A. La Rosa Lecture Notes

INTRODUCTION TO QUANTUM MECHANICS

PART-II MAKING PREDICTIONS in QUANTUM MECHANICS and the HEISENBERG's PRINCIPLE

CHAPTER-6
THE CONCEPT of AMPLITUDE PROBABILITY

References

R. Feynman, "The Feynman Lectures on Physics," **Volume III**, **Chapter 3.**

R. Eisberg and R. Resnick, "Quantum Physics," 2nd Edition, Wiley, 1985. **Sections 8-1 to 8-3.**

B. H. Bransdem& C.J. Joachin, "Quantum Mechanics" 2nd Ed. Prentice Hall, 2000. **Section 1.5**.

CHAPTER-6
THE AMPLITUDE PROBABILITY[1]

In Chapter 5 we emphasized the crucial aspect played by the measurement process in Quantum Mechanics, leading to the enunciation of the Heisenberg uncertainty principle. The determinism (characteristic of classical physics) was replaced by the probabilistic character (characteristic of quantum physics) to predict the behavior of a system of particles.

The wavefunction Ψ itself fully characterizing a given system (although, to be more precise, recall that Ψ represents the associated ensemble of identically prepared systems) does not have a direct interpretation in classical terms. It is $|\psi|^2$ what is interpreted as probability (the latter being a concept we can be understood in classical terms). To gain some familiarity with Ψ itself and to learn how to mathematically manipulate it in order to make predictions on the behavior of a system, in this chapter we will give emphasis to the description of Ψ itself and its spectral components (relative to a given base states.)

6.1 Quantum Mechanics Description in terms of Base-states

The concept of spectral components was introduced in Chapter 4. Given a pre-determined basis-set of functions $\{\ |\varphi_n\rangle;\ n = 1,\ 2,\ ...\}$, which satisfy $\langle\varphi_n|\varphi_m\rangle = \delta_{mn}$, any state of motion Ψ of the system can be described by a proper linear combination of the form,

$$\underset{\text{State}}{|\psi\rangle} = \sum_n \underbrace{|\varphi_n\rangle}_{\text{Base State}} A_n \tag{1}$$

Notice the generality of the notation, where the detailed dependence of Ψ on the coordinates x and t is put aside. Generality is gained because it allows the study of almost any quantum mechanical system, including the ones that do not admit a description in terms of spatial coordinates such as x. An example of the latter systems are the ones that involve spin (a property to be described at the end of this chapter.)

More about the scalar product, bra and ket notation

Before engaging in the description of amplitude probabilities, lets review a subtle manipulation of the inner product definition, which it will widely used in this and future chapters.)

- Given $\left|\psi\right\rangle = \sum_n \left|\varphi_n\right\rangle A_n$ and $\left|\chi\right\rangle = \sum_m \left|\varphi_m\right\rangle B_m$,

  how to calculate $\left\langle\chi\middle|\psi\right\rangle$? (2)

  We have $|\,\chi\rangle$ but we need $\langle\chi\,|$ so we can calculate $\langle\chi\,|\,|\,\Psi\rangle$

  Can we just affirm:

  $$\left|\chi\right\rangle = \sum_m \left|\varphi_m\right\rangle B_m \text{ implies } \left\langle\chi\right| = \sum_m \left\langle\varphi_m\right| B_m \;? \tag{3}$$

  If this statement were correct, we would obtain

  $$\left\langle\chi\middle|\psi\right\rangle = \left(\sum_m \left\langle\varphi_m\right| B_m\right)\left(\sum_n \left|\varphi_n\right\rangle A_n\right) = \sum_n \sum_m B_m A_n \left\langle\varphi_m\middle|\varphi_n\right\rangle$$

  However, this last expression is incorrect.

- To find out the correct value of $\left\langle\chi\middle|\psi\right\rangle$ let's use the original notation that uses parenthesis (as defined in Chapter 4).

  Taking $\psi = \sum_n \varphi_n\ A_n$ and $\chi = \sum_m \varphi_m\ B_m$, the definition of the scalar product $(\chi,\psi) \equiv \int_{-\infty}^{\infty} \chi^*(x)\psi(x)\,dx)$ leads us to the following,

$$(\chi,\psi) = \left(\sum_m \varphi_m\ B_m, \sum_n \varphi_n\ A_n\right) = \int_{-\infty}^{\infty} \left[\sum_m \varphi_m(x)\ B_m\right]^* \left[\sum_n \varphi_n(x)\ A_n\right] dx =$$

$$= \int_{-\infty}^{\infty} \left[\sum_m \varphi_m^*(x)\ B_m^*\right]\left[\sum_n \varphi_n(x)\ A_n\right] dx$$

$$= \sum_m \sum_n \int_{-\infty}^{\infty} \left[\varphi_m{}^*(x)\ B_m{}^*\right] \left[\varphi_n(x) A_n\right] dx$$

$$= \sum_m \sum_n B_m{}^* A_n \int_{-\infty}^{\infty} \left[\varphi_m{}^*(x)\ \right]\left[\varphi_n(x)\ \right] dx$$

$$= \sum_m \sum_n B_m{}^* A_n \left(\varphi_m , \varphi_n\right)$$

Thus,

$\psi = \sum_n \varphi_n\ A_n$ and $\chi = \sum_m \varphi_m\ B_m$ implies

$$\left(\chi, \psi\right) = \sum_m \sum_n B_m{}^* A_n \left(\varphi_m , \varphi_n\right) \tag{4}$$

- Translating this result into bra-ket notation,

If $\left|\chi\right\rangle = \sum_m \left|\varphi_m\right\rangle B_m$ and $\left|\psi\right\rangle = \sum_n \left|\varphi_n\right\rangle A_n$ then,

$$\left\langle\chi\middle|\psi\right\rangle = \sum_n \sum_m B_m{}^* A_n < \varphi_m , \varphi_n >$$

alternatively

$$\left\langle\chi\middle|\psi\right\rangle = \left(\sum_m B_m{}^* \left\langle\varphi_m\right|\right)\left(\sum_n A_n \left|\varphi_n\right\rangle\right) \tag{5}$$

This suggests the following rule:

If $\left|\chi\right\rangle = \sum_m \left|\varphi_m\right\rangle B_m$ then $\left\langle\chi\right| = \sum_m B_m{}^* \left\langle\varphi_m\right|$ (6)

where $B_m{}^*$ stands for the complex conjugate of B_m.

6.2 Definition of *Amplitude Probability*

In the expansion of $\left|\psi\right\rangle$ in terms of the base states $\{\left|\varphi_n\right\rangle$, $n = 1, 2,$ …}, given by $\left|\psi\right\rangle = \sum_n \left|\varphi_n\right\rangle A_n$, the complex number coefficients A_n (to be referred as the amplitude probabilities,) are obtained by means of the scalar product,

$$\langle \varphi_n | \psi \rangle = A_n \qquad \textit{Amplitude probability}$$

Interpretation:

In this formalism, the probability amplitude $\langle \varphi_n | \psi \rangle = A_n$ is a complex number whose magnitude squared $|\langle \varphi_n | \psi \rangle|^2 = A_n^{\ *} A_n = |A_n|^2$ gives the probability that a system initially in the state ψ will be found in the base state φ_n after a measurement.

