# Anonymous Card Shuffling and its Applications to Parallel Mixnets

Michael T. Goodrich
Dept. of Computer Science
University of California, Irvine
goodrich@ics.uci.edu

Michael Mitzenmacher
Dept. of Computer Science
Harvard University
michaelm@eecs.harvard.edu

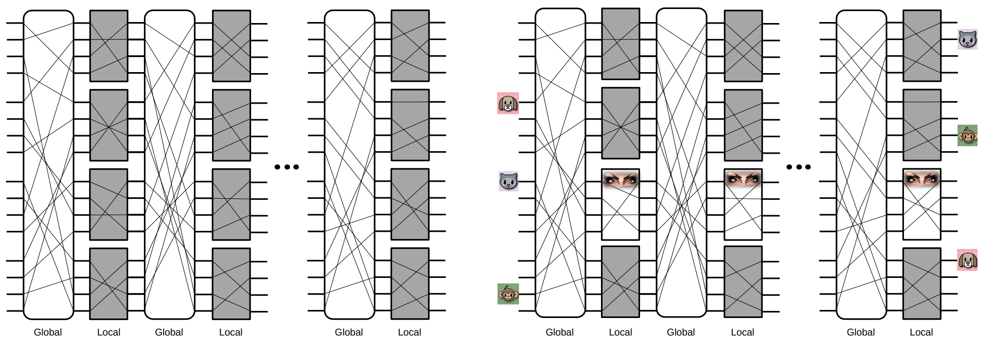
**Abstract**

We study the question of how to shuffle $n$ cards when faced with an opponent who knows the initial position of all the cards *and* can track every card when permuted, *except* when one takes $K < n$ cards at a time and shuffles them in a private buffer "behind your back," which we call *buffer shuffling*. The problem arises naturally in the context of parallel mixnet servers as well as other security applications. Our analysis is based on related analyses of load-balancing processes. We include extensions to variations that involve corrupted servers and adversarially injected messages, which correspond to an opponent who can peek at some shuffles in the buffer and who can mark some number of the cards. In addition, our analysis makes novel use of a sum-of-squares metric for anonymity, which leads to improved performance bounds for parallel mixnets and can also be used to bound well-known existing anonymity measures.

## 1 Introduction

Suppose an honest player, Alice, is playing cards with a card shark, Bob, who has a photographic memory and perfect vision. Not trusting Bob to shuffle, Alice insists on shuffling the deck for each hand they play. Unfortunately, Bob will only agree to this condition if he gets to scan through the deck of $n$ cards before she shuffles, so that he sees each card and its position in the deck, and if he also gets to watch her shuffle. It isn't hard to realize that, even though several well-known card shuffling algorithms, like random riffle shuffling [1], top-to-random shuffling [5], and Fisher-Yates shuffling [11], are great at placing cards in random order, they are terrible at obscuring that order from someone like Bob who has memorized the initial ordering of the cards and is watching Alice's every move. Thus, these algorithms on their own are of little use to Alice. What she needs is a way to shuffle that can place cards in random order in a way that hides that order from Bob. We refer to this as the *anonymous shuffling* problem. Our goal in this paper is to show that, as long as Alice has a private buffer where she can shuffle a subset of the cards, she can solve the anonymous shuffling problem.

Our main motivation for studying the anonymous shuffling problem in this paper comes from the problem of designing efficient parallel mixnets. A *parallel mix network* (or *mixnet*) is a distributed mechanism for connecting a set of $n$ inputs with a set of $n$ outputs in a way that hides the way the inputs and outputs are connected. This connection hiding is achieved by routing the $n$ inputs as messages through a set of $M$ *mix servers* in a series of synchronized rounds. In each round, the $n$ inputs are randomly assigned to servers so that each server is assigned $K = n/M$ messages. Then, each server randomly permutes the messages it receives and performs an encryption operation so that it is computationally infeasible for an eavesdropper watching the inputs and outputs of any (honest) server to determine which inputs are matched to the outputs. The mixnet repeats this process for a specific number of rounds. The goal of the adversary in this scenario is to determine (that is, link) one or more of the input messages with their corresponding outputs, while the




(a) (b)

Figure 1: (a) A parallel mixnet with $n = 16$ inputs and $M = 4$ mix servers. Shaded boxes illustrate mix servers, whose internal permutations are hidden from the adversary. The adversary is allowed to see the global permutation performed in each round. (b) A corrupted parallel mixnet, where $s = 3$ servers are not colluding with the adversary, who has injected $f = 3$ fake messages into the network.

mixnet shuffles so as to reduce the linkability between inputs and outputs to an acceptably small level. (See Figure 1a.)

Each of the servers is assumed to run the mixnet protocol correctly, which is enforced using cryptographic primitives and public sources of randomness (e.g., see [3, 7, 10, 14, 15]). In some cases, we also allow for a *corrupted* parallel mixnet, where some number $s \geq 1$ of the servers behave properly, but the remaining $M - s$ servers collude with the adversary so as to reveal how they are internally permuting the messages they receive. In addition, the adversary may also be allowed to inject some number, $f < n$, of fake messages that are marked in a way that allows the adversary to determine their placement at any point in the process, including in the final output ordering. (See Figure 1b.) In this paper, we are interested in studying a class of algorithms for anonymous shuffling, to show how the analysis of these algorithms can lead to improved protocols for uncorrupted and corrupted parallel mixnets.

## 1.1 Previous Related Work

Arkin *et al.* [2] describe an attack on online poker based on exploiting a poor shuffling algorithm, among other security weaknesses.

