A. La Rosa
PSU-Physics

Lecture Notes
PH 411/511

INTRODUCTION TO QUANTUM MECHANICS

PART-I TRANSITION from CLASSICAL to QUANTUM PHYSICS

CHAPTER 2 CLASSICAL PHYSICS

ELECTROMAGNETISM and RELATIVITY (REVIEW) [1,2]

2.1 ELECTROMAGNETISM

- **2.1.A Maxwell's Equations (ME)**
- **2.1.B Consequences of the Maxwell Equations**
  - **2.1.B.a The wave equation**
  - **2.1.B.b Light as electromagnetic radiation**
  - **2.1.B.c Independence of the motion of the source**
  - **2.1.B.d The Lorentz's Transformation**
- **2.1.C The hypothesis of the ether and the notion of absolute velocity**
  - **2.1.C.a The Michelson-Morley Experiment**
  - **2.1.C.b Lorentz' length-contraction hypothesis**
  - **2.1.C.c Transformation of coordinates based on the Lorentz' length-contraction hypothesis**

2.2 SPECIAL THEORY OF RELATIVITY

- **2.2.A Newton's Principle of Relativity**
  - **2.2.A.a The Galilean Transformation**
  - **2.2.A.b Electromagnetism and the Principle of Relativity**
- **2.2.B Einstein's Principles of Relativity**
  - **2.2.B.a The principles of relativity**
  - **2.2.B.b Consequences of Einstein's principles of relativity**
    - **2.2.B.b1 Relationship between the space-time coordinates in different inertial reference frames**
    - **2.2.B.b2 Relationship between the velocities**
- **2.2.C Required modification of the Mechanics laws to make them compatible with Einstein's relativity principles.**
  - **2.2.C.a Relativistic Momentum and Energy of a Particle**
  - **2.2.C.b Derivation of the relativistic mass**

2.2.C.c Equivalence of mass and energy
2.2.C.d Relationships involving p, E and v
2.2.D Four-components vectors and symmetry
2.2.D.a Transformation of space-time coordinates
2.2.D.b Transformation of energy-momentum coordinates

References

Richard Feynman, “The Feynman Lectures on Physics,”
Volume I, Chapter 15, 16, 17

R. Eisberg and R. Resnick, “Quantum Physics,” 2nd Edition, Wiley, 1985.
Appendix A

Chapter 2
CLASSICAL PHYSICS: ELECTROMAGNETISM and RELATIVITY

2.1 ELECTROMAGNETISM

2.1.A Maxwell's Equations (ME)

(Treatise on Electricity and Magnetism, published in 1873)

$$\left.\begin{array}{ll} \nabla.\mathbf{E} = \rho / \varepsilon_0 & \textit{Gauss' law} \\[2ex] \nabla.\mathbf{B} = 0 & \textit{No magnetic monopoles} \\[2ex] \nabla \mathrm{x} \mathbf{E} + \dfrac{\partial}{\partial t}\mathbf{B} = \mathbf{0} & \textit{Faraday's Law} \\[2ex] \nabla \mathrm{x} \mathrm{B} = \mu_0 \left(\mathbf{j} + \underbrace{\varepsilon_0 \dfrac{\partial \mathbf{E}}{\partial t}}\right) & \textit{Ampere's Law modified by Maxwell} \end{array}\right\} \quad (1)$$

Here,

$\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$ are the electromagnetic fields.

The vector $\mathbf{j}$ is the conduction current density in the medium,

The scalar ρ is the free charge density,

Both $\mathbf{j}$ and ρ subjected to the equation of continuity (or the conservation of charge) expressed as $\nabla \bullet \mathbf{j} + \partial\rho/\partial t = 0$.

2.1.B Consequences of the Maxwell Equations

2.1.B.a The wave equation

For the case $\mathbf{j} = \mathbf{0}$ and $\rho = 0$, it can be demonstrated that the components of the electromagnetic fields satisfy the wave equation

$$\frac{\partial^2 E_y}{\partial x^2} - \frac{1}{\mathrm{v_o}^2}\frac{\partial^2 E_Y}{\partial t^2} = 0 \qquad \text{where} \quad \mathrm{v_o} \equiv \frac{1}{\sqrt{\mu_o \varepsilon_o}} \qquad (2)$$

That is, the fields $\mathbf{E}$ and $\mathbf{B}$ travel at constant speed v_o.

2.1.B.b Light as an electromagnetic radiation

When Maxwell evaluated the **speed of the electromagnetic waves**, he found,

$\mathrm{v}_o \equiv 1/\sqrt{\mu_o \varepsilon_o} = 300{,}000 \ \ Km/s$. This is the speed of light **!!!**

With excitement Maxwell wrote:

> *"We can scarcely avoid the inference that light consists in transverse undulation of the same medium which is the cause of electric and magnetic phenomena."*

2.1.B.c Independence of the motion of the source

If the source of the disturbance is moving, the emitted light travels at the same speed in any direction.

2.1.B.d Lorentz's Transformation

H. A. Lorentz noticed a remarkable and curious transformation of coordinates that made the ME invariant:

$$x' = \frac{x - u_o t}{\sqrt{1-(u_o/c)^2}}\ , \quad y' = y\ , \quad z' = z\ , \quad t' = \frac{t - u_o x/c^2}{\sqrt{1-(u_o/c)^2}} \qquad (3)$$

Lorentz transformation

2.1.C The hypothesis of the ether and the notion of absolute velocity

At the end of the 19th century, it was still conceived that all waves needed a medium to travel. Accordingly, scientist saw a necessity in figuring out a medium in which light would propagate. It is in this context that the infamous "ether" was 'invented' as a medium permeating all the space, and in which light would propagate at speed $c = 300{,}000 \ \ Km/s$.

The ether hypothesis, together with the ME conception on the independence of the light speed relative to the source motion, created a possibility to measure the **absolute velocity** of an object (in particular the Earth) relative to the ether, as suggested in Fig. 2.1 below.

