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RESEARCH ARTICLE

Which coordinate representations can give correct Hawking temperature of Kerr black hole by tunneling approach

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Abstract Some authors found that, in different coordinates, the tunneling approach gives different Hawking temperature for the Schwarzschild black hole recently. In this paper, by studying the Hawking radiation of the Kerr black hole arising from the scalar and Dirac particles, we find that, to obtain the Hawking temperature by using tunneling effect, the coordinate representations for the stationary Kerr black hole should satisfy two conditions: (a) to keep the Killing vectors $\xi^{\mu}_{(t)}$ and $\xi^{\mu}_{(\varphi)}$ invariant; and (b) the radial coordinate transformation is a regular and non-zero function.

Keywords Black hole · Hawking temperature · Tunneling approach

1 Introduction

In recent years, a semi-classical method of controlling Hawking radiation as a tunneling effect has been developed and has excited a lot of interest [1–26]. In the tunneling approach, the particles are allowed to follow classically forbidden trajectories. The tunneling probability for the classically forbidden trajectory from inside to outside the event horizon is given by $\Gamma \propto e^{-2\mathrm{Im} I}$, which in turn is related to the Boltzmann factor for emission at the Hawking temperature, i.e. $\Gamma \propto e^{-2\mathrm{Im} I} = e^{-E/T_H}$, where T_H is the Hawking temperature of the black hole, E is the energy of the tunneling particles.

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There are two different approaches that are used to calculate the imaginary part of the action for the emitted particles. The first method is the null geodesic method presented by Parikh and Wilczek [4–7]. To derive the black hole's radiation, they used particles' null geodesic $\dot{r} = \frac{dr}{dt}$ in Painlevé coordinate representations. The other approach is the complex path analysis used by Angheben et al. [1] and Padmanabhan et al. [8–10]. This method does not need any coordinate transformation and just from the symmetries of the metric they pick an appropriate ansatz for the form of the action. For example, to the general spherically symmetrical black hole, the action is taken as $I = -Et + W(r) + J(x^i)$ [1]. So the emission, absorption and total probabilities are

$$\Gamma_{\text{out}} \propto e^{-2\text{Im}I} = e^{-2\text{Im}W_+}, \quad \Gamma_{\text{in}} \propto e^{-2\text{Im}I} = e^{-2\text{Im}W_-},$$
$$\Gamma = \Gamma_{\text{out}}/\Gamma_{\text{in}} \propto e^{-2[\text{Im}W_+ - \text{Im}W_-]},$$

where W_+ corresponds to the particles moving away from the black hole (outgoing particles) and W_- corresponds to particles moving toward the black hole (incoming particles).

This complex-path method has been extended by Mann et al. [11,12]. To ensure the probability is normalized, they used the boundary conditions for incoming particles which fall behind the horizon along classically permitted trajectories, i.e.

$$\Gamma_{\text{out}} \propto e^{-2\text{Im}I} = e^{-2[\text{Im}W_+ + \text{Im}K]}, \quad \Gamma_{\text{in}} \propto e^{-2\text{Im}I} = e^{-2[\text{Im}W_- + \text{Im}K]} = 1, \quad (1.1)$$

where the action is $I = -Et + W(r) + J(x^i) + K$, and K is a complex normalizing constant. So the total probability is

$$\Gamma = \Gamma_{\text{out}}/\Gamma_{\text{in}} \propto e^{-2[\text{Im}W_+ - \text{Im}W_-]}. \quad (1.2)$$

Do these tunneling methods work for any coordinates? The answer is no. For example, for the Schwarzschild black hole, introducing the isotropic coordinate [27]: $t \to t$, $\ln \rho = \int \frac{dr}{r\sqrt{1-\frac{2M}{r}}}$, the line element becomes

$$ds^2 = -\left(\frac{2\rho - M}{2\rho + M}\right)^2 dt^2 + \left(\frac{2\rho + M}{2\rho}\right)^4 d\rho^2 + \frac{(2\rho + M)^4}{16\rho^2} d\Omega^2. \quad (1.3)$$

The horizon is $\rho_H = M/2$. By using tunneling effect, it was found that the black hole's temperature is [27]

$$T_H = \frac{1}{16\pi M}, \quad (1.4)$$

which is one-half of the standard Hawking temperature $T_H = 1/8\pi M$. The example tell us that the invariance is missed in the isotropic coordinate by the tunneling method! The reason for the phenomenon comes from the coordinate transformation itself, i.e. the radial coordinate transformation has a singularity at the horizon $r = 2M$.