Chaum [4] introduced the concept of mix networks for achieving anonymity in messaging, and this work has led to a host of other papers on the topic (e.g., see [14, 15]).

Golle and Juels [7] study the parallel mixing problem, where mix servers process messages synchronously in parallel rounds, and discuss the cryptographic primitives sufficient to support parallel mixnet functionality. Their scheme has a total mixing time of $2n(M - s + 1)/M$ and a number of parallel mixing rounds that is $2(M - s + 1)$, assuming that $M^2$ divides $n$. It achieves a degree of anonymity "close" to $n - f$, using a specialized anonymity measure, $\mathsf{Anon}_t$, that they define (which we discuss in more detail in Section 2). Note that if $s$ is, say, $M/2$, then their protocol requires as many rounds as the number of servers, which diminishes the potential benefits of a *parallel* mixnet. In particular, their approach uses sequences of round-robin permutations (cyclic shifts) rather than the standard parallel mixing protocol described above and illustrated in Figure 1. Even then, Borisov [3] shows that their scheme can leak linkages between inputs and outputs (as can the standard parallel mixing protocol if the number of rounds is too small) if the vast majority of inputs are fake messages introduced by the adversary. Thus, it is reasonable to place realistic limits on how large $f$ can be, such as $f \leq n/2$, and require that the number of parallel mixing rounds is high enough to guarantee a high degree of anonymity. Klonowski and Kutylowski [10] also study the anonymity of parallel mixnets, characterizing it in terms of variation distance (which we discuss in the next section) for honest servers and for the case of a single corrupted server. They do not consider an adversary who can inject fake messages, however, and they only treat the case when $M^2$ is much less than $n$.

Goodrich, Mitzenmacher, Ohrimenko, and Tamassia [8] study a simple variant of the anonymous shuffling problem with no corrupted servers or fake cards, addressing a problem similar to parallel mixing in the context of oblivious storage. They show that when the number of cards per server each round is $K = n^{1/c}$, then $c + 1$ rounds are sufficient to hide any specific initial card, so that the adversary can guess its location with probability only $1/n + o(1/n)$. The current work provides a much more general and detailed result, using much more robust techniques.

Our techniques are based on work in dynamic load balancing by Ghosh and Muthukrishnan [6]. In their setting, tasks are balanced in a dynamic network by repeatedly choosing random matchings and balancing tasks across each edge. Here, we extend this work by choosing random subcollections of $K$ cards and balancing weights, corresponding to probabilites of a specific card being one of those $K$, among the $K$ cards via the shuffling.

### 1.2 Our Results

We study the problem of analyzing parallel mixnets in terms of a buffer-based solution to the anonymous shuffling problem, assuming, as with other works on parallel mixnets [3, 7, 8, 10], that cryptographic primitives exist to enforce re-encryption for each mix server, along with public sources of randomness and permutation verification so that servers must correctly follow the mixing protocol even if corrupted.

In the *buffer shuffling* algorithm [8, 10], Alice repeatedly performs a series of shuffling rounds, as in the parallel mixnet paradigm. That is, each round begins with Alice performing a random shuffle that places the cards in random order (albeit in a way that the adversary, Bob, can see). Then she splits the ordered cards into $M$ piles, with each pile getting $K = n/M$ cards. Finally, she randomly shuffles each pile, using a private buffer that Bob cannot see into. Once she has completed her private shuffles, she stacks up her piles, which become the working deck for the next round. She repeats these rounds until she is satisfied that the deck is sufficiently shuffled for the adversary. Note that during her shuffling, Bob can see cards go in and out of her buffer, but he cannot normally see cards while they are in the buffer. As we describe in more detail shortly, Alice's goal is to prevent Bob from being able to track a card; that is, Bob should only be able to guess the location of a card with probability $1/n + o(1/n)$, where generally we take the $o(1/n)$ term to be $O(1/n^b)$ for some $b \geq 1$.

To characterize the power of the adversary in the parallel mixnet framework, we consider buffer shuffling in a context where, for $M - s$ specific uses of Alice's buffer within each round, Bob is allowed to see how the cards are shuffled inside it. Likewise, we assume he is allowed to mark $f < n$ of the cards in a way that lets him determine their position in the deck at any time. We provide a novel analysis of this framework, and show how this analysis can be used to design improved methods for designing parallel mixnets. For instance, we show that buffer shuffling achieves our goal with $O(1)$ rounds even if the number of servers is relatively large and that buffer shuffling can be performed in $O(\log n)$ rounds even for high degrees of compromise. We summarize our results and how they compare with the previous related work in Table 1.

## 2 Anonymity Measures

We can model the anonymous shuffling problem in terms of probability distributions. Without loss of generality, we can assume that the initial ordering of cards is $[n] = (1, 2, \ldots, n)$. After Alice performs $t$ rounds of shuffling, let $w_i(t, c)$ denote the probability from the point of view of the adversary that the card in position $i$ at time $t$ is the card numbered $c$, and let $W(i, t)$ denote the distribution defined by these probabilities (we may drop the $i$ and $t$ if they are clear from the context). The ideal is for this probability to be $1/n$ for all $i$ and $c$, which corresponds to the uniform distribution, $U$.

| Solution | Corruption Tolerance | Server Restriction | Allows for Corrupt Servers | Allows for Fake Messages | Rounds |
|---|---|---|---|---|---|
| GJ [7] | high | $c = 2$ only | $\surd$ | $\surd$ | $O(M)$ |
| KK [10] | only one | $c = 2$ only | $\surd$ | – | $O(\log n)$ |
| GMOT [8] | none | const. $c$ | – | – | $O(1)$ |
| **Our Theorem 2** | medium | const. $c$ | $\surd$ | – | $O(1)$ |
| **Our Theorem 3** | high | const. $c$ | $\surd$ | $\surd$ | $O(\log n)$ |

Table 1: Summary of our results. We compare with the solutions of Golle and Juels [7], Klonowski and Kutylowski [10], and Goodrich *et al.* [8]. The server restriction column refers to the parameter $c$ in the inequality $K \geq n^{1/c}$. The bounds of "medium" and "high" for corruption tolerance are made more precise in the statement of the theorems.

