PART 3

TESTING

11

ORTHONORMAL POLYNOMIALS IN WAVEFRONT ANALYSIS

Virendra N. Mahajan*

The Aerospace Corporation
El Segundo, California

ABSTRACT

Zernike circle polynomials are in widespread use for wavefront analysis because they are orthogonal over a unit circle and represent balanced classical aberrations for imaging systems with circular pupils. However, they are not suitable for systems with noncircular pupils. Examples of such pupils are annular as in astronomical telescopes, elliptical as in the off-axis pupil of an otherwise rotationally symmetric system with a circular on-axis pupil, hexagonal as in the hexagonal segments of a large telescope, for example, Keck, and rectangular and square as in high-power laser beams. In this chapter, we list the orthonormal circle, annular, elliptical, hexagonal, rectangular, and square polynomials. The polynomials for a noncircular pupil can be obtained by orthogonalizing the circle polynomials over the pupil using the recursive Gram-Schmidt process or a nonrecursive matrix approach. These polynomials are unique in that they are not only orthogonal across such pupils, but also represent balanced classical aberrations for such pupils, just as the Zernike circle polynomials are unique in these respects for circular pupils. The polynomials are given in terms of the circle polynomials as well as in polar and Cartesian coordinates. The orthonormal polynomials for a one-dimensional slit pupil are given as a limiting case of a rectangular pupil. The polynomials corresponding to Seidel aberrations are illustrated isometrically, interferometrically, and with the corresponding point-spread functions (PSFs).

11.1 GLOSSARY

a half width of a unit rectangular pupil
a_j jth expansion coefficient
A area of pupil
b aspect ratio of a unit elliptical pupil

*The author is also an adjunct professor at the College of Optical Sciences, University of Arizona, Tucson, Arizona and Department of Optics and Photonics, National Central University, Chung Li, Taiwan. He gratefully acknowledges helpful discussions with Drs. Guang-ming Dai and Bill Swantner.

$E_j(x, y)$	orthonormal elliptical polynomial in Cartesian coordinates (x, y)
F	focal ratio of the image-forming light cone
$F_j(x, y)$	jth orthonormal polynomial
$H_j(x, y)$	orthonormal hexagonal polynomial
j	polynomial number
N_n	number of polynomials through an order n
$P_j(x)$	orthonormal slit polynomial along the x axis
$P_n(\cdot)$	Legendre polynomial of order n
$R_j(x, y)$	orthonormal rectangular polynomial
$R_n^m(\rho)$	Zernike circle radial polynomial
$R_n^m(\rho; \epsilon)$	Zernike annular radial polynomial
$S_j(x, y)$	orthonormal square polynomial
$W(x, y)$	wave aberration at a point (x, y)
$Z_j(\rho, \theta)$	orthonormal Zernike circle polynomial in polar coordinates (ρ, θ)
$Z_j(\rho, \theta; \epsilon)$	orthonormal Zernike annular polynomial
σ	standard deviation
σ^2	variance
ϵ	obscuration ratio of an annular pupil

11.2 INTRODUCTION

Optical systems generally have a circular pupil. The imaging elements of such systems have a circular boundary. Hence they also represent circular pupils in fabrication and testing. As a result, the Zernike circle polynomials have been in widespread use since Zernike introduced them in his phase contrast method for testing circular mirrors.[1] They are used in optical design and testing to understand the aberration content of a wavefront. They have also been used for analyzing the wavefront aberration introduced by atmospheric turbulence on a wave propagating through it.[2] Their utility stems from the fact that they are orthogonal over a unit circle and they represent balanced classical aberrations yielding minimum variance over a circular pupil.[3–6] They are unique in this respect since no other polynomials have these properties. Because of their orthogonality, when a wavefront is expanded in terms of them, the value of an expansion coefficient is independent of the number of polynomials used in the expansion. Hence, one or more polynomial terms can be added or subtracted without affecting the other coefficients. The piston coefficient represents the mean value of the aberration function and the variance of the function is given simply by the sum of the squares of the other expansion coefficients.[7]

For systems with noncircular pupils, the Zernike circle polynomials are neither orthogonal over such pupils nor do they represent balanced aberrations. Hence their special utility is lost. However, since they form a complete set, an aberration function over a noncircular wavefront can be expanded in terms of them. The expansion coefficients are no longer independent of each other and their values change as the number of polynomials used in the expansion changes. The piston coefficient does not represent the mean value of the aberration function, and the sum of the squares of the other coefficients does not yield the aberration variance.

The reflecting telescopes, such as the Hubble, have annular pupils and require polynomials that are orthogonal across an annulus to describe their aberrations.[8–11] The primary mirrors of large telescopes, such as the Keck, consist of hexagonal segments.[12] The wavefront analysis of such segments requires polynomials that are orthogonal over a hexagon. The pupil for off-axis imaging by a system with an axial circular pupil is vignetted, but can be approximated by an ellipse.[13] When a flat mirror is tested by shining a circular beam on it at some angle (other than normal incidence), the illuminated spot is elliptical. Similarly, the overlap region of two circular wavefronts that are displaced from each other, as in lateral shearing interferometry[14] or in the calculation of the optical transfer function of a system,[15] can also be approximated by an ellipse. In such cases we need polynomials that are orthogonal over an ellipse. In Refs. 14 and 15, the polynomials that are orthogonal over an elliptical region were obtained simply by scaling the Cartesian coordinates by its aspect ratio. However, such orthogonal polynomials cannot represent classical aberrations. For example, defocus, which varies as ρ^2, has the same scale for both the x and y coordinates. Similarly, they cannot represent balanced classical aberrations, for example, coma balanced with tilt. High-power laser beams have rectangular or square cross sections[16] and require polynomials that are orthogonal over a rectangle or a square, respectively.

The polynomials orthonormal over a unit annulus, hexagon, ellipse, rectangle, and a square inscribed inside a unit circle may be obtained from the circle polynomials by the recursive Gram-Schmidt orthogonalization process[17,18] or a nonrecursive matrix approach.[19] The orthonormal polynomials representing balanced aberrations for a slit pupil can be obtained as a limiting case of the rectangular polynomials, where one dimension of the rectangle approaches zero. They are the Legendre polynomials.[20] We use the circle polynomials as the basis functions for the orthogonalization process, so that the relationship of a noncircle polynomial to the circle polynomials is evident, since the former is a linear combination of the latter. We give the orthonormal form of the polynomials so that when an aberration function is expanded in terms of them, each expansion coefficient (with the exception of piston) represents the standard deviation of the corresponding expansion term. The noncircle polynomials are given not only in terms of the circle polynomials, but in polar and Cartesian coordinates as well. The circle, annular, hexagonal, and square polynomials are given up to the eighth order, and the elliptical and rectangular polynomials are given up to the fourth order. Just as the Zernike circle polynomials uniquely represent the orthogonal and balanced aberrations across circular pupils, similarly, the orthonormal polynomials for the noncircular pupils given in this chapter also uniquely represent the orthogonal and balanced aberrations across such pupils.

Orthogonal square polynomials were obtained by Bray by orthogonalizing the circle polynomials, but he chose a circle inscribed inside a square instead of the other way around.[21] Thus his square with a full width of unity has regions that fall outside the unit circle. Defining a unit square in this manner has the disadvantage that the coefficient of a term in a certain polynomial does not represent its peak value. Products of x and y Legendre polynomials,[17] which are orthogonal over a square pupil, have been suggested for analysis of square wavefronts.[22] But they do not represent classical or balanced aberrations. For example, defocus is represented by a term in $x^2 + y^2$. While it can be expanded in terms of a complete set of Legendre polynomials, it cannot be represented by a single two-dimensional Legendre polynomial (i.e., as a product of x and y Legendre polynomial). The same difficulty holds for spherical aberration and coma, and the like.

Although in many imaging applications, the amplitude across the pupil is uniform, such is not always the case, for example, a system with an apodized pupil. An example of such a pupil is the Gaussian pupil, where the amplitude has the form of a Gaussian due either to an amplitude filter placed at the pupil or to the wave incident on the pupil being Gaussian, as in the case of a Gaussian laser beam. Again, the balanced aberrations for a Gaussian pupil have a form that is different from the corresponding balanced aberrations for a uniform pupil due to the amplitude weighting of the pupil.[23–25] The amount of defocus to optimally balance spherical aberration, or the amount of wavefront tilt to optimally balance coma, for example, is different for a Gaussian pupil than its corresponding value for a uniform pupil.

11.3 ORTHONORMAL POLYNOMIALS

In Cartesian coordinates (x, y), the aberration function $W(x, y)$ for a certain pupil may be expanded in terms of J polynomials $F_j(x, y)$ that are orthonormal over the pupil:[26]

$$W(x, y) = \sum_{j=1}^{J} a_j F_j(x, y) \tag{1}$$

where a_j is an expansion or the aberration coefficient of the polynomial $F_j(x, y)$. The orthonormality of the polynomials is represented by

$$\frac{1}{A}\int_{\text{pupil}} F_j(x,y)F_{j'}(x,y)\,dx\,dy = \delta_{jj'} \tag{2}$$

where A is the area of the pupil inscribed inside a unit circle, the integration is carried out over the area of the pupil, and $\delta_{jj'}$ is a Kronecker delta. If $F_1 = 1$, then the mean value of each polynomial, except for $j = 1$, is zero, that is,

$$\frac{1}{A}\int_{\text{pupil}} F_j(x,y)\,dx\,dy = 0 \qquad \text{for } j \neq 1 \tag{3}$$

as may be seen by letting $j' = 1$ in Eq. (2). The aberration coefficients are given by

$$a_j = \frac{1}{A}\int_{\text{pupil}} W(x,y)F_j(x,y)\,dx\,dy \tag{4}$$

as may be seen by substituting Eq. (1) into Eq. (4) and using the orthonormality Eq. (2).

The mean and the mean square values of the aberration function are given by

$$\langle W(x,y)\rangle = a_1 \tag{5}$$

and

$$\langle W^2(x,y)\rangle = \sum_{j=1}^{J} a_j^2 \tag{6}$$

Accordingly, the variance σ^2 of the aberration function is given by

$$\sigma^2 = \langle W^2(x,y)\rangle - \langle W(x,y)\rangle^2 = \sum_{j=2}^{J} a_j^2 \tag{7}$$

where σ is the standard deviation of the aberration function. The number of polynomials J used in the expansion is a sufficiently large that the variance obtained from Eq. (6) equals the actual value obtained from the function $W(x, y)$ within some prescribed tolerance.

11.4 ZERNIKE CIRCLE POLYNOMIALS

An aberration function $W(\rho, \theta)$, across a *unit circle* can be expanded in terms of the orthonormal *Zernike circle polynomials* $Z_j(\rho, \theta)$ in the form[2,5]

$$W(\rho,\theta) = \sum_j a_j Z_j(\rho,\theta) \tag{8}$$

where (ρ, θ) are the polar coordinates of a point on the circle, $0 \le \rho \le 1$, $0 \le \theta < 2\pi$, and a_j are the expansion coefficients. The polynomials may be written in the form

$$Z_{\text{even}\,j}(\rho,\theta) = \sqrt{2(n+1)}\,R_n^m(\rho)\cos m\theta,\ m \neq 0 \tag{9a}$$

$$Z_{\text{odd}\,j}(\rho,\theta) = \sqrt{2(n+1)}\,R_n^m(\rho)\sin m\theta,\ m \neq 0 \tag{9b}$$

$$Z_j(\rho,\theta) = \sqrt{n+1}\,R_n^0(\rho),\ m = 0 \tag{9c}$$

where n and m are positive integers (including zero) and $n - m \geq 0$ and even. It is evident from Eqs. (9) that the circle polynomials are separable in the polar coordinates ρ and θ. A radial polynomial $R_n^m(\rho)$ is given by

$$R_n^m(\rho) = \sum_{s=0}^{(n-m)/2} \frac{(-1)^s (n-s)!}{s!\left(\frac{n+m}{2}-s\right)!\left(\frac{n-m}{2}-s\right)!}\rho^{n-2s} \tag{10}$$

with a degree n in ρ containing terms in ρ^n, ρ^{n-2}, …, and ρ^m. It is even or odd in ρ depending on whether n (or m) is even or odd. Also, $R_n^n(\rho) = \rho^n$, $R_n^m(1) = 1$, and $R_n^m(0) = \delta_{m0}$ for even $n/2$ and $-\delta_{m0}$ for odd $n/2$. The polynomials $R_n^m(\rho)$ obey the orthogonality relation

$$\int_0^1 R_n^m(\rho) R_{n'}^m(\rho)\rho d\rho = \frac{1}{2(n+1)}\delta_{nn'} \tag{11}$$

The orthogonality of the angular functions yields

$$\int_0^{2\pi} d\theta \begin{cases} \cos m\theta \cos m'\theta \ , & j \text{ and } j' \text{ are both even} \\ \cos m\theta \sin m'\theta \ , & j \text{ is even and } j' \text{ is odd} \\ \sin m\theta \cos m'\theta \ , & j \text{ is odd and } j' \text{ is even} \\ \sin m\theta \sin m'\theta \ , & j \text{ and } j' \text{ are both odd} \end{cases}$$

$$= \begin{cases} \pi(1+\delta_{m0})\delta_{mm'} \ , & j \text{ and } j' \text{ are both even} \\ \pi\delta_{mm'} \ , & j \text{ and } j' \text{ are both odd} \\ 0 \ , & \text{otherwise} \end{cases} \tag{12}$$

Therefore, the Zernike polynomials are orthonormal according to

$$\int_0^1 \int_0^{2\pi} Z_j(\rho,\theta) Z_{j'}(\rho,\theta)\rho d\rho d\theta \Big/ \int_0^1 \int_0^{2\pi} \rho d\rho d\theta = \delta_{jj'} \tag{13}$$

The expansion coefficients are given by

$$a_j = \frac{1}{\pi}\int_0^1 \int_0^{2\pi} W(\rho,\theta) Z_j(\rho,\theta)\rho d\rho d\theta \tag{14}$$

as may be seen by substituting Eq. (8) into Eq. (14) and using the orthonormality Eq. (13).

While the index n represents the radial *degree* or the *order* of a polynomial, since it represents the highest power of ρ in the polynomial, m is referred to as its *azimuthal frequency*. The index j is a *polynomial-ordering number* and is a function of both n and m. The polynomials are ordered such that an even j corresponds to a symmetric polynomial varying as $\cos m\theta$, while an odd j corresponds to an antisymmetric polynomial varying as $\sin m\theta$. A polynomial with a lower value of n is ordered first, and for a given value of n, a polynomial with a lower value of m is ordered first.

The Zernike circle polynomials are unique in that they are the only polynomials in two variables ρ and θ, which (a) are orthogonal over a circle, (b) are invariant in form with respect to rotation of the coordinate axes about the origin, and (c) include a polynomial for each permissible pair of n and m values.[4,27]

The orthonormal Zernike circle polynomials and the names associated with some of them when identified with classical aberrations are listed in Table 1*a* for $n \leq 8$. The polynomials independent of θ are the spherical aberrations, those varying as $\cos\theta$ are the coma aberrations, and those varying as $\cos 2\theta$ are the astigmatism aberrations. The variation of several radial polynomials $R_n^m(\rho)$ with ρ is illustrated in Fig. 1.

TABLE 1*a* Orthonormal Zernike *Circle* Polynomials $Z_j(\rho, \theta)$ Ordered Such That an Even j Corresponds to a Symmetric Polynomial Varying as $\cos m\theta$, While an Odd j Corresponds to an Antisymmetric Polynomial Varying as $\sin m\theta$

j	n	m	$Z_j(\rho, \theta)$	Aberration Name*
1	0	0	1	Piston
2	1	1	$2\rho\cos\theta$	x tilt
3	1	1	$2\rho\sin\theta$	y tilt
4	2	0	$\sqrt{3}(2\rho^2-1)$	Defocus
5	2	2	$\sqrt{6}\rho^2\sin 2\theta$	Primary astigmatism at 45°
6	2	2	$\sqrt{6}\rho^2\cos 2\theta$	Primary astigmatism at 0°
7	3	1	$\sqrt{8}(3\rho^3-2\rho)\sin\theta$	Primary y coma
8	3	1	$\sqrt{8}(3\rho^3-2\rho)\cos\theta$	Primary x coma
9	3	3	$\sqrt{8}\rho^3\sin 3\theta$	
10	3	3	$\sqrt{8}\rho^3\cos 3\theta$	
11	4	0	$\sqrt{5}(6\rho^4-6\rho^2+1)$	Primary spherical aberration
12	4	2	$\sqrt{10}(4\rho^4-3\rho^2)\cos 2\theta$	Secondary astigmatism at 0°
13	4	2	$\sqrt{10}(4\rho^4-3\rho^2)\sin 2\theta$	Secondary astigmatism at 45°
14	4	4	$\sqrt{10}\rho^4\cos 4\theta$	
15	4	4	$\sqrt{10}\rho^4\sin 4\theta$	
16	5	1	$\sqrt{12}(10\rho^5-12\rho^3+3\rho)\cos\theta$	Secondary x coma
17	5	1	$\sqrt{12}(10\rho^5-12\rho^3+3\rho)\sin\theta$	Secondary y coma
18	5	3	$\sqrt{12}(5\rho^5-4\rho^3)\cos 3\theta$	
19	5	3	$\sqrt{12}(5\rho^5-4\rho^3)\sin 3\theta$	
20	5	5	$\sqrt{12}\rho^5\cos 5\theta$	
21	5	5	$\sqrt{12}\rho^5\sin 5\theta$	
22	6	0	$\sqrt{7}(20\rho^6-30\rho^4+12\rho^2-1)$	Secondary spherical aberration
23	6	2	$\sqrt{14}(15\rho^6-20\rho^4+6\rho^2)\sin 2\theta$	Tertiary astigmatism at 45°
24	6	2	$\sqrt{14}(15\rho^6-20\rho^4+6\rho^2)\cos 2\theta$	Tertiary astigmatism at 0°
25	6	4	$\sqrt{14}(6\rho^6-5\rho^4)\sin 4\theta$	
26	6	4	$\sqrt{14}(6\rho^6-5\rho^4)\cos 4\theta$	
27	6	6	$\sqrt{14}\,6\rho^6\sin 6\theta$	
28	6	6	$\sqrt{14}\rho^6\cos 6\theta$	
29	7	1	$4(35\rho^7-60\rho^5+30\rho^3-4\rho)\sin\theta$	Tertiary y coma
30	7	1	$4(35\rho^7-60\rho^5+30\rho^3-4\rho)\cos\theta$	Tertiary x coma
31	7	3	$4(21\rho^7-30\rho^5+10\rho^3)\sin 3\theta$	
32	7	3	$4(21\rho^7-30\rho^5+10\rho^3)\cos 3\theta$	
33	7	5	$4(7\rho^7-6\rho^5)\sin 5\theta$	
34	7	5	$4(7\rho^7-6\rho^5)\cos 5\theta$	
35	7	7	$4\rho^7\sin 7\theta$	
36	7	7	$4\rho^7\cos 7\theta$	
37	8	0	$3(70\rho^8-140\rho^6+90\rho^4-20\rho^2+1)$	Tertiary spherical aberration
38	8	2	$\sqrt{18}(56\rho^8-105\rho^6+60\rho^4-10\rho^2)\cos 2\theta$	Quaternary astigmatism at 0°
39	8	2	$\sqrt{18}(56\rho^8-105\rho^6+60\rho^4-10\rho^2)\sin 2\theta$	Quaternary astigmatism at 45°
40	8	4	$\sqrt{18}(28\rho^8-42\rho^6+15\rho^4)\cos 4\theta$	

(*Continued*)

TABLE 1*a* Orthonormal Zernike *Circle* Polynomials $Z_j(\rho, \theta)$ Ordered Such That an Even j Corresponds to a Symmetric Polynomial Varying as $\cos m\theta$, While an Odd j Corresponds to an Antisymmetric Polynomial Varying as $\sin m\theta$ (*Continued*)

j	n	m	$Z_j(\rho, \theta)$	Aberration Name*
41	8	4	$\sqrt{18}(28\rho^8 - 42\rho^6 + 15\rho^4)\sin 4\theta$	
42	8	6	$\sqrt{18}(8\rho^8 - 7\rho^6)\cos 6\theta$	
43	8	6	$\sqrt{18}(8\rho^8 - 7\rho^6)\sin 6\theta$	
44	8	8	$\sqrt{18}\rho^8 \cos 8\theta$	
45	8	8	$\sqrt{18}\rho^8 \sin 8\theta$	

*The words *orthonormal Zernike circle* are to be associated with these names, e.g., *orthonormal Zernike circle primary astigmatism at* 0°.

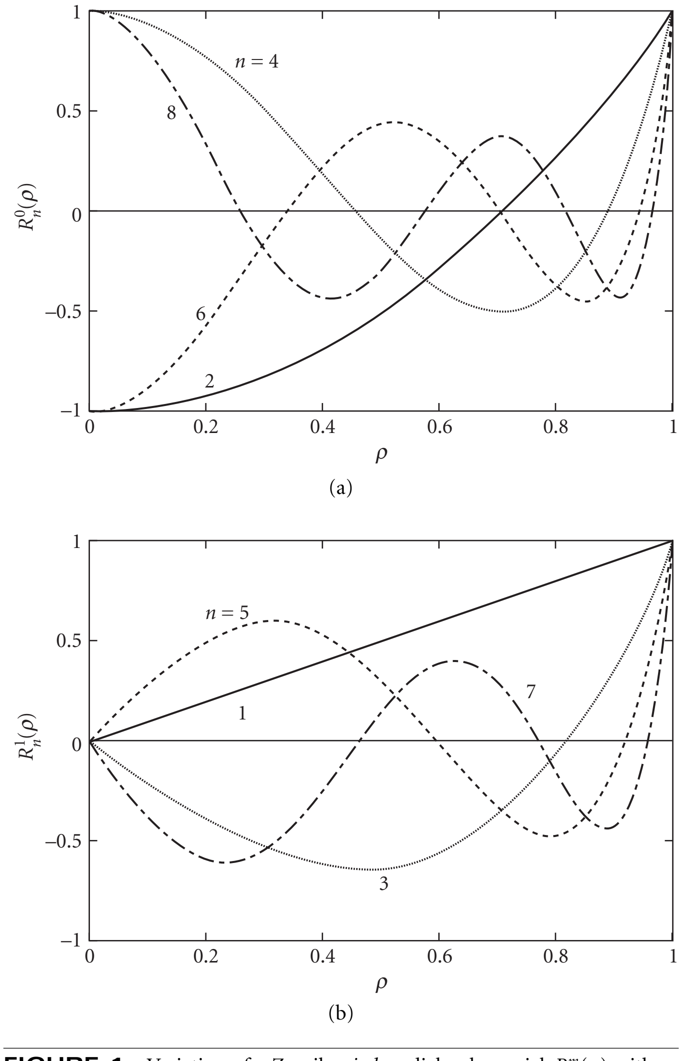
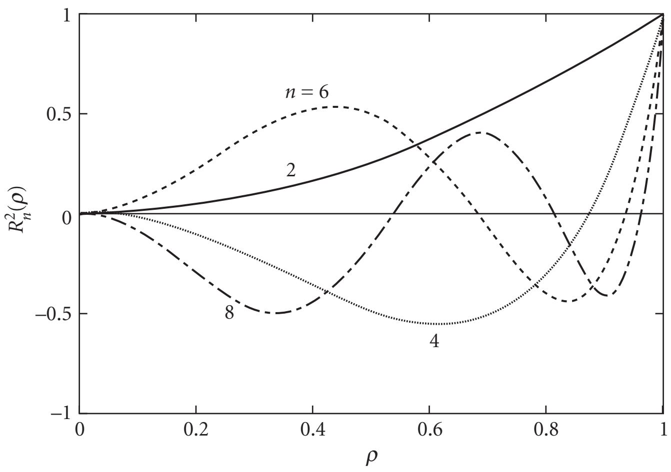


FIGURE 1 Variation of a Zernike *circle* radial polynomial $R_n^m(\rho)$ with ρ: (*a*) defocus and spherical aberrations; (*b*) tilt and coma; and (*c*) astigmatism.