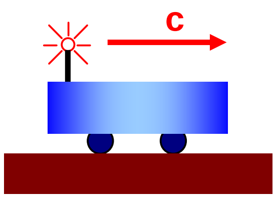


Fig. 2.1 *The light beam propagates with speed* ***c*** *regardless of the motion of the light source (c would be the velocity of light relative to the "ether.". If an observer in the car measured a speed* ***c'*** *for the light beam, then it would mean that the speed of the car is* ***u*** *=* ***c*** *-* ***c'.***

2.1.C.a The Michelson-Morley Experiment (1887)

The objective was to measure the speed of the Earth through the hypothetical ether.

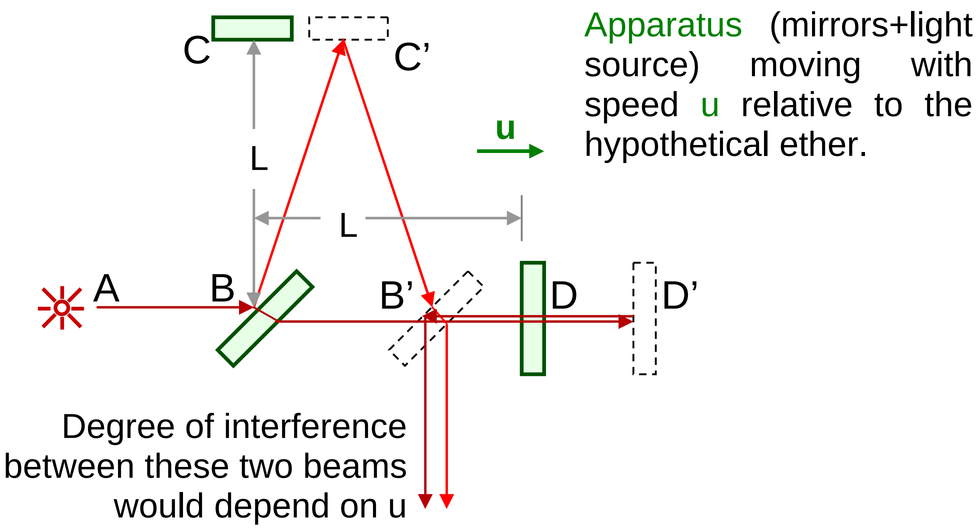


Fig. 2.2 *Schematic of the experimental arrangement in the Michelson-Morley experiment.*

Assuming that c is the speed of light relative to the hypothetical ether, we obtain the following expression:

Horizontal-beam travel time:

From B to D': $ct_1 = L + ut_1$, which gives $t_1 = L/(c-u)$

From D' to B': $ct_2 = L - ut_2$, which gives $t_2 = L/(c+u)$

$$t_{horizontal} = t_1 + t_2 = 2Lc/(c^2 - u^2) = \frac{2L/c}{1-(u/c)^2} \qquad (4)$$

Vertical-beam travel time:

From B to C': $(ct_3)^2 = L^2 + (ut_3)^2$, which gives $(t_3)^2 = L^2/(c^2 - u^2)$

From C' to B': same value as above. Hence

$$t_{vertical} = \frac{2L/c}{\sqrt{1-(u/c)^2}} \tag{5}$$

Notice in (4) and (5) that $t_{horizontal} = [1/\sqrt{1-(u/c)^2}]\ t_{vertical}$. That is,

$$t_{horizontal} > t_{vertical} \tag{6}$$

This difference in time travel (that depends on u) would result in an interference between the two beams, an information that could then be used to calculate the speed u of the Earth relative to the ether.

Michelson and Morley oriented the apparatus so that the line BD was nearly parallel to the Earth's motion in its orbits (at certain day and night). The orbital speed is about 30 Km/s, and the **apparatus** was **sensitive enough to detect the corresponding interference**. But, **it was never was detected**.

2.1.C.b Lorentz' length-contraction conjecture (1892)

As a way to explain Michelson and Morley's puzzling result, Lorentz proposed an ad hoc hypothesis that bodies contract when they are moving, and that the contraction occurs only in the direction they are moving. (That would make $t_{horizontal}$ smaller in the Michelson and Morley's experiment.)

Let L_o is the length of a body at rest (in the ether reference). Then when it moves with speed u along the horizontal axis, its new horizontal length $L_{//}$ would be smaller than L_o by a factor $\sqrt{1-(u/c)^2}$

This conjecture accounts for the Michelson and Morley experiment since indeed it makes equal the times $t_{vertical}$ and $t_{horizontal}$ given in (4) and (5).

$$t_{horizontal} = \frac{2L_{//}/c}{1-(u/c)^2} = \frac{2\ [L_o\sqrt{1-(u/c)^2}\ ]/c}{1-(u/c)^2} = \frac{2\ L_o/c}{\sqrt{1-(u/c)^2}}$$

$$t_{vertical} = \frac{2L_o/c}{\sqrt{1-(u/c)^2}}$$

Thus preditinng the experimentally obtained equality between $t_{vertical}$ and $t_{horizontal}$. This way, Lorentz was saving the hypothesis of the ether.

2.1.C.c Transformation of coordinates based on the Lorentz' length-contraction hypothesis

Observer O' measures the coordinate of a point **P** with a meter stick. He lays down the sticker x' times, so he affirms the distance is x' meters

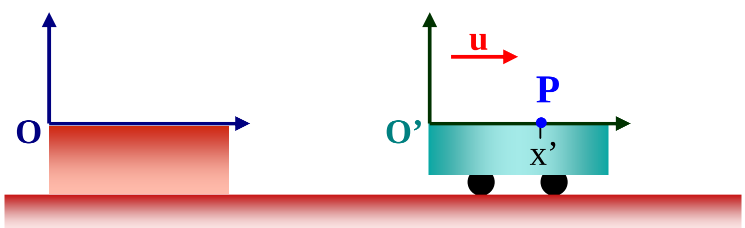


Fig. 2.3 *Measuring distances with sticks of different lengths.*

Observer O, however, considers that observer O' is using a foreshortened ruler, so the "real" distance is $x'\sqrt{1-(u/c)^2}$ meters.

