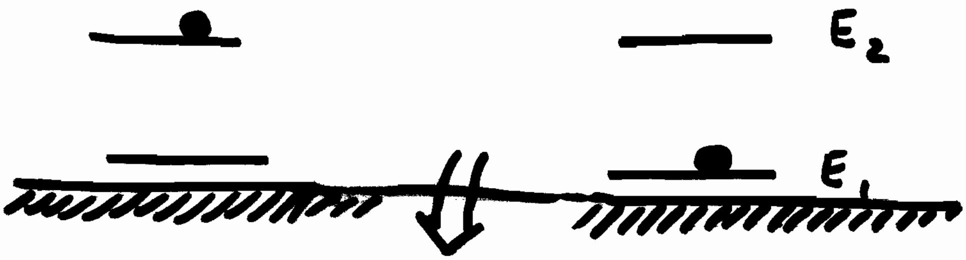


$$E_1 - E_0 = \hbar \omega_{01}$$

EXCITED STATE
- unstable
- short lived (10^{-8} s, in gases)

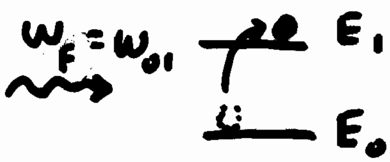
Re-emission of light

Alternativly



Energy ($E_2 - E_1$) converted to thermal energy Q through interatomic collissions (in solid and liquids)

Summary



leads to a) Light re-emission

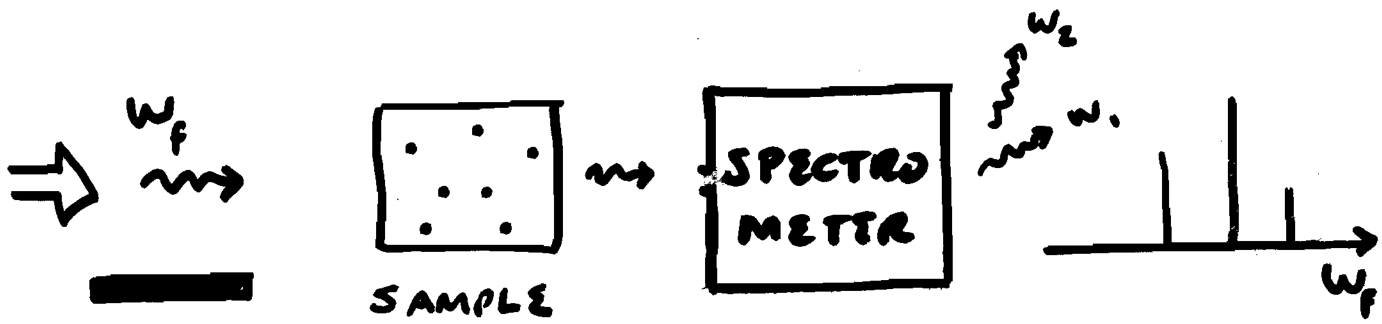
electromagnetic energy $\hbar\omega_{01}$

RESONANT ABSORPTION

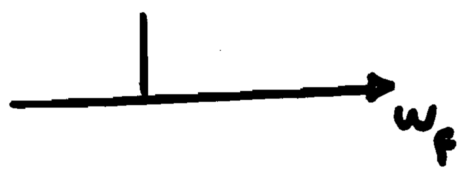
or

b)

thermal energy Q



<u>For GASES</u> (at low pressure)

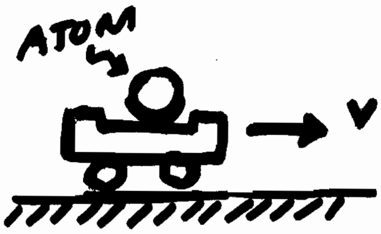


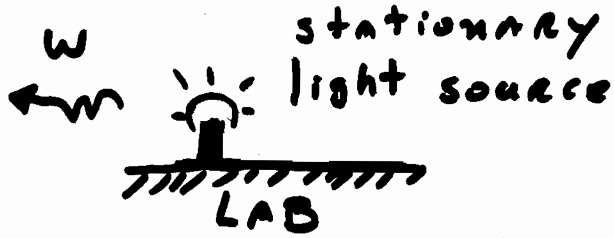
Atoms do not interact much

- Spectra consists of sharp lines.