(c)

FIGURE 1 (*Continued*)

The number of polynomials of a given order n is $n + 1$. Their number through a certain order n is given by

$$N_n = (n+1)(n+2)/2 \tag{15}$$

For a rotationally symmetric imaging system, each of the $\sin m\theta$ terms is zero.[4,28–32] Accordingly the number of polynomials of an even order is $(n/2) + 1$ and $(n + 1)/2$ for an odd order. Their number through an order n is given by

$$N_n = \left(\frac{n}{2}+1\right)^2 \quad \text{for even } n \tag{16a}$$

$$= (n+1)(n+3)/4 \quad \text{for odd } n \tag{16b}$$

Relationships among the Indices *n*, *m*, and *j*

The number of polynomials N_n through a certain order n represents the largest value of j. Since the number of terms with the same value of n but different values of m is equal to $n + 1$, the smallest value of j for a given value of n is $N_n - n$. For a given value of n and m, there are two j values, $N_n - n + m - 1$ and $N_n - n + m$. The even value of j represents the $\cos m\theta$ term and the odd value of j represents the $\sin m\theta$ term. The value of j with $m = 0$ is $N_n - n$. For example, for $n = 5$, $N_n = 21$, and $j = 21$ represents the $\sin 5\theta$ term. The number of the corresponding $\cos 5\theta$ term is $j = 20$. The two terms with $m = 3$, for example, have j values of 18 and 19 representing the $\cos 3\theta$ and the $\sin 3\theta$ terms, respectively.

For a given value of j, n is given by

$$n = [(2j-1)^{1/2} + 0.5]_{\text{integer}} - 1 \tag{17}$$

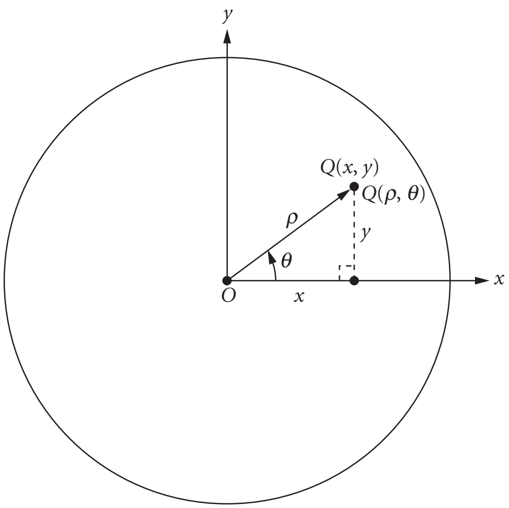


FIGURE 2 Cartesian and polar coordinates (x, y) and (ρ, θ), respectively, of a point Q in the plane of a unit circle representing the circular exit pupil of an imaging system.

where the subscript integer implies the integer value of the number in brackets. Once n is known, the value of m is given by

$$m = \begin{cases} 2\{[2j+1-n(n+1)]/4\}_{\text{integer}} & \text{when } n \text{ is even} \quad (18a) \\ 2\{[2(j+1)-n(n+1)]/4\}_{\text{integer}} - 1 & \text{when } n \text{ is odd} \quad (18b) \end{cases}$$

For example, suppose we want to know the values of n and m for the term $j = 10$. From Eq. (17), $n = 3$ and from Eq. (18b), $m = 3$. Hence, it is a $\cos 3\theta$ term.

The polar coordinates (ρ, θ) and the Cartesian coordinates (x, y) of a pupil point Q, as illustrated in Fig. 2, are related to each other according to

$$(x, y) = \rho(\cos\theta, \sin\theta) \tag{19}$$

The circle polynomials in the Cartesian coordinates (x, y) of a pupil point are listed in Table 1b. It is quite common in the optics literature to consider a point object lying along the y axis when imaged by a rotationally symmetric optical system, thus making the yz plane the tangential plane.[4,28–32] To maintain symmetry of the aberration function about this plane, the polar angle θ of a pupil point is accordingly defined as the angle made by its position vector $\overrightarrow{OQ}$ with the y axis, contrary to the standard convention as the angle with the x axis. We choose a point object along the x axis so that, for example, the coma aberration is expressed as $x(x^2 + y^2)$ and not as $y(x^2 + y^2)$. A positive value of our coma aberration yields a diffraction point spread function that is symmetric *about* the x axis (or symmetric *in* y) with its peak and centroid shifted to a positive value of x with respect to the Gaussian image point.

TABLE 1*b* Orthonormal Zernike *Circle* Polynomials $Z_j(x, y)$ in Cartesian Coordinates (x, y), Where $x = \rho\cos\theta$, $y = \rho\sin\theta$, and $0 \le \rho = \sqrt{x^2 + y^2} \le 1$

Polynomial	$Z_j(x, y)$
Z_1	1
Z_2	$2x$
Z_3	$2y$
Z_4	$\sqrt{3}(2\rho^2 - 1)$
Z_5	$2\sqrt{6}\,xy$
Z_6	$\sqrt{6}(x^2 - y^2)$
Z_7	$\sqrt{8}\,y(3\rho^2 - 2)$
Z_8	$\sqrt{8}\,x(3\rho^2 - 2)$
Z_9	$\sqrt{8}\,y(3x^2 - y^2)$
Z_{10}	$\sqrt{8}\,x(x^2 - 3y^2)$
Z_{11}	$\sqrt{5}(6\rho^4 - 6\rho^2 + 1)$
Z_{12}	$\sqrt{10}(x^2 - y^2)(4\rho^2 - 3)$
Z_{13}	$2\sqrt{10}\,xy(4\rho^2 - 3)$
Z_{14}	$\sqrt{10}(\rho^4 - 8x^2y^2)$
Z_{15}	$4\sqrt{10}\,xy(x^2 - y^2)$
Z_{16}	$\sqrt{12}\,x(10\rho^4 - 12\rho^2 + 3)$
Z_{17}	$\sqrt{12}\,y(10\rho^4 - 12\rho^2 + 3)]$
Z_{18}	$\sqrt{12}\,x(x^2 - 3y^2)(5\rho^2 - 4)$
Z_{19}	$\sqrt{12}\,y(3x^2 - y^2)(5\rho^2 - 4)$
Z_{20}	$\sqrt{12}\,x(16x^4 - 20x^2\rho^2 + 5\rho^4)$
Z_{21}	$\sqrt{12}\,y(16y^4 - 20y^2\rho^2 + 5\rho^4)$
Z_{22}	$\sqrt{7}(20\rho^6 - 30\rho^4 + 12\rho^2 - 1)$
Z_{23}	$2\sqrt{14}\,xy(15\rho^2 - 20\rho^2 + 6)$
Z_{24}	$\sqrt{14}(x^2 - y^2)(15\rho^4 - 20\rho^2 + 6)$
Z_{25}	$4\sqrt{14}\,xy(x^2 - y^2)(6\rho^2 - 5)$
Z_{26}	$\sqrt{14}(8x^4 - 8x^2\rho^2 + \rho^4)(6\rho^2 - 5)$
Z_{27}	$\sqrt{14}\,xy(32x^4 - 32x^2\rho^2 + 6\rho^4)$
Z_{28}	$\sqrt{14}(32x^6 - 48x^4\rho^2 + 18x^2\rho^4 - \rho^6)$
Z_{29}	$4y(35\rho^6 - 60\rho^4 + 30\rho^2 - 4)$
Z_{30}	$4x(35\rho^6 - 60\rho^4 + 30\rho^2 - 4)$
Z_{31}	$4y(3x^2 - y^2)(21\rho^4 - 30\rho^2 + 10)$
Z_{32}	$4x(x^2 - 3y^2)(21\rho^4 - 30\rho^2 + 10)$
Z_{33}	$4(7\rho^2 - 6)[4x^2y(x^2 - y^2) + y(\rho^4 - 8x^2y^2)]$
Z_{34}	$4(7\rho^2 - 6)[x(\rho^4 - 8x^2y^2) - 4xy^2(x^2 - y^2)]$
Z_{35}	$8x^2y(3\rho^4 - 16x^2y^2) + 4y(x^2 - y^2)(\rho^4 - 16x^2y^2)$
Z_{36}	$4x(x^2 - y^2)(\rho^4 - 16x^2y^2) - 8xy^2(3\rho^4 - 16x^2y^2)$
Z_{37}	$3(70\rho^8 - 140\rho^6 + 90\rho^4 - 20\rho^2 + 1)$
Z_{38}	$\sqrt{18}(56\rho^6 - 105\rho^4 + 60\rho^2 - 10)(x^2 - y^2)$
Z_{39}	$2\sqrt{18}\,xy(56\rho^6 - 105\rho^4 + 60\rho^2 - 10)$
Z_{40}	$\sqrt{18}(28\rho^4 - 42\rho^2 + 15)(\rho^4 - 8x^2y^2)$
Z_{41}	$4\sqrt{18}\,xy(28\rho^4 - 42\rho^2 + 15)(x^2 - y^2)$
Z_{42}	$\sqrt{18}(x^2 - y^2)(\rho^4 - 16x^2y^2)(8\rho^2 - 7)$
Z_{43}	$2\sqrt{18}\,xy(3\rho^4 - 16x^2y^2)$
Z_{44}	$2\sqrt{18}(\rho^4 - 8x^2y^2)^2 - \rho^8$
Z_{45}	$8\sqrt{18}\,xy(x^2 - y^2)(\rho^4 - 8x^2y^2)$

11.5 ZERNIKE ANNULAR POLYNOMIALS

The aberration function $W(\rho, \theta; \epsilon)$ across a *unit annulus* with an obscuration ratio ϵ, representing the ratio of its inner and outer radii, as illustrated in Fig. 3*a*, can be expanded in terms of a complete set of *Zernike annular polynomials* $Z_j(\rho, \theta; \epsilon)$ that are orthonormal over the unit annulus in the form[8–11]

$$W(\rho,\theta;\epsilon)=\sum_j a_j Z_j(\rho,\theta;\epsilon) \tag{20}$$

where a_j is an expansion coefficient of the polynomial, $\epsilon \le \rho \le 1$ and $0 \le \theta < 2\pi$. The annular polynomials are written in a manner similar to the circle polynomials. Thus

$$Z_{\text{even}\,j}(\rho,\theta;\epsilon)=\sqrt{2(n+1)}\,R_n^m(\rho;\epsilon)\cos m\theta, m\neq 0 \tag{21a}$$

$$Z_{\text{odd}\,j}(\rho,\theta;\epsilon)=\sqrt{2(n+1)}\,R_n^m(\rho;\epsilon)\sin m\theta, m\neq 0 \tag{21b}$$

$$Z_j(\rho,\theta;\epsilon)=\sqrt{n+1}\,R_n^0(\rho;\epsilon), m=0 \tag{21c}$$

where n and m are positive integers (including zero) and $n - m \ge 0$ and even. The radial annular polynomials $R_n^m(\rho;\epsilon)$ obey the orthogonality relation

$$\int_\epsilon^1 R_n^m(\rho;\epsilon)R_{n'}^m(\rho;\epsilon)\rho d\rho=\frac{1-\epsilon^2}{2(n+1)}\delta_{nn'} \tag{22}$$

Accordingly, the annular polynomials obey the orthonormality condition

$$\int_\epsilon^1\int_0^{2\pi} Z_j(\rho,\theta;\epsilon)Z_{j'}(\rho,\theta;\epsilon)\rho d\rho d\theta \bigg/ \int_\epsilon^1\int_0^{2\pi}\rho d\rho d\theta=\delta_{jj'} \tag{23}$$

The Zernike expansion coefficients are given by

$$a_j=\frac{1}{\pi(1-\epsilon)^2}\int_\epsilon^1\int_0^{2\pi} W(\rho,\theta;\epsilon)Z_j(\rho,\theta;\epsilon)\rho d\rho d\theta \tag{24}$$

as may be seen by substituting Eq. (20) into Eq. (24) and using Eq. (23) for the orthonormality of the polynomials.

The annular polynomials are similar to the circle polynomials, except that they are orthogonal over an annular pupil. They can be obtained from the circle polynomials by the Gram-Schmidt orthogonalization process.[17] The radial polynomials are accordingly given by

$$R_n^m(\rho;\epsilon)=N_n^m\left[R_n^m(\rho)-\sum_{i\ge 1}^{(n-m)/2}(n-2i+1)<R_n^m(\rho)R_{n-2i}^m(\rho;\epsilon)>R_{n-2i}^m(\rho;\epsilon)\right] \tag{25}$$

where

$$<R_n^m(\rho)R_{n'}^m(\rho;\epsilon)>=\frac{2}{1-\epsilon^2}\int_\epsilon^1 R_n^m(\rho)R_{n'}^m(\rho;\epsilon)\rho d\rho \tag{26}$$

and N_n^m is a normalization constant such that the radial polynomials satisfy the orthogonality Eq. (22). Thus, $R_n^m(\rho;\epsilon)$ is a radial polynomial of degree n in ρ containing terms in ρ^n, ρ^{n-2},…, and ρ^m with coefficients that depend on ϵ. The radial polynomials are even or odd in ρ depending on whether n

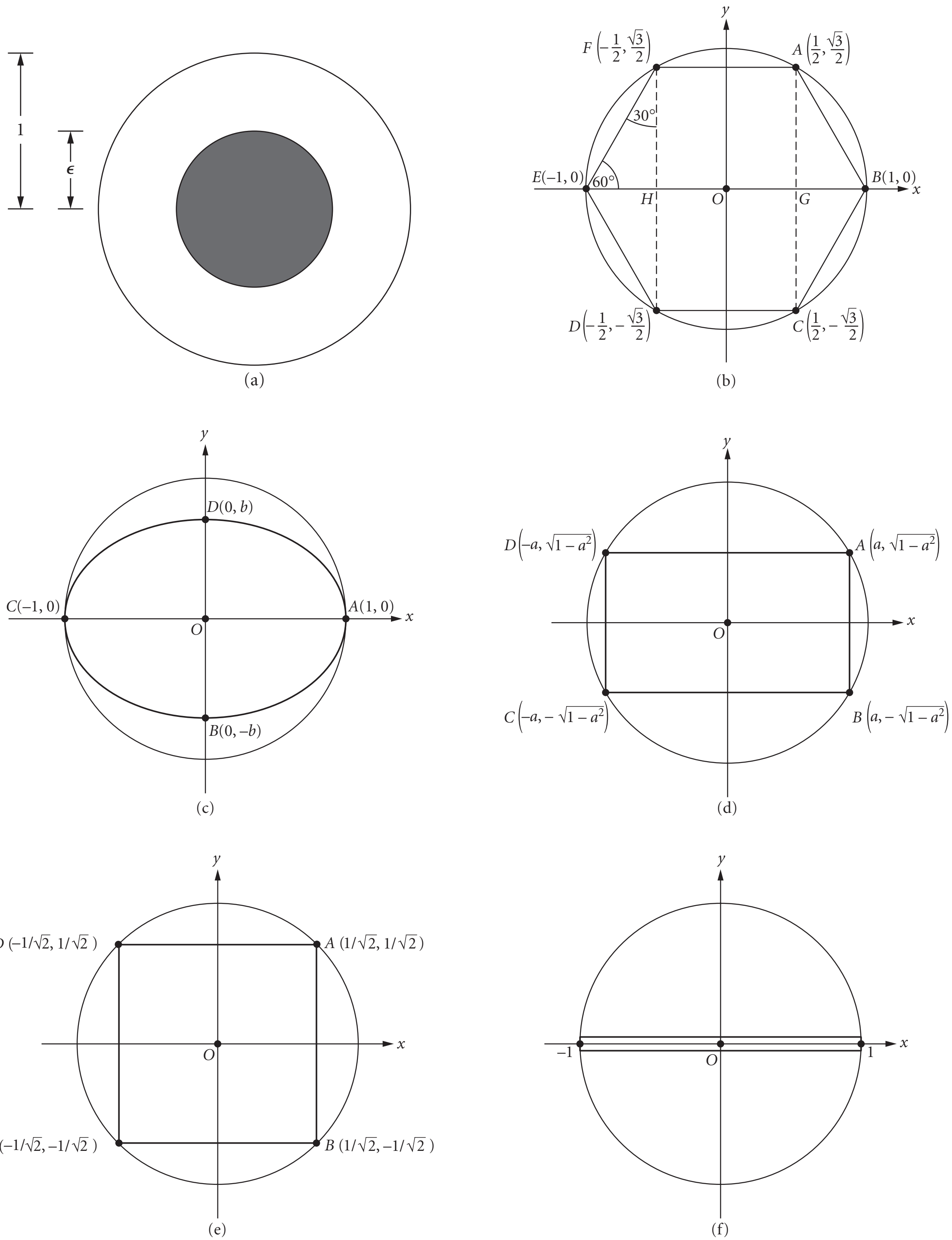


FIGURE 3 Unit pupils inscribed inside a unit circle: (*a*) annulus of obscuration ratio ϵ; (*b*) hexagon; (*c*) ellipse of aspect ratio *b*; (*d*) rectangle of half width *a*; (*e*) square of half width $1/\sqrt{2}$; and (*f*) slit.

(or m) is even or odd. For $m = 0$, the radial polynomials are equal to the Legendre polynomials $P_n(\cdot)$ according to

$$R_{2n}^0(\rho;\epsilon) = P_n\left[\frac{2(\rho^2-\epsilon^2)}{1-\epsilon^2}-1\right] \tag{27}$$

Thus, they can be obtained from the circle radial polynomials $R_{2n}^0(\rho)$ by replacing ρ by $[(\rho^2-\epsilon^2)/(1-\epsilon^2)]^{1/2}$, that is,

$$R_{2n}^0(\rho;\epsilon) = R_{2n}^0\left[\left(\frac{\rho^2-\epsilon^2}{1-\epsilon}\right)^{1/2}\right] \tag{28}$$

It can be seen from Eqs. (22) and (25) that

$$R_n^n(\rho;\epsilon) = \rho^n \Big/ \left(\sum_{i=0}^{n}\epsilon^{2i}\right)^{1/2} \tag{29}$$

$$= \rho^n\{(1-\epsilon^2)\big/[1-\epsilon^{2(n+1)}]\}^{1/2} \tag{30}$$

Moreover,

$$R_2^{n-2}(\rho;\epsilon) = \frac{n\rho^n-(n-1)[(1-\epsilon^{2n})/(1-\epsilon^{2(n-1)})]\rho^{n-2}}{\{(1-\epsilon^2)^{-1}[n^2(1-\epsilon^{2(n+1)})-(n^2-1)(1-\epsilon^{2n})^2/(1-\epsilon^{2(n-1)})]\}^{1/2}} \tag{31}$$

It is evident that the radial polynomial $R_n^n(\rho;\epsilon)$ differs from the corresponding circle polynomial $R_n^n(\rho)$ only in its normalization. We also note that

$$\begin{aligned} R_n^m(1;\epsilon) &= 1, \quad m=0 \\ &\neq 1, \quad m\neq 0 \end{aligned} \tag{32}$$

The variation of several Zernike annular radial polynomials with ρ is shown in Fig. 4 for $\epsilon = 0.5$.

It is evident from Eqs. (21) that the annular polynomials, like the circle polynomials, are separable in the polar coordinates ρ and θ. This is a consequence of the radial symmetry of the annular pupil. As may be evident from the Gram-Schmidt orthogonalization process, each annular polynomial is a linear combination of the circle polynomials.[33] Accordingly, each radial polynomial $R_n^m(\rho;\epsilon)$ can be written as a linear combination of the polynomials $R_n^m(\rho)$, $R_{n-2}^m(\rho)$, . . ., and $R_m^m(\rho)$. For example,

$$R_3^1(\rho;\epsilon) = \frac{1}{(1-\epsilon^2)(1+5\epsilon^2+5\epsilon^4+\epsilon^6)^{1/2}}[(1+\epsilon^2)R_3^1(\rho)-2\epsilon^4R_1^1(\rho)] \tag{33a}$$

and

$$R_4^0(\rho;\epsilon) = \frac{1}{(1-\epsilon^2)^2}[R_4^0(\rho)-3\epsilon^2R_2^0(\rho)+\epsilon^2(1+\epsilon^2)R_0^0(\rho)] \tag{33b}$$

The Zernike annular radial polynomials for $n \leq 8$ are listed in Table 2*a*. The number polynomials of a certain order or through a certain order n is given by the same expressions as in the case of Zernike circle polynomials. Table 2*b* lists the full annular polynomials illustrating their ordering. In Table 2*c*, they are given in the Cartesian coordinates.

