Scale invariance, unimodular gravity and dark energy

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Abstract

We demonstrate that the combination of the ideas of unimodular gravity, scale invariance, and the existence of an exactly massless dilaton leads to the evolution of the universe supported by present observations: inflation in the past, followed by the radiation and matter dominated stages and accelerated expansion at present. All mass scales in this type of theories come from one and the same source.

1. Introduction

The origin of different mass scales in particle physics is a mystery. The masses of quarks, leptons and intermediate vector bosons come from the vacuum expectation value (vev) of the Higgs field; the dimensionful parameters like the QCD scale Λ_{QCD} or the scales related to the running of all other dimensionless couplings of the Standard Model (SM) are believed to have nothing to do with the Higgs vev. Newton's gravitational constant provides yet another mass scale, very different from typical particle masses of the SM. The Higgs mass itself – where does it come from?

Is it possible that *all* these mass scales originate from one and the same source?[1] Indeed, it is not difficult to construct, on the *classical level*, a theory containing a new singlet field χ, which gives masses to all particles and fixes Newton's constant. Having in mind the SM extended by 3 light right-handed singlet fermions, the νMSM of [1, 2] (this theory – Neutrino Minimal

SM – unlike the SM, can explain neutrino masses and oscillations, dark matter and baryon asymmetry of the universe), one can write the Lagrangian realizing this idea in the following form[2]

$$\mathscr{L}_{\nu\text{MSM}} = \mathscr{L}_{\text{SM}[\text{M}\to 0]} + \mathscr{L}_G + \frac{1}{2}(\partial_\mu\chi)^2 - V(\varphi,\chi)$$
$$+ \left(\bar{N}_I i\gamma^\mu\partial_\mu N_I - h_{\alpha I}\bar{L}_\alpha N_I\tilde{\varphi} - f_I\bar{N}_I^c N_I\chi + \text{h.c.}\right), \quad (1)$$

where the first term is the SM Lagrangian without the Higgs potential, N_I (I=1,2,3) are the right-handed singlet leptons, φ and L_α ($\alpha = e,\mu,\tau$) are the Higgs and lepton doublets respectively, $h_{\alpha I}$ and f_I are the matrices of Yukawa coupling constants. The scalar potential is given by

$$V(\varphi,\chi) = \lambda\left(\varphi^\dagger\varphi - \frac{\alpha}{2\lambda}\chi^2\right)^2 + \beta(\chi^2 - \chi_0^2)^2, \quad (2)$$

and the gravity part is

$$\mathscr{L}_G = -\left(\xi_\chi\chi^2 + 2\xi_h\varphi^\dagger\varphi\right)\frac{R}{2}, \quad (3)$$

where R is the scalar curvature. We will only consider positive values for ξ_χ and ξ_h, for which the coefficient

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[1] We will refer to this possibility as "no-scale scenario".

[2] This expression is identical to the one in [3], Section 8, but uses different notations.

in front of the scalar curvature is positive, whatever values the scalar fields take. This is the Lagrangian of "induced gravity" going back to Refs. [4, 5] (see also [6, 7] in the νMSM context).

For positive λ and β the theory with potential (2) possesses a ground state[3]. It corresponds to the fields sitting at the minimum of the potential, i.e. $\chi = \chi_0$, $h = h_0$ with $h_0^2 = \frac{\alpha}{\lambda}\chi_0^2$ and a constant metric describing flat space-time. The field values at the potential minimum can be related to the Planck scale as $M_P^2 = \xi_\chi \chi_0^2 + \xi_h h_0^2$, $M_P = 2.44 \times 10^{18}$ GeV. Physics in this theory does not depend on a specific value of χ_0 – all dimensionful parameters are proportional to it – and only dimensionless ratios can be measured.

Although the aim of having one source for all mass scales is achieved by construction of the Lagrangian (1) (we stress that we are still discussing the classical theory) the solution is not satisfactory: the absence of explicit mass terms for the Higgs field and for singlet fermions, and the absence of a gravity scale along with the introduction of the dimensionful parameter χ_0, required to realize the scenario, are ad hoc and do not follow from any symmetry principle.