So it needs to discuss that in what kinds of coordinates can Hawking temperature keep invariant by the tunneling approach.

The purpose of this manuscript is to investigate the invariance of the Hawking temperature of the Kerr black hole from scalar and Dirac particles tunneling in some coordinate representations using the complex path method. In order to do that, we introduce the metrics of the Kerr black hole in the two coordinates: the Boyer–Lindquist and an new coordinate transform. The transform should satisfy two conditions: (a) keeping the Killing vectors $\xi^{\mu}_{(t)}$ and $\xi^{\mu}_{(\varphi)}$ invariant; (b) its radial coordinate transformation being a regular and non-zero function.

The paper is organized as follows. In Sect. 2 the different coordinate representations for the Kerr black hole are presented. In Sect. 3 the Hawking temperature of the Kerr black hole from scalar particles tunneling is investigated. In Sect. 4 the Hawking temperature of the Kerr black hole from Dirac particles tunneling is studied. The last section is devoted to a summary.

2 Coordinate representations for Kerr black hole

In the Boyer–Lindquist coordinate the line element for the Kerr black hole in four-dimensional spacetime is described by

$$\begin{aligned} ds^2 = &-\left(1-\frac{2Mr}{\rho^2}\right)dt_s^2 - \frac{4Mra\sin^2\theta}{\rho^2}dt_s d\varphi_s + \frac{\rho^2}{\triangle}dr^2 \\ &+\rho^2 d\theta^2 + \left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta d\varphi_s^2, \end{aligned} \tag{2.1}$$

with

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad \triangle = r^2 - 2Mr + a^2 = (r-r_+)(r-r_-),$$

where M is the mass of the black hole and a is the angular momentum parameter; r_- and r_+ are the inner and event horizons.

In order to keep the Killing vectors invariant in the spacetime, i.e. the timelike vector $\xi^{\mu}_{(t)}$ and the spacelike one $\xi^{\mu}_{(\varphi)}$, the general coordinate (v, u, θ, φ) that transformed from the Boyer–Lindquist coordinate (2.1) can be expressed as

$$du = F(r)dr, \quad dv = \eta dt_s + G(r)dr, \quad d\varphi = \delta d\varphi_s + H(r)dr, \tag{2.2}$$

where v, u and φ represent the time, radial, and angular coordinates respectively, θ remains the same; η and δ are arbitrary non-zero constants which re-scale the time and angle; G and H are arbitrary functions of r; F is a non-zero regular function of r (the reasons will be presented in Sect. 3.3). The line element (2.1) in the new coordinate becomes

$$ds^2 = -\frac{1}{\eta^2}\left(1-\frac{2Mr}{\rho^2}\right)\left(dv-\frac{G}{F}du\right)^2 - \frac{4Mra\sin^2\theta}{\eta\delta\rho^2} \times\left(dv-\frac{G}{F}du\right)\left(d\varphi-\frac{H}{F}du\right) + \frac{\rho^2}{\triangle F^2}du^2 + \rho^2 d\theta^2 + \left(r^2+a^2+\frac{2Mra^2\sin^2\theta}{\rho^2}\right)\frac{\sin^2\theta}{\delta^2}\left(d\varphi-\frac{H}{F}du\right)^2, \tag{2.3}$$

and its metric $g_{\bar{\mu}\bar{\nu}}$ are the functions of u and θ, i.e. $g_{\bar{\mu}\bar{\nu}} = g_{\bar{\mu}\bar{\nu}}(r(u),\theta)$ because r is now the function of u, i.e. $r = r(u)$.

In what follows, we show that two well-known coordinates, i.e. the Doran coordinate and the advanced Eddington–Finkelstein coordinate, are the special cases of the metric (2.3).

Doran coordinate representation. In the transformation (2.2), we take $\eta = \delta = 1$, $G(r) = \frac{\sqrt{2Mr(r^2+a^2)}}{\triangle}$, $H(r) = \frac{a}{\triangle}\sqrt{\frac{2Mr}{r^2+a^2}}$ and $F(r) = 1$, the line element (2.3) becomes the Doran coordinate representation [28,29] which has no singularity at $\triangle(r) = 0$.

