On the anabelian geometry of mixed-characteristic local fields

（混標数局所体の遠アーベル幾何学について）

Hyeon Seung-Hyeon

Abstract

Let K be a mixed-characteristic local field. For an integer $m \geq 0$, we denote by K^m/K the maximal m-step solvable extension of K, and denote by G_K^m the maximal m-step solvable quotient of the absolute Galois group G_K of K. We regard G_K and its quotients as filtered profinite groups, equipped with the respective ramification filtrations (in upper numbering). It is known from Mochizuki's previous result that the isomorphism class of K is determined by the isomorphism class of the filtered profinite group G_K. In this master's thesis, we prove that the isomorphism class of K is determined by the isomorphism class of the maximal metabelian quotient G_K^2 as a filtered profinite group, and furthermore, that K^m/K is determined functorially by the filtered profinite group G_K^{m+2} (resp. G_K^{m+3}) for $m \geq 2$ (resp. $m = 0, 1$).

On the anabelian geometry of mixed-characteristic local fields
（混標数局所体の遠アーベル幾何学について）

Hyeon Seung-Hyeon

Abstract

Let K be a mixed-characteristic local field. For an integer $m \geq 0$, we denote by K^m/K the maximal m-step solvable extension of K, and denote by G_K^m the maximal m-step solvable quotient of the absolute Galois group G_K of K. We regard G_K and its quotients as filtered profinite groups, equipped with the respective ramification filtrations (in upper numbering). It is known from Mochizuki's previous result that the isomorphism class of K is determined by the isomorphism class of the filtered profinite group G_K. In this master's thesis, we prove that the isomorphism class of K is determined by the isomorphism class of the maximal metabelian quotient G_K^2 as a filtered profinite group, and furthermore, that K^m/K is determined functorially by the filtered profinite group G_K^{m+2} (resp. G_K^{m+3}) for $m \geq 2$ (resp. $m = 0, 1$).

Contents

Part I
Introduction

1 Anabelian geometry and field arithmetic: A brief history

Anabelian geometry is a branch of arithmetic geometry that studies how the arithmetic information of a geometric object X is encoded in its (étale) fundamental group. The central philosophy, originally proposed by A. Grothendieck [6], is that certain geometric objects of an “anabelian” nature should have characterizations in the language of fundamental groups. This translates to the principle that a field k of a certain type should be determined by its absolute Galois group G_k (as a topological group), by setting

$$X = \operatorname{Spec} k.$$

The celebrated theorem of Neukirch-Uchida—which existed even before the term “anabelian” was coined—is one of the first validations of this philosophy. The theorem states that number fields can be determined by their absolute Galois groups [23, Corollary 2], i.e., for two number fields $F_\circ$ and $F_\bullet$, it holds that

$$F_\circ \cong F_\bullet \quad \Leftrightarrow \quad G_{F_\circ} \cong G_{F_\bullet}.$$

This remarkable result naturally led to investigations into the analogous statement in a local setting. That is, for two mixed-characteristic local fields $K_\circ$ and $K_\bullet$, it holds that

$$K_\circ \cong K_\bullet \quad \overset{?}{\Leftrightarrow} \quad G_{K_\circ} \cong G_{K_\bullet}.$$

It is known that this statement does *not* hold in general (cf., e.g., [25] for a counterexample). However, by attaching additional structures to the absolute Galois groups, S. Mochizuki [14] proved Theorem 2.1, which implies that

$$K_\circ \cong K_\bullet \quad \Leftrightarrow \quad G_{K_\circ} \cong_{\mathrm{filt}} G_{K_\bullet},$$

where $\cong_{\mathrm{filt}}$ means that the two objects are isomorphic as *filtered* profinite groups. (For the definition of filtered profinite groups and isomorphisms between them, cf. p. 6. Here, the Galois groups are regarded as filtered profinite groups by the *ramification groups in upper numbering*.) Later, V. Abrashkin [1], [2] extended this result—yet with a different method—to the case of general local fields (including equal-characteristic local fields).

On the other hand, the question of whether results analogous to the theorem of Neukirch-Uchida hold for various quotients of absolute Galois groups has been extensively studied. For number fields, Saïdi-Tamagawa [18] showed that number fields can be characterized by their maximal m-step solvable quotients (cf. p. 8) for $m \geq 3$. That is, for number fields $F_\circ$ and $F_\bullet$,

$$F_\circ \cong F_\bullet \quad \Leftrightarrow \quad G^3_{F_\circ} \cong G^3_{F_\bullet},$$

where G^m denotes the maximal m-step solvable quotient of a profinite group G. Their work demonstrated that these quotients, despite carrying less information than the full absolute Galois group, still retain enough arithmetic information to determine the field structure.

2 Main results

One of the principal results of this master’s thesis is as follows: For two mixed-characteristic local fields $K_\circ$ and $K_\bullet$,

$$K_\circ \cong K_\bullet \quad \Leftrightarrow \quad G^2_{K_\circ} \cong_{\mathrm{filt}} G^2_{K_\bullet}.$$

Let us begin by recalling Mochizuki’s result. Let $K_\circ$ (resp. $K_\bullet$) be a mixed-characteristic local field of residue characteristic $p_{K_\circ}$ (resp. $p_{K_\bullet}$). For each $\square \in \{\circ, \bullet\}$, we fix an algebraic closure $K_\square^{\mathrm{alg}}$ of $K_\square$, and regard the absolute Galois group $G_{K_\square} = \mathrm{Gal}(K_\square^{\mathrm{alg}}/K_\square)$ of $K_\square$ as a filtered profinite group by the ramification groups in upper numbering. Suppose we are given a field isomorphism $f\colon K_\circ \to K_\bullet$. Then we have $p_{K_\circ} = p_{K_\bullet}$ $(=: p)$ (cf., e.g., §5) and f is, in particular, a $\mathbf{Q}_p$-algebra isomorphism. We can choose an isomorphism $\theta\colon K_\circ^{\mathrm{alg}} \to K_\bullet^{\mathrm{alg}}$ that extends f, which defines an isomorphism

$$G_{K_\circ} \xrightarrow{\cong} G_{K_\bullet}; \quad \sigma \mapsto \theta \circ \sigma \circ \theta^{-1}$$

of profinite groups. One can check that the above isomorphism respects the filtration by using the fact that f preserves the p-adic valuation. We denote by

$$\eta(f) \in \mathrm{Out}_{\mathrm{filt}}(G_{K_\circ}, G_{K_\bullet}) := \mathrm{Inn}(G_{K_\bullet})\backslash \mathrm{Isom}_{\mathrm{filt}}(G_{K_\circ}, G_{K_\bullet})$$

the equivalence class of the above isomorphism modulo inner automorphisms of $G_{K_\bullet}$ (i.e., the *outer isomorphism* defined by the above isomorphism). Here, $\mathrm{Isom}_{\mathrm{filt}}(G_{K_\circ}, G_{K_\bullet})$ (resp. $\mathrm{Out}_{\mathrm{filt}}(G_{K_\circ}, G_{K_\bullet})$) denotes the set of isomorphisms (resp. outer isomorphisms) $G_{K_\circ} \to G_{K_\bullet}$ of *filtered* profinite groups. We see that $\eta(f)$ does not depend upon the choice of the extension θ; therefore, we obtain a natural map

$$\eta\colon \mathrm{Isom}_{\mathbf{Q}_p\text{-alg}}(K_\circ, K_\bullet) \to \mathrm{Out}_{\mathrm{filt}}(G_{K_\circ}, G_{K_\bullet}).$$

Theorem 2.1 (Mochizuki [14, Theorem 4.2])**.** *The map η is a bijection. Equivalently, for an isomorphism*

$$\alpha\colon G_{K_\circ} \xrightarrow{\cong} G_{K_\bullet}$$

of filtered *profinite groups, there exists a unique isomorphism $\theta\colon K_\circ^{\mathrm{alg}} \to K_\bullet^{\mathrm{alg}}$ such that*

$$\alpha(\sigma) = \theta \circ \sigma \circ \theta^{-1}$$

for every $\sigma \in G_{K_\circ}$. In particular, we have an isomorphism $\theta|_{K_\circ}\colon K_\circ \to K_\bullet$. ◇

Theorem 2.1 can be considered as one form of the *Grothendieck Conjecture* for mixed-characteristic local fields: The above theorem implies that the isomorphism class of a given mixed-characteristic local field K can be determined *functorially* from the isomorphism class of its absolute Galois group G_K (as a filtered profinite group).

We now turn to the results of Saïdi-Tamagawa. As mentioned in the previous section, their results refine the theorem of Neukirch-Uchida by focusing on the isomorphisms between the maximal m-step solvable quotients $G_{F_\circ}^m$ and $G_{F_\bullet}^m$, which carry less group-theoretic (and hence arithmetic) information compared to the full absolute Galois groups $G_{F_\circ}$ and $G_{F_\bullet}$, for two number fields $F_\circ$ and $F_\bullet$.

Theorem 2.2 (Saïdi-Tamagawa [18, Theorem 1])**.** *Assume that there exists an isomorphism*

$$A_3\colon G_{F_\circ}^3 \xrightarrow{\cong} G_{F_\bullet}^3$$

of profinite groups. Then there exists an isomorphism $h\colon F_\circ \xrightarrow{\cong} F_\bullet$. ◇

Let k be a field. For an integer $m \geq 0$, we denote by k^m/k the maximal m-step solvable extension of k, i.e., the subfield of k^{sep} fixed by

$$\mathrm{Ker}\,(G_k \twoheadrightarrow G_k^m),$$

so that $G_k^m = \mathrm{Gal}(k^m/k)$.

Theorem 2.3 (Saïdi-Tamagawa [18, Theorem 2])**.** *Let m be an integer ≥ 0. For an isomorphism $A_{m+4}\colon G_{F_\circ}^{m+4} \to G_{F_\bullet}^{m+4}$ of profinite groups, there exists an isomorphism $\Theta_{m+1}\colon F_\circ^{m+1} \to F_\bullet^{m+1}$ such that*

$$A_{m+1}(\sigma) = \Theta_{m+1} \circ \sigma \circ \Theta_{m+1}^{-1}$$

for every $\sigma \in G_{F_\circ}^{m+1}$, where $A_{m+1}\colon G_{F_\circ}^{m+1} \to G_{F_\bullet}^{m+1}$ is the isomorphism induced by A_{m+4}. Moreover,

- *if* $m \geq 1$*, the isomorphism* Θ_{m+1} *is uniquely determined by* A_{m+4}*;*
- *if* $m = 0$*, the isomorphism* $\Theta_{m+1}|_{F_\circ} \colon F_\circ \to F_\bullet$ *is uniquely determined by* A_{m+4}*.*

$\diamond$

The statement of Theorem 2.2 lacks functoriality, meaning that there is no clear description of how A_3 and h are related to each other, which makes it a *weak bi-anabelian* result. In contrast, the isomorphism class of a given number field F is *functorially* determined from the isomorphism class of G_F^4 in Theorem 2.3; hence one might claim that Theorem 2.3 is a *strong bi-anabelian* result.

We proceed to the local counterpart of these theorems by stating the main results in their precise form. For a mixed-characteristic local field K, we again regard $G_K^m = \mathrm{Gal}(K^m/K)$ as a filtered profinite group by the ramification groups in upper numbering. Let $K_\circ$ and $K_\bullet$ be two mixed-characteristic local fields.

Theorem 2.4. *Assume that there exists an isomorphism*

$$\alpha_2 \colon G_{K_\circ}^2 \xrightarrow{\cong} G_{K_\bullet}^2$$

of filtered *profinite groups. Then there exists an isomorphism* $f \colon K_\circ \xrightarrow{\cong} K_\bullet$*.* $\diamond$

Theorem 2.5. *Let* m *be an integer* ≥ 0*. For an isomorphism*

$$\alpha_{m+3} \colon G_{K_\circ}^{m+3} \xrightarrow{\cong} G_{K_\bullet}^{m+3}$$

of filtered *profinite groups, there exists an isomorphism* $\theta_{m+1} \colon K_\circ^{m+1} \to K_\bullet^{m+1}$ *such that*

$$\alpha_{m+1}(\sigma) = \theta_{m+1} \circ \sigma \circ \theta_{m+1}^{-1}$$

for every $\sigma \in G_{K_\circ}^{m+1}$*, where* $\alpha_{m+1} \colon G_{K_\circ}^{m+1} \to G_{K_\bullet}^{m+1}$ *is the isomorphism induced by* α_{m+3}*. Moreover,*

(i) if $m \geq 1$*, the isomorphism* θ_{m+1} *is uniquely determined by* α_{m+3}*;*

(ii) if $m = 0$*, the isomorphism* $\theta_{m+1}|_{K_\circ} \colon K_\circ \to K_\bullet$ *is uniquely determined by* α_{m+3}*.*

$\diamond$

In light of the developments so far, Theorem 2.4 and Theorem 2.5 can be viewed as

- local analogues of Theorem 2.2 and Theorem 2.3, respectively;
- refinements of Theorem 2.1 for maximal m-step solvable quotients ($m \geq 2$).

$$\begin{tikzcd}
& \boxed{\text{Neukirch-Uchida}} \arrow[dl, "\text{local analogue}"'] \arrow[dr, "m\text{-step solvable version } (m \geq 3)"] & \\
\boxed{\text{Mochizuki [14]}} \arrow[dr, "m\text{-step solvable version } (m \geq 2)"'] & & \boxed{\text{Saïdi-Tamagawa [18]}} \arrow[dl, "\text{local analogue}"] \\
& \boxed{\text{Theorems 2.4 and 2.5}} &
\end{tikzcd}$$

In parallel with the results of Saïdi-Tamagawa, one could claim that Theorem 2.4 is a *weak bi-anabelian* result, and Theorem 2.5 is a *strong bi-anabelian* result, since the isomorphism class of a given mixed-characteristic local field K is *functorially* determined from the isomorphism class of G_K^3 (as a filtered profinite group) in Theorem 2.5.

We prove Theorems 2.4 and 2.5 in Part II, §8. The proof of Theorem 2.4 can be thought of as an application of p-adic Hodge theory. In fact, we implement only a few adjustments to the method developed by Mochizuki. For instance, in §7, we show that the *Hodge-Tate numbers* of a given abelian p-adic representation of G_K can be determined by using only the invariants of K recoverable from the filtered profinite group G_K^2; this is a sharpening of the preceding result due to Mochizuki [14, Corollary 3.1].

For some of the invariants of K that we use in the proof, we will give explicit *group-theoretic algorithms* (cf. [7, §3]) to demonstrate that those invariants can be recovered entirely from the (filtered) profinite group structure of G_K^m for some $m \geq 1$ in §§5 and 6. The reader will further observe that some of those invariants can be recovered even without the filtration attached to the profinite group G_K^m, although the filtration is essential when we endow the G_K^2-module $K^{1,+}$ with the p_K-adic topology, which forces us to keep the additional conditions on filtration in Theorems 2.4 and 2.5 (see Proposition 6.3 for more details).

3 Terminology and notation

Sets, topological spaces and numbers.

- For a set X, we shall denote by $|X|$ the *cardinality* of X.
- For a topological space X and a subset $Y \subseteq X$, we shall denote by $\overline{Y}$ the *closure* of Y in X.
- We shall denote by $\mathfrak{Primes}$ the *set of prime numbers*.

Groups.

- For a group G and a set X on which G acts (on the left), we shall denote by X^G the *subset of G-invariant elements* of X.
- For a group G and a G-module M, we shall write $H^i(G, M)$ for the i^{th} *cohomology group* of G with coefficients in M.
- For a group G and a subset $S \subseteq G$, we shall denote by $Z_G(S)$ the *centralizer* of S in G, and write $Z(G) := Z_G(G)$ for the *center* of G. We shall say that G is *center-free* if $Z(G)$ is trivial.
- For a group G, we shall denote by $\widehat{G}$ (or $G^\wedge$) the *profinite completion* of G. For a group homomorphism $\alpha\colon G_1 \to G_2$, we shall denote by $\widehat{\alpha}$ (or $\alpha^\wedge$) the canonical homomorphism $\widehat{G_1} \to \widehat{G_2}$ induced by α.
- For a profinite group G and a set of prime numbers $\Sigma \subseteq \mathfrak{Primes}$, we shall denote by $G^{\text{pro-}\Sigma}$ the *maximal pro-Σ quotient* of G. For a prime number ℓ, we shall denote by $G^{\text{pro-}\ell}$ (resp. $G^{\text{prime-to-}\ell}$) the *maximal pro-ℓ (resp. pro-prime-to-ℓ) quotient* of G. For a homomorphism $\alpha\colon G_1 \to G_2$ of profinite groups, we shall write $\alpha^{\text{pro-}\Sigma}$ (resp. $\alpha^{\text{pro-}\ell}$, resp. $\alpha^{\text{prime-to-}\ell}$) for the homomorphism
$$G_1^{\text{pro-}\Sigma} \to G_2^{\text{pro-}\Sigma} \qquad (\text{resp. } G_1^{\text{pro-}\ell} \to G_2^{\text{pro-}\ell}, \quad \text{resp. } G_1^{\text{prime-to-}\ell} \to G_2^{\text{prime-to-}\ell})$$
induced by α.

Rings and modules.

- Throughout this master's thesis, the term *ring* shall mean a *commutative ring with identity element*. For a ring A, we denote by A^+ (resp. $A^\times$) the *additive (resp. multiplicative) group* of A.

- For an abelian group M and an integer n, we shall denote by M_{tor} (resp. $M[n]$) the *torsion (resp. n-torsion) subgroup* of M. We shall write $M_{/\mathrm{tor}}$ for the module M/M_{tor}.
- For a prime power q, we shall denote by $\mathbf{F}_q$ the *finite field of order q*.
- For a prime number p, we shall denote by $\mathbf{Z}_p$ (resp. $\mathbf{Q}_p$) the *ring of p-adic integers* (resp. *field of p-adic numbers*).
- For a field k, we shall denote by $\mathfrak{Primes}_{\times/k} \subseteq \mathfrak{Primes}$ the *set of prime numbers invertible in k*.
- We shall denote by $\widehat{\mathbf{Z}}$ the *ring of profinite integers*, i.e., the ring $\prod_{p\in\mathfrak{Primes}} \mathbf{Z}_p$. For a field k, we shall write $\widehat{\mathbf{Z}}_{\times/k}$ for the *maximal pro-$\mathfrak{Primes}_{\times/k}$ quotient of $\widehat{\mathbf{Z}}$*.
- For a field k, we shall denote by $\mu_n(k) = k^\times[n]$ the *group of n^{th} roots of unity* in k.
- For a field k, we shall fix an *algebraic closure* k^{alg} of k, and denote by $k^{\mathrm{sep}} \subseteq k^{\mathrm{alg}}$ the *separable closure*. A field k is said to be *perfect* if $k^{\mathrm{sep}} = k^{\mathrm{alg}}$.