Since the observe O' has traveled a distance ut away from O, observer O would say that the position of the same point P, measured in the O-coordinates is $x = ut + x'\sqrt{1-(u/c)^2}$, which leads to

$$x' = \frac{x - ut}{\sqrt{1-(u/c)^2}}$$

This is the first equation of the Lorentz transformation given in expression (3) above.

Still, Lorentz' contraction hypothesis was considered too artificial, as the ether hypothesis continued to get into other contradictions.

2.2 THE SPECIAL THEORY OF RELATIVITY

The origins of the **special theory of relativity** lie in the development of electromagnetism, from the time when scientist were trying to harmonize electromagnetism (theory and experiment) with the general principle of relativity. The development passed through Maxwell, Lorentz, Poincare and Einstein. It was Einstein, in 1905, who made the crucial generalization to all physical phenomena, not just electromagnetism.

(In 1915 Einstein published his **General Theory of Relativity**, which extends the special theory of relativity to the case of gravitation.)

2.2.A Newton's Principle of Relativity

Historically, the principle of relativity was stated by Newton (although it has been an hypothesis use in Mechanics since the days of Copernicus, if not before):

> " *The motion of bodies included in a given space is the same among themselves, whether that space is at rest or moves uniformly forward in a straight line.*" (7)

In all experiments performed inside a moving system the laws of physics will appear the same as they would if the system were standing still.[3]

Under what conditions is this principle valid? Let's review some history.

2.2.A.a The Galilean Transformation

Let's assume that the spatial and temporal coordinates of two reference systems (here denominated prime and no prime) are related by (here u_o is a constant)

$$x=x' + u_o t', \qquad y=y', \qquad z=z', \qquad t=t'. \tag{8}$$

Galilean Transformation

Notice that, since $(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}) = (\frac{d^2x'}{dt'^2}, \frac{d^2y'}{dt'^2}, \frac{d^2z'}{dt'^2})$,

$$\vec{F} = m_o \frac{d^2}{dt^2}(x,y,z) \quad \text{implies} \quad \boldsymbol{F} = m \frac{d^2}{dt'^2}(x', y', z') \tag{9}$$

That is, Newton's laws are of the same form in a moving system as in a stationary system.

The laws of mechanics (Newton's laws) then appear in agreement with the principle of relativity.

2.2.A.b Electromagnetism and the Principle of Relativity

The Maxwell Equations (ME), however, did not appear to satisfy the principle of relativity. That is, when using the Galilean transformation (8), the ME do not remain the same.[4]

Consequently, the prediction appeared to be that the electrical phenomena in a moving system of reference should be different from those in a stationary reference (which caused the ME into question).

Experimental evidence amounted to the ME to prevail. Something was wrong then, and the ME appeared not to be the culprit.

2.2.B The Special Theory of Relativity

2.2.B.a Einstein's Principles of Relativity

- By the end of the 19th century, there existed a few possibilities:[4]
  1. The ME were wrong
     The proper theory of electromagnetism would be one that is invariant under the Galilean transformation.
  2. Galilean transformation applied to classical mechanics, but electromagnetism had a preferred reference frame, the one in which the ether was at rest.
  3. There would exist a relativity principle valid for both classical mechanics and electromagnetism; but it was not one in which the Galilean transformation were valid. This would imply that the laws of mechanics were in need of modification.

- Einstein chose the third option. Einstein's Principles of Relativity is base on two postulates:

  The first postulate was enunciated already in (7) above.

  The second postulate states:

  The speed of light is finite and independent of the motion of its source. (10)

2.2.B.b Consequences of Einstein's Principles of Relativity

2.2.B.b1 Relationship between the space and times coordinates in different inertial reference frames

Consider two inertial references that move relative to each other with constant velocity u. The second postulate forces a peculiar relationship between the space and times coordinates (x, y, z, t) and (x', y', z', t') since both references have to measure the same speed of light.

If at t=0 the two origins O and O' coincide and a flash of light is emitted,

observer O will note that the light has reached point A in a time t, and will write $r = ct$, or, equivalently

$$x^2 + y^2 + z^2 = c^2 t^2 \tag{11a}$$

while observer O' will note that the light has reached the same point A in a time t', and will write $r' = ct'$, or, equivalently.

$$x'^2 + y'^2 + z'^2 = c^2 t'^2 \tag{11b}$$

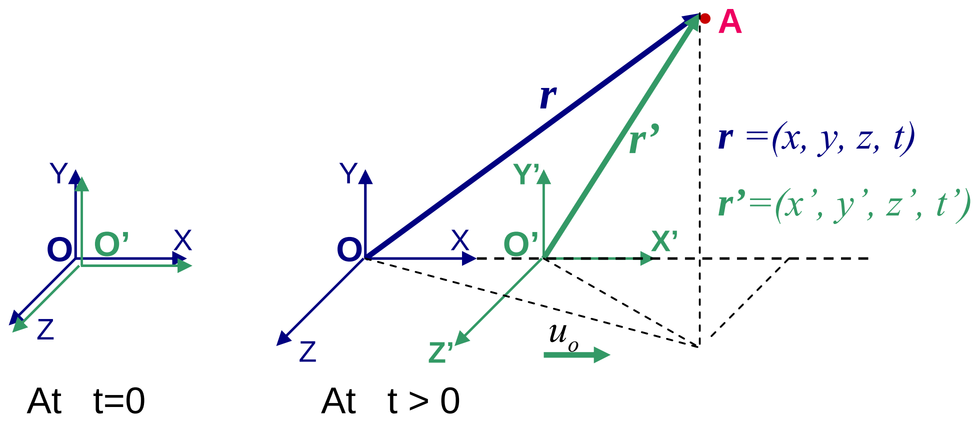


Fig. 2.4 Left: *Two references synchronize their clocks at t=0.* ***Right:*** *Monitoring a light beam from the two different references.*