Advanced Eddington–Finkelstein coordinate representation. In the transformation (2.2), we let $\eta = 1, \delta = -1$, $G(r) = \frac{r^2+a^2}{\triangle}$, $H(r) = -\frac{a}{\triangle}$ and $F(r) = 1$, the line element (2.3) becomes the advanced Eddington–Finkelstein representation [29,30] which has no coordinate singularity just as in the Doran coordinates.

3 Temperature of Kerr black hole from scalar particles tunneling

Now we investigate the scalar particles tunneling of the Kerr black hole in the Boyer–Lindquist coordinate (2.1) and the coordinate (2.3).

3.1 Scalar tunneling in the Boyer–Lindquist coordinate

Applying the WKB approximation

$$\phi(t,r,\theta,\varphi) = \exp\left[\frac{i}{\hbar}I(t,r,\theta,\varphi) + I_1(t,r,\theta,\varphi) + \mathcal{O}(\hbar)\right] \tag{3.1}$$

to the Klein–Gordon equation

$$\frac{1}{\sqrt{-g}}\partial_{\bar{\mu}}\left(\sqrt{-g}g^{\bar{\mu}\bar{\nu}}\partial_{\bar{\nu}}\phi\right) - \frac{\mu^2}{\hbar^2}\phi = 0, \tag{3.2}$$

then, to leading order in $\hbar$, we get the following relativistic Hamilton–Jacobi equation

$$g^{\bar{\mu}\bar{\nu}}(\partial_{\bar{\mu}}I\partial_{\bar{\nu}}I) + \mu^2 = 0. \tag{3.3}$$

From the symmetries of the metric (2.1), there exists a solution in the form

$$I = -Et_s + W(r) + m\varphi_s + J(\theta) + K. \tag{3.4}$$

Inserting (3.4) and the metric (2.1) into the Hamilton–Jacobi equation (3.3), we find

$$\triangle^2 W'^2(r) + \left[\triangle\mu^2 r^2 - a^2 m^2 + 4MramE - (r^2+a^2)^2 E^2\right] + \triangle\lambda = 0, \tag{3.5}$$

with

$$\lambda = \left(a^2 \sin^2\theta E^2 + J'^2(\theta) + \frac{m^2}{\sin^2\theta} + \mu^2 a^2 \cos^2\theta\right), \tag{3.6}$$

where $W'(r) = \frac{dW(r)}{dr}$, $J'(\theta) = \frac{dJ(\theta)}{d\theta}$. From it we can obtain

$$W'_{\pm}(r) = \pm\frac{1}{\triangle}\sqrt{(r^2+a^2)^2\left(E - \frac{ma}{r^2+a^2}\right)^2 - \triangle(\mu^2 r^2 + \lambda - 2maE)}. \tag{3.7}$$

One solution of the (3.7) corresponds to the scalar particles moving away from the black hole (i.e. "+" outgoing) and the other solution corresponds to particles moving toward the black hole (i.e. "−" incoming). Imaginary parts of the action can only come from the pole at the horizon. Integrating around the pole at the horizon leads to

$$\mathrm{Im}W_{\pm} = \pm\frac{\pi(r_+^2+a^2)}{2(r_+-M)}\,(E - m\Omega_+)\,, \tag{3.8}$$

where $\Omega_+ = \frac{a}{r_+^2+a^2}$ is the angular velocity of the horizon. Using (1.1) and (1.2), the probability of a particle tunneling from inside to outside the horizon is

$$\Gamma \propto \exp\left[-2(\mathrm{Im}W_+ - \mathrm{Im}W_-)\right] = \exp\left[-\frac{2\pi\left(r_+^2+a^2\right)}{(r_+-M)}\,(E - m\Omega_+)\right], \tag{3.9}$$

the relation between emission probability and the Hawking temperature for Kerr black hole is [3]

$$\Gamma \propto \exp\left[-\frac{E - m\Omega_+}{T_H}\right], \tag{3.10}$$

which is different from static case because the rotation of the Kerr black hole can cause the shifting of the radiating particles' energy when they tunneling across the event horizon, that is, $E \dashrightarrow E - m\Omega_+$. Combining (3.9) and (3.10), we obtain the Hawking temperature

$$T_H = \frac{r_+ - M}{2\pi(r_+^2+a^2)}, \tag{3.11}$$

which shows that the temperature of Kerr black hole is the same as previous work [3,11].

