Introduction to Quantum Computing

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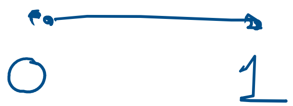
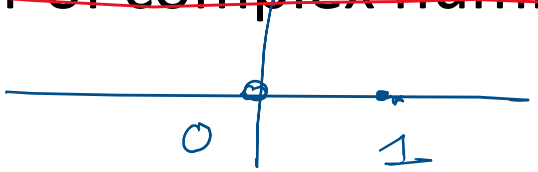
Qubits (Quantum (Bits)

binary digit 0, 1

- A quantum bit or a qubit is a fundamental unit of quantum information processing just as a bit is a fundamental unit of classical information processing.



- A single qubit state is represented by a pair of complex numbers $\begin{pmatrix} a \\ b \end{pmatrix}$ where $|a|^2 + |b|^2 = 1$.



- So a single qubit can exist in an infinite number of states whereas a bit can exist in either in 0 state or 1 state.

Bits

Quantum Bits Qubits

1 I should be able to set a bit to the 0 or 1, as I please. WRITE ✓

2. I should be able to read what I have written.

3. I should be able to store whatever I have written for ever(!) no matter how many times I read it

Our reading capability is extremely limited!

A single qubit is a quantum mechanical system that is completely specified by a pair of complex number $\begin{pmatrix} a \\ b \end{pmatrix}$, called its <u>state vector</u>.

Satisfy the equation $|a|^2 + |b|^2 = 1$

$a = \alpha_1 + i\alpha_2$, $\alpha_1, \alpha_2 \in \mathbb{R}$. $i^2 = -1$.

$|a|^2 = \underline{\alpha_1^2 + \alpha_2^2}$ $b = \beta_1 + i\beta_2$. $|b|^2 = \beta_1^2 + \beta_2^2$.

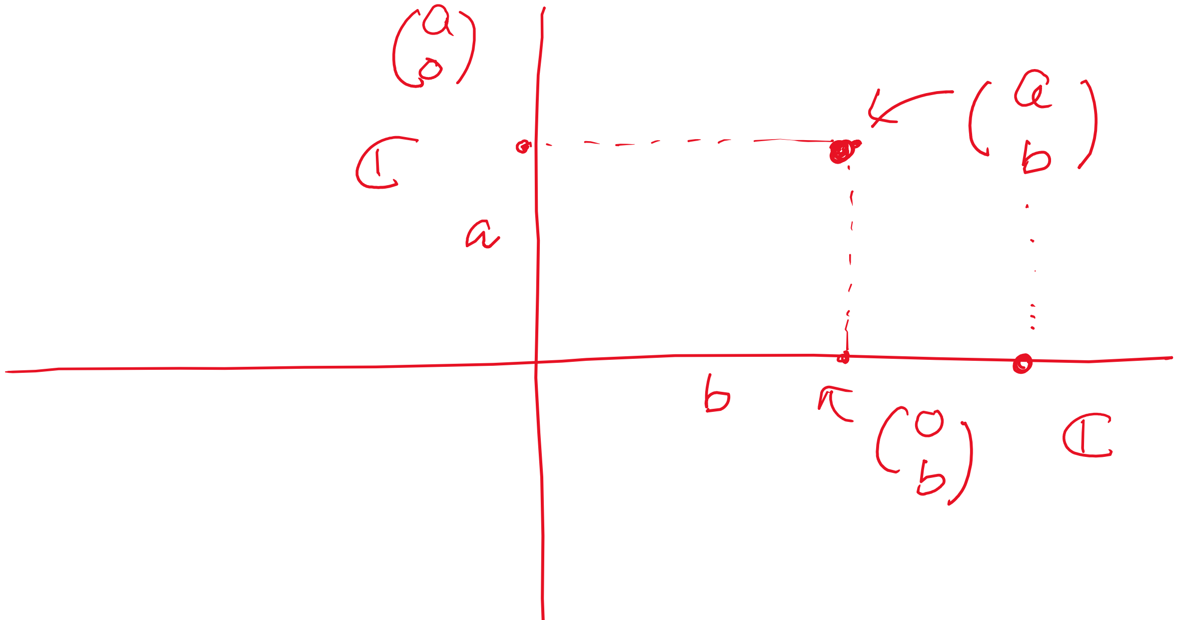
$\overline{a} = \alpha_1 - i\alpha_2$. $a.\overline{a} = (\alpha_1 + i\alpha_2)(\alpha_1 - i\alpha_2)$

$= \alpha_1^2 - i\cancel{\alpha_1\alpha_2} + i\cancel{\alpha_2\alpha_1} - i^2\alpha_2^2 = \alpha_1^2 + \alpha_2^2$.