Assuming that the relationship between the coordinates is linear, for example

$x' = k(x - ut)$ and $t' = a(t - bx)$,

where k, a, and b to be determined from the conditions 11a and 11b,

the following result is obtained,

$$x' = \frac{x - ut}{\sqrt{1-(u/c)^2}}\,, \quad y' = y\,, \quad z' = z\,, \quad t' = \frac{t - ux/c^2}{\sqrt{1-(u/c)^2}} \qquad (12)$$

The transformation given in (12) is nothing but the Lorentz transformation, mentioned above in (3), under which the Maxwell Equations are invariant. In other words,

The Maxwell's Equations of electromagnetism already inherently contain the relativistic effects. (13)

This means that what needs to be changed are rather the laws of mechanics *!!*

Finally, notice by symmetry that,

$$x=\frac{x'+u\,t'}{\sqrt{1-(u/c)^2}}\,,\qquad y=y'\,,\qquad z=z'\,,\qquad t=\frac{t'\;+u\,x'/c^2}{\sqrt{1-(u/c)^2}}\qquad (14)$$

2.2.B.b2 Relationship between the velocities

The following relationships acquire importance for their role in evaluating (later in this chapter) the linear momentum of a particle in different inertial references. The application of the conservation of the relativistic linear momentum leads to important and interesting consequences (as we will see in the next sections).

From (14)

$$\Delta x=\frac{\Delta x'+u\Delta t'}{\sqrt{1-(u/c)^2}}\,,\qquad \Delta y=\Delta y',\qquad \Delta z=\Delta z',\qquad \Delta t=\frac{\Delta t'+u_o(\Delta x')/c^2}{\sqrt{1-(u/c)^2}}$$

Since $V_x=\Delta x/\Delta t$, $V_{x'}=\Delta x'/\Delta t'$, … etc, we obtain

$$\left.\begin{aligned} V_x &= \frac{V_{x'}+u}{1+(u/c^2)V_{x'}} \\ V_y &= \frac{V_{y'}}{1+(u/c^2)V_{x'}}\sqrt{1-(u/c)^2} \\ V_z &= \frac{V_{z'}}{1+(u/c^2)V_{x'}}\sqrt{1-(u/c)^2} \end{aligned}\right\}\ \begin{array}{l}\textit{Velocities}\\ \textit{transformation}\end{array}\qquad (15)$$

2.2.C How to modify the laws of classical mechanics in order to make them compatible with the Einstein's principle of relativity

In other words, what modification should be introduced in order to make the laws of mechanics satisfy the Lorentz transformation?

It turns out (as we will demonstrate below) that the only required condition is that the mass m (assumed to be constant in the Newton's formulation) rather varies with the velocity of the particle.

$$m = \frac{m_o}{\sqrt{1-(\mathrm{v}/c)^2}} \tag{16}$$

Einstein's modification of Newton's Law

That is, the mass *m* increases with velocity. m_o represents the "rest mass" (the mass of the particle as measured standing still with respect to the observer.)

2.2.C.a Relativistic Momentum and Energy of a Particle

The derivation of expression (16) results from the requirement of the following conditions:

a) compatibility with the conservation laws of energy and momentum, and

b) compatibility with the Lorentz transformation.

A generalization of the momentum and energy, consistent with the Lorentz transformation, becomes necessary. To that effect, the following general expressions are considered

$$\vec{\mathbf{P}} = \underbrace{\mathcal{M}(\mathrm{v})}\ \vec{\mathbf{v}}$$

$$\mathrm{E} = \mathcal{E}(\mathrm{v}) \tag{17}$$

where $\mathcal{M}$ and $\mathcal{E}$ are scalar functions of the particle's velocity.

An elegant and simple derivation for $\mathcal{M}(\mathrm{v})$ and $\mathcal{E}(\mathrm{v})$ is obtained by Feynman[1] (which we reproduce in the next section below), while a more formal procedure is presented in Jackson's textbook.[5] The result is, $\mathcal{M}(\mathrm{v}) = m_o/\sqrt{1\text{-}(\mathrm{v}/c)^2}$ and $\mathrm{E} = \mathcal{E}(\mathrm{v}) = [\mathcal{M}(\mathrm{v})]c^2$. Written in a simpler form:

$$\vec{\mathbf{P}} = \frac{m_o}{\sqrt{1-(\mathrm{v}/c)^2}}\vec{\mathbf{v}} = m\ \vec{\mathbf{v}} \quad \text{(\textit{to be justified below})}$$

$$E = \frac{m_o}{\sqrt{1-(\mathrm{v}/c)^2}}c^2 = mc^2 \tag{18}$$

The energy $E = mc^2$ is the total energy of a body.

2.2.C.b Derivation of the relativistic mass

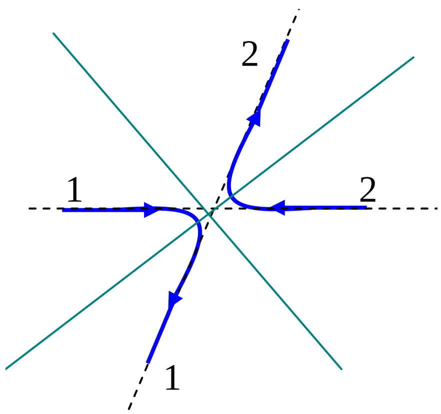
Let $\vec{\mathbf{p}} = [m(\mathrm{v})]\vec{\mathbf{v}}$

or, equivalently,

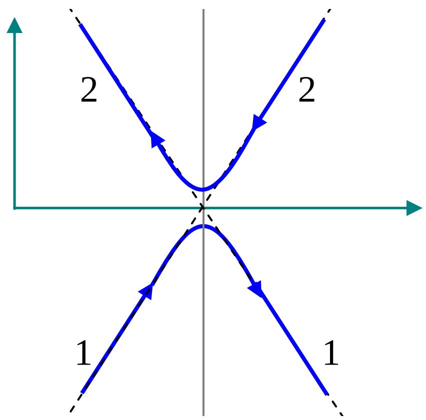
$$p_x = [m(\mathrm{v})]\, v_x \quad \text{and} \quad p_y = [m(\mathrm{v})]\, v_y$$

Our objective is to find the form of the function *m*(v)?