Representations of profinite groups.

- Let ℓ be a prime number. We shall say that V (resp. T) or (ρ, V) (resp. (ρ, T)) is an *ℓ-adic representation* (resp. a *$\mathbf{Z}_\ell$-representation*) of a profinite group G, when V (resp. T) is a $\mathbf{Q}_\ell$-vector space (resp. free $\mathbf{Z}_\ell$-module) of finite dimension (resp. rank) equipped with a *continuous* group homomorphism
$$\rho\colon G \to \mathrm{Aut}_{\mathbf{Q}_\ell}(V) \cong \mathbf{GL}_d(\mathbf{Q}_\ell) \quad (\text{resp. } \rho\colon G \to \mathrm{Aut}_{\mathbf{Z}_\ell}(T) \cong \mathbf{GL}_d(\mathbf{Z}_\ell)),$$
where d denotes $\dim_{\mathbf{Q}_\ell}(V)$ (resp. $\mathrm{rank}_{\mathbf{Z}_\ell}(T)$).
- For a Galois extension l/k, we shall denote by $\mathrm{Gal}(l/k)$ the *Galois group* of l/k, and write G_k for the *absolute Galois group* $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ of k. Unless otherwise stated, each Galois group will be endowed with the Krull topology, and hence regarded as a profinite group.
- For a field k, we shall write
$$\chi_{\mathrm{cycl},k}\colon G_k \to \mathrm{Aut}(\varprojlim_n \mu_n(k^{\mathrm{sep}})) \ \ (= (\widehat{\mathbf{Z}}_{\times/k})^\times)$$
for the ($\mathfrak{Primes}_{\times/k}$-adic) *cyclotomic character* of k. (The inverse limit is taken over the integers $n \geq 1$ whose prime factors belong to $\mathfrak{Primes}_{\times/k}$.) For $\ell \in \mathfrak{Primes}_{\times/k}$, we shall write
$$\chi^{(\ell)}_{\mathrm{cycl},k}\colon G_k \to \mathbf{Z}_\ell^\times$$
for the *ℓ-adic cyclotomic character* of k, i.e., the ℓ-part of $\chi_{\mathrm{cycl},k}$.

4 Preliminaries

Filtered profinite groups. Let G be a profinite group, and let $I \subseteq [0, +\infty)$ be a closed interval. We call a family $\{G(v)\}_{v\in[0,+\infty)}$ (resp. $\{G(v)\}_{v\in I}$) of closed normal subgroups of G a *filtration* (resp. an *I-filtration*) of G, if $G(v_1) \supseteq G(v_2)$ for any $v_1, v_2 \in [0, +\infty)$ (resp. $v_1, v_2 \in I$) with $v_1 \leq v_2$. We say that G is a *filtered* (resp. an *I-filtered*) profinite group if a filtration (resp. an I-filtration) is attached to it.

Let $G_\circ, G_\bullet$ be filtered (resp. *I-filtered*) profinite groups. We shall say that an isomorphism $\alpha\colon G_\circ \to G_\bullet$ (of profinite groups) is an isomorphism of *filtered* (resp. *I-filtered*) profinite groups if
$$\alpha(G_\circ(v)) = G_\bullet(v)$$

for all $v \in [0, +\infty)$ (resp. $v \in I$); we denote by

$$\mathrm{Isom}_{\mathrm{filt}}(G_\circ, G_\bullet) \quad (\text{resp. } \mathrm{Isom}_{I\text{-filt}}(G_\circ, G_\bullet))$$

the set of isomorphisms of filtered (resp. I-filtered) profinite groups from $G_\circ$ into $G_\bullet$. Note that the group $\mathrm{Inn}(G_\bullet)$ of *inner automorphisms* of $G_\bullet$ acts on $\mathrm{Isom}_{\mathrm{filt}}(G_\circ, G_\bullet)$ (resp. $\mathrm{Isom}_{I\text{-filt}}(G_\circ, G_\bullet)$), since $G_\bullet(v)$ is a normal subgroup of $G_\bullet$ for each v. Hence we can define the set of *outer isomorphisms* $G_\circ \to G_\bullet$:

$$\mathrm{Out}_{\mathrm{filt}}(G_\circ, G_\bullet) \coloneqq \mathrm{Inn}(G_\bullet)\backslash\mathrm{Isom}_{\mathrm{filt}}(G_\circ, G_\bullet) \quad (\text{resp. } \mathrm{Out}_{I\text{-filt}}(G_\circ, G_\bullet) \coloneqq \mathrm{Inn}(G_\bullet)\backslash\, \mathrm{Isom}_{I\text{-filt}}(G_\circ, G_\bullet)).$$

Mixed-characteristic local fields. We shall say that K is a *mixed-characteristic local field* if it is a finite extension of $\mathbf{Q}_p$ for some prime number p. Given a mixed-characteristic local field K, we write:

- $\mathcal{O}_K$ for the *ring of integers* of K;
- $\mathfrak{p}_K$ for the (unique) *maximal ideal* of $\mathcal{O}_K$;
- $\mathrm{ord}_K \colon K^\times \to \mathbf{Z}^+$ for the *normalized discrete valuation* on K;
- $U_K = U_K(0)$ for the *unit group* $\mathcal{O}_K^\times$ of $\mathcal{O}_K$;
- $U_K(n)$ for the n^{th} *higher unit group* $1 + \mathfrak{p}_K^n$ of $\mathcal{O}_K$ ($n \in \mathbf{Z}_{\geq 1}$);
- $\mathfrak{k}_K$ for the *residue field* $\mathcal{O}_K/\mathfrak{p}_K$ of K;
- p_K for the *residue characteristic* of K, i.e., the characteristic of $\mathfrak{k}_K$;
- $\varepsilon_K \coloneqq 1$ (resp. $\varepsilon_K \coloneqq 2$) if p_K is odd (resp. even);
- a_K for the largest integer ≥ 0 such that K contains a $(p_K^{a_K})^{\mathrm{th}}$ root of unity;
- d_K for the *absolute degree* $[K : \mathbf{Q}_{p_K}]$ of K;
- e_K for the *absolute ramification index* of K, so that $p_K\mathcal{O}_K = \mathfrak{p}_K^{e_K}$;
- f_K for the *absolute inertia degree* $[\mathfrak{k}_K : \mathbf{F}_{p_K}]$, so that $|\mathfrak{k}_K| = p_K^{f_K}$, and $d_K = e_K f_K$;
- $\chi_K = \chi_{\mathrm{cycl},K}^{(p_K)} \colon G_K \to \mathbf{Z}_{p_K}^\times$ for the p_K*-adic cyclotomic character* of K;
- K^{un} for the *maximal unramified extension* of K in K^{alg};
- K^{tame} for the *maximal tamely ramified extension* of K in K^{alg};
- $\mathrm{Frob}_K \in \mathrm{Gal}(K^{\mathrm{un}}/K)$ for the *arithmetic Frobenius* of K, so that $\mathrm{Frob}_K \mapsto (-)^{|\mathfrak{k}_K|}$ under the natural isomorphism $\mathrm{Gal}(K^{\mathrm{un}}/K) \to G_{\mathfrak{k}_K}$, and $\mathrm{Gal}(K^{\mathrm{un}}/K) = \mathrm{Frob}_K^{\widehat{\mathbf{Z}}^+} \cong \widehat{\mathbf{Z}}^+$.

For more details on (mixed-characteristic and general) local fields, cf., e.g., [4], [10], [12], [16], [21].

Ramification groups in upper numbering. For a mixed-characteristic local field K and any Galois extension F/K contained in K^{alg}, the Galois group $G = \mathrm{Gal}(F/K)$ is a profinite group equipped with the filtration defined by the *ramification groups in upper numbering* (cf. [21, Chap. IV, §3]); we denote by $G(v)$ the v^{th} ramification group for a real number $v \geq 0$. The upper numbering is compatible with quotients: If N is a closed normal subgroup of G, then

$$(G/N)(v) = G(v)N/N \tag{1}$$

for all $v \geq 0$ (cf. *loc. cit.*). Therefore, given a fundamental system $\mathscr{N}$ of neighborhoods of the identity element consisting of open normal subgroups of G, we have a natural isomorphism

$$G(v) \xrightarrow{\cong} \varprojlim_{N \in \mathscr{N}} (G/N)(v) \tag{2}$$

of profinite groups. Note that

$$G(0) = \mathrm{Gal}(F/(F \cap K^{\mathrm{un}})),$$
$$\overline{\bigcup_{v>0} G(v)} = \mathrm{Gal}(F/(F \cap K^{\mathrm{tame}})),$$

i.e., $G(0)$ (resp. $G(0+) := \overline{\bigcup_{v>0} G(v)}$) is precisely the *inertia subgroup* (resp. *wild inertia subgroup*) of G. (See also *loc. cit.*, Exercise 1.)

Suppose that L/K is a finite Galois subextension of F/K. We set $H := \mathrm{Gal}(F/L)$, so that $G/H = \mathrm{Gal}(L/K)$. We define the function $\phi = \phi_{L/K} : [0, +\infty) \to [0, +\infty)$ as the inverse function of

$$\psi(v) = \psi_{L/K}(v) := \int_0^v ((G/H) : (G/H)(w))\ dw.$$

It is clear from (1) that ϕ and ψ are determined by the groups H, G, and $G(v)$ for $v \geq 0$. Suppose that N is an open subgroup in H, and that $N \trianglelefteq G$. One can easily verify that

$$(H/N)(w) = (H/N) \cap (G/N)(\phi(w))$$

for all $w \geq 0$ (from, e.g., *loc. cit.*, Proposition 15), and derive the following lemma from (2).

Lemma 4.1. *For a real number $w \geq 0$, the w^{th} ramification group $H(w)$ of H is determined by the groups H, G, and $G(v)$ for $v \geq 0$: We have*

$$H(w) = \varprojlim_{N} \{(H/N) \cap (G/N)(\phi(w))\}$$

as a subset of $H = \varprojlim_N (H/N)$, where N runs through the open subgroups of H such that $N \trianglelefteq G$. $\diamond$

Solvable quotients of profinite groups. For a profinite group G, we denote by $\overline{[G, G]}$ the closed subgroup generated by the *commutators* of G, i.e., the elements of the form $\sigma\tau\sigma^{-1}\tau^{-1}$, where $\sigma, \tau \in G$. We inductively define the decreasing sequence

$$G = G^{[0]} \supseteq G^{[1]} \supseteq \cdots \supseteq G^{[m]} \supseteq \cdots$$

of closed normal subgroups of G, by $G^{[m+1]} = \overline{[G^{[m]}, G^{[m]}]}$. Note that $G^{[m]}$ are *characteristic subgroups* of G, i.e., every automorphism of G restricts to an automorphism of $G^{[m]}$. We say that a profinite group G is *m-step solvable* (resp. *abelian*, resp. *metabelian*) if $G^{[m]}$ (resp. $G^{[1]}$, resp. $G^{[2]}$) is trivial. We denote by G^m the quotient $G/G^{[m]}$, and call it the *maximal m-step solvable quotient* of G. We will often write G^{ab} (resp. G^{mab}) instead of G^1 (resp. G^2), and call it the *maximal abelian* (resp. *metabelian*) quotient or *abelianization* (resp. *metabelianization*) of G.

For a field k, we shall denote by k^m (resp. k^{ab}, resp. k^{mab}) the subextension of k^{sep}/k fixed by $G_k^{[m]}$ (resp. $G_k^{[1]}$, resp. $G_k^{[2]}$), and call it the *maximal m-step solvable (resp. abelian, resp. metabelian) extension* of k. In particular, we have

$$G_k^m = \mathrm{Gal}(k^m/k), \quad G_k^{\mathrm{ab}} = \mathrm{Gal}(k^{\mathrm{ab}}/k), \quad G_k^{\mathrm{mab}} = \mathrm{Gal}(k^{\mathrm{mab}}/k).$$

Definition 4.2. Let m be an integer ≥ 0, and let G be a profinite group. We shall say that G is a *profinite group of* MLF- *(resp.* MLF^m-*, resp.* MLF$^{\mathrm{ab}}$-*, resp.* MLF$^{\mathrm{mab}}$-*) type* if there exists an isomorphism of profinite groups between G and G_K (resp. G_K^m, resp. G_K^{ab}, resp. G_K^{mab}), for some mixed-characteristic local field K. We define *filtered* and *I-filtered profinite groups of* MLF- *(resp.* MLF^m-*, resp.* MLF$^{\mathrm{ab}}$-*, resp.* MLF$^{\mathrm{mab}}$-*) type* for a closed interval $I \subseteq [0, +\infty)$ in a similar way. $\diamond$

We prove the following lemma for later use.

Lemma 4.3. *Let m, n be integers ≥ 0.*

(1) Let Γ be a profinite group, H an open subgroup of Γ^{m+n} containing

$$(\Gamma^{m+n})^{[m]} = \mathrm{Ker}\,(\Gamma^{m+n} \twoheadrightarrow \Gamma^m) = \Gamma^{[m]}/\Gamma^{[m+n]}.$$

If we denote by $\tilde{H}$ the inverse image of H under the natural surjection $\Gamma \twoheadrightarrow \Gamma^{m+n}$, then the natural surjection $\tilde{H}^n \twoheadrightarrow H^n$ is injective.

(2) Let k be a field. For a finite extension l/k, we have

$$G_l^n = \mathrm{Gal}(k^{m+n}/l)^n$$

if l is contained in k^m. In particular, if G is a profinite group of MLF^{m+n}*-type (i.e., $G = \Gamma^{m+n}$ for some profinite group Γ of* MLF*-type), and H is an open subgroup of G containing $G^{[m]}$, then H^n is a profinite group of* MLF^n*-type.*

$\diamond$

Proof.

(1) Since the natural surjection $\tilde{H} \twoheadrightarrow H = \tilde{H}/\Gamma^{[m+n]}$ induces an isomorphism

$$\tilde{H}^{[n]}\Gamma^{[m+n]}/\Gamma^{[m+n]} = \tilde{H}^{[n]}/(\tilde{H}^{[n]} \cap \Gamma^{[m+n]}) \xrightarrow{\cong} (\tilde{H}/\Gamma^{[m+n]})^{[n]},$$

we have a natural isomorphism $\tilde{H}/\tilde{H}^{[n]}\Gamma^{[m+n]} \to H^n$. It follows from the hypothesis that $\tilde{H} \supseteq \Gamma^{[m]}$ (and that $\tilde{H}^{[n]} \supseteq \Gamma^{[m+n]}$), and hence the assertion holds.

(2) Apply (1) to the case $\Gamma = G_k$, $H = \mathrm{Gal}(k^{m+n}/l)$.

$\square$

Remark.

(1) If G is a profinite group of MLF-type, G is *prosolvable* [21, Chap. IV, Corollary 5 of Proposition 7]; hence

$$\bigcap_{m \geq 0} G^{[m]} = \{1\}.$$

However, G itself is *not solvable*, i.e., $G^{[m]} \neq \{1\}$ for every $m \geq 0$. This can be seen from the fact that, for every prime number p, the wild inertia subgroup of $G_{\mathbf{Q}_p}$ is isomorphic to a free pro-p group of countably infinite rank [17, Proposition 7.5.1], which is not solvable. Therefore, the sequence $\left\{G^{[m]}\right\}_{m \geq 0}$ is strictly decreasing.

(2) Let G be a profinite group of MLF^m-type for some integer $m \geq 0$. If we denote by $m(G)$ the minimal integer n such that $G^{[n]} = \{1\}$, then $m = m(G)$. In other words, $m(G)$—which is group-theoretically determined from the profinite group G—is the only integer $m \geq 0$ for which G is a profinite group of MLF^m-type: Assume that $G \cong G_K^m$ for some mixed-characteristic local field K and an integer m. Then obviously $G^{[m]} = \{1\}$, and it is clear from (1) that $G^{[n]} \neq \{1\}$ if $n < m$. Thus $m(G)$ equals m by definition.

$\diamond$

Part II

The m-step solvable anabelian geometry of mixed-characteristic local fields

Let m be an integer ≥ 1, and let $K, K_\circ, K_\bullet$ be mixed-characteristic local fields. In Part II, we work with (a (filtered) profinite group isomorphic to) the maximal m-step solvable quotient G_K^m of the absolute Galois group G_K, providing an analysis of what arithmetic information about K is retained by G_K^m, e.g.,

- In §5, we show that p_K, d_K, e_K and f_K can be determined entirely group-theoretically from the profinite group G_K^{ab}, and establish a group-theoretic algorithm that recovers χ_K from the profinite group G_K^{mab}.
- In §6, we determine group-theoretically the inertia and wild inertia group of G_K^{m+1}. Then we reconstruct the G_K-module $K^{m,+}$ from the profinite group structure of G_K^{m+1}, and recover its p_K-adic topology from the ramification groups attached to G_K^{m+1}.

With these results, we deduce the *Hodge-Tate preserving property* of an isomorphism $G_{K_\circ}^{\mathrm{mab}} \to G_{K_\bullet}^{\mathrm{mab}}$ of *filtered* profinite groups (see §7): If

$$G_{K_\bullet}^{\mathrm{ab}} \to \mathrm{Aut}_{\mathbf{Q}_{p_K}}(V) \tag{3}$$

is an abelian Hodge-Tate representation and $G_{K_\circ}^{\mathrm{mab}} \to G_{K_\bullet}^{\mathrm{mab}}$ is an isomorphism of filtered profinite groups, then the composition

$$G_{K_\circ}^{\mathrm{ab}} \to G_{K_\bullet}^{\mathrm{ab}} \to \mathrm{Aut}_{\mathbf{Q}_{p_K}}(V) \tag{4}$$

is also a Hodge-Tate representation, where the first arrow is the isomorphism induced by $G_{K_\circ}^{\mathrm{mab}} \to G_{K_\bullet}^{\mathrm{mab}}$. Moreover, the Hodge-Tate decompositions of the two representations (3) and (4) coincide. In §8, we apply this result (in a manner essentially identical to that of Mochizuki) to establish the main theorems (Theorems 2.4 and 2.5).

5 Restoration of the cyclotomic character

In the current section, we show that some invariants of a mixed-characteristic local field K (including the p_K-adic cyclotomic character χ_K) can be recovered "group-theoretically" from the maximal metabelian quotient G_K^{mab} of G_K.