A natural way to measure anonymity is to use a distance metric to determine how close the distribution $W$ is to $U$, for any particular card $i$ or in terms of a maximum taken over all the cards. The goal is for this metric to converge to 0 quickly as a function of $t$.

**Maximum difference.** The *maximum-difference* metric, which is also known as the $L_\infty$ metric, specialized to measure the distance between $W$ and $U$, is

$$\alpha(t) = \max_{i,c} |w_i(t,c) - 1/n| \,.$$

As mentioned above, the goal is to minimize $\alpha(t)$, getting close to 0 as quickly as possible.

Note that, in the case of buffer shuffling, the formula for $\alpha(t)$ can be simplified. In particular, since Alice starts each round with a random permutation, $w_i(t,c) = w_i(t,1)$. Thus, in our case, we can drop the $c$ and focus on $w_i(t)$, the probability that the $i$-th card is 1. In this case, we can simplify the definition as

$$\alpha(t) = \max_{i} |w_i(t) - 1/n|.$$

**The** $\mathsf{Anon}$ **measure of Golle and Juels.** In the context of parallel mixing, Golle and Juels [7] define a measure for anonymity, which, using the above notation, would be defined as follows:

$$\mathsf{Anon}_t = \min_i \left( \max_c \; w_i(t,c) \right)^{-1},$$

which they try to maximize. Note that $\max_c \; w_i(t,c) \geq 1/n$ for all $i$, so, to be consistent with the goals for other anonymity measures, which are all based on minimizations, we can use the following $\mathsf{Anon}'_t$ definition for an anonymity measure equivalent to that of Golle and Juels:

$$\mathsf{Anon}'_t = \max_i \left( \max_c \; w_i(t,c) \right) = \max_{i,c} w_i(t,c) = (\mathsf{Anon}_t)^{-1}.$$

The $\mathsf{Anon}'_t$ measure is not an actual distance metric, with respect to $U$, however, since its smallest value is $1/n$, not 0. In addition, it is biased towards the knowledge gained by the adversary for positive identifications and can downplay knowledge gained by ruling out possibilities. To see this, note that, if we let $W^+$ denote all the $w_i(t,c)$'s that are at least $1/n$ and $W^-$ denote all the $w_i(t,c)$ values less than $1/n$, then

$$\begin{aligned}\alpha(t) &= \max\{\max_{w_i(t,c)\in W^+}\{w_i(t,c)-1/n\}\,,\ \max_{w_i(t,c)\in W^-}\{1/n-w_i(t,c)\}\}\\ &= \max\{\mathsf{Anon}'_t - 1/n\,,\ \max_{w_i(t,c)\in W^-}\{1/n-w_i(t,c)\}\}.\end{aligned}$$

For example, suppose Alice shuffles the cards in a way that guarantees that the first card is not the Ace of Spades but is otherwise as close to uniform as possible. Then, $\alpha(t)x = 1/n$, since $w_1(t,1) = 0$, and $\mathsf{Anon}'_t = 1/(n-1)$ in this case. Alternatively, suppose Alice's shuffle is uniform except that it results in $w_1(t,1) = 1/(n-1)$ and $w_1(t,c) = 1/(n-1) - 1/(n-1)^2$, for $c \neq 1$. In this case, $\alpha(t) = 1/(n(n-1))$ while $\mathsf{Anon}'_t = 1/(n-1)$, as in the other example. The second example is much closer to uniform than the first and doesn't allow Bob to rule out any specific card as being the first card, but the $\mathsf{Anon}'_t$ measure (and, hence, the $\mathsf{Anon}_t$ measure) is the same in both cases. Therefore, we prefer to use anonymity measures that are based on metrics and are unbiased measures of the distance from $W$ to the uniform distribution, $U$.

**Variation Distance.** Li *et al.* [12] introduce a notion of anonymity called *threshold closeness* or *$t$-closeness*. For categorical data, as in card shuffling and mixnets, this metric amounts to the *variation distance* between the $W$-distribution defined by Alice's shuffling method and the (desired) uniform distribution, $U$, where each card occurs with probability $1/n$ (see also [10]). In particular, this metric would be defined as follows for buffer shuffling:

$$\beta(t) = \frac{1}{2}\sum_{i=1}^{n} |w_i(t) - 1/n|,$$

which is the same as half the $L_1$ distance between the $W$-distribution and the uniform distribution, $U$. As with other distance metrics, the goal is to minimize $\beta(t)$.

**The sum-of-squares metric.** In terms of measuring anonymity, an ideal metric is one that is sensitive to outliers while still being easy to work with. The reason that outlier sensitivity is useful is that focuses our attention on the areas where the adversary, Bob, can gain the most advantage. The above anonymity measures, related to the $L_1$ and $L_\infty$ metrics, have some sensitivity to outliers, but we would like to use a metric that is more sensitive than these.

