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PERTURBATIVE-VARIATIONAL CALCULATIONS OF GROUND-STATE ENERGIES OF LATTICE GAUGE THEORIES

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We apply the perturbative variational approximation scheme of Bessis and Villani to Z(2) gauge theory on a lattice in $2+1$ and $3+1$ dimensions and to a truncated version of SU(2) gauge theory in $3+1$ dimensions. We find that the BV scheme tends to distinguish between first- and second-order transition already in the first approximant.

1. Introduction

Recent Monte Carlo studies of lattice gauge theories [1] have greatly increased everyone's confidence in the existence of confinement in continuum non-abelian gauge theories. One of the most surprising features of the Monte Carlo calculations is their apparent ability to extract information about the continuum theory from a *finite* lattice system. At the $g_0 = 0$ fixed point which defines the continuum theory, the correlation length (as well as all particle Compton wavelengths) blows up like

$$e^{b_0/g_0^2}, \qquad \text{(in lattice units)}.$$

It seems clear that if we put the system in a box smaller than its correlation length we will distort its spectrum. Thus, Monte Carlo calculations, which for practical reasons are limited to rather small boxes, can only be successful at values of g_0 for which e^{b_0/g_0^2} is smaller than the size of the box. On the other hand, when g_0 is not small we might expect corrections to the continuum spectrum of the theory. Dimensionless physical quantities (e.g. mass ratios) are given by formulas of the type

$$R = R_0 + F(g_0),$$

where $F(0) = 0$ and R_0 is the continuum value. For values of g_0 for which a finite lattice calculation makes sense we have no *a priori* reason for believing that F is small.

The apparent successes of Monte Carlo calculations therefore indicate that the F appropriate to the ratio between the QCD scale parameter Λ and the Regge slope *is* small in the region of g_0 where the calculations are valid. If this result remains valid for other mass ratios (and it must if the Monte Carlo calculations are to succeed) then we can conclude that continuum QCD is well approximated by lattice QCD in a box

of size 5^4–10^4 at a value of g_0 below the weak to strong coupling transition region [2] but large enough so that the correlation length is smaller than the box length.

Assuming that this is true, one may envisage other methods of solving weak coupling QCD in a finite box. In four dimensions, exact numerical diagonalization of the hamiltonian is a hopeless task (even for a Z_2 gauge theory) for essentially the same reasons that Monte Carlo methods are successful. Instead we propose to use a perturbative-variational method invented by Bessis and Villani [3].

The BV method is remarkably accurate in few-particle quantum mechanics problems but it suffers from a serious defect when applied to infinite systems. In any finite order the ground-state energy is not automatically extensive and excitation energies are not intensive. Bessis and Villani have proposed that one exploit this property to get further variational bounds on the ground-state energy. For many systems one can prove that the ground-state energy density is monotonic in the volume. Finite order BV approximants violate this monotonicity. A simple graphical argument then shows that the most quickly convergent approximation to the infinite volume, ground-state energy density is obtained by evaluating the Nth BV approximant in a box of size V_N, where V_N is chosen to minimize the Nth-order energy density.

Last year, Kaplunovsky applied the BV method to an exactly soluble lattice system: the one-dimensional Ising model in a transverse field. His results for the ground-state energy density are impressive. The maximum error in this quantity is 0.1% even in the region of the phase transition (he uses 9th-order BV approximants). He found that the optimal value of V_N was always $N+1$ sites, i.e. one site more than the order of perturbation theory. Thus one must attribute the accuracy of his results to the fact that the ground-state energy density of the 10 site Ising model is already equal to that of the infinite lattice to an accuracy of one part in a thousand.

Although the result $V_N = N+1$ is not understood theoretically, one may expect it to hold true for other systems. In this case the BV method will only be useful for those properties of infinite volume systems which are well approximated by their restrictions to small volumes. If the Monte Carlo calculations are to be believed, this will be true for interesting quantities in QCD. In the present paper we apply the BV approximants to Z_2 lattice gauge theory hamiltonians in $2+1$ and $3+1$ dimensions. We compute low-order approximants but find interesting results. In particular, in $3+1$ dimensions we give further evidence that the Z_2 phase transition is first order as indicated by various other methods. The value of the transition coupling is improved with respect to mean field theory. In $2+1$ dimensions we find evidence that the transition is second order.

The rest of this paper is organized as follows. In sect. 2 we explain the BV method and choose our trial state for the lattice gauge theories. In sect. 3 we present our results for both the $2+1$ and $3+1$ dimensional theories. In sect. 4 we examine the result of dropping the variational part of the BV ansatz. We use the strong coupling vacuum as a trial state for arbitrary g. This would enable us to go to much higher

order. We apply our method to a truncated SU(2) theory recently introduced by Horn [4]. Calculations in this theory in 3 + 1 dimensions do not indicate a softening of the first-order transition found in mean field theory. In appendix A we discuss some controversial features of the mean field approximation.