BUT, there is always some frequency broadening due to:
- atomic collisions
- atomic motion (Doppler effect)

The Doppler effect





w' = frequency of the radiation, perceived by the atom in motion

w: frequency of the radiation, measured with respect to the stationary reference in the laboratory

$$w' = \frac{\sqrt{1+\frac{v}{c}}}{\sqrt{1-\frac{v}{c}}}\, w$$

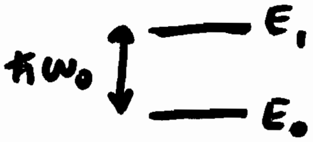
For small v/c: $w' \approx \dfrac{1+\frac{1}{2}\frac{v}{c}}{1-\frac{1}{2}\frac{v}{c}}\, w \approx \left(1+\frac{1}{2}\frac{v}{c}\right)\left(1-\frac{1}{2}\frac{v}{c}\right)^{-1} w$

$$\approx \left(1+\frac{1}{2}\frac{v}{c}\right)\left(1+\frac{1}{2}\frac{v}{c}\right) w \approx \left(1+\frac{v}{c}\right) w$$

$$w' \approx \left(1+\frac{v}{c}\right) w \quad \text{or} \quad \Delta w = (w'-w) = \frac{v}{c} w$$

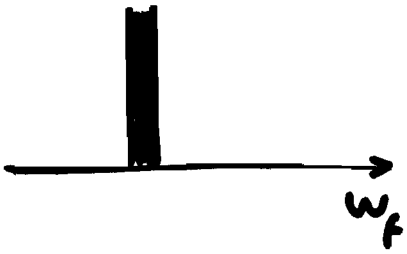
Doppler Effect and Spectral Line Broadening

If w_0 is the resonant absorption frequency of a stationary atom,



the frequency of the incident radiation has to be tuned a bit lower in order to be absorbed by atoms in motion

Different atoms in the gas, having different components of velocity in the direction of the incident light, have different apparent resonance frequencies.



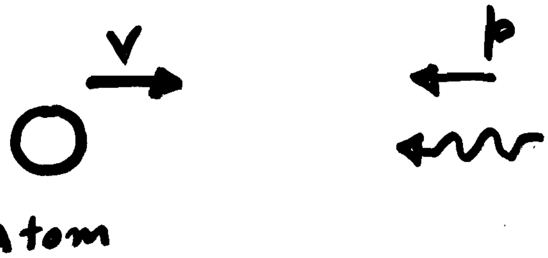
For the 632.8 nm (4.74×10^{14} Hz) He-Ne laser transition, the Doppler broadening is about 1500 MHz.

Doppler Effect and Laser Cooling

SECTION 3.4.4.

Photons do not have mass, still they do carry momentum: $p = \frac{h}{\lambda}$.

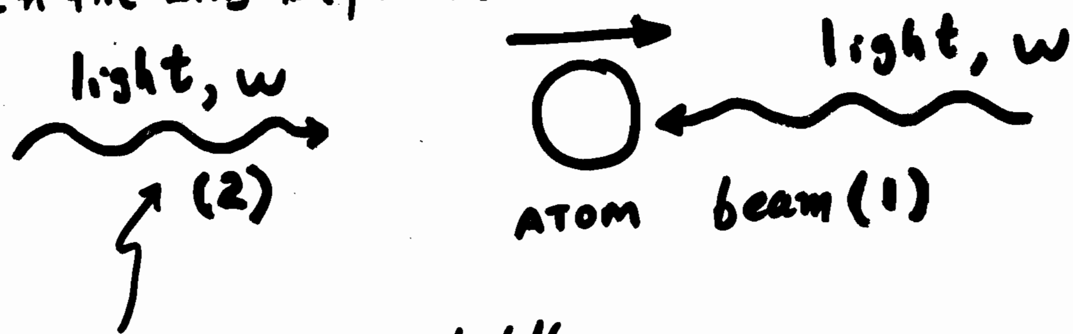
Photons can, therefore, exert forces on atoms. Hence, photons can be used to slow down atom (cooling)