For that purpose, let's consider a collision between two identical particles.



Collision between two identical particles



Same collision (seen from a rotated axis)

Fig 2.5 *The same collision observed from two different references.*

Let's further exploit the symmetry of the problem

Let's view the collision from a reference O' that is moving horizontally towards the right (with respect to the "green" reference in the figure above) at the same speed of the horizontal velocity component of particle 1. The resulting situation, seen by O', is shown in figure 2.6 below.

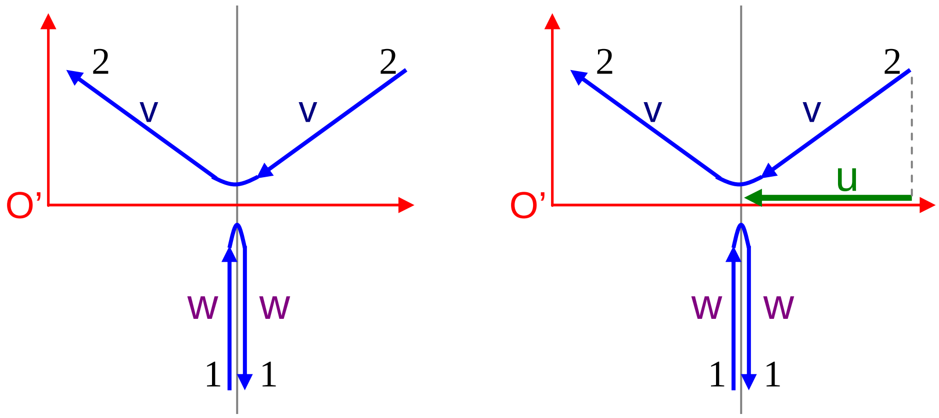


Fig 2.6 *Collision viewed in the reference O' where the horizontal velocity of particle-1 is zero. (The two graph above are identical.)*

With respect to observer O', we have the following

w: vertical velocity of particle-1.

v: magnitude of the incident (and receding) velocity of particle-2.

u: horizontal velocity component of particle-2

Conservation of the horizontal component of the linear momentum

Notice, due to the symmetry of the problem and since the particles are the same before and after the collision, that the horizontal component of the total momentum is conserved. In effect, in the reference O' we have (see Fig 2.6):

Before the collision: $p_{x'} = [m_2(\mathrm{v})]u + 0$

After the collision: $p_{x'} = [m_2(\mathrm{v})]u + 0$

That is, we verify that the horizontal component of the total linear momentum is conserved (even though we do not know yet the explicit form of m(v).)

Conservation of the vertical component of the linear momentum

What about the vertical component $p_{y'}$? How to apply the conservation of linear momentum?

First, we need to determine the vertical component $V_{y'}$ of particle 2 (see graph below). We do not know $V_{y'}$.

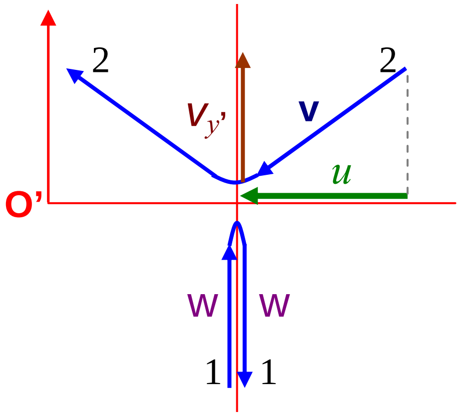


Fig. 2.7 *Collision as seen by observer O'. The component $V_{y'}$ needs to be determined in terms of v, u, and* w.

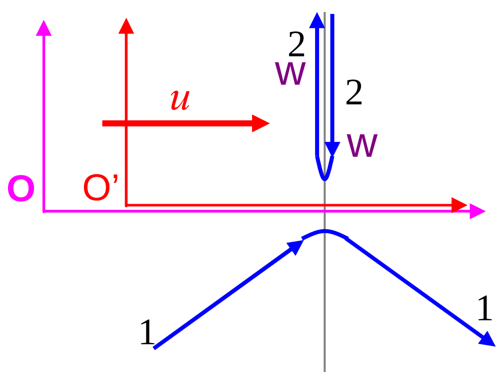
(We would be tempted to say that $V_{y'}$= w,

but that would be incompatible with the Lorentz transformation,

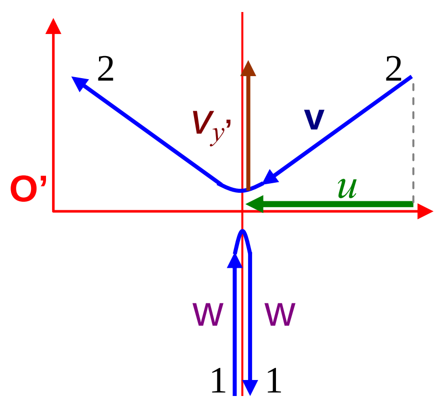
as we will see below).

To find the proper relationship between $V_{y'}$ and w let's view the collision in a reference **O** in which observer **O'** is seen to travel at speed u. Due to the symmetry of the problem, the collision in the references O and O' will look like in the Fig. 2.8 below (where, for comparison, the diagram of figure 2.7 is repeated.)

Notice that due to the symmetry of the problem, the diagram observed in **O** is simply a mirror of the diagram observed in the reference **O'**



Collision as observed in reference **O**. (Observer **O'** moves with speed u relative to **O**.)



