Le Físhe

Daniel Sapojnikov

Ori Uziel

Tom Bareket

Yair Maoz

April 4, 2021

Contents

Part I

Pure Mathematics

Chapter 1

Sequences and Series

In mathematics, a sequence is an enumerated collection of objects in which repetitions are allowed, with the goal of maintaining order. Similar to a set, it contains an amount of members, possibly even infinite. Those can be called "Elements" and "Terms". The number of elements in a sequence is considered it's length. However, unlike a set. Elements on a sequence may used multiple times, and the order of the elements does indeed matter.
Traditionally, a sequence is defined as a function whose domain is either the set of the natural numbers (for infinite sequences), or the set of the first n natural numbers (for a sequence of finite length n).
The nth element of a series a is usually written as a_n, and the first element of the series is usually a_1.

1.1 Arithmetic Series

Arithmetic series are some of the most well known types of series.
In arithmetic series, each element is equal to the previous one plus a constant, usually d.
For example:$\rightarrow (6, 16, 26, 36, 46, 56....)$
In this sequence, a value of 10 is added to each element in order to calculate the next one ($d = 10$).

The arithmetic mean (M) of two numbers a and b is defined as the number that makes the following arithmetic series:

$$(a, M, b)$$

By the definition of the arithmetic series, we can conclude that $b - M = M - a$.

Therefore, $M = \frac{a+b}{2}$.

This also leads to the following identity: For an arithmetic series a_n, $a_n + a_{n+2} = 2 \cdot a_{n+1}$.

In fact, $a_n + a_{n+2k} = 2 \cdot a_{n+k}$, for $k \in N$.

Calculating numbers in an arithmetic sequence:

$$a_n = a_1 + (n-1) \cdot d$$

This is the formula used to calculate the nth element of the arithmetic series a. To show how it is used, we shall examine it in action:

$$(9, 21, 33, 45, 57, 69)$$

For the sake of demonstration, we shall calculate the seventh element of the series.

First and foremost. It is a requirement to find the difference between each element, d. Simply by taking one arbitrary element, and subtracting

the value of the previous element from it, e.g $a_4 - a_3$. In this case this action will result in a difference of 12, thus $d = 12$.
Now that we know that $d = 12$, we can plug our new information into our formula:

$$a_n = a_1 + (n-1) \cdot d$$

Since we want to find the value of the seventh element of the sequence, we will set $n = 7$. $a_1 = 9$, due to it being the 1st element of the sequence.

$$a_7 = 9 + (7-1) \cdot 12 = 9 + 6 \cdot 12 = 9 + 72 = 81$$

By making a simple addition problem using the existing elements given to us we can find that the seventh element of this sequence is *81*, without manually calculating each element up to the seventh one.

Exercises :

1.2 Geometric Sequence

The geometric sequence are the same as their arithmetic brother with a few differences. It has a geometric mean just like the arithmetic mean, and is used in similar fashion. The geometric mean is solved via it's own original formula:

$$M = \sqrt{a \cdot b}$$

For the sake of demonstration we shall utilize the following sequence: $(5, 20, 80, 320, 1280)$

The goal of the question here is to find the geometric mean of 80 and 1280

$$M = \sqrt{80 \cdot 1280} = \sqrt{102,400} = 320$$

Between geometric and arithmetic sequences , instead. A common ratio exists. This is the common ratio, have a closer look at this example :

$$(5, 25, 125, 625)$$

In this scenario the common ratio 5. Seeing as every new elements in the sequence equals the previous one when multiplied by 5.

However, in spite of the fact we have a ratio we still have to find the following element. For that reason this formula is used:

$$a_n = a_1(R)^{n-1}$$

a_n represents the value and position of the given element on the sequence.It's intention is to find the value of a, with n representing it's position on the sequence
R is the ratio of the sequence.
Examples:

$$(6, 18, 54, 162)$$

The first thing we must do is finding the common ratio. In this case dividing 18 by the element. Which is 6. Reveals that the common ratio in this sequence is 3.
This question tasks us with finding the sixth element of the sequence. For this purpose we shall use the following formula:

$$a_n = a_1(R)^{n-1}$$

$$a_6 = 6 \cdot (3)^6 - 1 = 6 \cdot 3^5 = 6 \cdot 243 = 1458$$

This is our answer. In theory at least. If we want to find out whether our answer is correct or not. Simply by checking for the value of the fifth element using the same formula:

$$a_5 = 6 \cdot (3)^5 - 1 = 6 \cdot 3^4 = 6 \cdot 81 = 486$$

Now we shall multiply this fifth of elements using the value of R that we have acquired. In this case 3:

$$486 \cdot 3 = 1458$$

1.3 p- Series

Another type of series is the p- Series. The p- series is defined as:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

When p=1, the p- Series is called the harmonic series, which diverges. When $p \leq 1$, the p- Series diverges. However, when $p > 1$, the series is equal to $\zeta(p)$, where ζ is the Riemann zeta function. The p- Series has special values, such as:

- p=2: This is the Bazel problem, and the value of this p- Series is $\frac{\pi^2}{6}$.
- p=3: This p- Series converges to Apéry's constant, which is approximately equal to 1.202.

1.4 Introduction

1.5 Sigma Notation

What is the Sigma Notation ? The Sigma helps us in writing a sum or a series of sums:

$$\sum_{k=1}^{5} 2k = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) = 30$$

Breaking down The Sigma sign allows us to see how it looks:

$$\sum_{k=a}^{n}$$

k is the iterator of the sum. Though people might use other letters instead of it on occasion, however they share the same value and meaning.
a represents the first number k will iterate over. Meanwhile, n represents last number k will iterate over.
After every iteration, the value of k grows by 1.

The most essential piece of Sigma Notation is it's "expression". Read as the sum of the Sigma as it goes from start to finish.

Examples:

$$\sum_{k=1}^{7} k^3 = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 = 784$$

We shall begin from the number 1, and continue our spiral up until reaching then number 7 (1-7). Enlarging the value of k as we go up in

numbers.
Judging by the information previously given to us, we can determine that 1 represents the value of a while 7 represents the value of n. So k hops from 1 to 7.

1.5.1 Summation Formulas

$$\sum_{k=a}^{n} k = \frac{n(n+1)}{2}$$

$$\sum_{k=a}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{k=a}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$

These are the summation formulas. Which can provide aid in finding the sum of an expression or a function.

$$\sum_{k=1}^{3} 5k^2 = ?$$

how do we do that ? We can plug in the "k" value but we have to put some time into this However , we can use a useful method so Lets go over this we can put the 5 before the sigma Like this

$$5\sum_{k=1}^{3} k^2$$

Thus making the exercise more easy to solve , all we have to do is to use one of the summation formulas and take its solution then multiply it by 5 Lets do that!

$$5\sum_{k=1}^{3} k^2 = 5\frac{(3(3+1)(2\cdot 3+1))}{6} = 5\frac{(3(4)\cdot(7))}{6} = 5\cdot\frac{84}{6} = 5(14) = 70$$

We got the right solution , now we are going to go through more complex sigma's .

$$\sum_{k=1}^{7}(k-1)^2$$

So how do we solve that , there is a method which allows you to solve. the equation a little bit faster so lets get over it. we can break the sigma into 2 or 3 sigma's,let us see how it looks. But before then , we have to explain why we are doing this. Because we have here a Binomial Expression.

$$(k-1)^2$$

which is equal to

$$(k-1)(k-1)$$

or

$$k^2-2k+1$$

we have to break it because we have here 3 expressions.

$$k^2, 2k, 1$$

now that we have simplified the binomial expression we can break it into 3 sums.

$$\left[\sum_{k=1}^{7}(k^2)\right]+\left[\sum_{k=1}^{7}(-2k)\right]+\left[\sum_{k=1}^{7}(1)\right]$$

So now that we have our sums we have solve each of them separately with the summation formulas. For now let's skip and solve this later. Lets move on to some exercises before reaching the next section.a

$$\sum_{t=1}^{5} 7t^2$$

$$\sum_{k=1}^{3} (6k-2)^2$$

$$\sum_{k=1}^{7} \frac{1}{k}$$

$$\sum_{k=0}^{\infty} sin(\pi \cdot k)$$

$$\sum_{k=-1}^{\infty} k^3$$

1.6 Pi Notation

So we have learned so far how to solve the sigma notation series and now we are going to move on to the Pi Notation. So it has the same basics,the same idea and the same build. But instead of adding we are going to multiply it.

So lets see what we are going to do with $\prod$

$$\prod_{k=a}^{n}$$

so this is the Pi Notation, Now lets See The Method.

$$\prod_{k=1}^{\infty} \frac{1}{k}$$

As we can see, at $k = \infty$, $\frac{1}{k} = \frac{1}{\infty} = 0$ so we know that there is multiplication by 0, thus the answer is 0.

lets see another example :

$$\prod_{k=1}^{5} k$$

So Here the answer is simple :

$$\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 5! = 120$$

What is this exclamation mark supposed to mean ? Well its called the factorial series For example.

n! = n(n-1)(n-2)(n-3)......n

now lets plug in some numbers :

$$5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$$

$$3! = 1 \cdot 2 \cdot 3 = 6$$

$$0! = 1$$

It is wildly acceptable in the range of mathematics and its the right answers we are not going to go over it deeper we just wanted to show you the idea behind factorial numbers at the end you will see a combinatorics chapter there we are going to get deeper into factorial numbers but right now lets move on. So lets move one to the exercises.

$$\prod_{k=2}^{6} \frac{12}{k^2}$$

Find

$$\prod_{k=2}^{n} \frac{(k^2 - 1)}{k + 1}$$

in terms of n.

$$\prod_{k=0}^{7} \frac{6!}{k}$$

Chapter 2

Combinatorics

2.1 Introduction

Combinatorics is an area of mathematics primarily concerned with counting, both as a means and an end in obtaining results, and certain properties of finite structures. It is closely related to many other areas of mathematics and has many applications ranging from logic to statistical physics, from evolutionary biology to computer science, etc.

2.1.1 The Factorial

The factorial is one of the most basic tools in combinatorics. We usually use it to find the number of ways to order a list. n factorial (written $n!$) is defined as the number of unique ways to order a list with n elements. For example, we can reorder {1,2,3} in 6 different ways: {1,2,3}; {1,3,2}; {2,1,3}; {2,3,1}; {3,1,2}; {3,2,1} Therefore, $3! = 6$. The general formula for the factorial is as follows:

$$n! = \prod_{k=1}^{n} k = n(n-1)(n-2)\dots(2)(1)$$

Now, let's go through some exercises :

$$\text{Evaluate } \frac{4!}{(5-3)!}$$

$$\text{Evaluate } \frac{7! \cdot (5-4)!}{2}$$

Now we are going to talk about pascals triangle , and this triangle help us to solve binomials quickly and effectively. Here is the triangle :

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

For example lets solve this binomial expression : Here is what we are going to do and then we will explain it to you step-by-step.

$$(a+b)^2 = 1a^2 \cdot (b) + 2a^1 \cdot (b^1) + 1a^0 \cdot (b^2) = a^2 + 2ab + b^2$$

Now let us explain it to you. In theory pascals triangle has ∞ number of rows but we however are going to use less. So every power in any binomial expression indicates our row so when we see the power 5 we are going to go to the first row with a five on it. As you see, our power in this binomial expression is 2 so what are going to do is go to line number 2 in pascals triangle. Therefor we are going to use the second row which is 121. Now Using pascals triangle is all about the power of our binomial expression. For example if our power is 3 we are going to go to the third line and use it and so on. So now we have to understand how to use the LINES of Pascals triangle. In the previous example we used the second row which is 121 as you can see we put 121 as our coefficients. So that means, if our power is 5 for example so the line is going to be fifth line $1, 5, 10, 10, 5, 1$. now all we have to do is to put the row that we picked and plug it in as our coefficients. Its important to add that our powers at the equation are going to sum up to the power of the binomial expression.

so lets use this example :

Our power $= 5$ Coefficients $=1, 5, 10, 10, 5, 1$ And our powers in every expression are going to sum up to 5 since its our power.

$$(x+c)^5 = 1x^5 \cdot c^0 + 5x^4 \cdot c^1 + 10x^3 \cdot c^2 + 10x^2 \cdot c^3 + 5x^1 \cdot c^4 + 1x^0 \cdot c^5 =$$

$$x^5 + 5x^4c + 10x^3c^2 + 10x^2c^3 + 5xc^4 + c^5$$

So that is pascals triangle method for solving binomial expression , but we are going do to some exercises :

$$(a+b)^4$$

$$(\frac{1}{x}+x)^3$$

$$(Q+\sqrt{2})^7$$

Note : (leave the $\sqrt{2}$ as it is)

2.2 The binomial Theorem

Now that we know how to solve binomial expressions we have to talk about the binomial Theorem , this particular theorem allows us to solve binomial expressions too , so there are couple of ways to solve these and you can choose any of them.

So let us see the formula :

$$(a+b)^n = \sum_{k=0}^{n} \frac{n!}{(n-k)! \cdot k!} \cdot a^{n-k} \cdot b^k$$

But before jumping into conclusion we have to talk about this specific part

$$\rightarrow \frac{n!}{(n-k)! \cdot k!}$$

This Part can be seen like this $\binom{n}{k}$ so this $\rightarrow \frac{n!}{(n-k)! \cdot k!}$ is the exact the same as $\binom{n}{k}$. as we said earlier $0! = 1$ it is a mathematical fact. So in the binomial theorem the only thing that changes is the final solution and the k remember we have to plug in the k value until we reach our n value.

So let us see an example :

$$(a+b)^2 = \sum_{k=0}^{n} \frac{n!}{(n-k)! \cdot k!} \cdot a^{n-k} \cdot b^k$$

$$= \sum_{k=0}^{2} \frac{2!}{(2-k)! \cdot k!} \cdot a^{2-k} \cdot b^k$$

$$= \frac{2}{2! \cdot 0!} \cdot a^2 \cdot b^0 + \frac{2}{1! \cdot 1!} \cdot a^1 \cdot b^1 + \frac{2}{0! \cdot 2!} \cdot a^0 \cdot b^2 = a^2 + 2ab + b^2$$

now all we have to do is to add them together like so :

$$a^2 + 2ab + b^2$$

This way we proved the famous mathematical constant $(a + b)^2 = a^2 + 2ab + b^2$ and solved the binomial expression along the way.

So now lets solve these exercises by using the binomial theorem :

$$(\frac{1}{x} + x)^3$$

$$(Q + \sqrt{2})^7$$

Chapter 3

Trigonometry

3.1 Introduction

Trigonometry is a branch of mathematics that goes overs the connections between triangles, sides and angles.
If you ever used a scientific calculator (and chances are that you did) you might've seen some odd functions called: **sin, cos and tan**. These little devils are used in trigonometry expressions.
The basic use of trigonometric functions is for finding missing segment lengths of triangles using exclusively angles.