2. The Bessis–Villani approximation

The BV approximants are a general method for finding the discrete spectrum of a semibounded hamiltonian. Consider the expectation value of the resolvent operator in an arbitrary state $|\phi\rangle$:

$$R_\phi(Z) = \langle\phi|\frac{1}{Z-H}|\phi\rangle\,. \tag{1}$$

$R_\phi(Z)$ will have a pole for each Z in the discrete spectrum whose eigenstate has a non-trivial overlap with $|\phi\rangle$. Denote these eigenenergies by E_n, $E_0 \leqslant E_1 \leqslant E_2 \leqslant \cdots$. We can compute the function R_ϕ if we know the moments

$$\langle\phi|H^n|\phi\rangle = \mu_n \tag{2}$$

[and if they grow more slowly than $(2n)!$]. Define

$$R_\phi^N(Z) = [N-1/N, \mu_n](Z)\,,$$

the $[N-1/N]$ Padé approximant to $R_\phi(Z)$. Let $E_n^{N,\phi}$ be the poles of R_ϕ^N. Bessis and Villani have proven that

$$E_n \leqslant E_n^{N,\phi} \leqslant E_n^{N-1,\phi} \tag{3}$$

for each N and ϕ. Furthermore, the $E_n^{N,\phi}$ converge to E_n as $N \to \infty$.

This theorem has been proven for a very general class of hamiltonians which includes all *finite lattice* gauge theories. In many cases one can show that $|E_n - E_n^{N,\phi}| = \mathrm{O}(\mathrm{e}^{-N})$. Eq. (3) has an obvious variational meaning. If we allow ϕ to depend on some parameters, the best result in any given order is obtained by minimizing $E_n^{N,\phi}$ with respect to the parameters.

The lowest order approximation ($N = 1$) is easily seen to be equivalent to the Rayleigh–Ritz method with trial state $|\phi\rangle$. It gives an extensive ground-state energy for an infinite volume system if we choose a reasonable trial state. This is not necessarily true for the higher order approximants.

Actually, for higher order approximants one finds that $E_n^{N,\phi}$ becomes extensive in the limit $V \to \infty$ only for some choices of $|\phi\rangle$. It depends on V for finite values, in a rather complicated way. This property of the BV method can (and should) be turned into an advantage. Realizing the variational character of the approximation, one is led to the idea of using the volume of the system as a variational parameter, i.e. if we take for each N (the order of the approximant) a volume V_N which minimizes the energy density, we will get a better bound on the true energy density of the infinite system. Unfortunately, this is easier said than done. Our calculations are performed

with periodic boundary conditions (hypertorus) and if we allow our volume to become smaller than $V_N^* = (N+1)^d$, where N is the order of perturbation theory and d the dimensionality of space, we have to take into account graphs which are "self-overlapping" and calculations of phase space become practically impossible. Still, we can allow V_N to vary with the proviso $V_N \geq (N+1)^d$. Kaplunovski, doing this, found that for the Ising model in 1d the energy density decreases monotonically with V for the allowed range up to, and including, 9th order. In our calculation we find the same phenomenon and we believe it to be a general feature although we do not know how to prove it. It is, however, a reasonable hope that for $N \to \infty$ the "best volume" will rapidly approach the allowed range $[V_N \geq (N+1)^d]$. That this should be so must be clear from the discussion in the introduction (irrelevance of the boundary and so on). In fig. 1 we present, schematically, the dependence of the (ground-state) energy density upon the volume for increasing orders in the BV approximations.

The Z_2 lattice gauge theory hamiltonian is

$$H = -\sum_{x,\mu} \sigma_\mu^1(x) - g \sum_{x,\mu\nu} \sigma_\mu^2(x)\sigma_\nu^3(x+\mu)\sigma_\mu^3(x+\nu)\sigma_\nu^3(x)\,, \tag{4}$$

where $\sigma_\mu^i(x)$ is the gauge field variable on the link which emanates from the lattice site x in the positive μ direction. The gauge-invariant subspace is the set of vectors satisying

$$\sigma_\mu^1(x-\mu)\sigma_\nu^1(x-\nu)\sigma_\mu^1(x)\sigma_\nu^1(x)|\psi\rangle = |\psi\rangle\,. \tag{5}$$

We will choose our trial state to be a product state

$$|\phi\rangle = \prod_{x,\mu} \begin{vmatrix}\cos\theta\\ \sin\theta\end{vmatrix}_{x,\mu}\,. \tag{6}$$

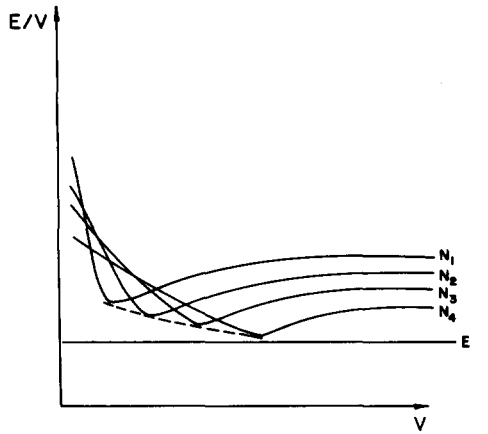

Fig. 1. Energy density versus volume for different orders in the BV approximation.

In zeroth order the BV variational principle for such a state produces the mean field approximation (MFA). This approximation, which was first introduced in lattice gauge theory by Wilson [5] and Balian, Drouffe and Itzykson [6] is widely considered to be an unreliable guide to the phase structure of lattice gauge theory. We do not agree with this estimate of the utility of the MFA (for an explanation of our views see the appendix). For the present purpose, however, one's political views of the MFA are irrelevant. In the BV method the trial state $|\phi\rangle$ is merely required to have non-trivial overlap with the true ground state. There is nothing in the nature of a product state which prevents such an overlap.

From this point onward everything is straightforward. To find the Nth BV approximant ($[N-1/N]$ Padé) we must compute the first $2N+1$ moments of the hamiltonian, find the poles of the $[N-1/N]$ Padé as functions of θ, and vary with respect to θ. Our results for the $2+1$ and $3+1$ dimensional Z_2 theories are described in sect. 3.