Collision as seen by observer **O'**.

Fig 2.8 *The same collision observed in references* ***O*** *(left diagram) and* ***O'*** *(right diagram.) Reference* ***O'*** *moves to the right with velocity* ***u*** *relative to observer* ***O****.*

The diagrams in Fig 2.8 allows finding $V_{y'}$ in terms of w.

Indeed, let's use expression (15) $V_y = \frac{V_{y'}}{1+(u/c^2)V_{x'}}\sqrt{1-(u/c)^2}$:

For the motion of particle 2 we identify:

According to reference **O** : V_y = w

According to reference **O'**: $V_{x'}$ = -u and $V_{y'} = V_{y'}$

This gives,

$$\text{w} = \frac{V_{y'}}{1+(u/c^2)(-u)}\sqrt{1-(u/c)^2} = \frac{V_{y'}}{\sqrt{1-(u/c)^2}} .$$

Thus

$$\boxed{V_{y'} = \text{w}\sqrt{1-(u/c)^2}} \qquad (19)$$

Let's return now to the diagram displayed in Fig. 2.7, and summarize our results:

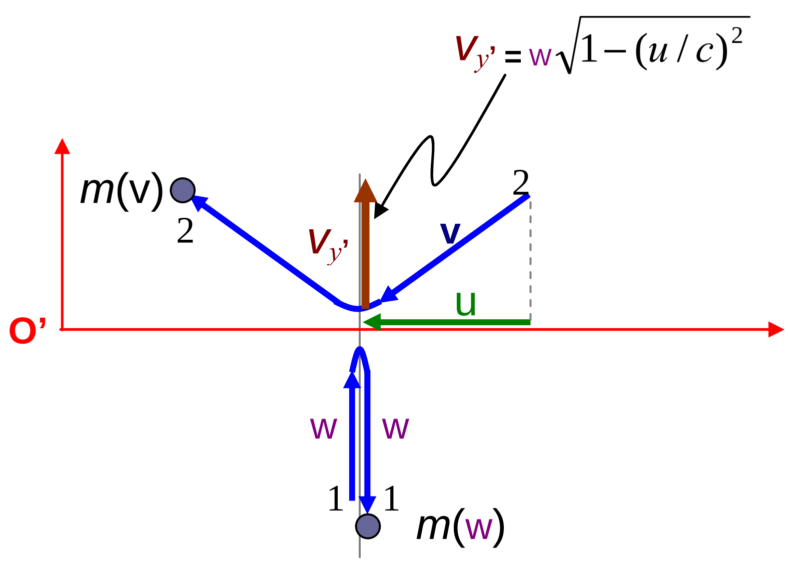


Fig 2.9 *Collision diagram showing the values of speeds compatible with the Lorentz transformation.*

Now we apply the conservation of (vertical component) of the linear momentum

$$\underbrace{[m(\mathrm{v})]\ v_{y'}}_{\text{Particle 2}} = \underbrace{[m(\mathrm{w})]\ \mathrm{w}}_{\text{Particle 1}}$$

$$\underbrace{[m(\mathrm{v})]\ \mathrm{w}\sqrt{1-(u/c)^2}}_{\text{Particle 2}} = \underbrace{[m(\mathrm{w})]\ \mathrm{w}}_{\text{Particle 1}}\ ,$$

which gives,

$$\frac{m(\mathrm{v})}{m(\mathrm{w})} = \frac{1}{\sqrt{1-(u/c)^2}} \qquad (20)$$

Now the question is:

What should be the form of the function *m* such that it satisfies the relationship (20)?

Case w~0

To gain some grasp abut the potential solution for the velocity dependent mass m satisfying (20) let's consider the case in which w~0 (glancing collision.)

When w→0, which also implies v→u), we obtain from (20),

$$m(u) = \frac{m(0)}{\sqrt{1-(u/c)^2}} \qquad (21)$$

Case: Arbitrary situation

Based on the result (21), obtained for the particular case of a glancing collision, what about considering that that expression is indeed the general relationship between mass and velocity?

In other words, would the expression

$$m(\mathrm{v}) = \frac{m(0)}{\sqrt{1-(\mathrm{v}/c)^2}}$$

be the solution that satisfies equation (20)?

The answer is affirmative and it is left as an exercise (see homework-1 assignment.)

2.2.C.c Equivalence of mass and energy $\Delta E = (\Delta m)c^2$

How much does the mass of a particle increases when its new speed is much smaller than c?

$$m = m_o[1-(\mathrm{v}/c)^2]^{-1/2} \sim m_o[1+\tfrac{1}{2}(\mathrm{v}/c)^2 + ...]$$

$$\sim m_o + \underbrace{\tfrac{1}{2}m_o\mathrm{v}^2}_{\Delta(\mathrm{KE})}[\frac{1}{c^2}] + ...$$

Change in the (Newtonian) kinetic energy

That is, $\Delta m \sim \dfrac{\Delta KE}{c^2}$. This observation led Einstein to the more general suggestion that E= mc^2. For instance, the expression above can be rewritten as,

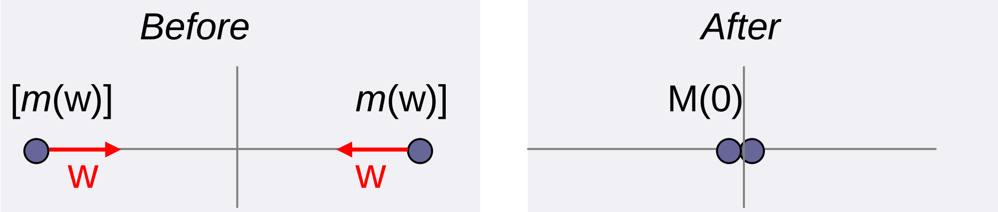
$$\underbrace{mc^2}_{\substack{\text{Interpreted as the}\\ \text{total energy of the}\\ \text{particle } E=mc^2}} = \underbrace{m_0\, c^2}_{\text{Intrinsic "rest energy"}} + ½\, m_0\, \mathrm{v}^2 + \ldots \qquad (22)$$

This theory of equivalence of mass and energy $E= mc^2$ has been verified by experiments in which matter is annihilated; that is mass at rest is converted totally to radiant energy:

> An electron and a positron come together at rest, each with a rest mass m_o. When they come together they disintegrate and two gamma rays emerge, each with the measured energy of $m_o c^2$.