The symmetry that forbids (on the classical level) the appearance of any dimensionful parameters is well known – it is the dilatational symmetry. Under dilatations, the scalar and fermionic fields change as $\phi(x) \to \sigma^n \phi(\sigma x)$ ($n = 1$ for scalars and $n = 3/2$ for fermions), while the metric transforms as $g_{\mu\nu}(x) \to g_{\mu\nu}(\sigma x)$. The action (1) is invariant under this symmetry, provided $\chi_0 = 0$, leading to the absence of all dimensionful parameters.

From now on we will require that a dilatation invariant theory should possess a ground state. Since this requirement is essential for our model, we will further discuss it in section 4. For the dilatation invariant theory to contain massive singlet and doublet fermions, the ground state should be such that $\chi \neq 0$ and $h \neq 0$. The only way to achieve this is to set $\beta = 0$. Thus, the no-scale scenario can only be realized if $\beta = 0$. In this case the potential (2) acquires the flat direction $h^2 - \frac{\alpha}{\lambda}\chi^2 = 0$, and the theory contains one exactly massless particle η – a certain mixture of the singlet χ and the Higgs field. The requirement $\beta = 0$ therefore leads to a theory with spontaneously broken scale invariance, where η appears as a Goldstone boson[4].

Equivalent arguments were given in [8].

The theory (1) with $\beta = 0$ (from now on only this choice of parameters will be considered) is rather peculiar[5]: not only is the physics independent of the value of $\chi \neq 0$ in the ground state, but the ground state is infinitely degenerate. The question "Who gives the mass to the dilaton ?" does not arise. It is massless, and the chain of questions "Who gives mass to whom?" terminates.

Not to any surprise, the classical scale-invariant theory constructed in this way does not contain a cosmological constant Λ. So, if we confront it with cosmological observations, it seems to fail, since the universe is in accelerated expansion, which requires the presence of dark energy with an equation of state close to that of the cosmological constant. This conclusion is certainly correct for standard General Relativity (GR), associated with the action

$$S_E = \int \sqrt{-g} d^4x \, \mathcal{L}_{\nu\text{MSM}} \;, \qquad (4)$$

where g is the determinant of the metric.

The aim of this Letter is to show that the situation is completely different if general relativity in (4) is replaced by Unimodular Gravity (UG). UG is a very modest modification of Einstein's theory: it adds a constraint $g = -1$ to the action principle defined by eq. (4) [9, 10, 11, 12, 13, 14, 15, 16]. UG is invariant under diffeomorphisms which conserve the 4-dimensional volume element. It contains the same number of dynamical degrees of freedom (massless graviton) as Einstein's theory. To the best of our knowledge, the consequences of scale-invariant UG with massless dilaton have not been considered previously.

The relevance of UG for the cosmological constant problem was realized long time ago [9, 10, 11, 12, 13]. If $g = -1$, adding a constant Λ to the Einstein-Hilbert action does not change the equations of motion. Still, the Λ problem is not solved, since the cosmological constant shows up again, but now as an initial condition for cosmological evolution in UG. We will see that for our case of a scale-invariant theory together with UG, the initial conditions lead to a non-trivial run-away effective potential for the dilaton rather than to a cosmological constant, and thus to dynamical dark energy. Moreover, it will turn out that both inflation and accel-

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[3] By a ground state we mean a constant solution of the equations of motion including gravity. The existence of such a ground state could be essential for a consistent quantization of the theory.

[4] Spontaneous breaking of the dilatational symmetry also occurs if the potential has a flat direction either along $\chi = 0$ or along $h = 0$. However, both cases are unsatisfactory. The first one corresponds to

a theory with no massive singlet fermions, whereas the second one is a theory with no electroweak symmetry breaking.

[5] We stress that this theory is *not invariant* under local conformal transformations. Conformal invariance requires the specific values for ξ_χ and ξ_h, $\xi_\chi = \xi_h = -\frac{1}{6}$.

erated expansion of the universe can be explained on the same footing.

The Letter is organized as follows. In Section 2 we show that scale-invariant UG with a massless dilaton is equivalent (on the classical level) to Einstein's theory with zero cosmological constant and a peculiar potential, the magnitude of which is fixed by initial conditions for all the fields. We continue in Section 3 with a discussion of the evolution of the universe in our model. The requirements a full quantum theory should satisfy for our findings to remain valid are formulated in Section 4. Section 5 is a summary of the results.