For this paper, we have chosen to focus on a metric for anonymity that is derived from a simple measure that is well-known for its sensitivity to outliers (which are undesirable in the context of anonymity). In this, the *sum-of-squares* metric, we take the sum of the squared differences between the given distribution and our desired ideal. In the context of buffer shuffling, this would be defined as follows:

$$\Phi(t) = \sum_{i=1}^{n} (w_i(t) - 1/n)^2,$$

which can be further simplified as follows:

$$\begin{aligned}\Phi(t) &= \sum_{i=1}^{n} (w_i^2(t) - 2w_i(t)/n + 1/n^2)\\ &= \left(\sum_{i=1}^{n} w_i^2(t)\right) - 1/n.\end{aligned}$$

This amounts to the square of the $L_2$-distance between the $W$-distribution and the uniform distribution, $U$. The goal is to minimize $\Phi(t)$.

Incidentally, this simplified form of the sum-of-squares metric, $\Phi(t)$, for the buffer shuffle, doesn't take into account possible correlations between pairs of items, but we show in this paper how to extend the sum-of-squares metric to these contexts as well.

**Relationships between anonymity measures.** Another benefit of the $\Phi(t)$ metric is that it can be used to bound other metrics and measures for anonymity, by well-known relationships among the $L_p$ norms. For instance, we can derive upper bounds for other metrics (which we leave as exercises for the interested reader), such as

$$\alpha(t) \leq \Phi(t)^{1/2} \quad \text{and} \quad \beta(t) \leq (n\Phi(t))^{1/2}/2.$$

And we can also derive lower bounds for other metrics, such as

$$\Phi(t)^{1/2}/2 \leq \beta(t) \quad \text{and} \quad \Phi(t)^{1/2}/n^{1/2} \leq \alpha(t).$$

In addition, even though $\mathsf{Anon}'_t$ is not a metric, we can derive the following bound for it, since $\mathsf{Anon}'_t \leq \alpha(t) + 1/n$:

$$\mathsf{Anon}'_t \leq \Phi(t)^{1/2} + 1/n.$$

So, for the remainder of this paper, we focus primarily on the $\Phi(t)$ metric.

# 3 Algorithms and Analysis

Our parallel mixing algorithm repeats the following steps:

1. Shuffle the cards, placing them according to a uniform permutation.
2. Under this ordering, divide the cards up into consecutive groups of $K = n/M$ cards.[1]
3. For each group of $K$ cards, shuffle their cards randomly, hidden from the adversary.

We refer to each repetition of the above steps as a *round*. In the parallel mixnet setting, each group of $K$ cards would be shuffled at a different server.

Let $w_i(t)$ be the probability that the $i$th card after $t$ rounds is the first card from time 0 from the point of view of the adversary. (We drop the dependence on $t$ where the meaning is clear.) Initially, $w_1 = 1$, and $w_2 \ldots w_n$ are all 0. Motivated by [6], let $\Phi(t)$ be a potential function $\Phi(t) = (\sum w_i(t)^2) - \frac{1}{n}$, based on the sum-of-squares metric, and let $\Delta\Phi(t) = \Phi(t) - \Phi(t+1)$. (Again, we drop the explicit dependence on $t$ where suitable.)

Our first goal is to prove the following theorem.

**Theorem 1:** *A non-corrupted parallel mixnet, designed as described above, has $\mathbf{E}[\Phi(t)] \leq K^{-t}$. In particular, such a mixnet, with $K \geq n^{1/c}$, can mix messages in $t = bc$ rounds so that the expected sum-of-squares error, $\mathbf{E}[\Phi(t)]$, between card-assignment probabilities and the uniform distribution is at most $1/n^b$, for any fixed $b \geq 1$.*

Before proving this theorem, we note some implications. From Theorem 1 and Markov's inequality, using $t = 2bc$ rounds, we can bound the probability that $\Phi(t) > 1/n^b$ to be at most $1/n^b$, for any fixed $b \geq 1$. So, taking $b = 2$ implies $\alpha_t \leq 1/n$ with probability $1 - 1/n^2$, taking $b = 3$ implies $\beta_t \leq 1/n$ with probability $1 - 1/n^3$, and taking $b = 2$ implies $\mathsf{Anon}'_t < (n-1)^{-1}$, with probability $1 - 1/n^2$, which achieves the anonymity goal of Golle and Juels [7] (who only treat the case $c = 2$). Therefore, a constant number of

[1] As we also study, we could alternatively assign each card uniformly at random to one of the $M = n/K$ piles, with each group getting $K = n/M$ cards in expectation.

rounds suffices for anonymously shuffling the inputs in a parallel mixnet, provided servers can internally mix $K \geq n^{1/c}$ items, for some constant $c \geq 1$.

We now move to the proof. Let $\Delta\Phi^*$ represent how the potential changes when a group of $K$ cards is shuffled during a round. For clarity, we examine the cases of $K = 2$ and 3 before the general case.