Generalization:

In a more general context,

$$\langle \chi | \psi \rangle \qquad \textit{Amplitude Probability} \tag{7}$$

Complex number whose magnitude squared $|\langle \chi | \psi \rangle|^2$ gives the probability that a system initially in the state $|\Psi\rangle$ will be found in the state $|\chi\rangle$ after the measurement.

6.3 General guiding principles to assign amplitude probabilities

To familiarize with the proper mathematical manipulation of amplitude probabilities, we'll follow Feynman's general guiding principles. For illustrative purposes, these principles are presented here along the description of the experiment described in Chapter 5 where particles (photons or electrons) pass through a couple of slits. It was mentioned there that we could not determined which aperture a particle uses before reaching the detection screen; when attempting to watch their trajectory, the mere observation affected dramatically the result. We would like to formalize the description of this experiment in terms of the *amplitude probability* tool.

Principle-1 How to assign amplitude probabilities

The amplitude probability *that a particle leaving the source S will arrive at the position x on the screen* can be represented by a complex number as follows,

$$\left\langle \textit{Particle arrives at } x \middle| \textit{particle leaves } s \right\rangle \tag{8}$$

To simplify the notation of this complex number in the particular case that we are considering, instead of (3) let's use,

$$\langle x | s \rangle \tag{9}$$

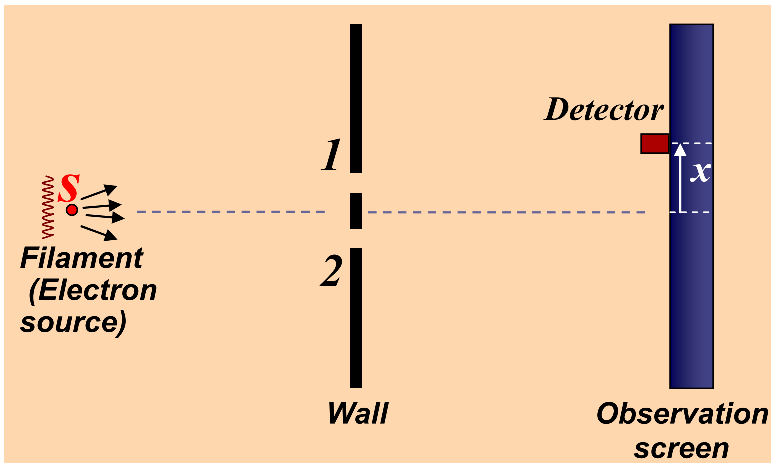


Fig. 6.1 Schematic set up of the two-slit experiment.

For a more general case, this notation will imply

$$\left\langle \textbf{\textit{final}} \textit{ state} \chi \middle| \textbf{\textit{starting}} \textit{ state } \psi \right\rangle \tag{10}$$

Amplitude probability that a system in a starting state "Ψ" ends up in a final state "χ".

Principle-2 When to add amplitude probabilities

When a particle can reach a given final state by two (or more) possible routes, the total amplitude for the process is the sum of the amplitudes associated to the two (or more) routes separately.

In other words, we add amplitudes when the corresponding possible routes have the same initial and final states.

In our example, since the event consisting of a particle leaving "s" and arriving at "x" can be realized via the aperture 1 or the aperture 2, the amplitude $\langle x|s\rangle$ will be given by,

$$\langle x|s\rangle = \underbrace{\langle x|s\rangle_{\text{via}-1}} + \underbrace{\langle x|s\rangle_{\text{via}-2}} \tag{11}$$

Both paths have the same initial and final states

Principle-3 When to multiply of amplitude probabilities

The amplitude for a given route can be written as the product of (the amplitude to go part of the way) times (the amplitude to go the rest of the way).

In our particular case, the amplitude to go from s to x via the aperture-1 will be written as

$$\langle x|s\rangle_{\text{via}-1} = \langle x|1\rangle\langle 1|s\rangle \tag{12}$$

Based on (11), expression (10) can be re-written as,

$$\langle x|s\rangle_{\text{both apertures open}} = \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle \tag{13}$$

We may complain at this stage that we do not know exactly what are the values of the terms we are dealing with (like, for example, what is the numerical value of $\langle x|1\rangle\langle 1|s\rangle$.) But be patient at this stage. We are not ready yet to calculate numerical values; rather we are focused in learning how to operate with those numbers. [We do not know what the lunar module is made of; but we are learning how to operate it in our quest to land on the Moon.]

6.4 Interference between Amplitude Probabilities

In this section we describe some specific examples that illustrate the simplicity and power of the amplitude probability formalism to correctly predict and understand quantum phenomena (in particular those that are not possible to explain with classical mechanics tools.) The key aspect to learn in this section is the occurrence of interference between amplitude probabilities terms associated to different events that produce the same final state. The first example is related to the two-slit experiment encountered in Chapter 5, which was a bit strange to understand with classical arguments. Being familiar with that experiment will help us to understand how to apply the concept of amplitude probabilities. The second example deals with the scattering of neutrons from a crystal, and emphasizes the role played by the spin character of the particle.

6.4.A Two-slit experiment: Watching electron's trajectories

Chapter 5 described how the act of using photons to watch the incident electrons, aiming to distinguish which of the two apertures a given electron passed through, affected substantially the pattern of intensity on the observing screen. In other words, the pattern of intensity on the screen is dramatically affected by the act of observing the electrons with a source of light. That example was used to illustrate the Heisenberg's uncertainty principle, which reflects the inherent limitations an observer must face when trying to measure the variables characterizing the motion of a particle. The act of measurement perturbs the system under analysis thus preventing the knowledge on the values of the physical parameter before the measurement.

Armed with the tool of *amplitude probability* (and its associated rules), let's describe the two slit experiment under such formalism and check what predictions it makes under different circumstances.

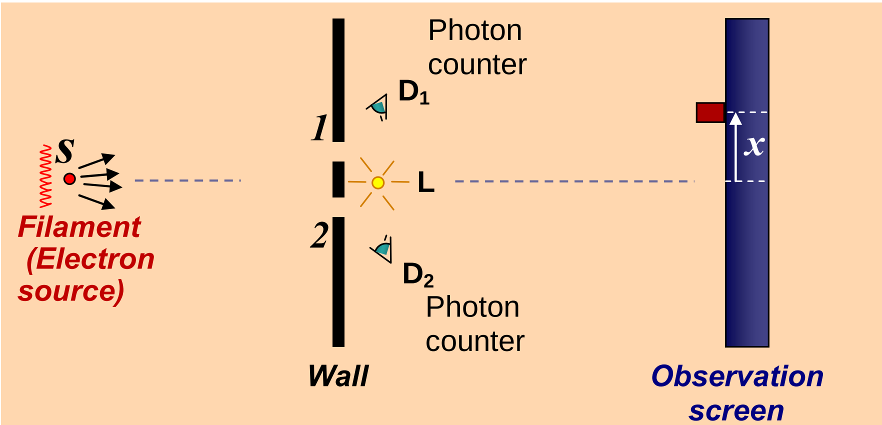


Fig. 6.2 Two slit experiment setup where electrons are being watched by photons (the latter monitored by two photon detectors.)