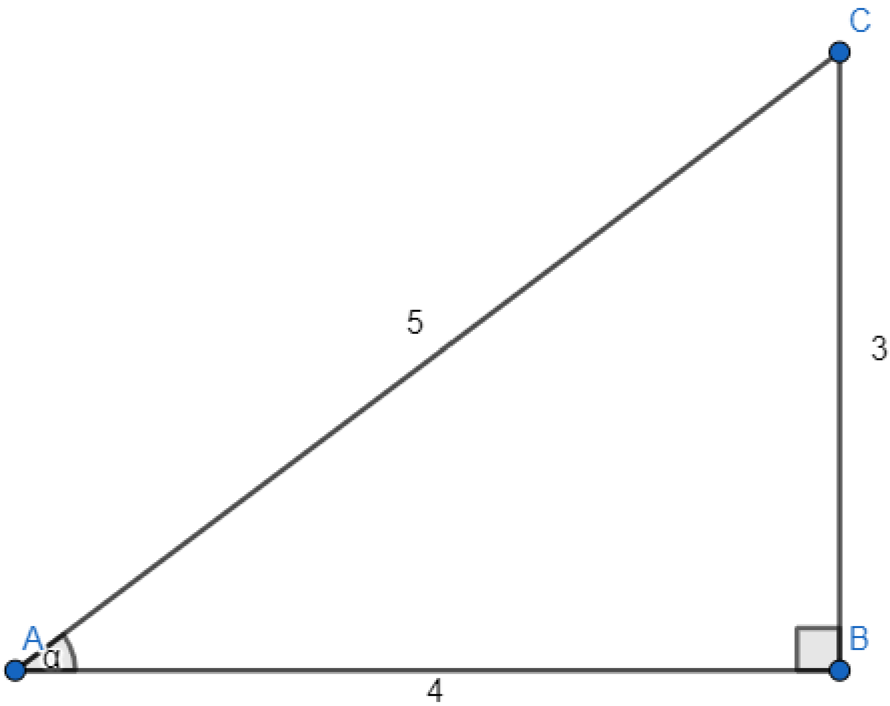
Here is the definition of the basic trigonometric ratios:

$$\sin = \frac{opposite}{hypotenuse}$$

$$\cos = \frac{adjacent}{hypotenuse}$$

$$\tan = \frac{opposite}{adjacent}$$

Disclaimer: The trigonometric functions are used in right triangles. And you use them on angles
For instance:



The sine (they lingual name of sin) of angle α is to be expressed as:

$$\sin\alpha$$

The value of $\sin\alpha$ would be the length of the line segment opposite to α (in this case 3), divided by the length of the hypotenuse (in this case 5).

$$\sin\alpha = \frac{3}{5} = 0.6$$

The value of $\cos\alpha$ would be the length of the line segment adjacent from α (in this case 4), divided by the length of the hypotenuse.

$$\cos\alpha = \frac{4}{5} = 0.8$$

The value of $\tan\alpha$ would be the length of the line segment opposite to α divided by the length of the line-segment adjacent to angle α.

$$\tan\alpha = \frac{3}{4} = 0.75$$

In order to remember which trigonometric functions are the ratio of which sides of a triangle, we highly recommend you remembering the following mnemonic: **soh cah toa**

$$\mathbf{s}in = \frac{\mathbf{o}pposite}{\mathbf{h}ypotenuse}$$
$$\mathbf{c}os = \frac{\mathbf{a}djacent}{\mathbf{h}ypotenuse}$$
$$\mathbf{t}an = \frac{\mathbf{o}pposite}{\mathbf{a}djacent}$$

3.2 Calculating missing right triangle sides using trigonometry

One very popular use of trigonometry is calculating missing sides of right triangles using an angle and a given side.

The reason they force the use of trigonometry in these riddles instead of the **Pythagorean theorem**. Is due to the lack of information regarding lengths of line-segments. Thus rendering the Pythagorean theorem useless.

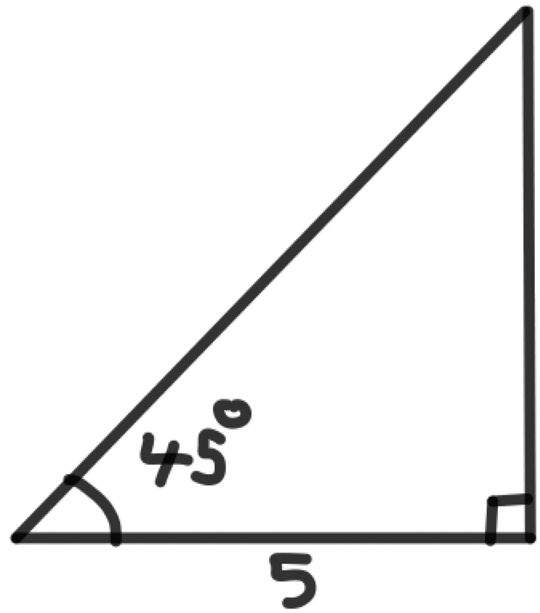
In order to do that, think about the definition of the trigonometric functions.

Take for example the trigonometric function:

$$\sin(\theta) = \frac{opposite}{hypotenuse}$$

If we know the sine of an angle and the hypotenuse length in the right triangle, we can calculate the length of the opposite line-segment. This comes straight out of the definition of sine. And the same goes for the other trigonometric functions.

Take this triangle for example:



In order to calculate the opposite side of the marked angle (which is 45°), we first need to find the right trigonometric function to use. To find it, we need to find a ratio that fits our current needs.

In this case, the ratio that fits is tan due to it containing both opposite (the side we want to find), as well as the adjacent side which is given to us already.

Using tan we can find the length of the opposite side. Simply by putting our given information into the tan:

$$\tan 45^\circ = \frac{opposite}{5}$$

By this logic, if we multiply the tan by the length of the adjacent side (which here is 5) we will get the length of the opposite side:

$$5tan45^\circ = opposite$$

Now all that is left is to calculate the value of $tan 45°$. Revealing that it equals to 1.

$$opposite = 5 \cdot 1 = 5$$

Now that we found the length of the opposite side we can easily calculate the length of the hypotenuse. Since now we have the value of 2 sides and can thus easily use the Pythagorean theorem.

3.2.1 Exercises

3.3 Inverse Trigonometric Functions

The inverse trigonometric functions are **arcsin, arccos and arctan**.

The trigonometric functions are also sometimes notated as: $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Note that the inverse trigonometric functions aren't things like: $\frac{1}{sin}$. But rather a way of notating the inverse function.

The inverse trigonometric functions are used to find the length of an angle within the ratio of line segments.

In order to find an angle, you need to know the value of at least 1 trigonometric ratio, in terms of the angle.

For instance, if you know that $\sin\theta = \frac{1}{2}$, we can use the calculator to find the angle θ. Using the calculator, we find that $\arcsin\theta = 30°$, meaning that $\theta = 30°$.

Chapter 4

Calculus

4.1 The definition of infinity

$$\infty - 1 = \infty$$

The riddle shown above can be used to best describe infinity. As it is not a number.

The term ∞? (Infinity) is used to describe the allegedly largest number there is. However, since there is no "largest number". ∞? is an undefined concept.

Take this example for instance:

$$\frac{1}{\infty} = 0$$

This expression doesn't necessarily equal 0. But due to the sheer quantity ∞ has, as well as the fact that it is undefined. Common sense would indicate the number the expression would realistically produce is unsurprisingly close to 0. But for the sake of simplicity due to how ∞ is undefined we instead say it just equals 0.

Think of it like how a basic division expression would be. The bigger the **absolute value** of the denominator the closer the answer will get to a 0 no matter what.
Think of it like this:

$$\frac{1}{1} = 1$$
$$\frac{1}{4} = 0.25$$
$$\frac{1}{20} = 0.05$$
$$\frac{1}{100} = 0.01$$
$$\vdots$$

∞ is also an odd case, primarily because it is undefined. For instance:

$$\infty - \infty = ?$$

This is an expression with no true answer, even though it's seemingly easy to solve with it resulting in 0.

The reason this is not the case is because of the lack of defined value of ∞. Meaning subtracting it from itself can still create any number in existence. Weird as it may, ∞ does not fit even with itself and is relatively separate from any expression or regular number.

In short this expression results in the same answer as dividing a number by 0. Being a practical error with no true answer.

4.2 The limit

4.2.1 Introduction

The "Limit" of a function is its value as its input "approaches" a certain number.

The accepted notation for the limit of a function f, as it's input approaches works as the following:

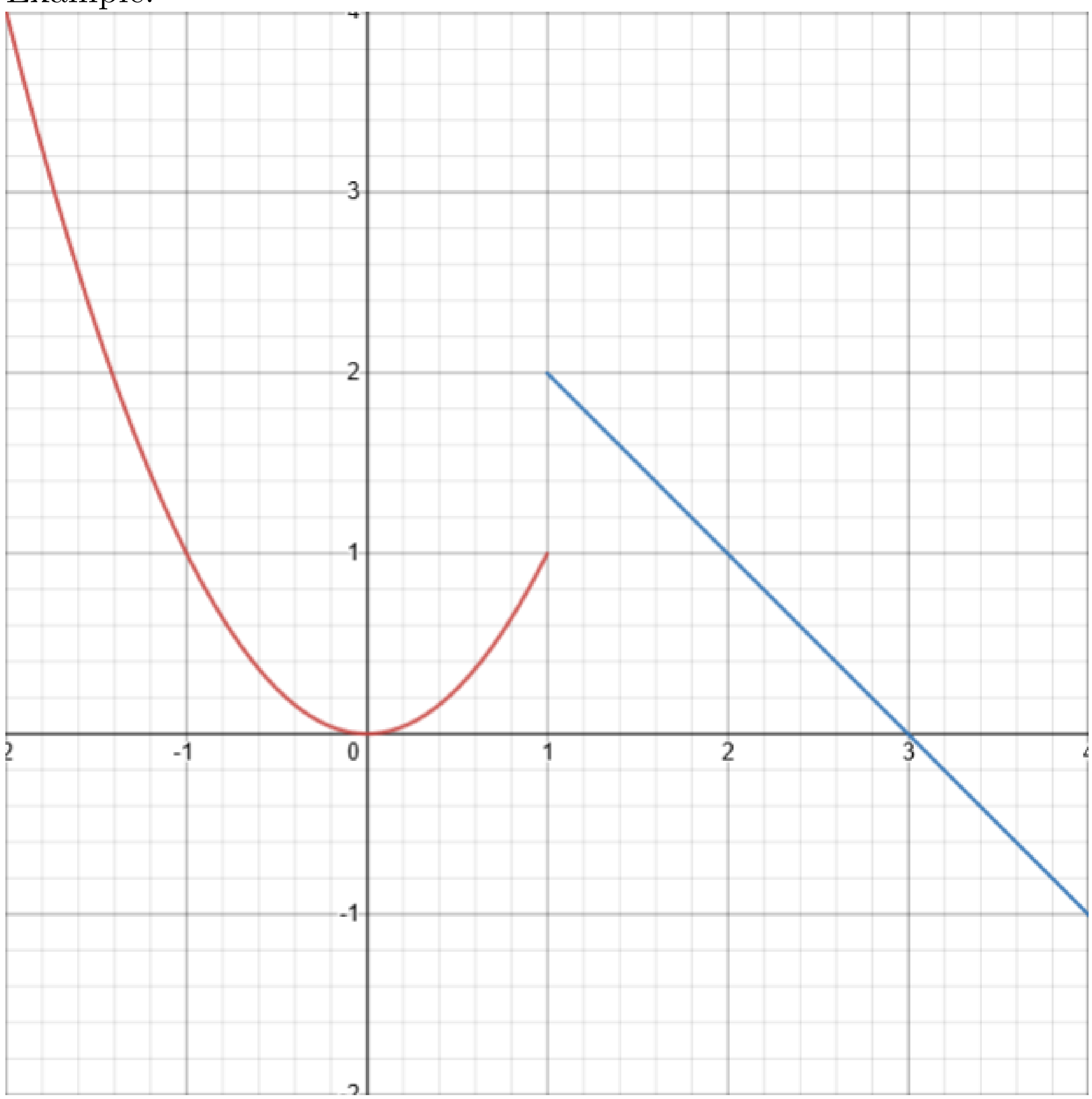
$$\lim_{x \to a} (f(x))$$

But with all of that approaching, we have to understand what sort of approaching we're dealing with. And indeed it is easy to answer:

When x approaches a, it is said that x is **infinitely** close to a.

Normally, a limit checks what value the function approaches as it's input gets closer to a certain value from **both** sides. Therefore, if the function approaches different values from different sides, it is said that the limit **does not exist**.

Example:



In this graph x approaches 1 from the negative direction. And with it the limit also approaches the number 1. However, from the positive direction, the limit approaches 2. Thus, the limit is said to **not** exist.

The notation for the limit of $f(x)$ as x approaches a from the negative side is $\lim_{x \to a^-} f(x)$. Similarly, the notation for the limit of $f(x)$ as x approaches a from the positive side is $\lim_{x \to a^+} f(x)$.

4.2.2 Logic Of Infinity And Zero

Sometimes while solving limits, you may get situations similar to $\lim_{x\to\infty} x^2$. The trick to these kinds of situations is to "think big and small". For example, notice that when $x \to \infty$, x is a very large number. As we know, when squaring a large number, we get a number that is even larger. Thus, as $x \to \infty$, we may conclude that x^2 will approach to ∞.

This trick comes handy in a lot of situations, and that is why it is crucial to understand.

Here are some examples:

- $$\lim_{x\to 0} \frac{1}{x} = \infty$$

  Explanation: when dividing by very small numbers, you get very large numbers.

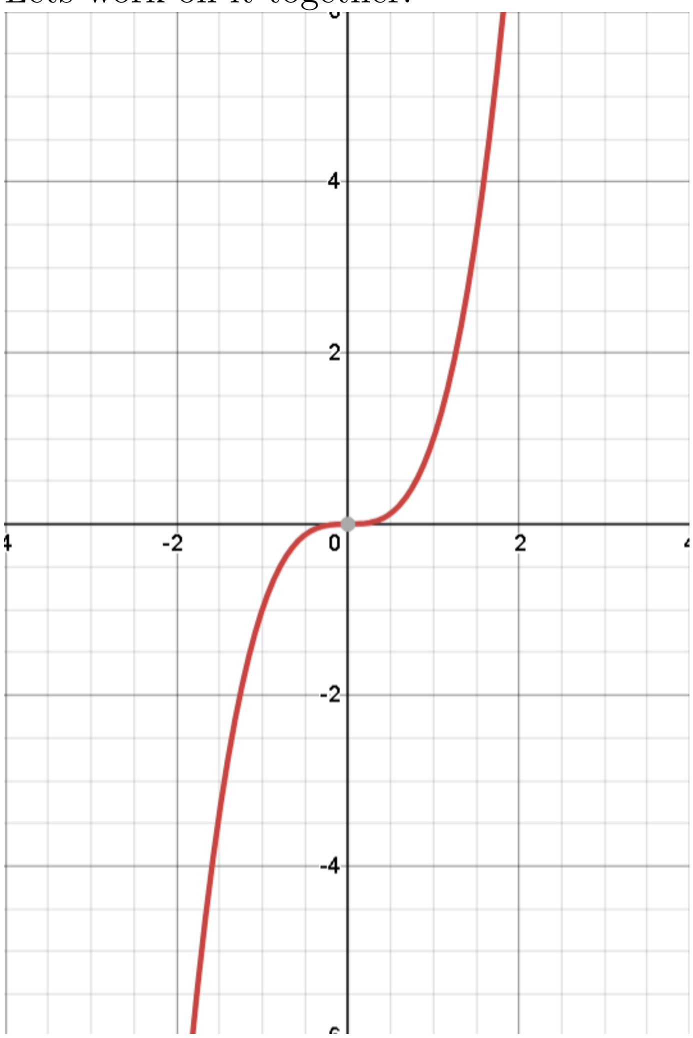
- $$\lim_{x\to \infty} \frac{1}{x} = 0$$

  Explanation: when dividing by very large numbers, you get very small numbers.

4.2.3 Exercises

- $$\lim_{x\to\infty} x^3$$

  Lets work on it together.

  As you can see, as x approaches ∞, the function also approaches ∞. Thus, $\lim_{x\to\infty} x^3 = \infty$

- $$\lim_{x \to 1} \frac{x^2+1}{x^2+4}$$
- $$\lim_{x \to 1} \frac{x^2+1}{x^2+2}$$
- $$\lim_{x \to 0} \frac{e^x}{\pi^x}$$
- $$\lim_{x \to \infty} \frac{ln(x)}{x}$$
- $$\lim_{x \to \infty} \frac{2x^2+x^2-1}{5x}$$
- $$\lim_{x \to 0} \frac{sin(x)}{cos(x)}$$
- $$\lim_{x \to 0} tan(x)$$
- $$\lim_{x \to e} ln(x)$$
- $$\lim_{x \to \infty} \frac{x \cdot tan^{-1}(x)}{\sqrt{sin(x)}}$$

4.2.4 Epsilon- Delta Definition

4.3 The Derivative

4.3.1 Introduction

The derivative is used to find the slope of the tangent line of any point on a function.