Suppose that G is a profinite group of MLF$^{\mathrm{ab}}$-type, i.e., there exists an isomorphism $G \to G_K^{\mathrm{ab}} = \mathrm{Gal}(K^{\mathrm{ab}}/K)$ of profinite groups for some mixed-characteristic local field K. We first observe the structure of the group $K^\times$. Let $\pi \in K^\times$ be a *uniformizer* of K, i.e., an element such that $\mathfrak{p}_K = \pi\mathcal{O}_K$. Then we have the isomorphisms of topological groups

$$\begin{aligned} K^\times &= U_K \cdot \pi^{\mathbf{Z}^+} \xrightarrow{\cong} U_K \oplus \mathbf{Z}^+, \\ U_K &= \mu_{|\mathfrak{k}_K|-1}(K) \cdot U_K(1) \xrightarrow{\cong} (\mathbf{Z}/(p_K^{f_K}-1)\mathbf{Z})^+ \oplus (\mathbf{Z}/p_K^{a_K}\mathbf{Z})^+ \oplus (\mathbf{Z}_{p_K}^+)^{\oplus d_K} \end{aligned}$$

(cf. [9, Chap. II, §2], [16, Chap. II, §5]). We recall from *local class field theory* (cf., e.g., [4], [9], [13], [16], [20], [21], [26]) that the *local reciprocity map* (or *local Artin map*) $\mathrm{Art}_K\colon K^\times \to G_K^{\mathrm{ab}}$ fits into the following commutative diagram (in which the rows are splitting exact sequences)

$$\begin{tikzcd} 1 \arrow[r] & U_K \arrow[r, "\subseteq"] \arrow[d, "{\cong, \mathrm{Art}_K|_{U_K}}"] & K^\times \arrow[r, "\mathrm{ord}_K"] \arrow[d, "\mathrm{Art}_K"] & \mathbf{Z}^+ \arrow[r] \arrow[d, "\mathrm{Frob}_K^{(-)}"] & 1 \\ 1 \arrow[r] & G_K^{\mathrm{ab}}(0) \arrow[r, "\subseteq"] & G_K^{\mathrm{ab}} \arrow[r, "(-)|_{K^{\mathrm{un}}}"] & \mathrm{Gal}(K^{\mathrm{un}}/K) \arrow[r] & 1 \end{tikzcd}$$

and yields an isomorphism of profinite groups

$$\widehat{K^\times}\ \ (\cong U_K \oplus \widehat{\mathbf{Z}}^+)\ \ \overset{\cong}{\longrightarrow}\ \ G_K^{\mathrm{ab}}\ \ (\cong G_K^{\mathrm{ab}}(0) \oplus \mathrm{Gal}(K^{\mathrm{un}}/K)),$$

by profinite completion. In particular, we have

$$G \cong (\mathbf{Z}/(p_K^{f_K}-1)\mathbf{Z})^+ \oplus (\mathbf{Z}/p_K^{a_K}\mathbf{Z})^+ \oplus (\mathbf{Z}_{p_K}^+)^{\oplus d_K} \oplus \widehat{\mathbf{Z}}^+ \tag{5}$$

as profinite groups. We denote by $p(G)$ the uniquely determined prime number ℓ such that

$$\log_\ell |G_{/\mathrm{tor}}/\ell \cdot G_{/\mathrm{tor}}| \geq 2.$$

Furthermore, we set:

- $\varepsilon(G) := 1$ (resp. $\varepsilon(G) := 2$) if $p(G)$ is odd (resp. even);
- $a(G) := \log_{p(G)} |(G_{\mathrm{tor}})^{\text{pro-}p(G)}|$;
- $d(G) := \log_{p(G)} |G_{/\mathrm{tor}}/p(G) \cdot G_{/\mathrm{tor}}| - 1$;
- $f(G) := \log_{p(G)}(|(G_{\mathrm{tor}})^{\text{prime-to-}p(G)}| + 1)$;
- $e(G) := d(G)/f(G)$.

Proposition 5.1. *Let K be a mixed-characteristic local field. Then we have*

$$p_K = p(G_K^{\mathrm{ab}}), \quad \varepsilon_K = \varepsilon(G_K^{\mathrm{ab}}), \quad a_K = a(G_K^{\mathrm{ab}}),$$
$$d_K = d(G_K^{\mathrm{ab}}), \quad e_K = e(G_K^{\mathrm{ab}}), \quad f_K = f(G_K^{\mathrm{ab}}).$$

$\diamond$

Intuitively speaking, $p_K, \varepsilon_K, a_K, d_K, e_K$ and f_K can be recovered entirely group-theoretically from the profinite group G_K^{ab}.

Proof. It follows from (5) that p_K is the only prime number ℓ such that

$$\log_\ell |(G_K^{\mathrm{ab}})_{/\mathrm{tor}}/\ell \cdot (G_K^{\mathrm{ab}})_{/\mathrm{tor}}| \geq 2.$$

Hence $p_K = p(G_K^{\mathrm{ab}})$, $\varepsilon_K = \varepsilon(G_K^{\mathrm{ab}})$ and

$$d(G_K^{\mathrm{ab}}) = \log_{p(G_K^{\mathrm{ab}})} |(G_K^{\mathrm{ab}})_{/\mathrm{tor}}/p(G) \cdot (G_K^{\mathrm{ab}})_{/\mathrm{tor}}| - 1 = \log_{p_K} |(G_K^{\mathrm{ab}})_{/\mathrm{tor}}/p_K \cdot (G_K^{\mathrm{ab}})_{/\mathrm{tor}}| - 1 = d_K.$$

We also see from (5) that the pro-prime-to-p_K (resp. pro-p_K) part of $(G_K^{\mathrm{ab}})_{\mathrm{tor}}$ has exactly $p_K^{f_K} - 1$ (resp. $p_K^{a_K}$) elements. Therefore, we obtain the third, fifth and sixth equalities. $\square$

Next, we give a reconstruction algorithm that takes as input a profinite group of MLF$^{\mathrm{mab}}$-type, say, G, and returns (the isomorphism class of) a $\mathbf{Z}_\ell$-representation of G of rank 1, for each prime number ℓ. Suppose that there exists an isomorphism $\alpha\colon G \to G_K^{\mathrm{mab}}$ of profinite groups for a mixed-characteristic local field K. We start by choosing a decreasing sequence

$$G = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_\nu \supseteq \cdots$$

of open normal subgroups of G such that, for each $\nu \in \mathbf{Z}_{\geq 0}$,

(i) $H_\nu^{\mathrm{ab}}[\ell^\nu] \cong (\mathbf{Z}/\ell^\nu\mathbf{Z})^+$;

(ii) G/H_ν is abelian.

(Note that G acts on $H_\nu^{\mathrm{ab}}[\ell^\nu]$ by conjugation.) Such a sequence $\{H_\nu\}_\nu$ exists: We can choose $H_\nu = \alpha^{-1}(\mathrm{Gal}(K^{\mathrm{mab}}/K(\zeta_{\ell^\nu})))$, where ζ_{ℓ^ν} is a primitive $(\ell^\nu)^{\mathrm{th}}$ root of unity. It follows immediately that $\{H_\nu\}_\nu$ satisfies the condition (ii). We can also verify that $\{H_\nu\}_\nu$ satisfies the condition (i), using the local reciprocity map

$$\mathrm{Art}_{K(\zeta_{\ell^\nu})}\colon K(\zeta_{\ell^\nu})^\times \to G^{\mathrm{ab}}_{K(\zeta_{\ell^\nu})}$$

and the fact that

$$H_\nu^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{mab}}/K(\zeta_{\ell^\nu}))^{\mathrm{ab}} = G^{\mathrm{ab}}_{K(\zeta_{\ell^\nu})},$$

which follows from Lemma 4.3.

We know from local class field theory that if $L_\square \subseteq K^{\mathrm{ab}}$ is the field fixed by $\alpha(H_\square)$ for each $\square \in \{\nu, \nu+1\}$, then the diagram

$$\begin{tikzcd}
H_\nu^{\mathrm{ab}} \arrow[r, "{\cong, \alpha_\nu}"] \arrow[d, "\mathrm{Ver}"] & \mathrm{Gal}(K^{\mathrm{mab}}/L_\nu)^{\mathrm{ab}} = G^{\mathrm{ab}}_{L_\nu} \arrow[d, "\mathrm{Ver}"] & L_\nu^\times \arrow[l, hook', "\mathrm{Art}_{L_\nu}"'] \arrow[d, "\subseteq"] \\
H_{\nu+1}^{\mathrm{ab}} \arrow[r, "{\cong, \alpha_{\nu+1}}"] & \mathrm{Gal}(K^{\mathrm{mab}}/L_{\nu+1})^{\mathrm{ab}} = G^{\mathrm{ab}}_{L_{\nu+1}} & L_{\nu+1}^\times \arrow[l, hook', "\mathrm{Art}_{L_{\nu+1}}"']
\end{tikzcd}$$

commutes, where Ver is the *transfer* map (cf., e.g., [21, Chap. VII, §8], [24, §6.7]) and $\alpha_\square$ is the isomorphism of profinite groups induced by α. Moreover, $\mathrm{Art}_{L_\square}$ restricts to an isomorphism $U_{L_\square} \to G^{\mathrm{ab}}_{L_\square}(0)$, and hence to an isomorphism $\mu_{\ell^\square}(L_\square) \to G^{\mathrm{ab}}_{L_\square}[\ell^\square]$. Therefore, Ver restricts to an injective homomorphism $H_\nu^{\mathrm{ab}}[\ell^\nu] \to H_{\nu+1}^{\mathrm{ab}}[\ell^{\nu+1}]$; we identify $H_\nu^{\mathrm{ab}}[\ell^\nu]$ with a subgroup of $H_{\nu+1}^{\mathrm{ab}}[\ell^{\nu+1}]$ via Ver. (We will see later that the transfer map Ver here is in fact an injective homomorphism—cf. Lemma A.3 (2).)

We have the inverse system

$$\cdots \xrightarrow{(-)^\ell} H_{\nu+1}^{\mathrm{ab}}[\ell^{\nu+1}] \xrightarrow{(-)^\ell} H_\nu^{\mathrm{ab}}[\ell^\nu] \xrightarrow{(-)^\ell} \cdots \xrightarrow{(-)^\ell} H_1^{\mathrm{ab}}[\ell]$$

of G-modules induced by the homomorphisms $H_{\nu+1}^{\mathrm{ab}} \xrightarrow{(-)^\ell} H_{\nu+1}^{\mathrm{ab}}$. By passage to the limit, we obtain

$$T_\ell(G) := \varprojlim_\nu H_\nu^{\mathrm{ab}}[\ell^\nu].$$

It will be implicitly shown in the proof of Proposition 5.2 that the isomorphism class of the G-module $H_\nu^{\mathrm{ab}}[\ell^\nu]$ for each ν (and hence the isomorphism class of $T_\ell(G)$) does not depend upon the choice of H_ν. We shall write

$$\chi^{(\ell)}(G)\colon G \to \mathrm{Aut}(T_\ell(G)) \ \ (= \mathbf{Z}_\ell^\times)$$

for the ℓ-adic character of G attached to $T_\ell(G)$, and we define $\chi(G)$ as follows:

$$\chi(G) := \chi^{(p(G^{\mathrm{ab}}))}(G).$$

Proposition 5.2. *Let K be a mixed-characteristic local field.*

(1) For each prime number ℓ, there exists an isomorphism

$$\mathbf{Z}_\ell(1) \xrightarrow{\cong} T_\ell(G_K^{\mathrm{mab}})$$

of G_K^{mab}-modules, where $\mathbf{Z}_\ell(1)$ denotes the first Tate twist *of the trivial G_K^{mab}-module $\mathbf{Z}_\ell$.*

(2) The cyclotomic character χ_K factors through $\chi(G_K^{\mathrm{mab}})$.

◇

Intuitively speaking, $\chi_{\mathrm{cycl},K}\colon G_K \to \widehat{\mathbf{Z}}^\times$ and χ_K can be recovered entirely group-theoretically from the profinite group G_K^{mab}.

Proof. (1) We take a decreasing sequence

$$G_K^{\mathrm{mab}} = H_{K,0} \supseteq H_{K,1} \supseteq \cdots \supseteq H_{K,\nu} \supseteq \cdots$$

of open normal subgroups of G_K^{mab} satisfying the above conditions (i) and (ii). We shall write L_ν for the corresponding fixed field $(K^{\mathrm{mab}})^{H_{K,\nu}}$ of $H_{K,\nu}$. By Lemma 4.3 and the condition (ii), we have $G_{L_\nu}^{\mathrm{ab}} = H_{K,\nu}^{\mathrm{ab}}$ for each ν, and thus we have a group homomorphism

$$r_\nu := \mathrm{Art}_{L_\nu} \colon L_\nu^\times \to H_{K,\nu}^{\mathrm{ab}}.$$

It is implied by the condition (i) that L_ν contains the $(\ell^\nu)^{\mathrm{th}}$ roots of unity. Moreover, it can be seen from local class field theory that r_ν respects the G_K^{mab}-action (cf. [4, Chap. IV, (4.2)]). We obtain by restriction the G_K^{mab}-module isomorphism

$$r_\nu \colon \ ((\mathbf{Z}/\ell^\nu\mathbf{Z})^+ \cong)\ \mu_{\ell^\nu}(L_\nu) \ \xrightarrow{\cong}\ ((\mathbf{Z}/\ell^\nu\mathbf{Z})^+ \cong)\ H_{K,\nu}^{\mathrm{ab}}[\ell^\nu],$$

and the commutative diagram

$$\begin{tikzcd}
L_{\nu+1}^\times \arrow[r, "(-)^\ell"] \arrow[d, "r_{\nu+1}"] & L_{\nu+1}^\times \arrow[d, "r_{\nu+1}"] \\
H_{K,\nu+1}^{\mathrm{ab}} \arrow[r, "(-)^\ell"] & H_{K,\nu+1}^{\mathrm{ab}}
\end{tikzcd}$$

of G_K^{mab}-modules. We also know from local class field theory that the diagram

$$\begin{tikzcd}
L_{\nu}^\times \arrow[r, "\subseteq"] \arrow[d, "r_{\nu}"] & L_{\nu+1}^\times \arrow[d, "r_{\nu+1}"] \\
H_{K,\nu}^{\mathrm{ab}} \arrow[r, "\mathrm{Ver}"] & H_{K,\nu+1}^{\mathrm{ab}}
\end{tikzcd}$$

commutes. Hence we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "(-)^\ell"] & \mu_{\ell^{\nu+1}}(L_{\nu+1}) \arrow[r, "(-)^\ell"] \arrow[d, "\cong{,}r_{\nu+1}"] & \mu_{\ell^\nu}(L_\nu) \arrow[r, "(-)^\ell"] \arrow[d, "\cong{,}r_\nu"] & \cdots \arrow[r, "(-)^\ell"] & \mu_\ell(L_1) \arrow[d, "\cong{,}r_1"] \\
\cdots \arrow[r, "(-)^\ell"] & H_{K,\nu+1}^{\mathrm{ab}}[\ell^{\nu+1}] \arrow[r, "(-)^\ell"] & H_{K,\nu}^{\mathrm{ab}}[\ell^\nu] \arrow[r, "(-)^\ell"] & \cdots \arrow[r, "(-)^\ell"] & H_{K,1}^{\mathrm{ab}}[\ell]
\end{tikzcd}.$$

By passage to the limit, we obtain the following isomorphism of G_K^{mab}-modules.

$$r := \varprojlim_\nu r_\nu \colon \quad \mathbf{Z}_\ell(1) = \varprojlim_\nu \mu_{\ell^\nu}(L_\nu) \ \xrightarrow{\cong}\ T_\ell(G_K^{\mathrm{mab}}) = \varprojlim_\nu H_{K,\nu}^{\mathrm{ab}}[\ell^\nu]$$

(2) It is clear from (1) and Proposition 5.1. □

Remark. Proposition 5.2 can be considered as a local analogue of [18, Proposition A.9]. ◇

6 Ramification groups in upper numbering

We keep the notation and hypotheses of §5. In this section, we recover the G_K-module structure of $K^{\mathrm{ab},+}$ and its p_K-adic completion from the *filtered* profinite group G_K^{mab}.

Assume that G is a profinite group of MLF^{m+1}-type for an integer $m \geq 1$. It follows directly from Lemma 4.3 that if H is an open subgroup of G containing $G^{[m]}$, then H^{ab} is a profinite group of MLF$^{\mathrm{ab}}$-type. We denote by $I(G)$ the intersection of open subgroups H such that $H \supseteq G^{[1]}$ and $e(H^{\mathrm{ab}}) = e(G^{\mathrm{ab}})$. We also denote by $P(G)$ the (necessarily unique) pro-$p(G^{\mathrm{ab}})$-Sylow subgroup of $I(G)$.

Lemma 6.1. *Let K be a mixed-characteristic local field, and let m be an integer ≥ 1. Then the* inertia group *(resp.* wild inertia group*) of G_K^{m+1} equals $I(G_K^{m+1})$ (resp. $P(G_K^{m+1})$). In particular, the inertia group (resp. wild inertia group) of G_K^{m+1} can be determined entirely group-theoretically, without the additional information on filtration.* ⋄

Proof. Keeping in mind that *every unramified extension of K is abelian*, one checks by using Lemma 4.3 and Proposition 5.1 that the open subgroups of G_K^{m+1} containing $I(G_K^{m+1})$ are precisely the ones corresponding to the finite unramified extensions over K; hence $I(G_K^{m+1})$ equals the inertia group. Since the wild inertia group is nothing but the unique pro-p_K-Sylow subgroup of the inertia group (For the finite order case, see [21, Chap. IV], Corollaries 1 and 3 of Proposition 7. One easily reduces to this case, since wild inertia groups are compatible with quotients, cf. *loc. cit.*, Exercise 1 of §2.), the assertion on the wild inertia group holds as well. □

Remark. If m is an integer ≥ 2 and H is an open subgroup of G containing $G^{[2]}$, H^{ab} is a profinite group of MLF$^{\mathrm{ab}}$-type as remarked above. Hence in the case $m \geq 2$, one could alternatively define $P(G)$ as the intersection of open subgroups H such that $H \supseteq G^{[2]}$ and $e(H^{\mathrm{ab}})/e(G^{\mathrm{ab}})$ is coprime to $p(G^{\mathrm{ab}})$: Keeping in mind that *every tamely ramified extension of K is metabelian*, one checks as in the above proof that the open subgroups of G_K^{m+1} containing $P(G_K^{m+1})$ are precisely the ones corresponding to the finite tamely ramified extensions over K. Thus $P(G_K^{m+1})$ equals the wild inertia group of G_K^{m+1}. ⋄

Once again, let G be a profinite group of MLF^{m+1}-type for an integer $m \geq 1$, and let $\mathscr{H}_m(G)$ denote the set of open normal subgroups of G containing $G^{[m]}$, ordered by reverse inclusion. For each $H \in \mathscr{H}_m(G)$, we denote by $U(H)$ the image of $H \cap P(G)$ under the natural map $H \twoheadrightarrow H^{\mathrm{ab}}$, then we see that G acts on $U(H)$ by conjugation.

