C.1 Einstein's Postulate of Quantized Radiation

Planck figured out that the spectral energy density require for equilibrium was

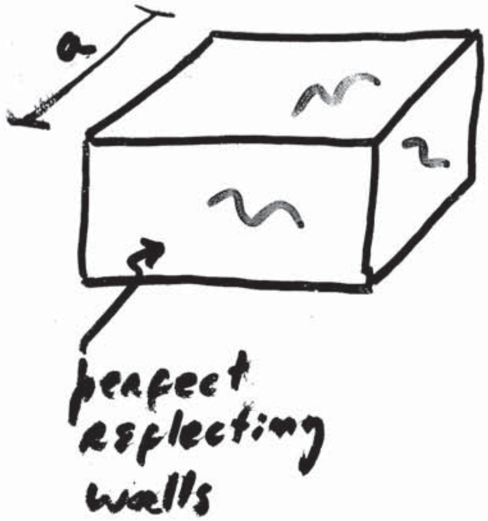
$$I(\omega)\, d\omega = \frac{\hbar \omega^3}{\pi^2 c^2} \frac{1}{e^{\hbar\omega/kT} - 1} d\omega$$

For Planck: - the matter was quantized
(the oscillator can take energy only in discrete amounts)
- light, once radiated, spreads out like a wave.

For Einstein - Radiant energy is quantized as well.

Calculation of the *Electromagnetic Energy Density U* inside the cavity

First, let's count the number of electromagnetic modes inside a cavity



$$E(x) = E_0 \sin\left(\frac{2\pi}{\lambda}x\right) \sin(\omega t)$$

where $\lambda \dfrac{\omega}{2\pi} = c$

The value of ω specifies the electromagnetic mode.

<u>Many</u> modes can exist inside the cavity (a cube of side "a"). But, not <u>all</u> values of ω are allowed.

The requirement that $E = 0$ at $x = a$ implies

$$\frac{2\pi}{\lambda} a = n \times \pi$$

$$\Longleftrightarrow$$

$$\frac{1}{c}\frac{\omega}{2\pi} = \frac{n}{2a} \quad \text{or} \quad \boxed{\omega = 2\pi\, n \frac{c}{2a} \equiv \omega_n}$$

$n = 1, 2, 3, \ldots$

modes allowed in the cavity

In a given interval $\Delta\omega$ how many modes there exists?

Answer: $\frac{1}{2\pi}\frac{2a}{c}\Delta\omega$ (one dimensional "cube")

In a three dimensional cube

$$\left[\frac{1}{2\pi}\times\frac{2a}{c}\right]^3\times\left[\frac{4\pi\omega^2}{8}\,\Delta\omega\right] =$$ Number of modes allowed (in a cube cavity of side "a") that have a frequency between ω and $\omega+\Delta\omega$

$$= \frac{V}{2\pi^2 c^3}\,\omega^2\,\Delta\omega$$, where $V \equiv a^3$

Since for each mode, 2 polarizations are possible,

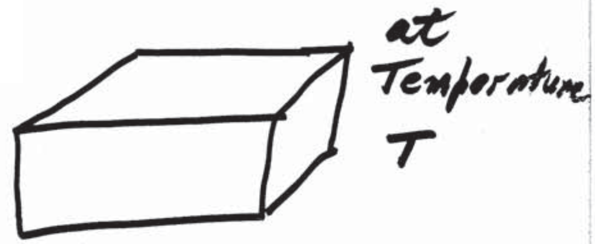
$$= \frac{V}{\pi^2 c^3}\,\omega^2\Delta\omega$$

Thus, we find that

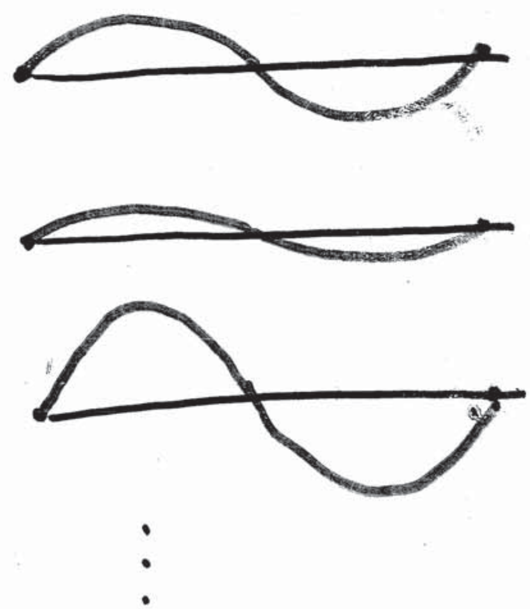
$$\frac{\omega^2}{\pi^2 c^3}\Delta\omega \equiv N(\omega)\Delta\omega$$

Number of modes per unit volume that have frequencies between ω and $\omega+\Delta\omega$

What is the energy stored in each mode?