3.2 Scalar tunneling in the coordinate (2.3)

Now we study the scalar tunneling in the coordinate (2.3). Employing the ansatz

$$I = -Ev + W(u) + m\varphi + J(\theta) + K \tag{3.12}$$

and substituting the metric (2.3) into the Hamilton–Jacobi equation (3.3), we obtain

$$\begin{aligned}&\triangle F^2 W'^2(u) - 2(GE - Hm)\triangle F W'(u) + \left[\triangle G^2 - \frac{\eta^2 \left(r^2+a^2\right)^2}{\triangle}\right] E^2 \\ &\quad + \left(\triangle H^2 - \frac{a^2\delta^2}{\triangle}\right) m^2 - 2\left(\triangle HG - \frac{2\eta\delta Mra}{\triangle}\right) mE + \mu^2 r^2 + \lambda = 0,\end{aligned} \tag{3.13}$$

with

$$\lambda = J'^2(\theta) + \eta^2 a^2 \sin^2\theta E^2 + \frac{\delta^2 m^2}{\sin^2\theta} + \mu^2 a^2 \cos^2\theta. \tag{3.14}$$

Then $W'(u)$ is

$$\begin{aligned}W'_\pm(u) = &\frac{GE - Hm}{F} \\ &\pm \frac{\sqrt{(r^2+a^2)^2\left[\eta E - \delta ma/(r^2+a^2)\right]^2 - \triangle(-2\eta\delta maE + r^2\mu^2 + \lambda)}}{F\triangle}.\end{aligned} \tag{3.15}$$

Without loss of generality, arbitrary functions G and H can be expressed as $G(r(u)) = \frac{A(r(u))}{\triangle(r(u))} + B(r(u))$, $H(r(u)) = \frac{C(r(u))}{\triangle(r(u))} + D(r(u))$ where $A(r(u))$, $B(r(u))$, $C(r(u))$, $D(r(u))$ are all regular functions. Equation (3.15) can be rewritten as

$$\begin{aligned}\mathrm{Im} W_\pm(u) = \mathrm{Im}\int_{u_+-\epsilon}^{u_++\epsilon} du \Bigg\{ &\frac{BE - Dm}{F} + \frac{AE - Cm}{F\triangle} \\ &\pm \frac{\sqrt{(r^2+a^2)^2\left[\eta E - \delta ma/(r^2+a^2)\right]^2 - \triangle(-2\eta\delta maE + r^2\mu^2 + \lambda)}}{F\triangle}\Bigg\},\end{aligned} \tag{3.16}$$

where u_+ is the event horizon, ϵ is small positive quantity. $\triangle(r(u))\big|_{u=u_+}=0$, and $\frac{d}{du}\triangle(r(u))\big|_{u=u_+}=\frac{d\triangle(r(u))}{dr(u)}\frac{dr(u)}{du}\Big|_{u=u_+}=\frac{2(r_+-M)}{F(r(u))}\Big|_{u=u_+}$ which is a non-zero finite value, so u_+ is first order zero point of $\triangle(r(u))$. We use the law of the residue and get

$$\mathrm{Im}W_\pm(u)=\left[\frac{A(r_+)E-C(r_+)m}{2(r_+-M)}\pm\frac{r_+^2+a^2}{2(r_+-M)}(\eta E-\delta m\Omega_+)\right]\pi. \quad (3.17)$$

According to functions A and C, (3.16) can be divided into four kinds of situations: (1) If $A=C=0$, both $G(r(u))$ and $H(r(u))$ are regular functions at the horizon; (2) If $A=0, C\neq 0$, $G(r(u))$ is a regular function while $H(r(u))$ is a non-regular one at the horizon; (3) If $A\neq 0, C=0$, $H(r(u))$ is a regular function while $G(r(u))$ is a non-regular one at the horizon; (4) If $A\neq 0, C\neq 0$, none of $G(r(u))$ and $H(r(u))$ is a regular function at the horizon. These four cases can all lead to the same probability of a particle tunneling from inside to outside the horizon

$$\Gamma\propto\exp[-2(\mathrm{Im}W_+-\mathrm{Im}W_-)]=\exp\left[-2\pi\frac{r_+^2+a^2}{(r_+-M)}(\eta E-\delta m\Omega_+)\right]. \quad (3.18)$$