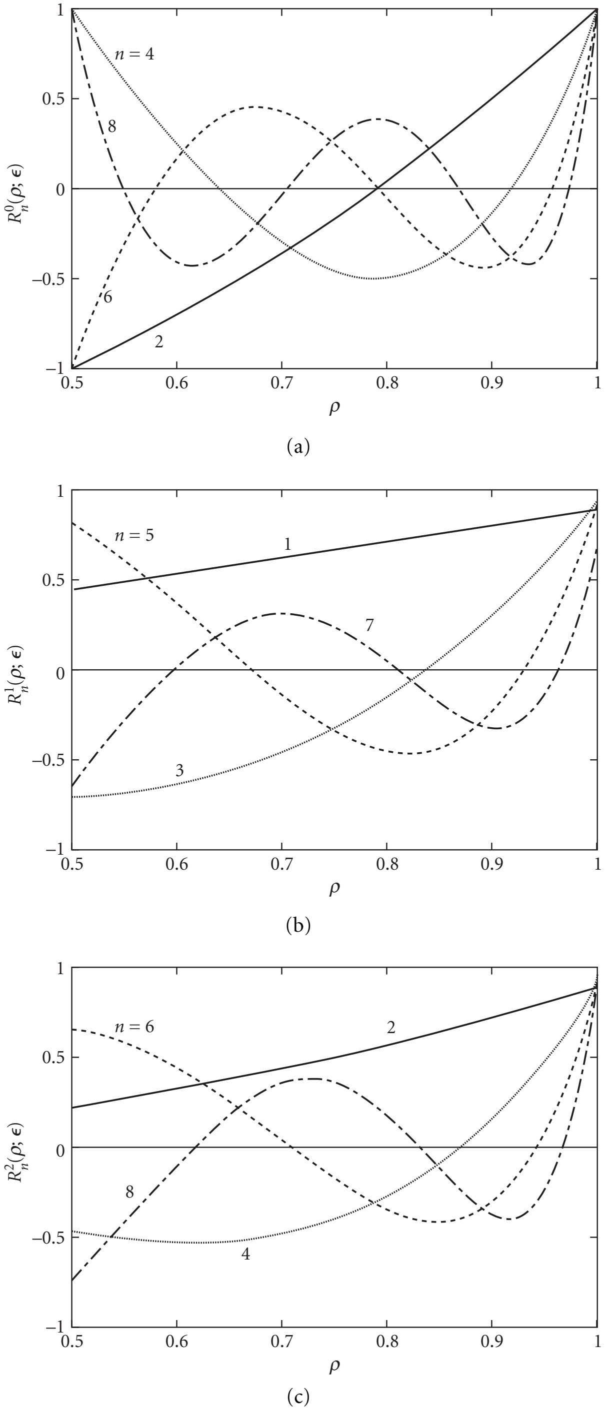


FIGURE 4 Variation of a Zernike *annular* radial polynomial $R_n^m(\rho;\epsilon)$ with ρ for $\epsilon = 0.5$: (*a*) defocus and spherical aberrations; (*b*) tilt and coma; and (*c*) astigmatism.

TABLE 2*a* Zernike *Annular* Radial Polynomials $R_n^m(\rho;\epsilon)$, Where ϵ Is the Obscuration Ratio of Annular Pupil and $\epsilon \le \rho \le 1$

n	m	$R_n^m(\rho;\epsilon)$
0	0	1
1	1	$\rho/(1+\epsilon^2)^{1/2}$
2	0	$(2\rho^2-1-\epsilon^2)/(1-\epsilon^2)$
2	2	$\rho^2/(1+\epsilon^2+\epsilon^2)^{1/2}$
3	1	$\dfrac{3(1+\epsilon^2)\rho^3-2(1+\epsilon^2+\epsilon^4)\rho}{(1-\epsilon^2)[(1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}}$
3	3	$\rho^3/(1+\epsilon^2+\epsilon^4+\epsilon^6)^{1/2}$
4	0	$[6\rho^4-6(1+\epsilon^2)\rho^2+1+4\epsilon^2+\epsilon^4]/(1-\epsilon^2)^2$
4	2	$\dfrac{4\rho^4-3[(1-\epsilon^8)/(1-\epsilon^6)]\rho^2}{\{(1-\epsilon^2)^{-1}[16(1-\epsilon^{10})-15(1-\epsilon^8)^2/(1-\epsilon^6)]^{1/2}\}}$
4	4	$\rho^4/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8)^{1/2}$
5	1	$\dfrac{10(1+4\epsilon^2+\epsilon^4)\rho^5-12(1+4\epsilon^2+4\epsilon^4+\epsilon^6)\rho^3+3(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)\rho}{(1-\epsilon^2)^2[(1+4\epsilon^2+\epsilon^4)(1+9\epsilon^2+9\epsilon^4+\epsilon^6)]^{1/2}}$
5	3	$\dfrac{5\rho^5-4[(1-\epsilon^{10})/(1-\epsilon^8)]\rho^3}{\{(1-\epsilon^2)^{-1}[25(1-\epsilon^{12})-24(1-\epsilon^{10})^2/(1-\epsilon^8)]\}^{1/2}}$
5	5	$\rho^5/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10})^{1/2}$
6	0	$[20\rho^6-30(1+\epsilon^2)\rho^4+12(1+3\epsilon^2+\epsilon^4)\rho^2-(1+9\epsilon^2+9\epsilon^4+\epsilon^6)]/(1-\epsilon^2)^3$
6	2	$\dfrac{15(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)\rho^6-20(1+4\epsilon^2+10\epsilon^4+10\epsilon^6+4\epsilon^8+\epsilon^{10})\rho^4 +6(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+10\epsilon^8+4\epsilon^{10}+\epsilon^{12})\rho^2}{(1+\epsilon^2)^2[(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)(1+9\epsilon^2+45\epsilon^4+65\epsilon^6+45\epsilon^8+9\epsilon^{10}+\epsilon^{12})]^{1/2}}$
6	4	$\dfrac{6\rho^6-5[(1-\epsilon^{12})/(1-\epsilon^{10})]\rho^4}{\{(1-\epsilon^2)^{-1}[36(1-\epsilon^{14})-35(1-\epsilon^{12})^2/(1-\epsilon^{10})]\}^{1/2}}$
6	6	$\rho^6/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12})^{1/2}$
7	1	$a_7^1\rho^7+b_7^1\rho^5+c_7^1\rho^3+d_7^1\rho$
7	3	$a_7^3\rho^7+b_7^3\rho^5+c_7^3\rho^3$
7	5	$\dfrac{7\rho^7-6[(1-\epsilon^{14})/(1-\epsilon^{12})]\rho^5}{\{(1-\epsilon^2)^{-1}[49(1-\epsilon^{16})-48(1-\epsilon^{14})^2/(1-\epsilon^{12})]\}^{1/2}}$
7	7	$\rho^7/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12}+\epsilon^{14})^{1/2}$
8	0	$\dfrac{70\rho^8-140(1+\epsilon^2)\rho^6+30(3+8\epsilon^2+3\epsilon^4)\rho^4-20(1+6\epsilon^2+6\epsilon^4+\epsilon^6)\rho^2+\epsilon_8^0}{(1-\epsilon^2)^4}$
8	2	$a_8^2\rho^8+b_8^2\rho^6+c_8^2\rho^4+d_8^2\rho^2$
8	4	$a_8^4\rho^8+b_8^4\rho^6+c_8^4\rho^4$
8	6	$a_8^6\rho^8+b_8^6\rho^6$
8	8	$\rho^8/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12}+\epsilon^{14}+\epsilon^{16})^{1/2}$

(*Continued*)

TABLE 2a Zernike *Annular* Radial Polynomials $R_n^m(\rho;\epsilon)$, Where ϵ Is the Obscuration Ratio of Annular Pupil and $\epsilon \le \rho \le 1$ (*Continued*)

$$a_7^1 = 35(1+9\epsilon^2+9\epsilon^4+\epsilon^6)/A_7^1$$
$$b_7^1 = -60(1+9\epsilon^2+15\epsilon^4+9\epsilon^6+\epsilon^8)/A_7^1$$
$$c_7^1 = 30(1+9\epsilon^2+25\epsilon^4+25\epsilon^6+9\epsilon^8+\epsilon^{10})/A_7^1$$
$$d_7^1 = -4(1+9\epsilon^2+45\epsilon^4+65\epsilon^6+45\epsilon^8+9\epsilon^{10}+\epsilon^{12})/A_7^1$$
$$A_7^1 = (1-\epsilon^2)^3(1+9\epsilon^2+9\epsilon^4+\epsilon^6)^{1/2}(1+16\epsilon^2+36\epsilon^4+16\epsilon^6+\epsilon^8)^{1/2}$$
$$a_7^3 = 21(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+10\epsilon^8+4\epsilon^{10}+\epsilon^{12})/A_7^3$$
$$b_7^3 = -30(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+20\epsilon^8+10\epsilon^{10}+4\epsilon^{12}+\epsilon^{14})/A_7^3$$
$$c_7^3 = 10(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+35\epsilon^8+20\epsilon^{10}+10\epsilon^{12}+4\epsilon^{14}+\epsilon^{16})/A_7^3$$
$$A_7^3 = (1-\epsilon^2)^2(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+10\epsilon^8+4\epsilon^{10}+\epsilon^{12})^{1/2}$$
$$\times(1+9\epsilon^2+45\epsilon^4+165\epsilon^6+270\epsilon^8+270\epsilon^{10}+165\epsilon^{12}+45\epsilon^{14}+9\epsilon^{16}+\epsilon^{18})^{1/2}$$
$$e_8^0 = 1+16\epsilon^2+36\epsilon^4+16\epsilon^6+\epsilon^8$$
$$a_8^2 = 56(1+9\epsilon^2+45\epsilon^4+65\epsilon^6+45\epsilon^8+9\epsilon^{10}+\epsilon^{12})/A_8^2$$
$$b_8^2 = -105(1+9\epsilon^2+45\epsilon^4+85\epsilon^6+85\epsilon^8+45\epsilon^{10}+9\epsilon^{12}+\epsilon^{14})/A_8^2$$
$$c_8^2 = 60(1+9\epsilon^2+45\epsilon^4+115\epsilon^6+150\epsilon^8+115\epsilon^{10}+45\epsilon^{12}+9\epsilon^{14}+\epsilon^{16})/A_8^2$$
$$d_8^2 = -10(1+9\epsilon^2+45\epsilon^4+165\epsilon^6+270\epsilon^8+270\epsilon^{10}+165\epsilon^{12}+45\epsilon^{14}+9\epsilon^{16}+\epsilon^{18})/A_8^2$$
$$A_8^2 = (1-\epsilon^2)^3(1+9\epsilon^2+45\epsilon^4+65\epsilon^6+45\epsilon^8+9\epsilon^{10}+\epsilon^{12})^{1/2}$$
$$\times(1+16\epsilon^2+136\epsilon^4+416\epsilon^6+626\epsilon^8+416\epsilon^{10}+136\epsilon^{12}+16\epsilon^{14}+\epsilon^{16})^{1/2}$$
$$a_8^4 = 28(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+35\epsilon^8+20\epsilon^{10}+10\epsilon^{12}+4\epsilon^{14}+\epsilon^{16})/A_8^4$$
$$b_8^4 = -42(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+35\epsilon^8+35\epsilon^{10}+20\epsilon^{12}+10\epsilon^{14}+4\epsilon^{16}+\epsilon^{18})/A_8^4$$
$$c_8^4 = 15(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+35\epsilon^8+56\epsilon^{10}+35\epsilon^{12}+20\epsilon^{14}+10\epsilon^{16}+4\epsilon^{18}+\epsilon^{20})/A_8^4$$
$$A_8^4 = (1-\epsilon^2)^2(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+35\epsilon^8+20\epsilon^{10}+10\epsilon^{12}+4\epsilon^{14}+\epsilon^{16})^{1/2}$$
$$\times(1+9\epsilon^2+45\epsilon^4+165\epsilon^6+495\epsilon^8+846\epsilon^{10}+994\epsilon^{12}+846\epsilon^{14}+495\epsilon^{16}+165\epsilon^{18}+45\epsilon^{20}+9\epsilon^{22}+\epsilon^{24})^{1/2}$$
$$a_8^6 = 8(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12})/A_8^6$$
$$b_8^6 = -7(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12}+\epsilon^{14})/A_8^6$$
$$A_8^6 = (1-\epsilon^2)(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12})^{1/2}$$
$$\times(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+35\epsilon^8+56\epsilon^{10}+84\epsilon^{12}+56\epsilon^{14}+35\epsilon^{16}+20\epsilon^{18}+10\epsilon^{20}+4\epsilon^{22}+\epsilon^{24})^{1/2}$$

TABLE 2*b* Orthonormal Zernike *Annular* Polynomials $Z_j(\rho, \theta; \epsilon)$, Ordered in the Same Manner as the Zernike Circle Polynomials in Table 1*a*

j	n	m	$Z_j(\rho, \theta; \epsilon)$*	Aberration Name*
1	0	0	$R_0^0(\rho;\epsilon)=1$	Piston
2	1	1	$2R_1^1(\rho;\epsilon)\cos\theta$	*x* tilt
3	1	1	$2R_1^1(\rho;\epsilon)\sin\theta$	*y* tilt
4	2	0	$\sqrt{3}R_2^0(\rho;\epsilon)$	Defocus
5	2	2	$\sqrt{6}R_2^2(\rho;\epsilon)\sin2\theta$	Primary astigmatism at 45°
6	2	2	$\sqrt{6}R_2^2(\rho;\epsilon)\cos2\theta$	Primary astigmatism at 0°
7	3	1	$\sqrt{8}R_3^1(\rho;\epsilon)\sin\theta$	Primary *y* coma
8	3	1	$\sqrt{8}R_3^1(\rho;\epsilon)\cos\theta$	Primary *x* coma
9	3	3	$\sqrt{8}R_3^3(\rho;\epsilon)\sin3\theta$	
10	3	3	$\sqrt{8}R_3^3(\rho;\epsilon)\cos3\theta$	
11	4	0	$\sqrt{5}R_4^0(\rho;\epsilon)$	Primary spherical aberration
12	4	2	$\sqrt{10}R_4^2(\rho;\epsilon)\cos2\theta$	Secondary astigmatism at 0°
13	4	2	$\sqrt{10}R_4^2(\rho;\epsilon)\sin2\theta$	Secondary astigmatism at 45°
14	4	4	$\sqrt{10}R_4^4(\rho;\epsilon)\cos4\theta$	
15	4	4	$\sqrt{10}R_4^4(\rho;\epsilon)\sin4\theta$	
16	5	1	$\sqrt{12}R_5^1(\rho;\epsilon)\cos\theta$	Secondary *x* coma
17	5	1	$\sqrt{12}R_5^1(\rho;\epsilon)\sin\theta$	Secondary *y* coma
18	5	3	$\sqrt{12}R_5^3(\rho;\epsilon)\cos3\theta$	
19	5	3	$\sqrt{12}R_5^3(\rho;\epsilon)\sin3\theta$	
20	5	5	$\sqrt{12}R_5^5(\rho;\epsilon)\cos5\theta$	
21	5	5	$\sqrt{12}R_5^5(\rho;\epsilon)\sin5\theta$	
22	6	0	$\sqrt{7}R_6^0(\rho;\epsilon)$	Secondary spherical aberration
23	6	2	$\sqrt{14}R_6^2(\rho;\epsilon)\sin2\theta$	Tertiary astigmatism at 45°
24	6	2	$\sqrt{14}R_6^2(\rho;\epsilon)\cos2\theta$	Tertiary astigmatism at 0°
25	6	4	$\sqrt{14}R_6^4(\rho;\epsilon)\cos4\theta$	
26	6	4	$\sqrt{14}R_6^6(\rho;\epsilon)\sin4\theta$	
27	6	6	$\sqrt{14}R_6^6(\rho;\epsilon)\sin6\theta$	
28	6	6	$\sqrt{14}R_6^6(\rho;\epsilon)\cos6\theta$	
29	7	1	$4R_7^1(\rho;\epsilon)\sin\theta$	
30	7	1	$4R_7^1(\rho;\epsilon)\cos\theta$	
31	7	3	$4R_7^3(\rho;\epsilon)\sin3\theta$	
32	7	3	$4R_7^3(\rho;\epsilon)\cos3\theta$	
33	7	5	$4R_7^5(\rho;\epsilon)\sin5\theta$	
34	7	5	$4R_7^5(\rho;\epsilon)\cos5\theta$	
35	7	7	$4R_7^7(\rho;\epsilon)\sin7\theta$	
36	7	7	$4R_7^7(\rho;\epsilon)\cos7\theta$	
37	8	0	$3R_8^0(\rho;\epsilon)$	Tertiary spherical aberration
38	8	2	$\sqrt{18}R_8^2(\rho;\epsilon)\cos2\theta$	Quaternary astigmatism at 0°
39	8	2	$\sqrt{18}R_8^2(\rho;\epsilon)\sin2\theta$	Quaternary astigmatism at 45°
40	8	4	$\sqrt{18}R_8^4(\rho;\epsilon)\cos4\theta$	
41	8	4	$\sqrt{18}R_8^4(\rho;\epsilon)\sin4\theta$	
42	8	6	$\sqrt{18}R_8^6(\rho;\epsilon)\cos6\theta$	
43	8	6	$\sqrt{18}R_8^6(\rho;\epsilon)\sin6\theta$	
44	8	8	$\sqrt{18}R_8^8(\rho;\epsilon)\cos8\theta$	
45	8	8	$\sqrt{18}R_8^8(\rho;\epsilon)\sin8\theta$	

*The words "orthonormal Zernike annular" should be added to the name, e.g., orthonormal Zernike annular primary spherical aberration.

TABLE 2c Orthonormal Zernike *Annular* Polynomials $Z_j(x, y; \epsilon)$ in Cartesian Coordinates (x, y), Where $x = \rho\cos\theta$, $y = \rho\sin\theta$, and $\epsilon \le \rho = \sqrt{x^2+y^2} \le 1$

Polynomial	$Z_j(x, y; \epsilon)$
Z_1	1
Z_2	$2x/(1+\epsilon^2)^{1/2}$
Z_3	$2y/(1+\epsilon^2)^{1/2}$
Z_4	$\sqrt{3}(2\rho^2-1-\epsilon^2)/(1-\epsilon^2)$
Z_5	$2\sqrt{6}xy/(1+\epsilon^2+\epsilon^4)^{1/2}$
Z_6	$\sqrt{6}(x^2-y^2)/(1+\epsilon^2+\epsilon^4)^{1/2}$
Z_7	$\dfrac{\sqrt{8}y[3(1+\epsilon^2)\rho^2-2(1+\epsilon^2+\epsilon^4)]}{(1-\epsilon^2)[1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}}$
Z_8	$\dfrac{\sqrt{8}x[3(1+\epsilon^2)\rho^2-2(1+\epsilon^2+\epsilon^4)]}{(1-\epsilon^2)[1+\epsilon^2)(1+4\epsilon^2+\epsilon^4)]^{1/2}}$
Z_9	$\sqrt{8}y(3x^2-y^2)/(1+\epsilon^2+\epsilon^4+\epsilon^6)^{1/2}$
Z_{10}	$\sqrt{8}x(x^2-3y^2)/(1+\epsilon^2+\epsilon^4+\epsilon^6)^{1/2}$
Z_{11}	$\sqrt{5}[6\rho^4-6(1+\epsilon^2)\rho^2+(1+4\epsilon^2+\epsilon^4)]/(1-\epsilon^2)^2$
Z_{12}	$\dfrac{\sqrt{10}(x^2-y^2)[4\rho^2-3(1-\epsilon^8)/(1-\epsilon^6)]}{\{(1-\epsilon^2)^{-1}[16(1-\epsilon^{10})-15(1-\epsilon^8)^2/(1-\epsilon^{12})]\}^{1/2}}$
Z_{13}	$\dfrac{2\sqrt{10}xy[4\rho^2-3(1-\epsilon^8)/(1-\epsilon^6)]}{\{(1-\epsilon^2)^{-1}[16(1-\epsilon^{10})-15(1-\epsilon^8)^2/(1-\epsilon^6)]\}^{1/2}}$
Z_{14}	$\sqrt{10}(\rho^4-8x^2y^2)/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8)^{1/2}$
Z_{15}	$4\sqrt{10}xy(x^2-y^2)/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8)^{1/2}$
Z_{16}	$\dfrac{\sqrt{12}x[10(1+4\epsilon^2+\epsilon^4)\rho^4-12(1+4\epsilon^2+4\epsilon^4+\epsilon^6)\rho^2+3(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)]}{(1-\epsilon^2)^2[(1+4\epsilon^2+\epsilon^4)(1+9\epsilon^2+9\epsilon^4+\epsilon^6)]^{1/2}}$
Z_{17}	$\dfrac{\sqrt{12}y[10(1+4\epsilon^2+\epsilon^4)\rho^4-12(1+4\epsilon^2+4\epsilon^4+\epsilon^6)\rho^2+3(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)]}{(1-\epsilon^2)^2[(1+4\epsilon^2+\epsilon^4)(1+9\epsilon^2+9\epsilon^4+\epsilon^6)]^{1/2}}$
Z_{18}	$\dfrac{\sqrt{12}x(x^2-3y^2)[5\rho^2-4(1-\epsilon^{10})/(1-\epsilon^8)]}{\{(1-\epsilon^2)^{-1}[25(1-\epsilon^{12})-24(1-\epsilon^{10})^2/(1-\epsilon^8)]\}^{1/2}}$
Z_{19}	$\dfrac{\sqrt{12}y(3x^2-y^2)[5\rho^2-4(1-\epsilon^{10})/(1-\epsilon^8)]}{\{(1-\epsilon^2)^{-1}[25(1-\epsilon^{12})-24(1-\epsilon^{10})^2/(1-\epsilon^8)]\}^{1/2}}$
Z_{20}	$\sqrt{12}x(16x^4-20x^2\rho^2+5\rho^4)/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10})^{1/2}$
Z_{21}	$\sqrt{12}y(16y^4-20y^2\rho^2+5\rho^4)/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10})^{1/2}$
Z_{22}	$\sqrt{7}[20\rho^6-30(1+\epsilon^2)\rho^4+12(1+3\epsilon^2+\epsilon^4)\rho^2-(1+9\epsilon^2+9\epsilon^4+\epsilon^6)]/(1-\epsilon^2)^3$
Z_{23}	$\dfrac{2\sqrt{14}xy[15(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)\rho^4-20(1+4\epsilon^2+10\epsilon^4+10\epsilon^6+4\epsilon^8+\epsilon^{10})\rho^2+6(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+10\epsilon^8+4\epsilon^{10}+\epsilon^{12})]}{(1-\epsilon^2)^2[(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)(1+9\epsilon^2+45\epsilon^4+65\epsilon^6+45\epsilon^8+9\epsilon^{10}+\epsilon^{12})]^{1/2}}$
Z_{24}	$\dfrac{\sqrt{14}(x^2-y^2)[15(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)\rho^4-20(1+4\epsilon^2+10\epsilon^4+10\epsilon^6+4\epsilon^8+\epsilon^{10})\rho^2+6(1+4\epsilon^2+10\epsilon^4+20\epsilon^6+10\epsilon^8+4\epsilon^{10}+\epsilon^{12})]}{(1-\epsilon^2)^2(1+4\epsilon^2+10\epsilon^4+4\epsilon^6+\epsilon^8)(1+9\epsilon^2+45\epsilon^4+65\epsilon^6+45\epsilon^8+9\epsilon^{10}+\epsilon^{12})^{1/2}}$

(*Continued*)

TABLE 2c Orthonormal Zernike *Annular* Polynomials $Z_j(x, y; \epsilon)$ in Cartesian Coordinates (x, y), Where $x = \rho\cos\theta$, $y = \rho\sin\theta$, and $\epsilon \le \rho = \sqrt{x^2+y^2} \le 1$ (*Continued*)

Polynomial	$Z_j(x, y; \epsilon)$
Z_{25}	$\dfrac{4\sqrt{14}xy(x^2-y^2)[6\rho^2-5(1-\epsilon^{12})/(1-\epsilon^{10})]}{\{(1-\epsilon^2)^{-1}[36(1-\epsilon^{14})-35(1-\epsilon^{12})^2/(1-\epsilon^{10})]\}^{1/2}}$
Z_{26}	$\dfrac{\sqrt{14}(8x^4-8x^2\rho^2+\rho^4)[6\rho^2-5(1-\epsilon^{12})/(1-\epsilon^{10})]}{\{(1-\epsilon^2)^{-1}[36(1-\epsilon^{14})-35(1-\epsilon^{12})^2/(1-\epsilon^{10})]\}^{1/2}}$
Z_{27}	$\sqrt{14}xy(32x^4-32x^2\rho^2+6\rho^4)/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12})^{1/2}$
Z_{28}	$\sqrt{14}(32x^6-48x^4\rho^2+18x^2\rho^4-\rho^6)/(1+\epsilon^2+\epsilon^4+\epsilon^6+\epsilon^8+\epsilon^{10}+\epsilon^{12})^{1/2}$

11.6 HEXAGONAL POLYNOMIALS

Figure 3*b* shows a *unit hexagon* inscribed inside a unit circle. Each side of the hexagon has a length of unity. The area of the hexagon is $A = 3\sqrt{3}/2$. The orthonormality of the hexagonal polynomials $H_j(x, y)$ implies that[26]

$$\frac{2}{3\sqrt{3}} \int_{\text{hexagon}} H_j(x, y) H_{j'}(x, y)\, dx\, dy = \delta_{jj'} \tag{34}$$

The orthonormal hexagonal polynomials are given in Tables 3 up to the eighth order in three different but equivalent forms. In Table 3*a*, each hexagonal polynomial is written in terms of the circle polynomials, thus illustrating the relationship between the two. In particular, it helps determine the potential error made when a hexagonal aberration function is expanded in terms of the circle polynomials.[34] The polynomials up to H_{19} are given in their analytical form, but those with $j > 19$ are written in a numerical form because of the increasing complexity of the coefficients of the circle polynomials. In Table 3*b*, the hexagonal polynomials are given in polar coordinates, showing one-to-one correspondence with the circle polynomials, but illustrating the difference from them. This form is convenient for analytical calculations because of the integration of trigonometric functions over symmetric limits. Finally, in Table 3*c*, they are given in Cartesian coordinates, as they would be used for any quantitative numerical analysis of, say, an interferogram.