3. The ground-state energy of the Z_2 gauge theory in 3+1 and 2+1 dimensions

The Z_2 gauge theory has been studied extensively, both analytically [7] and numerically [8]. The $(2+1)$d theory is dual to the 2d Ising model and consequently has a 2nd-order phase transition at $g = 3.125$. The $(3+1)$d theory is self-dual and has a single 1st-order phase transition at the self-dual point $g = 1$.

In the mean field approximation (or the zeroth BV approximant) both theories exhibit a first-order phase transition. Using our trial wave function we get

$$E(g, \theta) = \langle\phi|H|\phi\rangle = -N_{\mathrm{L}} \sin(2\theta) - gN_{\mathrm{P}} \cos^4(2\theta), \tag{7}$$

with N_{L} = number of links and N_{P} = number of plaquets. In d space dimensions $N_{\mathrm{L}} = [2/(\mathrm{d}-1)]N_{\mathrm{P}}$ and $N_{\mathrm{P}} = 2dV$; thus.

$$E(g, \theta) = V\left(-\frac{4d}{d-1}\sin(2\theta) - 2dg\cos^4(2\theta)\right). \tag{8}$$

The mean field ground-state energy is $E_{\mathrm{MF}}(g) = \mathrm{Min}_\theta E(g, \theta)$ and elementary calculation yields a 1st-order phase transition at $g^* = 27/[16(d-1)]$ which is $g = 1.69$ for $d = 2$ and $g = 0.84$ for $d = 3$. The energy is constant for $g < g^*$ and almost linear in g for $g > g^*$.

We have calculated the next order BV approximant ($N = 3$). The results (ground-state energy density) are plotted in figs. 2 ($d = 2$) and 3 ($d = 3$) against the MF result of the same model. Our results show a clear cut difference between the 2- and 3-dimensional model. In the 2d model one observes two important features: a sizable improvement in the critical coupling value (1.69 MFT in comparison to 2.11) and a considerable softening of the first-order phase transition with respect to the MFT

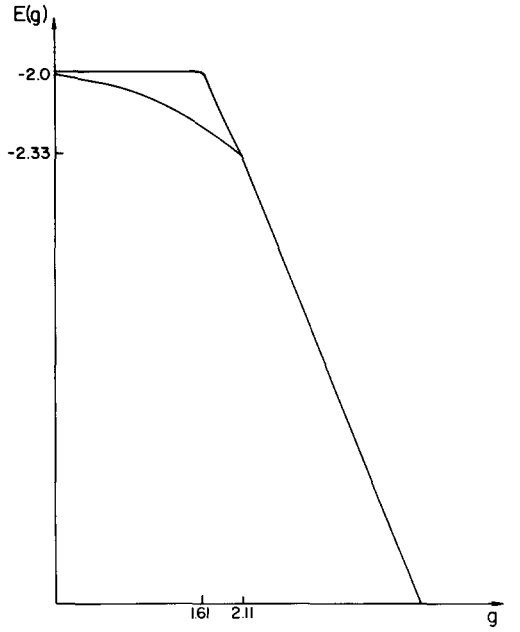

Fig. 2. Z(2) in 2d 1st BV approximant versus MF for ground-state energy.

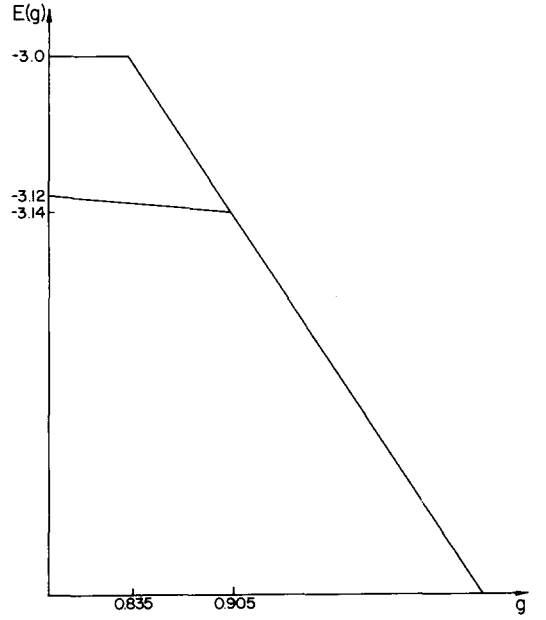

Fig. 3. Z(2) in 3d 1st BV approximant versus MF for ground-state energy.

result, or in statistical mechanics terminology, a decrease in the latent heat of the transition. For the 3d theory, one observes again a shift of the transition point (from 0.835 to 0.905) towards the exact result but, in contrast to the 2d case, there is no decrease in the latent heat. It is gratifying to see that already the first non-trivial BV approximant can distinguish so nicely between 1st- and 2nd-order transitions although the zeroth-order approximant (MFT) predicts 1st-order transition for both cases. We believe that this is a general feature of the BV method and consequently we believe it may be a valuable tool to check MFT results whenever it appears to predict a 1st-order transition.

Finally, we observe that in the strong and weak coupling regions the results coincide with MFT results as one would expect.

To summarize, the BV approximation for the ground-state energy density seems to have the strongest effect in the problematic region of the coupling constant near the critical point. Its effect seems to make a noticeable improvement on the MFT approximation with the aforementioned distinction between 1st- and 2nd-order transitions.