Energy has inertia

Consider an inelastic collision



where $m(\mathrm{w}) = \dfrac{m(0)}{\sqrt{1-(\mathrm{w}/c)^2}}$

Energy= 2 [m(w)]c^2 Energy= [M(0)] c^2

Conservation of energy implies

M(0) = 2 m(w) > 2 m(0) (23)

Notice that, even though the two individual masses come to rest after the collision, the mass after the inelastic collision M(0) is greater than 2[m(0)]:

M(0) > 2 m(0)

[2m(w) - 2m(0)]c^2 is the kinetic energy brought in. (24)

Because of the kinetic energy involved in the collision, the resulting object M(0) will be heavier. M(0) > 2 m(0)

"When we put the two masses together gently they make something whose mass is 2 m_o; when we put them together forcefully, they make something whose mass is greater than 2m_o."

This is different than in Newtonian mechanics, where

two particles collide inelastically and form an object of mass 2m_o, which is in no way different form the one resulting putting them together slowly. There is more kinetic energy inside, but that does not affect the mass.

In Einstein's Mechanics, the mass of a system composed of two particles depend on how the particles were brought together (gently or violently). However, we can not always identify the “parts” inside an object of mass M. M could disintegrate in two, or in three particles.

It is not convenient, and often no possible, to separate the total energy Mc^2 of an object into *a)* rest energy of the inside pieces, *b)* kinetic energy of the pieces, *c)* potential energy of the pieces; instead we simply speak of the total energy of the particle.

2.2.C.d Relationships involving p, E and v

From the expressions for the relativistic energy in (18) we obtain,

$$E^2 = \frac{{m_o}^2}{1-(\mathrm{v}/c)^2}c^4 \quad \text{or} \quad \frac{E^2}{c^2} = \frac{{m_o}^2}{1-(\mathrm{v}/c)^2}c^2 \tag{25}$$

From the relativistic momentum expression we obtain,

$$\mathrm{P}^2 = \frac{{m_o}^2}{1-(\mathrm{v}/c)^2}\mathrm{v}^2 \tag{26}$$

Subtracting (26) from (25) we obtain $\frac{E^2}{c^2} - \mathrm{P}^2 = \frac{{m_o}^2}{1-(\mathrm{v}/c)^2}(c^2 - \mathrm{v}^2)$,

which gives,

$$\frac{E^2}{c^2} - \mathrm{P}^2 = {m_o}^2 c^2 \tag{27}$$

On the other hand, from $\vec{\mathbf{P}} = m\ \vec{\mathbf{v}}$ one can obtain $\vec{\mathbf{P}} = \frac{m\ c^2}{c^2}\vec{\mathbf{v}}$, or

$$\boldsymbol{P} = (E/c^2)\mathbf{v} \tag{28}$$

2.2.D Four-components vectors and symmetry

2.2.D.a Transformation of space-time coordinates

Notice the relativistic transformation of coordinates given in (12)

$$x' = \frac{x-ut}{\sqrt{1-(u/c)^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-ux/c^2}{\sqrt{1-(u/c)^2}}$$

can be expressed also as,

$$x' = \frac{x - (\frac{u}{c})\ ct}{\sqrt{1-(u/c)^2}}\ , \quad y' = y\ , \quad z' = z\ , \quad ct' = \frac{ct - \frac{u}{c}x}{\sqrt{1-(u/c)^2}} \quad (29)$$

If we denote the coordinates as

$x_1 = x$, $\quad x_2 = y$, $\quad x_3 = z$, $\quad x_o = ct$,

the transformation of coordinates

$(x_1,\ x_2\ ,\ x_3\ ,\ x_0) \quad \rightarrow \quad (x_1',\ x_2',\ x_3'\ , x_o'\ )$

adopts a more symmetric form,

$$x_1' = \frac{x_1 - \frac{u}{c}\ x_o}{\sqrt{1-(u/c)^2}}\ ; \quad x_2' = x_2\ ; \quad x_3' = x_3\ ; \quad x_o' = \frac{x_o - \frac{u}{c}x_1}{\sqrt{1-(u/c)^2}} \quad (30)$$

2.2.D.b Transformation of the energy-momentum coordinates

Starting from the transformation of velocities

$$V_{x'} = \frac{V_x - u}{1-(u\ /c^2)V_x} \qquad V_{y'} = \frac{V_y}{1-(u/c^2)V_x}\sqrt{1-(u/c)^2}$$

$$V_{z'} = \frac{V_z}{1-(u/c^2)V_x}\sqrt{1-(u/c)^2}$$

one can obtain (it is left as an exercise)

$$\frac{1}{\sqrt{1-\frac{1}{c^2}(V')^2}} = \frac{1-(u/c^2)\,V_x}{\sqrt{1-(u/c)^2}}\frac{1}{\sqrt{1-(V^2/c^2)}} \quad (31)$$

Note: Try to obtain (31) on your own. You may want to check your answer with the one given at the end of this chapter.

- The first energy-momentum transformation is obtained from the expression $m' = \dfrac{m_o}{\sqrt{1 - \dfrac{1}{c^2}(V')^2}}$.