A light source "L" is used in an attempt to watch what specific aperture an electron chooses to pass through.

Simplification: We will assume that when an electron passes through a given aperture it will scatter a photon towards either the photon-detector D_1 or the photon detector D_2. (That is, we assume a scattered photon does not go undetected.) There are different possible results. Let's analyze them in more detail.

Monitoring the experiment with the photon counter D_2

Event: Electron starts at "s", ends up at the position "x" on the screen, and a photon is detected at counter D_1

What is the amplitude probability for this event to happen?

- There is an amplitude probability that an electron goes from the source s to the aperture-1: $\langle 1 | s \rangle$

  While the electron passes through the slit-1, there is an amplitude probability that it scatters a photon into the photon counter D_1; let's call this amplitude a

  There is an amplitude probability that the electron goes from the slit-1 to the position x on the screen: $\langle x | 1 \rangle$

  Accordingly the **amplitude probability** that an electron goes from s to x via the aperture-1, AND scatters a photon into the photon counter D_1 is given by: $\langle x | 1 \rangle \, a \, \langle 1 | s \rangle$

- An electron passing through the slit-2 could produce the same result. That is,

  There is an amplitude probability that an electron goes from the source s to the aperture-2: $\langle 2 | s \rangle$

  While the electron passes through the slit-2, there is an amplitude probability that it scatters a photon into the photon counter D_1; let's call this amplitude b

  [It is possible that the magnitude of b could be very small compared to the magnitude of a, since the photon detector D_1 is farther away from aperture-2.]

There is an amplitude probability that the electron goes from the slit-2 to the position x on the screen: $\langle x | 2 \rangle$

Accordingly: the **amplitude probability** that an electron goes from s to x via the aperture-2, AND scatters a photon into the photon counter D_1 is given by: $\langle x | 2 \rangle \, b \, \langle 2 | s \rangle$

- Interference between events that produce the same final state:

  Notice in the two cases mentioned above that the net result is an electron going from s to x, AND a photon from the source L scattered into the photon counter D_1. That is, the final state produced by these two events can not be distinguished. Accordingly, we add these two amplitudes to describe the following event:

$$\left\langle \begin{matrix} electron\ at\ x \\ photon\ in\ D_1 \end{matrix} \middle| \begin{matrix} electron\ \ from\ s \\ photon\ \ from\ L \end{matrix} \right\rangle = \underbrace{\langle x|1\rangle\, a\, \langle 1|s\rangle} + \underbrace{\langle x|2\rangle\, b\, \langle 2|s\rangle} \qquad (14)$$

*These two events have the **same initial state** and follow a path that (respectively) leads to the **same final state**. Accordingly, we add up their corresponding amplitude probabilities.*

Monitoring the experiment with the photon counter D_2

Event: Electron starts at "s", ends up at the position "x" on the screen, and a photon is detected at counter D_2

What is the amplitude probability for this event to happen?

- By symmetry, the amplitude that an electron scatters a photon into the photon counter D_2 while passing through the aperture-2 should be equal to a.

  Also, the amplitude that an electron scatters a photon into the photon counter D_2 while passing through the aperture-1 should be equal to b. Accordingly,

$$\left\langle \begin{matrix} electron\ at\ x \\ photon\ in\ D_2 \end{matrix} \middle| \begin{matrix} electron\ from\ s \\ photon\ from\ L \end{matrix} \right\rangle = \underbrace{\langle x|2\rangle\, a\, \langle 2|s\rangle} + \underbrace{\langle x|1\rangle\, b\, \langle 1|s\rangle} \qquad (15)$$

*These two events have the **same initial state** and follow a path that (respectively) leads to the **same final state**. Accordingly, we add up their corresponding amplitude probabilitues.*

Expressions (14) and (15) should be compared with result (13) above $\langle x|s\rangle = \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle$ (obtained for the case where no light source was used). Notice that,

> *the effect of using light introduces "weighting" complex factors a and b into the participating amplitudes $\langle x|1\rangle\langle 1|s\rangle$ and $\langle x|2\rangle\langle 2|s\rangle$.*

To understand the effect of light, let's calculate the probability for different situations.

When to add "*probabilities*" and not "*amplitude-probabilities*"

Event: Electron starts at "s", ends up at the position "x" on the screen (regardless whether or not the scattered photon ended up at D_1 or D_2**)** (16)

What is the amplitude probability for this event to happen?

The event described in (16) appears to be composed of the two events treated in the section above. Accordingly, should we add the amplitude probabilities given in (14) and (15)?

Paraphrasing Feynman, the answer is NO!

> *You must never add amplitudes corresponding to events that have distinct final states*.

Indeed, compare (14) and (15) and notice that they refer to distinct final states. In one case the photon went to D_1; in the other case the photon went to D_2.

$$\left\langle \underbrace{\begin{matrix} electron\ at\ x \\ photon\ in\ D_1 \end{matrix}} \middle| \begin{matrix} electron\ from\ s \\ photon\ from\ L \end{matrix} \right\rangle$$

$$\left\langle \underbrace{\begin{matrix} electron\ at\ x \\ photon\ in\ D_2 \end{matrix}} \middle| \begin{matrix} electron\ from\ s \\ photon\ from\ L \end{matrix} \right\rangle$$

*These two **final states**.are different*

We may say,

> "*why not just ignore the information available from the photon counters. That way we have a situation similar to the one when we did not have the light source nor the photon counter*."

But, looking or not to the photon counters *"is your business … Nature does not know what you are looking at, and behaves the way she is going to behave whether or not you bother to take a look to the photon counters*." …

> *"Once a photon is accepted by one of the photon counters, we can always determine which alternative occurred if we want, **without causing any further disturbance to the system.**"*

(This latter expression from Feynman is key!!!) That is, we can find out which of the two cases occurred without disturbing the system.

Accordingly, in the two-slit experiment with a light source present in the set up

> we shouldn't be asking about the "*amplitude-probability"* that an electron arrives at x regardless of which photon counter the photon went;

instead

> we should ask abut *the "probability"* for such an event to occur.