In figs. 4 and 5 we present the $N = 3$ BV results for the ground-state energy densities of the 2d and 3d Z_2 gauge theories with different values of the volume. The results demonstrate our statement from sect. 2 that this quantity increases monotonically with the volume and that the minimal volume allowed by our method of calculation (in the present case $V = 4^d$) gives the best result.

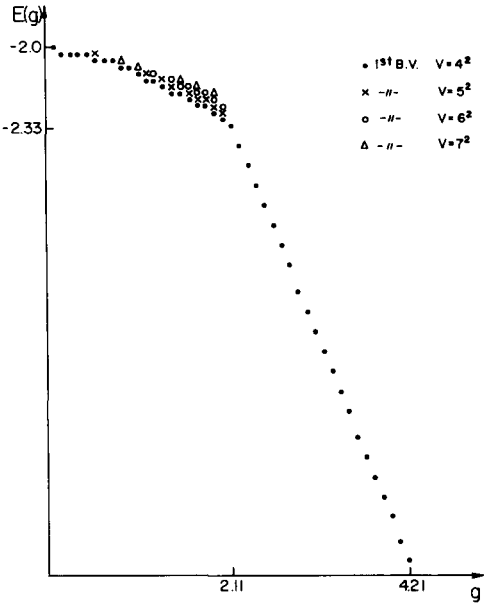

Fig. 4. Ground-state energy in the 2d Z(2) theory for different volumes.

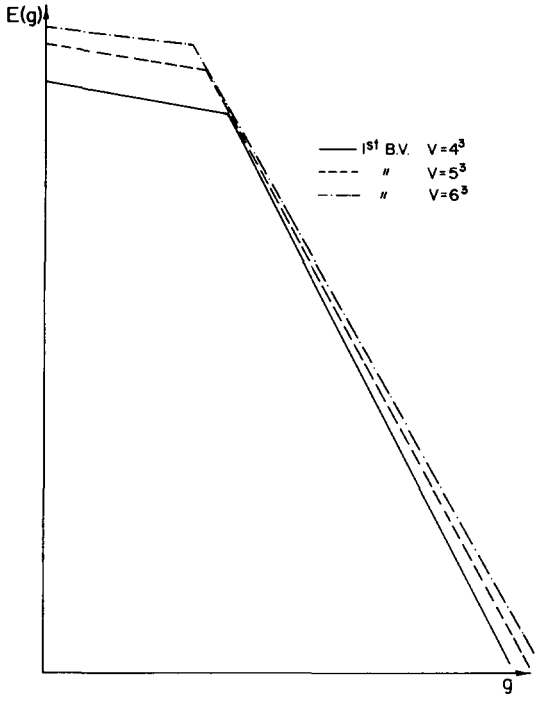

Fig. 5. Ground-state energy in the 3d Z(2) theory for different volumes.

In calculating higher order BV approximants, one of course improves the bound on the ground-state energy, but there is also a bonus. Each order provides a bound on a new energy level (and improves the bounds on previous ones). As a rule, the results for excited states are considerably worse than for the ground state. In fig. 6 we present our results for the energy of the first excited state. We believe that one has to go to higher order approximants before one can draw any conclusions from the behaviour of this quantity.

4. The BV approximation with non-variational wave function

It goes without saying that it would be desirable to push the calculation to higher orders. However, this is a rather difficult proposition. Our choice of variational wave function makes the evaluation of matrix elements, and phase-space factors much more tedious and complicated (and thus error prone) than similar strong coupling perturbation calculations. To demonstrate it we note that calculating μ_1 involves 2 terms, μ_2 6 terms, μ_3 17 terms, μ_4 48 terms, μ_5 136 terms. Pushing the approximation every step further involves calculating two additional moments (from 1st to 2nd it means going from 25 to 209 terms) and that the higher the moment one calculates,

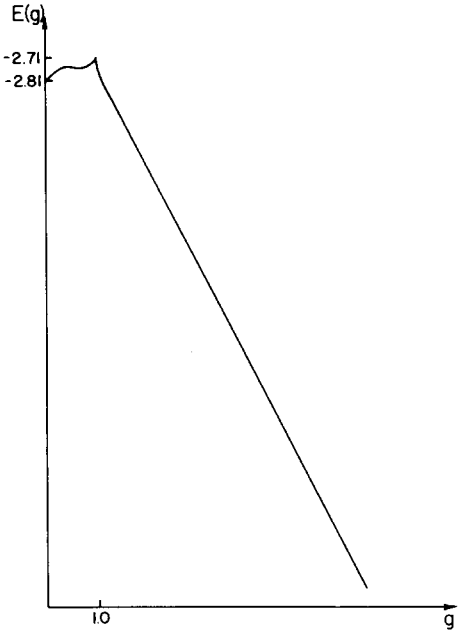

Fig. 6. First excited state in the 2d Z(2) theory.

the more complicated are the phase-space factors. One is led to look for a simpler scheme.

An obvious simplification is to abandon the variational aspect and to work with a fixed and simple wave function. One still gets bound on the energy and there may be a hope that the price one pays in precision may be compensated by the ability to push the calculation to a higher order.

We thus calculated the 1st non-trivial BV approximant for $(2+1)$d Z_2 gauge theory with a fixed wave function. We tried both weak and strong coupling ($g \to \infty$ and $g \to 0$, respectively, in our convention) wave functions. The calculation with weak coupling wave functions gives bad results; it does not exhibit any phase transition. The calculation with the strong coupling wave function is somewhat better, it exhibits a phase transition at $g = 0.81$. The first-order phase transition is softer than mean field theory (small latent heat) but the large-g behaviour is wrong and we do not know whether this is responsible for the softness of transition.