The relation between the emission probability and the Hawking temperature is

$$\Gamma\propto\exp\left[-\frac{(\eta E-\delta m\Omega_+)}{T_H}\right], \quad (3.19)$$

and we can recover Hawking temperature (3.11). Equation (3.19) shows that the radiating particles' energy also shifts from $E-ma/(r_+^2+a^2)$ to $\eta E-\delta ma/(r_+^2+a^2)$ when we rescale the time t by ηt and the angle φ by $\delta\varphi$ (it is reasonable because object's energy is dependent on the choice of coordinate). Therefore, rescale of the t and φ does not affect the Hawking temperature.

From above discussions we know that the Hawking temperature of the Kerr black hole arising from the scalar particles tunneling is invariant in the coordinate (2.3).

3.3 Why F must be a regular and non-zero function

Why the function F is taken as a regular and non-zero function in the coordinate change (2.2)? Now we give the reasons as follows:

(i) If F is non-regular function at horizon, for example, the isotropic coordinate transformation $\ln 2u=\int\frac{dr}{\sqrt{\triangle}}$ [1], then

$$\frac{du}{u}=\frac{dr}{\sqrt{\triangle}},\quad F=\frac{u}{\sqrt{\triangle}}. \quad (3.20)$$

So (3.16) becomes

$$\begin{aligned}&\mathrm{Im}W_{\pm}(u)\\&=\mathrm{Im}\int\limits_{u_+-\epsilon}^{u_++\epsilon}du\left\{\frac{(BE-Dm)\sqrt{\triangle}}{u}+\frac{AE-Cm}{u\sqrt{\triangle}}\right.\\&\quad\left.\pm\frac{\sqrt{(r^2+a^2)^2\left[\eta E-\delta ma/(r^2+a^2)\right]^2-\triangle[-2\eta\delta maE+r^2\mu^2+\lambda]}}{u\sqrt{\triangle}}\right\},\end{aligned} \tag{3.21}$$

where $u_+=\frac{1}{2}(r_+-M)$. $\sqrt{\triangle(r(u))}\big|_{u=u_+}=0$, and $\frac{d}{du}\sqrt{\triangle(r(u))}\big|_{u=u_+}=2$, so u_+ is the first order zero point of $\sqrt{\triangle(r(u))}$. From the law of the residue, we obtain

$$\mathrm{Im}W_{\pm}(u)=2\left[\frac{A(r_+)E-C(r_+)m}{2(r_+-M)}\pm\frac{r_+^2+a^2}{2(r_+-M)}(\eta E-\delta m\Omega_+)\right]\pi. \tag{3.22}$$

Using (1.2) and (3.19), we obtain the result which is one-half of Hawking temperature (3.11)! This factor $\frac{1}{2}$ also is the same as in [27].

(ii) If F is regular but equal to zero at horizon, i.e. $F=\triangle X(r)$, where X is non-zero regular functions, the (3.16) becomes

$$\begin{aligned}&\mathrm{Im}W_{\pm}(u)\\&=\mathrm{Im}\int\limits_{u_+-\epsilon}^{u_++\epsilon}du\left\{\frac{BE-Dm}{\triangle X}+\frac{AE-Cm}{\triangle^2 X}\right.\\&\quad\left.\pm\frac{\sqrt{(r^2+a^2)^2\left[\eta E-\delta ma/(r^2+a^2)\right]^2-\triangle[-2\eta\delta maE+r^2\mu^2+\lambda]}}{\triangle^2 X}\right\},\end{aligned} \tag{3.23}$$

where $[\triangle(r(u))]^2\big|_{u=u_+}=0$, $\frac{d}{du}[\triangle^2(r(u))]\big|_{u=u_+}=\frac{4(r_+-M)}{X(r(u))}\Big|_{u=u_+}$. Which shows that u_+ is the first order zero point of $\triangle^2(r(u))$. The Laurent expansions of $\frac{1}{\triangle^2(r(u))}$ is

$$\frac{1}{\triangle^2(r(u))}=a_{-1}(u-u_+)^{-1}+\sum_{n=0}^{\infty}a_n(u-u_+)^n, \tag{3.24}$$

with

$$a_{-1}=\lim_{u\to u_+}\left[(u-u_+)\frac{1}{\triangle^2(r(u))}\right]=\frac{X(r(u_+))}{4(r_+-M)}. \tag{3.25}$$