We first claim that, for $H_1, H_2 \in \mathscr{H}_m(G)$ with $H_1 \supseteq H_2$, the transfer map $\mathrm{Ver}\colon H_1^{\mathrm{ab}} \to H_2^{\mathrm{ab}}$ restricts to $U(H_1) \to U(H_2)$, and that $\{U(H)\}_{H \in \mathscr{H}_m(G)}$ forms a direct system of G-modules, together with $V_{1,2} := \mathrm{Ver}\,|_{U(H_1)}\colon U(H_1) \to U(H_2)$ for each pair $H_1 \supseteq H_2$. Suppose that there exists an isomorphism $\alpha\colon G \to G_K^{m+1}$ of profinite groups for some mixed-characteristic local field K, and that, for each $\square \in \{1,2\}$, the image of $H_\square$ equals $\mathrm{Gal}(K^{m+1}/L_\square)$, where $L_\square/K$ is a finite Galois subextension of K^m/K. Note that $\mathrm{Gal}(K^{m+1}/L_\square)^{\mathrm{ab}} = G_{L_\square}^{\mathrm{ab}}$ by Lemma 4.3 (and hence $H_\square^{\mathrm{ab}}$ is of MLF$^{\mathrm{ab}}$-type). The isomorphism $\alpha_\square\colon H_\square^{\mathrm{ab}} \to \mathrm{Gal}(K^{m+1}/L_\square)^{\mathrm{ab}}$ induced by α indeed fits into the following commutative diagram:

$$\begin{array}{ccccc}
H_1^{\mathrm{ab}} & \xrightarrow{\cong,\alpha_1} & \mathrm{Gal}(K^{m+1}/L_1)^{\mathrm{ab}} = G_{L_1}^{\mathrm{ab}} & \xleftarrow{\mathrm{Art}_{L_1}} & L_1^\times \\
\big\downarrow{\scriptstyle \mathrm{Ver}} & & \big\downarrow{\scriptstyle \mathrm{Ver}} & & \big\downarrow{\scriptstyle \subseteq} \\
H_2^{\mathrm{ab}} & \xrightarrow{\cong,\alpha_2} & \mathrm{Gal}(K^{m+1}/L_2)^{\mathrm{ab}} = G_{L_2}^{\mathrm{ab}} & \xleftarrow{\mathrm{Art}_{L_2}} & L_2^\times
\end{array}$$

We see from Lemma 6.1 that $H_\square \cap P(G) \subseteq G$ is mapped onto

$$\mathrm{Gal}(K^{m+1}/L_\square) \cap P(G_K^{m+1}) = \mathrm{Gal}(K^{m+1}/L_\square) \cap G_K^{m+1}(0+) = \mathrm{Gal}(K^{m+1}/L_\square)(0+)$$

under α; hence $U(H_\square) \subseteq H_\square^{\mathrm{ab}}$ is mapped onto $G_{L_\square}^{\mathrm{ab}}(0+)$ under $\alpha_\square$. Therefore, it suffices to show that the middle vertical arrow restricts to $G_{L_1}^{\mathrm{ab}}(0+) \to G_{L_2}^{\mathrm{ab}}(0+)$. But by local class field theory and the theorem of Hasse-Arf (cf. [21, Chap. V]), $U_{L_\square}(1)$ is mapped onto $G_{L_\square}^{\mathrm{ab}}(0+)$ under the local reciprocity map $\mathrm{Art}_{L_\square}$, and hence

$$\mathrm{Ver}(G_{L_1}^{\mathrm{ab}}(0+)) = \mathrm{Ver}(\mathrm{Art}_{L_1}(U_{L_1}(1))) \subseteq \mathrm{Art}_{L_2}(U_{L_2}(1)) = G_{L_2}^{\mathrm{ab}}(0+).$$

In particular, the restriction of $\mathrm{Art}_{L_\square}$ to $U_{L_\square}(1) \to G_{L_\square}^{\mathrm{ab}}(0+)$ is an isomorphism. It follows immediately that $\{U(H)\}_{H \in \mathscr{H}_m(G)}$ is a direct system induced by the direct system $\{U_L(1)\}_{L/K}$, where L/K runs through the finite Galois subextensions of K^m/K; each $U(H)$ is a (topological) $\mathbf{Z}_p$-module of finite rank, where $p := p(G^{\mathrm{ab}}) = p_K$. Hence we obtain a direct system $\left\{U(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\right\}_{H \in \mathscr{H}_m(G)}$ of G-modules; we set

$$k^{m,+}(G) := \varinjlim_{H \in \mathscr{H}_m(G)} \left(U(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\right).$$

Proposition 6.2. *Let K be a mixed-characteristic local field, and let m be an integer ≥ 1. Then there exists an isomorphism*

$$k^{m,+}(G_K^{m+1}) \xrightarrow{\cong} K^{m,+}$$

of G_K^{m+1}-modules. $\diamond$

Speaking from an intuitive level, the G_K^{m+1}-module $K^{m,+}$ can be recovered entirely group-theoretically from the profinite group G_K^{m+1}.

Proof. Let $H_\square = \mathrm{Gal}(K^{m+1}/L_\square) \in \mathscr{H}_m(G_K^{m+1})$ for each $\square \in \{1,2\}$, and assume that $H_1 \supseteq H_2$. By construction, $U(H_\square) = \mathrm{Gal}(K^{m+1}/L_\square)^{\mathrm{ab}}(0+) = G_{L_\square}^{\mathrm{ab}}(0+)$. The p_K-adic logarithm (cf. [16, Chap. II, §5], [10, Chap. IV, §2]) gives the following commutative diagram

$$\begin{tikzcd}
G_{L_1}^{\mathrm{ab}}(0+) \otimes_{\mathbf{Z}_{p_K}} \mathbf{Q}_{p_K} \arrow[r, "{\cong,\mathrm{Art}_{L_1}^{-1}}"] \arrow[d, "V_{1,2}\otimes\mathrm{id}"] & U_{L_1}(1) \otimes_{\mathbf{Z}_{p_K}} \mathbf{Q}_{p_K} \arrow[r, "{\cong,\log}"] \arrow[d, "\subseteq\otimes\,\mathrm{id}"] & L_1^+ \arrow[d, "\subseteq"] \\
G_{L_2}^{\mathrm{ab}}(0+) \otimes_{\mathbf{Z}_{p_K}} \mathbf{Q}_{p_K} \arrow[r, "{\cong,\mathrm{Art}_{L_2}^{-1}}"] & U_{L_2}(1) \otimes_{\mathbf{Z}_{p_K}} \mathbf{Q}_{p_K} \arrow[r, "{\cong,\log}"] & L_2^+
\end{tikzcd} \tag{6}$$

of G_K^{m+1}-modules. By passage to the limit, we obtain the desired isomorphism

$$k^{m,+}(G_K^{m+1}) = \varinjlim_{L/K} \left(G_L^{\mathrm{ab}}(0+) \otimes_{\mathbf{Z}_{p_K}} \mathbf{Q}_{p_K}\right) \xrightarrow{\cong} K^{m,+} = \varinjlim_{L/K} L^+,$$

where L/K runs through the finite Galois subextensions of K^m/K. $\square$

For an infinite algebraic extension F/K, we shall denote by $\mathscr{C}_F$ or $\mathscr{C}(F)$ the p_K-adic completion of F, and often write $\mathbf{C}_{p_K}$ instead of $\mathscr{C}_{K^{\mathrm{alg}}}$. In [14, Proposition 2.2], it is shown that the isomorphism class of G_K-module $\mathbf{C}_{p_K}^+$ can be recovered group-theoretically from the *filtered* profinite group G_K; we will prove an "m-step solvable analogue" of this result. To do so, we further assume that G is a *filtered* profinite group of MLF^{m+1}-type for an integer $m \geq 1$, i.e., there exists an isomorphism $\alpha^{\mathrm{filt}} \colon G \to G_K^{m+1}$ of *filtered* profinite groups for some mixed-characteristic local field K.

For any closed normal subgroup N of G, we shall equip G/N with the filtration defined by

$$(G/N)(v) := G(v)N/N$$

for each $v \geq 0$.

Suppose that $H \in \mathscr{H}_m(G)$. Then there exists a finite Galois subextension L/K of K^m/K such that $\alpha^{\mathrm{filt}}(H) = \mathrm{Gal}(K^{m+1}/L)$. We define the functions $\phi_H, \psi_H \colon [0,+\infty) \to [0,+\infty)$ by

$$\psi_H(v) := \int_0^v ((G/H) : (G/H)(w))\, dw,$$
$$\phi_H(w) := \psi_H^{-1}(w).$$

We regard H as a filtered profinite group by setting

$$H(w) := \varprojlim_N \{(H/N) \cap (G/N)(\phi_H(w))\} \qquad \left(\subseteq H = \varprojlim_N (H/N)\right)$$

for each $w \geq 0$, where N runs through the open subgroups of H such that $N \trianglelefteq G$. As a direct consequence of Lemma 4.1, $\alpha^{\mathrm{filt}}|_H \colon H \to \mathrm{Gal}(K^{m+1}/L)$ is an isomorphism of *filtered* profinite groups.

We denote by $U(H,w)$ the image of $H(w)$ under the natural map $H \twoheadrightarrow H^{\mathrm{ab}}$. Then we see that G acts on $U(H,w)$ by conjugation. We claim that, for $H_1, H_2 \in \mathscr{H}_m(G)$ with $H_1 \supseteq H_2$, the transfer map $\mathrm{Ver} \colon H_1^{\mathrm{ab}} \to H_2^{\mathrm{ab}}$ restricts to

$$U(H_1, \varepsilon(G)e(H_1^{\mathrm{ab}})) \to U(H_2, \varepsilon(G)e(H_2^{\mathrm{ab}})),$$

and that if we denote by $U'(H)$ the group $U(H, \varepsilon(G)e(H^{\mathrm{ab}}))$ for each $H \in \mathscr{H}_m(G)$,

$$\{U'(H)\}_{H \in \mathscr{H}_m(G)}$$

forms a direct system of G-modules, together with $V'_{1,2} \coloneqq \mathrm{Ver}|_{U'(H_1)} \colon U'(H_1) \to U'(H_2)$ for each pair $H_1 \supseteq H_2$. Suppose that, for each $\square \in \{1, 2\}$, the image of $H_\square$ equals $\mathrm{Gal}(K^{m+1}/L_\square)$, where $L_\square/K$ is a finite Galois subextension of K^m/K. Then the isomorphism $\alpha^{\mathrm{filt}}_\square \colon H^{\mathrm{ab}}_\square \to \mathrm{Gal}(K^{m+1}/L_\square)^{\mathrm{ab}}$ induced by α^{filt} fits into the following commutative diagram:

$$\begin{tikzcd}
H_1^{\mathrm{ab}} \arrow[r, "{\cong, \alpha_1^{\mathrm{filt}}}"] \arrow[d, "\mathrm{Ver}"] & \mathrm{Gal}(K^{m+1}/L_1)^{\mathrm{ab}} = G_{L_1}^{\mathrm{ab}} \arrow[d, "\mathrm{Ver}"] & L_1^\times \arrow[l, "\mathrm{Art}_{L_1}"'] \arrow[d, "\subseteq"] \\
H_2^{\mathrm{ab}} \arrow[r, "{\cong, \alpha_2^{\mathrm{filt}}}"] & \mathrm{Gal}(K^{m+1}/L_2)^{\mathrm{ab}} = G_{L_2}^{\mathrm{ab}} & L_2^\times \arrow[l, "\mathrm{Art}_{L_2}"']
\end{tikzcd}$$

As we have seen above, $H_\square(w)$ is mapped onto $\mathrm{Gal}(K^{m+1}/L_\square)(w)$ under $\alpha^{\mathrm{filt}}|_{H_\square}$ for all $w \geq 0$; thus $U'(H_\square) \subseteq H^{\mathrm{ab}}_\square$ is mapped onto $G^{\mathrm{ab}}_{L_\square}(\varepsilon(G)e(H^{\mathrm{ab}}_\square))$ under $\alpha^{\mathrm{filt}}_\square$. Therefore, in order to prove the claim, it suffices to show that the middle vertical arrow restricts to

$$G^{\mathrm{ab}}_{L_1}(\varepsilon_K e_{L_1}) \to G^{\mathrm{ab}}_{L_2}(\varepsilon_K e_{L_2}).$$

We have

$$\mathfrak{p}_{L_1}^{\varepsilon_K e_{L_1}} \subseteq \left(\mathfrak{p}_{L_1}\mathcal{O}_{L_2}\right)^{\varepsilon_K e_{L_1}} = \left(\mathfrak{p}_{L_2}^{e_{L_2}/e_{L_1}}\right)^{\varepsilon_K e_{L_1}} = \mathfrak{p}_{L_2}^{\varepsilon_K e_{L_2}},$$

and it follows that $U_{L_1}(\varepsilon_K e_{L_1}) \subseteq U_{L_2}(\varepsilon_K e_{L_2})$. Together with the fact that $\mathrm{Art}_{L_\square}$ restricts to an isomorphism $U_{L_\square}(w) \to G^{\mathrm{ab}}_{L_\square}(w)$ for all $w \geq 0$ [20, p. 155, Theorem 1], we conclude that

$$\mathrm{Ver}(G^{\mathrm{ab}}_{L_1}(\varepsilon_K e_{L_1})) = \mathrm{Ver}(\mathrm{Art}_{L_1}(U_{L_1}(\varepsilon_K e_{L_1}))) \subseteq \mathrm{Art}_{L_2}(U_{L_2}(\varepsilon_K e_{L_2})) = G^{\mathrm{ab}}_{L_2}(\varepsilon_K e_{L_2}).$$

Therefore, we obtain a direct system $\{U'(H)\}_{H \in \mathscr{H}_m(G)}$ of G-modules in a way similar to the way in which we obtained $\{U(H)\}_{H \in \mathscr{H}_m(G)}$. We again put $p \coloneqq p(G^{\mathrm{ab}}) = p_K$. Note that, for each $H \in \mathscr{H}_m(G)$ and the subextension L/K fixed by $\alpha^{\mathrm{filt}}(H)$, the natural map $U_L(\varepsilon_K e_L) \to U_L(1) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$ (and hence the natural map $U'(H) \to U(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$) is injective since $U_L(\varepsilon_K e_L)$ is contained in the non-torsion part of U_L (as $\varepsilon_K e_L > e_L/(p-1)$ and the p-adic logarithm restricts to an isomorphism $U_L(\varepsilon_K e_L) \to \mathfrak{p}_L^{\varepsilon_K e_L}$). We identify $U'(H)$ with a submodule of $U(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$; we set

$$\mathcal{O}^+_{k^m}(G) \coloneqq p^{-\varepsilon(G)}\left(\varinjlim_{H \in \mathscr{H}_m(G)} U'(H)\right) \quad \left(\subseteq k^{m,+}(G) = \varinjlim_{H \in \mathscr{H}_m(G)} \left(U(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\right)\right),$$

$$\mathscr{C}^+_{k^m}(G) \coloneqq \left(\varprojlim_n \left(\mathcal{O}^+_{k^m}(G)/p^n\mathcal{O}^+_{k^m}(G)\right)\right) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p.$$

Proposition 6.3. *Let K be a mixed-characteristic local field, and let m be an integer ≥ 1. Then the isomorphism of Proposition 6.2 restricts to an isomorphism*

$$\mathcal{O}^+_{k^m}(G^{m+1}_K) \xrightarrow{\cong} \mathcal{O}^+_{K^m}$$

of G^{m+1}_K-modules, where $\mathcal{O}_{K^m}$ denotes the integral closure of $\mathcal{O}_K$ in K^m. In particular, there exists an isomorphism

$$\mathscr{C}^+_{k^m}(G^{m+1}_K) \xrightarrow{\cong} \mathscr{C}^+_{K^m}$$

of G^{m+1}_K-modules. ◇

Speaking from an intuitive level, the G_K^{m+1}-module $\mathscr{C}^+_{K^m}$ can be recovered entirely group-theoretically from the *filtered* profinite group G_K^{m+1}.