Fig. 7a presents the results of this calculation against the result of the variational calculation. We calculated the next BV approximant with the strong coupling wave function. The result is presented in fig. 7b superimposed on the 1st-order BV approximant. It is plain to see that the improvement is not dramatic but it is instructive to observe that corrections are large where the approximation fails (large g) and small where it is already good (small g). This is certainly a promising feature and there is no reason why it should not persist to higher orders.

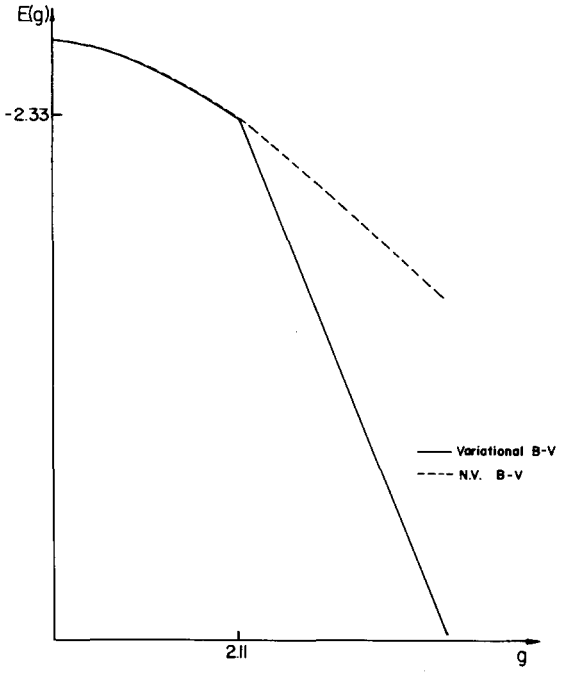

Fig. 7a. Non-variational versus variational ground-state energy in the 2d Z(2) theory.

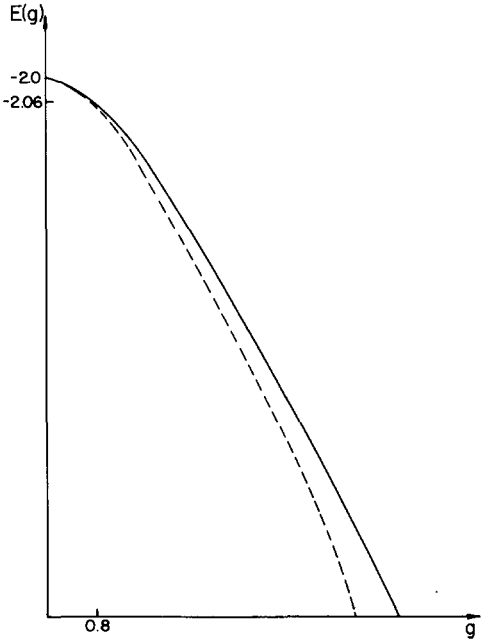

Fig. 7b. 1st and 2nd BV approximant for the ground-state energy in the 2d Z(2) theory with non-variational (strong coupling) wave function.

To summarize this section, we observe that the first BV approximant with variational wave function is far better than the second with fixed (strong coupling) wave function. However, it is easier to calculate the fourth approximant with the strong coupling wave function than even the second one with the variational wave function so, as far as we have gone it is not clear which is the best approach.

5. Finite matrix representation of SU(2) gauge theory

Recently Horn introduced a truncated version of the SU(2) Kogut-Susskind hamiltonian. The idea was originated by Barber and Richardson for spin systems on the lattice. The rationale of this idea goes along the following lines. Consider a spin system hamiltonian, e.g. O(3),

$$H = \sum_x \boldsymbol{J}^2(x) - g \sum_{x,\mu} \boldsymbol{n}(x) \cdot \boldsymbol{n}(x+\mu) \tag{9}$$

$$|\boldsymbol{n}(x)| = 1\,, \qquad [\hat{\boldsymbol{\alpha}} \cdot \boldsymbol{J}(x)_i \boldsymbol{n}(x')] = \hat{\boldsymbol{\alpha}} \times \boldsymbol{n}(x) \delta_{x,x'}\,. \tag{10}$$

Now consider a (O(3) invariant) truncation of H to states with l less than some $l_{\max}$. This is equivalent to finding a finite matrix representation for the commutation relation with the spectrum of $\boldsymbol{J}^2$ being the allowed values of the Casimir on each site. Then, one can use renormalization group arguments to claim that the large-distance behaviour of the truncated theory will not be different from the original theory. The large-distance behaviour (near a critical point) is controlled by large spin blocks interacting as effective degrees of freedom. These degrees of freedom may have arbitrarily high spin even though the individual sites have finite angular momentum.

The Kogut–Susskind hamiltonian for the SU(2) gauge theory is

$$H = g^2 \sum_{x,\mu} \boldsymbol{E}_\mu^2(x) - \frac{1}{g^2} \sum_{x,\mu\nu} \operatorname{tr}\left(U_\mu(x) U_\nu(x+\mu) U_\mu^+(x+\nu) U_\nu^+(x)\right), \tag{11}$$

with the well-known commutation relations

$$[E_\mu^a(x),\, U_\mu(x')] = \tfrac{1}{2}\sigma^a U_\mu(x')\, \delta_{x,x'}\,, \tag{12}$$

$$[E_{-\mu}^a(x+\mu),\, U_\mu(x')] = U_\mu(x') \tfrac{1}{2}\sigma^a\, \delta_{x,x'}\,, \tag{13}$$

$$[E_{\pm\mu}^a(x),\, E_{\pm\mu}^b(x')] = i\varepsilon^{abc} E_{\pm\mu}^c(x')\, \delta_{x,x'}\,. \tag{14}$$

Horn found a finite dimensional representation of these commutation relations. His solution contains one singlet and two doublets [there must be two doublets, the symmetry of the link variables on the lattice is SU(2)×SU(2)]. We will now briefly present Horn's version of the finite dimensional SU(2) hamiltonian.