Now, let's talk about how we check when a function is differentiable and continuous.

4.3.2 Continuity & Differentiability

If a function is differentiable at a point, it means you can take the derivative of the function on that point.

Every point on a function that is both **continuous**, doesn't have "sharp turns" and its tangent line is not vertical, is differentiable at that point.

Luckily for us, limits come into play when we need to check if the function is continuous at a certain point.

To check if the limit of a function exists at a certain point, we construct 2 single-sided limits (one from negative and one from positive direction) approaching the x value of the function where we want to check the continuity of.

If the 2 one-sided limits are equal, then the limit at the point exists.

If the one-sided limits aren't equal to each other, the function at that point isn't continuous.

To check if the function is continuous if the limit exists, we need to check if

$$\lim_{x \to c} f(x) = f(c)$$

If the statement is true, then the function is continuous.

Here's an example:

$$\lim_{x \to -2^-} x^2 = 4$$

$$\lim_{x \to -2^+} x^2 = 4$$

In this case, both sides of the limit are equal to each other.

This means that the limit at this point exists.

Now, we need to check if the value of the limit is equal to the value of the function at the point.

For example:

$$\lim_{x \to -2} x^2 = 4$$

In this case, the limit when x is approaching -2 on the function

$$y = x^2$$

is actually equal to the value of the function at this point.

This means that the function is continuous.

The function

$$x^2$$

is a continuous function. It doesn't have any "sharp turns" or vertical tangents at the point x = -2.

Therefore, the function

$$y = x^2$$

is differential at

$$x = -2$$

Here's another example:

$$\lim_{x \to -1^-} \frac{x+1}{x+1} = 1$$

$$\lim_{x \to -1^+} \frac{x+1}{x+1} = 1$$

The one-sided limits are equal, therefore the limit exists.

Now we need to check if

$$\lim_{x \to -1} = f(-1)$$

So now we need to determine f(-1).

After a quick check, we can see that f(-1) isn't defined.

Therefore, the function isn't continuous, nor differentiable.

4.3.3 Differentiability & Continuity Exercises

Check continuity for each function at the specified points:

- x^3 at $x = -3$
- $tan(x)$ at $x = \frac{\pi}{2}$
- $\frac{x+1}{x}$ at $x = 1$
- $|x - 3|$ at $x = 3$
- $sin(x)$ at $x = \pi$
- $sin(x)$ at $x = \frac{3\pi}{2}$
- $\frac{1}{x}$ at $x = 0$
- $\frac{x+3}{x-1}$ at $x = -3$
- $|x + 5|$ at $x = 5$
- $|x - 3|$ at $x = 3$

4.3.4 Notation

In the previous sections, we have discussed about the idea of derivatives, differentiability and continuity.

It's about time now that we also learn how to notate derivatives.

The derivative has 3 main notations:

Lagrange's Notation

To notate the derivative of $f(x)$ in Lagrange's notation, you add a $'$ after the function name.

$f'(x)$ - The derivative of f(x).

To find the derivative at any x on f(x), you input your x in the function.

For example:

$f'(3)$ - The derivative at $x = 3$ on the function f(x).

To notate the nth derivative, you add another ' after the function.

- $f''(x)$ - The second derivative of $f(x)$.
- $f'''(x)$ - The third derivative of $f(x)$.

Usually beyond third derivative, the nth derivative is notated as $f^{(n)}(x)$

Newton's notation

To notate the derivative of f(x) in Newton's notation, you put a dot above the function:

$\dot{f}(x)$ - The derivative of f(x).

To find the derivative at any x on f(x), you input your x in the function.

For example:

$\dot{f}(-4)$ - The derivative at $x = -4$ on the function $f(x)$

To notate the nth derivative, you add more dots above the function.

For example:

- $\ddot{f}(x)$ - The second derivative of f(x)
- $\dddot{f}(x)$ - The third derivative of f(x)

Usually beyond the third derivative, the nth derivative is notated as $\overset{\overset{n}{\cdot}}{f}(x)$

Leibniz Notation

Leibniz notation is one of the most vital notations for derivatives. The notation itself is very logical and can be used to simplify or think of things more logically.

To notate the derivative in Leibniz notation on f(x):

$$\frac{df}{dx}$$

df stands for **difference in f**, and **dx** stands for **in respect to x**.

$\frac{df}{dx}$ - The derivative of f in respect to x (can be rewritten as f'(x) in Lagrange's notation)

To find the derivative at any x on f(x), you add parentheses with the number in them.

For example:

$\frac{df}{dx}(3)$ - The derivative at $x = 3$ on the function f(x)

To notate the nth derivative:

$$\frac{d^n f}{dx^n}$$

For example:

$\frac{d^2 f}{dx^2}$ - The second derivative of f(x)

Euler's Notation

4.3.5 Calculating the derivative

Now that we have finished learning notations and checking differentiation, let's finally get to the reason you've been learning this all along.

Now we can finally calculate the derivative!

But first, let's define the derivative more extensively.

The derivative

The derivative is a function that is defined as the tangent line to any differentiable function.

The derivative is also often referred to as the instantaneous rate of change of a function at a certain point.

A tangent line, is a straight line that "just touches" a curve at a specified point.

Recall: Secant line - a line that passes through 2 points on a curve;

The formula to calculate the slope of a secant line:

$$m = \frac{\Delta y}{\Delta x}$$

We have the fitting formula to calculate secant lines.

If we want to calculate the instantaneous rate of change of a function, the points from which we want to calculate the secant line from need to be as close as possible.

If you can recall, the function that can do it the best?

Of course, limits.

With limits, we can calculate the secant line, when the 2 points we calculate them with are infinitely close to each other.

From our secant slope calculation formula, we can derive the formula to calculate the derivative.

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

With this formula (also named the formal definition of the derivative), we can calculate the slope of the tangent line of any differentiable function.

Let's do some calculating examples:

Here we just input our values into the formula. Remember that $f(x) = x^2$.

$$\frac{d}{dx}[x^2] = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h}$$

Expanding terms:

$$\frac{d}{dx}[x^2] = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

Combining like-terms:

$$\frac{d}{dx}[x^2] = \lim_{h \to 0} \frac{2xh + h^2}{h}$$

Divide by h:

$$\frac{d}{dx}[x^2] = \lim_{h \to 0} (2x + h)$$

Set h = 0 (as described in the limit):

$$\frac{d}{dx}[x^2] = 2x + 0 = \mathbf{2x}$$

By using the derivative formal definition, we have found that the derivative of $x^2 = 2x$.

What does that mean?

This means that the slope of any point on the function x^2, is $2 \cdot x$.

Here's an example for clarity:

The slope of the tangent line on the function $f(x) = x^2$ at $x = 3$ will be $3 \cdot 2 = 6$.

Lets calculate the derivative of x^3 now:

$$\frac{d}{dx}[x^3] = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

$$\frac{d}{dx}[x^3] = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 - x^3}{h}$$

$$\frac{d}{dx}[x^3] = \lim_{h \to 0} \frac{3x^2h + 3xh^2}{h}$$

$$\frac{d}{dx}[x^3] = \lim_{h \to 0} (3x^2 + 3xh)$$

$$\frac{d}{dx}[x^3] = 3x^2$$

By now, you might be starting to recognize a pattern.

Let me introduce you to the power rule.
The power rule says that

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

So for example, the derivative of x^4 is $4x^3$.

The second derivative rule that you need to remember is the sum rule:

$$\frac{d}{dx}[f(x) + g(x)] = \frac{df}{dx} + \frac{dg}{dx}$$

The sum rule is one of the most common rules using derivatives, and will help you a lot in your calculus journey.

Here's an example for clarity:

$$\frac{d}{dx}[x^2 + x] = \frac{d}{dx}[x^2] + \frac{d}{dx}[x]$$
$$\frac{d}{dx}[x^2 + x] = 2x + 1$$

The last rule for now: The multiplication by constant rule:

$$\frac{d}{dx}[cf(x)] = c\frac{df}{dx}$$

where c is a constant.

For example:

$$\frac{d}{dx}[3x^2] = 3\frac{d}{dx}[x^2]$$

$$\frac{d}{dx}[3x^2] = 3 \cdot 2x = \mathbf{6x}$$

Exercises:

1. Finding the derivative of the following expressions:

- $$\frac{d}{dx}[2x + 1] =$$
- $$\frac{d}{dx}[3x^2 + 3x] =$$
- $$\frac{d}{dx}[5x^3 + 3x^2 + x] =$$
- $$\frac{d}{dx}[5x^2 + 7x^2 + 6] =$$
- $$\frac{d}{dx}[3x^3 + 2x + 5] =$$

Find the equation of a tangent line

In order to find the equation of the tangent line to a function, we need to first take the derivative of the function.

For example, take the function $y = x^2$.
We can find the derivative of the function using the power rule.
We get $\frac{d}{dx}[x^2] = 2x$.

Let's take for example the tangent of the curve at $x = 3$.
We input the x into the derivative, and get that the slope of the tangent line is 6.

To find the point on which the tangent line intersects the curve, we input $x = 3$ in the function x^2 to get the y value of the point.

After calculations, we get that the point is (3, 9).

Recall, the equation of the line when the slope and a point are given is

$$y - y_1 = m(x - x_1)$$

Now, we input our values to find the equation of the tangent line.

$$y - 9 = 6(x - 3)$$

Now we expand the terms and combine like-terms.

$$y = 6x + 9$$

4.3.6 Rules

Basic Rules

$$\frac{d}{dx}\left[x^n\right] = nx^{n-1} \tag{4.1}$$

$$\frac{d}{dx}\left[f(x) + g(x)\right] = f'(x) + g'(x) \tag{4.2}$$

$$\frac{d}{dx}\left[cf\right] = c \cdot f'(x) \tag{4.3}$$

$$\frac{df}{dx}\frac{dg}{dx} = f'(x) \cdot g(x) + g'(x) \cdot f(x) \tag{4.4}$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g(x)}{g^2(x)} \tag{4.5}$$

$$\frac{dy}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx} \tag{4.6}$$

Exponential & Logarithmic Derivatives

$$\frac{d}{dx}\left[e^x\right] = e^x \tag{4.7}$$

$$\frac{d}{dx}\left[lnx\right] = \frac{1}{x} \tag{4.8}$$

$$\frac{d}{dx}\left[a^x\right] = a^x lna \tag{4.9}$$

$$\frac{d}{dx}\left[log_a x\right] = \frac{1}{xlna} \tag{4.10}$$

Trigonometric Derivatives

$$\frac{d}{dx}[sinx] = \cos x \tag{4.11}$$

$$\frac{d}{dx}[cosx] = -\sin x \tag{4.12}$$

$$\frac{d}{dx}[tanx] = \sec^2 x \tag{4.13}$$

$$\frac{d}{dx}[cscx] = -\cot x \cdot \csc x \tag{4.14}$$

$$\frac{d}{dx}[secx] = \sec x \cdot \tan x \tag{4.15}$$

$$\frac{d}{dx}[cotx] = -\csc^2 x \tag{4.16}$$

$$\frac{d}{dx}[arcsinx] = \frac{1}{\sqrt{1-x^2}} \tag{4.17}$$

$$\frac{d}{dx}[arccosx] = -\frac{1}{\sqrt{1-x^2}} \tag{4.18}$$

$$\frac{d}{dx}[arctanx] = \frac{1}{1+x^2} \tag{4.19}$$

$$\frac{d}{dx}[arccscx] = -\frac{1}{x\sqrt{1-x^2}} \tag{4.20}$$

$$\frac{d}{dx}[arcsecx] = \frac{1}{x\sqrt{1-x^2}} \tag{4.21}$$

$$\frac{d}{dx}[arccotx] = -\frac{1}{1+x^2} \tag{4.22}$$

L'Hôpital's Rule

L'Hôpital's Rule states that if the limit is of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, we may differentiate the top and bottom of the quotient.

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

As long as

$$\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = c, c \in \{0, \infty\}$$

For example:

$$\lim_{x \to 0} \frac{sin(x)}{x} \overset{\frac{0}{0}}{\underset{}{=}}^{L'H} \lim_{x \to 0} \frac{[sin(x)]'}{[x]'} = \lim_{x \to 0} \frac{cos(x)}{1} = \lim_{x \to 0} cos(x) = cos(0) = 1$$

This is because of what we had discussed at section 3.2 "Logic Of Infinity And Zero". When dividing two infinitely big or small numbers, the result will be calculated using orders of magnitude. We can show that one way of doing so is using the rate of change, aka derivative. Therefore, one alternative of this rule is logic. For example:

$$\lim_{x \to \infty} \frac{2x^2 + x}{x^2} \overset{\overset{L'H}{\frac{\infty}{\infty}}}{=} \lim_{x \to \infty} \frac{4x + 1}{2x} \overset{\overset{L'H}{\frac{\infty}{\infty}}}{=} \frac{4}{2} = 2$$

Alternatively, we could say that for x infinitely large, x is negligible compared to x^2. Thus the limit is equal to

$$\lim_{x \to \infty} \frac{2x^2}{x^2} = 2$$

4.3.7 Partial Derivative

Consider the function $f(x_0, x_1)$. As you can see, this function takes in two parameters. Unlike in normal derivatives, we would like to know how the function changes as we change only one of those parameters, for example, x_0. When calculating the partial derivative of a function with respect to some parameter, we differentiate it normally, except that we treat all other parameters as if they were constants. This is because when taking the partial derivative, they do not change.

Leibniz notation

The most well known notation for the partial derivative is similar to the Leibniz notation, except the "d" is usually replaced with a "∂". For this example, we will write:

$$\frac{\partial f}{\partial x_0}$$

Similarly, if we wanted to see the change of f as x_1 changes, we would use:

$$\frac{\partial f}{\partial x_1}$$

Let us use an example:

$$\frac{\partial}{\partial a}\left(a^2 + b^2\right) = \frac{\partial}{\partial a}\left(a^2\right) + \frac{\partial}{\partial a}\left(b^2\right) = 2a + 0 = 2a$$

Euler's notation

Euler's notation for partial derivatives is also very similar to Euler's notation for normal derivatives, with only a few differences. The first one is that instead of the "D", we usually use "∂". The second one is that we write the parameter by which we want to differentiate in the subscript of the ∂ sign. Let us use an example:

$$\partial_x [xy] = y \cdot \partial_x [x] = y \cdot 1 = y$$

4.3.8 Exercises

4.4 The Integral

4.4.1 Introduction

Have you ever wondered if there was an inverse to the derivative?
Regardless, it exists!

The indefinite integral is known as the anti-derivative.
As we continue, we will go over the different types of integrals.

Here's the explanation for the anti-derivative:

$$\int f'(x)dx = f(x) + c$$

Now we will go over the notation of the integral.

That fancy S ($\int$) is how you notate an integral.

You might notice that dx on the end of the integral sign.
That dx signifies we are solving the integral in respect to x; exactly how it is in the Leibniz notation of the derivative too.
But wait! What is this $+c$ at the end of the solution?
c stands for any constant.
Recall: Using the power rule, the derivative of any constant is 0
This means that we cant know what that constant was. We notate that constant as c.

We will further learn the uses of integrals on upcoming sections.

4.4.2 Types of Integrals

There is a total of 2 types of integrals.

The first one is the indefinite integral.

The indefinite integral is the integral we went over on in the previous section, the anti-derivative.