Calculation of "*probability*" instead of "*amplitude probability"* applies in the case indicated in (16) because it involves a collection (in this case two) of different final states.	(17)

Accordingly, the probability that an electron leaving from s arrives at x (while being watched by photons) is given by :

$$P(x;s)_{when\ a\ ligh\ source\ is\ present} =$$

$$= \left| \left\langle \begin{matrix} electron\ at\ x \\ photon\ in\ D_1 \end{matrix} \middle| \begin{matrix} electron\ from\ s \\ photon\ from\ L \end{matrix} \right\rangle \right|^2 + \left| \left\langle \begin{matrix} electron\ at\ x \\ photon\ in\ D_2 \end{matrix} \middle| \begin{matrix} electron\ from\ s \\ photon\ from\ L \end{matrix} \right\rangle \right|^2$$

$$= \left| \langle x|1\rangle a \langle 1|s\rangle + \langle x|2\rangle b \langle 2|s\rangle \right|^2 + \left| \langle x|1\rangle b \langle 1|s\rangle + \langle x|2\rangle a \langle 2|s\rangle \right|^2 \qquad (18)$$

For comparison purposes, let's re-write expression (13), which gives the probability that an electron leaving from s arrives at x (without being perturbed by photons), in the following way,

$$P(x;s)_{no\ light\ source\ present} =$$

$$= \left| \langle x|s\rangle_{no\ light\ source\ present} \right|^2 = \left| \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle \right|^2 \qquad (19)$$

Thus,

$$\left| P(x;s)_{no\ light\ source\ present} \right|^2 \neq P(x;s)_{light\ source\ present}$$

Analyzing particular cases

Case: Aperture-2 is closed

$$P(\ x;s) = \left| \langle x|1\rangle a \langle 1|s\rangle \right|^2 + \left| \langle x|1\rangle b \langle 1|s\rangle \right|^2$$

$$= (|a|^2 + |b|^2) \left| \langle x|1\rangle\langle 1|s\rangle \right|^2$$

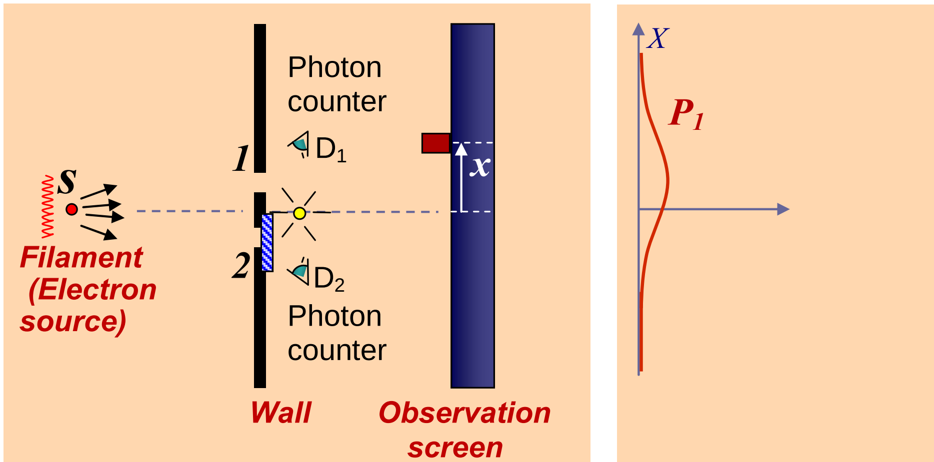


Fig. 6.3

Case: Aperture-1 is closed

$$P(\ x;s) = \left|\ \langle x|2\rangle b\langle 2|s\rangle\ \right|^2 + \left|\ \langle x|2\rangle a\langle 2|s\rangle\ \right|^2$$

$$= (\ |a|^2 + |b|^2\ )\ \left|\ \langle x|2\rangle\langle 2|s\rangle\ \right|^2$$

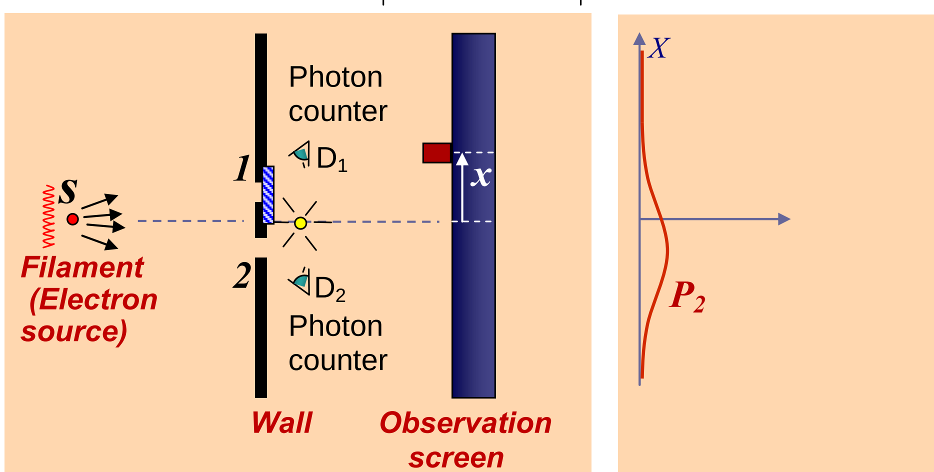


Fig. 6.4

Case: Photon of very long wavelength (i.e. p=h/λ very small) as to minimize the perturbation of the electron's motion.

For photons of long wavelength, the lateral resolution becomes so poor that a photon arriving at D_1, for example, could equally come from either an electron passing through aperture 1 or aperture-2.

Alternatively we could say that the photons are of so low energy that they could easily be scattered in any direction by a electron; that is, a photon could end up with equal probability in the photon detector-1 or the photon detector-2.

Better said, the coefficients a and b should have approximately the same magnitude.

If we take $a = b$ (that is, amplitude probability that a photon is scattered into the photon counter D_1 is the same as the amplitude probability that a photon is scattered into the photon counter D_2), expression (18) becomes,

$$
\begin{aligned}
P(\ x; s) = |a|^2 \ \big|\ \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle \ \big|^2 &+ \\
+ |a|^2 \ \big|\ \langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle \ \big|^2 & \\
= 2|a|^2 \ \big|\ \underbrace{\langle x|1\rangle\langle 1|s\rangle + \langle x|2\rangle\langle 2|s\rangle}_{\text{Interference term}} \ \big|^2 & \qquad (20)
\end{aligned}
$$

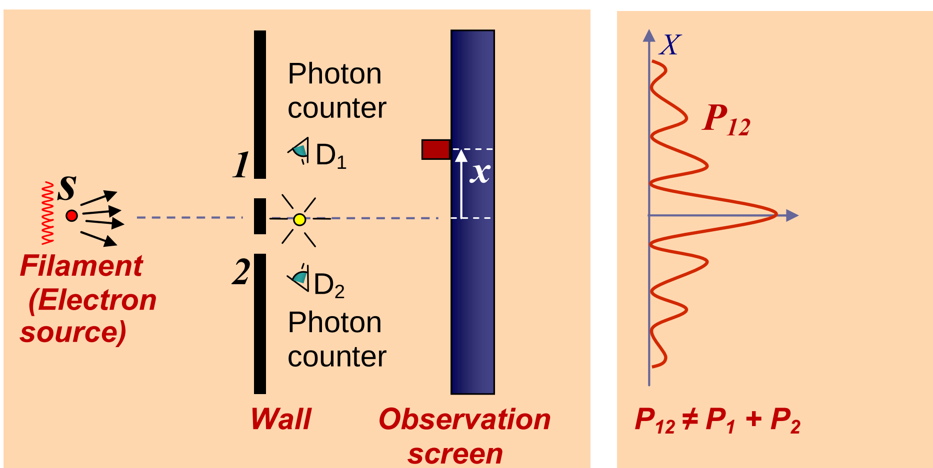


Fig. 6.5

Thus, when we do not know which aperture an electron passes through we get a wavy pattern of intensity on the screen.

Case: Using photons of small wavelength, as to improve the lateral resolution and thus identify which aperture an electron passed through.