- For 2 cards with incoming weights $w_i$ and $w_j$ (outgoing weights are the average):

$$\begin{aligned} \Delta\Phi^* &= w_i^2 + w_j^2 - 2((w_i + w_j)/2)^2 \\ &= (w_i - w_j)^2/2. \end{aligned}$$

- For 3 cards with incoming weights $w_i$, $w_j$, and $w_k$:

$$\begin{aligned} \Delta\Phi^* &= w_i^2 + w_j^2 + w_k^2 - 3((w_i + w_j + w_k)/3)^2 \\ &= (w_i - w_j)^2/3 + (w_j - w_k)^2/3 + (w_k - w_i)^2/3. \end{aligned}$$

- For $K$ cards with weights $w_{i1}, w_{i2}, \ldots, w_{iK}$:

$$\begin{aligned} \Delta\Phi^* &= \sum_{k=1}^{K} w_{ik}^2 - K\left(\frac{\sum_{k=1}^{K} w_{ik}}{K}\right)^2 \\ &= \frac{1}{K} \sum_{1 \leq j < k \leq K}^{K} (w_{ij} - w_{ik})^2 . \end{aligned}$$

We now proceed to bound $\mathbf{E}[\Phi(t)]$ by making use of $\Delta\Phi$.

$$\begin{aligned} \mathbf{E}[\Delta\Phi] &= \frac{1}{K} \sum_{1 \leq i < j \leq n} \Pr\left((i,j) \text{ are in the same set of } K \text{ cards}\right) (w_i - w_j)^2 \\ &= \frac{K-1}{K(n-1)} \sum_{i<j} (w_i - w_j)^2 \\ &= \frac{K-1}{2K(n-1)} \sum_{1 \leq i,j \leq n} (w_i - w_j)^2. \end{aligned}$$

Also,

$$\mathbf{E}[\Delta\Phi/\Phi] = \frac{K-1}{2K(n-1)} \frac{\sum_{i,j} \left((w_i - 1/n) - (w_j - 1/n)\right)^2}{\sum_k (w_k - 1/n)^2}.$$

Let $x_i = w_i - 1/n$ to get

$$\mathbf{E}[\Delta\Phi/\Phi] = \frac{K-1}{2K(n-1)} \frac{\sum_{i,j} (x_i - x_j)^2}{\sum_k x_k^2}.$$

Interestingly, when $K = n$, we should have mixing in one step, so in this case $\mathbf{E}[\Delta\Phi/\Phi]$ should be 1. Notice if that is the case, then perhaps surprisingly the above expression is independent of the actual $x_i$ values, and then we have immediately:

$$\mathbf{E}[\Delta\Phi/\Phi] = \frac{n(K-1)}{K(n-1)}.$$

We can in fact confirm this easily. Since $\sum_k x_k = 0$, we have

$$\sum_{i,j}(x_i - x_j)^2 = \sum_{i,j}(x_i - x_j)^2 + 2\left(\sum_k x_k\right)^2 = 2n\sum_k x_k^2,$$

and cancellation gives the desired result.

This analysis also gives us fast convergence to the uniform distribution in the general case. Let $\gamma = \frac{n(K-1)}{K(n-1)}$, and note $\gamma \leq 1$. In particular,

$$1 - \gamma = \frac{n-K}{K(n-1)} < \frac{1}{K}.$$

Also note $\Phi(0) < 1$. So we have

$$\mathbf{E}[\Phi(t+1)] = (1-\gamma)\mathbf{E}[\Phi(t)],$$

and a simple induction yields

$$\mathbf{E}[\Phi(t)] = (1-\gamma)^t\Phi(0) \leq K^{-t}.$$

The rest of the theorem follows easily.

### 3.1 Extensions to Mixnets with Corrupted Servers

In the case of there being corrupted servers, Bob will know the permutation for the cards assigned to each such server. In terms of the analysis, we can treat the permutation for each corrupted server as the identity operation, since Bob can simply undo that permutation. Let us suppose, then, that there are $M = n/K$ servers, so that each obtains $K$ cards in each round, and that $1 \leq s \leq n/K$ servers are uncorrupted. Following our previous analysis, we find

$$\begin{aligned}
\mathbf{E}[\Delta\Phi] &= \frac{1}{K}\sum_{1\leq i<j\leq n} \Pr\left((i,j) \text{ are in the same uncorrupted server}\right)(w_i - w_j)^2 \\
&= \frac{(K-1)}{K(n-1)}\frac{s}{n/K}\sum_{i<j}(w_i - w_j)^2 \\
&= \frac{s(K-1)}{2n(n-1)}\sum_{1\leq i,j\leq n}(w_i - w_j)^2.
\end{aligned}$$

Again, based on our previous analysis, we have

$$\mathbf{E}[\Delta\Phi/\Phi] = \frac{s(K-1)}{n-1}.$$

Now let $\gamma' = \frac{s(K-1)}{(n-1)}$; if, for example, $s = \epsilon\frac{n-1}{K-1}$ then $1 - \gamma' = 1 - \epsilon$. In that case,

$$\mathbf{E}[\Phi(t)] = (1-\epsilon)^t.$$

**Theorem 2:** *A corrupted parallel mixnet, designed as described above, with $s \geq \epsilon(n-1)/(K-1)$ non-corrupted servers, for $\epsilon \geq 1/2$, can mix messages in $t = b\log n$ rounds so that the expected sum-of-squares error, $\mathbf{E}[\Phi(t)]$, between card-assignment probabilities and the uniform distribution is at most $1/n^b$, for any fixed $b \geq 1$. Likewise, if there are at most $\frac{n^{-1/c}(n-1)}{K-1} - \frac{n-K}{K(K-1)}$ corrupted servers, with $K \geq n^{1/c}$ for some constant $c \geq 1$, then in $t = bc$ rounds it is also the case that $\mathbf{E}[\Phi(t)]$ is at most $1/n^b$, for any fixed $b \geq 1$.*

Thus, by Markov's inequality, using $t = 2b \log n$ or $t = 2bc$ rounds, depending on the number of uncorrupted servers, $s$, we can bound the probability that $\Phi(t) > 1/n^b$ to itself be at most $1/n^b$, for any fixed $b \geq 1$.