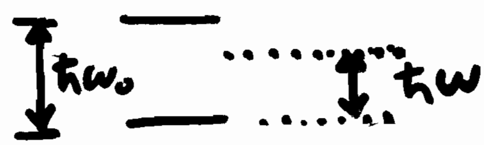
If the atom absorbs one photon
V changes by only 1 cm/s

A more efficient way to slow down atoms uses the Doppler Effect:

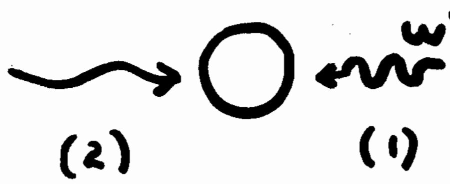
. In the LAB Reference

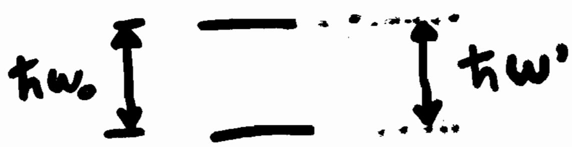


frequency ω slightly lower than ω_0



. IN the ATOM REFERENCE



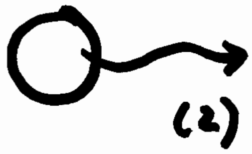


From the atom perspective, beam (1) has a frequency ω' higher than ω. (The opposite occurs for beam (2))

CONSEQUENCE:

Atom is more likely to absorb photons from beam (1)

Atom experience a net force opposite to its motion (it slows down).



Atom is transparent to beam (2)

The re-emission of the absorbed light (1) pushes the atom in an umpredictable direction

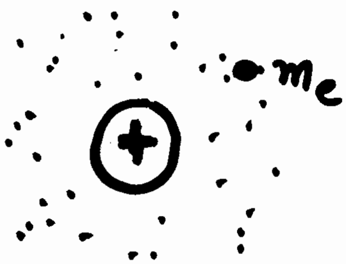
But, since the whole processs repeats many times, the average recoil amounts to ZERO.

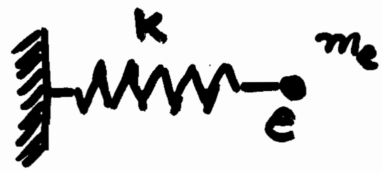
Hence, the NET EFFECT is a SLOW DOWN of the atom.

Because, in this scheme, light acts as a "viscous force", the beams are called optical molasses.

Temperatures as low as 240×10^{-6} K have been obtained through optical molases (and 3×10^{-6} K under certain conditions).

- Analogy between
  electronic excitation in an atom
  and
  the motion of a mechanically forced oscillator.

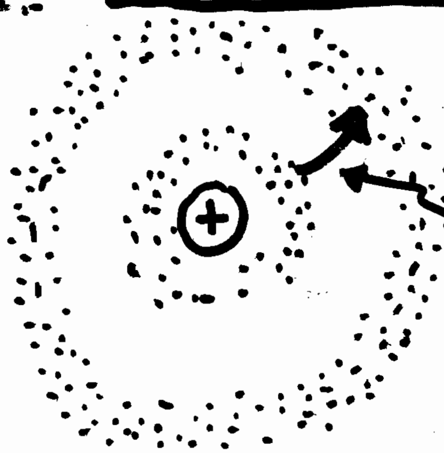




$m_e = 9.1 \times 10^{-31}$ Kg

But, how to choose K ?