One can argue that the energy of a photon is so high that a photon will be scattered more towards the forward direction; thus it is more likely that it will end up in the closer photon counter than in the farther away. In such a case, we expect $|a| >> |b|$.

The phenomena will be then influenced mainly by a and minimally by b. Taking b=0 in (18) we obtain,

$$P(\ x;s) = \left|\ \langle x|1\rangle a\langle 1|s\rangle\ \right|^2 + \left|\ \langle x|2\rangle a\langle 2|s\rangle\ \right|^2$$
$$= |a|^2 \left(\left|\ \langle x|1\rangle\langle 1|s\rangle\ \right|^2 + \left|\ \langle x|2\rangle\langle 2|s\rangle\ \right|^2 \right) \qquad (21)$$

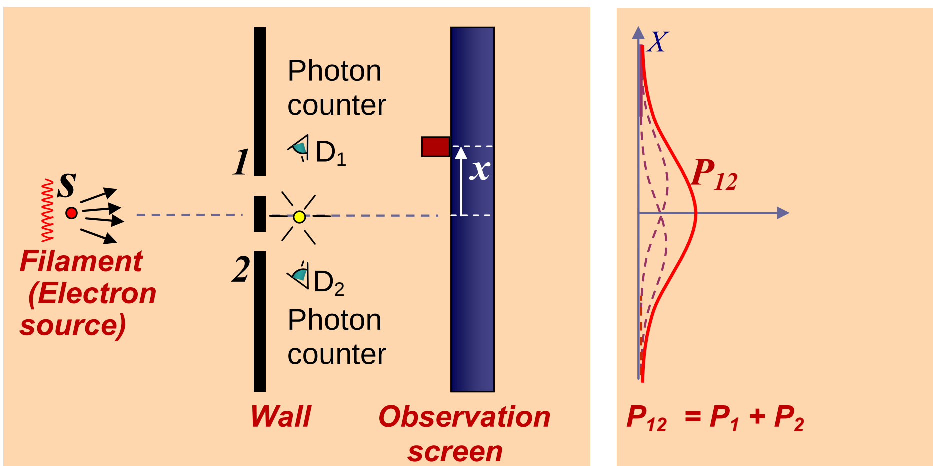


Fig. 6.6

We do know which aperture an electron passed through, and we get a pattern of intensity on the screen that is simply the addition of the probabilities P_1 and P_2 (no interference occurs.)

6.4.B Scattering from a crystal by particles with or without spin

6.4.B.a Angular Momentum and Magnetic Dipole

Orbital angular momentum

In a classical hydrogenic atom model:

- An electron occupies a circular orbit, rotating with an orbital **angular momentum** $\vec{L} = m_e \vec{r} x\ \vec{\mathrm{v}}$

- Since a moving charge gives rise to an electric current I, an electron moving in a close orbit of radius r forms an equivalent **magnetic dipole** $\vec{\mu}$ of magnitude $\mu = (I)(\pi r^2)$

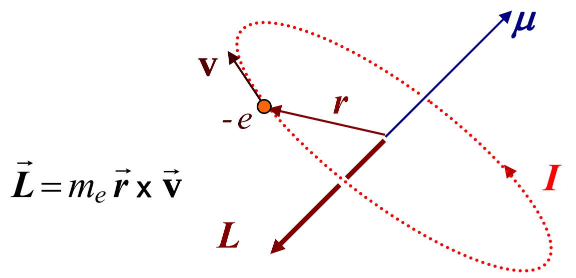


Fig. 6.7 *Orbital motion of an electron with associated angular momentum and magnetic dipole.*

Let's quantify these two quantities.

Magnetic dipole:

$$\mu = (current)(area) = I\pi\, r^2$$

$$= \frac{e}{orbit\ period}\pi\, r^2$$

Orbital angular momentum:

$$\mathrm{L} = m_e \mathrm{r\, v}$$

$$= m_e \mathrm{r\, v} = m_e \mathrm{r} \frac{\text{orbit length}}{\text{orbit period}}$$

$$= m_e \mathrm{r\, v} = m_e \mathrm{r} \frac{2\pi \mathrm{r}}{\text{orbit period}}$$

$$= m_e 2 \frac{\pi \mathrm{r}^2}{\text{orbit period}}$$

From these expressions the following relationship is obtained,

$$\boxed{\vec{\mu} = -\frac{e}{2m_e}\vec{L}} \qquad (22)$$

6.4.B.b The Stern-Gerlach experiment: Measurement of the magnetic dipole $\vec{\mu}$.

Interaction with a Magnetic Field

A magnetic dipole $\vec{\mu}$ inside a magnetic field $\vec{B}$ experiences a force, whose components are given by,

$$F_x = \vec{\mu} \cdot \frac{\partial \vec{B}}{\partial x} \ , \quad F_y = \vec{\mu} \cdot \frac{\partial \vec{B}}{\partial y} \ , \quad F_z = \vec{\mu} \cdot \frac{\partial \vec{B}}{\partial z} \qquad (23)$$

In 1921, Stern suggested that magnetic moments $\vec{\mu}$ of atoms could be measured by detecting the deflection of an atomic beam by an inhomogeneous magnetic field $\vec{B}$.

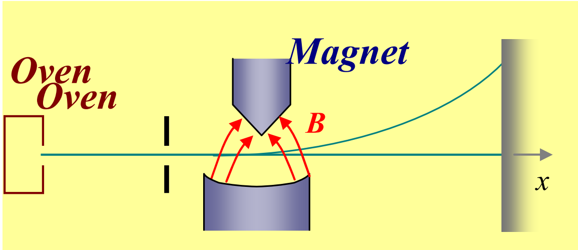
The experiment was made using atoms of silver

- A sample of silver metal is vaporized in an oven.
- A fraction of atoms emerging from a small hole is collimated by the apparatus slits, which enter the magnetic field region.
- The poles are designed in order to provide a more pronounced variation of the magnetic field in the z-direction.

  (Notice in the schematic diagram above, that the change of $\vec{B}$ along the x and y directions is much less, if not null, than in the z-direction). Thus a net force acting on the dipole is given by,

$$F_z = \vec{\mu} \cdot \frac{\partial \vec{B}}{\partial z} \approx \mu_z \frac{\partial B_z}{\partial z} \qquad (24)$$

- The atoms in the incident beam are deflected according to the value of their magnetic moment, and detected upon on a collecting plate.



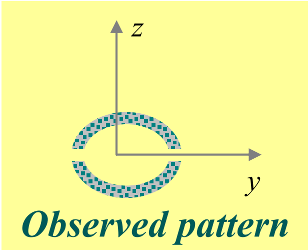


Fig. 6.8 Left: *Schematic apparatus to measure the magnetic moment of atoms.* ***Right:*** *Spatial distribution of the detected atoms.*

Classical prediction

Since the orientation of the silver atoms in the incident beam is random, the corresponding values of $\boldsymbol{\mu}_z$ should be spread over a continuum range

$$-M < \boldsymbol{\mu}_z < M, \tag{25}$$

where M is the magnitude of $\vec{\boldsymbol{\mu}}$.

Consequently, the spots on the collecting plate should distribute over a continuum filled spot, centered around the non-deflection direction.