Proof. Let $H_\Box = \mathrm{Gal}(K^{m+1}/L_\Box) \in \mathscr{H}_m(G_K^{m+1})$ for each $\Box \in \{1,2\}$, and assume that $H_1 \supseteq H_2$. By construction, we have $U'(H_\Box) = G^{\mathrm{ab}}_{L_\Box}(\varepsilon_K e_{L_\Box})$. Under the isomorphism $\mathrm{Art}^{-1}_{L_\Box}\colon G^{\mathrm{ab}}_{L_\Box}(0+) \to U_{L_\Box}(1)$, the subgroup $G^{\mathrm{ab}}_{L_\Box}(\varepsilon_K e_{L_\Box})$ is mapped onto $U_{L_\Box}(\varepsilon_K e_{L_\Box})$, which is again mapped onto $\mathfrak{p}_{L_\Box}^{\varepsilon_K e_{L_\Box}} = p_K^{\varepsilon_K}\mathscr{O}^+_{L_\Box}$ under the p_K-adic logarithm

$$U_{L_\Box}(1) \otimes_{\mathbf{Z}_{p_K}} \mathbf{Q}_{p_K} \xrightarrow{\cong,\log} L^+_\Box.$$

Hence we have the following commutative diagram

$$\begin{array}{ccccc}
G^{\mathrm{ab}}_{L_1}(\varepsilon_K e_{L_1}) & \xrightarrow{\cong,\mathrm{Art}^{-1}_{L_1}} & U_{L_1}(\varepsilon_K e_{L_1}) & \xrightarrow{\cong,\log} & p_K^{\varepsilon_K}\mathscr{O}^+_{L_1} \\
\downarrow{\scriptstyle V'_{1,2}} & & \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} \\
G^{\mathrm{ab}}_{L_2}(\varepsilon_K e_{L_2}) & \xrightarrow{\cong,\mathrm{Art}^{-1}_{L_2}} & U_{L_2}(\varepsilon_K e_{L_2}) & \xrightarrow{\cong,\log} & p_K^{\varepsilon_K}\mathscr{O}^+_{L_2}
\end{array}$$

of G_K^{m+1}-modules, which is compatible with the above commmutative diagram (6). By passage to the limit, we see that the isomorphism of Proposition 6.2 restricts to the isomorphism

$$p_K^{\varepsilon_K}\mathscr{O}^+_{k^m}(G_K^{m+1}) = \varinjlim_{L/K} G^{\mathrm{ab}}_L(\varepsilon_K e_L) \ \xrightarrow{\cong}\ p_K^{\varepsilon_K}\mathscr{O}^+_{K^m} = \varinjlim_{L/K} p_K^{\varepsilon_K}\mathscr{O}^+_L,$$

where L/K runs through the finite Galois subextensions of K^m/K. Therefore, we obtain the desired isomorphism, by multiplying both sides by $p_K^{-\varepsilon_K}$. $\Box$

7 Abelian p-adic representations

Hodge-Tate numbers

Let G be a profinite group, and (ρ, V) an ℓ-adic representation of G for a prime number ℓ. For an ℓ-adic character $\chi\colon G \to \mathbf{Z}_\ell^\times$, we shall write $\mathbf{Z}_\ell(\chi)$ for the $\mathbf{Z}_\ell$-representation $(\chi, \mathbf{Z}_\ell^+)$ of G, and $V(\chi)$ for the ℓ-adic representation $V \otimes_{\mathbf{Q}_\ell} (\mathbf{Q}_\ell \otimes_{\mathbf{Z}_\ell} \mathbf{Z}_\ell(\chi))$ of G.

Let K be a mixed-characteristic local field. Recall that, for a p_K-adic representation (ρ, V) of G_K and an integer i, the i^{th} *Hodge-Tate number* $d^i_{\mathrm{HT},K}(\rho, V)$ of (ρ, V) is the dimension of the K-vector space

$$\left(\mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K},$$

where $V(-i) = (\rho(-i), V(-i))$ denotes the $(-i)^{\mathrm{th}}$ Tate twist $V(\chi_K^{-i})$ of V. From the theory of p-adic representations, it is known that

$$\sum_{i\in\mathbf{Z}} d^i_{\mathrm{HT},K}(\rho, V) \le \dim_{\mathbf{Q}_{p_K}}(V),$$

and we say that (ρ, V) is *Hodge-Tate* when the equality holds (cf. [5, §5.1]).

Lemma 7.1. *Let (ρ, V) be a p_K-adic representation of G_K. Then we have*

$$\left(\mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V\right)^{G_K} = \left(\mathscr{C}((K^{\mathrm{alg}})^{\mathrm{Ker}(\rho)}) \otimes_{\mathbf{Q}_{p_K}} V\right)^{G_K}. \tag{7}$$

$\diamond$

Proof. We choose a basis $v_1, \dots, v_n$ of V. For each $\sigma \in G_K$, we shall write $(a_{ij}(\sigma)) \in \mathbf{GL}_n(\mathbf{Q}_{p_K})$ for the matrix of the linear transformation $\rho_\sigma := \rho(\sigma)$ with respect to the basis $v_1, \dots, v_n$, so that

$$(\rho_\sigma(v_1) \ \cdots \ \rho_\sigma(v_n)) = (v_1 \ \cdots \ v_n)(a_{ij}(\sigma)).$$

Suppose that $c_1, \dots, c_n \in \mathbf{C}_{p_K}$, and that $c_1 \otimes v_1 + \cdots + c_n \otimes v_n$ belongs to the left hand side of (7). Then for all $\sigma \in G_K$,

$$\begin{aligned} c_1 \otimes v_1 + \cdots + c_n \otimes v_n &= \sigma(c_1) \otimes \rho_\sigma(v_1) + \cdots + \sigma(c_n) \otimes \rho_\sigma(v_n) \\ &= \left(\sum_{j=1}^n \sigma(c_j) a_{1j}(\sigma)\right) \otimes v_1 + \cdots + \left(\sum_{j=1}^n \sigma(c_j) a_{nj}(\sigma)\right) \otimes v_n, \end{aligned}$$

and hence

$$\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = (a_{ij}(\sigma)) \begin{pmatrix} \sigma(c_1) \\ \vdots \\ \sigma(c_n) \end{pmatrix}.$$

In particular, we have

$$c_1 \otimes v_1 + \cdots + c_n \otimes v_n \in \mathbf{C}_{p_K}^{\mathrm{Ker}\,(\rho)} \otimes_{\mathbf{Q}_{p_K}} V = \mathscr{C}((K^{\mathrm{alg}})^{\mathrm{Ker}\,(\rho)}) \otimes_{\mathbf{Q}_{p_K}} V,$$

since it holds that $\sigma(c_1) = c_1, \dots, \sigma(c_n) = c_n$ for all $\sigma \in \mathrm{Ker}\,(\rho)$. (Note that, for any closed subgroup H of G_K,

$$\mathbf{C}_{p_K}^H = \mathscr{C}((K^{\mathrm{alg}})^H)$$

by the theorem of Ax-Sen-Tate—cf. [22], [3], [5, Chap. 3].) □

Definition 7.2. Let G be a profinite group, and let (ρ, V) be an ℓ-adic representation of G for a prime number ℓ. We shall say that (ρ, V) is an *m-step solvable* ℓ-adic representation of G for an integer $m \geq 0$ if ρ annihilates $G^{[m]}$. We shall also say that (ρ, V) is *abelian* (resp. *metabelian*) if it is a 1-step (resp. 2-step) solvable ℓ-adic representation of G. ◇

Let G be a filtered profinite group of MLF^{m+1}-type for an integer $m \geq 1$; we set $p := p(G^{\mathrm{ab}})$. Let (ρ, V) and χ be a p-adic representation and a p-adic character of G, respectively. We shall write

$$d^i_{\mathrm{HT},m}(G, \chi, \rho, V) := \dim_{\mathbf{Q}_p} \left(\mathscr{C}^+_{k^m}(G) \otimes_{\mathbf{Q}_p} V(\chi^{-i})\right)^G / d(G^{\mathrm{ab}})$$

for each $i \in \mathbf{Z}$.

Proposition 7.3. *Let K be a mixed-characteristic local field, and let m be an integer ≥ 1. Given an m-step solvable p_K-adic representation (ρ, V) of G_K, it holds that*

$$d^i_{\mathrm{HT},m}(G_K^{m+1}, \chi(G_K^{\mathrm{mab}}), \rho, V) = d^i_{\mathrm{HT},K}(\rho, V)$$

for each $i \in \mathbf{Z}$. ◇

Intuitively speaking, the i^{th} *Hodge-Tate number* of such (ρ, V) (and hence the issue of *whether or not such (ρ, V) is Hodge-Tate*) can be determined group-theoretically from the *filtered* profinite group G_K^{m+1} and its action on V.

Proof. We have

$$\begin{aligned} d^i_{\mathrm{HT},m}(G_K^{m+1}, \chi(G_K^{\mathrm{mab}}), \rho, V) &= \dim_{\mathbf{Q}_{p_K}} \left(\mathscr{C}(K^m) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} / d_K \\ &= \dim_K \left(\mathscr{C}(K^m) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} \end{aligned}$$

from Propositions 5.1, 5.2 and 6.3. Since both ρ and χ_K annihilates $G_K^{[m]}$, we see that

$$\left(\mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} \supseteq \left(\mathscr{C}(K^m) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} \supseteq \left(\mathscr{C}((K^{\mathrm{alg}})^{\mathrm{Ker}\,(\rho(-i))}) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K},$$

and deduce the desired equality from Lemma 7.1. □

Uniformizing representations

Let (ρ, V) be a p_K-adic representation of G_K for a mixed-characteristic local field K, and E/K a finite extension such that $E/\mathbf{Q}_{p_K}$ is Galois. Suppose that V is an E-vector space of dimension 1 and the G_K-action on V is E-linear. Then we see that $\rho\colon G_K \to \mathrm{Aut}_{\mathbf{Q}_{p_K}}(V)$ factors through $\rho\colon G_K^{\mathrm{ab}} \to E^\times$. In particular, (ρ, V) is an *abelian* representation. We say that a representation (ρ, V) of this type is *uniformizing* if there exist an open subgroup $I \subseteq U_K$ and a field homomorphism $\iota\colon K \to E$ such that

$$(\rho \circ \mathrm{Art}_K)|_I = \iota^\times|_I,$$

where $\iota^\times\colon K^\times \to E^\times$ is the group homomorphism induced by ι.

Example 7.4. Given any finite extension E/K such that $E/\mathbf{Q}_{p_K}$ is Galois, $V = E^+$ can be regarded as a uniformizing representation by local class field theory. More precisely, we define the G_K-action on V via the composition

$$\rho\colon\ G_K^{\mathrm{ab}} \xrightarrow{\mathrm{Ver}} G_E^{\mathrm{ab}}\,(\cong G_E^{\mathrm{ab}}(0) \oplus \mathrm{Gal}(E^{\mathrm{un}}/E)) \twoheadrightarrow G_E^{\mathrm{ab}}(0) \to U_E$$

of continuous homomorphisms, where the second (resp. third) arrow is the natural surjection (resp. the isomorphism Art_E^{-1}). ◇

Proposition 7.5. *Let K be a mixed-characteristic local field, and let E/K be a finite extension such that $E/\mathbf{Q}_{p_K}$ is Galois. Suppose that (ρ, V) is an E-linear representation of G_K, of E-dimension* 1. *Then (ρ, V) is a uniformizing representation if and only if*

$$d^i_{\mathrm{HT},K}(\rho, V) = \begin{cases} [E:K]([K:\mathbf{Q}_{p_K}]-1) & i = 0 \\ [E:K] & i = 1 \end{cases}.$$

◇

Proof. cf. [19, Chap. III, A5], [14, §3]. □

Corollary 7.6. *Let $K_\circ$ and $K_\bullet$ be mixed-characteristic local fields, $\alpha_2\colon G_{K_\circ}^{\mathrm{mab}} \to G_{K_\bullet}^{\mathrm{mab}}$ an isomorphism of* filtered *profinite groups. Suppose that $E/\mathbf{Q}_{p_{K_\circ}} (= \mathbf{Q}_{p_{K_\bullet}})$ is a finite Galois extension containing both $K_\circ$ and $K_\bullet$, and that (ρ, V) is an E-linear representation of $G_{K_\circ}$, of E-dimension* 1. *Then $(\rho \circ \alpha_1^{-1}, V)$ is a uniformizing representation of $G_{K_\bullet}$ if and only if (ρ, V) is a uniformizing representation of $G_{K_\circ}$, where $\alpha_1\colon G_{K_\circ}^{\mathrm{ab}} \to G_{K_\bullet}^{\mathrm{ab}}$ is the isomorphism induced by α_2.* ◇

Proof. It follows immediately from Propositions 7.3 and 7.5. □

8 Proofs of the main theorems

Lemma 8.1 ([14, Lemma 4.1])**.** *Let K be a mixed-characteristic local field, and I an open subgroup of U_K. Then the sub-$\mathbf{Q}_{p_K}$-vector space generated by I in K equals K.* ◇

Proof. We denote by W the sub-$\mathbf{Q}_{p_K}$-vector space generated by I in K. First, observe that I is an open subset of K, since U_K is open in K. Then note that, for each $w \in W$, $w + I \subseteq W$; hence W is also an open subgroup of K. Therefore, the $\mathbf{Q}_{p_K}$-vector space K/W ($\cong \mathbf{Q}_{p_K}^{\oplus d}$, where $d = d_K - \dim_{\mathbf{Q}_{p_K}}(W)$) is discrete, meaning that $d_K = \dim_{\mathbf{Q}_{p_K}}(W)$. □

Proof of Theorem 2.4. We denote by α_1 the isomorphism $G_{K_\circ}^{\mathrm{ab}} \to G_{K_\bullet}^{\mathrm{ab}}$ induced by α_2. We set $p := p_{K_\circ} = p_{K_\bullet}$, and choose a finite Galois extension $E/\mathbf{Q}_p$ that contains both $K_\circ$ and $K_\bullet$. As we have seen in Example 7.4, we have the natural uniformizing representation $(\rho_\circ, V := E^+)$ of $G_{K_\circ}$; it is clear from Corollary 7.6 that $(\rho_\bullet := \rho_\circ \circ \alpha_1^{-1}, V)$ is also a uniformizing representation of $G_{K_\bullet}$. Hence there exist an open subgroup $I_\circ$ (resp. $I_\bullet$) of $U_{K_\circ}$ (resp. $U_{K_\bullet}$) and a field homomorphism $\iota_\circ \colon K_\circ \to E$ (resp. $\iota_\bullet \colon K_\bullet \to E$) such that $\alpha_1(I_\circ) = I_\bullet$ and the diagram

$$\begin{tikzcd}
 & K_\circ^\times \arrow[rrr, bend left, "\iota_\circ^\times"] & & & \\
I_\circ \arrow[ur, "\subseteq"] \arrow[r, "\subseteq"] \arrow[d, "\cong{,}\alpha_1|_{I_\circ}"] & U_{K_\circ} \arrow[r, "\mathrm{Art}_{K_\circ}"] \arrow[d, "\cong{,}\alpha_1|_{U_{K_\circ}}"] & G_{K_\circ}^{\mathrm{ab}} \arrow[r, "\rho_\circ"] \arrow[d, "\cong{,}\alpha_1"] & E^\times \arrow[d, equal] \\
I_\bullet \arrow[dr, "\subseteq"'] \arrow[r, "\subseteq"] & U_{K_\bullet} \arrow[r, "\mathrm{Art}_{K_\bullet}"] & G_{K_\bullet}^{\mathrm{ab}} \arrow[r, "\rho_\bullet"] & E^\times \\
 & K_\bullet^\times \arrow[rru, bend right, "\iota_\bullet^\times"'] & & &
\end{tikzcd}$$

commutes. In particular, $\iota_\bullet|_{I_\bullet} \circ \alpha_1|_{I_\circ} = \iota_\circ|_{I_\circ}$, and by Lemma 8.1, $\iota_\circ(K_\circ) = \iota_\bullet(K_\bullet)$ in E. Therefore, we have the following field isomorphism:

$$f \colon K_\circ \xrightarrow{\cong, \iota_\circ} \iota_\circ(K_\circ) = \iota_\bullet(K_\bullet) \xrightarrow{\cong, \iota_\bullet^{-1}} K_\bullet.$$

□

Proof of Theorem 2.5. We keep the notation and hypotheses of the proof of Theorem 2.4.

Existence. We suppose that, for each $i \in \{1, 2\}$,

- $L_{i,\square}$ is a finite Galois extension of $K_\square$ contained in $K_\square^{m+1}$ for each $\square \in \{\circ, \bullet\}$;
- $H_{i,\square} = \mathrm{Gal}(K_\square^{m+3}/L_{i,\square})$ ($\supseteq (G_{K_\square}^{m+3})^{[m+1]}$) for each $\square \in \{\circ, \bullet\}$;
- $H_{i,\bullet} = \alpha_{m+3}(H_{i,\circ})$,

and that $L_{1,\circ} \subseteq L_{2,\circ}$. Then $\alpha_{m+3}|_{H_{i,\circ}} \colon H_{i,\circ} \to H_{i,\bullet}$ is an isomorphism of *filtered* profinite groups by Lemma 4.1. Hence $\alpha_{m+3}|_{H_{i,\circ}}$ induces an isomorphism $\alpha_{2,i} \colon G_{L_{i,\circ}}^{\mathrm{mab}} = H_{i,\circ}^{\mathrm{mab}} \to G_{L_{i,\bullet}}^{\mathrm{mab}} = H_{i,\bullet}^{\mathrm{mab}}$ of filtered profinite groups. As we have seen in the proof of Theorem 2.4, there exist an open subgroup $I_{i,\circ}$ (resp. $I_{i,\bullet}$) of $U_{L_{i,\circ}}$ (resp. $U_{L_{i,\bullet}}$) and a field isomorphism $\theta_{L_{i,\circ}} \colon L_{i,\circ} \to L_{i,\bullet}$ such that $\alpha_{1,i}(I_{i,\circ}) = I_{i,\bullet}$ and $\theta_{L_{i,\circ}}$ (set-theoretically) extends the group isomorphism $\alpha_{1,i}|_{I_{i,\circ}} \colon I_{i,\circ} \to I_{i,\bullet}$, where $\alpha_{1,i} \colon G_{L_{i,\circ}}^{\mathrm{ab}} \to G_{L_{i,\bullet}}^{\mathrm{ab}}$ is the isomorphism induced by $\alpha_{2,i}$. We can assume without loss of generality that $I_{1,\circ} \subseteq I_{2,\circ}$, replacing $I_{1,\circ}$ with $I_{1,\circ} \cap I_{2,\circ}$ if necessary; the diagram

$$\begin{tikzcd}
I_{1,\circ} \arrow[r, "\subseteq"] \arrow[d, "\alpha_{1,1}|_{I_{1,\circ}}"] & I_{2,\circ} \arrow[d, "\alpha_{1,2}|_{I_{2,\circ}}"] \\
I_{1,\bullet} \arrow[r, "\subseteq"] & I_{2,\bullet}
\end{tikzcd}$$

commutes by definition. It follows immediately from Lemma 8.1 that $\theta_{L_{2,\circ}}$ restricts to $\theta_{L_{1,\circ}}$; by passage to the limit, we obtain

$$\theta_{m+1} \colon K_\circ^{m+1} \to K_\bullet^{m+1}.$$

It remains to check that θ_{m+1} satisfies the stated condition: Assume without loss of generality that $I_{i,\square}$ is $G_{K_\square}$-stable for each $\square \in \{\circ, \bullet\}$. By Lemma 8.1, it is reduced to showing that, for all $\gamma_\circ \in G^{m+3}_{K_\circ}$ and $x \in I_{i,\circ}$,

$$\alpha_{1,i}(\gamma_\circ(x))\,(=\theta_{L_{i,\circ}}(\gamma_\circ(x))\,) \;=\; \gamma_\bullet(\alpha_{1,i}(x))\,(=\gamma_\bullet(\theta_{L_{i,\circ}}(x))\,),$$

where $\gamma_\bullet = \alpha_{m+3}(\gamma_\circ)$. This holds since we are regarding $I_{i,\square}$ as a subgroup of $G^{\mathrm{ab}}_{L_{i,\square}}$ via $\mathrm{Art}_{L_{i,\square}}$, and

$$\alpha_{1,i}(\gamma_\circ|_{L^{\mathrm{ab}}_{i,\circ}} \circ \sigma \circ \gamma_\circ^{-1}|_{L^{\mathrm{ab}}_{i,\circ}}) = \gamma_\bullet|_{L^{\mathrm{ab}}_{i,\bullet}} \circ \alpha_{1,i}(\sigma) \circ \gamma_\bullet^{-1}|_{L^{\mathrm{ab}}_{i,\bullet}}$$

for all $\sigma \in G^{\mathrm{ab}}_{L_{i,\circ}}$.