  Together with (31) it becomes,

$$m' = \frac{[1-(u/c^2)V_x]}{\sqrt{1-(u/c)^2}} \frac{m_o}{\sqrt{1-(V^2/c^2)}} = \frac{m-(u/c^2)P_x}{\sqrt{1-(u/c)^2}}$$

  Hence $E' = m'c^2 = \dfrac{mc^2 - (u)\ P_x}{\sqrt{1-(u/c)^2}}$, or

$$E' = \frac{E - uP_x}{\sqrt{1-(u/c)^2}} \tag{32}$$

- Another energy-momentum transformation is obtained using $P_{x'} = m'V_{x'} = \dfrac{m_o}{\sqrt{1 - \dfrac{1}{c^2}(V')^2}} V_{x'}$.

  With the help of (31) and $V_{x'} = \dfrac{V_x - u}{1-(u\ /c^2)V_x}$ one obtains,

$$P_{x'} = \frac{P_x - (u/c^2)E}{\sqrt{1-(u/c)^2}} \tag{33}$$

- The remaining two expressions are more straightforward to demonstrate

$$P_{y'} = P_y\,, \quad P_{z'} = P_z \tag{34}$$

They are obtained from, for example, using

$P_{y'} = m' V_{y'} = \dfrac{m_o}{\sqrt{1-\frac{1}{c^2}(V')^2}} V_{y'}$ and then expression (31), etc.

Note: You are encouraged to verify on your own those last expressions.)

In summary,

$$(P_x,\ P_y,\ P_z,\ m) \rightarrow (P_x',\ P_y',\ P_z',\ m')$$

$$P_{x'} = \frac{P_x - um}{\sqrt{1-(u/c)^2}}, \quad P_{y'} = P_y, \quad P_{z'} = P_z, \quad m' = \frac{m-(u/c^2)P_x}{\sqrt{1-(u/c)^2}} \quad (35)$$

Notice the similarity with the coordinates transformation

$$(x,\ y,\ z, t) \rightarrow (t', x',\ y',\ z', t')$$

$$x' = \frac{x-ut}{\sqrt{1-(u/c)^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-(u/c^2)x}{\sqrt{1-(u/c)^2}}$$

The symmetry is more straightforward using the variables *mc* and *ct,*

$$(P_x,\ P_y,\ P_z,\ mc) \rightarrow (P_x',\ P_y',\ P_z',\ m'c)$$

$$P_{x'} = \frac{P_x - (\frac{u}{c})mc}{\sqrt{1-(\frac{u}{c})^2}}, \quad P_{y'} = P_y, \quad P_{z'} = P_z, \quad m'c = \frac{mc-(\frac{u}{c})P_x}{\sqrt{1-(\frac{u}{c})^2}} \quad (24)$$

$$(x,\ y,\ z, ct) \rightarrow (t', x',\ y',\ z', ct')$$

$$x' = \frac{x-(\frac{u}{c})\ ct}{\sqrt{1-(\frac{u}{c})^2}}, \quad y' = y, \quad z' = z, \quad ct' = \frac{ct-(\frac{u}{c})x}{\sqrt{1-(\frac{u}{c})^2}}$$

APPENDIX-1 Optional procedure to obtain expression (31)

1. Starting from the velocities transformation,

$$V_{x'}=\frac{V_x-u}{1-(u\ /c^2)V_x}\;;\qquad V_{y'}=\frac{\sqrt{1-(u/c)^2}}{1-(u/c^2)V_x}V_y\quad;\qquad V_{z'}=\frac{\sqrt{1-(u/c)^2}}{1-(u/c^2)V_x}V_z$$

1A. Demonstrate $1-\frac{1}{c^2}(V_{x'})^2=\frac{1-(u/c)^2}{[1-(u/c^2)V_x]^2}[1-(V_x{}^2/c^2)]$.

1B. Demonstrate $1-\frac{1}{c^2}(V')^2=\frac{1-(u/c)^2}{[1-(u/c^2)V_x]^2}[1-(V^2/c^2)]$,

or equivalently

$$\frac{1}{\sqrt{1-\frac{1}{c^2}(V')^2}}=\frac{1-(u/c^2)\,V_x}{\sqrt{1-(u/c)^2}}\frac{1}{\sqrt{1-(V^2/c^2)}}\,.$$

Solution

$$1-\frac{1}{c^2}(V_{x'})^2=1-\frac{1}{c^2}\left[\frac{V_x-u_o}{1-(u/c^2)V_x}\right]^2=$$

$$=\frac{c^2[1-2(u/c^2)V_x+(u/c^2)^2V_x{}^2]\ -\ [V_x{}^2-2uV_x+u^2]}{c^2[1-(u/c^2)V_x]^2}$$

$$=\frac{c^2[1+(u/c^2)^2V_x{}^2]-[V_x{}^2+u^2]}{c^2[1-(u/c^2)V_x]^2}=\frac{[c^2+(u/c)^2V_x{}^2]-[V_x{}^2+u^2]}{c^2[1-(u/c^2)V_x]^2}$$

$$=\frac{[c^2-u^2]-[1-(u/c)^2]V_x{}^2}{c^2[1-(u/c^2)V_x]^2}=\frac{[1-(u/c)^2][1-(V_x{}^2/c^2)]}{[1-(u/c^2)V_x]^2}$$

From the last expression one obtains,

$$1-\frac{1}{c^2}(V')^2=\frac{[1-(u/c)^2][1-(V^2/c^2)]}{[1-(u/c^2)V_x]^2}$$

Or,

$$\frac{1}{\sqrt{1-\frac{1}{c^2}(V')^2}} = \frac{[1-(u/c^2)V_x]}{\sqrt{1-(u/c)^2}\sqrt{1-(V^2/c^2)}}$$

REFERENCES

[1] This description follows closely Feynman Lectures, Vol - I, Chapters 15 and 16.
[2] See also "Appendix A" of this course's textbook R. Eisberg and R. Resnick, "Quantum Physics,"
[3] Ref 1 (Feynman Lectures) Vol - I, p.15-1.
[4] J. D. Jackson, "Classical Electrodynamics," 3rd Ed.; John Wiley and Sons; page 516
[5] Ref. 4, page 533.