Lastly we find

$$
\begin{aligned}
\mathrm{Im} W_{\pm}(u) = &- 2\sqrt{\epsilon a_{-1}} \cdot \frac{BE - Dm}{X}\bigg|_{u=u_+} \\
&+ \frac{1}{2}\left[\frac{A(r_+)E - C(r_+)m}{2(r_+ - M)} \pm \frac{r_+^2 + a^2}{2(r_+ - M)}(\eta E - \delta m \Omega_+)\right]\pi,
\end{aligned}
\tag{3.26}
$$

or

$$
\begin{aligned}
\mathrm{Im} W_{\pm}(u) = &\ 2\sqrt{\epsilon a_{-1}} \cdot \frac{BE - Dm}{X}\bigg|_{u=u_+} \\
&+ \frac{1}{2}\left[\frac{A(r_+)E - C(r_+)m}{2(r_+ - M)} \pm \frac{r_+^2 + a^2}{2(r_+ - M)}(\eta E - \delta m \Omega_+)\right]\pi,
\end{aligned}
\tag{3.27}
$$

where $\int_{u_+-\epsilon}^{u_++\epsilon}(u-u_+)^{-1}du = i\pi$ and $\int_{u_+-\epsilon}^{u_++\epsilon}(u-u_+)^{-1/2}du = \pm 2\sqrt{\epsilon}(i+1)$ are used. From (1.2) and (3.19) we obtain the result which is twice of Hawking temperature (3.11)!

Therefore, to keep the invariance of the Hawking temperature, the radial transformation function F of the (2.2) should be a non-zero and regular function.

4 Temperature of Kerr black hole from the Dirac particles tunneling

In this section, we study the Dirac particles tunneling of the Kerr black hole in the coordinates (2.1) and (2.3).

4.1 Dirac particles tunneling in the Boyer–Lindquist coordinate

For a general spacetime, the Dirac equation is [31]

$$
\left[\gamma^{\alpha} e_{\alpha}^{\bar{\mu}}(\partial_{\bar{\mu}} + \Gamma_{\bar{\mu}}) + \frac{\mu}{\hbar}\right]\psi = 0,
\tag{4.1}
$$

with

$$
\Gamma_{\bar{\mu}} = \frac{1}{8}[\gamma^a, \gamma^b] e_a^{\bar{\nu}} e_{b\bar{\nu};\bar{\mu}},
$$

where γ^a are the Dirac matrices and $e_a^{\bar{\mu}}$ is the inverse tetrad defined by $\{e_a^{\bar{\mu}}\gamma^a, e_b^{\bar{\nu}}\gamma^b\} = 2g^{\bar{\mu}\bar{\nu}} \times 1$. For the Kerr metrics in the Boyer–Lindquist coordinate (2.1), the tetrad $e_a^{\bar{\mu}}$

can be taken as

$$
\begin{aligned}
e_a^{t_s} &= \left(\sqrt{\frac{(r^2+a^2)^2-\triangle a^2\sin^2\theta}{\rho^2\triangle}},\ 0,\ 0,\ 0\right),\\
e_a^{r} &= \left(0,\ \frac{\sqrt{\triangle}}{\rho},\ 0,\ 0\right),\\
e_a^{\theta} &= \left(0,\ 0,\ \frac{1}{\rho},\ 0\right),\\
e_a^{\varphi_s} &= \left(\frac{2Mra}{\sqrt{\rho^2\triangle\left[(r^2+a^2)^2-\triangle a^2\sin^2\theta\right]}},\ 0,\ 0,\ \frac{\rho}{\sin\theta\sqrt{(r^2+a^2)^2-\triangle a^2\sin^2\theta}}\right).
\end{aligned}
\tag{4.2}
$$

