Single Qubit states and their visualization

Introduction to Quantum Computing

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May 7, 2023

Outline

What we know so far

- In quantum computing, the fundamental unit of information is a quantum bit (qubit).
- A qubit can be used to represent the state of many physical objects.
- We considered photons, where the qubit was used to represent the polarization state of the photon.
- Vertical ($|0\rangle$) and horizontal ($|1\rangle$) polarization states were considered as the basis for these qubits.

What we know so far

- A real superposition (linear combination) of $|0\rangle$ and $|1\rangle$ formed oblique polarization states.
- Complex superposition also accounts for circular and elliptical polarization states.
- A vertically aligned polarizer can be used to "measure" the polarization state of these photons.
- If a photon passes through this polarizer its polarization is measured to be vertically aligned. If a photon is blocked (absorbed) by this polarizer , its polarization is measured to be horizontally aligned.

What we know so far

- Measuring a qubit state with respect to a basis will change the quantum state of the photon.
- After measurement, the state of the photon changes to one of the basis states.
- The outcome of a quantum measurement is in general represented by a real number.
- In the case of a qubit, the measurement outcome is represented by a single classical bit.

The vector space $\mathbb{C}^2$

- A vector in the space $\mathbb{C}^2$ is represented as follows:

$$|\psi\rangle = \begin{pmatrix} a \\ b \end{pmatrix} \; ; \; a, b \in \mathbb{C}$$

- If $|\phi\rangle = \begin{pmatrix} c \\ d \end{pmatrix}$, the inner products $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$ are defined as:

$$\langle\phi|\psi\rangle = \begin{pmatrix} \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a\bar{c} + b\bar{d}$$

and

$$\langle\psi|\phi\rangle = \left\{ \begin{pmatrix} \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \right\}^{\dagger} = \bar{a}c + \bar{b}d = \overline{\langle\phi|\psi\rangle}$$

Normalization and the Qubit state

- Consider the quantity $\langle\psi|\psi\rangle$ which has a value

$$\langle\psi|\psi\rangle = |a|^2 + |b|^2$$

- Therefore a vector $|\psi\rangle$ is a qubit state if $\langle\psi|\psi\rangle = 1$.
- A general vector $|\phi\rangle$ can be converted to a qubit state $|\tilde{\phi}\rangle$ as follows:

$$|\tilde{\phi}\rangle = \frac{1}{\langle\phi|\phi\rangle^{\frac{1}{2}}}\,|\phi\rangle$$

- The above process is referred to as normalization and the quantity $\langle\phi|\phi\rangle^{\frac{1}{2}}$ is called the norm of the vector $|\phi\rangle$.
- Qubit states can now be formally defined as vectors in $\mathbb{C}^2$ with a unit norm.

Basis States

- Defining the vectors $|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $|1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, the vector $|\psi\rangle$ is now expressed as:

  $$|\psi\rangle \;=\; a\,|0\rangle + b\,|1\rangle$$

  It is possible to represent any vector in $\mathbb{C}^2$ is the above manner.

- The set $\{|0\rangle, |1\rangle\}$ is called the standard or the computational basis and is said to span $\mathbb{C}^2$.

- The inner products have the values; $\langle 0|0\rangle = \langle 1|1\rangle = 1$ and $\langle 0|1\rangle = \langle 1|0\rangle = 0$.

- A basis satisfying the above property is known as an orthonormal basis. It is worth noting that orthonormal basis vectors are valid qubit states.

Coordinates and Projections

- Considering the vector $|\psi\rangle$ and the computational basis $\{|0\rangle, |1\rangle\}$, the following is true

$$\langle 0|\psi\rangle = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = a \; ; \; \langle 1|\psi\rangle = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = b$$

- The coordinates of the vector $|\psi\rangle$ can be defined in terms of the inner product with the basis vectors.

- The inner product of $|\psi\rangle$ with a basis vector is known as the projection of $|\psi\rangle$ along that basis vector.