The State Space under consideration is $\mathbb{C} \times \mathbb{C}$

$|a|^2+|b|^2=1$

$\{0,1\}$



$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

Basis

$$\begin{pmatrix} a \\ b \end{pmatrix} = a \underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} + b \underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

Writing conventions

$\begin{pmatrix}1\\0\end{pmatrix}$ $\begin{pmatrix}0\\1\end{pmatrix}$

$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}$

$= a|0\rangle + b|1\rangle.$

- It is customary to write $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.

Ket 0

Ket 1

- Then, a single qubit state is

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \underline{a|0\rangle + b|1\rangle}$$

- We must not forget that

$$\boxed{|a|^2 + |b|^2 = 1}$$

Recalling complex numbers

- A complex number is written as $z = x + \mathbf{i}\, y$ where x, y are real numbers, and $\mathbf{i}^2 = -1$.

- The conjugate of z is $\bar{z} = x - \mathbf{i}y$.

- The modulus of a complex number is $|z|$ where

$$|z|^2 = z\bar{z} = x^2 + y^2$$

A single-bit system

- A single-bit system can exist in one of the two states: 0 and 1. Such a system can be visualized as

- In a classical computer it is possible to set a bit to the 0 or 1 state. It is also possible to read (measure) that state, and reading from a bit does not change its state.
- On a quantum computer it is possible to create a single-qubit state, but it is not possible to measure it without changing the state.

Quantum bits – Qubits

- Quantum bits – Qubits :

  A qubit is the fundamental unit of quantum information just as a bit is the fundamental unit of classical information.

  - A bit can exist in two states: 0 and 1.

  - A qubit is a vector having two complex components.

    Consider the vector space $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{C} \right\}$.

    A vector of the form $\begin{pmatrix} a \\ b \end{pmatrix}$ defines a state of a qubit if and only if

$$|a|^2 + |b|^2 = 1.$$

$\mathbb{C}^2 = \mathbb{C} \times \mathbb{C} = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} : a, b \in \mathbb{C} \right\}$. $\quad c \in \mathbb{C} \quad c\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ca \\ cb \end{pmatrix}$

Basis of $\mathbb{C}^2$

• Linear dependence and independence.

- The set of vectors $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is said to be a basis is $\mathbb{C}^2$ since any element in $\mathbb{C}^2$ can be ~~written~~ uniquely as a linear combination

$$\begin{pmatrix}a\\b\end{pmatrix} = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix}.$$

$a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$\Rightarrow a = 0, b = 0$

- Any set of vectors with this property is said to be a basis of $\mathbb{C}^2$.

For example: $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}$, $\left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\mathbf{i}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-\mathbf{i}\end{pmatrix}\right\}$

where $\mathbf{i}^2 = -1$.

$\frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \quad \frac{|0\rangle - i|1\rangle}{\sqrt{2}}$

$\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle \qquad \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$

$$B_1 = \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\} = \{ |0\rangle, |1\rangle \}$$

$$\sqrt{2}\, |+\rangle = |0\rangle + |1\rangle \longrightarrow$$

$$B_2 = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix}1\\-1\end{pmatrix} \right\} = \{ |+\rangle, |-\rangle \}$$

$$\sqrt{2}\, |-\rangle = |0\rangle - |1\rangle \longrightarrow$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle$$

$$= \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix}1\\-1\end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix}0\\-1\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

$$2|0\rangle = \sqrt{2} \left(|+\rangle + |-\rangle \right)$$

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}$$

$$2|1\rangle = \sqrt{2} \left(|+\rangle - |-\rangle \right)$$

$$|1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

$$a|0\rangle + b|1\rangle = a \left(\frac{|+\rangle + |-\rangle}{\sqrt{2}} \right) + b \left(\frac{|+\rangle - |-\rangle}{\sqrt{2}} \right)$$

$$= \left(\frac{a+b}{\sqrt{2}} \right) |+\rangle + \left(\frac{a-b}{\sqrt{2}} \right) |-\rangle .$$

Inner Product

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} = a|0\rangle + b|1\rangle$$

$$|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix} = c|0\rangle + d|1\rangle .$$