Recall:

$$\int f(x)dx = f(x) + c$$

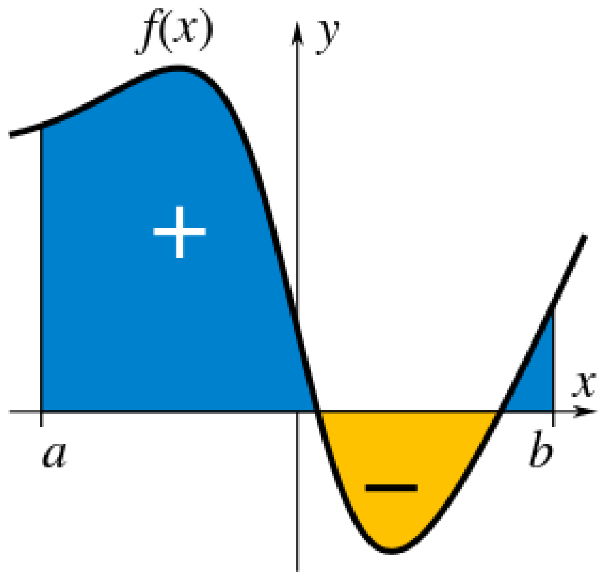
The definite integral is a bounded integral.

What does that mean?

To understand that, we first need to understand the definition of definite integrals.

The definite integral gives us the ”summed area” under a curve between the defined boundaries.

The area that you get by integration which is **below** the x-axis, is negative. Here’s a visualisation of that:



Say for example, if we have an integral that has an area of 20 on top of the x-axis, and an area of 4 below the x-axis, by integrating the function on the specified boundaries, you will get $20 + (-4) = 16$

The notation of the definite integral

This is how the notation of the definite integral looks like:

$$\int_a^b f(x)dx$$

a and b signify the interval on which the integral runs on.

For clarity, the interval of the integral is [a,b].

Usually, $a < b$ on the integral bounds.

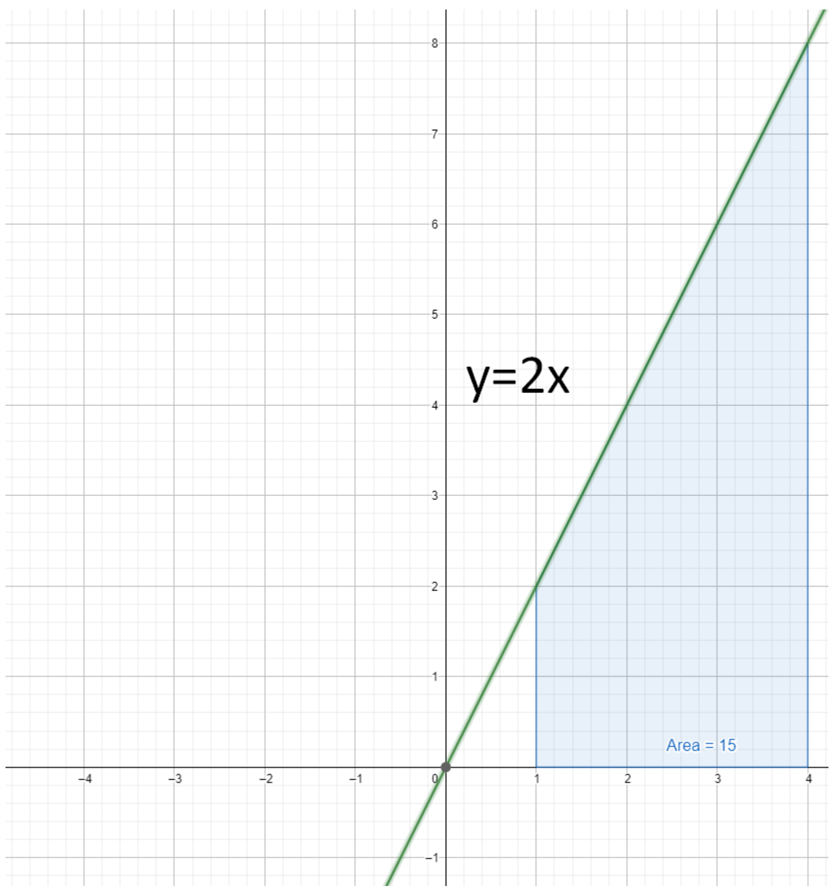
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

When solving for definite integrals, the constant c cancels out so there is no need to include it.

Here's an example for how to solve a definite integral (later we will learn how to solve integrals, for now, know arbitrarily that the integral of $2x$ is x^2):

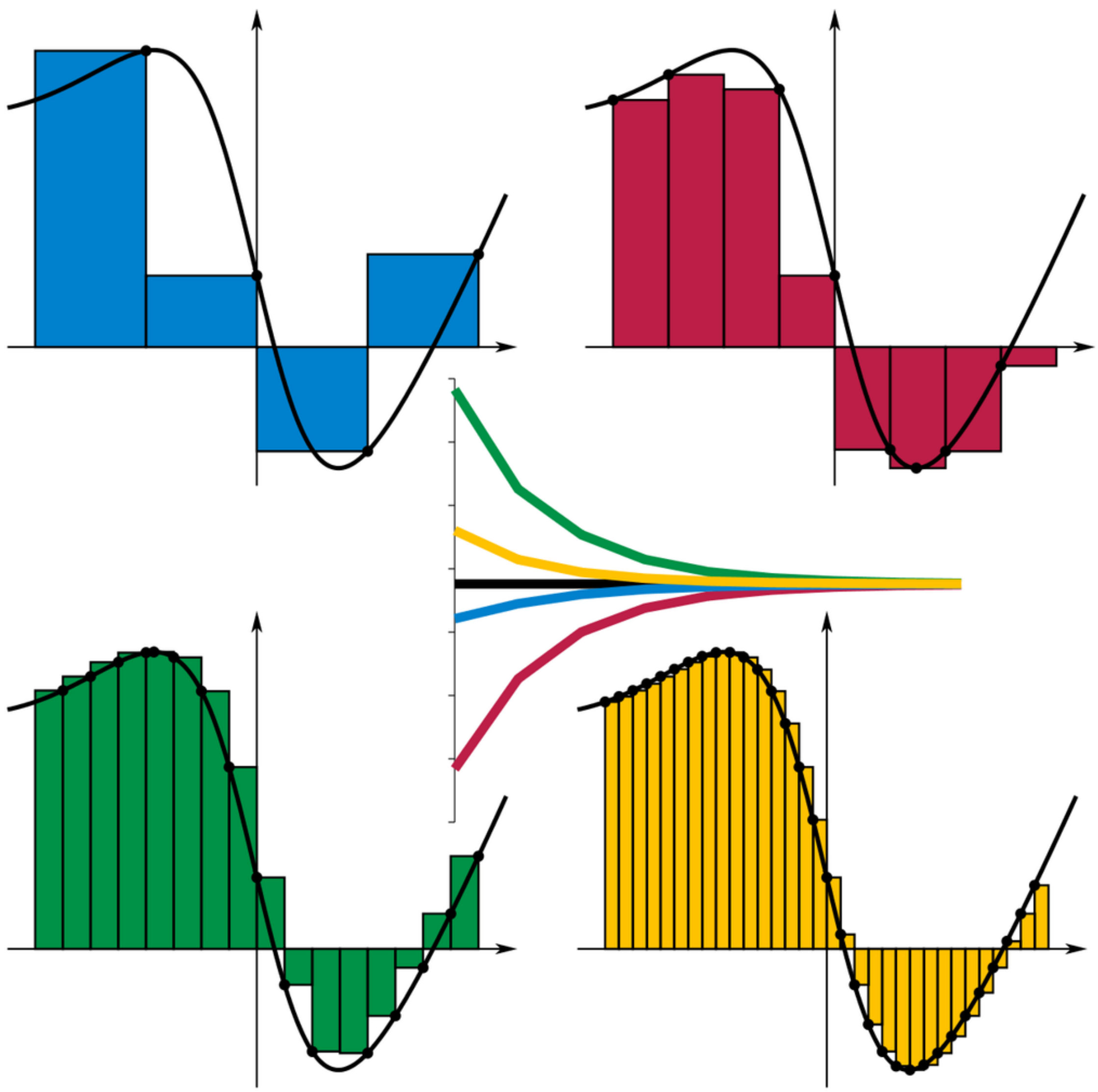
$$\int_1^4 2x dx = x^2 \Big|_1^4$$

$$4^2 - 1^2 = 16 - 1 = 15$$



4.4.3 Riemann Sum

The Riemann sum is an approximation of a definite integral by a finite sum. It is calculated by adding the area's of different shapes (usually rectangles) confined in the shape of the integrand. As the number of shapes approaches infinity, the sum of their area's approaches the integral, as shown in the following picture:



From here, we can derive the following formula:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

where: $\Delta x_i = x_i - x_{i-1}$, and: $x_i^* \in [x_{i-1}, x_i]$

Most of the time, for basic integrals:

$$x_i^* = a + i\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

This integration technique is not used very often in basic calculus, however, it is very important to know, and it is used more frequently in advanced calculus.

Let us try an example:

$$\int_1^3 x^2 - x + 1\, dx$$

$$\text{In this case, } \Delta x = \frac{3-1}{n} = \frac{2}{n}$$

$$\text{And therefore, } x_i^* = 1 + i \cdot \frac{2}{n} = 1 + \frac{2i}{n}$$

$$\int_1^3 x^2 - x + 1\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[\left(1 + \frac{2i}{n}\right)^2 - \left(1 + \frac{2i}{n}\right) + 1 \right] \cdot \frac{2}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n} \cdot \sum_{i=1}^{n} \left(1 + \frac{2i}{n} + \frac{4i^2}{n^2}\right)$$

$$= \lim_{n\to\infty} \frac{2}{n} \cdot \left[\sum_{i=1}^{n} (1) + \sum_{i=1}^{n} \left(\frac{2i}{n}\right) + \sum_{i=1}^{n} \left(\frac{4i^2}{n^2}\right) \right]$$

$$= \lim_{n\to\infty} \frac{2}{n} \cdot \left[n + \frac{2}{n} \cdot \frac{n \cdot (n+1)}{2} + \frac{4}{n^2} \cdot \frac{n \cdot (n+1) \cdot (2n+1)}{6} \right]$$

$$= \lim_{n\to\infty} \left[2 + \frac{2n+2}{n} + \frac{4}{3} \cdot \frac{(n+1)(2n+1)}{n^2} \right]$$

$$= 2 + \lim_{n\to\infty} 2 + \frac{2}{n} + \frac{4}{3} \cdot \frac{2n^2 + 3n + 1}{n^2}$$

Notice that $\frac{2}{n} \to 0$, and $\frac{3n+1}{n^2} = 0$. Therefore:

$$= 4 + \lim_{n\to\infty} \frac{4}{3} \cdot 2$$

At this point, we can remove the limit sign, as the expression is no longer dependant on n. After calculating, we get $\frac{20}{3}$. Thus:

$$\int_1^3 x^2 - x + 1 \, dx = \frac{20}{3} = 6\frac{2}{3}$$

4.4.4 Basic Integral Rules

Now that you know have the knowledge of calculating definite integrals using Riemann sums, let's go over how we can do it way more easily.

By this point, you must be familiar with some derivative rules.
As you can guess from the title, there are also integral rules that make solving both indefinite integrals and definite integrals much more easy.

Before going into some integral rules, you need to know that solving indefinite integrals is also possible without using the integration rules.

For example:

$$\int 2xdx =$$

To calculate that, you need to think about what number you need to differentiate to get $2x$.
If you have already mastered differentiation, you probably know at the top of your head that you'll get $2x$ when you differentiate x^2.

This means that $\int 2xdx = x^2 + c$.

But this method is not the most reliable method to calculate integrals. That's where integral rules come into action.

You might be familiar with the derivative power rule, $\frac{d}{dx}[x^n] = nx^{n-1}$.

The opposite rule to that is the integral power rule. By manipulation of the derivative power rule, we can find that the integral power rule is:

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \text{ when } n \neq -1$$

For example:

$$\int x^3\,dx = \frac{x^{3+1}}{3+1} + c = \frac{x^4}{4} + c$$

Here are some more integration rules that are often used:

$$\int f'(x)\,dx = f(x) + C$$

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

$$\int c \cdot f(x)\,dx = c \cdot \int f(x)\,dx$$

$$\int x^n = \frac{x^{n+1}}{n+1} + C$$

$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\int \frac{1}{x}\,dx = ln|x| + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \cdot \tan x \, dx = \sec x + C$$

$$\int \csc x \cdot \cot x \, dx = -\csc x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \sin^{-1} x + C$$

$$\int \frac{1}{x \cdot \sqrt{x^2-1}} \, dx = \sec^{-1} x + C$$

$$\int \frac{1}{\sqrt{a^2+x^2}} \, dx = \frac{1}{a} \cdot \tan^{-1} \frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{1+x^2}} \, dx = \tan^{-1} x + C$$

Exercises

A. Find the indefinite integral of the following expressions:

1. $$\int 3x^2 dx$$

2. $$\int (7x^2 + 4) dx$$

3. $$\int (4x^2 + 3x + 1) dx$$

4. $$\int (2x^3 + x + 4) dx$$

5. $$\int (9x^8 + 6x + 2) dx$$

B. Find the definite integral of the following expressions:

1. $$\int_0^6 3x^2 dx$$

2. $$\int_3^8 2x^3 dx$$

3. $$\int_4^5 (x^3 + 2x + 1) dx$$

4.
$$\int_7^4 (x^4 + 2x + 5)dx$$

5.
$$\int_2^5 (3x^3 + 2x^2 + 6x + 2)dx$$

4.4.5 u - Substitution

The u-Substitution is a method to find the integral of a function, and is the anti- derivative version of the chain rule for differentiation.

To evaluate an integral using this method, we use the following steps: First, you have to decide what's your u. It has to be a function of the variable of integration (for example, x).

Then you need to calculate dx in terms of du.

$$\frac{du}{dx} = u'(x)$$

Thus,

$$\frac{du}{u'(x)} = dx$$

For example:

$$\int (2x + 1) \cdot (x^2 + x)\, dx$$

Let $u = x^2 + x$
Then, $\frac{du}{2x+1} = dx$
Plugging into our integral, we get:

$$\int (2x + 1) \cdot u \cdot \frac{du}{2x + 1} = \int u\, du$$

$$= \frac{u^2}{2} + c = \frac{(x^2 + x)^2}{2} + c$$

Exercises

$$\int (x^3)(2x + x^4)dx$$

$$\int \frac{(x)}{(x^2+1)}dx$$

$$\int sec(x)tan(x)dx$$

$$\int \frac{1}{e^x+1}$$

$$\int \frac{cos(x)sin(x)}{2}dx$$

$$\int \frac{x^3+2x}{\sqrt{x^2+1}}$$

4.4.6 Integration By Parts

Integration by parts is a method of integrating the product of two functions by their derivative and anti- derivative.

To use this method, we let $u = u(x)$ and $v = v(x)$.

The formula states that:

$$\int u\,dv = uv - \int v\,du$$

For example, let's evaluate the following integral:

$$\int x \cdot \cos x\,dx$$

Let $v(x) = x$, and $u(x) = \sin x$.

Thus, $v'(x) = 1$, and $u'(x) = \cos x$.

Plugging into the formula, we get:

$$\int x \cdot \cos x\,dx = (x)(\sin x) - \int (\sin x)(1 \cdot\ dx)$$

$$= x \sin x - \int \sin x\,dx$$

$$= x \sin x + \cos x + c$$

DI method for integration by parts

Integration by parts may be difficult for some people to remember, so there are other methods that do the same thing, but are easier to remember and use. One such method is the DI method.