1. RESONANT ABSORPTION (Section 3.4.4)

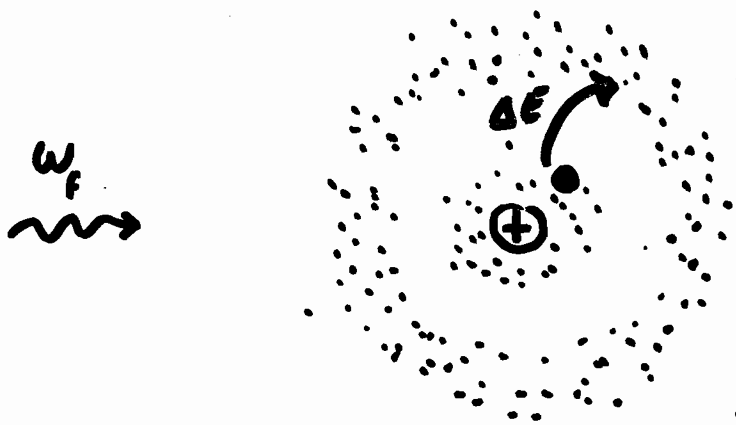


transition between energy levels

$\Delta E = \hbar \omega_0$

$$\omega_{res} = \sqrt{\frac{K}{m_e}}$$

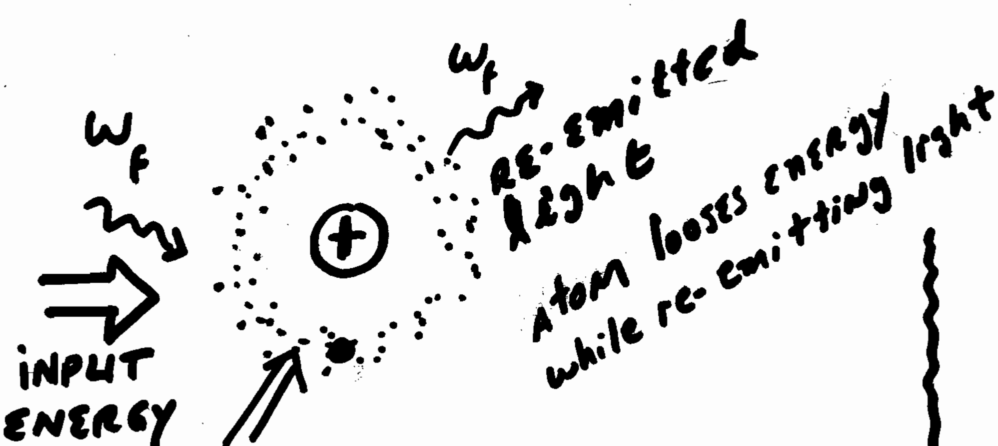
$$K \equiv m_e \omega_0^2$$



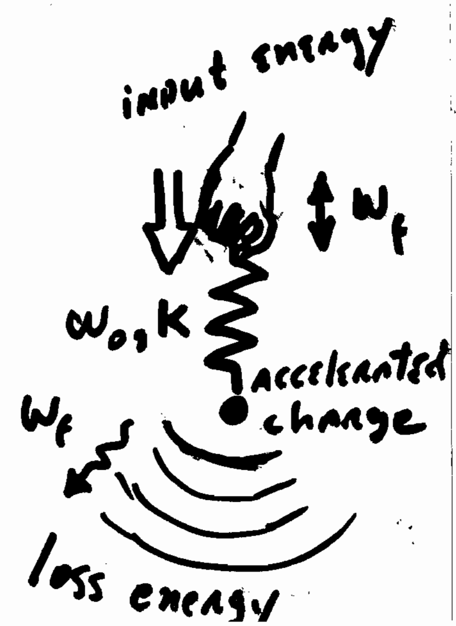
$\Delta E = \hbar \omega_0$

If $\omega_f = \omega_0$ the incident photon is absorbed by the atom

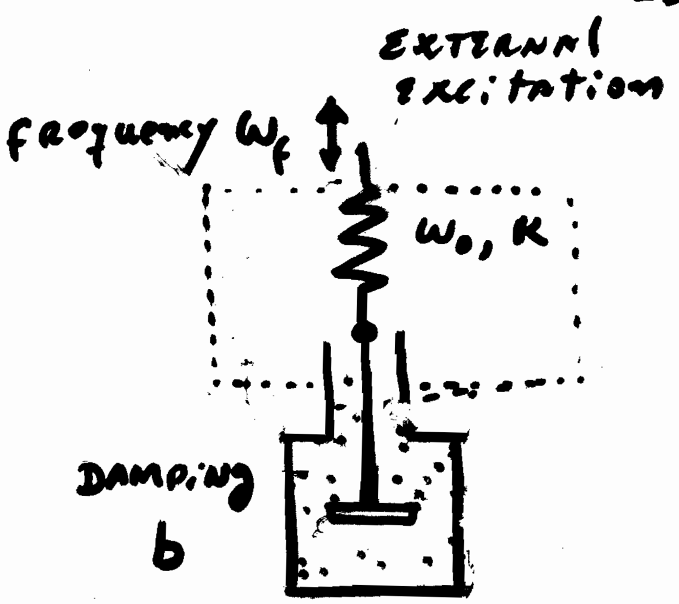
Resonant absorption



Atom driven by the external radiation of frequency ω_f



The emission of electromagnetic waves by the accelerated charge, can be considered as a channel of "dissipative" energy



Light-atom interaction by a resonance absorption process, can therefore be modelled by a equation similar to a harmonically forced damped oscillator.

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + Kx = F_0\cos(\omega_f t) \qquad (1)$$

electron's mass

x is the "position" of the electron

Related to the rate at which the accelerated electron re-emit light.
$b = b(\omega_f, m, \dots)$

We'll give later a specific expression of b

$K = m\omega_0^2$

ω_0 is one of the atom's discrete resonant absorption frequencies.