As an instructive specific example, suppose $K = M = \sqrt{n}$, and there are a constant $z$ servers that are corrupted. Then

$$1 - \epsilon = 1 - (\sqrt{n} - z)\frac{K-1}{n-1} = 1 - \frac{\sqrt{n} - z}{\sqrt{n}+1} = \frac{z+1}{\sqrt{n}+1}.$$

Hence, in this specific case,

$$\mathbf{E}[\Phi(t)] = \left(\frac{z+1}{\sqrt{n}+1}\right)^t < \frac{(z+1)^t}{n^{t/2}},$$

and for any constant $b$ after $4b$ rounds we have that $\Phi(t) \leq n^{-b}$ with probability $P(n^{-b})$.

As our expressions become less clean in our remaining settings, we state a general theorem which can be applied to these settings in a straightforward way:

**Theorem 3:** *Given a parallel mixnet with corrupted servers or adversarially generated inputs, let $\gamma = \mathbf{E}[\Delta\Phi/\Phi]$ in that setting. Then in $t = b \log_{1/(1-\gamma)} n$ rounds the expected sum-of-squares error, $\mathbf{E}[\Phi(t)]$, between card-assignment probabilities and the uniform distribution is at most $1/n^b$, for any fixed $b \geq 1$. In particular, if $\gamma \geq 1/2$, at most $b \log n$ rounds are required; if $\gamma \geq 1 - n^{-1/c}$, at most $bc$ rounds are required.*

## 3.2 Extensions to Mixnets with Corrupted Inputs

For the case of corrupted inputs, Bob will be able to track those cards throughout the shuffle process. In the shuffling setting, we can think of some number of the cards as being marked—no matter what we do, Bob knows the locations of those cards. In terms of the analysis, we can treat this in the following way: when we have a group of $K$ cards, it is as though we are shuffling only $K' \leq K$ cards, where $K'$ is the number of unmarked cards in the collection of $K$ cards. Let us suppose that $f \leq n - 2$ cards are marked. Note that we may think of $w_i$ as being 0 for any cards in a marked position; alternatively, without loss of generality, let us calculate at each step as though $w_i$ is non-zero only for $i = 1$ to $n - f$. (Think of $w_i$ as being the appropriate value for the $i$th unmarked card.) Note that, for consistency, we must have

$$\Phi(t) = \left(\sum_{i=1}^{n-f} w_i(t)^2\right) - \frac{1}{n-f}.$$

Following our previous analysis, we find

$$\begin{aligned}
\mathbf{E}[\Delta\Phi] &= \sum_{K'=2}^{K} \frac{1}{K'} \sum_{1 \leq i < j \leq n-f} \Pr\left(\begin{array}{l}(i,j) \text{ are in the same set of } K' \\ \text{out of } K \text{ unmarked cards}\end{array}\right)(w_i - w_j)^2 \\
&= \sum_{K'=2}^{K} \frac{\binom{n-f}{K'}\binom{f}{K-K'}}{\binom{n}{K}} \frac{K'-1}{K'(n-1)} \sum_{i<j} (w_i - w_j)^2 \\
&= \sum_{K'=2}^{K} \frac{\binom{K}{K'}\binom{n-K}{n-f-K'}}{\binom{n}{f}} \frac{K'-1}{K'(n-1)} \sum_{i<j} (w_i - w_j)^2 \\
&= \frac{1}{2(n-1)\binom{n}{f}} \left(\sum_{K'=2}^{K} \binom{K}{K'}\binom{n-K}{n-f-K'} \frac{K'-1}{K'}\right) \sum_{1 \leq i,j \leq n-f} (w_i - w_j)^2.
\end{aligned}$$

We can then compute $\mathbf{E}[\Delta\Phi/\Phi]$ as

$$\frac{1}{2(n-1)\binom{n}{f}}\left(\sum_{K'=2}^{K}\binom{K}{K'}\binom{n-K}{n-f-K'}\frac{K'-1}{K'}\right)\frac{\sum_{1\le i,j\le n-f}(x_i-x_j)^2}{\sum_{1\le k\le n-f}x_k^2}.$$

Following the same computations as previously, we have

$$\mathbf{E}[\Delta\Phi/\Phi] \;=\; \frac{n-f}{(n-1)\binom{n}{f}}\left(\sum_{K'=2}^{K}\binom{K}{K'}\binom{n-K}{n-f-K'}\frac{K'-1}{K'}\right).$$

Note the $n-f$ term in the numerator in place of an $n$.

Now let $\nu$ equal the right hand side above; then we have

$$\mathbf{E}[\Phi(t)] = (1-\nu)^t.$$

In particular, it is clear that $\nu \le \frac{n(K-1)}{K(n-1)}$, so the convergence of $\mathbf{E}[\Phi(t)]$ to 0 happens more slowly than in the case without corrupted inputs, as expected. Nevertheless, we can still derive a theorem analogous to Theorem 2 using Theorem 3 and the above characterization of $\mathbf{E}[\Phi(t)]$. We omit a full restatement for space reasons.