For example, let us solve the integral:

$$\int x^4 \cdot \ln x \, dx$$

First, we draw a table with the titles D, and I and write the function we wish to integrate under the I, and the function we wish to differentiate under the D, like so:

D	I
$\ln x$	x^4

Then, we differentiate the function under the D, and write it under the original function. Similarly, we integrate the function under the I, and write it under the original function:

D	I
$\ln x$	x^4
$\frac{1}{x}$	$\frac{x^5}{5}$

We continue this process until we reach one of three states:

1. When we get 0 under D or I.
2. When the product of the two functions in the bottom row is integrable.
3. When the integral of the product of the bottom row is a multiple of the original integral.

In this case, we reached the second state, as the integral of $\frac{x^4}{5}$ is easy to evaluate.

Then, we write plus and minus signs (alternating, starting with plus) before each element in the D column:

	D	I
$+$	$\ln x$	x^4
$-$	$\frac{1}{x}$	$\frac{x^5}{5}$

Then, we add the product of each element in the D column, and the element in the next row, in the I column, and finally, we add the integral of the product of the two last elements:

$$\begin{aligned}\int x^4 \cdot \ln x\, dx &= +(\ln x) \cdot \frac{x^5}{5} + \left(\int -\frac{1}{x} \cdot \frac{x^5}{5}\, dx\right) \\ &= \frac{x^5}{5} \cdot \ln x - \int \frac{x^4}{5}\, dx \\ &= \frac{x^5}{5} \cdot \ln x - \frac{1}{25} \cdot x^5 + c\end{aligned}$$

Exercises - DI method.

$$\int x^2 sin(x)dx$$

$$\int xln(x)dx$$

$$\int (x+2)e^x dx$$

$$\int \sqrt{x}ln(x)dx$$

$$\int cos(x)e^3xdx$$

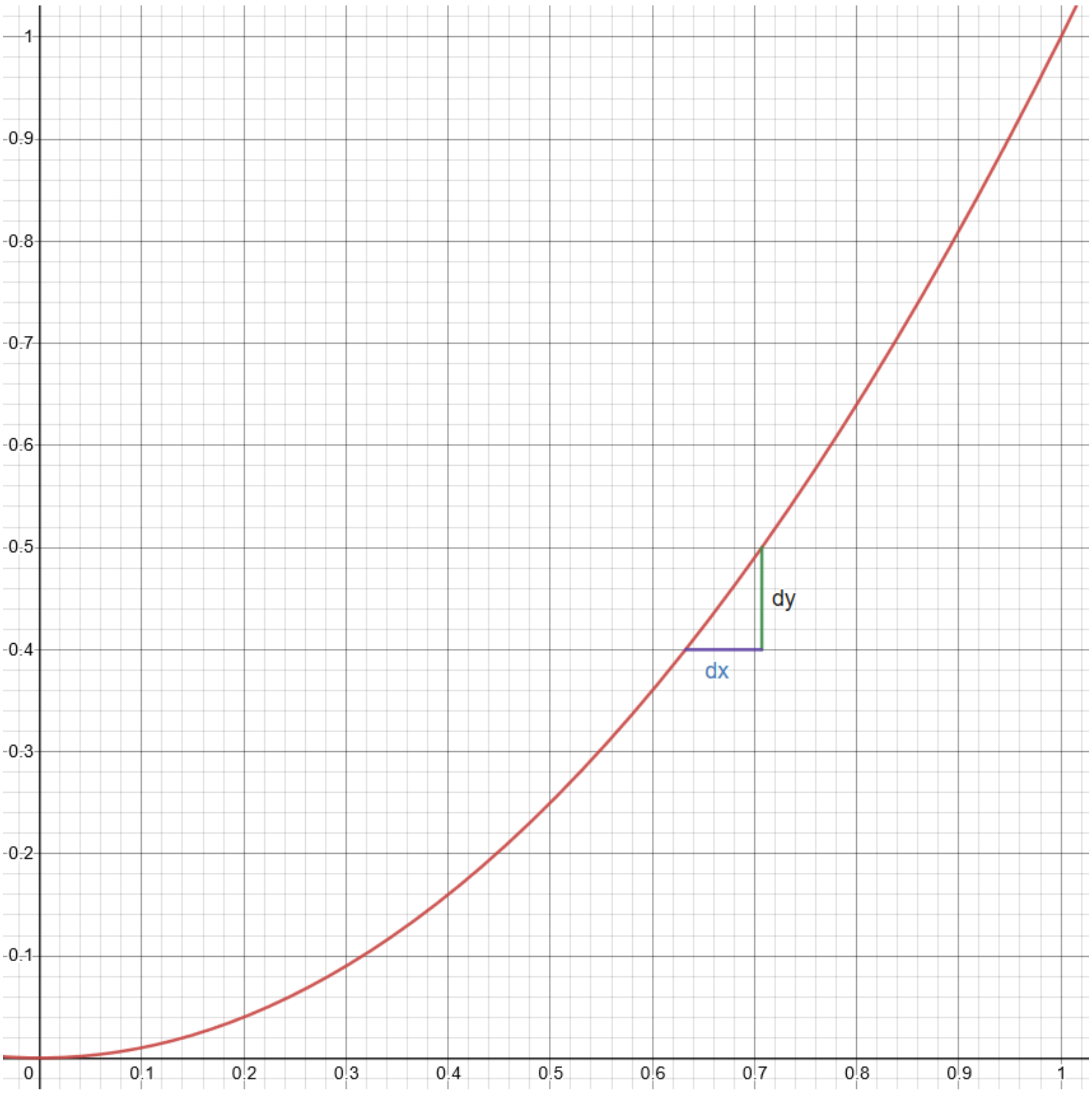
$$\int \frac{ln(x)}{x^3}dx$$

4.4.7 Arc-length Formula

When plotting a function on Cartesian coordinates, you will get a curve. In some situations, finding the length of the curve (also called the arc), you may use the following formula:

$$\text{Arc-length of } f(x)\Big|_a^b = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx$$

But how would one derive it? Well, let us look at the graph of $y = x^2$, from 0 to 1:



1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
dy
dx

As you can see, we can construct a right triangle with sides dy, dx. Clearly, we can approximate the length of the arc's parts confined by the sides of the triangle with the triangle's hypotenuse. By the Pythagorean theorem, the hypotenuse's length is equal to $\sqrt{dx^2 + dy^2}$. We can factor out dx^2 to get:

$$\sqrt{dx^2\left(1+\left(\frac{dy}{dx}\right)^2\right)} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\cdot dx$$

Notice that when dx and dy approach zero, the approximation is exact. We can integrate this from a to b, to sum up all of the arc's parts, getting:

$$\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

Notice that the $\frac{dy}{dx}$ term is just the derivative of $f(x)$.

Now, let us find the arc-length of $y = x^2$ from 0 to 0.5.

$$\int_0^{0.5} \sqrt{1+(2x)^2}\,dx$$

Let us introduce the following substitution:

$$tan(\theta) = 2x$$

We get:

$$\frac{1}{2}\int_0^{\frac{\pi}{4}} \sec^3\theta d\theta$$

$$= \frac{1}{4}\left[\sec\theta\cdot\tan\theta + \ln|\sec\theta+\tan\theta|\right]\Bigg|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{4}(\sqrt{2}\cdot 1 + \ln(\sqrt{2}+1) - 1\cdot 0 + \ln(1+0))$$

$$= \frac{\sqrt{2} + \ln(\sqrt{2} + 1)}{4} \approx 0.5739$$

4.5 Taylor And Maclaurin Series

The Taylor series is an infinite sum of elements that are expressed through the nth derivative at a single point, usually at $x = a$.
The more elements you add, the closer the series to the true function around that particular point.

The Taylor expansion is the partial sum of the full Taylor series, and is an approximation of the original function around $x = a$.

In summary, the Taylor series converts itself to the function around $x = a$ (and on occasion in the entire function).

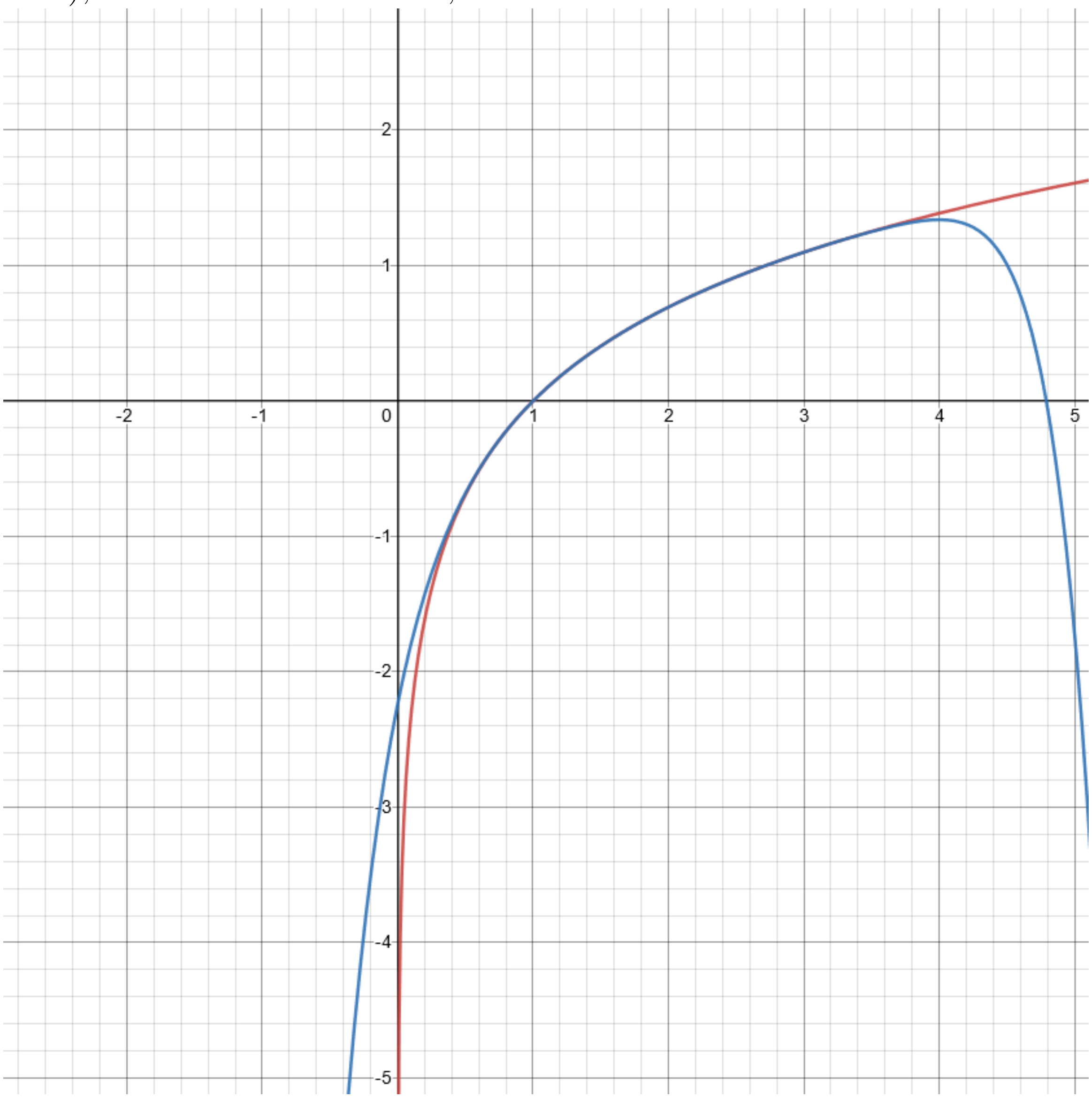
It works by setting the series' nth derivative in order for it to be equal to the original function's nth derivative, for all derivatives up until the number of elements minus one, at $x = a$ (with the 0th derivative being the original function).

The radius of convergence is the distance in which the Taylor series converges around it's center.

The general formula for the Taylor series centered at $x = a$, of $f(x)$ (of k terms) is:

$$\sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

Here is a graph of $\ln(x)$ (coloured red) and it's Taylor series (coloured blue), centered around $x = 2$, for $k = 10$:



Inspecting this graph reveals that the Taylor series is nearly equal to the original function around $x = 2$. In fact, its radius of convergence is 2, which means that it converges in the range $(0, 4)$. Meanwhile the Maclaurin series of a function is its Taylor series, centered around $x = 0$.

4.5.1 Taylor And Maclaurin Series Of Some Functions

The Taylor series of $\sin(x)$ is:

$$\sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n+1}}{(2n+1)!}$$

And it converges for $x \in R$.

The Taylor series of $\cos(x)$ is:

$$\sum_{n=0}^{\infty}(-1)^n \cdot \frac{x^{2n}}{(2n)!}$$

And it converges for $x \in R$.

The Taylor series of e^x is:

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

And it converges for $x \in R$.

The Taylor series of $\frac{1}{1-x}$ is:

$$\sum_{n=0}^{\infty}x^n$$

And it converges for $x \in (-1, 1)$.

The Taylor series of $\ln(1+x)$ is:

$$\sum_{n=1}^{\infty}(-1)^{(n-1)} \cdot \frac{x^n}{n}$$

And it converges for $x \in (-1, 1]$.

The Taylor series of $\tan^{-1}(1+x)$ is:

$$\sum_{n=0}^{\infty}(-1)^{n} \cdot \frac{x^{2n+1}}{2n+1}$$

And it converges for $x \in [-1, 1]$.

Chapter 5

Linear Algebra

5.1 Introduction

Linear algebra is the branch of mathematics concerning linear equations such as:

$$a_1x_1 + \cdots + a_nx_n = b$$

linear maps such as:

$$(x_1, \ldots, x_n) \mapsto a_1x_1 + \cdots + a_nx_n$$

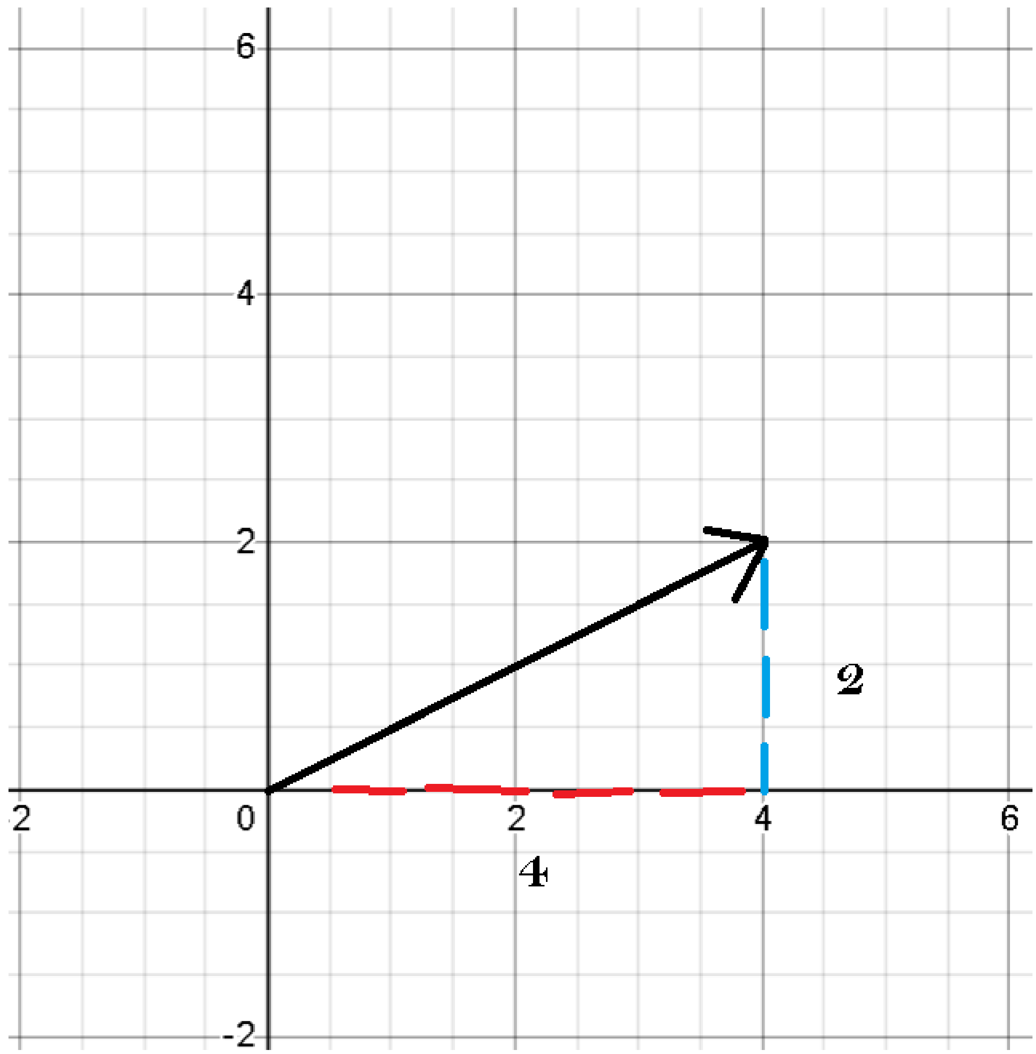
and their representations in vector spaces and through matrices.