$F_0 = -eE_0$

external electric field amplitude

Solving Eq (1) using complex variable.

First, we apply a trick:

Let $y = y(t)$ be the solution of

$$m \frac{d^2 y}{dt^2} + b \frac{dy}{dt} + Ky = F_0 \sin(\omega_f t) \qquad (2)$$

Multiply (2) by i

$$m \frac{d^2}{dt^2}(iy) + b \frac{d}{dt}(iy) + K(iy) = F_0 \left[i \sin(\omega_f t) \right] \qquad (3)$$

(1) + (3) gives

$$\boxed{m \frac{d^2 z}{dt^2} + b \frac{dz}{dt} + Kz = F_0 e^{i\omega_f t} \qquad \text{where} \quad z(t) \equiv x(t) + y(t)} \qquad (4)$$

If we find a solution in (4), let's say $z(t)$ then a solution to Eq (1) can be obtained by taking $x(t) = \text{Real}\{z(t)\}$

Solving Eq ④

Since the driving force is harmonic of frequency ω_f, we guess the displacement $z(t)$ will also be harmonic of frequency ω_f. Potentially, there may be phase difference between $F_0 e^{i\omega_f t}$ and $z(t)$.

Thus, we proposse a solution of the form,

$$z(t) = A\, e^{i(\omega_f t + \phi)} \qquad ⑤$$

where A and ϕ can depend on ω_f and other parameters

⑤ in ④ gives,

$$\left(-\omega_f^2 + \omega_0^2 + i\frac{b}{m}\omega_f\right) A e^{i\phi} = \frac{F_0}{m}$$