- The vector $|\psi\rangle$ can now be represented in terms of the projections as follows:

$$|\psi\rangle = \langle 0|\psi\rangle |0\rangle + \langle 1|\psi\rangle |1\rangle$$

"Non-standard" Basis

- The idea of coordinates and projections is true for any orthonormal basis.
- Consider the basis orthonormal basis $\{|+\rangle, |-\rangle\}$, where $|+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$ and $|-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$.
- In this basis, the vector $|\psi\rangle$ is represented as:

$$|\psi\rangle = \langle+|\psi\rangle\,|+\rangle + \langle-|\psi\rangle\,|-\rangle$$

evaluating the inner products give the result

$$|\psi\rangle = \frac{a+b}{\sqrt{2}}\,|+\rangle + \frac{a-b}{\sqrt{2}}\,|-\rangle$$

Projections and Photons

- The state of an obliquely polarized photon $|\chi\rangle = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$, this state can be represented in the standard basis as:

$$|\chi\rangle = \cos\theta\,|0\rangle + \sin\theta\,|1\rangle$$

where the basis vectors $|0\rangle$, $|1\rangle$ represent the vertical and horizontal polarization states.

- The probability that this photon is transmitted by a polarized aligned along the $|0\rangle$ is given by $|\cos\theta|^2$.

- The transmission probability can now be correctly reinterpreted as $|\langle 0|\chi\rangle|^2$ and this result maybe used to calculate the transmission probabilities for circularly polarized light as well.

- The above result can be generalised to transmission probabilities for a polarizer oriented along any direction using the same method as described before.

Polar Coordinates

- A complex number, $z = x + \mathbf{i}\, y$ can be represented in the polar form as, $z = \mathrm{r}e^{\mathbf{i}\phi}$ where

$$\mathrm{r} = \sqrt{x^2 + y^2}\ ;\ \phi = \arctan\left(\frac{y}{x}\right)$$

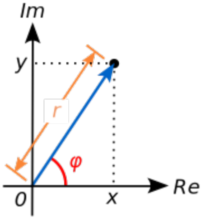

Figure 1: Figure showing the representations of a complex number. Source: Wikipedia[2]

[2] Complex_number_illustration.svg: The original uploader was Wolfkeeper at English Wikipedia. derivative work: Kan8eDie (talk) `https://commons.wikimedia.org/wiki/File:Complex_number_illustration_modarg.svg`, "Complex number illustration modarg", `https://creativecommons.org/licenses/by-sa/3.0/legalcode`

The global phase

- The unit complex number $e^{\mathbf{i}\phi}$ is also referred to as a phase factor.
- Multiplying a photon state $|\psi\rangle$ with a phase factor gives a state $e^{\mathbf{i}\phi}|\psi\rangle$. The phase factor is now referred to as a global phase factor.
- The projection of this new vector with respect to a basis state (say $|0\rangle$) is given by $e^{\mathbf{i}\phi}\langle 0|\psi\rangle$.
- While this is different from $\langle 0|\psi\rangle$, it should be noted that this new state will have the same transmission probabilities as that of $|\psi\rangle$.
- It is therefore not possible to distinguish $e^{\mathbf{i}\phi}|\psi\rangle$ from $|\psi\rangle$ by performing polarization measurements.
- Therefore, states that differ from each other by only a global phase are considered to be equivalent.

Qubit state in polar coordinates

- Consider the state $|\psi\rangle$ in the standard basis with the coordinates represented in polar coordinates. $a = \mathrm{r}_0 e^{\mathbf{i}\phi_0}$ and $b = \mathrm{r}_1 e^{\mathbf{i}\phi_1}$
- The state may now be expressed as follows:

$$\begin{aligned} |\psi\rangle &= \mathrm{r}_0 e^{\mathbf{i}\phi_0} |0\rangle + \mathrm{r}_1 e^{\mathbf{i}\phi_1} |1\rangle \\ &= e^{\mathbf{i}\phi_0} \left(\mathrm{r}_0 |0\rangle + \mathrm{r}_1 e^{\mathbf{i}(\phi_1 - \phi_0)} |1\rangle \right) \\ &\equiv \mathrm{r}_0 |0\rangle + \mathrm{r}_1 e^{\mathbf{i}(\phi_1 - \phi_0)} |1\rangle \end{aligned}$$

setting $\phi_1 - \phi_0 = \phi$,

$$|\psi\rangle = \mathrm{r}_0 |0\rangle + \mathrm{r}_1 e^{\mathbf{i}\phi} |1\rangle$$

additionally,

$$\langle\psi|\psi\rangle = 1 \;\Rightarrow\; \mathrm{r}_0^2 + \mathrm{r}_1^2 = 1$$