Let us define the matrices:

$$(L_m)_{\alpha\beta} = -i\varepsilon_{m\alpha\beta}\,, \qquad (N_m)_{\alpha\beta} = i(\delta_{\alpha,4}\,\delta_{\beta,m} - \delta_{\alpha,m}\,\delta_{\beta,4})\,. \tag{15}$$

(Latin indices take values 1, 2, 3, whereas Greek indices run from 0 to 4). One finds

$$[L_i, L_j] = i\varepsilon_{ijk}L_k\,, \qquad [L_i, N_j] = i\varepsilon_{ijk}N_k\,, \qquad [N_i, N_j] = i\varepsilon_{ijk}L_k\,. \tag{16}$$

It follows that

$$J_i = \tfrac{1}{2}(L_i + N_i)\,, \qquad K_i = \tfrac{1}{2}(L_i - N_i) \tag{17}$$

are generators of two commuting SU(2) groups.

Now define the 4 hermitian matrices

$$(M_k)_{\alpha\beta} = \delta_{\alpha,0}\,\delta_{\beta,k} + \delta_{\alpha,k}\,\delta_{\beta,0}\,, \qquad (M_4)_{\alpha,\beta} = \delta_{\alpha,0}\,\delta_{\beta,4} + \delta_{\alpha,4}\,\delta_{\beta,0}\,, \tag{18}$$

and

$$V = M_4 + i\sigma_k M_k\,. \tag{19}$$

It is straightforward to verify that

$$[J_i, V] = \tfrac{1}{2}\sigma_i V, \qquad [K_i, V] = V\tfrac{1}{2}\sigma_i\,. \tag{20}$$

The hamiltonian in terms of these variables is

$$H = \sum_{\text{links}} J(\ell)^2 - g \sum_{\text{plaq}} (W(\mathrm{p}) + W^+(\mathrm{p})) \tag{21}$$

$$W(\mathrm{p}) = \mathrm{Tr}_\sigma(V(1)V(2)V^+(3)V^+(4))\,, \qquad J^2 = \begin{pmatrix} 0 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & \ddots \end{pmatrix}, \tag{22}$$

where Tr_σ indicates trace over the Pauli matrices.

We applied the BV approximation to this model and discovered that the first-order transition found in MFT is considerably "harder" in the next order approximation (fig. 8). We have also calculated the one-plaquette Wilson loop (fig. 9) and the same phenomenon manifests itself in increase of the discontinuity of the Wilson loop across the transition.

There are several remarks in order. If the intuitive argument presented at the beginning of this section is correct, one would expect the truncated theory to have the same phase stucture as the full SU(2) theory and the same critical indices. Consequently, one would expect that a good approximation scheme will smooth a first-order transition and push the critical point towards the weak coupling region ($g = \infty$ in our convention) as one goes to higher orders. This then may be interpreted as an indication that the theory has a critical point at $g = \infty$. The BV approximation does not bear out those expectations. There may be several explanations to this. One can observe that together with the increase of latent heat (from 0.10 to 0.25) the critical point moves towards $g = 0$ (strong coupling). This may be taken as an indication that what we have found in MF (and in the 1st BV approximant) is a