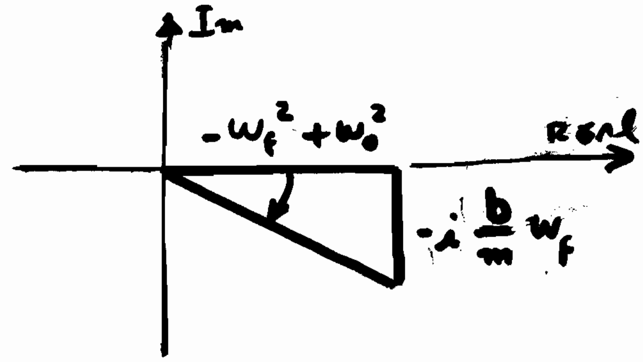
$$\Rightarrow \quad A e^{i\phi} = \frac{1}{-\omega_f^2 + \omega_0^2 + i\frac{b}{m}\omega_f}\,\frac{F_0}{m}$$

$$= \frac{(-\omega_f^2 + \omega_0^2) - i\frac{b}{m}\omega_f}{(-\omega_f^2 + \omega_0^2)^2 + \left(\frac{b}{m}\omega_f\right)^2}\,\frac{F_0}{m}$$

Notice, the complex number in the numerator can be expressed as,

$$(-\omega_f^2+\omega_0^2)+i\frac{b}{m}\omega_f =$$

$$=\left[(-\omega_f^2+\omega_0^2)^2+\left(\frac{b}{m}\omega_f\right)^2\right]^{1/2} e^{i\tan^{-1}\frac{-\frac{b}{m}\omega_f}{-\omega_0^2+\omega_f^2}}$$

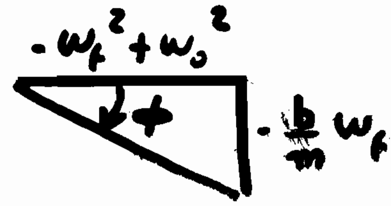


$\Rightarrow$

$$A e^{i\phi} = \frac{F_0/m}{\left[(-\omega_f^2+\omega_0^2)^2+\left(\frac{b}{m}\omega_f\right)^2\right]^{1/2}}\, e^{i\tan^{-1}\frac{-\frac{b}{m}\omega_f}{-\omega_f^2+\omega_0^2}} \tag{5}$$

$$A(\omega_f) = \frac{F_0/m}{\left[(-\omega_f^2+\omega_0^2)^2+\left(\frac{b}{m}\omega_f\right)^2\right]^{1/2}}$$

$$\phi(\omega_f) = \tan^{-1}\frac{-\frac{b}{m}\omega_f}{-\omega_f^2+\omega_0^2}$$



Summary.