From Table 3*a*, we note that each hexagonal polynomial consists of cosine or sine terms, but not both. Unlike the circle,[3–6] annular,[8–11] or Gauss[23,24] polynomials, the hexagonal polynomials are generally not separable in ρ and θ due to the lack of radial symmetry of the hexagonal pupil. The first 13 polynomials, that is, up to H_{13}, are separable, but H_{14} and H_{15} are not; H_{16} through H_{19} are separable, but H_{20} and H_{21} are not. Accordingly, the notion of two indices n and m with dependence on m in the form of $\cos m\theta$ or $\sin m\theta$, as in the case of circle polynomials, loses significance. For example, the Zernike polynomial Z_{14} for $n = 4$ and $m = 4$ varies as $\cos 4\theta$, but H_{14} has a term in $\cos 2\theta$ also. Hence, the hexagonal polynomials can be ordered by a single index only. While the polynomials H_{11} and H_{22} representing the balanced primary and secondary spherical aberrations are radially symmetric, the polynomial H_{37} representing the balanced tertiary spherical aberration is not, since it consists of an angle-dependent term in Z_{28} or $\cos 6\theta$ also. If this term is not included in the polynomial H_{37}, the standard deviation of the aberration increases from a value of unity to 1.3339.

11.7 ELLIPTICAL POLYNOMIALS

Figure 3*c* shows a *unit ellipse* of an aspect ratio b inscribed inside a unit circle. The semimajor and semiminor axes of the ellipse have lengths of unity and b, respectively. Of course, a unit ellipse is not unique, since b can have any value between 0 and 1. It is represented by an equation

$$x^2 + y^2/b^2 = 1 \tag{35a}$$

TABLE 3a Orthonormal *Hexagonal* Polynomials H_j in Terms of Zernike Circle Polynomials Z_j

$H_1 = Z_1$
$H_2 = \sqrt{6/5}Z_2$
$H_3 = \sqrt{6/5}Z_3$
$H_4 = \sqrt{5/43}Z_1 + (2\sqrt{15/43})Z_4$
$H_5 = \sqrt{10/7}Z_5$
$H_6 = \sqrt{10/7}Z_6$
$H_7 = 16\sqrt{14/11055}Z_3 + 10\sqrt{35/2211}Z_7$
$H_8 = 16\sqrt{14/11055}Z_2 + 10\sqrt{35/2211}Z_8$
$H_9 = (2\sqrt{5}/3)Z_9$
$H_{10} = 2\sqrt{35/103}Z_{10}$
$H_{11} = (521/\sqrt{1072205})Z_1 + 88\sqrt{15/214441}Z_4 + 14\sqrt{43/4987}Z_{11}$
$H_{12} = 225\sqrt{6/492583}Z_6 + 42\sqrt{70/70369}Z_{12}$
$H_{13} = 225\sqrt{6/492583}Z_5 + 42\sqrt{70/70369}Z_{13}$
$H_{14} = -2525\sqrt{14/297774543}Z_6 - (1495\sqrt{70/99258181}/3)Z_{12} + (\sqrt{378910/18337}/3)Z_{14}$
$H_{15} = 2525\sqrt{14/297774543}Z_5 + (1495\sqrt{70/99258181}/3)Z_{13} + (\sqrt{378910/18337}/3)Z_{15}$
$H_{16} = 30857\sqrt{2/3268147641}Z_2 + (49168/\sqrt{3268147641})Z_8 + 42\sqrt{1474/1478131}Z_{16}$
$H_{17} = 30857\sqrt{2/3268147641}Z_3 + (49168/\sqrt{3268147641})Z_7 + 42\sqrt{1474/1478131}Z_{17}$
$H_{18} = 386\sqrt{770/295894589}Z_{10} + 6\sqrt{118965/2872763}Z_{18}$
$H_{19} = 6\sqrt{10/97}Z_9 + 14\sqrt{5/291}Z_{19}$
$H_{20} = -0.71499593Z_2 - 0.72488884Z_8 - 0.46636441Z_{16} + 1.72029850Z_{20}$
$H_{21} = 0.71499594Z_3 + 0.72488884Z_7 + 0.46636441Z_{17} + 1.72029850Z_{21}$
$H_{22} = 0.58113135Z_1 + 0.89024136Z_4 + 0.89044507Z_{11} + 1.32320623Z_{22}$
$H_{23} = 1.15667686Z_5 + 1.10775599Z_{13} + 0.43375081Z_{15} + 1.39889072Z_{23}$
$H_{24} = 1.15667686Z_6 + 1.10775599Z_{12} - 0.43375081Z_{14} + 1.39889072Z_{24}$
$H_{25} = 1.31832566Z_5 + 1.14465174Z_{13} + 1.94724032Z_{15} + 0.67629133Z_{23} + 1.75496998Z_{25}$
$H_{26} = -1.31832566Z_6 - 1.14465174Z_{12} + 1.94724032Z_{14} - 0.67629133Z_{24} + 1.75496998Z_{26}$
$H_{27} = 2\sqrt{77/93}Z_{27}$
$H_{28} = -1.07362889Z_1 - 1.52546162Z_4 - 1.28216588Z_{11} - 0.70446308Z_{22} + 2.09532473Z_{28}$
$H_{29} = 0.97998834Z_3 + 1.16162002Z_7 + 1.04573775Z_{17} + 0.40808953Z_{21} + 1.36410394Z_{29}$
$H_{30} = 0.97998834Z_2 + 1.16162002Z_8 + 1.04573775Z_{16} - 0.40808953Z_{20} + 1.36410394Z_{30}$
$H_{31} = 3.63513758Z_9 + 2.92084414Z_{19} + 2.11189625Z_{31}$
$H_{32} = 0.69734874Z_{10} + 0.67589740Z_{18} + 1.22484055Z_{32}$
$H_{33} = 1.56189763Z_3 + 1.69985309Z_7 + 1.29338869Z_{17} + 2.57680871Z_{21} + 0.67653220Z_{29}$
$\quad +1.95719339Z_{33}$
$H_{34} = -1.56189763Z_2 - 1.69985309Z_8 - 1.29338869Z_{16} + 2.57680871Z_{20} - 0.67653220Z_{30}$
$\quad +1.95719339Z_{34}$
$H_{35} = -1.63832594Z_3 - 1.74759886Z_7 - 1.27572528Z_{17} - 0.77446421Z_{21} - 0.60947360Z_{29}$
$\quad -0.36228537Z_{33} + 2.24453237Z_{35}$
$H_{36} = -1.63832594Z_2 - 1.74759886Z_8 - 1.27572528Z_{16} + 0.77446421Z_{20} - 0.60947360Z_{30}$
$\quad +0.36228537Z_{34} + 2.24453237Z_{36}$
$H_{37} = 0.82154671Z_1 + 1.27988084Z_4 + 1.32912377Z_{11} + 1.11636637Z_{22} - 0.54097038Z_{28}$
$\quad +1.37406534Z_{37}$
$H_{38} = 1.54526522Z_6 + 1.57785242Z_{12} - 0.89280081Z_{14} + 1.28876176Z_{24} - 0.60514082Z_{26}$
$\quad +1.43097780Z_{38}$
$H_{39} = 1.54526522Z_5 + 1.57785242Z_{13} + 0.89280081Z_{15} + 1.28876176Z_{23} + 0.60514082Z_{25}$
$\quad +1.43097780Z_{39}$
$H_{40} = -2.51783502Z_6 - 2.38279377Z_{12} + 3.42458933Z_{14} - 1.69296616Z_{24} + 2.56612920Z_{26}$
$\quad -0.85703819Z_{38} + 1.89468756Z_{40}$
$H_{41} = 2.51783502Z_5 + 2.38279377Z_{13} + 3.42458933Z_{15} + 1.69296616Z_{23} + 2.56612920Z_{25}$
$\quad +0.85703819Z_{39} + 1.89468756Z_{41}$

(*Continued*)

TABLE 3a Orthonormal *Hexagonal* Polynomials H_j in Terms of Zernike Circle Polynomials Z_j (*Continued*)

$$H_{42} = -2.72919646Z_1 - 4.02313214Z_4 - 3.69899239Z_{11} - 2.49229315Z_{22} + 4.36717121Z_{28} - 1.13485132Z_{37} + 2.52330106Z_{42}$$
$$H_{43} = 1362\sqrt{77/20334667}Z_{27} + (260/3)\sqrt{341/655957}Z_{43}$$
$$H_{44} = -2.76678413Z_6 - 2.50005278Z_{12} + 1.48041348Z_{14} - 1.62947374Z_{24} + 0.95864121Z_{26} - 0.69034812Z_{38} + 0.40743941Z_{40} + 2.56965299Z_{44}$$
$$H_{45} = -2.76678413Z_5 - 2.50005278Z_{13} - 1.48041348Z_{15} - 1.62947374Z_{23} - 0.95864121Z_{25} - 0.69034812Z_{39} - 0.40743941Z_{41} + 2.56965299Z_{45}$$

TABLE 3b Orthonormal *Hexagonal* Polynomials $H_j(\rho, \theta)$ in Polar Coordinates

$$H_1 = 1$$
$$H_2 = 2\sqrt{6/5}\rho\cos\theta$$
$$H_3 = 2\sqrt{6/5}\rho\sin\theta$$
$$H_4 = \sqrt{5/43}(-5 + 12\rho^2)$$
$$H_5 = 2\sqrt{15/7}\rho^2\sin 2\theta$$
$$H_6 = 2\sqrt{15/7}\rho^2\cos 2\theta$$
$$H_7 = 4\sqrt{42/3685}(-14\rho + 25\rho^3)\sin\theta$$
$$H_8 = 4\sqrt{42/3685}(-14\rho + 25\rho^3)\cos\theta$$
$$H_9 = (4\sqrt{10}/3)\rho^3\sin 3\theta$$
$$H_{10} = 4\sqrt{70/103}\rho^3\cos 3\theta$$
$$H_{11} = (3/\sqrt{1072205})(737 - 5140\rho^2 + 6020\rho^4)$$
$$H_{12} = (30/\sqrt{492583})(-249\rho^2 + 392\rho^4)\cos 2\theta$$
$$H_{13} = (30/\sqrt{492583})(-249\rho^2 + 392\rho^4)\sin 2\theta$$
$$H_{14} = (10/3)\sqrt{7/99258181}[10(297 - 598\rho^2)\rho^2\cos 2\theta + 5413\rho^4\cos 4\theta]$$
$$H_{15} = (10/3)\sqrt{7/99258181}[-10(297 - 598\rho^2)\rho^2\sin 2\theta + 5413\rho^4\sin 4\theta]$$
$$H_{16} = 2\sqrt{6/1089382547}(70369\rho - 322280\rho^3 + 309540\rho^5)\cos\theta$$
$$H_{17} = 2\sqrt{6/1089382547}(70369\rho - 322280\rho^3 + 309540\rho^5)\sin\theta$$
$$H_{18} = 4\sqrt{385/295894589}(-3322\rho^3 + 4635\rho^5)\cos 3\theta$$
$$H_{19} = 4\sqrt{5/97}(-22\rho^3 + 35\rho^5)\sin 3\theta$$
$$H_{20} = (-2.17600248\rho + 13.23551876\rho^3 - 16.15533716\rho^5)\cos\theta + 5.95928883\rho^5\cos 5\theta$$
$$H_{21} = (2.17600248\rho - 13.23551876\rho^3 + 16.15533716\rho^5)\sin\theta + 5.95928883\rho^5\sin 5\theta$$
$$H_{22} = -2.47059083 + 33.14780774\rho^2 - 93.07966445\rho^4 + 70.01749250\rho^6$$
$$H_{23} = (23.72919095\rho^2 - 90.67126833\rho^4 + 78.51254738\rho^6)\sin 2\theta + 1.37164051\rho^4\sin 4\theta$$
$$H_{24} = (23.72919095\rho^2 - 90.67126833\rho^4 + 78.51254738\rho^6)\cos 2\theta - 1.37164051\rho^4\cos 4\theta$$
$$H_{25} = (7.55280798\rho^2 - 36.13018255\rho^4 + 37.95675688\rho^6)\sin 2\theta + (-26.67476754\rho^4 + 39.39897852\rho^6)\sin 4\theta$$
$$H_{26} = (-7.55280798\rho^2 + 36.13018255\rho^4 - 37.95675688\rho^6)\cos 2\theta + (-26.67476754\rho^4 + 39.39897852\rho^6)\cos 4\theta$$
$$H_{27} = 14\sqrt{22/93}\rho^6\sin 6\theta$$
$$H_{28} = 0.56537219 - 10.44830313\rho^2 + 38.71296332\rho^4 - 37.27668254\rho^6 + 7.83998727\rho^6\cos 6\theta$$
$$H_{29} = (-15.56917599 + 130.07864353\rho^2 - 288.33220017\rho^4 + 190.97455178\rho^6)\rho\sin\theta + 2.82732724\rho^5\sin 3\theta + 1.41366362\rho^5\sin 5\theta$$
$$H_{30} = (-15.56917599 + 130.07864353\rho^2 - 288.33220017\rho^4 + 190.97455178\rho^6)\rho\cos\theta + 2.82732724\rho^5\cos 3\theta + 1.41366362\rho^5\cos 5\theta$$
$$H_{31} = (54.28516840 - 202.83704634\rho^2 + 177.39928561\rho^4)\rho^3\sin 3\theta$$
$$H_{32} = (41.60051295 - 135.27397959\rho^2 + 102.88660624\rho^4)\rho^3\cos 3\theta$$
$$H_{33} = (-3.87525156 + 41.84243767\rho^2 - 193.65605837\rho^4 + 204.31733848\rho^6)\rho\sin\theta + (76.09262860 - 109.60283027\rho^2)\rho^5\sin 3\theta + (38.04631430 - 54.80141514\rho^2)\rho^5\sin 5\theta$$
$$H_{34} = (3.87525156 - 41.84243767\rho^2 + 117.56342977\rho^4 - 94.71450820\rho^6)\rho\cos\theta + (-76.09262860 + 109.60283027\rho^2)\rho^5\cos 3\theta + (38.04631430 - 54.80141514\rho^2)\rho^5\cos 5\theta$$

(*Continued*)

TABLE 3*b* Orthonormal *Hexagonal* Polynomials $H_j(\rho, \theta)$ in Polar Coordinates (*Continued*)

$$H_{35} = (3.10311187 - 34.93479698\rho^2 + 114.10529848\rho^4 - 87.65802721\rho^6)\rho \sin\theta + (12.02405243 - 2.33172188\rho^2)\rho^5 \sin 3\theta + (12.02405243 + 3.68030434\rho^2)\rho^5 \sin 5\theta + 6.01202622\rho^7 \sin 7\theta$$
$$H_{36} = (3.10311187 - 34.93479698\rho^2 + 114.10529848\rho^4 - 87.65802721\rho^6)\rho \cos\theta + (12.02405243 - 2.33172188\rho^2)\rho^5 \cos 3\theta + (12.02405243 + 3.68030434\rho^2)\rho^5 \sin 5\theta + 6.01202622\rho^7 \cos 7\theta$$
$$H_{37} = 2.74530738 - 60.39881618\rho^2 + 300.22087475\rho^4 - 518.03488742\rho^6 + 288.55372176\rho^8 - 2.02412582\rho^6 \cos 6\theta$$
$$H_{38} = (-42.96232789 + 287.78381063\rho^2 - 565.13651608\rho^4 + 339.98298180\rho^4)\rho^2 \cos 2\theta + (8.49786414 - 13.58537785\rho^2)\rho^4 \cos 4\theta$$
$$H_{39} = (-42.96232789 + 287.78381063\rho^2 - 565.13651608\rho^4 + 339.98298180\rho^4)\rho^2 \sin 2\theta + (8.49786414 - 13.58537785\rho^2)\rho^4 \sin 4\theta$$
$$H_{40} = (14.79181046 - 121.61654135\rho^2 + 286.77354559\rho^4 - 203.62188574\rho^6)\rho^2 \cos 2\theta + (83.39879886 - 280.00664075\rho^2 + 225.07739907\rho^4)\rho^4 \cos 4\theta$$
$$H_{41} = (-14.79181046 + 121.61654135\rho^2 - 286.77354559\rho^4 + 203.62188574\rho^6)\rho^2 \sin 2\theta + (83.39879886 - 280.00664075\rho^2 + 225.07739907\rho^4)\rho^4 \sin 4\theta$$
$$H_{42} = -0.84269170 + 24.65387703\rho^2 - 158.21741244\rho^4 + 344.75780000\rho^6 - 238.31877895\rho^8 + (-58.59775991 + 85.64367812\rho^2)\rho^6 \cos 6\theta$$
$$H_{43} = 2\sqrt{22/20334667}(-23443 + 32240\rho^2)\rho^6 \sin 6\theta$$
$$H_{44} = (9.64776957 - 85.41873843\rho^2 + 216.08041438\rho^4 - 164.01834750\rho^6)\rho^2 \cos 2\theta + (12.67622930 - 51.08055822\rho^2 + 48.40133344\rho^4)\rho^4 \cos 4\theta + 10.90211434\rho^8 \cos 8\theta$$
$$H_{45} = (9.64776957 - 85.41873843\rho^2 + 216.08041438\rho^4 - 164.01834750\rho^6)\rho^2 \sin 2\theta - (12.67622930 - 51.08055822\rho^2 + 48.40133344\rho^4)\rho^4 \sin 4\theta + 10.90211434\rho^8 \sin 8\theta$$

TABLE 3*c* Orthonormal *Hexagonal* Polynomials $H_j(x, y)$ in Cartesian Coordinates, Where $\rho^2 = x^2 + y^2$

$$H_1 = 1$$
$$H_2 = 2\sqrt{6/5}x$$
$$H_3 = 2\sqrt{6/5}y$$
$$H_4 = \sqrt{5/43}(-5 + 12\rho^2)$$
$$H_5 = 4\sqrt{15/7}xy$$
$$H_6 = 2\sqrt{15/7}(x^2 - y^2)$$
$$H_7 = 4\sqrt{42/3685}(-14 + 25\rho^2)y$$
$$H_8 = 4\sqrt{42/3685}(-14 + 25\rho^2)x$$
$$H_9 = (4/3)\sqrt{10}(3x^2y - y^3)$$
$$H_{10} = 4\sqrt{70/103}(x^3 - 3xy^2)$$
$$H_{11} = (3/\sqrt{1072205})(737 - 5140\rho^2 + 6020\rho^4)$$
$$H_{12} = (30/\sqrt{492583})(392\rho^2 - 249)(x^2 - y^2)$$
$$H_{13} = (60/\sqrt{492583})(392\rho^2 - 249)xy$$
$$H_{14} = -(10/3)\sqrt{7/99258181}[567x^4 + 32478x^2y^2 - 11393y^4 - 2970(x^2 - y^2)]$$
$$H_{15} = (40/3)\sqrt{7/99258181}(-1485 + 8403x^2 - 2423y^2)xy$$
$$H_{16} = 2\sqrt{2/3268147641}(211107 - 966840\rho^2 + 928620\rho^4)x$$
$$H_{17} = 2\sqrt{2/3268147641}(211107 - 966840\rho^2 + 928620\rho^4)y$$
$$H_{18} = 4\sqrt{385/295894589}(-3322 + 4635\rho^2)(x^3 - 3xy^2)$$
$$H_{19} = 4\sqrt{5/97}(-22 + 35\rho^2)(3x^2y - y^3)$$
$$H_{20} = (-2.17600247 + 13.23551876\rho^2 - 10.19604832x^4 - 91.90356268x^2y^2 + 13.64110702y^4)x$$
$$H_{21} = (2.17600247 - 13.23551876\rho^2 + 45.95178134x^4 - 27.28221405x^2y^2 + 22.11462599y^4)y$$
$$H_{22} = -2.47059083 + 33.14780774\rho^2 - 93.07966445\rho^4 + 70.01749250\rho^6$$
$$H_{23} = (47.45838189 - 175.85597460x^2 - 186.82909872y^2 + 157.02509476x^4 + 314.05018953x^2y^2 + 157.02509476y^4)xy$$
$$H_{24} = (23.72919094 - 92.04290884x^2 + 78.51254738x^4)x^2 + (-23.72919094 + 8.22984309x^2 + 89.29962781y^2 + 78.51254738x^4 - 78.51254738x^2y^2 - 78.51254738y^4)y^2$$