The inner product of $|\psi\rangle$ and $|\phi\rangle$

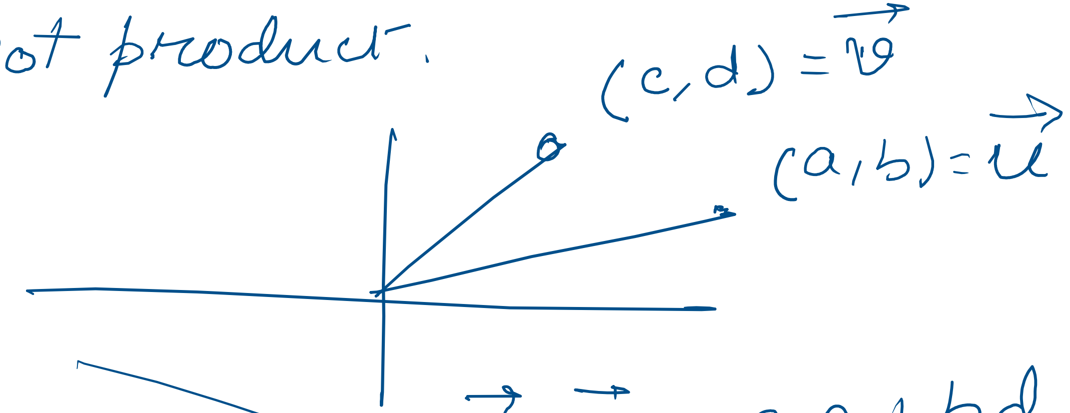
$$\langle \psi | \phi \rangle = \bar{a}c + \bar{b}d .$$

$$\langle \psi | \psi \rangle = \bar{a}a + \bar{b}b$$

$$= |a|^2 + |b|^2$$

$$|a|^2 + |b|^2 = 1$$

Dot product.



$$\vec{u} \cdot \vec{v} = ac + bd .$$

$$\vec{u} \cdot \vec{u} = a^2 + b^2$$

Norm

$$\|\psi\| = \sqrt{\langle \psi | \psi \rangle}$$

$$= \sqrt{|a|^2 + |b|^2} = 1$$

Dirac's bra/ket notation.

$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$

ket ψ

$|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$

ket ϕ

$\langle\psi| = (\bar{a} \;\; \bar{b})$

bra ψ

$$\langle\psi|\,|\phi\rangle = (\bar{a} \;\; \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix}$$

$$= \bar{a}c + \bar{b}d$$

braket ψ, ϕ $= \langle\psi|\phi\rangle$

Inner product on $\mathbb{C}^2$

$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$, $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$

- Inner product of two vectors $\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \in \mathbb{C}^2$ is

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d.$$

- Two vector are said to be orthogonal if

$$\underbrace{\begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix}}_{\langle\psi|\phi\rangle} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d = 0.$$

$|\psi\rangle$ and $|\phi\rangle$ will be said to be orthogonal vectors.

$= \bar{a}c + \bar{b}d = 0$

Orthonormal basis of $\mathbb{C}^2$

Such a basis is called an orthonormal basis.

$|\psi\rangle \quad |\phi\rangle$

- Suppose $\left\{\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix}\right\}$ is a basis such that

$$\begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d = 0$$

and

$$\langle\psi|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} a \\ b \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} a \\ b \end{pmatrix} = \bar{a}a + \bar{b}b = |a|^2 + |b|^2 = 1 \checkmark$$

$$\langle\phi|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{c} \quad \bar{d}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{c}c + \bar{d}d = |c|^2 + |d|^2 = 1 \checkmark$$

$$\langle\psi|\phi\rangle = \langle\phi|\psi\rangle = 0$$

Orthonormal basis of $\mathbb{C}^2$

$|0\rangle, |1\rangle$

- Computational basis: $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\bar{1} \quad \bar{0}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{1}0 + \bar{0}1 = \underline{0}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} 1 \\ 0 \end{pmatrix} = (\bar{1} \quad \bar{0}) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \bar{1}1 + \bar{0}0 = \underline{1}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}^\dagger \begin{pmatrix} 0 \\ 1 \end{pmatrix} = (\bar{0} \quad \bar{1}) \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{0}0 + \bar{1}1 = \underline{1}$$

Orthonormal basis of $\mathbb{C}^2$: Examples

- Hadamard basis: $\mathcal{H} = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\right\}$.
- Nega-Hadamard basis: $\mathcal{N} = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\\mathbf{i}\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-\mathbf{i}\end{pmatrix}\right\}$.
- Verify that $\mathcal{H}$ and $\mathcal{N}$ are orthonormal bases.