$$m\frac{d^2 z}{dt^2} + b\frac{dz}{dt} + Kz = F_0 e^{i\omega_f t}$$

this equation admits solutions of the form

$$z(t) = A e^{i(\omega_f t + \phi)}$$

where $A = A(\omega_f)$ and $\phi = \phi(\omega_f)$

are given in expression 5

Accordingly,

The solution to Eq

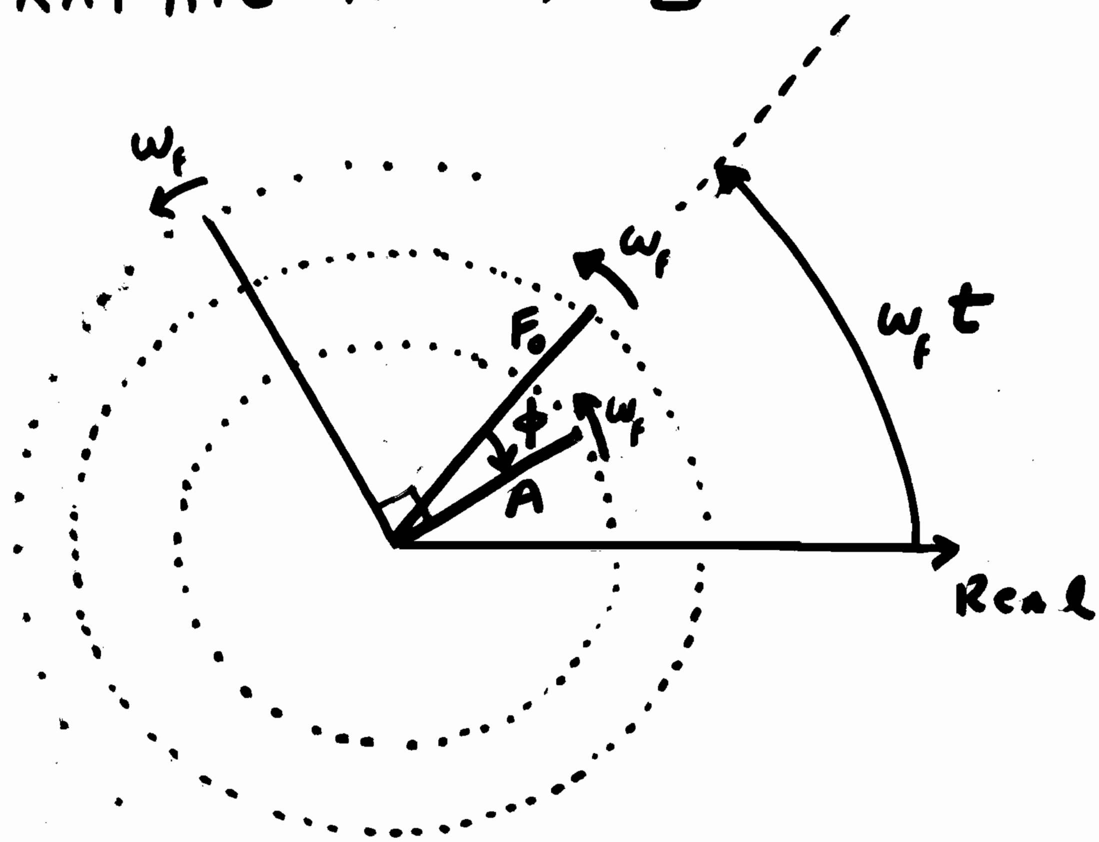
$$m\frac{d^2 x}{dt^2} + b\frac{dx}{dt} + Kx = F_0 \cos(\omega_f t)$$

is given by

$$x(t) = \text{Real}\{z(t)\}$$

$$= A(\omega_f)\cos(\omega_f t + \phi)$$

GRAPHIC ANALYSIS



Phasor force : $F_0 e^{i\omega_f t}$

Phasor position: $A e^{i(\omega_f t + \phi)}$

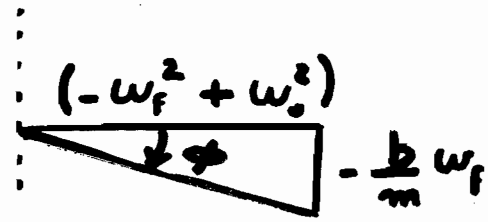
Phasor velocity : $A\omega_f e^{i(\omega_f t + \phi + \pi/2)}$

Notice :

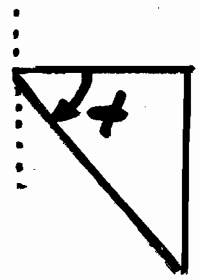
Since ϕ is always negative, the position phasor always lags the force phasor.