2. Scale-invariant unimodular gravity : the classical theory

In this Letter we want to bring together several a priori separate ideas. One of them is unimodular gravity, which has appeared many times in the literature [9, 10, 11, 13, 14, 15, 16]. In unimodular gravity one reduces the dynamical components of the metric $g_{\mu\nu}$ by one, imposing that the metric determinant $g \equiv \det(g_{\mu\nu})$ takes some fixed constant value.[6] Conventionally one takes $|g| = 1$, hence the name. Fixing the metric determinant to one is not a strong restriction, in the sense that the family of metrics satisfying this requirement can still describe all possible geometries. For pure gravity, things are very simple and well known. The analog of the Einstein-Hilbert Lagrangian for unimodular gravity is

$$\mathcal{L}_{EH} = -M_P^2 \frac{\hat{R}}{2} . \tag{5}$$

Writing quantities with a hat, like $\hat{R}$, we mean that they depend on the metric with $g = -1$. These quantities transform like tensors under the group of volume preserving diffeomorphisms, i.e. coordinate transformations $x^\mu \to \xi^\mu(x)$, with the condition $\hat{\nabla}_\mu \xi^\mu = 0$.[7] Doing variations of this action that keep the metric determinant fixed, since it is not a dynamical variable, yields the equations of motion

$$\hat{G}_{\mu\nu} = -\Lambda \hat{g}_{\mu\nu} , \tag{6}$$

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[6] In principle one can fix $g = a(x)$, where $a(x)$ is a fixed external field, and the results are the same.

[7] It is important to distinguish UG from theories constructed on the simple requirement of invariance under restricted coordinate transformations $x^\mu \to \xi^\mu(x)$, with $\hat{\nabla}_\mu \xi^\mu = 0$ (sometimes called TDiff gravity) [12, 17, 18]. The latter theories contain in general a third dynamical degree of freedom for the metric. In addition, they have field equations, which depend on the choice of coordinates. In UG the constraint on the metric determinant is essential. It is responsible for the absence of a third metric degree of freedom and guarantees that the equations of motion do not depend on the coordinate choice.

where Λ is an integration constant given by initial conditions. Now, these are also the equations for standard Einstein gravity with an added cosmological constant, for a choice of coordinates such that the metric determinant is equal to one, which is always possible [9]. Therefore, the two theories are classically equivalent, except that in the standard theory the cosmological constant appears in the action, whereas in unimodular gravity it is an integration constant. It has been shown [9, 13, 14, 15] that if one adds a matter sector that couples minimally to gravity, and therefore has a covariantly conserved energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$, the application of UG also results in the appearance of an integration constant that plays the role of an additional cosmological constant. We now want to find a similar statement for a more general case, in particular the one in which Newton's constant is generated dynamically.

The action for unimodular gravity and any other fields, which couple to gravity in an arbitrary way, has the following functional dependence:

$$\Sigma = \int d^4x \mathcal{L}(\hat{g}_{\mu\nu}, \partial\hat{g}_{\mu\nu}, \Phi, \partial\Phi), \tag{7}$$

where Φ stands for all non-gravitational fields. If we want to derive the equations of motion for this theory, we have to vary the action keeping the constraint on the determinant. This is done using the Lagrange multiplier method. We add an additional variable, whose equation of motion will be the constraint. So, the following Lagrangian is equivalent to the former one:

$$\tilde{\mathcal{L}} = \underbrace{\sqrt{-g}\Big(\mathcal{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \Phi, \partial\Phi) + \Lambda(x)\Big)}_{A} \underbrace{-\Lambda(x)}_{B} . \tag{8}$$

Here, apart from the usual symmetry requirement $g_{\mu\nu} = g_{\nu\mu}$, $g_{\mu\nu}$ is unconstrained (the initial Lagrangian was multiplied by a factor $\sqrt{-g}$, which does not change the theory because of the unimodular constraint).