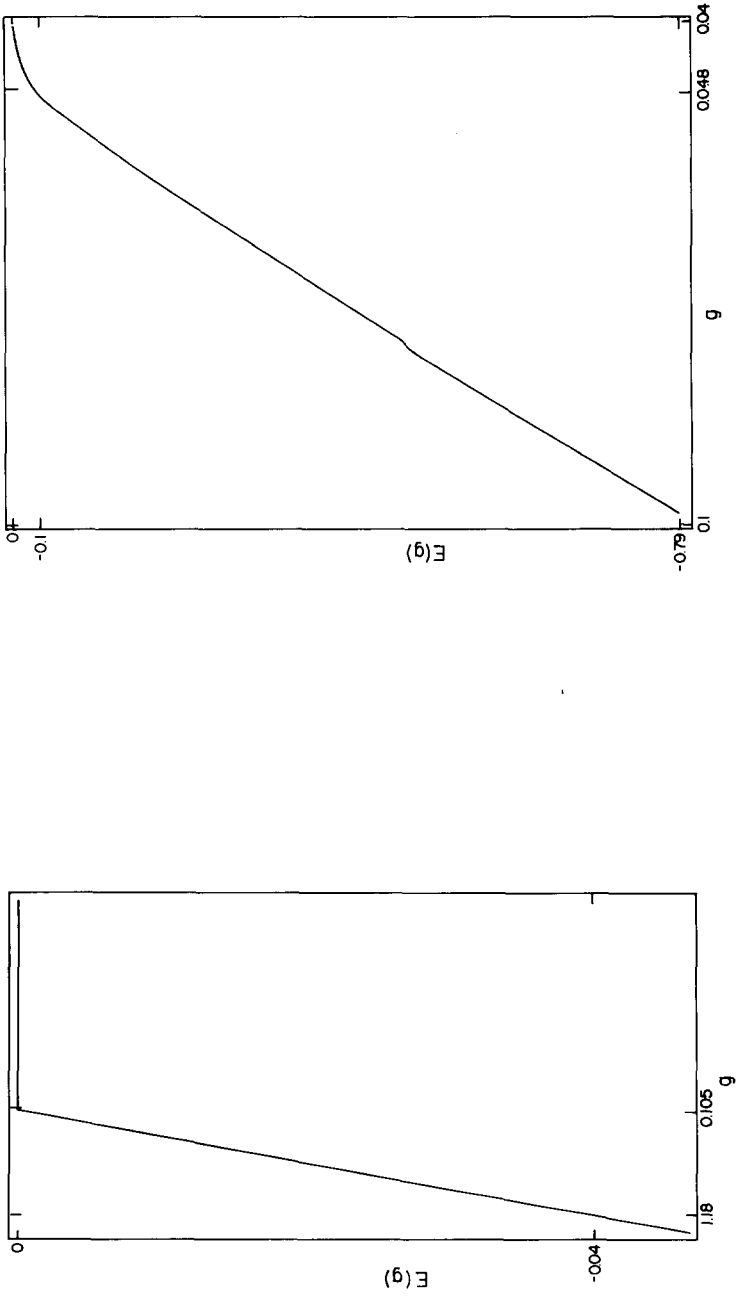

Fig. 8. (a) MF and (b) 1st BV approximant for ground-state energy in the truncated SU(2) theory.

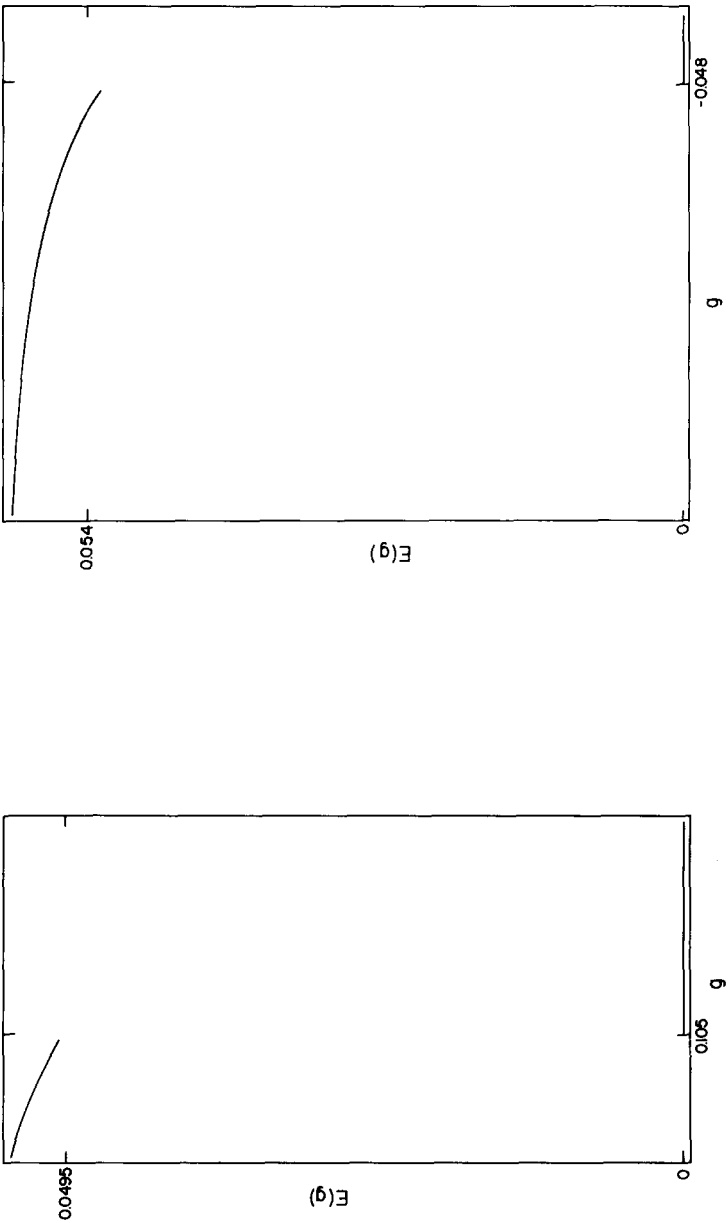

Fig. 9. One-plaquette Wilson loop in the truncated SU(2) theory. (a) MF (b) 1st BV approx.

fictitious first-order phase transition which will be pushed towards infinite coupling ($g = 0$) as we go to higher orders and will disappear from the theory. In this scenario one assumes that the transition found in MF theory has nothing to do with the critical point of the full SU(2) gauge theory at $g = \infty$.

Another possibility is that this transition is a real 1st-order transition of the truncated theory. This does not make the argument at the beginning of this section invalid, it merely states that the phase structure of the truncated theory is more complex than the phase structure of the conventional one [i.e. if the SU(2) gauge theory does not have a 1st-order transition] and that the BV approximation does not find the postulated critical point.

There may, however, be still a third possibility. One may argue that the transition found is indeed the $g = \infty$ critical point and that the "wrong" behaviour of the 1st BV approximant will be "corrected" in the next order. The reason for this may rest in the fact that the magnetic term in the hamiltonian is not a unitary operator [see eq. (19)]. Consequently, it has eigenvalues inside the unit circle which should be irrelevant near the critical point and are not so for finite order approximation. This may cause the convergence to the critical point to oscillate instead of being monotonous. Horn and Katznelson [10], using mean field methods for the same model found the oscillatory behaviour we envision. To decide the matter one should calculate the next BV approximant to this model.