Variation of ϕ

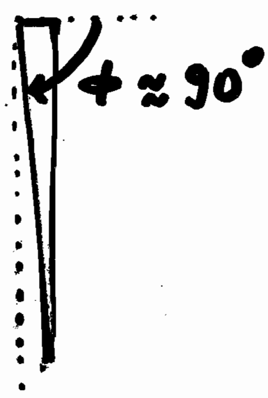
At low ω_f



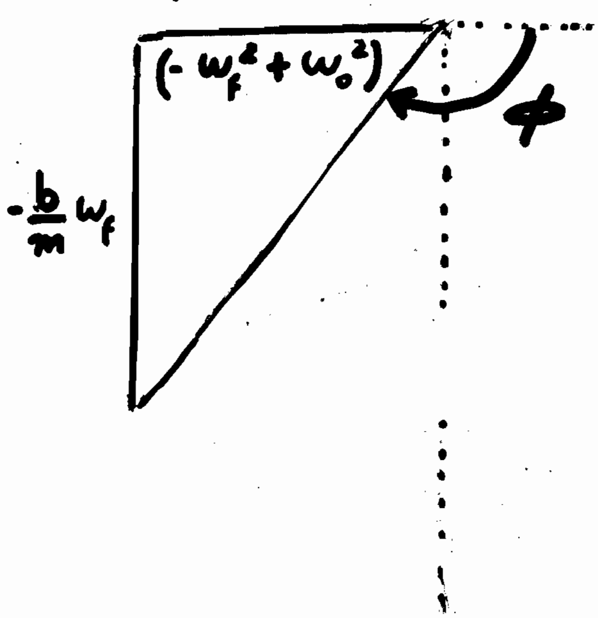
At higher ω_f
but $\omega_f < \omega_0$

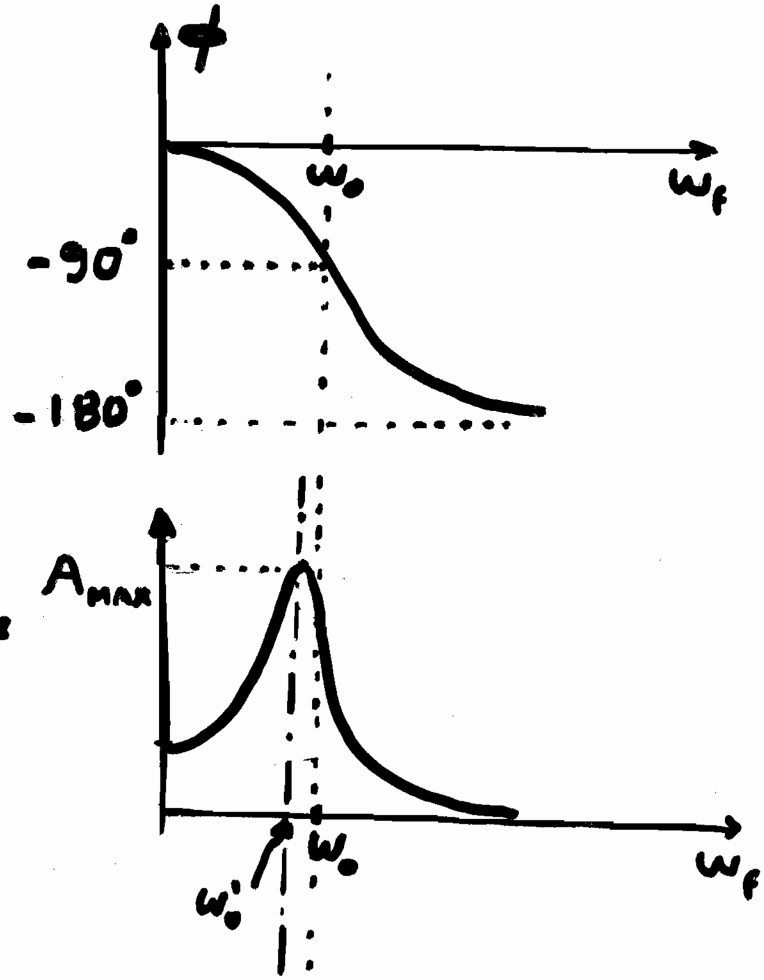


$\omega_f \approx \omega_0$



At $\omega_f > \omega_0$





$$A_{MAX} = \frac{F_0/K}{\frac{1}{Q}\left(1-\frac{1}{4Q^2}\right)^{1/2}}$$

FOR $Q \gg 1$

$$A_{MAX} \approx Q\,\frac{F_0}{K}$$

$$\omega_0' = \sqrt{\omega_0^2 - \frac{1}{2}\left(\frac{b}{m}\right)^2}$$

$$= \omega_0\sqrt{1-\frac{1}{2Q^2}} \qquad \left(Q \equiv \frac{m}{b}\omega_0\right)$$

$\omega_0' \approx \omega_0$ for high Q