The equations of motion are

$$\frac{\delta A}{\delta g_{\mu\nu}} = 0 , \tag{9}$$

$$\frac{\delta A}{\delta \Phi} = 0 , \tag{10}$$

$$\frac{\delta(A+B)}{\delta \Lambda} = 0 = (\sqrt{-g} - 1) . \tag{11}$$

We observe that $\int d^4x A(x)$ is invariant under the full group of diffeomorphisms. The infinitesimal transformations are

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$$g_{\mu\nu} \to g_{\mu\nu} + \delta_\xi g_{\mu\nu} \, ,$$
$$\Phi \to \Phi + \delta_\xi \Phi \, ,$$
$$\Lambda \to \Lambda + \delta_\xi \Lambda \, , \tag{12}$$

where δ_ξ depends on the nature of the fields, i.e. scalar, vector, etc. If, for instance, we take Φ to be a scalar field, the δ_ξ's are given by

$$\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu \, ,$$
$$\delta_\xi \Phi = \partial_\mu \Phi \xi^\mu \, ,$$
$$\delta_\xi \Lambda = \partial_\mu \Lambda \xi^\mu \, . \tag{13}$$

Due to this symmetry, the following relation holds.

$$\int d^4 x \left(\frac{\delta A}{\delta g_{\mu\nu}} \delta_\xi g_{\mu\nu} + \frac{\delta A}{\delta \Phi} \delta_\xi \Phi + \frac{\delta A}{\delta \Lambda} \delta_\xi \Lambda \right) = 0 \, . \tag{14}$$

The coefficients of the first two terms are zero because of the equations of motion and the last coefficient yields $\frac{\delta A}{\delta \Lambda} = \sqrt{-g}$. The equation reduces to

$$\int d^4 x \sqrt{-g} (\partial_\mu \Lambda) \xi^\mu = 0 \, . \tag{15}$$

Since this holds for all possible functions $\xi^\mu(x)$, we can conclude that

$$\partial_\mu \Lambda(x) = 0 \, , \tag{16}$$

and hence that Λ is a constant of motion. Its value can be determined by the field equations together with the initial conditions for all fields. Knowing this, let us again look at the equations (9)

$$\frac{\delta A}{\delta g_{\mu\nu}} = \frac{\delta \{ \sqrt{-g} (\mathscr{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \Phi, \partial \Phi) + \Lambda(x)) \}}{\delta g_{\mu\nu}} = 0 \, .$$

These equations along with the constraint $\sqrt{-g} = 1$ are the field equations for unimodular gravity plus other fields. From (16) we know that Λ is an integration constant. We conclude that the theory given by (7) is classically equivalent to a fully diffeomorphism invariant theory described by the Lagrangian

$$\mathscr{L}_{\text{diff}} = \sqrt{-g} \left(\mathscr{L}(g_{\mu\nu}, \partial g_{\mu\nu}, \Phi, \partial \Phi) + \Lambda \right) , \tag{17}$$

apart from the different ways in which the parameter Λ appears [8]. The quantity Λ plays the role of a cosmological constant in the theory with explicit Planck mass. However, as we will see shortly, this is not the case if Newton's constant is induced dynamically.

We now want to combine the ideas of UG and scale invariance. Considering only the gravitational and the

[8] In [12] the authors presented a proof of the same statement for TDiff theories a using similar type of arguments. However, to our understanding, this proof is in fact not valid for general TDiff theories.

scalar sectors, a general Lagrangian containing scalar fields ϕ_i has the form:

$$\mathscr{L} = -\frac{1}{2} K_{ij} \phi_i \phi_j \hat{R} + \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i - U_{ijkl} \phi_i \phi_j \phi_k \phi_l \, . \tag{18}$$

The result derived above tells us that the solutions of UG with this Lagrangian are equivalent to the solutions of GR with Lagrangian

$$\mathscr{L} = \sqrt{-g} \left(-\frac{1}{2} K_{ij} \phi_i \phi_j R + \frac{1}{2} \sum_i g^{\mu\nu} \partial_\mu \phi_i \partial_\nu \phi_i \right.$$
$$\left. - U_{ijkl} \phi_i \phi_j \phi_k \phi_l - \Lambda \right) . \tag{19}$$

Let us finally add to unimodular gravity and scale invariance the requirement that the scalar potential should have a flat direction. The potential for a theory containing the Higgs field h and an additional scalar field χ is then given by

$$V(h, \chi) = \frac{\lambda}{4} \left(h^2 - \frac{\alpha}{\lambda} \chi^2 \right)^2 \, . \tag{20}$$