Now what is a Vector ? a vector is an object with a magnitude and direction , for example a ball rolling to the south with $5kg$ is a vector. we have another quantity which is a scalar. a scalar is a vector but without a direction , meaning it has a magnitude. let us take a look on a vector.

$$\begin{pmatrix} x \\ y \end{pmatrix}$$

so here we have a "vector" now lets plug in some numbers and see how it looks.

$$\begin{pmatrix} 4 \\ 2 \end{pmatrix}$$



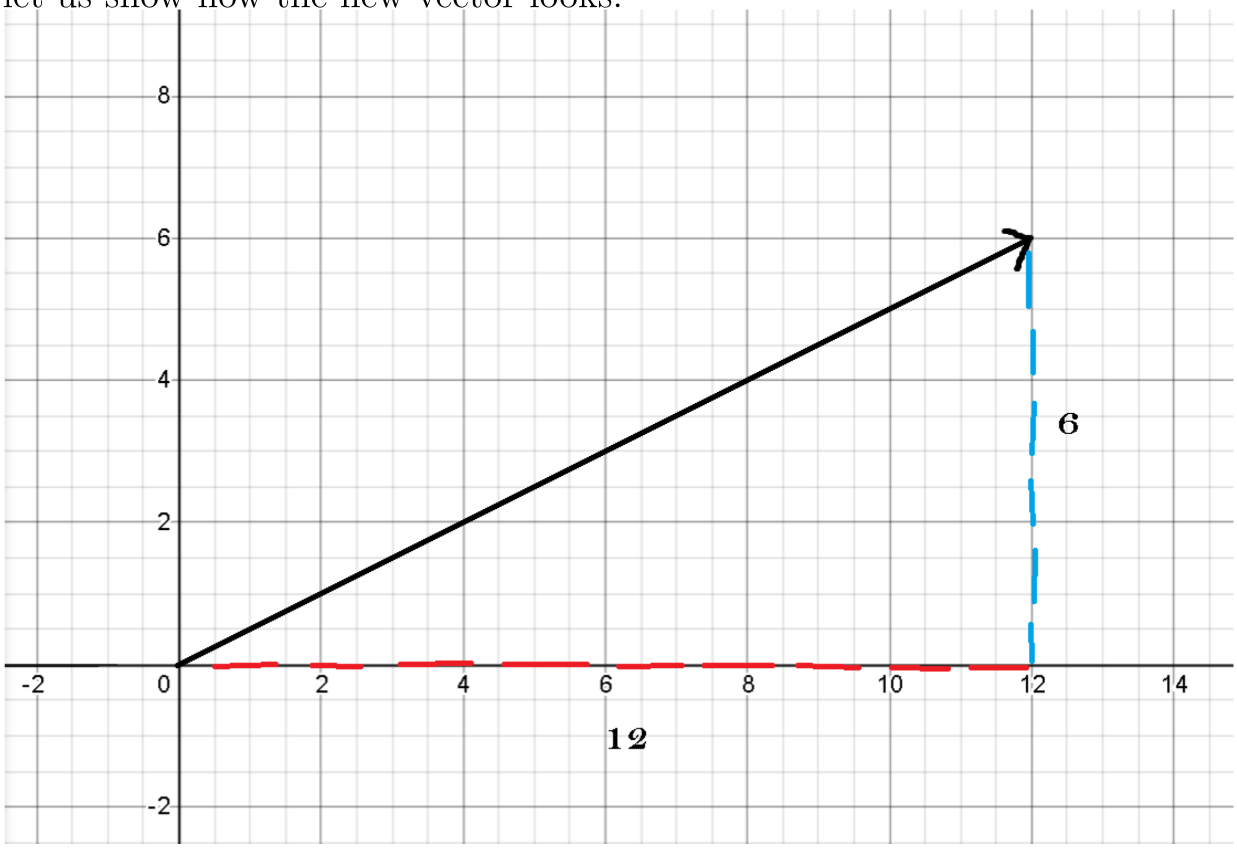
we see vectors as arrows pointing to some direction. so as you can see , this vector is going 4 points in the x-axis and 2 in the y-axis. let us show you how to multiply a scalar with a vector. so lets take a simplest scalar 3 and lets take the vector $\binom{4}{2}$. all we have to do is to multiply the x and y value with 3.

5.2 vectors

as we discussed earlier , we have to multiply the vector by 3.

$$3 \cdot \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix}$$

let us show how the new vector looks.



as you can see the vector changed , not in the position but in its length. but do we know its length? well for that we have to use the Pythagorean theorem. lets calculate the length of the vector before the multiplication. to see the vectors in our calculation we put an arrow on top of a variable , lets calculate the length of $\overrightarrow{v} = \binom{4}{2}$. so lets use the Pythagorean theorem to evaluate the length of the vector.

$$\overrightarrow{v} = \sqrt{2^2 + 4^2} = \sqrt{20} \approx 4.47$$

now lets calculate the length of the vector after the multiplication.

$$\overrightarrow{v} = \sqrt{6^2 + 12^2} = \sqrt{180} \approx 13.41$$

now lets check our answer by simply dividing 13.41 by 4.47.

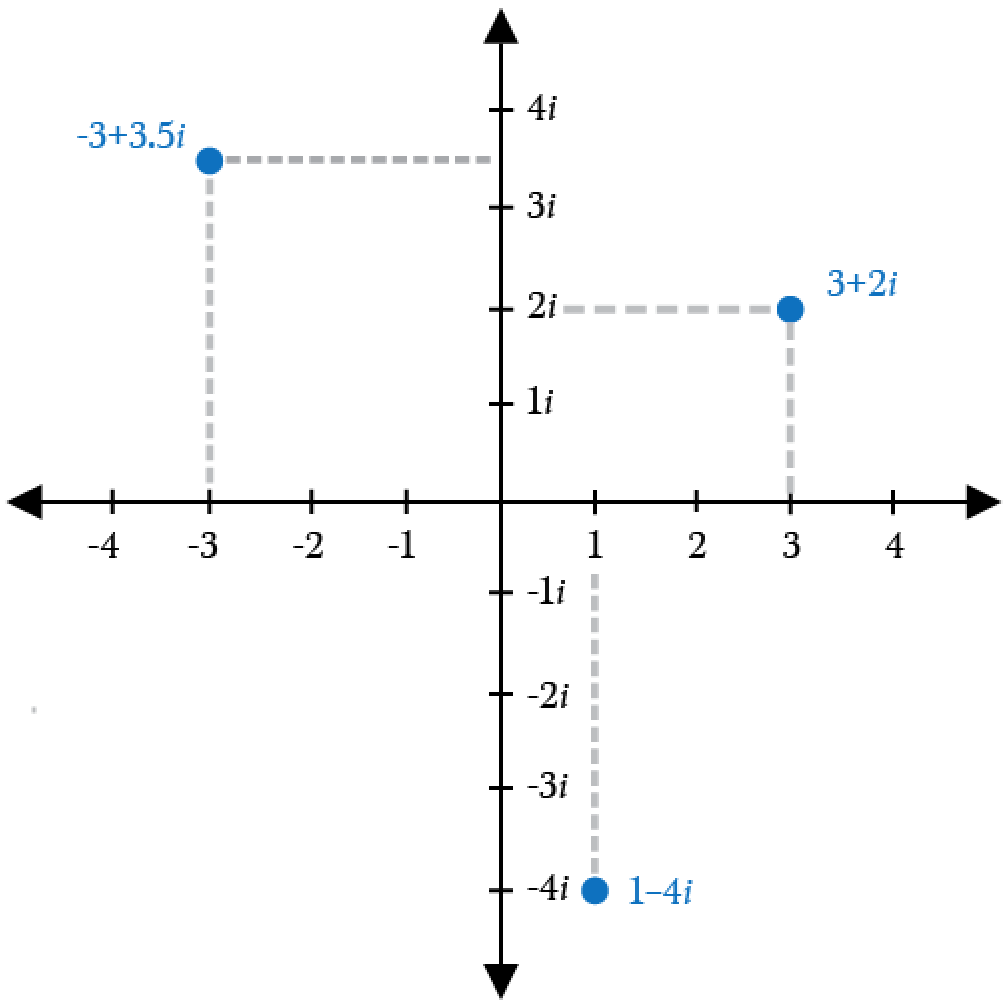
$$\frac{13.41}{4.47} = 3$$

. we got the right answer.

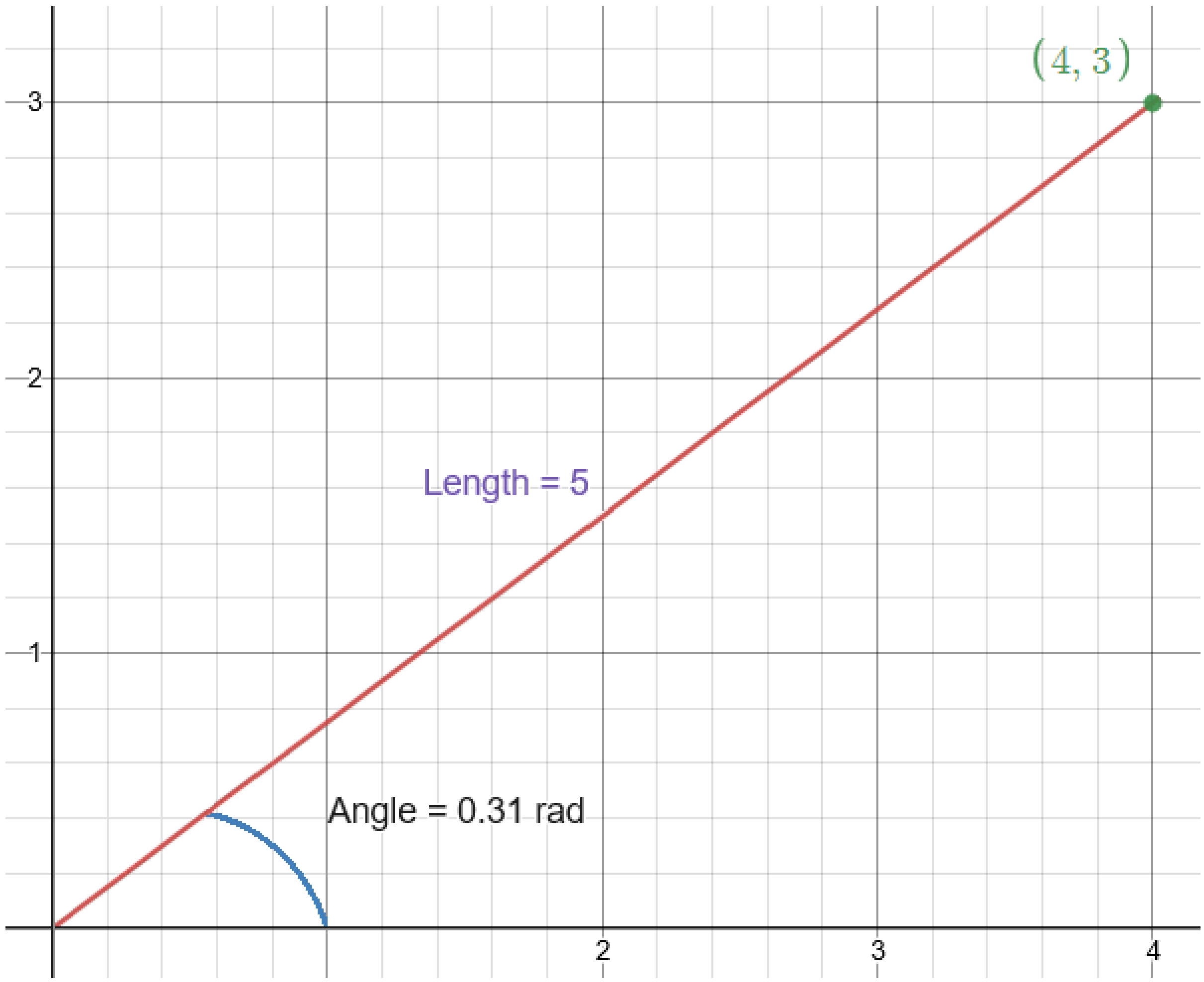
Chapter 6

Complex Numbers

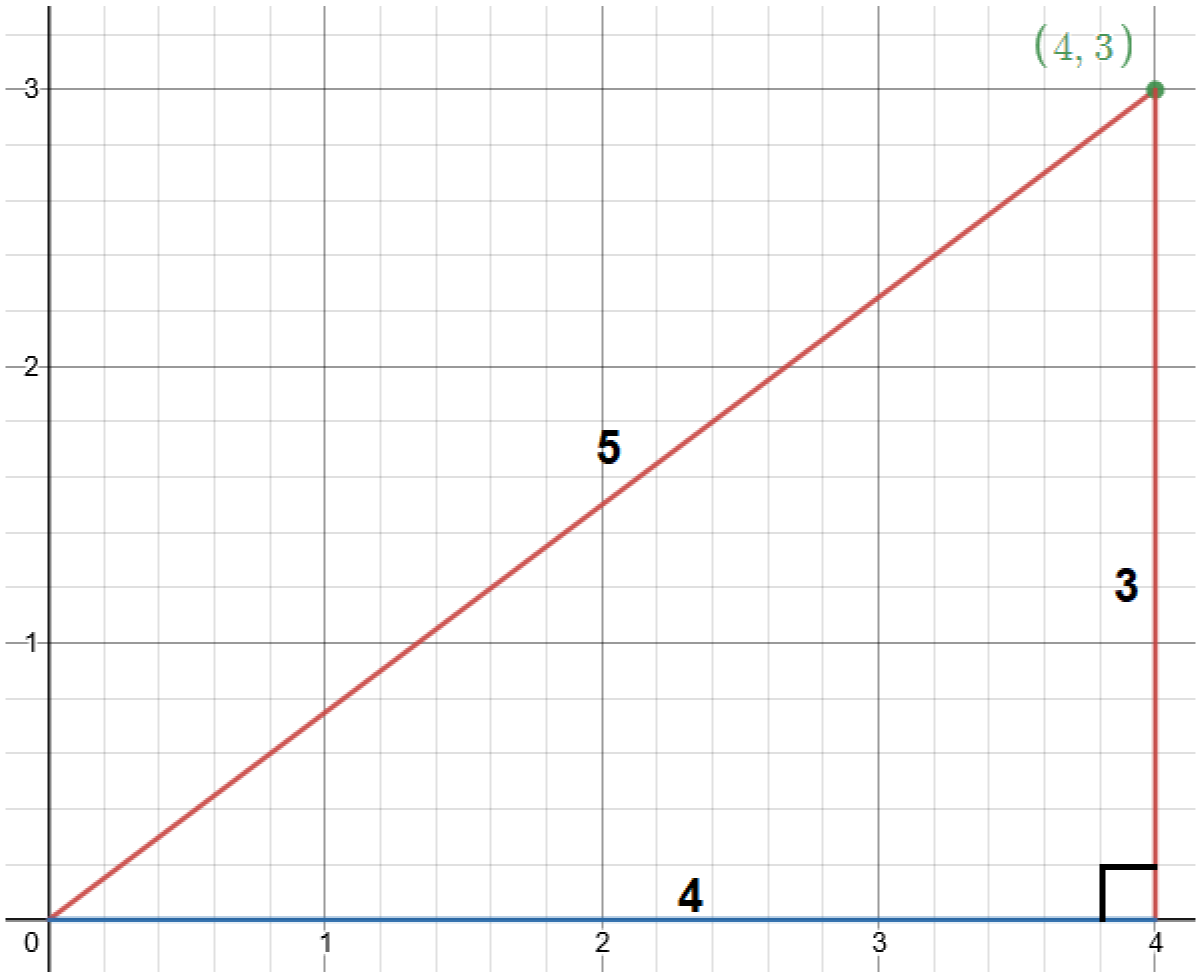
Usually, we assume that the square root of any negative number is undefined. However, it turns out that it is perfectly defined. The square root of -1 is usually written as i, or in engineering, j. By using basic root laws, it can be shown that $\sqrt{-a^2} = i \cdot a$. These numbers are called imaginary numbers, and complex numbers are the sum of real and imaginary numbers ($z = a + i \cdot b$). They are not on the normal number line. Instead, we use the Gauss plane (also called the complex plane). Here is an example of complex numbers on the complex plane:



As you can see, when placing a complex number on the complex plane, we use the real part as the x value of the point, and the imaginary part as the y function of the graph, as if the complex plane was the Cartesian x, y plane. Similarly to a vector, complex numbers have angles and amplitudes, and the amplitude of a complex number can be written as $|z|$, for a complex number z. For example, look at the complex number $4 + 3i$:



In this case, the angle is about 0.31 radians, and the amplitude is 5. The angle can be calculated with the inverse tangent, and the the amplitude can be found by constructing a right triangle, with one on the x axis from the origin to the real part of $4+3i$, the other one on the y axis from the origin to the imaginary part of $4+3i$, and the hypotenuse as the line going from the original to $4+3i$, as such:



The sides are equal to 3 and 4, because those are the real and imaginary parts of the number $4 + 3i$. By using the Pythagorean theorem, we get the the amplitude is equal to $\sqrt{4^2 + 3^2} = 5$. We can get the angle by taking the inverse tangent of the opposite side divided by the adjacent side:
$\theta = \tan^{-1}(\frac{3}{4}) \approx 0.31$ rad.

Thus, the general formulas are:

$$\text{For a complex number } z = a + i \cdot b\text{:}$$

$$\text{Angle } = \tan^{-1}(\frac{b}{a})$$

$$\text{Amplitude} = \ |a + i \cdot b| = \sqrt{a^2 + b^2}$$

Part II
Physics

6.1 Introduction

So what is physics? Physics is the natural science that studies matter, its motion and behavior through space and time, and the related entities of energy and force. Physics is one of the most fundamental scientific disciplines, and its main goal is to understand how the universe behaves.

6.2 Newtonian Mechanics

6.2.1 Conservation Of Energy And Mass

The law of conservation of energy is one of the most fundamental laws in physics. It states that the sum of the energy in a closed system cannot change under any circumstances. Energy may take on different forms, for example, heat energy may be converted into height energy, but the sum of all of the energies in the system must remain the same.

Similarly, the law of conservation of mass states that the sum of the mass in a closed system cannot change under any circumstances. The mass may change shape, position etc., but the total sum of the masses in a closed system must remain the same.

6.2.2 Height Energy

Now that we've the basic understanding of physics we can start working on it and the first subject that we are going to go through is The Height energy.

So what is Height energy ? The height energy is the energy of an object caused by mostly falling out of an high place. we can calculate the Height energy of an object by this formula :

$$Eh = W \cdot h$$

Or

$$Eh = m \cdot g \cdot h$$

Now what are these symbols even mean ?

Eh stands for Height Energy,the W stands for The weight of an Object,the h stands for the height,the g stands for the gravity in the planet (we will go through these in a second) and the M stands for the mass of the Object.

now we need to understand something,what is the difference between W and M ? well the most official difference is that the unit of M is kilograms and its the mass of the object,However the w Stands for the weight of the Object and its unit is Newton.

Now that we have a general understanding of height energy lets go through the formulas :

$$W = m \cdot g$$

$$W = \frac{Eh}{h}$$

$$h = \frac{Eh}{W}$$

$$m = \frac{Eh}{g \cdot h}$$

$$g = \frac{Eh}{m \cdot h}$$

$$h = \frac{Eh}{m \cdot g}$$

Good , we have got the knowledge of the formulas now we can work on couple of examples :

Disclaimer: Gravity can also be written in the unit $\frac{N}{kg}$

- An Object is falling out of the sky with the force of $2(N)$, consider that his mass is $5(kg)$ and his gravity is $(9.8m/s^2)$what is the height energy of the Object?

  So This Example is Fairly Simple all we need to do is to plug in our givens into this equation $\rightarrow Eh = W \cdot g \cdot m$

  lets plug in our givens : $Eh = 2 \cdot 9.8 \cdot 5 = 98(J)$

Great ,now that we did this example , we have to go through couple more of these thus learning more in the field of physics. But before then , we have to discuss more about g "Gravity , what is that even supposed to mean ? So Every planet has its own gravity , for example Earth has the gravity of $9.81m/s^2$ whereas Mars has the gravity of $3.711m/s^2$. So lets go through our "Gravity table" therefore comes the question , should we memories these numbers ? the answer is simple : you don't have to but it will help you.

Planet	Gravity
Earth	$9.81m/s^2$
Mars	$3.711m/s$
Jupiter	$24.8m/s^2$
Saturn	$10.5m/s^2$
Uranus	$8.9m/s^2$
Neptune	$11.4m/s^2$
The moon	$1.6m/s^2$
Mercury	$3.7m/s$

Now that we know the gravity of each planet lets talk about the mass of the Object . There is a rule in physics that state this following statements : No matter what , the mass of the object cannot change. Make a simulation in your head , imagine an astronaut going to earth to mars , now the mass of the astronaut on earth is $40kg$ when he arrives at mars his mass won't change and will remain the same as $40kg$ as it was on earth. In addition we have to talk about converting $grams$ to kg and $centimeters$ to $meters$ In order to convert $grams$ into kg we have to divide our mass by 1000 in order to get a normal equation since we can't plug in $grams$. For example this one :

Given$\rightarrow$ $5grams$ now we have to put $kilograms$ into our equation so let do this :

$$\frac{5}{1000} = 0.005kg$$

Exercises

1. A ball is 15m above the ground.
   The mass of the ball is 500g.
   Assume the ball is on planet Earth (round the gravity to an integer).

   Calculate the height energy of the ball.

2. A girl walked up the stairs to the second floor of a building.
   A boy walked up the stairs to the second floor of the same building.

   Will their height energy be the same? Why?

3. A ball made of steel weighs 1500g.
   Another ball made of rubber weighs 100g.
   The steel ball is 1m above the ground.
   The rubber ball is 15m above the ground.

   Do the 2 balls have the same height energy? Prove your answer with a calculation.

6.2.3 Kinetic energy

What is kinetic energy ? kinetic energy is the energy of a moving system , for example A kid runs at the street or the ball is rolling around. These are good examples of kinetic energy so lets go through its formula's :

$$Ek = \frac{m \cdot v^2}{2}$$

You may see this formula in its different way $\rightarrow Ek = \frac{1}{2} \cdot m \cdot v^2$
let us introduce you to the units of the formula v stands for the velocity of the Object m/s, and m stands for the mass of the Object kg. So lets go over some formula's before we continue over to kinetic energy at roads.

$$v = \sqrt{\frac{Ek}{m}}$$

$$m = \frac{2 \cdot Ek}{v^2}$$

So lets go over some exercises:

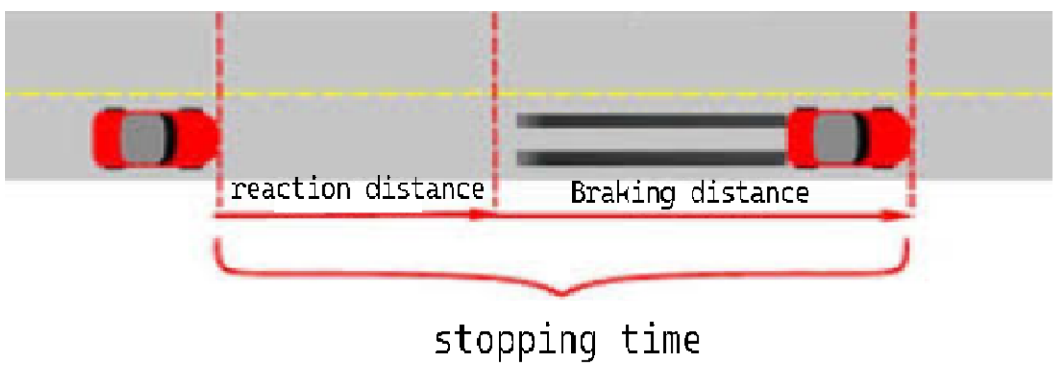
1. a ball is rolling on earth at the velocity of $5m/s$ and his kinetic energy is $20J$ what is his height energy if the ball is rolling on a shelf standing 2 meters above the ground ? (round the gravity to an integer)
2. The height energy of an object is $56J$ his weight is $7N$
   First calculate his height,then Consider that his mass is $5kg$ and his kinetic energy is $45J$ explain us (with a calculation) what is his velocity.

6.3 kinetic energy in roads

Kinetic energy in roads has the same meaning, however we have to be familiar with new facts.

- Braking distance : braking distance is the distance the vehicle travels until he stops from emergency braking.
- Reaction distance : The reaction time is the distance that the vehicle travels during the drivers reaction.
- Stopping distance : the distance that the vehicle travels from the moment of the reaction of the driver until official stopping.

Lets think about it this way.



So lets go through some formula's :

Stopping distance = Reaction distance + Braking distance

$$Reactiondistance = v \cdot reactiontime$$

Braking distance = the kinetic energy of the vehicle divided by a given

Lets think about the braking distance :

1. what is the braking distance of a vehicle that his kinetic energy is $100J$ and every meter that hes passing the friction is taking $2J$ what is his braking distance ?

so lets see what we should do. so lets divide his kinetic energy by the given.

$$\frac{100}{2} = 50meters$$

6.4 Thermodynamics

What is Thermodynamics ? Thermodynamics is a branch of physics that deals with heat, work, and temperature, and their relation to energy, radiation, and physical properties of matter. The behavior of these quantities is governed by the four laws of thermodynamics which convey a quantitative description using measurable macroscopic physical quantities, but may be explained in terms of microscopic constituents by statistical mechanics. Thermodynamics applies to a wide variety of topics in science and engineering, especially physical chemistry, biochemistry, chemical engineering and mechanical engineering, but also in other complex fields such as meteorology.

So what are the four thermodynamics laws ?

1. Zeroth Law : The Zeroth Law of Thermodynamics states that if two systems are in thermodynamic equilibrium with a third system, the two original systems are in thermal equilibrium with each other. Basically, if system A is in thermal equilibrium with system C and system B is also in thermal equilibrium with system C, system A and system B are in thermal equilibrium with each other.
2. The first Law : The First Law of Thermodynamics states that energy can be converted from one form to another with the interaction of heat, work and internal energy, but it cannot be created nor destroyed, under any circumstances.
3. The second Law : The Second Law of Thermodynamics states that the state of entropy of the entire universe, as an isolated system, will always increase over time. The second law also states that the changes in the entropy in the universe can never be negative.

4. The third law : The 3rd law of thermodynamics will essentially allow us to quantify the absolute amplitude of entropies. It says that when we are considering a totally perfect (100 pure) crystalline structure, at absolute zero (0 Kelvin), it will have no entropy (S). Note that if the structure in question were not totally crystalline, then although it would only have an extremely small disorder (entropy) in space, we could not precisely say it had no entropy. we are going to through these

in a bit but we shall move on to some processes that you need to know :

5. Isochoric process = An isochoric process, also called a constant-volume process, an isovolumetric process, or an isometric process, is a thermodynamic process during which the volume of the closed system undergoing such a process remains constant. now lets see the formula of the work done at an Isochoric process.

$$W = P \cdot \Delta V$$

Now because ΔV is a constant (0) at Isochoric process we can get this equation.

$$W = P \cdot 0$$

therefore the work done by an Isochoric process is 0.

Now lets go through some subjects : Entropy : To make it short entropy is a measure of disorder of a system. It has its own formula which looks like this :

$$S = k_b ln(\Omega)$$

so (S) represents entropy , the k_b represents the Boltzmann constant which looks like this
$\rightarrow 1.3806485210 - 23m2kgs - 2K - 1.$ and Ω is the number of micro states the system can occupy. and the ln represents the natural log of Omega. Before we get deeper into this material we want to show you 3 main concepts that you have to know.

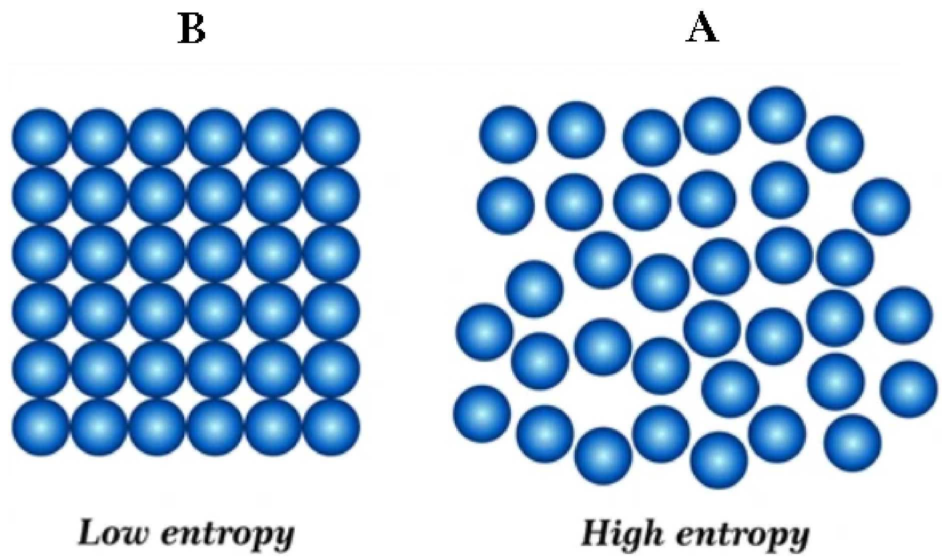
Enthalpy :

$$\Delta H = E + PV$$

Gibbs free energy :

$$\Delta G = \Delta H - T \cdot \Delta S$$

So lets go through these ”complex” formula’s and meanings. lets start by discussing on entropy,what is it ? well entropy is a measure of disorder and to understand it better we can look at this picture.