Dirac's bra/ket notation

- A vector $\begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{C}^2$ is written as $|\psi\rangle$ read as "*ket psi*".
- The vector $\begin{pmatrix} a \\ b \end{pmatrix}^\dagger = (\bar{a} \quad \bar{b})$ is written as $\langle\psi|$.
- Inner product of two vectors $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$, and $|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$ is $\langle\psi|\phi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \quad \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d.$

  The order in the which $|\phi\rangle$ and $|\psi\rangle$ appear matters. This is the inner product of $|\phi\rangle$ and $|\psi\rangle$ and not $|\psi\rangle$ and $|\phi\rangle$.

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |a|^2 + |b|^2 = 1$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \qquad |a|^2 + |b|^2 = 1.$$

$$|\psi\rangle, \ |\phi\rangle \qquad |\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\langle\psi| = \begin{pmatrix} a \\ b \end{pmatrix}^\dagger = (\bar{a}, \bar{b})$$

$$\langle\psi|\phi\rangle = \begin{pmatrix} a \\ b \end{pmatrix}^\dagger \begin{pmatrix} c \\ d \end{pmatrix} = (\bar{a} \ \ \bar{b}) \begin{pmatrix} c \\ d \end{pmatrix} = \bar{a}c + \bar{b}d$$

braket ψ ϕ

Computational, Hadamard and Nega-Hadamard Bases in Dirac's notation

- Computational basis: $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

- Hadamard basis: $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{|0\rangle+|1\rangle}{\sqrt{2}}, |-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{|0\rangle-|1\rangle}{\sqrt{2}}$

- Nega-Hadamard basis:

  $|\mathbf{i}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ \mathbf{i} \end{pmatrix} = \frac{|0\rangle+\mathbf{i}|1\rangle}{\sqrt{2}}, |-\mathbf{i}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathbf{i} \end{pmatrix} = \frac{|0\rangle-\mathbf{i}|1\rangle}{\sqrt{2}}$

- Verify that all the above bases are orthonormal.

$|0\rangle$ $|1\rangle$ bit qubit

$0 \longrightarrow |0\rangle$

$1 \longrightarrow |1\rangle$

Superposition of states

- The state of a single-qubit is of the form

A single qubit can exist in any one of the state

$$|\psi\rangle = \binom{a}{b} = a\binom{1}{0} + b\binom{0}{1} = \underline{a|0\rangle + b|1\rangle}$$

$a=1$ $b=0$ $a=0$ $b=1$

where $\underline{|a|^2 + |b|^2 = 1.}$

$\underline{|\psi\rangle = a|0\rangle + b|1\rangle}$

- If $a \neq 0$ and $b \neq 0$ the qubit is said to be in the superposition of two states $|0\rangle$ and $|1\rangle$.

$a \neq 0$, $b \neq 0$

$|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

Once a superposition, always a superposition?

NO

- $|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}}$ is a superposition of two states $|0\rangle$, and $|1\rangle$.

- We say that $|\psi\rangle$ is in superposition with respect to the basis $\{|0\rangle, |1\rangle\}$.

$|\psi\rangle = \frac{|0\rangle+|1\rangle}{\sqrt{2}} = |+\rangle$

- However, the representation of $|\psi\rangle$ with respect to the basis $\mathcal{H} = \{|+\rangle, |-\rangle\}$ is $|\psi\rangle = |+\rangle$.

Hadamard Basis

- Therefore, $|\psi\rangle$ is not in superposition with respect to the basis $\mathcal{H}$.