However, this is not what is observed, as seen in the right side schematic of Fig. 6.8.

Bohr's quantization of the angular momentum

A quantum interpretation could be explored according to the Bohr's model of a hydrogenic atom. Bohr had postulated that the orbital angular momentum L of the electron can take only discrete values given by,

$$L = n\hbar = n\frac{h}{2\pi} \tag{26}$$

with n being the quantum number, $n = 1,\ 2, 3, \ldots$

and h is the Planck's constant.

Since the angular momentum is, according to Bohr, given in units of $\hbar$, it is usual to express the magnetic dipole, given in expression (22) above, in terms of the Bohr's magneton $\boldsymbol{\mu}_B$

$$\vec{\boldsymbol{\mu}}_{orbital} = -\underbrace{\frac{e\hbar}{2m_e}}_{\mu_B = 9.27 \times 10^{-24} \frac{Joule}{Tesla}} \frac{\vec{\boldsymbol{L}}}{\hbar} \tag{27}$$

Bohr's magneton

Accordingly, a discrete variation of $\vec{\boldsymbol{L}}$ would imply a discrete variation of $\vec{\boldsymbol{\mu}}_{orbital}$.

Quantization of L_z

If we further postulate that the component L_z is also quantized ("space quantization"), then L_z would change within the range,

$$-L_{\max} < L_z < L_{\max}$$

More specifically,

for a given orbit of angular momentum $L = \ell \hbar$,
the possible values of L_z would be

$$L_z = m\hbar, \tag{28}$$

with m taking values within the range,

$-\ell$, $-\ell+1$, ... , $\ell-1$, ℓ

Thus, for a given ℓ the multiplicity for the possible values of L_z is $2\ell+1$

For $\ell = 0$ the multiplicity is $2\ell+1=1$

$\ell = 1$ $\quad 2\ell+1=3$

$\ell = 2$ $\quad 2\ell+1=5$

However the results from the Stern and Gerlach experiment does not fit this scheme; the experimental result displayed in Fig 6.8 above indicates a multiplicity equal to 2, which requires $\ell = 1/2$.

Electron Spin, $\vec{\mathbf{S}}_{spin}$

In 1925, S. Gousmit and G. E. Uhlenbeck suggested that, in addition to the magnetic moment produced by the orbital motion, $\vec{\boldsymbol{\mu}}_{orbital}$,

*Electrons could also posses an **intrinsic magnetic moment** $\vec{\boldsymbol{\mu}}_{spin}$, where its components in a given direction can take only two values $\pm\ \mu_{spin}$.*

(Hence the split of the beam in two, after passing through the magnetic filed,= as shown in Fig 6.8.)

Associated with $\vec{\boldsymbol{\mu}}_{spin}$ one also postulates an **intrinsic angular momentum**, or **spin** of the electron, $\vec{\boldsymbol{S}}_{spin}$.

IF, in analogy to the orbital angular momentum l, we introduce a *spin quantum number* s so that the multiplicity of $(\boldsymbol{S}_z)_{spin}$ along the z direction be equal to $2s+1$, THEN for an electron we must have $s=1/2$

The postulates mentioned above are summarized in the following expression (by analogy with expression (27)),

$$\vec{\mu}_{spin} = -g_s \frac{e\hbar}{2m_e} \frac{\vec{\mathbf{S}}_{spin}}{\hbar} \tag{29}$$

with $S_z = -\frac{1}{2}\hbar$ *or* $\frac{1}{2}\hbar$ (*for the case of one electron*)

[If we had used an expression similar to (27) by simply replacing $\vec{\mu}_{orbital}$ and $\vec{\mathbf{L}}_{orbital}$ *with* $\vec{\mu}_{spin}$ and $\vec{\mathbf{S}}_{spin}$ respectively, the resultant relationship $\vec{\mu}_{spin} = -\frac{e\hbar}{2m} \frac{\vec{\mathbf{S}}_{spin}}{\hbar}$ would predict an intrinsic magnetic moment too small by a factor of 2, compared to what is measured experimentally. Hence the need to ad-hoc introduce the factor g_S. The origin of g_S is in fact a relativistic quantum effect.]

In the expression above,

$$g_S = 2.00232 \tag{30}$$

and it is called the "*spin gyromagnetic ratio.*"

Using expressions (27) and (29), the total magnetic moment for an electron in an atom is give by,

$$\vec{\mu}_{total} = -\frac{e\hbar}{2m_e} \frac{\vec{L}_{orbital} + g_S \vec{S}_{spin}}{\hbar} \tag{31}$$

$$\mu_B \equiv \frac{e\hbar}{2m_e} = 9.27 \times 10^{-24} \frac{Joule}{Tesla} \qquad Bohr's\ magneton$$

Nuclear spin $\vec{\mathbf{I}}_{spin}$

Similar to expression (31), the nuclear magnetic moment $\vec{\mu}_{nuclear}$ and the nuclear spin $\vec{\mathbf{I}}_{spin}$ of a given nucleon are related by,[2]

$$\vec{\mu}_{nuclear} = g \underbrace{\frac{e\hbar}{2m_{nucleon}}}_{\mu_N \text{ nuclear magneton}} \frac{\vec{I}_{spin}}{\hbar} \tag{32}$$

$$\mu_N \equiv 5.05084\text{x}10^{-27} \text{ eV/T}$$

For a free proton: $I_{proton} = \hbar/2$, $g_{proton} = 5.58569$, $\mu_{proton} = 2.79284\mu_N$

For a free neutron: $I_{neutronn} = \hbar/2$, $g_{neutron} = -3.82608$,

$$\mu_{neutron} = -1.9130418\mu_N$$

Even the uncharged neutron has a large magnetic moment. This suggests that there is internal structure involving the motion of charged particles.

6.4.B.b Scattering form a crystal of atoms of *spin zero*

Consider,

a) A crystal: a periodic array of atoms with nuclei at their centers, The atoms are labeled by the index i =1, 2, 3, … N.

b) A beam of neutrons incident on the crystal.

The objective is to calculate the probability that a neutron gets into the counter at a given angular location, as shown in the figure below.

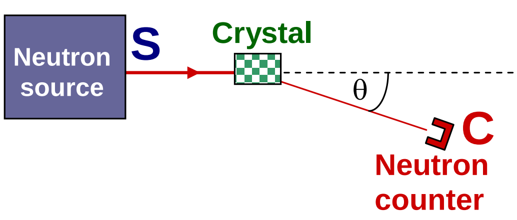


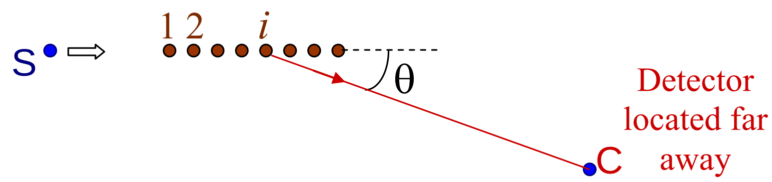
Fig. 6.9 *Schematic experimental arrangement for the diffraction of protons by a crystal.*

Let's assume the incident neutrons have a relatively low energy so that they do not cause any major disturbance or modification on the crystal (i.e. they do not leave track of their interaction.)