Parameter Selection

- Since, $0 \leq \mathrm{r}_0, \mathrm{r}_1 \leq 1$ and $\mathrm{r}_0^2 + \mathrm{r}_1^2 = 1$ it is possible to represent $\mathrm{r}_0 = \cos(\theta/2)$ and $\mathrm{r}_1 = \sin(\theta/2)$ where, $\theta \in [0, \pi]$

- Since, $e^{\mathbf{i}\phi} = \cos\phi + \mathbf{i}\sin\phi \;\Rightarrow\; \phi \in [0, 2\pi)$. This angle is known as the relative phase.

- These parameters are identical to the angle variable in spherical polar coordinates. Therefore, each qubit state defined using these parameters corresponds to a point on a unit sphere.

- The qubit state in terms of this parameter is represented as

$$|\psi\rangle \;=\; \cos(\theta/2)\,|0\rangle + e^{\mathbf{i}\phi}\sin(\theta/2)\,|1\rangle$$

- The sphere on which the point corresponding to the state is present is known as the *Bloch Sphere*.

Qubit states on the Bloch Sphere

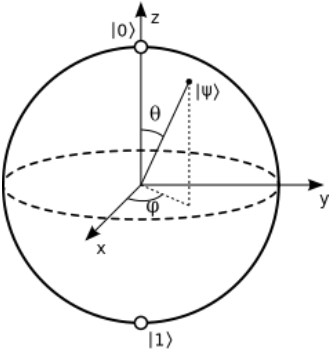

Figure 2: Figure showing a point on the Bloch Sphere. Source: Wikipedia[3]

[3] Smite-Meister
(https://commons.wikimedia.org/wiki/File:Bloch_sphere.svg), "Bloch sphere",
(https://creativecommons.org/licenses/by-sa/3.0/legalcode)

Qubit states on the Bloch Sphere

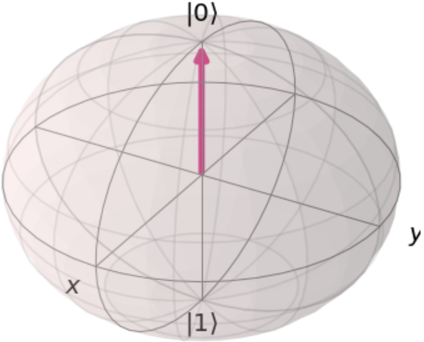

Figure 3: Figure showing the the point corresponding to $|0\rangle$

Qubit states on the Bloch Sphere

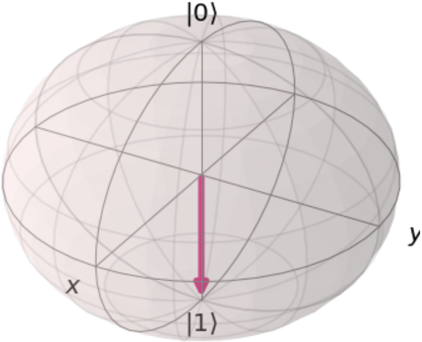

Figure 4: Figure showing the point corresponding to $|1\rangle$

Qubit states on the Bloch Sphere

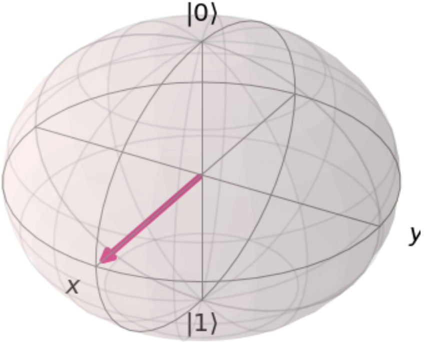

Figure 5: Figure showing the point corresponding to $|+\rangle$