Question 1

For some p_{i0}, p_{i1} let

$$M_i = \begin{pmatrix} 1 & 0 & p_{i0} & -p_{i1} \\ 0 & 1 & p_{i1} & p_{i0} \end{pmatrix}$$

1. It follows that $B_i = M_i^T M_i$ is symmetric as
$$B_i = M_i^T M_i = (M_i^T M_i)^T = B_i^T$$

2. To show that B_i is positive semi-definite we must show that real number $x^T B_i x =\geq 0$ for any $x \in \mathbb{R}^4$. It follows that
$$x^T B_i x = x^T M_i^T M_i x = (M_i x)^T (M_i x) = \|M_i x\|^2 \geq 0$$
And therefore B_i is positive semi-definite.

Question 2

Considering the optimization problem

$$x^* = \text{argmin}_{x \in \mathbb{R}^4} \sum_{i=1}^{n} \|M_i x - \pi_i\|_2^2 \quad \text{ s.t. } \quad x_3^2 + x_4^2 = 1$$

1. We can manipulate the problem as follows
$$\begin{aligned} x^* &= \text{argmin}_{x \in \mathbb{R}^4} \sum_{i=1}^{n} \|M_i x - \pi_i\|_2^2 \\ &= \text{argmin}_{x \in \mathbb{R}^4} \sum_{i=1}^{n} (M_i x - \pi_i)^T (M_i x - \pi_i) \\ &= \text{argmin}_{x \in \mathbb{R}^4} \sum_{i=1}^{n} x^T M_i^T M_i x - 2\pi_i^T M_i x + \|\pi_i\|^2 \\ &= \text{argmin}_{x \in \mathbb{R}^4} x^T \left(\sum_{i=1}^{n} M_i^T M_i \right) x + \left(\sum_{i=1}^{n} -2\pi_i^T M_i \right) x \end{aligned}$$
We can ignore constants as they do not affect minimization. Therefore we can say that
$$M = \sum_{i=1}^{n} M_i^T M_i \qquad g = \sum_{i=1}^{n} -2\pi_i^T M_i \qquad W = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
With the given W it follows that
$$\begin{aligned} x^T W x &= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & x_3 & x_4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \\ &= x_3^2 + x_4^2 \end{aligned}$$

Question 2 continued on next page. . .

Therefore our problem becomes

$$x^* = \text{argmin}_{x \in \mathbb{R}^4} \, x^T M x + g^T x \quad \text{s.t. } x^T W x = 1$$

as required.

2. We will now show that M, W are positive semi-definite.

   (a) As M is a composition of positive semi-definite matrices it follows that

   $$x^T M x = x^T \left(\sum_{i=1}^{n} M_i^T M_i \right) x = \sum_{i=1}^{n} x^T M_i^T M_i x = \sum_{i=1}^{n} x^T B_i x \geq 0$$

   As B_i are positive semi-definite it follows that the summation of values $x^T B_i x$ which are strictly non-negative would be strictly non-negative as well. Therefore M must be positive semi-definite.

   (b) Given above we know that $x^T W x = x_3^2 + x_4^2 \geq 0$ as squared values are strictly non-negative. Therefore W must be positive semi-definite.