6. Conclusions

In this work we examined the BV approximation by applying it to the Z_2 gauge theories in $2+1$ and $3+1$ dimensions and to Horn's truncated SU(2) hamiltonian. As far as numerical results go, we seem to obtain quite good numbers even in the low order to which we are calculating (in comparison to strong coupling and mean field calculations). But far more interesting are the qualitative aspects of our calculations. It seems that the 1st BV approximant is already very sensitive to the behaviour of the investigated theory near its critical point(s). This gives us hope that application of the BV scheme to the SU(2) Kogut–Susskind Hamiltonian may produce a sizeable improvement on the mean field and strong coupling perturbation calculation. It may be hoped that pushing the calculation with a variational wave function to higher orders will produce results comparable to those of the Monte Carlo calculations. In order to do that, one has to find a good variational wave function. It is essential that the wave function will interpolate between the strong coupling and weak coupling wave functions and, for practical reasons, will not depend on too many parameters. Still, remembering that the SU(2) hamiltonian has infinitely many states per link, the BV scheme may not work as well as in the present case. We hope to investigate the SU(2) theory in a future publication.

Appendix

THE MEAN FIELD APPROXIMATION

The MFA is widely regarded with suspicion because it is not a gauge-invariant approximation and particularly because it violates Elitzur's theorem, which requires all non-gauge-invariant operators to have zero expectation value. We would like to exhibit the MFA as a limit of a perfectly sensible gauge-invariant approximation scheme. To this end consider the Z_2 Higgs model,

$$H = g^2 \Sigma \sigma_\mu^1 + \frac{1}{g^2} \Sigma \sigma_{\mu\nu}^3 + \frac{1}{h^2} \Sigma \tau_x^1 + h^2 \Sigma \tau_x^3 \sigma_\mu^3 \tau_{x+\mu}^3 . \tag{A.1}$$

Now make a Stueckelberg transformation,

$$\Sigma_\mu^1 = \sigma_\mu^1 , \qquad \Sigma_\mu^3(x) = \tau_x^3 \sigma_\mu^3(x) \tau_{x+\mu}^3 , \tag{A.2}$$

$$T^3 = \tau^3 . \tag{A.3}$$

Note that Σ_μ is a gauge-invariant variable. On the subspace of gauge-invariant states which satisfy

$$\prod_\mu \sigma_\mu^1(x-\mu)\sigma_\mu^1(x)|\psi\rangle = \tau^1(x)|\psi\rangle , \tag{A.4}$$

the hamiltonian can be written entirely in terms of Σ_μ

$$H = g^2 \sum_{x,\mu} \Sigma_\mu^1(x) + \frac{1}{g^2} \sum_{x,\mu\nu} \Sigma_{\mu\nu}^3(x) + \frac{1}{h^2} \sum_x \prod_\mu \Sigma_\mu^1(x-\mu)\Sigma_\mu^1(x) + h^2 \sum_{x,\mu} \Sigma_\mu^3(x) . \tag{A.5}$$

We now formulate a simple minded approximation for this *gauge-invariant* hamiltonian. For large h^2, and finite g^2, the term $h^2 \Sigma_\mu^3$ dominates the hamiltonian and its ground state is a product state of the form (6). For large h^2 and g^2 it is still a one-link hamiltonian and its ground state is again a product. For small g^2 and large h^2 we can diagonalize Σ_μ^3 on each link and the *ground state* is again a product. Thus, throughout the large h^2 region, (6) is a sensible ansatz for the ground state. Applying the variational principle we obtain a gauge-invariant MFA for the Higgs model.

The phase diagram resulting from this approximation is shown in fig. 10b. It gives a line of first-order transitions ending in a critical point. Note in particular that the Higgs ($h^2 \gg 1$, $g^2 \ll 1$) and confinement ($h^2 \ll 1$, $g^2 \gg 1$) phases are analytically connected. This rather surprising sounding result was originally discovered by much more sophisticated methods.

The actual phase diagram of this theory is believed to be that of fig. 10a. The critical couplings on the axes are of order $g_c^2 \propto d\, h_c^2 \propto 1/d$. Thus, for large d the region where the MFA is untrustworthy is very small. At $h = 0$ the approximation predicts a vacuum expectation for U_μ which violates Elitzur's theorem. Nonetheless,

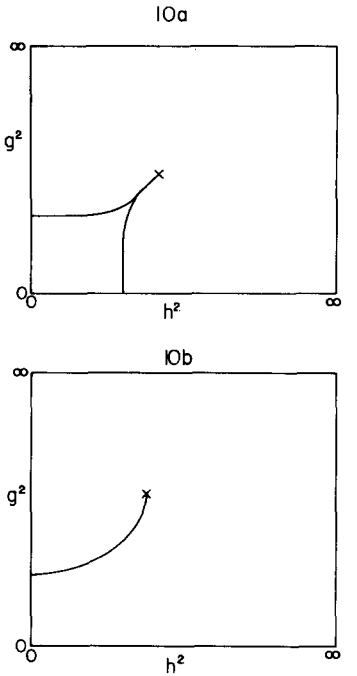

Fig. 10. (a) Phase diagram for the Z(2) Higgs model in the MFA. (b) The actual Phase diagram for the Z(2) Higgs model.

for large d the $h = 0$ region is close to the region of validity of the approximation so by continuity the MFA vacuum energy should be a good approximation even for $h = 0$.

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