TABLE 3c Orthonormal *Hexagonal* Polynomials $H_j(x, y)$ in Cartesian Coordinates, Where $\rho^2 = x^2 + y^2$ (*Continued*)

$$\begin{aligned}
H_{25} &= (15.10561596 - 178.95943525x^2 + 34.43870505y^2 + 233.50942786x^4 \\
&\quad +151.82702751x^2y^2 - 81.68240034y^4)xy \\
H_{26} &= (-7.55280798 + 9.45541501x^2 + 1.44222164x^4)x^2 + (7.55280798 + 160.04860523x^2 \\
&\quad -62.80495008y^2 - 234.95164950x^4 - 159.03813574x^2y^2 + 77.35573540y^4)y^2 \\
H_{27} &= (40.85537039x^4 - 136.18456799x^2y^2 + 40.85537039y^4)xy \\
H_{28} &= 0.56537219 - 10.44830312\rho^2 + 38.71296332x^4 + 77.42592664x^2y^2 + 38.71296332y^4 \\
&\quad -29.43669525x^6 - 229.42985678x^4y^2 + 5.76976155x^2y^4 - 45.11666981y^6 \\
H_{29} &= (-15.56917599 + 7.06831810x^4 - 14.13663621x^2y^2 + 1.41366362y^4 + 130.07864353\rho^2 \\
&\quad -291.15952741\rho^4 + 190.97455178\rho^6)y \\
H_{30} &= (-15.56917599 - 1.41366362x^4 + 14.13663621x^2y^2 - 7.06831810y^4 + 130.07864353\rho^2 \\
&\quad -291.15952741\rho^4 + 190.97455178\rho^6)x \\
H_{31} &= 162.85550520x^2 - 54.28516840y^2 - 608.51113904x^2\rho^2 + 202.83704634y^2\rho^2 \\
&\quad +532.19785685x^2\rho^4 - 177.39928561y^2\rho^4)y \\
H_{32} &= [(41.60051295 - 135.27397959x^2 + 102.88660624x^4)x^2 + (-124.80153887 + 270.54795919x^2 \\
&\quad +405.82193879y^2 - 102.88660624x^4 - 514.43303123x^2y^2 - 308.65981874y^4)y^2]x \\
H_{33} &= [-3.87525156 + (41.84243767 - 307.79500129x^2 + 368.72158389x^4)x^2 + (41.84243767 \\
&\quad +145.33628349x^2 - 155.60974407y^2 + 10.13644892x^4 - 209.06921162x^2y^2 + 149.51592334y^4)y^2]y \\
H_{34} &= [3.87525156 + (-41.84243767 + 79.51711547x^2 - 39.91309306x^4)x^2 + (-41.84243767 \\
&\quad +615.59000259x^2 - 72.66814174y^2 - 777.35626084x^4 - 558.15060029x^2y^2 + 179.29256748y^4)y^2]x \\
H_{35} &= [3.10311187 + (-34.93479698 + 132.14137712x^2 - 73.19935100x^4)x^2 + (-34.93479698 \\
&\quad +144.04222993x^2 + 108.09327226y^2 - 519.49349681x^4 + 23.85771799x^2y^2 - 104.44842531y^4)y^2]y \\
H_{36} &= [3.10311187 + (-34.93479698 + 96.06921983x^2 - 66.20418535x^4)x^2 + (-34.93479698 \\
&\quad +264.28275425x^2 + 72.02111496y^2 - 535.81555000x^4 + 7.53566481x^2y^2 - 97.45325965y^4)y^2]x \\
H_{37} &= 2.74530738 - 60.39881618\rho^2 + 300.22087475\rho^4 + 288.55372176\rho^8 - 520.05901324x^6 \\
&\quad -1523.74277487x^4y^2 - 1584.46654966x^2y^4 - 516.01076159y^6 \\
H_{38} &= (-42.96232789 + 296.28167478x^2 - 578.72189394x^4 + 339.98298180x^6)x^2 + (42.96232789 \\
&\quad -50.98718488x^2 - 279.28594648y^2 - 497.20962679x^4 + 633.06340537x^2y^2 + 551.55113822y^4 \\
&\quad +679.96596360x^6 - 679.96596360x^2y^4 - 339.98298180y^6)y^2 \\
H_{39} &= [-85.92465579 + (541.57616468 - 1075.93152073x^2 + 679.96596360x^4)x^2 + (609.55907786 \\
&\quad -2260.54606433x^2 - 1184.61454360y^2 + 2039.89789081x^4 + 2039.89789081x^2y^2 + 679.96596360y^4)y^2]xy \\
H_{40} &= (14.79181046 - 38.21774249x^2 + 6.76690483x^4 + 21.45551332x^6)x^2 + (-14.79181046 \\
&\quad -500.39279319x^2 + 205.01534022y^2 + 1686.80674937x^4 + 1113.25965819x^2y^2 - 566.78018634y^4 \\
&\quad -1307.55336779x^6 - 2250.77399075x^4y^2 - 493.06582480x^2y^4 + 428.69928482y^6)y^2 \\
H_{41} &= [-29.58362093 + (576.82827818 - 1693.57365421x^2 + 1307.55336779x^4)x^2 + (-90.36211274 \\
&\quad -1147.09418236x^2 + 546.47947184y^2 + 2122.04091078x^4 + 321.42171817x^2y^2 - 493.06582480y^4)y^2]xy \\
H_{42} &= -0.84269170 + (24.65387703 - 158.21741244x^2 + 286.16004008x^4 - 152.67510082x^6)x^2 \\
&\quad +(24.65387703 - 316.43482489x^2 - 158.21741244y^2 + 1913.23979875x^4 + 155.30700127x^2y^2 \\
&\quad +403.35555992y^4 - 2152.28660953x^6 - 1429.91267370x^4y^2 + 245.73637792x^2y^4 - 323.96245707y^6)y^2 \\
H_{43} &= 2\sqrt{22/20334667}(6x^5y - 20x^3y^3 + 6xy^5)(-23443 + 32240\rho^2) \\
H_{44} &= (9.64776957 - 72.74250912x^2 + 164.99985615x^4 - 104.71489971x^6)x^2 + (-9.64776957 \\
&\quad -76.05737585x^2 + 98.09496774y^2 + 471.48320551x^4 + 39.32237674x^2y^2 - 267.16097261y^4 \\
&\quad -826.90123032x^6 + 279.13466933x^4y^2 - 170.82784030x^2y^4 + 223.32179529y^6)y^2 \\
H_{45} &= [19.29553915 + (-221.54239411 + 636.48306167x^2 - 434.42511407x^4)x^2 + (-120.13255963 \\
&\quad +864.32165754x^2 + 227.83859586y^2 - 1788.23382186x^4 - 179.98634818x^2y^2 - 221.64827593y^4)y^2]xy
\end{aligned}$$

or

$$y = \pm b\sqrt{1 - x^2} \tag{35b}$$

Its area is equal to πb. The orthonormality of the elliptical polynomials $E_j(x, y)$ is represented by[26]

$$\frac{1}{\pi b}\int_{-1}^{1} dx \int_{-b\sqrt{1-x^2}}^{b\sqrt{1-x^2}} E_j(x, y)E_{j'}(x, y)dy = \delta_{jj'} \tag{36}$$

The orthonormal elliptical polynomials up to the fourth order are given in Tables 4 in three different but equivalent forms, as in the case of hexagonal polynomials. As in the case of a hexagonal

TABLE 4a Orthonormal *Elliptical* Polynomials E_j in terms of Zernike Circle Polynomials Z_j, Which Reduce to the Corresponding Circle Polynomials as the Aspect Ratio $b \to 1$

$$E_1 = Z_1$$
$$E_2 = Z_2$$
$$E_3 = Z_3/b$$
$$E_4 = (1/\sqrt{3-2b^2+3b^4})[\sqrt{3}(1-b^2)Z_1 + 2Z_4]$$
$$E_5 = Z_5/b$$
$$E_6 = [1/(2\sqrt{2}b^2\sqrt{3-2b^2+3b^4})][-\sqrt{3}(3-4b^2+b^4)Z_1 - 3(1-b^4)Z_4 + \sqrt{2}(3-2b^2+3b^4)Z_6]$$
$$E_7 = [1/(b\sqrt{5-6b^2+9b^4})][6(1-b^2)Z_3 + 2\sqrt{2}Z_7]$$
$$E_8 = (2/\sqrt{9-6b^2+5b^4})[(1-b^2)Z_2 + \sqrt{2}Z_8]$$
$$E_9 = [1/(2\sqrt{2}b^3\sqrt{5-6b^2+9b^4})][-2\sqrt{2}(5-8b^2+3b^4)Z_3 - (5-2b^2-3b^4)Z_7 + (5-6b^2+9b^4)Z_9]$$
$$E_{10} = [1/(2\sqrt{2}b^3\sqrt{9-6b^2+5b^4})][-2\sqrt{2}(3-4b^2+b^4)Z_2 - (3+2b^2-5b^4)Z_8 + (9-6b^2+5b^4)Z_{10}]$$
$$E_{11} = (1/\alpha)[\sqrt{5}(7-10b^2+3b^4)Z_1 + 4\sqrt{15}(1-b^2)Z_4 - 2\sqrt{30}(1-b^2)Z_6 + 8Z_{11}]$$
$$\begin{aligned} E_{12} = &-\sqrt{5/8}b^{-2}(195-475b^2+558b^4-422b^6+159b^8-15b^{10})\beta^{-1}Z_1 \\ &-\sqrt{15/8}b^{-2}(105-205b^2+194b^4-114b^6+5b^8+15b^{10})\beta^{-1}Z_4 \\ &+\sqrt{15/4}(75-155b^2+174b^4-134b^6+55b^8-15b^{10})\beta^{-1}Z_6 \\ &-10\sqrt{2}b^{-2}(3-2b^2+2b^6-3b^8)\beta^{-1}Z_{11} + b^{-2}\alpha\gamma^{-1}Z_{12} \end{aligned}$$
$$E_{13} = [1/(b\sqrt{5-6b^2+5b^4})][\sqrt{15}(1-b^2)Z_5 + 2Z_{13}]$$
$$\begin{aligned} E_{14} = &(\sqrt{5/2}/4)(1-b^2)^2b^{-4}(35-10b^2-b^4)\gamma^{-1}Z_1 + +(5\sqrt{15/2}/8)(1-b^2)^2b^{-4}(7+2b^2-b^4)\gamma^{-1}Z_4 \\ &-(\sqrt{15}/8)(35-70b^2+56b^4-26b^6+5b^8)\gamma^{-1}Z_6 + [5/(8\sqrt{2})](1-b^2)^2b^{-4}(7+10b^2+7b^4)\gamma^{-1}Z_{11} \\ &-(5/8)b^{-4}(7-6b^2+6b^6-7b^8)\gamma^{-1}Z_{12} + [\gamma/(8b^4)]Z_{14} \end{aligned}$$
$$E_{15} = -(\sqrt{15}/4)b^{-3}(5-8b^2+3b^4)\delta^{-1}Z_5 - (5/4)(1-b^4)b^{-3}\beta^{-1}Z_{13} + [\delta/(2b^3)]Z_{15}$$

$$\alpha = (45-60b^2+94b^4-60b^6+45b^8)^{1/2}$$
$$\beta = (1575-4800b^2+12020b^4-17280b^6+21066b^8-17280b^{10}+12020b^{12}-4800b^{14}+1575b^{16})^{1/2}$$
$$\gamma = (35-60b^2+114b^4-60b^6+35b^8)^{1/2}$$
$$\delta = (5-6b^2+5b^4)^{1/2}$$

TABLE 4b Orthonormal *Elliptical* Polynomials $E_j(\rho, \theta)$ in Polar Coordinates

$$E_1 = 1$$
$$E_2 = 2\rho\cos\theta$$
$$E_3 = (2\rho\sin\theta)/b$$
$$E_4 = \sqrt{3/(3-2b^2+3b^4)}(-1-b^2+4\rho^2)$$
$$E_5 = (\sqrt{6}/b)\rho^2\sin 2\theta$$
$$E_6 = [1/(2b^2)]\sqrt{6/(3-2b^2+3b^4)}[2b^2(1-b^2) - 3(1-b^4)\rho^2 + (3-2b^2+3b^4)\rho^2\cos 2\theta]$$
$$E_7 = [4/(b\sqrt{5-6b^2+9b^4})][-(1+3b^2)\rho + 6\rho^3]\sin\theta$$
$$E_8 = (4/\sqrt{9-6b^2+5b^4})[-(3+b^2)\rho + 6\rho^3]\cos\theta$$
$$E_9 = [1/(b^3\sqrt{5-6b^2+9b^4})]\{3[4b^2(1-b^2)\rho - (5-2b^2-3b^4)\rho^3]\sin\theta + (5-6b^2+9b^4)\rho^3\sin 3\theta]\}$$
$$E_{10} = [1/(b^2\sqrt{9-6b^2+5b^4})]\{3[4b^2(1-b^2)\rho - (3+2b^2-5b^4)\rho^3]\cos\theta + (9-6b^2+5b^4)\rho^3\cos 3\theta]\}$$
$$E_{11} = \sqrt{5}[3+2b^2+3b^4-24(1+b^2)\rho^2+48\rho^4-12(1-b^2)\rho^2\cos 2\theta]\alpha$$
$$\begin{aligned} E_{12} = &[\sqrt{10}\alpha/(\gamma b^2)](-3\rho^2+4\rho^4)\cos 2\theta + [\sqrt{5/2}/(2b^2\beta)][-12b^2(5-2b^2+2b^6-5b^8) \\ &+6(15+125b^2-194b^4+194b^6-125b^8-15b^{10})\rho^2 + 240(-3+2b^2-2b^6+3b^8)\rho^4 \\ &+6(75-155b^2+174b^4-134b^6+55b^8-15b^{10})\rho^2\cos 2\theta] \end{aligned}$$
$$E_{13} = (\sqrt{10}/b)\delta^{-1}[-3(1+b^2)\rho^2+8\rho^4]\sin 2\theta$$
$$\begin{aligned} E_{14} = &[\sqrt{10}/(8b^4\gamma)]\{3(1-b^2)^2[8b^4-40b^2(1+b^2)\rho^2+5(7+10b^2+7b^4)\rho^4] \\ &+4[6b^2(5-7b^2+7b^4-5b^6)-5(7-6b^2+6b^6-7b^8)\rho^2]\rho^2\cos 2\theta + (35-60b^2+114b^4 \\ &-60b^6+35b^8)\rho^4\cos 4\theta\} \end{aligned}$$
$$E_{15} = (\sqrt{10}/b^3)\delta^{-1}\{[6b^2(1-b^2)-5(1-b^4)\rho^2]\rho^2\sin 2\theta + [(5-6b^2+5b^4)/2]\rho^4\sin 4\theta\}$$

TABLE 4c Orthonormal *Elliptical* Polynomials $E_j(x, y)$ in Cartesian Coordinates, Where $\rho^2 = x^2 + y^2, -1 \le x \le 1$, and $-\sqrt{1-b^2x^2} \le y \le \sqrt{1-b^2x^2}$

$$E_1 = 1$$
$$E_2 = 2x$$
$$E_3 = 2y/b$$
$$E_4 = (\sqrt{3}/\sqrt{3 - 2b^2 + 3b^4})(-1 - b^2 + 4\rho^2)$$
$$E_5 = (2\sqrt{6}/b)xy$$
$$E_6 = [\sqrt{3}/(b^2\sqrt{6 - 4b^2 + 6b^4})][b^2(1 - b^2) + b^2(3b^2 - 1)x^2 - (3 - b^2)y^2]$$
$$E_7 = [4/(b\sqrt{5 - 6b^2 + 9b^4})][-(1 + 3b^2) + 6\rho^2]y$$
$$E_8 = (4/\sqrt{9 - 6b^2 + 5b^4})[-(3 + b^2) + 6\rho^2]x$$
$$E_9 = [4/(b^3\sqrt{5 - 6b^2 + 9b^4})][3b^2(3b^2 - 1)x^2 - (5 - 3b^2)y^2 + 3b^2(1 - b^2)]y$$
$$E_{10} = [4/(b^2\sqrt{9 - 6b^2 + 5b^4})][b^2(5b^2 - 3)x^2 - 3(3 - b^2)y^2 + 3b^2(1 - b^2)]x$$
$$E_{11} = (\sqrt{5}/\alpha)[48\rho^4 - 12(3 + b^2)x^2 - 12(1 + 3b^2)y^2 + 3 + 2b^2 + 3b^4]$$
$$E_{12} = [\sqrt{10}\alpha/(b^2\gamma)][(x^2 - y^2)(4\rho^2 - 3) + [\sqrt{5}/(2\sqrt{2}b^2\beta)][240(-3 + 2b^2 - 2b^6 + 3b^8)\rho^4 \\ -60(-9 + 3b^2 + 2b^4 - 6b^6 + 7b^8 + 3b^{10})x^2 - 24(15 - 70b^2 + 92b^4 - 82b^6 + 45b^8)y^2 \\ +12b^2(-5 + 2b^2 - 2b^6 + 5b^8)]$$
$$E_{13} = [2\sqrt{10}/(b\delta)](8\rho^2 - 3 - 3b^2)xy$$
$$E_{14} = [\sqrt{10}/(b^4\gamma)][b^4(3 - 30b^2 + 35b^4)x^4 + 6b^2(5 - 18b^2 + 5b^4)x^2y^2 + (35 - 30b^2 + 3b^4)y^4 \\ -6b^4(1 - 6b^2 + 5b^4)x^2 - 6b^2(5 - 6b^2 + b^4)y^2 + 3b^4(1 - b^2)^2]$$
$$E_{15} = [4\sqrt{10}/(b^3\delta)][b^2(5b^2 - 3)x^2 - (5 - 3b^2)y^2 + 3b^2(1 - b^2)]xy$$

pupil, each elliptical polynomial consists of either cosine or sine terms, but not both. For example, E_6 is a linear combination of Z_6, Z_4, and Z_1. It also shows that the balancing defocus for (zero-degree) Seidel astigmatism is different for an elliptical pupil compared to that for a circular,[3–6] annular,[8–11] or a Gaussian pupil.[23–25] Moreover, E_{11} is a linear combination of Z_{11}, Z_6, Z_4, and Z_1. Thus, spherical aberration ρ^4 is balanced with not only defocus ρ^2 but astigmatism $\rho^2 \cos^2\theta$ as well. The elliptical polynomials are generally more complex in that they are made up of a larger number of circle polynomials. These results are a consequence of the fact that the *x* and *y* dimensions of the elliptical pupil are not equal. As expected, the elliptical polynomials reduce to the circle polynomials as $b \to 1$, that is, as the unit ellipse approaches a unit circle.

11.8 RECTANGULAR POLYNOMIALS

Figure 3*d* shows a *unit rectangle* inscribed inside a unit circle. While the distance of a corner point of the rectangle, such as *A*, from its center *O* is unity, the half widths of the rectangle along the *x* and *y* axes are *a* and $\sqrt{1-a^2}$, respectively. Accordingly, the aspect ratio of the rectangle is $\sqrt{1-a^2}/a$, and its area is $4a\sqrt{1-a^2}$. As in the case of a unit ellipse, a unit rectangle is also not unique, since *a* can have any value between 0 and 1. The orthonormality of the rectangular polynomials $R_j(x, y)$ is represented by[26]

$$\frac{1}{4a\sqrt{1-a^2}} \int_{-\sqrt{1-a^2}}^{\sqrt{1-a^2}} dy \int_{-a}^{a} R_j(x, y) R_{j'}(x, y) dx = \delta_{jj'} \tag{37}$$

The rectangular polynomials thus obtained up to the fourth order are given in Tables 5 in the same manner as the hexagonal and elliptical polynomials. As in the case of hexagonal and elliptical polynomials, each rectangular polynomial also consists of either cosine or sine terms, but not both. Like the elliptical polynomials, the rectangular polynomials also consist of a larger number of circle polynomials. The rectangular polynomial R_{11}, like the elliptical polynomials E_{11}, representing a balanced primary spherical aberration is not radially symmetric, since it consists of a term in astigmatism Z_6 or $\cos 2\theta$. As discussed below, the rectangular polynomials reduce to the square polynomials as $a \to 1/\sqrt{2}$, and the slit polynomials for a slit pupil parallel to the x axis as $a \to 1$.