So, our requirements lead us to the scalar and gravitational parts of the action (1) with $\beta = 0$ and standard gravity replaced by UG. The corresponding Lagrangian, invariant under all diffeomorphisms (a particular case of (19)), is

$$\mathscr{L} = \sqrt{-g} \left(-\frac{1}{2} (\xi_\chi \chi^2 + \xi_h h^2) R + \right.$$
$$\left. \frac{1}{2} (\partial_\mu \chi)^2 + \frac{1}{2} (\partial_\mu h)^2 - V(h, \chi) - \Lambda \right) . \tag{21}$$

Now, in order to facilitate the physical interpretation of this Lagrangian, we can do a change of variables (conformal or Weyl transformation) of the following type

$$g_{\mu\nu} = \Omega(x)^2 \tilde{g}_{\mu\nu} \, . \tag{22}$$

If we choose Ω such that $(\xi_\chi \chi^2 + \xi_h h^2) \Omega^2 = M_P^2$, the action (21) in terms of the new metric $\tilde{g}_{\mu\nu}$ reads

$$\mathscr{L}_E = \sqrt{-\tilde{g}} \left(-M_P^2 \frac{\tilde{R}}{2} + K - U_E(h, \chi) \right) , \tag{23}$$

and is said to be in the Einstein frame (see e.g. [19]). Here K is a complicated non-linear kinetic term for the scalar fields, given by

$$K = \Omega^2 \left(\frac{1}{2} (\partial_\mu \chi)^2 + \frac{1}{2} (\partial_\mu h)^2 \right) - 3 M_P^2 (\partial_\mu \Omega)^2 \, . \tag{24}$$

In our case where $\xi_\chi, \xi_h > 0$, the kinetic form K is positive-definite, which guarantees the absence of ghosts. The Einstein-frame potential $U_E(h, \chi)$ is given by

$$U_E(h, \chi) = \frac{M_P^4}{(\xi_\chi \chi^2 + \xi_h h^2)^2} [V(h, \chi) + \Lambda] \, , \tag{25}$$

where the parameter Λ is related to initial conditions for scalar fields and gravity and does not depend on space-time coordinates. It is not a cosmological constant but rather the strength of a peculiar potential.

3. Dark energy, inflation and cosmological constant

We now want to analyze the cosmological consequences of the theory (1), working in the Einstein frame. Although the dynamics of the general system described by (23) is very complicated due to the non-canonical form[9] of the kinetic term K, we can gain some insight into the evolution of the system looking at the potential part only.

The phenomenologically interesting domain of parameters, explained below, corresponds to $\xi_h \gg 1$, $\xi_\chi \ll 1$, $\lambda \sim 1$, $\alpha \simeq \lambda \xi_\chi \frac{v^2}{M_P^2} \ll 1$ (here $v \simeq 250$ GeV is the Higgs vev). For this case the kinetic mixing of two fields in K is indeed not essential. The potential $U_E(h, \chi)$ for $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$ is shown in Fig. 1. For $\Lambda = 0$ it has two valleys around the lines $h = \pm \sqrt{\alpha/\lambda}\,\chi$, corresponding to the exact zero mode. As soon as $\Lambda \neq 0$ the valleys get a slope. If $\Lambda > 0$, the potential is positive for all field values and does not have any minima but decreases if one moves away from the origin along a valley. On the contrary, for $\Lambda < 0$ the potential is negative for small χ and h and has a singularity at the origin.

For $0 < \Lambda \lesssim M_P^4$ a typical behavior of the scalar fields is as follows. Like in the chaotic inflation scenario [21], it is expected that initially both fields are displaced from their ground state values and are generically larger than the Planck scale: the first term in (25) dominates. Moreover, by assumption, $\xi_h \gg \xi_\chi$, meaning that for $\chi \sim h$ the dynamics is mainly driven by the Higgs field, moving the system towards the valley. This corresponds to inflation due to the Higgs field, suggested in [22]. When the value of the Higgs field becomes of the order of the Planck scale, it is trapped by the valley and oscillates there, producing particles of the SM (this process is studied in detail in [23]). The correct spectrum of perturbations is generated if $\xi_h \sim 20000$ [22], for which the reheating temperature is $T_{rh} \sim 10^{13}$ GeV. This part of the evolution is quite similar to the hybrid inflation scenario [24]. The later evolution of the universe depends crucially on the sign of Λ, which is defined by initial

conditions in UG. We would expect that with 50% probability the universe was born in the state with $\Lambda > 0$. In this case it will evolve along the valley towards a state with χ, $h = \infty$ with zero cosmological constant. At any finite evolution time the universe must contain dark energy.