Changing a Qubit representation from computational to Hadamard basis

- $|\psi\rangle = a|0\rangle + b|1\rangle$ is a single-qubit state written in computational basis.
- The Hadamard basis vectors in terms of computational basis vectors are:

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \qquad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

- Solving for $|0\rangle$ and $|1\rangle$ yields:

$$|0\rangle = \frac{|+\rangle + |-\rangle}{\sqrt{2}}, \qquad |1\rangle = \frac{|+\rangle - |-\rangle}{\sqrt{2}}.$$

- $|\psi\rangle = a\left(\frac{|+\rangle+|-\rangle}{\sqrt{2}}\right) + b\left(\frac{|+\rangle-|-\rangle}{\sqrt{2}}\right) = \frac{a+b}{\sqrt{2}}\,|+\rangle + \frac{a-b}{\sqrt{2}}\,|-\rangle.$

$|\psi\rangle = a|0\rangle + b|1\rangle \qquad |a|^2 + |b|^2 = 1$

Global phase versus relative phase

$e^{i\theta}|\psi\rangle = e^{i\theta}a|0\rangle + e^{i\theta}b|1\rangle \qquad |e^{i\theta}a|^2 + |e^{i\theta}b|^2 = |e^{i\theta}|^2|a|^2 + |e^{i\theta}|^2|b|^2 = |a|^2+|b|^2 = 1$

- Two single-qubit states $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\phi\rangle = c|0\rangle + d|1\rangle$ are said to differ by the global phase θ if

$$|\psi\rangle = a|0\rangle + b|1\rangle = e^{\mathbf{i}\theta}(c|0\rangle + d|1\rangle) = e^{\mathbf{i}\theta}\,|\phi\rangle.$$

- If two quantum states differ by a global phase, they are considered to be same. We write $|\psi\rangle \sim |\phi\rangle$.
- The relative phase of a single-qubit state $|\psi\rangle = a|0\rangle + b|1\rangle$ is a number φ which satisfies the equation

$$\frac{a}{b} = e^{\mathbf{i}\varphi}\,\frac{|a|}{|b|}.$$

- Two quantum states with different relative phases are not the same quantum state.

$|\psi\rangle = a|0\rangle + b|1\rangle \qquad \frac{a}{b} = e^{i\varphi}\frac{|a|}{|b|}.$

Examples of qubits differing by a global phase

- Consider: $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{\frac{\mathbf{i}\pi}{4}}|1\rangle\right)$ and $\frac{1}{\sqrt{2}}\left(e^{-\frac{\mathbf{i}\pi}{4}}|0\rangle + |1\rangle\right)$

- The qubit state $\frac{1}{\sqrt{2}}\left(e^{-\frac{\mathbf{i}\pi}{4}}|0\rangle + |1\rangle\right) = \frac{e^{-\frac{\mathbf{i}\pi}{4}}}{\sqrt{2}}\left(|0\rangle + e^{\frac{\mathbf{i}\pi}{4}}|1\rangle\right)$

- Therefore, these two quantum states are the same.

Examples of qubits differing by relative phases

- Consider: $\frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(-|0\rangle + \mathbf{i}|1\rangle)$

- Let $$a|0\rangle + b|1\rangle = \frac{1}{\sqrt{2}}(\,|0\rangle + |1\rangle)$$
  and $$a'|0\rangle + b'|1\rangle = \frac{1}{\sqrt{2}}(-|0\rangle + \mathbf{i}|1\rangle).$$
  $$\frac{a}{b} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{1} = e^{0\mathbf{i}}\frac{|a|}{|b|},\quad \text{and} \quad \frac{a'}{b'} = -\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\imath} = -\frac{1}{\mathbf{i}} = \mathbf{i} = e^{\frac{\pi\mathbf{i}}{2}}\frac{|a'|}{|b'|}.$$

  By definition the relative phase of the first qubit is 0 and the relative phase of the second qubit is $\frac{\pi}{2}$. Since they have different relative phases they are different quantum states.

Single qubit measurement

- A single-qubit measurement, M is associated to an orthonormal basis

$$\{|\Phi_1\rangle, |\Phi_2\rangle\}$$

- Measuring $|\Psi\rangle = a|0\rangle + b|1\rangle$ by M outputs either $|\Phi_1\rangle$ or $|\Phi_2\rangle$.

- The probability of outcome $|\Phi_1\rangle$ is $|\langle\Phi_1|\Psi\rangle|^2$

- The probability of outcome $|\Phi_2\rangle$ is $|\langle\Phi_2|\Psi\rangle|^2$

Example 1

- Consider the single-qubit state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \mathbf{i}|1\rangle)$ and the measurement basis $\{|0\rangle, |1\rangle\}$.