Uniqueness. Suppose that both isomorphisms $\theta_{m+1,1}, \theta_{m+1,2} \colon K^{m+1}_\circ \to K^{m+1}_\bullet$ satisfy the condition. Then there exist isomorphisms $\theta_1, \theta_2 \colon K^{\mathrm{alg}}_\circ \to K^{\mathrm{alg}}_\bullet$ that respectively extend $\theta_{m+1,1}, \theta_{m+1,2}$; we have

$$(\,\gamma \;:=)\;(\theta_2)^{-1} \circ \theta_1 \in \mathrm{Gal}(K^{\mathrm{alg}}_\circ/\mathbf{Q}_p)$$

and

$$\gamma|_{K^{m+1}_\circ} \circ \sigma \circ \gamma^{-1}|_{K^{m+1}_\circ} = (\theta_{m+1,2})^{-1} \circ \theta_{m+1,1} \circ \sigma \circ (\theta_{m+1,1})^{-1} \circ \theta_{m+1,2} = \sigma, \quad \text{for all } \sigma \in G^{m+1}_{K_\circ}. \tag{8}$$

Hence we see that, for all $x \in K^\times_\circ$,

$$\gamma|_{K^{\mathrm{ab}}_\circ} \circ \mathrm{Art}_{K_\circ}(x) \circ \gamma^{-1}|_{K^{\mathrm{ab}}_\circ} = \mathrm{Art}_{K_\circ}(x),$$

and $\gamma(x) = x$ by local class field theory; it follows that $\gamma \in \mathrm{Gal}(K^{\mathrm{alg}}_\circ/K_\circ)$ (i.e., $\theta_{m+1,1}|_{K_\circ} = \theta_{m+1,2}|_{K_\circ}$). Furthermore, $\gamma|_{K^{m+1}_\circ} \in Z(G^{m+1}_{K_\circ})$ by (8); together with the fact that $Z(G^{m+1}_{K_\circ})$ is trivial for $m \geq 1$ (cf. Proposition A.1), we conclude that $\gamma|_{K^{m+1}_\circ} = 1$ (i.e., $\theta_{m+1,1} = \theta_{m+1,2}$) if $m \geq 1$. □

A Center-freeness of G^m_K, $m \geq 2$

This appendix is devoted to the proof of the following proposition.

Proposition A.1. *Let K be a mixed-characteristic local field. Then*

$$Z(G^{m+1}_K) = \{1\}$$

for all integer $m \geq 1$. ◇

Remark. Proposition A.1 has been originally proved by S. Ladkani [11] for the case $m = 1$. (It is known that the assertion for general m follows from the case $m = 1$ by induction, cf. [18, Proof of Proposition 1.1 (ix)].) In this appendix, we provide an alternative proof of the proposition. ◇

To prove Proposition A.1, we first give a proof of a weaker statement.

Lemma A.2. *Let K be a mixed-characteristic local field. Then*

$$Z(G^{m+1}_K) \subseteq \mathrm{Gal}(K^{m+1}/K^m)$$

for all integer $m \geq 0$. ◇

Proof. Suppose that $\gamma \in Z(G^{m+1}_K)$. Then $\gamma \circ \sigma \circ \gamma^{-1} = \sigma$ for all $\sigma \in G^{m+1}_K$. Let L/K be a finite Galois subextension of K^m/K, so that $L^{\mathrm{ab}} \subseteq K^{m+1}$. We see that, for all $x \in L^\times$,

$$\gamma|_{L^{\mathrm{ab}}} \circ \mathrm{Art}_L(x) \circ \gamma^{-1}|_{L^{\mathrm{ab}}} = \mathrm{Art}_L(x),$$

and $\gamma(x) = x$ by local class field theory. Therefore, $\gamma \in \mathrm{Gal}(K^{m+1}/L)$, and the assertion follows as the subextension L/K is arbitrary. □

Remark. For the case in which the base field is *torally Kummer-faithful* (cf., e.g., [15, Definition 1.5], for the definition of (torally) Kummer-faithful fields), a claim similar to that of Lemma A.2 holds: Let k be a torally Kummer-faithful field, and m an integer ≥ 0. Then

$$Z(G_k^{m+1}) \cap \mathrm{Ker}\,(\chi_{\mathrm{cycl},k} \colon G_k^{m+1} \to (\widehat{\mathbf{Z}}_{\times/k})^{\times}) \subseteq \mathrm{Gal}(k^{m+1}/k^m)$$

holds. (Compare [8, Proposition 1.5].)

There is nothing to show if $m = 0$. For the case $m \geq 1$, we give a proof by contradiction. Assume that there exists an element $\gamma \in Z(G_k^{m+1}) \cap \mathrm{Ker}\,(\chi_{\mathrm{cycl},k})$ which does not belong to $\mathrm{Gal}(k^{m+1}/k^m)$; let $\tilde{\gamma} \in G_k$ be a lifting of γ. Then we can choose a finite Galois subextension l/k of k^m/k, such that the corresponding open normal subgroup $\mathrm{Gal}(k^{m+1}/l)$ does not contain γ.

Since k is torally Kummer-faithful, we have the *injective* homomorphism

$$l^{\times} \to \varprojlim_n l^{\times}/(l^{\times})^n \cong H^1(G_l, \varprojlim_n \mu_n(k^{\mathrm{sep}}))$$

of G_k-modules. (n runs through the integers ≥ 1 whose prime factors belong to $\mathfrak{Primes}_{\times/k}$.) We deduce a contradiction by claiming that $\tilde{\gamma}$ acts trivially on $H^1(G_l, \varprojlim_n \mu_n(k^{\mathrm{sep}}))$, and hence on $l^{\times}$.

For all $\sigma \in G_l$, we have

$$(\xi_\sigma :=)\ \sigma^{-1}\tilde{\gamma}^{-1}\sigma\tilde{\gamma} \in G_k^{[m+1]}\ \ (\subseteq G_k^{[m]} \subseteq G_l),$$

since $\gamma \in Z(G_k^{m+1})$. On the other hand, the action of $\tilde{\gamma}$ on $H^1(G_l, \mu_n(k^{\mathrm{sep}}))$ for each n is determined as follows: For each 1-cocycle (i.e., crossed homomorphism) $\omega \colon G_l \to \mu_n(k^{\mathrm{sep}})$,

$$\tilde{\gamma}\omega(-) = \tilde{\gamma} \cdot \omega(\tilde{\gamma}^{-1} \cdot - \cdot \tilde{\gamma}) = \omega(\tilde{\gamma}^{-1} \cdot - \cdot \tilde{\gamma}).$$

Therefore, it suffices to show that

$$\tilde{\gamma}\omega(-)/\omega(-) = \omega(\tilde{\gamma}^{-1} \cdot - \cdot \tilde{\gamma})/\omega(-) = \omega(- \cdot \xi_{(-)})/\omega(-) = (-) \cdot \omega(\xi_{(-)})$$

is a 1-coboundary. This is straightforward, since it holds that $\omega(\xi) = 1$ for all 1-cocycle ω and $\xi \in G_k^{[m+1]}$, which follows from the fact that $\omega|_{G_k^{[m]}}$ is a group homomorphism (as $G_k^{[m]}$ acts trivially on $\mu_n(k^{\mathrm{sep}})$). $\diamond$

Lemma A.3. *Let K be a mixed-characteristic local field, and let L/K be a finite extension with inclusion map $\iota \colon K \to L$.*

(1) The group homomorphism $\widehat{\iota^{\times}} \colon \widehat{K^{\times}} \to \widehat{L^{\times}}$ is injective.

(2) The transfer map $\mathrm{Ver} \colon G_K^{\mathrm{ab}} \to G_L^{\mathrm{ab}}$ is injective.

(3) It holds that $(\widehat{L^{\times}})^{G_K} = \widehat{K^{\times}}$.

$\diamond$

Proof. (1) We set $e := e_L/e_K$. We consider the following commutative diagram (of abelian groups) with exact rows:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_K & \longrightarrow & K^{\times} & \xrightarrow{\mathrm{ord}_K} & \mathbf{Z}^{+} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow{\scriptstyle \iota^{\times}} & & \downarrow{\scriptstyle e\cdot(-)} & & \\ 1 & \longrightarrow & U_L & \longrightarrow & L^{\times} & \xrightarrow{\mathrm{ord}_L} & \mathbf{Z}^{+} & \longrightarrow & 1 \end{array} \quad . \tag{9}$$

By profinite completion, we obtain the following commutative diagram, in which all rows are exact:

$$
\begin{tikzcd}
 & 1 \arrow[r] & U_K \arrow[r] \arrow[d] \arrow[ddl, dashed] & K^\times \arrow[r] \arrow[d, "\iota^\times"] \arrow[ddl, dashed] & \mathbf{Z}^+ \arrow[r] \arrow[d, "e\cdot(-)"] \arrow[ddl, dashed] & 1 \\
 & 1 \arrow[r] & U_L \arrow[r] \arrow[ddl, dashed] & L^\times \arrow[r] \arrow[ddl, dashed] & \mathbf{Z}^+ \arrow[r] \arrow[ddl, dashed] & 1 \\
1 \arrow[r] & U_K \arrow[r] \arrow[d] & \widehat{K^\times} \arrow[r] \arrow[d, "\widehat{\iota^\times}"] & \widehat{\mathbf{Z}}^+ \arrow[r] \arrow[d, "e\cdot(-)"] & 1 & \\
1 \arrow[r] & U_L \arrow[r] & \widehat{L^\times} \arrow[r] & \widehat{\mathbf{Z}}^+ \arrow[r] & 1 &
\end{tikzcd}
. \tag{10}
$$

Since e is not a zero-divisor in $\widehat{\mathbf{Z}}$, we can conclude that the map $\widehat{\iota^\times}\colon \widehat{K^\times} \to \widehat{L^\times}$ is injective.

(2) Art_K and Art_L respectively induce the isomorphisms $\widehat{\mathrm{Art}_K}\colon \widehat{K^\times} \to G_K^{\mathrm{ab}}$ and $\widehat{\mathrm{Art}_L}\colon \widehat{L^\times} \to G_L^{\mathrm{ab}}$ (cf. p. 11); these fit into the following commutative diagram:

$$
\begin{tikzcd}
\widehat{K^\times} \arrow[r, "\cong"] \arrow[d, "\widehat{\iota^\times}"] & G_K^{\mathrm{ab}} \arrow[d, "\mathrm{Ver}"] \\
\widehat{L^\times} \arrow[r, "\cong"] & G_L^{\mathrm{ab}}
\end{tikzcd}
.
$$

Hence the injectivity of Ver is implied by that of $\widehat{\iota^\times}$, which we have already seen in (1).

(3) By (1), we can assume without loss of generality that L/K is a finite Galois extension, with Galois group $G = \mathrm{Gal}(L/K)$. Then it follows that (9) and (10) are also commutative diagrams of G-modules; in (10), we see that the second and fourth rows from the top induce the long exact sequences, and make the following diagram commutative:

$$
\begin{tikzcd}
1 \arrow[r] & U_K \arrow[r] \arrow[d, equal] & K^\times \arrow[r, "\mathrm{ord}_L|_{K^\times}"] \arrow[d, dashed] & \mathbf{Z}^+ \arrow[r, "\delta"] \arrow[d, dashed] & H^1(G, U_L) \arrow[r] \arrow[d, equal] & H^1(G, L^\times) \arrow[d, dashed] \\
1 \arrow[r] & U_K \arrow[r] & (\widehat{L^\times})^G \arrow[r] & \widehat{\mathbf{Z}}^+ \arrow[r, "\delta"] & H^1(G, U_L) \arrow[r] & H^1(G, \widehat{L^\times})
\end{tikzcd}
.
$$

The connecting homomorphism $\delta\colon \mathbf{Z}^+ \to H^1(G, U_L)$ induces the injective homomorphism

$$
(\,\mathrm{Coker}\,(\mathrm{ord}_L|_{K^\times})\ =)\ (\mathbf{Z}/e\mathbf{Z})^+ \ \to\ H^1(G, U_L),
$$

—which is an isomorphism by Hilbert's Theorem 90—and as a result, $e\widehat{\mathbf{Z}}^+$ is annihilated by $\delta\colon \widehat{\mathbf{Z}}^+ \to H^1(G, U_L)$. Therefore, the diagram

$$
\begin{tikzcd}
1 \arrow[r] & U_K \arrow[r] \arrow[d, equal] & \widehat{K^\times} \arrow[r] \arrow[d] & \widehat{\mathbf{Z}}^+ \arrow[r] \arrow[d, "e\cdot(-)"] & 1 \arrow[d] \\
1 \arrow[r] & U_K \arrow[r] & (\widehat{L^\times})^G \arrow[r] & \widehat{\mathbf{Z}}^+ \arrow[r, "\delta"] & H^1(G, U_L)
\end{tikzcd}
$$

with exact rows commutes, and yields the exact sequence of cokernels:

$$
1 \to (\widehat{L^\times})^G/\widehat{K^\times} \to (\widehat{\mathbf{Z}}/e\widehat{\mathbf{Z}})^+ \xrightarrow{\cong} H^1(G, U_L).
$$

Hence $(\widehat{L^\times})^G = \widehat{K^\times}$. □

Proof of Proposition A.1. By definition, $Z(G_K^{m+1})$ is precisely the set of G_K-invariant elements in G_K^{m+1}, if we let G_K act on G_K^{m+1} by conjugation. Hence it follows from Lemma A.2 that

$$Z(G_K^{m+1}) = \mathrm{Gal}(K^{m+1}/K^m)^{G_K}.$$

To demonstrate that $\mathrm{Gal}(K^{m+1}/K^m)^{G_K}$ is trivial, we first note that $\mathrm{Gal}(K^{m+1}/K^m)$ can be written as an inverse limit of profinite groups:

$$\begin{aligned}\mathrm{Gal}(K^{m+1}/K^m) &= G_K^{[m]}/G_K^{[m+1]} = (G_K^{[m]})^{\mathrm{ab}} \\ &= \left(\bigcap_{H\in\mathscr{H}_m(G_K)} H\right)^{\mathrm{ab}} = \left(\varprojlim_{H\in\mathscr{H}_m(G_K)} H\right)^{\mathrm{ab}} = \varprojlim_{H\in\mathscr{H}_m(G_K)} H^{\mathrm{ab}},\end{aligned}$$

where $\mathscr{H}_m(G_K)$ is the set of open normal subgroups of G_K containing $G_K^{[m]}$, ordered by reverse inclusion. (For the last equality, cf., e.g., [16, Chap. IV, §2, Exercise 6].) Suppose that L_1/K, L_2/K are finite Galois subextensions of K^m/K with $L_1 \subseteq L_2$. It is clear from local class field theory that the diagram

$$\begin{tikzcd} G_{L_1}^{\mathrm{ab}} & L_1^\times \arrow[l, "\mathrm{Art}_{L_1}"'] \\ G_{L_2}^{\mathrm{ab}} \arrow[u] & L_2^\times \arrow[l, "\mathrm{Art}_{L_2}"'] \arrow[u, "\mathrm{N}_{L_2/L_1}"'] \end{tikzcd}$$

commutes, where the left vertical arrow is induced by the inclusion map $G_{L_2} \to G_{L_1}$, and N_{L_2/L_1} is the *norm* map. We obtain the commutative diagram

$$\begin{tikzcd} G_{L_1}^{\mathrm{ab}} & \widehat{L_1^\times} \arrow[l, "\cong"'] \\ G_{L_2}^{\mathrm{ab}} \arrow[u] & \widehat{L_2^\times} \arrow[l, "\cong"'] \arrow[u, "\widehat{\mathrm{N}_{L_2/L_1}}"'] \end{tikzcd}$$

by profinite completion, and the isomorphism

$$\varprojlim_{H\in\mathscr{H}_m(G_K)} H^{\mathrm{ab}} \xrightarrow{\cong} \varprojlim_{L/K} \widehat{L^\times}$$

which respects the G_K-action, by taking inverse limits. Hence

$$\mathrm{Gal}(K^{m+1}/K^m)^{G_K} \cong \varprojlim_{L/K} \widehat{K^\times} \tag{11}$$

by Lemma A.3 (3). (Note that the limit is taken over the inverse system in which the homomorphism

$$(-)^{[L_2:L_1]}:\ \ \widehat{K^\times}\ (=\ (\widehat{L_2^\times})^{G_K}) \ \ \to\ \ \widehat{K^\times}\ (=\ (\widehat{L_1^\times})^{G_K})$$

is assigned to each pair $L_1 \subseteq L_2$.)

It remains to show that if $x = \{x_L\}_{L/K}$ belongs to the right hand side of (11), then $x_L = 1$ for all L/K. This can be verified as follows: For all $n \geq 1$, we can always find a finite extension $L_{(n)}/L$ such that $[L_{(n)} : L] = n$ and $L_{(n)} \subseteq K^m$, e.g.,

$$L_{(n)} := L(\mu_{|\mathfrak{k}_L|^n-1}(K^{\mathrm{alg}})).$$

Therefore, $x_L = (x_{L_{(n)}})^n$ for all $n \geq 1$, and hence

$$x_L \in \bigcap_{n\geq 1}(\widehat{K^\times})^n = \{1\},$$

which proves the claim. □

B Notes on pro-p and pro-Σ quotients

Let k be a field. For a subset $\Sigma \subseteq \mathfrak{Primes}$, we denote by $k^{\text{pro-}\Sigma}$ the maximal pro-Σ extension of k, i.e., the subfield of k^{sep} fixed by the kernel of

$$G_k \twoheadrightarrow G_k^{\text{pro-}\Sigma},$$

so that $G_k^{\text{pro-}\Sigma}$ equals $\mathrm{Gal}(k^{\text{pro-}\Sigma}/k)$. We also denote by $k^{m,\,\text{pro-}\Sigma}$ the maximal m-step solvable pro-Σ extension of k, i.e., the subfield of k^{sep} fixed by the kernel of

$$G_k \twoheadrightarrow G_k^{m,\,\text{pro-}\Sigma},$$

so that $G_k^{m,\,\text{pro-}\Sigma}$ equals $\mathrm{Gal}(k^{m,\,\text{pro-}\Sigma}/k)$, for an integer $m \geq 0$. We will often write

$$k^{\text{ab},\,\text{pro-}\Sigma} \quad (\text{resp. } k^{\text{mab},\,\text{pro-}\Sigma})$$

for the maximal abelian (resp. metabelian) pro-Σ extension $k^{1,\,\text{pro-}\Sigma}$ (resp. $k^{2,\,\text{pro-}\Sigma}$) of k.