Present cosmological observations allow to pin down the value of the non-minimal coupling of the field χ. For the late time evolution and $\alpha \ll 1$ ($\Rightarrow h \ll \chi$) the dilaton field η with (almost) canonical kinetic term is related to χ as

$$\chi = M_P \exp\left(\frac{\gamma \eta}{4 M_P}\right), \quad \gamma = \frac{4}{\sqrt{6 + \frac{1}{\xi_\chi}}}. \quad (26)$$

Its dynamics is practically decoupled from the dynamics of the Higgs field (the deviation of it from the vev will be denoted by ϕ in the Einstein frame). The corresponding equations of motion have the form

$$\ddot{\eta} + 3H\dot{\eta} + \frac{dU_\eta}{d\eta} = I_\eta\,, \quad (27)$$

$$\ddot{\phi} + 3H\dot{\phi} + m_h^2 \phi = I_\phi\,, \quad (28)$$

where m_h is the Higgs mass and

$$U_\eta = \frac{\Lambda}{\xi_\chi^2} \exp\left(-\frac{\gamma \eta}{M_P}\right)\,. \quad (29)$$

The source terms $I_{\eta,\phi}$ originate from the kinetic mixing K in eq. (24),

$$I_\eta \propto \frac{1}{\xi_h}\left(\frac{\alpha \xi_h}{\lambda \xi_\chi}\right)^{\frac{1}{2}}\left(\ddot{\phi} + 3H\dot{\phi}\right),$$

$$I_\phi \propto \left(\frac{\alpha \xi_h}{\lambda \xi_\chi}\right)^{\frac{1}{2}}\left(\ddot{\eta} + 3H\dot{\eta} - \frac{\gamma}{4M_P}\dot{\eta}^2\right)\,, \quad (30)$$

and can be safely neglected. The Hubble constant H is given by the standard expression

$$H^2 = \frac{1}{3M_P^2}\left(\frac{1}{2}\dot{\eta}^2 + U_\eta + \frac{C_\gamma}{a^4} + \frac{C_M}{a^3}\right)\,, \quad (31)$$

where the two last terms correspond to radiation and matter contributions to the energy density, and a is the scale factor. It is amazing that here the exponential potential, proposed for a quintessence field a long time ago in [25, 26, 27], appears automatically, though with Λ not being a fundamental parameter but rather a random initial condition.

The dynamics of the universe described by eq. (27) with $I_\eta = 0$ has been studied in a number of works (for a recent review see [28]). In [26, 30] it was shown that for $\gamma > \sqrt{3}$ this model possesses attractors corresponding

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[9] For the system containing just one scalar field the field transformation leading to a canonically normalized kinetic term can be found easily (see, e.g. [20]). For multiple fields we did not manage to find the required transformation.
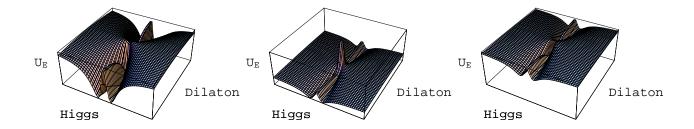
Fig. 1. Potential for the Higgs field and dilaton in the Einstein frame. Left: $\Lambda = 0$, middle: $\Lambda > 0$, right $\Lambda < 0$.

to scaling solutions. In that case the energy density of the scalar field eventually scales like the dominant component of the universe. Therefore, those models cannot describe accelerated expansion. The situation is different for $\gamma < \sqrt{3}$: the scalar component changes slower than radiation and matter and eventually starts dominating. If γ is in this region, the dynamics of η is that of a "thawing" quintessence field [29, 30]. In that scenario the scalar field at early times is nearly constant and has $\omega \simeq -1$. When the Hubble friction gets weaker, the field starts rolling down the exponential potential. At the same time ω grows and moves away, although extremely slowly, from $\omega = -1$.