We employ the following ansatz for the Dirac field

$$
\begin{aligned}
\psi_\uparrow &= \begin{pmatrix} A(t_s,r,\theta,\varphi_s)\xi_\uparrow \\ B(t_s,r,\theta,\varphi_s)\xi_\uparrow \end{pmatrix} \exp\left(\frac{i}{\hbar}I_\uparrow(t_s,r,\theta,\varphi_s)\right)\\
&= \begin{pmatrix} A(t_s,r,\theta,\varphi_s) \\ 0 \\ B(t_s,r,\theta,\varphi_s) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar}I_\uparrow(t_s,r,\theta,\varphi_s)\right),\\
\psi_\downarrow &= \begin{pmatrix} C(t_s,r,\theta,\varphi_s)\xi_\downarrow \\ D(t_s,r,\theta,\varphi_s)\xi_\downarrow \end{pmatrix} \exp\left(\frac{i}{\hbar}I_\downarrow(t_s,r,\theta,\varphi_s)\right)\\
&= \begin{pmatrix} 0 \\ C(t_s,r,\theta,\varphi_s) \\ 0 \\ D(t_s,r,\theta,\varphi_s) \end{pmatrix} \exp\left(\frac{i}{\hbar}I_\downarrow(t_s,r,\theta,\varphi_s)\right),
\end{aligned}
\tag{4.3}
$$

where "↑" and "↓" represent the spin up and spin down cases, and $\xi_\uparrow$ and $\xi_\downarrow$ are the eigenvectors of σ^3. Inserting (4.2) and (4.3) into the Dirac equation (4.1) and employing the ansatz

$$
I_\uparrow = -Et_s + W(r) + m\varphi_s + J(\theta) + K, \tag{4.4}
$$

to the lowest order in $\hbar$ we obtain

$$
-A\left[e_0^{t_s}E - e_0^{\varphi_s}m\right] + e_1^r BW'(r) + \mu A = 0, \tag{4.5}
$$

$$
B\left[e_2^{\theta}J'(\theta) + ie_3^{\varphi_s}m\right] = 0, \tag{4.6}
$$

$$
B\left[e_0^{t_s}E - e_0^{\varphi_s}m\right] - e_1^r AW'(r) + \mu B = 0, \tag{4.7}
$$

$$
-A\left[e_2^{\theta}J'(\theta) + ie_3^{\varphi_s}m\right] = 0, \tag{4.8}
$$

where we consider the positive frequency contributions. Equations (4.6) and (4.8) both yield $(e_2^{\theta}J'(\theta) + ie_3^{\varphi_s}m) = 0$ regardless of A or B, then from (4.5) to (4.8), we can obtain

$$\triangle^2 W'^2(r) + \left[\triangle\mu^2 r^2 - a^2m^2 + 4MramE - (r^2+a^2)^2E^2\right] + \triangle\lambda = 0, \quad (4.9)$$

with

$$\lambda = \left(a^2\sin^2\theta E^2 + J'^2(\theta) + \frac{m^2}{\sin^2\theta} + \mu^2a^2\cos^2\theta\right), \quad (4.10)$$

which are the same as the (3.5), so the Hawking temperature (3.11) is again recovered.

The spin-down calculation is very similar to the spin-up case discussed above. Other than some changes of sign, the equations are of the same form as the spin up case. For both the massive and massless spin down cases the Hawking temperature (3.11) is obtained, implying that both spin up and spin down particles are emitted at the same temperature.

4.2 Dirac particles tunneling in the coordinate (2.3)

Without loss of the generality, we take the following ansatz

$$\psi_\uparrow = \begin{pmatrix} A(v,u,\theta,\varphi) \\ 0 \\ B(v,u,\theta,\varphi) \\ 0 \end{pmatrix} \exp\left(\frac{i}{\hbar}I_\uparrow(v,u,\theta,\varphi)\right),$$
$$I_\uparrow = -Ev + W(u) + m\varphi + J(\theta) + K,$$