Scattering via atom i :

The amplitude probability that a neutron arrives at C scattered by the atom i in the crystal can be represented by,

$$< neutron\ at\ C\,|\,neutron\ from\ S >_{via\ i} = \langle C|\,i\rangle\langle\,i|\,S\rangle$$



When a neutron arrives at the counter, it could have come scattered from any atom in the crystal, and there is no way to know which atom produced the scattering. We have therefore different routes with the same initial and final state, which will therefore interfere as to give,

$$\langle neutron\ at\ C\,\big|\,neutron\ from\ S\rangle = \sum_{i}^{N}\langle C|\,i\,\rangle\langle\,i\,|S\rangle \qquad (33)$$

Since we are adding amplitudes of scattering associated to atoms at different space locations, the amplitudes will have different phases and will produce an interference pattern similar to the diffraction grating studied in chapter 5.

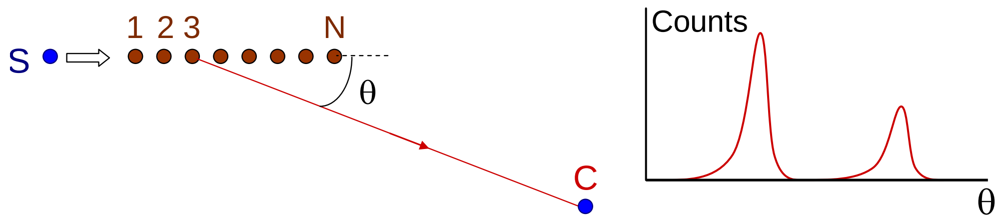


Fig. 6.10 *The neutron intensity as a function of the angle θ results from constructive and destructive interference of the different amplitudes associated to the different possible paths that join the same initial state (S) and the same final state (C).*

The intensity pattern shown in Fig. 6.10 above, however, is different when using other crystals made out of atoms that do have spin. The spin character of the atoms will be reflected in the different intensity pattern response as explain below.

6.4.B.c Scattering form a crystal of atoms with *spin 1/2*

So far, we have not considered the fact that,

a neutron has spin ½ (34)

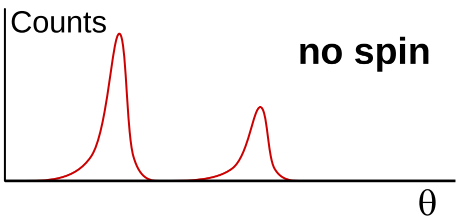
(as discussed in Section 6.4.B.a above.)

This gives two spatial directions to consider for the nuclear dipole moment:

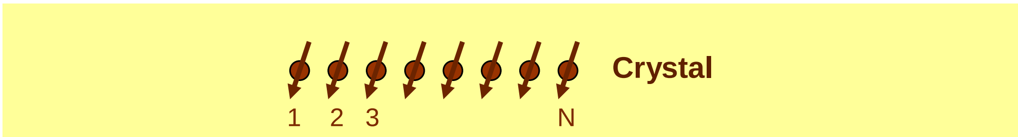
up or **down**

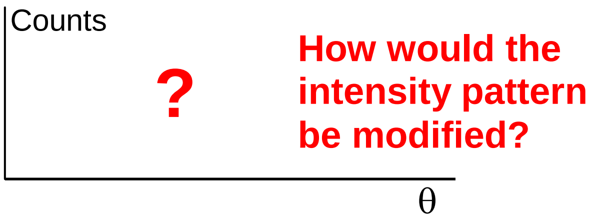
(along, for example, the direction of the axis perpendicular to the page in figure 6.9.)

- CASE-1. If the nuclei of the crystal had **no spin**, the spin of the incident neutrons would have no effect in the experiment. Thus an intensity pattern similar to the one shown in Fig 6.10 would be obtained.



- CASE-2. But, if the nuclei in the crystal had, for example, spin ½ a different intensity pattern is obtained due to the different possible interchange of spin between the nuclei and the incident neutrons.



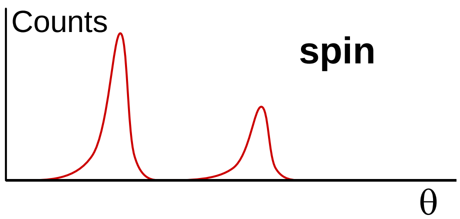


Let's consider CASE-2 in more detail.

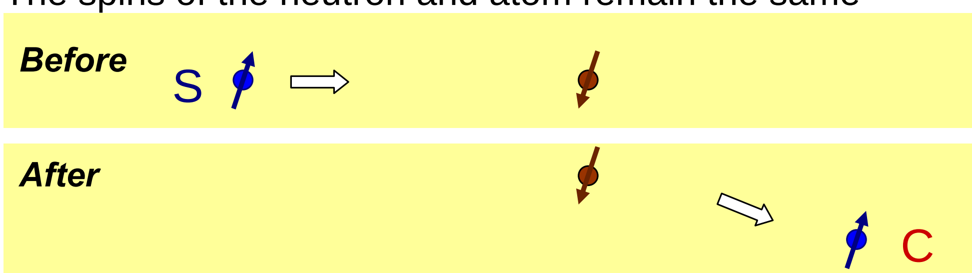
In this discussion we will apply a rule similar to the **conservation of angular momentum**:

- **2A.** If the neutron and the atomic nucleus have the same spin, then the scattering will produce no change in the spin of either particle.

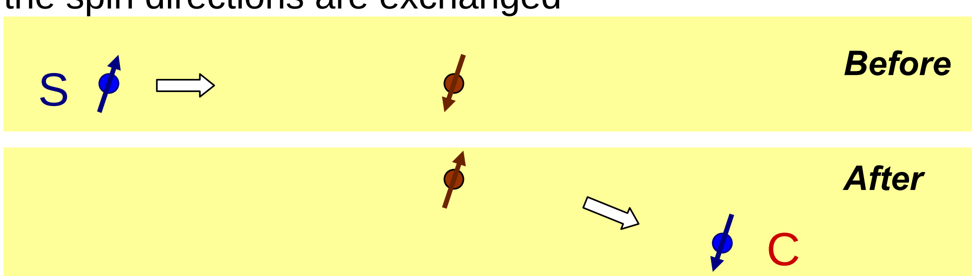
This case would be similar then to the one discussed in section 6.4.B.b above (results displayed in Fig. 6.10.)



- **2B** If the neutron and the atomic nucleus have opposite spin, then scattering can occur by two processes:
  - The spins of the neutron and atom remain the same



  - the spin directions are exchanged



We'll consider these two possible cases separately.

Incident neutron and scatterer nuclei having opposite spin

Let's define,

a = amplitude probability that there is no interchange of spin due to the scattering

b = amplitude probability that there is exchange of spin (35)

To start gaining some understanding of the problem, we will assume that **all the nuclei are set up with spin in one direction**, pointing down for example.