Let us find a constraint on the parameter ξ_χ. The most convenient for us is the result of [31], which gives the relationship between the parameter ω of the equation of Ω_χ, valid for the thawing scenario realized by an exponential potential with $\gamma < \sqrt{3}$. It reads [31]:

$$1 + \omega = \frac{\gamma^2}{3} \left[\frac{1}{\sqrt{\Omega_\chi}} - \frac{1}{2} \left(\frac{1}{\Omega_\chi} - 1 \right) \log \frac{1 + \sqrt{\Omega_\chi}}{1 - \sqrt{\Omega_\chi}} \right]^2 . \tag{32}$$

Taking the results of WMAP [32] for the equation of state $-0.14 < 1 + \omega < 0.12$ and identifying Ω_χ with the dark energy abundance $\Omega_{DM} \simeq 0.73$, we find that the value of ξ_χ must be in the interval

$$0 < \xi_\chi < 0.07 . \tag{33}$$

Hence, $\gamma \lesssim 1 < \sqrt{3}$ which shows that γ is indeed in the correct parameter region for the thawing scenario.

As for the value of Λ, it cannot be determined unambiguously, since the value of the dilaton field is unknown. For typical values appearing in run-away scenarios, $\eta \sim M_P \log(t M_P)/2\sqrt{\xi_\chi}$ [25, 26] (t is the present age of the universe), the initial value of Λ can be as large as M_P^4.

It has been shown in [26, 30], that in this model (for $\Lambda > 0$) the universe becomes dark energy dominated at late times, i.e. $\Omega_\chi \to 1$. In this limit the parameter of the

equation of state becomes $\omega \to \frac{\gamma^2}{3} - 1$ and the universe will expand according to the power law $a(t) \propto t^{2/\gamma^2}$.

An important comment is now in order. The change of η with time *does not lead* to any visible time variation of Newton's constant or of particle masses. In the Einstein frame the dilaton practically decouples from all the fields of the SM, and the amplitude of the time-dependent corrections to masses, from eq. (28), is of the order of

$$\frac{\phi}{v} \sim \frac{I_\phi}{m_h^2 v} , \tag{34}$$

which is too small to be tested in any observations.

To conclude, the cosmological evolution of a *classical* scale-invariant theory with unimodular gravity and an exactly massless dilaton typically leads to initial inflation (large $\xi_h \gg 1$ is required to be in accordance with observations), then to the heating of the universe and eventually, with a 50% chance, to accelerated expansion in the late stages ($\xi_\chi \ll 1$ is required to fit observations).

4. Quantum theory

The analysis of the two previous sections was entirely classical. Therefore, we will formulate the conditions, which should be satisfied for the results to be valid in the quantum case as well. Since the theory (21) is not renormalizable, the discussion in this section will be on the level of wishful thinking and does not pretend to any rigor (see, however, [33, 34]).

As the dilatational symmetry of the theory is easier to see in the Jordan frame, we will use it for the present discussion, which in a number of respects resembles the one in [25, 26, 35]. Clearly, the classical results survive if the dilatational symmetry remains exact on the quantum level and if the dilaton is still massless in full quantum field theory. Like in the classical case, the exact dilaton degeneracy of the ground state will guarantee

the absence of the cosmological constant, whereas the unimodular character of gravity would induce, through initial conditions, the run-away behavior (for $\Lambda > 0$) of the dilaton field at a late time in the expansion of the universe. If true, *all* dimensionful parameters of the SM, including those coming from dimensional transmutation like Λ_{QCD} will change in the same way during the run-away of the dilaton field. The deviation of dimensionless ratios (only they are relevant for physics in a scale-invariant theory) from constants, due to the cosmological evolution, will be strongly suppressed as in (34) and thus be invisible. The dilaton will only have derivative couplings to the fields of the νMSM, being a Goldstone boson related to the spontaneous breaking of dilatational invariance and thus evade all the constraints [25, 26, 35] considered for the Brans-Dicke field [36].

The required dependence of low-energy parameters such as Λ_{QCD} on the dilaton field would appear if the following strategy is applied to the computation of radiative corrections in the constant χ, h backgrounds [33, 34] (see also [25, 26]). Use the field-dependent cut-off Q^2, related to the effective Planck scale in the Jordan frame as $Q^2 = \xi_\chi \chi^2 + \xi_h h^2$, and assume that the values of all dimensionless couplings at this scale do not depend on Q^2. With this prescription the non-trivial dimensionful parameters, appearing as a result of the renormalization procedure, would acquire the necessary dependence on the dilaton field. It is this prescription, which was effectively used in [22] to discuss radiative corrections to the Higgs-inflaton potential. Note that the renormalization procedure of [37] is completely different, and there is no surprise that the results and conclusions of our present work and of [22] are not the same as those of [37].