and for the line element (2.3), we chose the tetrad $e_a^{\bar{\mu}}$

$$\begin{aligned}
e_a^v &= \left(\frac{\sqrt{\chi - \triangle^2G^2}}{\rho\sqrt{\triangle}},\ 0,\ 0,\ 0\right),\\
e_a^u &= \left(-\frac{1}{\rho\sqrt{\triangle}}\frac{\triangle^2FG}{\sqrt{\chi-\triangle^2G^2}},\ \frac{1}{\rho\sqrt{\triangle}}\frac{\triangle F\sqrt{\chi}}{\sqrt{\chi-\triangle^2G^2}},\ 0,\ 0\right),\\
e_a^\theta &= \left(0,\ 0,\ \frac{1}{\rho},\ 0\right),\\
e_a^\varphi &= \left(\frac{1}{\rho\sqrt{\triangle}}\frac{2\eta\delta Mra - \triangle^2HG}{\sqrt{\chi-\triangle^2G^2}},\ \frac{1}{\rho\sqrt{\triangle}}\frac{\triangle(H\chi - 2\eta\delta MraG)}{\sqrt{\chi(\chi-\triangle^2G^2)}},\ 0,\ \frac{\eta\delta\rho}{\sin\theta\sqrt{\chi}}\right),
\end{aligned} \quad (4.11)$$

with

$$\chi = \eta^2\left[\left(r^2+a^2\right)^2 - \triangle a^2\sin^2\theta\right]. \quad (4.12)$$

Then the Dirac equation (4.1) can be expressed as

$$\left[-e_0^v E + e_0^u W'(u) + e_0^\varphi m\right] A + B\left[e_1^u W'(u) + e_1^\varphi m\right] + \mu A = 0, \quad (4.13)$$

$$B\left[e_2^\theta J'(\theta) + i e_3^\varphi m\right] = 0, \quad (4.14)$$

$$-\left[-e_0^v E + e_0^u W'(u) + e_0^\varphi m\right] B - A\left[e_1^u W'(u) + e_1^\varphi m\right] + \mu B = 0, \quad (4.15)$$

$$-A\left[e_2^\theta J'(\theta) + i e_3^\varphi m\right] = 0. \quad (4.16)$$

Eqs. (4.14) and (4.16) both yield $\left[e_2^\theta J'(\theta) + i e_3^\varphi m\right] = 0$ regardless of A or B. Then substituting tetrad elements (4.11) into (4.13)–(4.16), after tedious calculating, we can obtain

$$\triangle F^2 W'^2(u) - 2(GE - Hm)\triangle F W'(u) + \left[\triangle G^2 - \frac{\eta^2\left(r^2+a^2\right)^2}{\triangle}\right]E^2 + \left(\triangle H^2 - \frac{a^2\delta^2}{\triangle}\right)m^2 - 2\left(\triangle HG - \frac{2\eta\delta Mra}{\triangle}\right)mE + \mu^2 r^2 + \lambda = 0, \quad (4.17)$$

with

$$\lambda = J'^2(\theta) + \eta^2 a^2 \sin^2\theta E^2 + \frac{\delta^2 m^2}{\sin^2\theta} + \mu^2 a^2 \cos^2\theta. \quad (4.18)$$

which are the same as (3.13). Taking the same method used in the Sect. 3.2, it is easy to get the Hawking temperature (3.11).

Above discussions show us that the Hawking temperature of the Kerr black hole arising from the Dirac particles tunneling is also invariant in the coordinate (2.3).

5 Summary

By studying the Hawking radiation of the stationary Kerr black hole arising from the scalar and Dirac particles in the Boyer–Lindquist and some coordinate representations, we find that, to obtain the Hawking temperature by using a tunneling effect, the coordinate transformation should satisfies two conditions: (a) keep the timelike vector $\xi^\mu_{(t)}$ and the spacelike one $\xi^\mu_{(\varphi)}$ invariant; and (b) its radial coordinate transformation is a regular and non-zero function.

Our results indicate additional interesting conclusions: (1) The rotation of the Kerr black hole can cause the shift of the radiating particles' energy when they tunneling across the event horizon, i.e., $E \dashrightarrow E - m\Omega_+$, and then the relation between the probability and the Hawking temperature becomes $\Gamma \propto \exp[-(E - m\Omega_+)/T_H]$. (2) When the time coordinate transforms from t_s to ηt_s, and the angular coordinate transforms from φ_s to $\delta\varphi_s$, i.e., we re-scale the time and angle, the corresponding energy shift from $E - ma/(r_+^2 + a^2)$ to $\eta E - \delta ma/(r_+^2 + a^2)$, therefore the remeasurements of the t and φ do not affect the Hawking temperature. (3) The mass and the angular quantum number of the particles do not affect the Hawking temperature for both the scalar and Dirac particles. (4) The black hole radiates the scalar and Dirac particles at the same temperature.

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