When the incident neutron has spin up, two different scenarios can occur:

CASE 2B.1: No interchange of spin

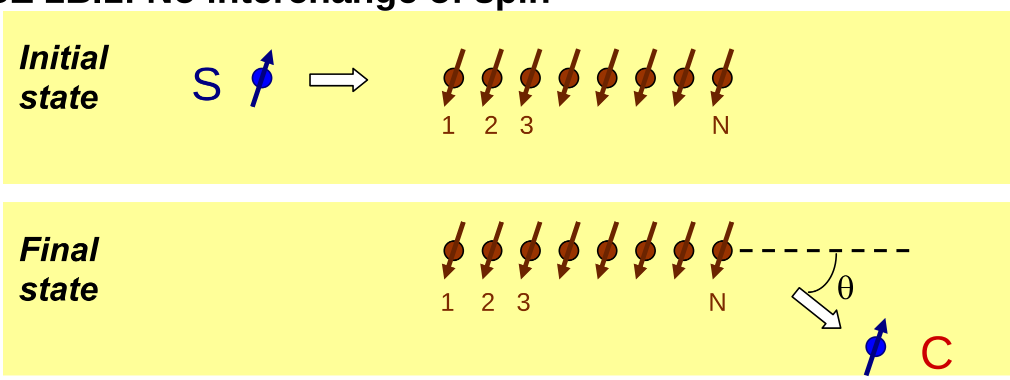


Fig. 6.11

Amplitude probability that the neutron is scattered toward C via the nucleus-*i*:

$$\left\langle C_{up;} all\ nuclei\ \mathrm{down} \middle| S_{up}; all\ nuclei\ \mathrm{down} \right\rangle_{via\ nucleus\ i} = \langle C | i \rangle \, a \langle i | s \rangle \tag{35}$$

Amplitude probability that the neutron is scattered toward C via the nucleus-k:

$$\left\langle C_{up;} all\ nuclei\ \mathrm{down} \middle| S_{up}; all\ nuclei\ \mathrm{down} \right\rangle_{via\ nucleus\ k} = \langle C | k \rangle \, a \langle k | s \rangle \tag{36}$$

Notice, the two path-alternatives alluded in (35) and (36) leave the crystal intact; also, both paths, k and i, have the same initial and final state. In fact, the various possible path-alternatives (scattering from any of the N nuclei) produce the same initial and final state.

Accordingly:

Amplitude probability that the neutron is scattered toward C,

$$\left\langle C_{up}; \textit{all nuclei down} \,\middle|\, S_{up}; \textit{all nuclei down} \right\rangle = \sum_{i=1}^{N} \langle C|i\rangle\, a\langle i|S\rangle \qquad (37)$$

Probability that the neutron is scattered toward C,

$$Prob\left(C_{up}; \textit{all nuclei down} \,\middle|\, S_{up}; \textit{all nuclei down}\right) =$$

$$= \left| \sum_{i=1}^{N} \langle C|i\rangle a\langle i|S\rangle \right|^2 \qquad (38)$$

The interference among the different paths is reflected in the peaks of interference shown in Fig. 6.12.

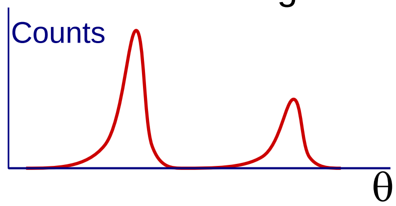


Fig. 6.12

CASE-2B.2 (interchange of spin)

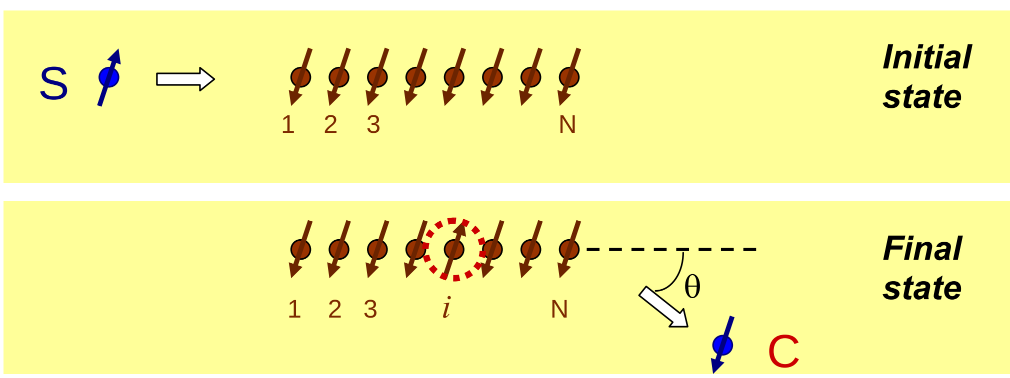


Fig. 6.13

Amplitude probability that the neutron is scattered towards C via the nucleus-i:

$$\left\langle C_{down}; \begin{matrix} \textit{nucleus } i \text{ up} \\ \text{all the other nucleus down} \end{matrix} \middle| S_{up}; \textit{all nuclei down} \right\rangle_{\textit{via nucleus } i} =$$

$$= \langle C|i\rangle\, b\langle i|S\rangle \qquad (39)$$

The difference between (35) and (39) is that in the latter case, the interaction has left a signature indicating which nucleus participated in the interaction. In other words, it can be distinguished from the event in which the neutron is scattered by, for example, the nucleus k. (We can go and verify which nucleus made the flip without affecting the information detected by the counter.)

Amplitude probability that the neutron is scattered towards C via the nucleus-k:

$$\left\langle C_{down};\, \begin{matrix} \textit{nucleus } k \text{ up} \\ \text{all the other nucleus down} \end{matrix} \middle| S_{up};\, \textit{all nuclei } \text{down} \right\rangle_{\textit{via nucleus } k} =$$

$$= \langle C | k \rangle \; \mathrm{b} \langle k | \mathrm{S} \rangle \quad (40)$$

Each case, (39) and (40), leaves the crystal in a different configuration. This means that the final states represented in (39) and (40) are different. Since paths with different final states do not interfere, we have to calculate first the individual probabilities $|\langle C|i\rangle \, \mathrm{b}\langle i|\mathrm{S}\rangle|^2$ for $i = 1, 2, 3, \ldots$, N; and then add them up

Probability that the neutron is scattered towards C:

$$Prob(C_{down};\textit{one of the nuclei } \text{up} \mid S_{up};\textit{all nuclei } \text{down}) =$$

$$= \sum_{i=1}^{N} \left| \langle C|i\rangle \mathrm{b} \langle i|\mathrm{S}\rangle \right|^2 \qquad (41)$$

Since all the phases are gone, no peaks of constructive interference will be displayed; rather a smooth distribution will be detected by the counter (as shown in Fig. 6.14.)

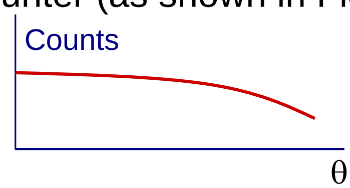


Fig. 6.14

When cases 1 and 2 combine, the result will be a diagram like the one depicted in figure 6.15.

Fig. 6.15

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[1] Richard Feynman, "The Feynman Lecture on Physics," Vol-III, Addison-Wesley, 1963; Chapter 3.

[2] http://hyperphysics.phy-astr.gsu.edu/hbase/nuclear/nspin.html