TABLE 5a Orthonormal *Rectangular* Polynomials R_j in Terms of Zernike Circle Polynomials Z_j Which Reduce to the Corresponding Square Polynomials as $a \to 1/\sqrt{2}$

$$R_1 = Z_1$$
$$R_2 = [\sqrt{3}/(2a)]Z_2$$
$$R_3 = [\sqrt{3}/(2\sqrt{1-a^2})]Z_3$$
$$R_4 = [\sqrt{5}/(4\sqrt{1-2a^2+2a^4})](Z_1+\sqrt{3}Z_4)$$
$$R_5 = [\sqrt{3/2}/(2a\sqrt{1-a^2})]Z_5$$
$$R_6 = \{\sqrt{5}/[8a^2(1-a^2)\sqrt{1-2a^2+2a^4}]\}[(3-10a^2+12a^4-8a^6)Z_1+\sqrt{3}(1-2a^2)Z_4 +\sqrt{6}(1-2a^2+2a^4)Z_6]$$
$$R_7 = [\sqrt{21}/(4\sqrt{2}\sqrt{27-81a^2+116a^4-62a^6})][\sqrt{2}(1+4a^2)Z_3+5Z_7]$$
$$R_8 = [\sqrt{21}/(4\sqrt{2}a\sqrt{35-70a^2+62a^4})][\sqrt{2}(5-4a^2)Z_2+5Z_8]$$
$$R_9 = \{\sqrt{5/2}\sqrt{(27-54a^2+62a^4)/(1-a^2)}/[16a^2(27-81a^2+116a^4-62a^6)]\}[2\sqrt{2}(9-36a^2 +52a^4-60a^6)Z_3+(9-18a^2-26a^4)Z_7+(27-54a^2+62a^4)Z_9]$$
$$R_{10} = \{\sqrt{5/2}/[16a^3(1-a^2)\sqrt{35-70a^2+62a^4}]\}[2\sqrt{2}(35-112a^2+128a^4-60a^6)Z_2 +(35-70a^2+26a^4)Z_8+(35-70a^2+62a^4)Z_{10}]$$
$$R_{11} = [1/(16\mu)][8(3+4a^2-4a^4)Z_1+25\sqrt{3}Z_4+10\sqrt{6}(1-2a^2)Z_6+21\sqrt{5}Z_{11}]$$
$$R_{12} = \{3\mu/[16a^2\nu\eta]\}\{(105-550a^2+1559a^4-2836a^6+2695a^8-1078a^{10})Z_1 +5\sqrt{3}(14-74a^2+205a^4-360a^6+335a^8-134a^{10})Z_4+(5\sqrt{3}/2)(35-156a^2 +421a^4-530a^6+265a^8)Z_6+21\sqrt{5}(1-4a^2+6a^4-4a^6)Z_{11}+[(7/2)\sqrt{5/2}\eta/(1-a^2)]Z_{12}\}$$
$$R_{13} = [\sqrt{21}/(16\sqrt{2}a\sqrt{1-3a^2+4a^4-2a^6})](\sqrt{3}Z_5+\sqrt{5}Z_{13})$$
$$R_{14} = \tau[6(245-1400a^2+3378a^4-4452a^6+3466a^8-1488a^{10}+496a^{12})Z_1 +15\sqrt{3}(49-252a^2+522a^4-540a^6+270a^8)Z_4+15\sqrt{6}(49-252a^2+534a^4-596a^6 +360a^8-144a^{10})Z_6+3\sqrt{5}(49-196a^2+282a^4-172a^6+86a^8)Z_{11} +147\sqrt{10}(1-4a^2+6a^4-4a^6)Z_{12}+3\sqrt{10}\nu^2 Z_{14}]$$
$$R_{15} = \{1/[32a^3(1-a^2)(1-3a^2+4a^4-2a^6)^{1/2}]\}[3\sqrt{7/2}(5-18a^2+24a^4-16a^6)Z_5 +\sqrt{105/2}(1-2a^2)Z_{13}+\sqrt{210}(1-2a^2+2a^4)Z_{15}]$$

$$\mu = (9-36a^2+103a^4-134a^6+67a^8)^{1/2}$$
$$\nu = (49-196a^2+330a^4-268a^6+134a^8)^{1/2}$$
$$\tau = 1/[128\nu a^4(1-a^2)^2]$$
$$\eta = 9-45a^2+139a^4-237a^6+210a^8-67a^{10}$$

TABLE 5*b* Orthonormal *Rectangular* Polynomials $R_j(\rho, \theta)$ in Polar Coordinates

$$
\begin{aligned}
R_1 &= 1 \\
R_2 &= (\sqrt{3}/a)\rho\cos\theta \\
R_3 &= \sqrt{3/(1-a^2)}\rho\sin\theta \\
R_4 &= [\sqrt{5}/(2\sqrt{1-2a^2+2a^4})](3\rho^2-1) \\
R_5 &= [3/(2a\sqrt{1-a^2})]\rho^2\sin 2\theta \\
R_6 &= \{\sqrt{5}/[4a^2(1-a^2)\sqrt{1-2a^2+2a^4}]\}[3(1-2a^2+2a^4)\rho^2\cos 2\theta+3(1-2a^2)\rho^2 \\
&\quad -2a^2(1-a^2)(1-2a^2)] \\
R_7 &= [\sqrt{21}/(2\sqrt{27-81a^2+116a^4-62a^6})](15\rho^2-9+4a^2)\rho\sin\theta \\
R_8 &= [\sqrt{21}/(2a\sqrt{35-70a^2+62a^4})](15\rho^2-5-4a^2)\rho\cos\theta \\
R_9 &= \{\sqrt{5}\sqrt{(27-54a^2+62a^4)/(1-a^2)}/[8a^2(27-81a^2+116a^4-62a^6)]\}\{(27-54a^2+62a^4) \\
&\quad \times\rho^3\sin 3\theta-3[4a^2(3-13a^2+10a^4)-(9-18a^2-26a^4)\rho^2]\rho\sin\theta\} \\
R_{10} &= \{\sqrt{5}/[8a^3(1-a^2)\sqrt{35-70a^2+62a^4}]\}\{(35-70a^2+62a^4)\rho^3\cos 3\theta \\
&\quad -3[4a^2(7-17a^2+10a^4)-(35-70a^2+26a^4)\rho^2]\rho\cos\theta\} \\
R_{11} &= [1/(8\mu)][315\rho^4+30(1-2a^2)\rho^2\cos 2\theta-240\rho^2+27+16a^2-16a^4] \\
R_{12} &= [3\mu/(8a^2\nu\eta)][315(1-2a^2)(1-2a^2+2a^4)\rho^4+5(7\mu^2\rho^2-21+72a^2-225a^4+306a^6 \\
&\quad -153a^8)\rho^2\cos 2\theta-15(1-2a^2)(7+4a^2-71a^4+134a^6-67a^8)\rho^2 \\
&\quad +a^2(1-a^2)(1-2a^2)(70-233a^2+233a^4)] \\
R_{13} &= [\sqrt{21}/(4a\sqrt{1-3a^2+4a^4-2a^6})](5\rho^2-3)\rho^2\sin 2\theta \\
R_{14} &= 6\tau\{5\nu^2\rho^4\cos 4\theta-20(1-2a^2)[6a^2(7-16a^2+18a^4-9a^6)-49(1-2a^2+2a^4)\rho^2]\rho^2\cos 2\theta \\
&\quad +8a^4(1-a^2)^2(21-62a^2+62a^4)-120a^2(7-30a^2+46a^4-23a^6)\rho^2 \\
&\quad +15(49-196a^2+282a^4-172a^6+86a^8)\rho^4\} \\
R_{15} &= \{\sqrt{21}/[8a^3(1-a^2)^{3/2}(1-2a^2+2a^4)^{1/2}]\}[-(1-2a^2)(6a^2-6a^4-5\rho^2)\rho^2\sin 2\theta \\
&\quad +(5/2)(1-2a^2+2a^4)\rho^4\sin 4\theta]
\end{aligned}
$$

TABLE 5*c* Orthonormal *Rectangular* Polynomials $R_j(x, y)$ in Cartesian Coordinates, Where $\rho^2 = x^2 + y^2, -a \le x \le a,$ and $-\sqrt{1-a^2} \le y \le \sqrt{1-a^2}$

$$
\begin{aligned}
R_1 &= 1 \\
R_2 &= (\sqrt{3}/a)x \\
R_3 &= \sqrt{3/(1-a^2)}y \\
R_4 &= [\sqrt{5}/(2\sqrt{1-2a^2+2a^4})](3\rho^2-1) \\
R_5 &= [3/(a\sqrt{1-a^2})]xy \\
R_6 &= \{\sqrt{5}/[2a^2(1-a^2)\sqrt{1-2a^2+2a^4}]\}[3(1-a^2)^2x^2-3a^4y^2-a^2(1-3a^2+2a^4)] \\
R_7 &= [\sqrt{21}/(2\sqrt{27-81a^2+116a^4-62a^6})](15\rho^2-9+4a^2)y \\
R_8 &= [\sqrt{21}/(2a\sqrt{35-70a^2+62a^4})](15\rho^2-5-4a^2)x \\
R_9 &= \{\sqrt{5}\sqrt{(27-54a^2+62a^4)/(1-a^2)}/[2a^2(27-81a^2+116a^4-62a^6)]\}[27(1-a^2)^2x^2 \\
&\quad -35a^4y^2-a^2(9-39a^2+30a^4)]y \\
R_{10} &= \{\sqrt{5}/[2a^3(1-a^2)\sqrt{35-70a^2+62a^4}]\}[35(1-a^2)^2x^2-27a^4y^2-a^2(21-51a^2+30a^4)]x \\
R_{11} &= [1/(8\mu)][315\rho^4-30(7+2a^2)x^2-30(9-2a^2)y^2+27+16a^2-16a^4] \\
R_{12} &= [3\mu/(8a^2\nu\eta)][35(1-a^2)^2(18-36a^2+67a^4)x^4+630(1-2a^2)(1-2a^2+2a^4)x^2y^2 \\
&\quad -35a^4(49-98a^2+67a^4)y^4-30(1-a^2)(7-10a^2-12a^4+75a^6-67a^8)x^2 \\
&\quad -30a^2(7-77a^2+189a^4-193a^6+67a^8)y^2+a^2(1-a^2)(1-2a^2)(70-233a^2+233a^4)] \\
R_{13} &= [\sqrt{21}/(2a\sqrt{1-3a^2+4a^4-2a^6})](5\rho^2-3)xy \\
R_{14} &= 16\tau[735(1-a^2)^4x^4-540a^4(1-a^2)^2x^2y^2+735a^8y^4-90a^2(1-a^2)^3(7-9a^2)x^2 \\
&\quad +90a^6(1-a^2)(2-9a^2)y^2+3a^4(1-a^2)^2(21-62a^2+62a^4)] \\
R_{15} &= \{\sqrt{21}/[2a^3(1-a^2)\sqrt{1-3a^2+4a^4-2a^6}]\}[5(1-a^2)^2x^2-5a^4y^2-a^2(3-9a^2+6a^4)]xy
\end{aligned}
$$

11.9 SQUARE POLYNOMIALS

Figure 3*e* shows a *unit square* inscribed inside a unit circle, as in the case of a rectangle. The distance of a corner point of the square, such as A, from its center O is unity, but each of its sides has a length of $\sqrt{2}$ and its area is 2. The orthonormality of the square polynomials $S_j(x, y)$ is represented by[26]

$$\frac{1}{2}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} dy \int_{-1/\sqrt{2}}^{1/\sqrt{2}} S_j(x, y) S_{j'}(x, y) dx = \delta_{jj'} \tag{38}$$

The square polynomials through the eighth order are given in terms of the Zernike polynomials in Table 6*a*. The first 15 polynomials are given in their analytical form, but those with $j > 15$ are written in a numerical form because of the increasing complexity of the coefficients of the circle polynomials. The corresponding polynomials in polar and Cartesian coordinates are given in Tables 6*b* and 6*c*, respectively. Of course, up to the fourth order, they can be obtained simply from the rectangular polynomials $R_j(x, y)$ given in Tables 5 by letting $a = 1/\sqrt{2}$. The square polynomial S_{11} representing the balanced primary spherical aberration is radially symmetric, but the polynomial S_{22} representing the balanced secondary spherical aberration is not, since it consists of a term in Z_{14} or $\cos 4\theta$ also. Similarly, the polynomial S_{37} representing the balanced tertiary spherical aberration is also not radially symmetric, since it consists of terms in Z_{14} and Z_{26} both varying as $\cos 4\theta$.

11.10 SLIT POLYNOMIALS

By letting $a \to 1$ in the rectangular pupil, we obtain a *unit slit* pupil that is parallel to the x axis, as illustrated in Figure 3*f*. The corresponding orthonormal polynomials representing balanced aberrations for such pupils can be obtained from the rectangular polynomials $R_j(x, y)$ given in Table 5*c* by letting $y \to 0$ and $a \to 1$. Half of the rectangular polynomials thus reduce to zero. Some of the other polynomials are redundant. For example, the one-dimensional defocus and astigmatism can not be distinguished from each other. The slit polynomials are orthonormal according to[26]

$$\frac{1}{2}\int_{-1}^{1} P_j(x) P_{j'}(x) dx = \delta_{ij} \tag{39}$$

The relevant orthonormal slit polynomials are listed in Table 7. They are the Legendre polynomials,[17] which represent balanced aberrations uniquely.[20,26] Since the pupil is one dimensional along the x axis, the aberrations vary with x only.

11.11 ABERRATION BALANCING AND TOLERANCING, AND DIFFRACTION FOCUS

For small aberrations, the Strehl ratio of the image of a point object is approximately given by $1 - \sigma^2$ or $\exp(-\sigma^2)$ when the standard deviation σ of the aberration is in units of radians.[4,5,35] The Zernike circle and annular polynomials are separable in ρ and θ. The balanced spherical aberrations for these radially symmetric pupils are radially symmetric, and the balanced primary astigmatism for them has the same form. This is also true of a Gaussian circular or annular pupil, again because of the radial symmetry of the pupil and the amplitude across it.[23–25] From the orthonormal form H_4 of defocus for a hexagonal pupil, the sigma of the defocus aberration ρ^2 is given by $(1/12)\sqrt{43/5}$. The hexagonal polynomials H_5 and H_6 show that the balanced astigmatism has the same form as the circle polynomials Z_5 and Z_6, respectively. Thus the relative amount of defocus ρ^2 that balances classical or Seidel astigmatism $\rho^2\cos^2\theta$ is the same for a hexagonal pupil as for a circular pupil. Hence, for

TABLE 6a Orthonormal *Square* Polynomials S_j in Terms of Zernike Circle Polynomials Z_j

$S_1 = Z_1$
$S_2 = \sqrt{3/2}Z_2$
$S_3 = \sqrt{3/2}Z_3$
$S_4 = (\sqrt{5/2}/2)Z_1 + (\sqrt{15/2}/2)Z_4$
$S_5 = \sqrt{3/2}Z_5$
$S_6 = (\sqrt{15}/2)Z_6$
$S_7 = (3\sqrt{21/31}/2)Z_3 + (5\sqrt{21/62}/2)Z_7$
$S_8 = (3\sqrt{21/31}/2)Z_2 + (5\sqrt{21/62}/2)Z_8$
$S_9 = -(7\sqrt{5/31}/2)Z_3 - (13\sqrt{5/62}/4)Z_7 + (\sqrt{155/2}/4)Z_9$
$S_{10} = (7\sqrt{5/31}/2)Z_2 + (13\sqrt{5/62}/4)Z_8 + (\sqrt{155/2}/4)Z_{10}$
$S_{11} = (8/\sqrt{67})Z_1 + (25\sqrt{3/67}/4)Z_4 + (21\sqrt{5/67}/4)Z_{11}$
$S_{12} = (45\sqrt{3}/16)Z_6 + (21\sqrt{5}/16)Z_{12}$
$S_{13} = (3\sqrt{7}/8)Z_5 + (\sqrt{105}/8)Z_{13}$
$S_{14} = 261/(8\sqrt{134})Z_1 + (345\sqrt{3/134}/16)Z_4 + (129\sqrt{5/134}/16)Z_{11} + (3\sqrt{335}/16)Z_{14}$
$S_{15} = (\sqrt{105}/4)Z_{15}$
$S_{16} = 1.71440511Z_2 + 1.71491497Z_8 + 0.65048499Z_{10} + 1.52093102Z_{16}$
$S_{17} = 1.71440511Z_3 + 1.71491497Z_7 - 0.65048449Z_9 + 1.52093102Z_{17}$
$S_{18} = 4.10471345Z_2 + 3.45884077Z_8 + 5.34411808Z_{10} + 1.51830574Z_{16} + 2.80808005Z_{18}$
$S_{19} = -4.10471345Z_3 - 3.45884078Z_7 + 5.34411808Z_9 - 1.51830575Z_{17} + 2.80808005Z_{19}$
$S_{20} = 5.57146696Z_2 + 4.44429264Z_8 + 3.00807599Z_{10} + 1.70525179Z_{16} + 1.16777987Z_{18} + 4.19716701Z_{20}$
$S_{21} = 5.57146696Z_3 + 4.44429264Z_7 - 3.00807599Z_9 + 1.70525179Z_{17} - 1.16777988Z_{19} + 4.19716701Z_{21}$
$S_{22} = 1.33159935Z_1 + 1.94695912Z_4 + 1.74012467Z_{11} + 0.65624211Z_{14} + 1.50989174Z_{22}$
$S_{23} = 0.95479991Z_5 + 1.01511643Z_{13} + 1.28689496Z_{23}$
$S_{24} = 9.87992565Z_6 + 7.28853095Z_{12} + 3.38796312Z_{24}$
$S_{25} = 5.61978925Z_{15} + 2.84975327Z_{25}$
$S_{26} = 11.00650275Z_1 + 14.00366597Z_4 + 9.22698484Z_{11} + 13.55765720Z_{14} + 3.18799971Z_{22} + 5.11045000Z_{26}$
$S_{27} = 4.24396143Z_5 + 2.70990074Z_{13} + 0.84615108Z_{23} + 5.17855026Z_{27}$
$S_{28} = 17.58672314Z_6 + 11.15913268Z_{12} + 3.57668869Z_{24} + 6.44185987Z_{28}$
$S_{29} = 2.42764289Z_3 + 2.69721906Z_7 - 1.56598064Z_9 + 2.12208902Z_{17} - 0.93135653Z_{19} + 0.25252773Z_{21} + 1.59017528Z_{29}$
$S_{30} = 2.42764289Z_2 + 2.69721906Z_8 + 1.56598064Z_{10} + 2.12208902Z_{16} + 0.93135653Z_{18} + 0.25252773Z_{20} + 1.59017528Z_{30}$
$S_{31} = -9.10300982Z_3 - 8.79978208Z_7 + 10.69381427Z_9 - 5.37383385Z_{17} + 7.01044701Z_{19} - 1.26347272Z_{21} - 1.90131756Z_{29} + 3.07960207Z_{31}$
$S_{32} = 9.10300982Z_2 + 8.79978208Z_8 + 10.69381427Z_{10} + 5.37383385Z_{16} + 7.01044701Z_{18} + 1.26347272Z_{20} + 1.90131756Z_{30} + 3.07960207Z_{32}$
$S_{33} = 21.39630883Z_3 + 19.76696884Z_7 - 12.70550260Z_9 + 11.05819453Z_{17} - 7.02178756Z_{19} + 15.80286172Z_{21} + 3.29259996Z_{29} - 2.07602718Z_{31} + 5.40902889Z_{33}$
$S_{34} = 21.39630883Z_2 + 19.76696884Z_8 + 12.70550260Z_{10} + 11.05819453Z_{16} + 7.02178756Z_{18} + 15.80286172Z_{20} + 3.29259996Z_{30} + 2.07602718Z_{32} + 5.40902889Z_{34}$
$S_{35} = -16.54454462Z_3 - 14.89205549Z_7 + 22.18054997Z_9 - 7.94524849Z_{17} + 11.85458952Z_{19} - 6.18963457Z_{21} - 2.19431441Z_{29} + 3.24324400Z_{31} - 1.72001172Z_{33} + 8.16384008Z_{35}$
$S_{36} = 16.54454462Z_2 + 14.89205549Z_8 + 22.18054997Z_{10} + 7.94524849Z_{16} + 11.85458952Z_{18} + 6.18963457Z_{20} + 2.19431441Z_{30} + 3.24324400Z_{32} + 1.72001172Z_{34} + 8.16384008Z_{36}$

(*Continued*)

TABLE 6*a* Orthonormal *Square* Polynomials $S_j(\rho, \theta)$ in Terms of Zernike Circle Polynomials (*Continued*)

$$S_{37} = 1.75238960Z_1 + 2.72870567Z_4 + 2.76530671Z_{11} + 1.43647360Z_{14} + 2.12459170Z_{22} + 0.92450043Z_{26} + 1.58545010Z_{37}$$
$$S_{38} = 19.24848143Z_6 + 16.41468913Z_{12} + 9.76776798Z_{24} + 1.47438007Z_{28} + 3.83118509Z_{38}$$
$$S_{39} = 0.46604820Z_5 + 0.84124290Z_{13} + 1.00986774Z_{23} - 0.42520747Z_{27} + 1.30579570Z_{39}$$
$$S_{40} = 28.18104531Z_1 + 38.52219208Z_4 + 30.18363661Z_{11} + 36.44278147Z_{14} + 15.52577202Z_{22} + 19.21524879Z_{26} + 4.44731721Z_{37} + 6.00189814Z_{40}$$
$$S_{41} = (369/4)\sqrt{35/3574}Z_{15} + [11781/(32\sqrt{3574})]Z_{25} + (2145/32)\sqrt{7/3574}Z_{41}$$
$$S_{42} = 85.33469748Z_6 + 64.01249391Z_{12} + 30.59874671Z_{24} + 34.09158819Z_{28} + 7.75796322Z_{38} + 9.37150432Z_{42}$$
$$S_{43} = 14.30642479Z_5 + 11.17404702Z_{13} + 5.68231935Z_{23} + 18.15306055Z_{27} + 1.54919583Z_{39} + 5.90178984Z_{43}$$
$$S_{44} = 36.12567424Z_1 + 47.95305224Z_4 + 35.30691679Z_{11} + 56.72014548Z_{14} + 16.36470429Z_{22} + 26.32636277Z_{26} + 3.95466397Z_{37} + 6.33853092Z_{40} + 12.38056785Z_{44}$$
$$S_{45} = 21.45429746Z_{15} + 9.94633083Z_{25} + 2.34632890Z_{41} + 10.39130049Z_{45}$$

TABLE 6*b* Orthonormal *Square* Polynomials $S_j(\rho, \theta)$ in Polar Coordinates

$$S_1 = 1$$
$$S_2 = \sqrt{6}\rho\cos\theta$$
$$S_3 = \sqrt{6}\rho\sin\theta$$
$$S_4 = \sqrt{5/2}(3\rho^2 - 1)$$
$$S_5 = 3\rho^2\sin 2\theta$$
$$S_6 = 3\sqrt{5/2}\rho^2\cos 2\theta$$
$$S_7 = \sqrt{21/31}(15\rho^2 - 7)\rho\sin\theta$$
$$S_8 = \sqrt{21/31}(15\rho^2 - 7)\rho\cos\theta$$
$$S_9 = (\sqrt{5/31}/2)[31\rho^3\sin 3\theta - 3(13\rho^2 - 4)\rho\sin\theta]$$
$$S_{10} = (\sqrt{5/31}/2)[31\rho^3\cos 3\theta + 3(13\rho^2 - 4)\rho\cos\theta]$$
$$S_{11} = [1/(2\sqrt{67})](315\rho^4 - 240\rho^2 + 31)$$
$$S_{12} = [15/(2\sqrt{2})](7\rho^2 - 3)\rho^2\cos 2\theta$$
$$S_{13} = \sqrt{21/2}(5\rho^2 - 3)\rho^2\sin 2\theta$$
$$S_{14} = [3/(8\sqrt{134})](335\rho^4\cos 4\theta + 645\rho^4 - 300\rho^2 + 22)$$
$$S_{15} = (5/2)\sqrt{21/2}\rho^4\sin 4\theta$$
$$S_{16} = \sqrt{55/1966}[11\rho^3\cos 3\theta + 3(19 - 97\rho^2 + 105\rho^4)\rho\cos\theta]$$
$$S_{17} = \sqrt{55/1966}[-11\rho^3\sin 3\theta + 3(19 - 97\rho^2 + 105\rho^4)\rho\sin\theta]$$
$$S_{18} = (1/4)\sqrt{3/844397}[5(-10099 + 20643\rho^2)\rho^3\cos 3\theta + 3(3128 - 23885\rho^2 + 37205\rho^4)\rho\cos\theta]$$
$$S_{19} = (1/4)\sqrt{3/844397}[5(-10099 + 20643\rho^2)\rho^3\sin 3\theta - 3(3128 - 23885\rho^2 + 37205\rho^4)\rho\sin\theta]$$
$$S_{20} = (1/16)\sqrt{7/859}[2577\rho^5\cos 5\theta - 5(272 - 717\rho^2)\rho^3\cos 3\theta + 30(22 - 196\rho^2 + 349\rho^4)\rho\cos\theta]$$
$$S_{21} = (1/16)\sqrt{7/859}[2577\rho^5\sin 5\theta + 5(272 - 717\rho^2)\rho^3\sin 3\theta + 30(22 - 196\rho^2 + 349\rho^4)\rho\sin\theta]$$
$$S_{22} = (1/4)\sqrt{65/849}(1155\rho^6 + 30\rho^4\cos 4\theta - 1395\rho^4 + 453\rho^2 - 31)$$
$$S_{23} = (1/2)\sqrt{33/3923}(471 - 1820\rho^2 + 1575\rho^4)\rho^2\sin 2\theta$$
$$S_{24} = (21/4)\sqrt{65/1349}(27 - 140\rho^2 + 165\rho^4)\rho^2\cos 2\theta$$
$$S_{25} = (7/4)\sqrt{33/2}(9\rho^2 - 5)\rho^4\sin 4\theta$$
$$S_{26} = [1/(16\sqrt{849})][5(-98 + 2418\rho^2 - 12051\rho^4 + 15729\rho^6) + 3(-8195 + 17829\rho^2)\rho^4\cos 4\theta]$$
$$S_{27} = [1/(16\sqrt{7846})][27461\rho^6\sin 6\theta + 15(348 - 2744\rho^2 + 4487\rho^4)\rho^2\sin 2\theta]$$
$$S_{28} = [21/(32\sqrt{1349})][1349\rho^6\cos 6\theta + 5(196 - 1416\rho^2 + 2247\rho^4)\rho^2\cos 2\theta]$$