Since the perfectly reflecting walls are kept at temperature T, we can picture a given mode interchanging energy with the walls. That is, sometimes the mode will have small amplitudes, other times large amplitudes, etc. But it will have an average amplitude, and, hence, an average energy. Let's call it W.



(W could depend on T and the frequency ω of the mode)

Accordingly, the energy density (J/m^3) in the cavity, contributed by the modes whose angular frequency lie between ω and $\omega+\Delta\omega$

$$N(\omega)\, W\, \Delta\omega =$$

$$= \frac{\omega^2}{\pi^2 c^3} \underbrace{W} \Delta\omega \equiv u(\omega)\, \Delta\omega$$

$$u(\omega) = \frac{\omega^2}{\pi^2 c^3} \underbrace{W}_{\text{average energy of the mode of frequency } \omega} \qquad \textbf{(1)}$$

Notice the similarity between expression (1), and the epression for the sprectral light intensity obtained in the previous section, expression (32)
that reads

$$I(\omega) = \omega^2\, W/(3\pi^2 c^2)$$

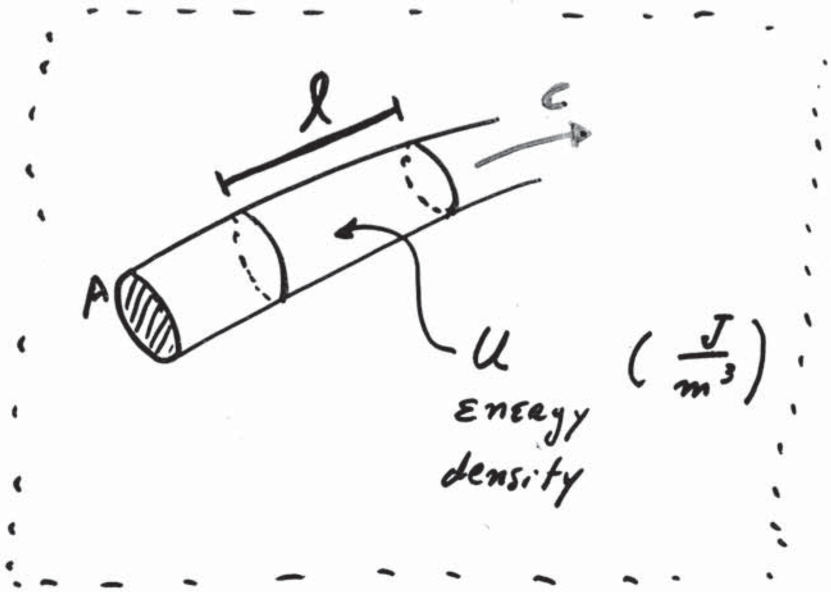
$$I(\omega) = \frac{\omega^2}{\pi^2 c^2} \underbrace{W}_{\text{average energy of the oscillator (atom) at temperature } T} \tag{2}$$

$I(\omega)\,d\omega$ ↳ has the units of intensity or energy flux (or power) density $\frac{J}{m^2 s}$

↳ this contribution is from waves having an angular frequency between ω and $\omega + d\omega$

INFRARED visible UV $\longrightarrow \omega$

Relationship between $\mathcal{U}$ and I



Take a look to the "cylinder" of lenght ℓ

and cross section area A:

$\Delta u =$ Total energy inside the cylinder $= \mathcal{U} \, \ell \, A$

Δu will cross a section A in $\Delta t = \frac{\ell}{c}$ seconds

$$\frac{\text{Travsling energy}}{\text{per unit time}} = \frac{\Delta u}{\Delta t} = \mathcal{U} A c$$

So

$$I = \frac{\text{travsling energy}}{\text{per unit area} \times \text{per unit time}} = \mathcal{U} c$$

How tocalculate the average energy of a mode of frequency ω?

To answer this question, we should realize the analogy between these two problems:

i) a charged oscillator of natural frequency ω in equilibrium with radiation in a box at temperature T (as studied in the previous sections)

and

ii) an electromagnetic mode ω in thermal equilibrium inside a box at temperature T.

For the first case, Planck derived a formula. to calculate the average energy of the oscillator.

For the second case Einstein extended Planck's ideas about quantized energies of an oscillator to the electromagnetic radiation. Yes, the energy of the radiation would be also quantized.

Einstein used, then, Plank's formula

to calculate the average energy of a electromagnetic mode

$$\underbrace{W(\omega)}_{\text{average energy of a mode of frequency } \omega} = \frac{\hbar\omega}{e^{\frac{1}{KT}\hbar\omega} - 1} \equiv W_{Einstein}$$

[that is, it does make no differance wheter we have an atom (or oscillator) of natural frequency ω, or an electromagnetic mode of frequency ω.]

$$U(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{\hbar\omega}{e^{\frac{1}{KT}\hbar\omega} - 1}$$

$$I(\omega) = \frac{\hbar\omega^3}{\pi^2 c^2} \frac{1}{e^{\frac{1}{KT}\hbar\omega} - 1}$$

Einstein used this result to obtain new information about radiation-matter interaction.