Let K be a mixed-characteristic local field. In this appendix, we sharpen several results from §§5 and 6 by presenting explicit group-theoretic algorithms that recover key arithmetic invariants of K from

$$G_K^{m,\,\text{pro-}\Sigma_K} \quad \text{or} \quad G_K^{m,\,\text{pro-}\Sigma'_K},$$

where Σ_K (resp. Σ'_K) is a subset of $\mathfrak{Primes}$ containing all prime factors of p_K (resp. $p_K(p_K-1)$). Then we conclude this appendix by demonstrating Theorem B.10, which is a refinement of Theorem 2.4.

Definition B.1. Let $\star$ be an element of $\{\emptyset, m, \text{ab}, \text{mab}\}$, where m is an integer ≥ 0. Let G be a profinite group. We shall say that G is a profinite group of

$$\mathrm{MLF}^{\star,\,\text{pro-}\Sigma}\text{-} \quad (\text{resp. } \mathrm{MLF}^{\star,\,\text{pro-}\Sigma'}\text{-})$$

type if there exists an isomorphism of profinite groups between G and

$$G_K^{\star,\,\text{pro-}\Sigma_K} \quad (\text{resp. } G_K^{\star,\,\text{pro-}\Sigma'_K})$$

for some mixed-characteristic local field K and some subset Σ_K (resp. Σ'_K) of $\mathfrak{Primes}$ containing all prime factors of p_K (resp. $p_K(p_K-1)$). We define filtered and I-filtered profinite groups of

$$\mathrm{MLF}^{\star,\,\text{pro-}\Sigma}\text{-} \quad (\text{resp. } \mathrm{MLF}^{\star,\,\text{pro-}\Sigma'}\text{-})$$

type for a closed interval $I \subseteq [0, +\infty)$ in a similar way. ◇

Suppose that G is a profinite group of $\mathrm{MLF}^{\text{ab},\,\text{pro-}\Sigma}$-type, i.e., there exists an isomorphism

$$G \xrightarrow{\cong} G_K^{\text{ab},\,\text{pro-}\Sigma_K}$$

of profinite groups for some mixed-characteristic local field K and some subset Σ_K of $\mathfrak{Primes}$ containing p_K. We denote by $p(G)$ the uniquely determined prime number ℓ such that

$$\log_\ell |G_{/\text{tor}}/\ell \cdot G_{/\text{tor}}| \geq 2$$

(cf. p. 11). Furthermore, we set:

- $\varepsilon(G) := 1$ (resp. $\varepsilon(G) := 2$) if $p(G)$ is odd (resp. even);
- $a(G) := \log_{p(G)} |(G_{\text{tor}})^{\text{pro-}p(G)}|$;
- $d(G) := \log_{p(G)} |G_{/\text{tor}}/p(G) \cdot G_{/\text{tor}}| - 1$.

Proposition B.2. *Let K be a mixed-characteristic local field. We have*

$$p_K = p(G_K^{\mathrm{ab,\,pro}\text{-}\Sigma_K}), \quad \varepsilon_K = \varepsilon(G_K^{\mathrm{ab,\,pro}\text{-}\Sigma_K}), \quad a_K = a(G_K^{\mathrm{ab,\,pro}\text{-}\Sigma_K}), \quad d_K = d(G_K^{\mathrm{ab,\,pro}\text{-}\Sigma_K}),$$

for any subset Σ_K of $\mathfrak{Primes}$ containing p_K. $\diamond$

Intuitively speaking, p_K, ε_K, a_K and d_K can be recovered entirely group-theoretically from the profinite group $G_K^{\mathrm{ab,\,pro}\text{-}\Sigma_K}$.

Proof. The proof is parallel to that of Proposition 5.1. $\square$

Restoration of the cyclotomic character

Definition B.3.

(1) A mixed-characteristic local field K is said to be of *p-cyclotomic type* if K contains a primitive $(p_K)^{\mathrm{th}}$ root of unity.

(2) Let $\star$ be an element of $\{\emptyset, m, \mathrm{ab}, \mathrm{mab}\}$, where m is an integer ≥ 1. A profinite group G is said to be of p-cycl-MLF$^{\star,\,\mathrm{pro}\text{-}\Sigma}$*-type* if there exists an isomorphism of profinite groups between G and $G_K^{\star,\,\mathrm{pro}\text{-}\Sigma_K}$ for some mixed-characteristic local field K of p-cyclotomic type and some subset Σ_K of $\mathfrak{Primes}$ containing p_K. We define filtered and I-filtered profinite groups of p-cycl-MLF$^{\star,\,\mathrm{pro}\text{-}\Sigma}$-*type* for a closed interval $I \subseteq [0, +\infty)$ in a similar way.

$\diamond$

Suppose that G is a profinite group of either MLF$^{\mathrm{mab,\,pro}\text{-}\Sigma'}$-type or p-cycl-MLF$^{\mathrm{mab,\,pro}\text{-}\Sigma}$-type. We put $p := p(G^{\mathrm{ab}})$, and choose a decreasing sequence

$$G = H_0 \supseteq H_1 \supseteq \cdots \supseteq H_\nu \supseteq \cdots$$

of open normal subgroups of G such that

(i) $H_\nu^{\mathrm{ab}}[p^\nu] \cong (\mathbf{Z}/p^\nu\mathbf{Z})^+$;

(ii) G/H_ν is abelian,

for each $\nu \in \mathbf{Z}_{\geq 0}$.

- If G is of MLF$^{\mathrm{mab,\,pro}\text{-}\Sigma'}$-type, then there exists an isomorphism

$$\alpha' : G \xrightarrow{\cong} G_K^{\mathrm{mab,\,pro}\text{-}\Sigma'_K}$$

of profinite groups for some mixed-characteristic local field K and some subset Σ'_K of $\mathfrak{Primes}$ containing all prime factors of $p_K(p_K - 1)$. Such a sequence $\{H_\nu\}_\nu$ satisfying (i) and (ii) exists: Let $\zeta_{p_K^\nu} \in K^{\mathrm{alg}}$ be a primitive $(p_K^\nu)^{\mathrm{th}}$ root of unity for each $\nu \geq 0$. Then $[K(\zeta_{p_K^\nu}) : K]$ divides $p_K^{\nu-1}(p_K - 1)$ for all $\nu \geq 1$. Hence $K(\zeta_{p_K^\nu}) \subseteq K^{\mathrm{ab,\,pro}\text{-}\Sigma'_K}$, and we can choose

$$H_\nu = \alpha'^{-1}(\mathrm{Gal}(K^{\mathrm{mab,\,pro}\text{-}\Sigma'_K}/K(\zeta_{p_K^\nu}))).$$

For each $\square \in \{\nu, \nu+1\}$, let $L_\square$ be the field fixed by $\alpha'(H_\square)$. Then $L_\square$ is contained in $K^{\mathrm{ab,\,pro}\text{-}\Sigma'_K}$, and we have the following commutative diagram:

$$\begin{array}{ccccc} (L_\nu^\times)^{\wedge,\,\mathrm{pro}\text{-}\Sigma'_K} & \xrightarrow{\cong,\,r'_\nu} & G_{L_\nu}^{\mathrm{ab,\,pro}\text{-}\Sigma'_K} & \xrightarrow{\cong,\,\alpha_\nu'^{-1}} & H_\nu^{\mathrm{ab}} \\ \big\downarrow{\scriptstyle \subseteq^{\wedge,\mathrm{pro}\text{-}\Sigma'_K}} & & & & \big\downarrow{\scriptstyle \mathrm{Ver}} \\ (L_{\nu+1}^\times)^{\wedge,\,\mathrm{pro}\text{-}\Sigma'_K} & \xrightarrow{\cong,\,r'_{\nu+1}} & G_{L_{\nu+1}}^{\mathrm{ab,\,pro}\text{-}\Sigma'_K} & \xrightarrow{\cong,\,\alpha_{\nu+1}'^{-1}} & H_{\nu+1}^{\mathrm{ab}} \end{array},$$

where $\alpha'_\square$ is the isomorphism of profinite groups induced by α' and $r'_\square$ denotes the isomorphism $(\mathrm{Art}_{L_\square})^{\wedge,\,\mathrm{pro}\text{-}\Sigma'_K}$.

- If G is of p-cycl-MLF$^{\text{mab, pro-}\Sigma}$-type, then there exists an isomorphism

$$\alpha\colon G \xrightarrow{\cong} G_K^{\text{mab, pro-}\Sigma_K}$$

  of profinite groups for some mixed-characteristic local field K of p-cyclotomic type and some subset Σ_K of $\mathfrak{Primes}$ containing p_K. Such a sequence $\{H_\nu\}_\nu$ satisfying (i) and (ii) exists: Let $\zeta_{p_K^\nu} \in K^{\text{alg}}$ be a primitive $(p_K^\nu)^{\text{th}}$ root of unity for each $\nu \geq 0$. Since $\zeta_{p_K} \in K$, $K(\zeta_{p_K^\nu})/K$ is an abelian p_K-extension, and we can choose

$$H_\nu = \alpha^{-1}(\mathrm{Gal}(K^{\text{mab, pro-}\Sigma_K}/K(\zeta_{p_K^\nu}))).$$

  For each $\square \in \{\nu, \nu+1\}$, let $L_\square$ be the field fixed by $\alpha(H_\square)$. Then $L_\square$ is contained in $K^{\text{ab, pro-}\Sigma_K}$, and we have the following commutative diagram:

$$\begin{tikzcd}
(L_\nu^\times)^{\wedge,\text{ pro-}\Sigma_K} \arrow[r, "\cong{,}r_\nu"] \arrow[d, "\subseteq^{\wedge,\text{pro-}\Sigma_K}"] & G_{L_\nu}^{\text{ab, pro-}\Sigma_K} \arrow[r, "\cong{,}\alpha_\nu^{-1}"] & H_\nu^{\text{ab}} \arrow[d, "\mathrm{Ver}"] \\
(L_{\nu+1}^\times)^{\wedge,\text{ pro-}\Sigma_K} \arrow[r, "\cong{,}r_{\nu+1}"] & G_{L_{\nu+1}}^{\text{ab, pro-}\Sigma_K} \arrow[r, "\cong{,}\alpha_{\nu+1}^{-1}"] & H_{\nu+1}^{\text{ab}}
\end{tikzcd},$$

  where $\alpha_\square$ is the isomorphism of profinite groups induced by α and $r_\square$ denotes the isomorphism $(\mathrm{Art}_{L_\square})^{\wedge,\text{ pro-}\Sigma_K}$.

We see that the transfer map $\mathrm{Ver}\colon H_\nu^{\text{ab}} \to H_{\nu+1}^{\text{ab}}$ restricts to an injective homomorphism $H_\nu^{\text{ab}}[p^\nu] \to H_{\nu+1}^{\text{ab}}[p^{\nu+1}]$ in both cases; we obtain the inverse system

$$\cdots \xrightarrow{(-)^p} H_{\nu+1}^{\text{ab}}[p^{\nu+1}] \xrightarrow{(-)^p} H_\nu^{\text{ab}}[p^\nu] \xrightarrow{(-)^p} \cdots \xrightarrow{(-)^p} H_1^{\text{ab}}[p]$$

of G-modules by identifying $H_\nu^{\text{ab}}[p^\nu]$ with a subgroup of $H_{\nu+1}^{\text{ab}}[p^{\nu+1}]$. By passage to the limit, we obtain

$$T(G) \coloneqq \varprojlim_\nu H_\nu^{\text{ab}}[p^\nu].$$

We shall write

$$\chi(G)\colon G \to \mathrm{Aut}(T(G)) \ \ (= \mathbf{Z}_p^\times)$$

for the p-adic character of G attached to $T(G)$. The proof of the following proposition (and that $T(G)$ is well-defined up to isomorphism) follows the same steps as that of Proposition 5.2.

Proposition B.4. *Let K be a mixed-characteristic local field, and let Σ_K (resp. Σ_K') be any subset of $\mathfrak{Primes}$ containing all prime factors of p_K (resp. $p_K(p_K-1)$).*

(1) The p_K-adic cyclotomic character χ_K factors through $\chi(G_K^{\text{mab, pro-}\Sigma_K'})$.

(2) If K is of p-cyclotomic type, then χ_K factors through $\chi(G_K^{\text{mab, pro-}\Sigma_K})$.

◇

Intuitively speaking, χ_K can be recovered entirely group-theoretically from:

- the profinite group $G_K^{\text{mab, pro-}\Sigma_K'}$;
- the profinite group $G_K^{\text{mab, pro-}\Sigma_K}$, if K is of p-cyclotomic type.

Ramification groups in upper numbering: Wild inertia groups

We fix a real number $\delta \in (0,1]$ throughout this appendix. Let m be an integer ≥ 1, and let G be a $[0,\delta]$-filtered profinite group of $\mathrm{MLF}^{m+1,\,\text{pro-}\Sigma}$-type. We set

$$G(0+) := \overline{\bigcup_{v \in (0,\delta]} G(v)}.$$

Let $\mathscr{H}_m(G)$ denote the set of open normal subgroups of G containing $G^{[m]}$, ordered by reverse inclusion. For each $H \in \mathscr{H}_m(G)$, we denote by $U(H)$ the image of $H \cap G(0+)$ under the natural map $H \twoheadrightarrow H^{\mathrm{ab}}$.

We first claim that, for $H_1, H_2 \in \mathscr{H}_m(G)$ with $H_1 \supseteq H_2$, the transfer map $\mathrm{Ver}\colon H_1^{\mathrm{ab}} \to H_2^{\mathrm{ab}}$ restricts to $U(H_1) \to U(H_2)$, and that $\{U(H)\}_{H \in \mathscr{H}_m(G)}$ forms a direct system of G-modules, together with $V_{1,2} := \mathrm{Ver}\,|_{U(H_1)}\colon U(H_1) \to U(H_2)$ for each pair $H_1 \supseteq H_2$. Suppose that:

- there exists an isomorphism $\alpha^{[0,\delta]}\colon G \to G_K^{m+1,\,\text{pro-}\Sigma_K}$ of $[0,\delta]$-filtered profinite groups for some mixed-characteristic local field K and some subset Σ_K of $\mathfrak{Primes}$ containing p_K;
- for each $\square \in \{1,2\}$, the image of $H_\square$ under $\alpha^{[0,\delta]}$ equals $\mathrm{Gal}(K^{m+1,\,\text{pro-}\Sigma_K}/L_\square)$, where $L_\square/K$ is a finite Galois subextension of $K^{m,\,\text{pro-}\Sigma_K}$.

Note that

$$\mathrm{Gal}(K^{m+1,\,\text{pro-}\Sigma_K}/L_\square)^{\mathrm{ab}} = \mathrm{Gal}(K^{m+1}/L_\square)^{\mathrm{ab},\,\text{pro-}\Sigma_K} = G_{L_\square}^{\mathrm{ab},\,\text{pro-}\Sigma_K}$$

by Lemma 4.3 (and hence $H_\square^{\mathrm{ab}}$ is a profinite group of $\mathrm{MLF}^{\mathrm{ab},\,\text{pro-}\Sigma}$-type). The isomorphism

$$\alpha_\square^{[0,\delta]}\colon H_\square^{\mathrm{ab}} \to G_{L_\square}^{\mathrm{ab},\,\text{pro-}\Sigma_K}$$

induced by $\alpha^{[0,\delta]}$ fits into the following commutative diagram:

$$\begin{array}{ccccc}
H_1^{\mathrm{ab}} & \xrightarrow{\cong,\,\alpha_1^{[0,\delta]}} & G_{L_1}^{\mathrm{ab},\,\text{pro-}\Sigma_K} & \xrightarrow{\cong,\,r_1^{-1}} & (L_1^\times)^{\wedge,\,\text{pro-}\Sigma_K} \\
\big\downarrow{\scriptstyle \mathrm{Ver}} & & & & \big\downarrow{\scriptstyle \subseteq^{\wedge,\,\text{pro-}\Sigma_K}} \\
H_2^{\mathrm{ab}} & \xrightarrow{\cong,\,\alpha_2^{[0,\delta]}} & G_{L_2}^{\mathrm{ab},\,\text{pro-}\Sigma_K} & \xrightarrow{\cong,\,r_2^{-1}} & (L_2^\times)^{\wedge,\,\text{pro-}\Sigma_K}
\end{array},$$

where $r_\square$ denotes the isomorphism $(\mathrm{Art}_{L_\square})^{\wedge,\,\text{pro-}\Sigma_K}$. Since $H_\square \cap G(0+) \subseteq G$ is mapped onto

$$\mathrm{Gal}(K^{m+1,\,\text{pro-}\Sigma_K}/L_\square) \cap G_K^{m+1,\,\text{pro-}\Sigma_K}(0+) = \mathrm{Gal}(K^{m+1,\,\text{pro-}\Sigma_K}/L_\square)(0+)$$

under $\alpha^{[0,\delta]}$, $U(H_\square) \subseteq H_\square^{\mathrm{ab}}$ is mapped onto $G_{L_\square}^{\mathrm{ab},\,\text{pro-}\Sigma_K}(0+)$ under $\alpha_\square^{[0,\delta]}$. It follows that

$$(r_\square^{-1} \circ \alpha_\square^{[0,\delta]})(U(H_\square)) = r_\square^{-1}(G_{L_\square}^{\mathrm{ab},\,\text{pro-}\Sigma_K}(0+)) = U_{L_\square}(1)$$

for each $\square \in \{1,2\}$; the claim holds, since $U_{L_1}(1)$ is mapped into $U_{L_2}(1)$ under the right vertical arrow. It also follows immediately that $\{U(H)\}_{H \in \mathscr{H}_m(G)}$ is a direct system induced by the direct system $\{U_L(1)\}_{L/K}$, where L/K runs through the finite Galois subextensions of $K^{m,\,\text{pro-}\Sigma_K}/K$; each $U(H)$ is a (topological) $\mathbf{Z}_p$-module of finite rank, where $p := p(G^{\mathrm{ab}}) = p_K$. Hence we obtain a direct system

$$\{U(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\}_{H \in \mathscr{H}_m(G)}$$

of G-modules; we set

$$k^{m,\,\text{pro-}\Sigma,+}(G) := \varinjlim_{H \in \mathscr{H}_m(G)} \Big(U(H) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p\Big).$$

The proof of the following proposition follows the same steps as that of Proposition 6.2.