(*Continued*)

TABLE 6*b* Orthonormal *Square* Polynomials $S_j(\rho, \theta)$ in Polar Coordinates (*Continued*)

$$
\begin{aligned}
S_{29} &= (-13.79189793\rho + 125.49411319\rho^3 - 308.13074909\rho^5 + 222.62454035\rho^7)\sin\theta \\
&\quad +(8.47599260\rho^3 - 16.13156842\rho^5)\sin 3\theta + 0.87478174\rho^5 \sin 5\theta \\
S_{30} &= (-13.79189793\rho + 125.49411319\rho^3 - 308.13074909\rho^5 + 222.62454035\rho^7)\cos\theta \\
&\quad +(-8.47599260\rho^3 + 16.13156842\rho^5)\cos 3\theta + 0.87478174\rho^5 \cos 5\theta \\
S_{31} &= (6.14762642\rho - 79.44065626\rho^3 + 270.16115026\rho^5 - 266.18445920\rho^7)\sin\theta \\
&\quad +(56.29115383\rho^3 - 248.12774426\rho^5 + 258.68657393\rho^7)\sin 3\theta - 4.37679791\rho^5 \sin 5\theta \\
S_{32} &= (-6.14762642\rho + 79.44065626\rho^3 - 270.16115026\rho^5 + 266.18445920\rho^7)\cos\theta \\
&\quad +(56.29115383\rho^3 - 248.12774426\rho^5 + 258.68657393\rho^7)\cos 3\theta + 4.37679791\rho^5 \cos 5\theta \\
S_{33} &= (-6.78771487\rho + 103.15977419\rho^3 - 407.15689696\rho^5 + 460.96399558\rho^7)\sin\theta \\
&\quad +(-21.68093294\rho^3 + 127.50233381\rho^5 - 174.38628345\rho^7)\sin 3\theta \\
&\quad +(-75.07397471\rho^5 + 151.45280913\rho^7)\sin 5\theta \\
S_{34} &= (-6.78771487\rho + 103.15977419\rho^3 - 407.15689696\rho^5 + 460.96399558\rho^7)\cos\theta \\
&\quad +(21.68093294\rho^3 - 127.50233381\rho^5 + 174.38628345\rho^7)\cos 3\theta \\
&\quad +\rho^5(-75.07397471 + 151.45280913\rho^2)\cos 5\theta \\
S_{35} &= (3.69268433\rho - 59.40323317\rho^3 + 251.40397826\rho^5 - 307.20401818\rho^7)\sin\theta \\
&\quad +(28.20381860\rho^3 - 183.86176738\rho^5 + 272.43249673\rho^7)\sin 3\theta \\
&\quad +(19.83875817\rho^5 - 48.16032819\rho^7)\sin 5\theta + 32.65536033\rho^7 \sin 7\theta \\
S_{36} &= (-3.69268433\rho + 59.40323317\rho^3 - 251.40397826\rho^5 + 307.20401818\rho^7)\cos\theta \\
&\quad +(28.20381860\rho^3 - 183.86176738\rho^5 + 272.43249673\rho^7)\cos 3\theta \\
&\quad +(-19.83875817\rho^5 + 48.16032819\rho^7)\cos 5\theta + 32.65536033\rho^7 \cos 7\theta \\
S_{37} &= 2.34475558 - 55.32128002\rho^2 + 296.53777290\rho^4 - 553.46621887\rho^6 + 332.94452229\rho^8 \\
&\quad +(-12.75329096\rho^4 + 20.75498320\rho^6)\cos 4\theta \\
S_{38} &= (-51.83202694\rho^2 + 451.93890159\rho^4 - 1158.49126888\rho^6 + 910.24313983\rho^8)\cos 2\theta \\
&\quad +5.51662508\rho^6 \cos 6\theta \\
S_{39} &= (-39.56789598\rho^2 + 267.47071204\rho^4 - 525.02362247\rho^6 + 310.24123146\rho^8)\sin 2\theta \\
&\quad -1.59098067\rho^6 \sin 6\theta \\
S_{40} &= 1.21593465 - 45.42224477\rho^2 + 373.41167834\rho^4 - 1046.32659847\rho^6 + 933.93661610\rho^8 \\
&\quad +(137.71626496\rho^4 - 638.10242034\rho^6 + 712.98912399\rho^8)\cos 4\theta \\
S_{41} &= (9/8)\sqrt{7/1787}(1455 - 5544\rho^2 + 5005\rho^4)\rho^4 \sin 4\theta \\
S_{42} &= (-40.45171657\rho^2 + 494.75561036\rho^4 - 1738.64589491\rho^6 + 1843.19802390\rho^8)\cos 2\theta \\
&\quad +(-150.76043598\rho^6 + 318.07940431\rho^8)\cos 6\theta \\
S_{43} &= (-9.12193686\rho^2 + 110.47679089\rho^4 - 371.21215287\rho^6 + 368.07015240\rho^8)\sin 2\theta \\
&\quad +(-107.35168289\rho^6 + 200.31338972\rho^8)\sin 6\theta \\
S_{44} &= 0.58427150 - 25.29433513\rho^2 + 242.54313549\rho^4 - 795.02011474\rho^6 + 830.47943579\rho^8 \\
&\quad +(90.22533813\rho^4 - 538.44320774\rho^6 + 752.97905752\rho^8)\cos 4\theta + 52.52630092\rho^8 \cos 8\theta \\
S_{45} &= (31.08509142\rho^4 - 194.79990628\rho^6 + 278.72965314\rho^8)\sin 4\theta + 44.08655427\rho^8 \sin 8\theta
\end{aligned}
$$

TABLE 6*c* Orthonormal *Square* Polynomials $S_j(x, y)$ in Cartesian Coordinates, Where $\rho^2 = x^2 + y^2, -1/\sqrt{2} \le x \le 1/\sqrt{2}$, and , $-1/\sqrt{2} \le y \le 1/\sqrt{2}$

$$
\begin{aligned}
S_1 &= 1 \\
S_2 &= \sqrt{6}x \\
S_3 &= \sqrt{6}y \\
S_4 &= \sqrt{5/2}(3\rho^2 - 1) \\
S_5 &= 6xy \\
S_6 &= 3\sqrt{5/2}(x^2 - y^2) \\
S_7 &= \sqrt{21/31}(15\rho^2 - 7)y \\
S_8 &= \sqrt{21/31}(15\rho^2 - 7)x \\
S_9 &= \sqrt{5/31}(27x^2 - 35y^2 + 6)y \\
S_{10} &= \sqrt{5/31}(35x^2 - 27y^2 - 6)x \\
S_{11} &= [1/(2\sqrt{67})](315\rho^4 - 240\rho^2 + 31) \\
S_{12} &= [15/(2\sqrt{2})](x^2 - y^2)(7\rho^2 - 3) \\
S_{13} &= \sqrt{42}(5\rho^2 - 3)xy
\end{aligned}
$$

(*Continued*)

TABLE 6c Orthonormal *Square* Polynomials $S_j(x, y)$, in Cartesian Coordinates, Where $\rho^2 = x^2 + y^2$, $-1/\sqrt{2} \le x \le 1/\sqrt{2}$, and , $-1/\sqrt{2} \le y \le 1/\sqrt{2}$ (*Continued*)

$$S_{14} = [3/(4\sqrt{134})][10(49x^4 - 36x^2y^2 + 49y^4) - 150\rho^2 + 11]$$
$$S_{15} = 5\sqrt{42}(x^2 - y^2)xy$$
$$S_{16} = \sqrt{55/1966}(315\rho^4 - 280x^2 - 324y^2 + 57)x$$
$$S_{17} = \sqrt{55/1966}(315\rho^4 - 324x^2 - 280y^2 + 57)y$$
$$S_{18} = (1/2)\sqrt{3/844397}[105(1023x^4 + 80x^2y^2 - 943y^4) - 61075x^2 + 39915y^2 + 4692]x$$
$$S_{19} = (1/2)\sqrt{3/844397}[105(943x^4 - 80x^2y^2 - 1023y^4) - 39915x^2 + 61075y^2 - 4692]y$$
$$S_{20} = (1/4)\sqrt{7/859}[6(693x^4 - 500x^2y^2 + 525y^4) - 1810x^2 - 450y^2 + 165]x$$
$$S_{21} = (1/4)\sqrt{7/859}[6(525x^4 - 500x^2y^2 + 693y^4) - 450x^2 - 1810y^2 + 165]y$$
$$S_{22} = (1/4)\sqrt{65/849}[1155\rho^6 - 15(91x^4 + 198x^2y^2 + 91y^4) + 453\rho^2 - 31]$$
$$S_{23} = \sqrt{33/3923}(1575\rho^4 - 1820\rho^2 + 471)xy$$
$$S_{24} = (21/4)\sqrt{65/1349}(165\rho^4 - 140\rho^2 + 27)(x^2 - y^2)$$
$$S_{25} = 7\sqrt{33/2}(9\rho^2 - 5)xy(x^2 - y^2)$$
$$S_{26} = [1/(8\sqrt{849})][42(1573x^6 - 375x^4y^2 - 375x^2y^4 + 1573y^6) - 60(707x^4 - 225x^2y^2 + 707y^4) + 6045\rho^2 - 245]$$
$$S_{27} = [1/(2\sqrt{7846})][14(2673x^4 - 2500x^2y^2 + 2673y^4) - 10290\rho^2 + 1305]xy$$
$$S_{28} = [21/(8\sqrt{1349})][3146x^6 - 2250x^4y^2 + 2250x^2y^4 - 3146y^6 - 1770(x^4 - y^4) + 245(x^2 - y^2)]$$
$$S_{29} = (-13.79189793 + 150.92209099x^2 + 117.01812058y^2 - 352.15154565x^4 - 657.27245247x^2y^2 - 291.12439892y^4 + 222.62454035x^6 + 667.87362106x^4y^2 + 667.87362106x^2y^4 + 222.62454035y^6)y$$
$$S_{30} = (-13.79189793 + 117.01812058x^2 + 150.92209099y^2 - 291.12439892x^4 - 657.27245247x^2y^2 - 352.15154565y^4 + 222.62454035x^6 + 667.87362106x^4y^2 + 667.87362106x^2y^4 + 222.62454035y^6)x$$
$$S_{31} = (6.14762642 + 89.43280522x^2 - 135.73181009y^2 - 496.10607212x^4 + 87.83479115x^2y^2 + 513.91209661y^4 + 509.87526260x^6 + 494.87949207x^4y^2 - 539.86680367x^2y^4 - 524.87103314y^6)y$$
$$S_{32} = (-6.14762642 + 135.73181009x^2 - 89.43280522y^2 - 513.91209661x^4 - 87.83479115x^2y^2 + 496.10607212y^4 + 524.87103314x^6 + 539.86680367x^4y^2 - 494.87949207x^2y^4 - 509.87526260y^6)x$$
$$S_{33} = (-6.78771487 + 38.11697536x^2 + 124.84070714y^2 - 400.01976911x^4 + 191.43062089x^2y^2 - 609.73320550y^4 + 695.06919087x^6 - 246.30347616x^4y^2 - 154.56957886x^2y^4 + 786.80308817y^6)y$$
$$S_{34} = (-6.78771487 + 124.84070714x^2 + 38.11697536y^2 - 609.73320550x^4 + 191.43062089x^2y^2 - 400.01976911y^4 + 786.80308817x^6 - 154.56957886x^4y^2 - 246.30347616x^2y^4 + 695.06919087y^6)x$$
$$S_{35} = (3.69268433 + 25.20822264x^2 - 87.60705178y^2 - 200.98753298x^4 - 63.30315999x^2y^2 + 455.10450382y^4 + 497.87935336x^6 - 461.58554163x^4y^2 + 470.02596297x^2y^4 - 660.45220344y^6)y$$
$$S_{36} = (-3.69268433 + 87.60705178x^2 - 25.20822264y^2 - 455.10450382x^4 + 63.30315999x^2y^2 + 200.98753298y^4 + 660.45220344x^6 - 470.02596297x^4y^2 + 461.58554163x^2y^4 - 497.87935336y^6)x$$
$$S_{37} = 9.37902233 - 221.28512011\rho^2 + 1186.15109160\rho^4 - 2213.86487550\rho^6 + 1331.77808917\rho^8 + 0.0190064(x^4 - 6x^2y^2 + y^4)(-671 + 1092\rho^2)$$
$$S_{38} = (-51.83202694 + 451.93890159x^2 - 1152.97464379x^4 + 910.24313983x^6)x^2 + (51.83202694 - 451.93890159y^2 - 1241.24064523x^4 + 1241.24064523x^2y^2 + 1152.97464379y^4 + 1820.48627967x^6 - 1820.48627967x^2y^4 - 910.24313983y^6)y^2$$
$$S_{39} = (-79.13579197 + 534.94142408x^2 + 534.94142408y^2 - 1059.59312899x^4 - 2068.27487642x^2y^2 - 1059.59312899y^4 + 620.48246292x^6 + 1861.44738877x^4y^2 + 1861.44738877x^2y^4 + 620.48246292y^6)xy$$
$$S_{40} = 1.21593465 + (-45.42224477 + 511.12794331x^2 - 1684.42901882x^4 + 1646.92574009x^6)x^2 + (-45.42224477 - 79.47423312x^2 + 511.12794331y^2 + 51.53230630x^4 + 51.53230630x^2y^2 - 1684.42901882y^4 + 883.78996844x^6 - 1526.27154329x^4y^2 + 883.78996844x^2y^4 + 1646.92574009y^6)y^2$$
$$S_{41} = (409.79084415x^2 - 409.79084415y^2 - 1561.42985567x^4 + 1561.42985567y^4 + 1409.62417525x^6 + 1409.62417525x^4y^2 - 1409.62417525x^2y^4 - 1409.62417525y^6)xy$$
$$S_{42} = (-40.45171657 + 494.75561036x^2 - 1889.40633090x^4 + 2161.27742821x^6)x^2 + (40.45171657 - 494.75561036y^2 + 522.76064491x^4 - 522.76064491x^2y^2 + 1889.40633090y^4 - 766.71561254x^6 + 766.71561254x^2y^4 - 2161.27742821y^6)y^2$$
$$S_{43} = (-18.24387372 + 220.95358178x^2 + 220.95358178y^2 - 1386.53440310x^4 + 662.18504631x^2y^2 - 1386.53440310y^4 + 1938.02064313x^6 - 595.96654168x^4y^2 - 595.96654168x^2y^4 + 1938.02064313y^6)xy$$
$$S_{44} = 0.58427150 + (-25.29433513 + 332.76847363x^2 - 1333.46332249x^4 + 1635.98479424x^6)x^2 + (-25.29433513 - 56.26575785x^2 + 332.76847363y^2 + 307.15569451x^4 + 307.15569451x^2y^2 - 1333.46332249y^4 - 1160.73491284x^6 + 1129.92710444x^4y^2 - 1160.73491284x^2y^4 + 1635.98479424y^6)y^2$$
$$S_{45} = (124.34036571x^2 - 124.34036571y^2 - 779.19962514x^4 + 779.19962514y^4 + 1467.61104674x^6 - 1353.92842666x^4y^2 + 1353.92842666x^2y^4 - 1467.61104674y^6)xy$$

TABLE 7 Orthonormal Polynomials for a Unit *Slit* Pupil

j	Aberration	Orthonormal Polynomials
1	Piston	1
2	Tilt	$\sqrt{3}x$
3	Defocus	$(\sqrt{5}/2)(3x^2-1)$
4	Coma	$(\sqrt{7}/2)(5x^3-3x)$
5	Spherical aberration	$(3/8)(35x^4-30x^2+3)$
6	Secondary coma	$(\sqrt{11}/8)(63x^5-70x^3+15x)$
7	Secondary spherical aberration	$(\sqrt{13}/16)(231x^6-315x^4+105x^2-5)$

a small amount of astigmatism, the diffraction focus for an inscribed hexagonal pupil is the same as for a circular pupil.[4,5] For an image with a focal ratio of F, it lies along the z axis at a distance of $-8F^2$ times the amount of the balancing defocus from the Gaussian image point. However, the hexagonal polynomials H_7 and H_8 show that the relative amount of tilt $\rho\cos\theta$ that optimally balances classical or Seidel coma $\rho^3\cos\theta$ is $-14/25 \approx -0.56$ compared to $-2/3$ for a circular pupil. The diffraction focus in this case lies along the x axis at a distance of $-2F$ times the amount of tilt from the Gaussian image point. Similarly, the hexagonal polynomial H_{11} shows that the relative amount of defocus that optimally balances classical primary or Seidel spherical aberration ρ^4 is $-257/301 \approx -0.85$ compared to a value of -1 for a circular pupil. It has the consequence that the diffraction focus lies closer to the Gaussian image point in the case of coma, and closer to the Gaussian image plane in the case of spherical aberration, compared to their corresponding locations for a circular pupil. While the balanced primary and secondary spherical aberrations H_{11} and H_{22} are radially symmetric, the balanced tertiary spherical aberration H_{37} is not. The tertiary spherical aberration ρ^8 is balanced not only by defocus ρ^2 and primary and secondary spherical aberrations ρ^4 and ρ^6, but by a term in Z_{28} or $\rho^6 \cos 6\theta$ as well.

In the case of an elliptical pupil, the sigma of Seidel astigmatism $\rho^2\cos\theta$ is given by $\sigma_a = 1/4$, independent of its aspect ratio b, and thus equal to that for a circular pupil. Since Seidel astigmatism x^2 varies only along the x axis for which the unit ellipse has the same length as a unit circle, the sigma is independent of b. The amount of balancing defocus ρ^2 for astigmatism is different in the case of an elliptical or a rectangular pupil from the value of $-1/2$ for a circular pupil. Moreover, for these pupils, spherical aberration ρ^4 is balanced not only by defocus ρ^2 but astigmatism $\rho^2\cos^2\theta$ as well. This is a consequence of the fact that the x and y dimensions of the these pupils are not equal.

A square pupil is a special case of a rectangular pupil for which $a = 1/\sqrt{2}$. It is evident from the square polynomials S_5 and S_6 that they have the same form as the corresponding circle polynomials. Thus there is no additional defocus for balancing astigmatism, as may be seen by the absence of a Z_4 term in the expression for S_6. Hence, the diffraction focus of a system does not change when its circular pupil is replaced by an inscribed square pupil. Unlike an elliptical or a rectangular pupil, the primary spherical aberration in a square pupil is balanced by defocus only, as is evident from the radially symmetric expression for S_{11}. However, the balanced secondary and tertiary spherical aberrations are not radially symmetric, since they contain angle-dependent terms varying as $\cos 4\theta$. From the polynomials S_7, S_8, and S_{11}, the diffraction foci in the case of coma and spherical aberration are closer to the Gaussian image point and the Gaussian image plane, respectively, compared to their corresponding locations for a circular pupil.