### 3.3 Extensions to Mixnets with Corrupted Servers and Inputs

One nice aspect of our analysis is that combinations of corrupted servers and inputs are entirely straightforward. In this setting, we have

$$\begin{aligned}
\mathbf{E}[\Delta\Phi] &= \sum_{K'=2}^{K}\frac{1}{K'}\sum_{1\le i<j\le n-f}\Pr\begin{pmatrix}(i,j)\text{ are in the same set of }K'\\ \text{out of }K\text{ unmarked cards at an}\\ \text{uncorrupted server}\end{pmatrix}(w_i-w_j)^2\\
&= \sum_{K'=2}^{K}\frac{s}{n/K}\frac{\binom{n-f}{K'}\binom{f}{K-K'}}{\binom{n}{K}}\frac{K'-1}{K'(n-1)}\sum_{i<j}(w_i-w_j)^2\\
&= \frac{sK}{n(n-1)}\sum_{K'=2}^{K}\frac{\binom{K}{K'}\binom{n-K}{n-f-K'}}{\binom{n}{f}}\frac{K'-1}{K'}\sum_{i<j}(w_i-w_j)^2\\
&= \frac{sK}{2n(n-1)\binom{n}{f}}\left(\sum_{K'=2}^{K}\binom{K}{K'}\binom{n-K}{n-f-K'}\frac{K'-1}{K'}\right)\sum_{1\le i,j\le n-f}(w_i-w_j)^2.
\end{aligned}$$

Hence

$$\mathbf{E}[\Delta\Phi/\Phi] \;=\; \frac{sK(n-f)}{n(n-1)\binom{n}{f}}\left(\sum_{K'=2}^{K}\binom{K}{K'}\binom{n-K}{n-f-K'}\frac{K'-1}{K'}\right).$$

Given this bound, we can then derive a theorem analogous to Theorem 2 for the case when mix servers can be corrupted and the adversary can inject fake messages using Theorem 3. We omit a full restatement for space reasons.

### 3.4 Extensions to Mixnets with a Less Powerful Global Step

Under our model, we can also consider a weaker global step between rounds, where the cards must be split evenly into $n/K$ groups of size $K$ for the $n/K$ servers. Instead, let us assume simply that each server obtains each card independently with probability $K/n$; this requires less synchronization of the messages between rounds. For example, previously we have assumed that the encryption step taken by each server on each round is used to provide the random permutation that maps messages to servers, and that a global step sorted the re-encrypted messages in order for each server to obtain $n/K$ messages. Assume instead that the encrypted messages are mapped to values, which can be assumed to be uniformly distributed over their range, and the range for such messages is divided into $n/K$ equally sized subranges, one for each server. Then the server for each message for each round is determined without a complete sort (indeed, at the end of reach round the behavior is like the first step of a radix sort on random inputs). Assuming $K$ is sufficiently large (e.g., at least $c \log n$ for a suitable constant $c$) then Chernoff bounds yield that each server will obtain at most $(1+\epsilon)K$ cards in each round with high probability; hence, the maximum shuffle size can still be bounded.

Extending the analysis above, we have that $\mathbf{E}[\Delta\Phi]$ equals

$$
\begin{aligned}
&\sum_{J=2}^{n}\binom{n}{J}\left(\frac{K}{n}\right)^J\left(1-\frac{K}{n}\right)^J\sum_{K'=2}^{J}\frac{1}{K'}\sum_{1\le i<j\le n-f}\Pr\left(\begin{array}{l}(i,j) \text{ are in the same set of } K' \text{ out of } J \text{ unmarked cards at an uncorrupted server}\end{array}\right)(w_i-w_j)^2\\
=\;&\sum_{J=2}^{n}\binom{n}{J}\left(\frac{K}{n}\right)^J\left(1-\frac{K}{n}\right)^J\sum_{K'=2}^{J}\frac{s}{n/J}\frac{\binom{n-f}{K'}\binom{f}{J-K'}}{\binom{n}{J}}\frac{K'-1}{K'(n-1)}\sum_{i<j}(w_i-w_j)^2\\
=\;&\sum_{J=2}^{n}\binom{n}{J}\left(\frac{K}{n}\right)^J\left(1-\frac{K}{n}\right)^J\frac{sJ}{n(n-1)}\sum_{K'=2}^{J}\frac{\binom{J}{K'}\binom{n-J}{n-f-K'}}{\binom{n}{f}}\frac{K'-1}{K'}\sum_{i<j}(w_i-w_j)^2\\
=\;&\sum_{J=2}^{n}\binom{n}{J}\left(\frac{K}{n}\right)^J\left(1-\frac{K}{n}\right)^J\frac{sK}{2n(n-1)\binom{n}{f}}\left(\sum_{K'=2}^{K}\binom{K}{K'}\binom{n-K}{n-f-K'}\frac{K'-1}{K'}\right)\\
&\cdot\sum_{1\le i,j\le n-f}(w_i-w_j)^2.
\end{aligned}
$$

Generally, since for $K$ sufficiently large we will have concentration of the number of cards arounds its mean, the slowdown from the random distribution can be bounded by a small constant factor.

# 4 Conclusion and Open Problems

In this paper, we have provided a comprehensive analysis of buffer shuffling and shown that this leads to improved algorithms for achieving anonymity and unlinkability in parallel mixnets. An interesting direction for future research could be to extend this analysis to other topologies, including hypercubes and expander graphs.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## References


[1] D. Aldous and P. Diaconis. Shuffling cards and stopping times. *Amer. Math. Monthly*, 93(5):333–348, 1986.

[2] B. Arkin, F. Hill, S. Marks, M. Schmid, T. J. Walls, and G. McGraw. How we learned to cheat at online poker: A study in software security. *Developer.com: Tech Focus*, September 1999.