- The measurement outcome is $|0\rangle$ with probability

$$|\langle 0|\Psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

- The measurement outcome is $|1\rangle$ with probability

$$|\langle 1|\Psi\rangle|^2 = \left|\mathbf{i}\,\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Calculations

- $\langle 0|\Psi\rangle = \langle 0| \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}|1\rangle\right) = \frac{1}{\sqrt{2}}\langle 0|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}\langle 0|1\rangle = \frac{1}{\sqrt{2}}.$

- $\langle 0|\Psi\rangle = \langle 1| \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}|1\rangle\right) = \frac{1}{\sqrt{2}}\langle 1|0\rangle + \frac{1}{\sqrt{2}}\mathbf{i}\langle 1|1\rangle = \frac{1}{\sqrt{2}}\mathbf{i}.$

Example 2

- Consider the single-qubit state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + \mathbf{i}|1\rangle)$ and the measurement basis $\{|+\rangle, |-\rangle\}$.

- The measurement outcome is $|+\rangle$ with probability
$$|\langle +|\Psi\rangle|^2 = \left|\frac{1}{2}(1+\mathbf{i})\right|^2 = \frac{1}{2}.$$

- The measurement outcome is $|-\rangle$ with probability
$$|\langle -|\Psi\rangle|^2 = \left|\frac{1}{2}(1-\mathbf{i})\right|^2 = \frac{1}{2}.$$

Calculations

- $\langle + | \Psi \rangle = \left(\frac{1}{\sqrt{2}} (\langle 0 | + \langle 1 |) \right) \left(\frac{1}{\sqrt{2}} (| 0 \rangle + \mathbf{i} | 1 \rangle) \right) = \frac{1}{2} (1 + \mathbf{i})$.

- $\langle - | \Psi \rangle = \left(\frac{1}{\sqrt{2}} (\langle 0 | - \langle 1 |) \right) \left(\frac{1}{\sqrt{2}} (| 0 \rangle + \mathbf{i} | 1 \rangle) \right) = \frac{1}{2} (1 - \mathbf{i})$.

- $| \langle + | \Psi \rangle |^2 = \left| \frac{1}{2} (1 + \mathbf{i}) \right|^2 = \frac{1}{2}$.

- $| \langle - | \Psi \rangle |^2 = \left| \frac{1}{2} (1 - \mathbf{i}) \right|^2 = \frac{1}{2}$.

Inner Product

An ***inner product*** $\langle v_2|v_1\rangle$, or ***dot product***, on a complex vector space V is a complex function defined on pairs of vectors $|v_1\rangle$ and $|v_2\rangle$, satisfying

- $\langle v|v\rangle$ is non-negative real,
- $\langle v_2|v_1\rangle = \overline{\langle v_1|v_2\rangle}$, and
- $(a\langle v_2| + b\langle v_3|)|v_1\rangle = a\langle v_2|v_1\rangle + b\langle v_3|v_1\rangle$
- where $\bar{z}$ is the complex conjugate $\bar{z} = a - \imath b$ of $z = a + \imath b$.

Orthogonality of vector

- Two vectors $|v_1\rangle$ and $|v_2\rangle$ are said to be ***orthogonal*** if $\langle v_1|v_2\rangle = 0$.
- A set of vectors is orthogonal if all of its members are orthogonal to each other.
- The ***length***, or norm, of a vector $|v\rangle$ is $||v\rangle| = \sqrt{\langle v|v\rangle}$.
- Since all vectors representing quantum states are of unit length, $\langle x|x\rangle = 1$ for any state vector $|x\rangle$.

Orthonormal bases

- A set of vectors is said to be ***orthonormal*** if all of its elements are of length one, and orthogonal to each other: a set of vectors $B = \{|\beta_1\rangle, |\beta_2\rangle, \dots, |\beta_n\rangle\}$ is orthonormal if $\langle\beta_i|\beta_j\rangle = \delta_{ij}$ for all i, j, where
- $\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherewise.} \end{cases}$
- A basis of a vector space consisting of orthonormal vectors is said to be an ***orthonormal basis***.

For the $\boldsymbol{n}$-dimensional space over $\mathbb{C}$

- In general, a vector $|v\rangle$ in an n dimensional space is a column vector $v = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$. The conjugate transpose of ket is called ***bra*** and is written as $\langle v|$.

- The matrix corresponding to $\langle v|$ is $v^\dagger = (\bar{a}_1, \dots, \bar{a}_n)$.

The Inner Product

- If $|a\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$ and $|b\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$, the inner product

$$\langle a|b\rangle = \langle a||b\rangle = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \cdots & \bar{a}_n \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} \bar{a}_i b_i$$