The sigma of Seidel aberrations with and without balancing are listed in Table 8 for elliptical and rectangular pupils. The corresponding values for a circular, hexagonal, square, and a slit pupil are listed in Table 9.[26] As expected, the results for an elliptical pupil reduce to those for a circular pupil as $b \to 1$, and the results for a rectangular pupil reduce to those for a square pupil as $a \to 1/\sqrt{2}$. As the area of a unit pupil decreases in going from a circular to a hexagonal to a square pupil (from π to $3\sqrt{3}/2 \approx 2.6$ to 2), the sigma of an aberration decreases and its tolerance for a certain Strehl ratio

TABLE 8 Standard Deviation or Sigma of a Primary and a Balanced Primary Aberration for Elliptical and Rectangular Pupils

Sigma	Elliptical	Rectangular
σ_d	$(1/4)[(3-2b^2+3b^4)/3]^{1/2}$	$(2/3)[(1-2a^2+2a^4)/5]^{1/2}$
σ_a	$1/4$	$2a^2/(3\sqrt{5})$
σ_{ba}	$b^2/[6(3-2b^2+3b^4)]^{1/2}$	$2a^2(1-a^2)/\{3[5(1-2a^2+2a^4)]^{1/2}\}$
σ_c	$(5+2b^2+b^4)^{1/2}/8$	$a[(7+8a^4)/105]^{1/2}$
σ_{bc}	$(9-6b^2+5b^4)^{1/2}/24$	$2a(35-70a^2+62a^4)^{1/2}/(15\sqrt{21})$
σ_s	$(225+60b^2-58b^4+60b^6+225b^8)^{1/2}/(24\sqrt{10})$	$4(63-162a^2+206a^4-88a^6+44a^8)^{1/2}/(45\sqrt{7})$
σ_{bc}	$(45-60b^2+94b^4-60b^6+45b^8)^{1/2}/(48\sqrt{5})$	$(8/315)(9-36a^2+103a^4-134a^6+67a^8)^{1/2}$

TABLE 9 Standard Deviation or Sigma of a Primary and a Balanced Primary Aberration for Circular, Hexagonal, Square, and Slit Pupils

Sigma	Circle	Hexagon	Square	Slit
σ_d	$1/(2\sqrt{3})$	$(1/12)\sqrt{43/5}$	$(1/3)\sqrt{2/5}$	$2/(3\sqrt{5})$
	$=1/3.464$	$=1/4.092$	$=1/4.743$	$=1/3.354$
σ_a	$1/4$	$(1/24)\sqrt{127/5}$	$1/(3\sqrt{5})$	-
		$=1/4.762$	$=1/6.708$	-
σ_{ba}	$1/(2\sqrt{6})$	$(1/4)\sqrt{7/15}$	$1/(3\sqrt{10})$	-
	$=1/4.899$	$=1/5.855$	$=1/9.487$	-
σ_c	$1/(2\sqrt{2})$	$(1/4)\sqrt{83/70}$	$\sqrt{3/70}$	$1\sqrt{7}$
	$=1/2.828$	$=1/3.673$	$=1/4.831$	$=1/2.646$
σ_{bc}	$1/(6\sqrt{2})$	$(1/20)\sqrt{737/210}$	$(1/15)\sqrt{31/21}$	$2/(5\sqrt{7})$
	$=1/8.485$	$=1/10.676$	$=1/12.346$	$=1/6.614$
σ_s	$2/(3\sqrt{5})$	$(1/6)\sqrt{59/35}$	$(2/45)\sqrt{101/7}$	$4/15$
	$=1/3.354$	$=1/4.621$	$=1/5.923$	$=1/3.750$
σ_{bs}	$1/(6\sqrt{5})$	$(1/84)\sqrt{4987/215}$	$(2/315)\sqrt{67}$	$8/105$
	$=1/13.416$	$=1/17.441$	$=1/19.242$	$=1/13.125$

increases. The slit pupil is more sensitive compared to a circular pupil, except for spherical aberration for which it is slightly less sensitive. To obtain the Seidel coefficients from the orthonormal coefficients of a noncircular wavefront, all significant coefficients that contain a Seidel term must be taken into account, just as in the case of Zernike coefficients.[36]

11.12 ISOMETRIC, INTERFEROMETRIC, AND PSF PLOTS FOR ORTHONORMAL ABERRATIONS

The aberration-free point-spread functions (PSFs) for unit pupils considered in this chapter are shown in Fig. 5, illustrating their symmetry, for example, 6-fold symmetry for a hexagonal pupil. Their linear scale is such that the first zero of the PSF, for example, for a square pupil occurs at unity in units of λF (corresponding to 1.22 for a circular pupil). Here λ is the wavelength of the object radiation and F is the focal ratio of the image-forming light cone. These PSFs are the ultimate goal of fabrication and testing. The obscuration ratio of the annular pupil in Fig. 5*b* is $\epsilon = 0.5$; the aspect ratio of the elliptical pupil in Fig. 5*d* is $b = 0.85$; and the half width of the rectangular pupil in Fig. 5*e*

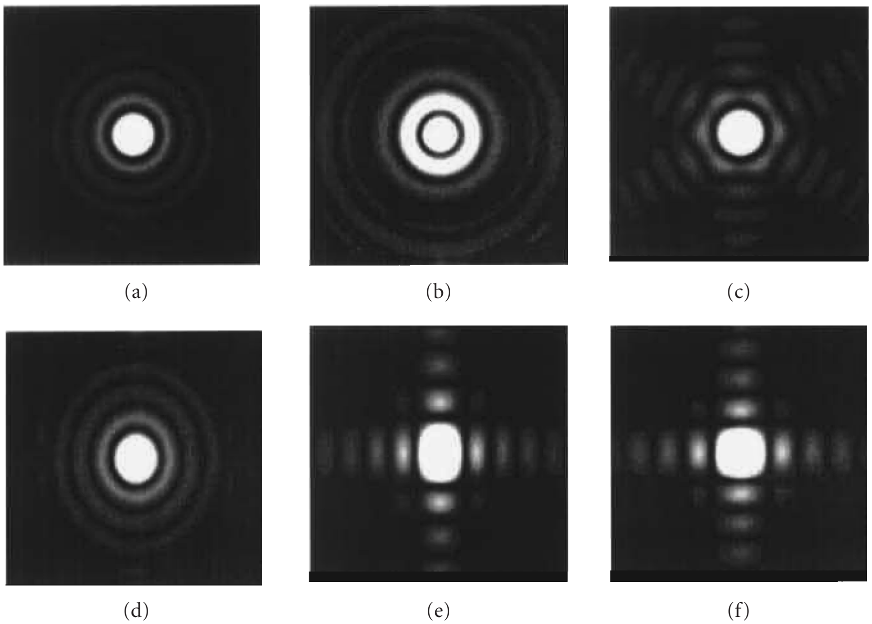


FIGURE 5 Aberration-free PSFs for different unit pupils: (*a*) Circular; (*b*) annular with obscuration ratio $\epsilon = 0.5$; (*c*) hexagonal; (*d*) elliptical with aspect ratio $b = 0.85$; (*e*) rectangular with half width $a = 0.8$; and (*f*) square.

is $a = 0.8$. The orthonormal polynomials corresponding to a Seidel aberration for a hexagonal, elliptical, rectangular, and square pupils are illustrated in three different but equivalent ways in Fig. 6.[7] In Fig. 6*d*, as in Fig. 5*a* the aspect ratio of the elliptical pupil is $b = 0.85$. In Fig. 6*e*, as in Fig. 5*e*, the half width of the rectangular pupil is $a = 0.8$. For each polynomial, the isometric plot at the top illustrates its shape as produced, for example, in a deformable mirror. The standard deviation of each polynomial aberration in the figure is one wave. An interferogram, as in optical testing, is shown on the left. The number of fringes, which is equal to the number of times the aberration changes by one wave as we move from the center to the edge of a pupil, is different for the different polynomials. Each fringe represents a contour of constant phase or aberration. The fringe is dark when the phase is an odd multiple of π or the aberration is an odd multiple of $\lambda/2$. On the right for each polynomial are shown the PSFs, which represent the images of a point object in the presence of a polynomial aberration.

11.13 USE OF CIRCLE POLYNOMIALS FOR NONCIRCULAR PUPILS

Since the Zernike circle polynomials form a complete set, any wavefront, regardless of the shape of the pupil (which defines the perimeter of the wavefront) can be expanded in terms of them.[34] However, unless the pupil is circular, advantages of orthogonality and aberration balancing are lost. For example, the mean value of a Zernike circle polynomial across a noncircular pupil is not zero, the Zernike piston coefficient does not represent the mean value of the aberration, the other Zernike coefficients do not represent the standard deviation of the corresponding aberration terms, and the variance of the aberration is not equal to the sum of the squares of these other coefficients. Moreover, the value of a Zernike coefficient changes as the number of polynomials used in the

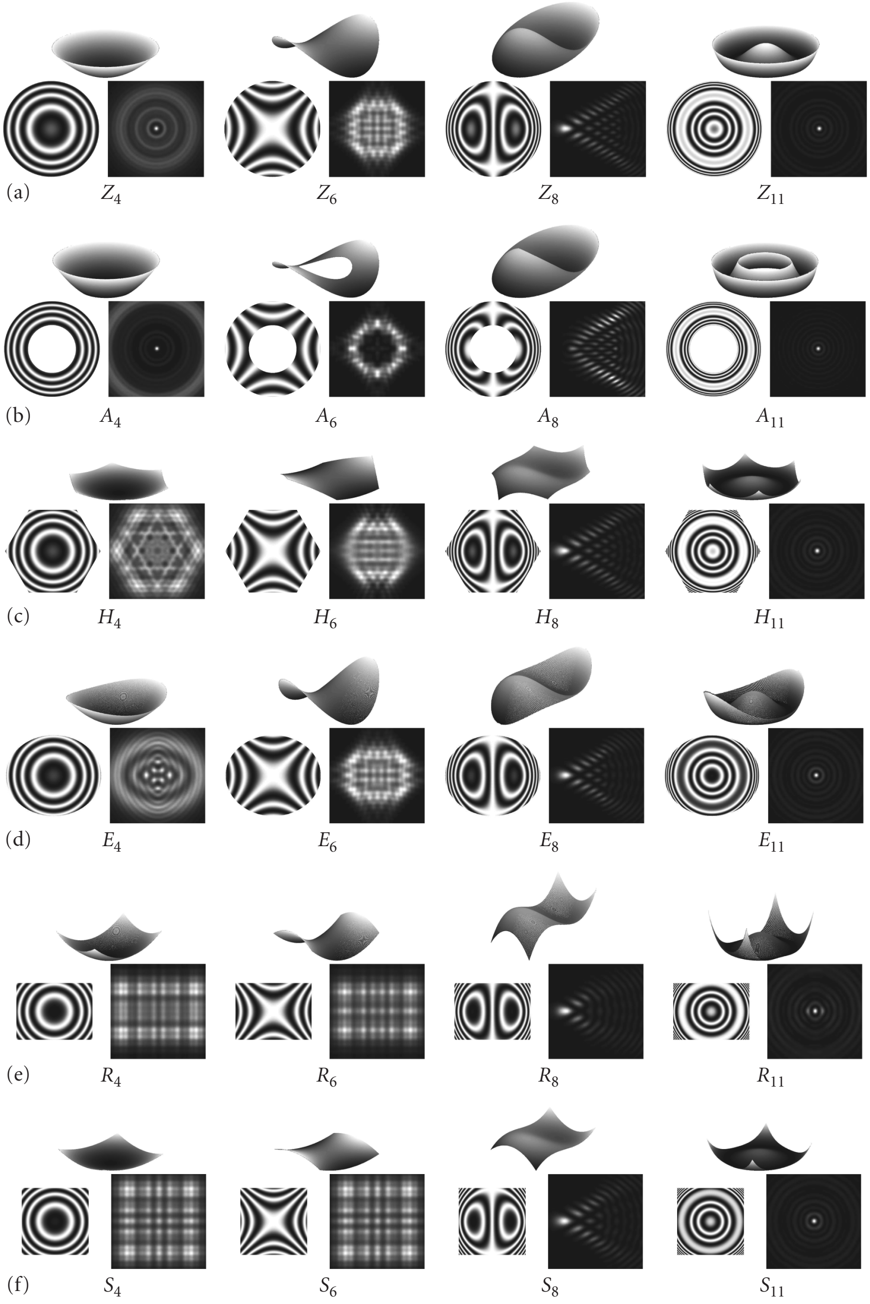


FIGURE 6 Isometric plots, interferograms, and PSFs for defocus ($j = 4$), astigmatism ($j = 6$), coma ($j = 8$), and spherical aberration ($j = 11$) in unit pupils. (*a*) Circular; (*b*) annular with $\epsilon = 0.5$; (*c*) hexagonal; (*d*) elliptical with aspect ratio $b = 0.85$; (*e*) rectangular with half width $a = 0.8$; and (*f*) square.

ORTHONORMAL POLYNOMIALS IN WAVEFRONT ANALYSIS **11.39**

expansion of an aberration function changes. Hence, the circle polynomials are not appropriate for analysis of noncircular wavefronts. The polynomials given in this chapter for various pupils uniquely represent balanced classical aberrations that are also orthogonal across those pupils, just like the Zernike circle polynomials are for a circular pupil. Since each orthonormal polynomial is a linear combination of the Zernike circle polynomials, the wavefront fitting is as complete with the latter as it is with the former. However, since the circle polynomials do not represent the balanced classical aberrations for a noncircular pupil, the Zernike coefficients do not have the physical significance of their orthonormal counterparts. But the tip/tilt and defocus values in an interferometrically obtained aberration function, representing the lateral and longitudinal errors of an interferometer setting, obtained from the corresponding Zernike circle coefficients when the function is approximated with only the first four circle polynomials in a least square sense are identically the same as those obtained from the corresponding orthonormal coefficients. Accordingly, the aberration function obtained by subtracting the tip/tilt and defocus values from the measured aberration function is independent of the nature of the polynomials used in the expansion, regardless of the domain of the function or the shape of the pupil, so long as the nonorthogonal expansion is in terms of only the first four circle polynomials. The difference function is what is provided to the optician to zero out from the surface under fabrication by polishing.

11.14 DISCUSSION AND CONCLUSIONS

The Zernike circle polynomials are in widespread use for wavefront analysis in optical design and testing, because they are orthogonal over a unit circle and represent balanced aberrations of systems with circular pupils. When an aberration function of a circular wavefront is expanded in terms of them, the value of an expansion coefficient is independent of the number of polynomials used in the expansion. Accordingly, one or more terms can be added or subtracted without affecting the other coefficients. The piston coefficient represents the mean value of the aberration function and the other coefficients represent the standard deviation of the corresponding terms. The variance of the aberration is given simply by the sum of the squares of those other aberration coefficients.

We have also listed the orthonormal polynomials for analyzing the wavefronts across noncircular pupils, such as annular, hexagonal, elliptical, rectangular, and square. These polynomials are for unit pupils inscribed inside a unit circle. Such a choice keeps the maximum value of the distance of a point on the pupil from its center to be unity, thus easily identifying the peak of value of a classical aberration across it. Each orthonormal polynomial for the pupils considered consists of either the cosine or the sine terms, but not both due to the biaxial symmetry of the pupils. Whereas the circle and annular polynomials are separable in their dependence on the polar coordinates ρ and θ of a pupil point due to the radial symmetry of the pupils, only some of the polynomials for other pupils are separable. Hence polynomial numbering with two indices n and m, as for circular and annular polynomials, loses significance, and must be numbered with a single index j. The hexagonal polynomials H_{11} and H_{22} representing the balanced primary and secondary spherical aberrations are radially symmetric, but the polynomial H_{37} representing the balanced tertiary spherical aberration is not, since it contains an angle-dependent term in Z_{28} or $\cos 6\theta$ also. A hexagonal pupil has two distinct configurations where the hexagon in one is rotated by 30° with respect to that in the other. Only some of the polynomials are the same for the two configurations.[26] While the balancing defocus to optimally balance Seidel astigmatism for a hexagonal or a square pupil is the same as that for circular and annular pupils, it is different for the elliptical and rectangular pupils. For the elliptical and rectangular pupils, the Seidel or primary aberration ρ^4 is balanced not only by defocus but astigmatism as well. The square polynomial S_{11} representing the balanced primary spherical aberration is radially symmetric, but the square polynomial S_{22} representing the balanced secondary spherical aberration is not, since it contains a term in Z_{14} or $\cos 4\theta$ also. Similarly, the polynomial S_{37} representing the balanced tertiary spherical aberration is also not radially symmetric, since it consists of terms in Z_{14} and Z_{26} both varying as $\cos 4\theta$. We have illustrated orthonormal polynomials in three different but equivalent ways: isometrically, interferometrically, and by the corresponding aberrated PSFs.

The sigma of a Seidel aberration with and without balancing decreases as the area of a unit pupil decreases in going from a circular to a hexagonal to a square pupil. The sigma for Seidel astigmatism $\rho^2\cos\theta$ for an elliptical pupil is independent of its aspect ratio and, therefore, is the same as for a circular pupil. This is due to the fact that the aberration is one dimensional along the dimension for which the unit ellipse has the same length as the unit circle. Since a slit pupil is one dimensional, there is no distinction between defocus and astigmatism. It is more sensitive to a Seidel aberration with or without balancing compared to a circular pupil, except for spherical aberration for which it is slightly less sensitive.

When the aberration function is known only at a discrete set of points, as in a digitized interferogram, the integral for determining the aberration coefficients reduces to a sum and the orthonormal coefficients thus obtained may be in error, since the polynomials are not orthonormal over the discrete points of the aberration data set. The magnitude of the error decreases as the number of points increases. This is not a serious problem when the wavefront errors are determined by, say, phase-shifting interferometry,[37] since the number of points can be very large. However, when the number of data points is small, or the pupil is irregular in shape due to vignetting, then ray tracing or testing of the system yields wavefront error data at an array of points across a region for which closed-form orthonormal polynomials are not available. In such cases, we can determine the coefficients of an expansion in terms of numerical polynomials that are orthogonal over the data set, obtained by the Gram-Schmidt orthogonalization process.[7,38] However, if we just want to determine the values of tip/tilt and defocus terms, yielding the errors in interferometer settings, they can be obtained by least squares fitting the aberration function data with only these terms.

11.15 REFERENCES

1. F. Zernike, "Diffraction Theory of Knife-Edge Test and Its Improved Form, the Phase Contrast Method," *Mon. Not. R. Astron. Soc.* **94:** 377–384 (1934).
2. R. J. Noll, "Zernike Polynomials and Atmospheric Turbulence," *J. Opt. Soc. Am.* **66:** 207–211 (1976).
3. B. R. A. Nijboer, "The Diffraction Theory of Optical Aberrations. Part II: Diffraction Pattern in the Presence of Small Aberrations," *Physica.* **13:** 605–620 (1947).
4. M. Born and E. Wolf, *Principles of Optics,* 7th ed., Oxford, New York (1999).
5. V. N. Mahajan, *Optical Imaging and Aberrations,* Part II: Wave Diffraction Optics, SPIE Press, Bellingham, Washington (Second Printing 2004).
6. V. N. Mahajan, "Zernike Polynomials and Aberration Balancing," SPIE Proc. **5173:** 1–17 (2003).
7. V. N. Mahajan, "Zernike Polynomials and Wavefront Fitting," in *Optical Shop Testing,* 3rd ed., D. Malacara, ed., Wiley, New York, pp. 498–546 (2007).
8. V. N. Mahajan, "Zernike Annular Polynomials for Imaging Systems with Annular Pupils," *J. Opt. Soc. Am.* **71:** 75–85(1981).
9. V. N. Mahajan, "Zernike Annular Polynomials for Imaging Systems with Annular Pupils," *J. Opt. Soc. Am.* **71:** 1408 (1981).
10. V. N. Mahajan, "Zernike Annular Polynomials for Imaging Systems with Annular Pupils," *J. Opt. Soc. Am. A* **1:** 685 (1984).
11. V. N. Mahajan, "Zernike Annular Polynomials and Optical Aberrations of Systems with Annular Pupils," *Appl. Opt.* **33:** 8125–8127 (1994).
12. http://scikits.com/KFacts.html
13. W. B. King, "The Approximation of Vignetted Pupil Shape by an Ellipse," *Appl. Opt.* **7:** 197–201 (1968).
14. G. Harbers, P. J. Kunst, and G. W. R. Leibbrandt, "Analysis of Lateral Shearing Interferograms by Use of Zernike Polynomials," *Appl. Opt.* **35:** 6162–6172 (1996).
15. H. Sumita, "Orthogonal Expansion of the Aberration Difference Function and Its Application to Image Evaluation," *Jpn. J. Appl. Phys.* **8:** 1027–1036 (1969).

16. K. N. LaFortune, R. L. Hurd, S. N. Fochs, M. D. Rotter, P. H. Pax, R. L. Combs, S. S. Olivier, J. M. Brase, and R. M. Yamamoto, "Technical Challenges for the Future of High Energy Lasers," SPIE Proc. **6454:** 1–11(2007).
17. G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers,* McGraw–Hill, New York, (1968).
18. V. N. Mahajan and G.-m. Dai, "Orthonormal Polynomials for Hexagonal Pupils," *Opt. Lett.* **31:** 2462–2465 (2006).
19. G.-m. Dai and V. N. Mahajan, "Nonrecursive Orthonormal Polynomials with Matrix Formulation," *Opt. Lett.* **32:** 74–76 (2007).
20. R. Barakat and L. Riseberg, "Diffraction Theory of the Aberrations of a Slit Aperture," *J. Opt. Soc. Am.* **55:** 878–881 (1965). There is an error in their polynomial S_2, which should read as $x^2 - 1/3$.
21. M. Bray, "Orthogonal Polynomials: A Set for Square Areas," *SPIE Proc.* **5252:** 314–320 (2004).
22. J. L. Rayces, "Least-Squares Fitting of Orthogonal Polynomials to the Wave-Aberration Function," *Appl. Opt.* **31:** 2223–2228 (1992).
23. V. N. Mahajan, "Uniform Versus Gaussian Beams: a Comparison of the Effects of Diffraction, Obscuration, and Aberrations," *J. Opt. Soc. Am.* **A3:** 470–485 (1986).
24. V. N. Mahajan, "Zernike-Gauss Polynomials and Optical Aberrations of Systems with Gaussian Pupils," *Appl. Opt.* **34:** 8057–8059 (1995).
25. S. Szapiel, "Aberration Balancing Techniques for Radially Symmetric Amplitude Distributions; a Generalization of the Maréchal Approach," *J. Opt. Soc. Am.* **72:** 947–956 (1982).
26. V. N. Mahajan and G.-m Dai, "Orthonormal Polynomials in Wavefront Analysis: Analytical Solution," *J Opt. Soc. Am.* **A24:** 2994–3016 (2007).
27. A. B. Bhatia and E. Wolf, "On the Circle Polynomials of Zernike and Related Orthogonal Sets," in *Proc. Camb. Phil. Soc.* **50:** 40–48 (1954).
28. V. N. Mahajan, *Optical Imaging and Aberrations,* Part I: *Ray Geometrical Optics,* SPIE Press, Bellingham, Washington (Second Printing 2001).
29. W. T. Welford, *Aberrations of the Symmetrical Optical System,* Academic Press, New York (1974).
30. R. R. Shannon, *The Art and Science of Optical Design,* Cambridge University Press, New York (1997).
31. P. Mouroulis and J. Macdonald, *Geometrical Optics and Optical Design,* Oxford, New York (1997).
32. D. Malacara and Z. Malacara, *Handbook of Lens Design,* Dekkar, New York (1994).
33. G.-m Dai and V. N. Mahajan, "Zernike Annular Polynomials and Atmospheric Turbulence," *J Opt. Soc. Am.* **A24:** 139–155 (2007).
34. G.-m Dai and V. N. Mahajan, "Orthonormal Polynomials in Wavefront Analysis: Error Analysis," *Appl. Opt.* **47:** 3433–3445 (2008).
35. V. N. Mahajan, "Strehl Ratio for Primary Aberrations in Terms of Their Aberration Variance," *J. Opt. Soc. Am.* **73:** 860–861 (1983).
36. V. N. Mahajan and W. H. Swantner, "Seidel Coefficients in Optical Testing," *Asian J. Phys.* **15:** 203–209 (2006).
37. K. Creath, "Phase-Measurement Interferometry Techniques," *Progress in Optics*, E. Wolf, ed., Elsevier, New York, **26:** 349–393 (1988).
38. D. Malacara, J. M. Carpio-Valdéz, and J. Javier Sánchez-Mondragón, "Wavefront Fitting with Discrete Orthogonal Polynomials in a Unit Radius Circle," *Opt. Eng.* **29:** 672–675 (1990).