As you can see the organized system has less entropy than the less organized one. so if we call them as system A and system B we can say that.

$$S(A) > S(B)$$

Lets take a look on the second formula of entropy :

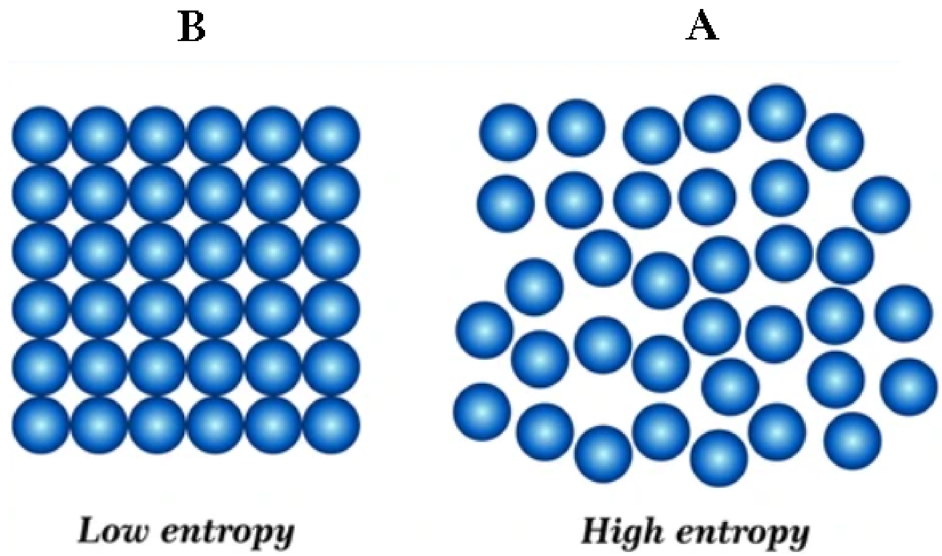
$$S = -k_b \sum_i p_i ln(p_i)$$

But its not that important since it as the same meaning as this one $\rightarrow S = k_b ln(\Omega)$.

but since this formula is not used anymore and we use this formula

$$S = k_b ln(\Omega)$$

we don't have to think about this formula and use it , so we shall move on. now lets speak about it more thoroughly. so lets think about the last simulation of entropy in depth. Recall :



so why does system A have more entropy ? the answer is simple , since the system A is less organized its measure of disorder is higher than the one in system B therefore its entropy is bigger, the main reason is that the particles at system A are more "Free" than the ones at system B. So from that statement we can say for sure that solids have less entropy than liquids or gases. so lets see this chemical form.

$$S(Solid) < S(Liquid) < S(Gas)$$

So now lets take a look on a concept that you have to know , Δ The Delta is a Greek sign that signifies a change in the area of physics and mathematics. So lets take a look on couple of these.

$$\Delta S$$

The Change of the entropy on the system And etc. lets talk about some basic thermodynamic systems that we can use on our example or question. First thing that we want to know is the basic example of the frying pan. what happens to the entropy of the water if we boil it ? so lets think about it this way , we know that liquids hold the average amount of entropy. now if we boil it , that means that we input more heat into the system (in that case its the water). thus the electrons flow more rapidly and quickly, which means that the structure of the water is spreading. because of that information we can say that the amount of entropy has increased due to the motion of the electrons. so now we know that if we boil water into hot water the amount of entropy will increase.

so now lets work on some examples :

6. Present the entropy on a system that occupies 3 micro-states.
7. A system has occupied 7 micro-states , but in ten minutes she will occupy 10 micro-states , what is the difference in the entropy between this moment and the moment in 10 minutes. describe the change with a calculation.

we shall move on to some processes that you need to know :

8. Isochoric process = An isochoric process, also called a constant-volume process, an isovolumetric process, or an isometric process, is a thermodynamic process during which the volume of the closed system undergoing such a process remains constant. now lets see the formula of the work done at an Isochoric process.

$$W = P \cdot \Delta V$$

Now because ΔV is a constant (0) at Isochoric process we can get this equation.

$$W = P \cdot 0$$

therefore the work done by an Isochoric process is 0.

lets move on to the complex stuff. lets talk about heat which takes a big place is thermodynamics , so we have an equation for heat.

$$Q = nC_V \Delta T$$

as we know Q Is Heat , now n represents the number of molls. C_V is the molar heat capacity , and ΔT is the change of the temperature. so what is a mole ?

The mole is the unit of measurement for amount of substance in the International System of Units (SI). It is defined as exactly $6.02214076 \cdot 10^{23}$ particles, which may be atoms, molecules, ions, or electrons. but we don't have to input this number into our calculation since mole is a unit.

what is ΔV ? V is the volume of the system , so ΔV is the change of the volume. therefore the work(W) done at isochoric process is always 0 (its a constant). Lets talk about the change of the internal energy of a system , as you can see in the formula below.

$$\Delta U = Q - W$$

W is always 0 at a isochoric process system thus the formula for the change of the internal energy of isochoric process system is as follows :

$$\Delta U = Q$$

So if we find the heat , we can find the internal energy of the system (at an isochoric process). we shall start talking about the isobaric process. In thermodynamics, an isobaric process is a type of thermodynamic process in which the pressure of the system stays constant: $\Delta P = 0$. The heat transferred to the system does work, but also changes the internal energy (U) of the system. This article uses the physics sign convention for work, where positive work is work done by the system. Using this convention, by the first law of thermodynamics.

Disclaimer : ΔP is NOT P. lets go through some formulas.

$$W = P \cdot \Delta V$$

$$Q = nC_p \Delta T$$

now what is the internal energy at a isobaric process ? well we have to know one equation for the internal energy.

$$\Delta U = nC_v \Delta T$$

lets talk about the last process that we need to know and its the isothermic process. In thermodynamics, an isothermal process is a type of thermodynamic process in which the temperature of the system remains constant: $\Delta T = 0$. This typically occurs when a system is in contact with an outside thermal reservoir, and the change in the system will occur slowly enough to allow the system to continue to adjust to the temperature of the reservoir through heat exchange. we have to know that at isothermal process the change of the temperature is a constant ($\Delta T = 0$)

lets go through some formula's and then we are going to move over to some serious problems.

$$\Delta U = nC_v \Delta T$$

$$\Delta U = Q - W$$

these formula's are the same , feel free to use them both. But since the temperature in isothermal process is a constant (0) the internal energy of a system running at a isothermal process is https://www.overleaf.com/project/604e5ea28ee4e26e4bd2f9f6Zero. think about it this way.

$$\Delta U = nC_v \cdot 0 =$$

$$\Delta U = n \cdot 0 =$$

$$\Delta U = 0$$

Now because of that we can say that.

$$W = Q$$

qqqWhy is that ? , so here is the answer.

$$\Delta U = Q - W$$

Lets plug in the 0.

$$0 = Q - W$$

Now we can move the work to the other side , and then we will get this.

$$W = Q$$

Thus the work at a isothermal process equals to the heat. lets go through some more formula's that you need to know.

6.5 Gibbs free energy

lets talk about Gibbs free energy. so make it short and understandable we say that Gibbs free energy is the amount of useful work left that the system can do. so here is the formula.

$$\Delta G = \Delta H - T \cdot \Delta S$$

we are going to explain what enthalpy is in the later sections but for now we are going to give it to you as a given and not as something to calculate. lets move to some examples , and exercises. Lets make it simple , while going through this exercise.

Take a system , and consider that the change of the entropy is 3.789323
consider that the temperature of the system is $21 Kelvin$. Let $\Delta H < 0$

Give an approximation for Gibbs free energy in the system.
Present the answer with a clear statement and with only ONE calculation.

SInce that our change in enthalpy is lower than Zero , we can plug in numbers that are lower than Zero to make our approximation more sharp. Ok lets plug in our givens and see what we get.

$$\Delta G = x < 0 - 21 \cdot 3.789323$$

Lets pick a number that is lower than Zero , for instance -2.

$$\Delta G = -2 - 21 \cdot 3.789323$$

$$\Delta G = -2 - \approx 80 = \approx -82$$

Thus we can state that for every number that is lower than Zero , the change in Gibbs free energy is definitely lower than Zero which means that the change is negative.

Note : (the statement is true for only this particular example.)

Chapter 7

Lagrangian mechanics

Have you ever thought of using energy's for your benefit ? well in classical mechanics it is true. but firstly , what is Lagrangian mechanics? Unlike Newtonian mechanics , Lagrangian mechanics is a concept of which we use energy's for our will , unlike Newtonian mechanics where we use forces. you don't have to use forces at Lagrangian mechanics at all , we are going to introduce you to some concepts and equations. so let us talk about the Lagrangian. the lagrangian is defined as follows.

$$\mathcal{L} = T - V$$

Now we are not going to introduce you to new concepts however we have to clear the minds on the symbols. That fancy L represents our Lagrange to find , whereas T represents the kinetic energy of our object. That V represents the potential energy of the object. now lets see formulas due to some misunderstandings.

$$T = \frac{m\dot{x}^2}{2}$$

Now is this $\dot{x}$? , in Newtonian it is represented by v , however in Lagrangian mechanics we use $\dot{x}$ as our velocity.

Note : (they hold the same meaning). Now lets go through the formula of the potential energy.

$$v = m \cdot g \cdot x$$

As stated previously it has the same meaning , therefore in Lagrangian mechanics the x is the h in potential energy. lets think about another term that you should be familiar with and its the acceleration. the acceleration can be written as

$$\ddot{x}$$

In other words.

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - (m \cdot g \cdot x)$$

lets go over to the fun topic. lets talk about the famous equation used in the field.

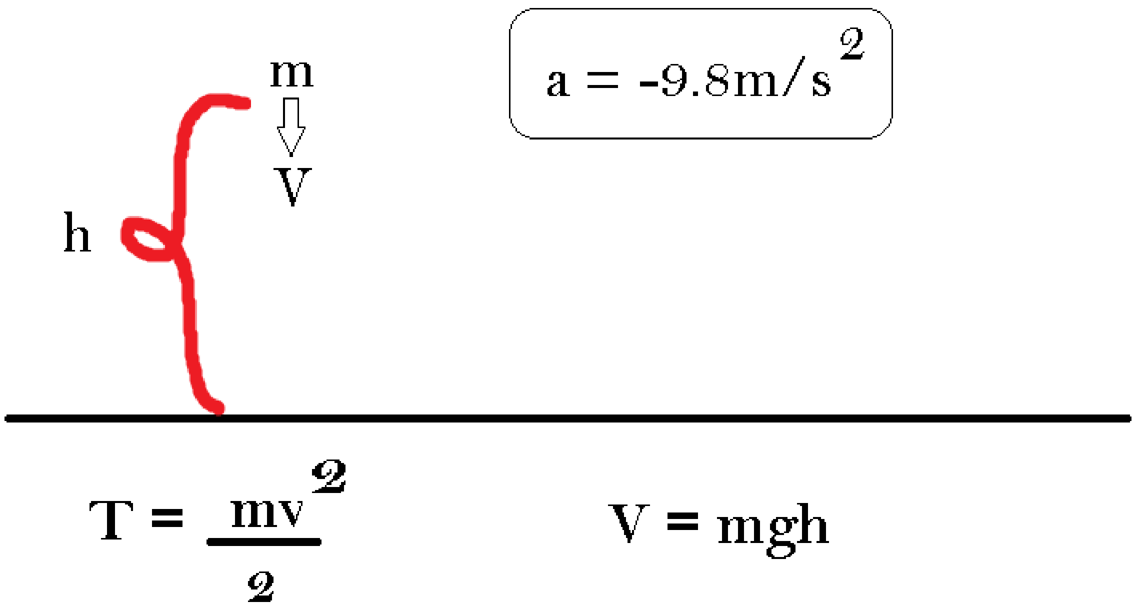
$$\frac{d}{dt} \cdot \left(\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) - \frac{\partial \mathcal{L}}{\partial x} = 0$$

now , what are they signifying ? well as we know $\frac{d}{dt} \cdot \frac{\partial \mathcal{L}}{\partial \dot{x}}$ is the first derivative of time of the partial derivative of the Lagrangian with respect to x , that is the first part of the equation. now what is the second part ? well the second part means is minus the partial derivative of the Lagrangian with respect to x.

we shall show you an illustration of the equation in action.

7.0.1 Free fall

Lets take this example.



as you can see we have an Object that has the quantity of mass m falling to the ground with the velocity v. the acceleration is assumed to be $-9.8m/s^2$. Recall :

$$T = \frac{m \cdot \dot{x}^2}{2} = \frac{m \cdot v^2}{2}$$

$$V = m \cdot g \cdot h = m \cdot g \cdot x$$

lets calculate our Lagrangian.

$$\mathcal{L} = \frac{m \cdot \dot{x}^2}{2} - (m \cdot g \cdot x)$$

So lets take the partial derivative of our Lagrangian with respect to $\dot{x}$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

Since every other term is a constant.
Let us take the partial derivative of our Lagrangian with respect to x

$$\frac{\partial \mathcal{L}}{\partial x} = (-m \cdot g)$$

before moving forward lets look at these meanings.

$$\dot{x} = \frac{dx}{dt}$$

$$\ddot{x} = \frac{d^2x}{dt^2}$$

As we know from newtons notation the number of dots indicates our derivative.

Now lets see what does this mean.

$$\frac{d}{dt} \cdot \left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)$$

In other words

$$\frac{d}{dt} \cdot \left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{d}{dt} \cdot m\dot{x}$$

and we know that this is equal to $m\ddot{x}$ so lets plug everything in.

$$m\ddot{x} - (-mg) = 0$$

$$m\ddot{x} + mg = 0$$

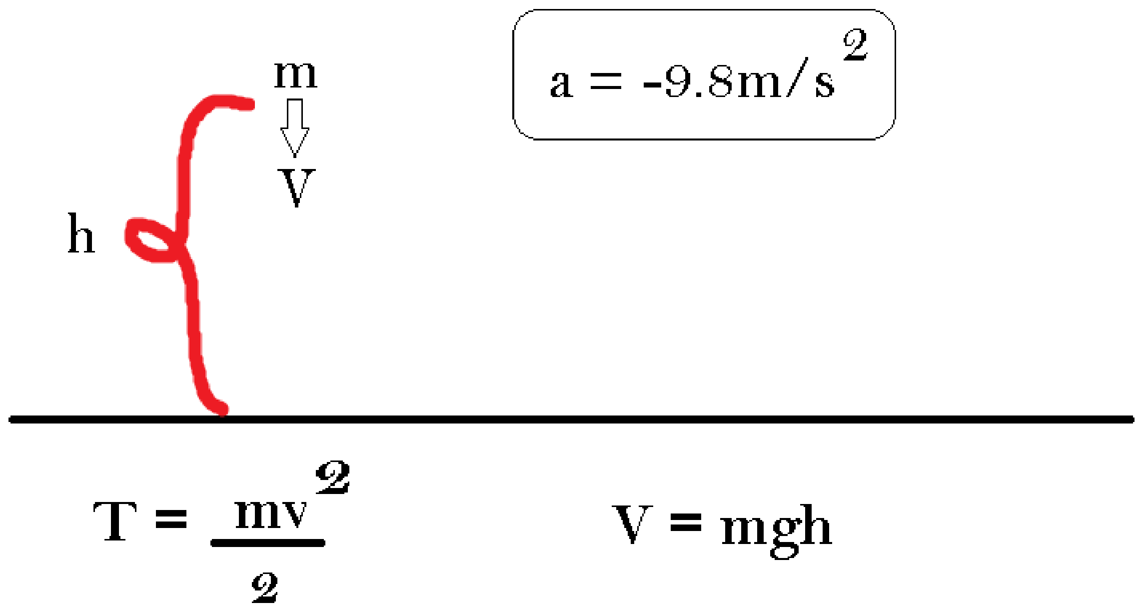
$$m\ddot{x} = -mg$$

lets reconstruct this equation.

$$-mg = m\ddot{x}$$

Now why is the gravity negative ?

Recall :



As you can see the Object is falling down which means that the force that the gravity is applying to the Object is negative.
Now , $(-mg)$ can be replaced with $\mathcal{F}$. and $m\ddot{x}$ can be replaced with acceleration. thus we get the infamous physics equation in Newtonian mechanics.

$$F = ma$$

So in this specific case , we got $F = ma$ as you can see in Lagrangian mechanics we don't have to use forces , we can use energy's , as we used here , (kinetic and potential).