[3] N. Borisov. An analysis of parallel mixing with attacker-controlled inputs. In G. Danezis and D. Martin, editors, *Privacy Enhancing Tech.*, volume 3856 of *LNCS*, pages 12–25. Springer, 2006.

[4] D. L. Chaum. Untraceable electronic mail, return addresses, and digital pseudonyms. *Commun. ACM*, 24(2):84–90, Feb. 1981.

[5] P. Diaconis, J. A. Fill, and J. Pitman. Analysis of top to random shuffles. *Combinatorics, Probability and Computing*, 1(02):135–155, 1992.

[6] B. Ghosh and S. Muthukrishnan. Dynamic load balancing in parallel and distributed networks by random matchings. In *6th ACM Symp. Par. Alg. & Arch. (SPAA)*, pages 226–235, 1994.

[7] P. Golle and A. Juels. Parallel mixing. In *11th ACM Conf. on Comp. and Comm. Security (CCS)*, pages 220–226, 2004.

[8] M. T. Goodrich, M. Mitzenmacher, O. Ohrimenko, and R. Tamassia. Practical oblivious storage. In *2nd ACM Conf. on Data and App. Sec. & Priv. (CODASPY)*, pages 1–10, 2011.

[9] N. Hopper and E. Y. Vasserman. On the effectiveness of k-anonymity against traffic analysis and surveillance. In *5th ACM Work. on Priv. in Elec. Society (WPES)*, pages 9–18, 2006.

[10] M. Klonowski and M. Kutylowski. Provable anonymity for networks of mixes. In M. Barni, editor, *Information Hiding*, volume 3727 of *LNCS*, pages 26–38. Springer, 2005.

[11] D. E. Knuth. *Art of Computer Programming, Volume 2: Seminumerical Algorithms (3rd Edition)*, volume 2. Addison-Wesley, 1997.

[12] N. Li, T. Li, and S. Venkatasubramanian. t-closeness: Privacy beyond k-anonymity and l-diversity. In *23rd IEEE Int. Conf. on Data Engineering (ICDE)*, pages 106–115, 2007.

[13] A. Machanavajjhala, D. Kifer, J. Gehrke, and M. Venkitasubramaniam. L-diversity: Privacy beyond k-anonymity. *ACM Trans. Knowl. Discov. Data*, 1, 2007.

[14] J. Ren and J. Wu. Survey on anonymous communications in computer networks. *Computer Communications*, 33(4):420–431, 2010.

[15] K. Sampigethaya and R. Poovendran. A survey on mix networks and their secure applications. *Proc. of the IEEE*, 94(12):2142–2181, dec 2006.

[16] L. Sweeney. k-anonymity: a model for protecting privacy. *Int. J. Uncertain. Fuzziness Knowl.-Based Syst.*, 10(5):557–570, 2002.

[17] L. von Ahn, A. Bortz, and N. J. Hopper. k-anonymous message transmission. In *10th ACM Conf. on Comp. and Comm. Security (CCS)*, pages 122–130, 2003.

[18] P. Wang, P. Ning, and D. S. Reeves. A k-anonymous communication protocol for overlay networks. In *2nd ACM Symp. Info., Comp. & Comm. Sec. (ASIACCS)*, pages 45–56, 2007.

# A Additional Anonymity Measures

There are several additional measures that Alice can use to determine when she is close to placing her cards in random order in a fashion that obscures that order from Bob. As with the $\mathsf{Anon}_t$ measure, the ones we review here are also not metrics.

## A.1 $k$-Anonymity

One measure of obfuscation is $k$*-anonymity* [16]. In the context of the anonymous shuffling problem, we would say that Alice's shuffling algorithm achieves $k$-anonymity if, from Bob's perspective, every card in Alice's final output has at least $k$ input cards that have a possibility of being mapped to that card during the shuffling. This measure has been used in several applications in computer security and privacy, including mixnets [17, 18]. A well-known weakness of $k$-anonymity (e.g., see [9, 12, 13]), unfortunately, is that doesn't take probabilities into consideration. So, for example, if Bob has a 90% certainty about the identity of the top card after Alice's shuffle, with all the other cards sharing the remaining 10%, then we would say that the top card's identity had achieved $n$-anonymity (i.e., the maximum possible), even though Bob can be confident about which card is on top. Thus, we feel that $k$-anonymity is insufficient to use as a measure of obscurity for the anonymous shuffling problem.

## A.2 Relative entropy

Relative entropy measures, in bits, the amount of information that exists between two probability measures. For probability distributions $P$ and $Q$, it is defined as

$$D(P||Q) = \sum_i P_i \log_2 \frac{P_i}{Q_i}.$$

In the context of buffer shuffling, and the information present in the distribution, $W$, for a single card, compared to the uniform distribution, $U$, this amounts to

$$\begin{aligned} D(W||U) &= \sum_i w_i(t) \log_2 \frac{w_i(t)}{1/n} \\ &= \log_2 n \; - \left| \sum_i w_i(t) \log_2 w_i(t) \right|. \end{aligned}$$

The goal for Alice would be to minimize the relative entropy, $D(W||U)$. While relative entropy is more sensitive to outliers than the distance notations, $\alpha_t$ and $\beta_t$, related to the $L_\infty$ and $L_1$ metrics, and it measures information leakage in bits, which is a useful quantity, it is not always easy to work with. Moreover, it is not actually a metric, since it doesn't satisfy the symmetric property.