The requirement that the dilaton remains exactly massless on the quantum level, or in other words that the quantum effective potential has a flat direction (in classical theory this corresponds to $\beta = 0$) is crucial for our findings. It is highly non-trivial as it is not a consequence of scale symmetry. Still, the quantum field theories constructed near the scale-invariant ground state $\langle h \rangle = \langle \chi \rangle = 0$ and the state with spontaneously broken scale invariance are completely different at the quantitative level. The theory with $\langle h \rangle = \langle \chi \rangle = 0$ in general does not have asymptotic scattering states [10] (for a review see e.g. [40]). In other words, it does not have particles at all and thus cannot be accepted as a realistic theory. On the contrary, if the quantum scale invariance is *spontaneously* broken, the theory does

describe particles and thus may be phenomenologically relevant. These considerations single out the theories with degenerate ground state, corresponding to $\beta = 0$ at the classical level [11].

In the cosmological setup the relevance of these arguments is not obvious. Indeed, suppose that the flat direction is lifted, i.e. $\beta > 0$ and consider late time classical evolution corresponding to initial conditions with $\Lambda > 0$. A non-zero value of β leads to a positive vacuum energy in the Einstein frame, $E_{vac} \sim \beta M_P^4$. As in Section 3, the scalar field η has a run-away behavior, leading to the breaking of scale invariance, whereas the universe is expanding exponentially with the Hubble constant determined by E_{vac}. Certainly, there is nothing wrong with the classical solution of this type, except the fact that it is unstable against scalar field fluctuations which grow as in the inflationary stage [41]. Whether this picture survives quantum-mechanically, is an open issue. If it does, and if the notion of particles can be defined, we will get a universe with non-zero cosmological constant, loosing thus an explanation of its absence. However, the classical picture may happen to be misleading. Indeed, in the full quantum field theory, any state including cosmological solutions, can be considered as a superposition of quantum excitations above the vacuum state. As no particles can be defined for the scale-invariant theory with scale-invariant ground state [12], we would expect that the effective particle-like excitations about the classical background can exist only for a finite time, presumably of the order of the inverse Hubble constant. If true, the requirement that the scale-invariant quantum theory must be able to describe particles again singles out the case corresponding to $\beta = 0$.

5. Conclusions

In this Letter we showed that the scale-invariant classical SM and the νMSM coupled to unimodular gravity have all necessary ingredients to describe the evolution of the universe, including early inflation and late acceleration. The requirement of scale invariance and of the existence of a massless dilaton leads to a theory in which all mass scales, including that of gravity, orig-

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[10] If it does, the exact propagators coincide with the free ones and the theory is likely to be trivial in this case [38, 39].

[11] When quantum effects are included, the flat direction is not necessary associated with $\beta = 0$. Nevertheless, to simplify the discussion, we will refer to the case with spontaneously broken scale invariance on the quantum level still quoting the classical value $\beta = 0$.

[12] The case when particles can be defined corresponds to free field theory, see footnote 10. Then the Lagrangian (1) is simply a very complicated way to describe this trivial theory, and the classical cosmological solution has nothing to do with the exact quantum solution.

inate from one and the same source. The unimodular character of gravity leads to the generation of an exponential potential for the dilaton, ensuring the existence of dark energy. If the full quantum field theory exhibits the same symmetries (for the explicit construction see [33, 34]), it will have the same properties. Moreover, the argument can be reverted – the observation of an accelerated universe with a dark energy component may tell that the underlying quantum field theory should be scale-invariant, and that this scale invariance should be broken spontaneously, leading to the massless dilaton.

The theory (1) with $\beta = 0$ and UG happens to be very rich in applications. It can address all confirmed signals which suggest that the SM is not complete: neutrino masses and oscillations, existence of dark and baryonic matter in the universe, early inflation and late acceleration, leading to an extra argument against the necessity of new physics between the electroweak and the Planck scale [42] (see also [43]). The discussion of particle physics experiments and astrophysical observations that can confirm or rule out this theory can be found in [44, 45, 46, 47, 48]. Our findings here indicate that the equation of state parameter ω for dark energy must be different from that of the cosmological constant, but also that $\omega > -1$, adding an extra cosmological test that could rule out the νMSM.

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