Proposition B.5. *Let K be a mixed-characteristic local field, and let Σ_K be any subset of $\mathfrak{Primes}$ containing p_K. Let m be an integer ≥ 1. Then there exists an isomorphism*

$$k^{m,\,\text{pro-}\Sigma,+}(G_K^{m+1,\,\text{pro-}\Sigma_K}) \xrightarrow{\cong} K^{m,\,\text{pro-}\Sigma_K,+}$$

of $G_K^{m+1,\,\text{pro-}\Sigma_K}$-modules. ◇

Speaking from an intuitive level, the $G_K^{m+1,\,\text{pro-}\Sigma_K}$-module $K^{m,\,\text{pro-}\Sigma_K,+}$ can be recovered entirely group-theoretically from the $[0,\delta]$-filtered profinite group $G_K^{m+1,\,\text{pro-}\Sigma_K}$.

Restoration of the absolute ramification index

Let G be a $[0, 1+\delta]$-filtered profinite group of $\mathrm{MLF}^{\mathrm{ab},\,\text{pro-}\Sigma}$-type. We set:

- $f(G) \coloneqq \log_{p(G)}|G(1)/G(1+\delta)|$;
- $e(G) \coloneqq d(G)/f(G)$.

Proposition B.6. *Let K be a mixed-characteristic local field. We have*

$$e_K = e(G_K^{\mathrm{ab},\,\text{pro-}\Sigma_K}), \quad f_K = f(G_K^{\mathrm{ab},\,\text{pro-}\Sigma_K}),$$

for any subset Σ_K of $\mathfrak{Primes}$ containing p_K. ◇

Intuitively speaking, e_K and f_K can be recovered entirely group-theoretically from the $[0, 1+\delta]$-filtered profinite group $G_K^{\mathrm{ab},\,\text{pro-}\Sigma_K}$.

Proof. We have

$$|G_K^{\mathrm{ab},\,\text{pro-}\Sigma_K}(1)/G_K^{\mathrm{ab},\,\text{pro-}\Sigma_K}(1+\delta)| = |(1+\mathfrak{p}_K)/(1+\mathfrak{p}_K^2)| = |\mathfrak{k}_K^+| = p_K^{f_K}.$$

Hence the second equality follows. The first equality follows from the second equality, together with Proposition B.2. □

Ramification groups in upper numbering: Higher ramification groups

We now assume that G is a filtered profinite group of $\mathrm{MLF}^{m+1,\,\text{pro-}\Sigma}$-type for an integer $m \geq 1$, i.e., there exists an isomorphism $\alpha^{\mathrm{filt}}\colon G \to G_K^{m+1,\,\text{pro-}\Sigma_K}$ of filtered profinite groups for some mixed-characteristic local field K and some subset Σ_K of $\mathfrak{Primes}$ containing p_K. For any closed normal subgroup N of G and any $H \in \mathscr{H}_m(G)$, we shall regard G/N and H as filtered profinite groups in the manners described in §6. We shall also regard H^{ab} as a filtered profinite group (of $\mathrm{MLF}^{\mathrm{ab},\,\text{pro-}\Sigma}$-type) by setting

$$H^{\mathrm{ab}}(w) \coloneqq H(w)H^{[1]}/H^{[1]}$$

for each $w \geq 0$. We denote by $U(H, w)$ the group $H^{\mathrm{ab}}(w)$.

We claim that, for $H_1, H_2 \in \mathscr{H}_m(G)$ with $H_1 \supseteq H_2$, the transfer map $\mathrm{Ver}\colon H_1^{\mathrm{ab}} \to H_2^{\mathrm{ab}}$ restricts to

$$U(H_1, \varepsilon(G)e(H_1^{\mathrm{ab}})) \to U(H_2, \varepsilon(G)e(H_2^{\mathrm{ab}})),$$

and that if we denote by $U'(H)$ the group $U(H, \varepsilon(G)e(H^{\mathrm{ab}}))$ for each $H \in \mathscr{H}_m(G)$,

$$\{U'(H)\}_{H\in\mathscr{H}_m(G)}$$

forms a direct system of G-modules, together with $V'_{1,2} \coloneqq \mathrm{Ver}|_{U'(H_1)}\colon U'(H_1) \to U'(H_2)$ for each pair $H_1 \supseteq H_2$. Suppose that, for each $\square \in \{1,2\}$, the image of $H_\square$ under α^{filt} equals $\mathrm{Gal}(K^{m+1,\,\text{pro-}\Sigma_K}/L_\square)$, where $L_\square/K$ is a finite Galois subextension of $K^{m,\,\text{pro-}\Sigma_K}$. Then the isomorphism $\alpha^{\text{filt}}_\square\colon H^{\text{ab}}_\square \to G^{\text{ab},\,\text{pro-}\Sigma_K}_{L_\square}$ induced by α^{filt} fits into the following commutative diagram:

$$\begin{array}{ccccc} H_1^{\text{ab}} & \xrightarrow{\cong,\alpha_1^{\text{filt}}} & G_{L_1}^{\text{ab},\,\text{pro-}\Sigma_K} & \xrightarrow{\cong,\,r_1^{-1}} & (L_1^\times)^{\wedge,\,\text{pro-}\Sigma_K} \\ \big\downarrow{\scriptstyle \text{Ver}} & & & & \big\downarrow{\scriptstyle \subseteq^{\wedge,\,\text{pro-}\Sigma_K}} \\ H_2^{\text{ab}} & \xrightarrow{\cong,\alpha_2^{\text{filt}}} & G_{L_2}^{\text{ab},\,\text{pro-}\Sigma_K} & \xrightarrow{\cong,\,r_2^{-1}} & (L_2^\times)^{\wedge,\,\text{pro-}\Sigma_K} \end{array},$$

where $r_\square$ denotes the isomorphism $(\text{Art}_{L_\square})^{\wedge,\,\text{pro-}\Sigma_K}$. Since $H_\square(w)$ is mapped onto

$$\text{Gal}(K^{m+1,\,\text{pro-}\Sigma_K}/L_\square)(w)$$

under $\alpha^{\text{filt}}|_{H_\square}$ for all $w \geq 0$, $U'(H_\square) \subseteq H^{\text{ab}}_\square$ is mapped onto

$$G^{\text{ab},\,\text{pro-}\Sigma_K}_{L_\square}(\varepsilon(G)e(H^{\text{ab}}_\square))$$

under $\alpha^{\text{filt}}_\square$. It follows from local class field theory that

$$(r_\square^{-1}\circ\alpha^{\text{filt}}_\square)(U'(H_\square)) = r_\square^{-1}(G^{\text{ab},\,\text{pro-}\Sigma_K}_{L_\square}(\varepsilon(G)e(H^{\text{ab}}_\square))) = r_\square^{-1}(G^{\text{ab},\,\text{pro-}\Sigma_K}_{L_\square}(\varepsilon_K e_{L_\square})) = U_{L_\square}(\varepsilon_K e_{L_\square})$$

for each $\square \in \{1,2\}$; the claim holds, since $U_{L_1}(\varepsilon_K e_{L_1})$ is mapped into $U_{L_2}(\varepsilon_K e_{L_2})$ under the right vertical arrow. Therefore, we obtain a direct system $\{U'(H)\}_{H\in\mathscr{H}_m(G)}$ of G-modules in a way similar to the way in which we obtained $\{U(H)\}_{H\in\mathscr{H}_m(G)}$. We again put $p := p(G^{\text{ab}}) = p_K$. We set:

$$\mathscr{O}^+_{k^{m,\,\text{pro-}\Sigma}}(G) := p^{-\varepsilon(G)}\left(\varinjlim_{H\in\mathscr{H}_m(G)} U'(H)\right)\ \left(\subseteq k^{m,\,\text{pro-}\Sigma,+}(G) = \varinjlim_{H\in\mathscr{H}_m(G)}\Big(U(H)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p\Big)\right),$$

$$\mathscr{C}^+_{k^{m,\,\text{pro-}\Sigma}}(G) := \left(\varprojlim_n\Big(\mathscr{O}^+_{k^{m,\,\text{pro-}\Sigma}}(G)/p^n\mathscr{O}^+_{k^{m,\,\text{pro-}\Sigma}}(G)\Big)\right)\otimes_{\mathbf{Z}_p}\mathbf{Q}_p.$$

The proof of the following proposition follows the same steps as that of Proposition 6.3.

Proposition B.7. *Let K be a mixed-characteristic local field, and let Σ_K be any subset of $\mathfrak{Primes}$ containing p_K. Let m be an integer ≥ 1. Then the isomorphism of Proposition B.5 restricts to an isomorphism*

$$\mathscr{O}^+_{k^{m,\,\text{pro-}\Sigma}}(G_K^{m+1,\,\text{pro-}\Sigma_K}) \xrightarrow{\cong} \mathscr{O}^+_{K^{m,\,\text{pro-}\Sigma_K}}$$

of $G_K^{m+1,\,\text{pro-}\Sigma_K}$-modules, where $\mathscr{O}_{K^{m,\,\text{pro-}\Sigma}}$ denotes the integral closure of $\mathscr{O}_K$ in $K^{m,\,\text{pro-}\Sigma}$. In particular, there exists an isomorphism

$$\mathscr{C}^+_{k^{m,\,\text{pro-}\Sigma}}(G_K^{m+1,\,\text{pro-}\Sigma_K}) \xrightarrow{\cong} \mathscr{C}^+_{K^{m,\,\text{pro-}\Sigma_K}}$$

of $G_K^{m+1,\,\text{pro-}\Sigma_K}$-modules. ◇

Speaking from an intuitive level, the $G_K^{m+1,\,\text{pro-}\Sigma_K}$-module $\mathscr{C}^+_{K^{m,\,\text{pro-}\Sigma_K}}$ can be recovered entirely group-theoretically from the filtered profinite group $G_K^{m+1,\,\text{pro-}\Sigma_K}$.

Abelian pro-Σ representations

Let G be a filtered profinite group of $\text{MLF}^{m+1,\,\text{pro-}\Sigma}$-type for an integer $m \geq 1$; we set $p := p(G^{\text{ab}})$. Let (ρ, V) and χ be a p-adic representation and a p-adic character of G, respectively. We shall write

$$d^i_{\text{HT},m,\,\text{pro-}\Sigma}(G,\chi,\rho,V) := \dim_{\mathbf{Q}_p}\left(\mathscr{C}^+_{k^{m,\,\text{pro-}\Sigma}}(G)\otimes_{\mathbf{Q}_p}V(\chi^{-i})\right)^G/d(G^{\text{ab}})$$

for each $i \in \mathbf{Z}$.

Proposition B.8. *Let K be a mixed-characteristic local field, Σ_K (resp. Σ'_K) a subset of $\mathfrak{Primes}$ containing all prime factors of p_K (resp. $p_K(p_K-1)$), and m an integer ≥ 1. Suppose that (ρ, V) is a p_K-adic representation of G_K.*

(1) If ρ factors through $G_K^{m,\,\text{pro-}\Sigma'_K}$, then

$$d^i_{\mathrm{HT},m,\,\text{pro-}\Sigma}(G_K^{m+1,\,\text{pro-}\Sigma'_K}, \chi(G_K^{\mathrm{mab},\,\text{pro-}\Sigma'_K}), \rho, V) = d^i_{\mathrm{HT},K}(\rho, V)$$

for each $i \in \mathbf{Z}$.

(2) If K is of p-cyclotomic type and ρ factors through $G_K^{m,\,\text{pro-}\Sigma_K}$, then

$$d^i_{\mathrm{HT},m,\,\text{pro-}\Sigma}(G_K^{m+1,\,\text{pro-}\Sigma_K}, \chi(G_K^{\mathrm{mab},\,\text{pro-}\Sigma_K}), \rho, V) = d^i_{\mathrm{HT},K}(\rho, V)$$

for each $i \in \mathbf{Z}$.

$\diamond$

The assertions of Proposition B.8 can be translated into intuitive terms as follows:

(1) If ρ factors through $G_K^{m,\,\text{pro-}\Sigma'_K}$, then the i^{th} *Hodge-Tate number* of (ρ, V) (and hence the issue of *whether or not (ρ, V) is Hodge-Tate*) can be determined entirely group-theoretically from the *filtered* profinite group $G_K^{m+1,\,\text{pro-}\Sigma'_K}$ and its action on V;

(2) If K is of p-cyclotomic type and ρ factors through $G_K^{m,\,\text{pro-}\Sigma_K}$, then the i^{th} *Hodge-Tate number* of (ρ, V) (and hence the issue of *whether or not (ρ, V) is Hodge-Tate*) can be determined entirely group-theoretically from the *filtered* profinite group $G_K^{m+1,\,\text{pro-}\Sigma_K}$ and its action on V;

Proof. (1) We have

$$d^i_{\mathrm{HT},m,\,\text{pro-}\Sigma}(G_K^{m+1,\,\text{pro-}\Sigma'_K}, \chi(G_K^{\mathrm{mab},\,\text{pro-}\Sigma'_K}), \rho, V) = \dim_K \left(\mathscr{C}(K^{m,\,\text{pro-}\Sigma'_K}) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K}$$

from Propositions B.2, B.4 and B.7. Since both ρ and χ_K annihilates

$$\mathrm{Ker}\,(G_K \twoheadrightarrow G_K^{m,\,\text{pro-}\Sigma'_K}),$$

we see that

$$\left(\mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} \supseteq \left(\mathscr{C}(K^{m,\,\text{pro-}\Sigma'_K}) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} \supseteq \left(\mathscr{C}((K^{\mathrm{alg}})^{\mathrm{Ker}\,(\rho(-i))}) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K}.$$

By applying Lemma 7.1, we deduce the desired equality.

(2) Similarly, we have

$$d^i_{\mathrm{HT},m,\,\text{pro-}\Sigma}(G_K^{m+1,\,\text{pro-}\Sigma_K}, \chi(G_K^{\mathrm{mab},\,\text{pro-}\Sigma_K}), \rho, V) = \dim_K \left(\mathscr{C}(K^{m,\,\text{pro-}\Sigma_K}) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K}.$$

Since both ρ and χ_K annihilates

$$\mathrm{Ker}\,(G_K \twoheadrightarrow G_K^{m,\,\text{pro-}\Sigma_K}),$$

we see that

$$\left(\mathbf{C}_{p_K} \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} \supseteq \left(\mathscr{C}(K^{m,\,\text{pro-}\Sigma_K}) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K} \supseteq \left(\mathscr{C}((K^{\mathrm{alg}})^{\mathrm{Ker}\,(\rho(-i))}) \otimes_{\mathbf{Q}_{p_K}} V(-i)\right)^{G_K}.$$

By applying Lemma 7.1 again, we deduce the desired equality. $\square$

Proposition B.9. *For each $\square \in \{\circ, \bullet\}$, let $K_\square$ be a mixed-characteristic local field, $\Sigma_{K_\square}$ (resp. $\Sigma'_{K_\square}$) a subset of $\mathfrak{Primes}$ containing all prime factors of $p_{K_\square}$ (resp. $p_{K_\square}(p_{K_\square} - 1)$).*

(1) Assume that

$$\alpha_{2,\,\text{pro-}\Sigma'} \colon G_{K_\circ}^{\text{mab, pro-}\Sigma'_{K_\circ}} \xrightarrow{\cong} G_{K_\bullet}^{\text{mab, pro-}\Sigma'_{K_\bullet}}$$

is an isomorphism of filtered *profinite groups, and $E/\mathbf{Q}_{p_{K_\circ}} (= \mathbf{Q}_{p_{K_\bullet}})$ is a finite Galois extension containing both $K_\circ$ and $K_\bullet$. If (ρ, V) is a 1-dimensional E-linear representation of $G_{K_\circ}$ such that ρ factors through*

$$G_{K_\circ}^{\text{ab, pro-}\Sigma'_{K_\circ}},$$

then $(\rho \circ (\alpha_{1,\,\text{pro-}\Sigma'})^{-1}, V)$ is a uniformizing representation of $G_{K_\bullet}$ if and only if (ρ, V) is a uniformizing representation of $G_{K_\circ}$, where

$$\alpha_{1,\,\text{pro-}\Sigma'} \colon G_{K_\circ}^{\text{ab, pro-}\Sigma'_{K_\circ}} \xrightarrow{\cong} G_{K_\bullet}^{\text{ab, pro-}\Sigma'_{K_\bullet}}$$

is the isomorphism induced by $\alpha_{2,\,\text{pro-}\Sigma'}$.

(2) Assume that $K_\circ$ and $K_\bullet$ are of p-cyclotomic type,

$$\alpha_{2,\,\text{pro-}\Sigma} \colon G_{K_\circ}^{\text{mab, pro-}\Sigma_{K_\circ}} \xrightarrow{\cong} G_{K_\bullet}^{\text{mab, pro-}\Sigma_{K_\bullet}}$$

is an isomorphism of filtered *profinite groups, and $E/\mathbf{Q}_{p_{K_\circ}} (= \mathbf{Q}_{p_{K_\bullet}})$ is a finite Galois extension containing both $K_\circ$ and $K_\bullet$. If (ρ, V) is a 1-dimensional E-linear representation of $G_{K_\circ}$ such that ρ factors through*

$$G_{K_\circ}^{\text{ab, pro-}\Sigma_{K_\circ}},$$

then $(\rho \circ (\alpha_{1,\,\text{pro-}\Sigma})^{-1}, V)$ is a uniformizing representation of $G_{K_\bullet}$ if and only if (ρ, V) is a uniformizing representation of $G_{K_\circ}$, where

$$\alpha_{1,\,\text{pro-}\Sigma} \colon G_{K_\circ}^{\text{ab, pro-}\Sigma_{K_\circ}} \xrightarrow{\cong} G_{K_\bullet}^{\text{ab, pro-}\Sigma_{K_\bullet}}$$

is the isomorphism induced by $\alpha_{2,\,\text{pro-}\Sigma}$.

$\diamond$

Proof. It follows immediately from Propositions B.8 and 7.5. $\square$

Theorem B.10.

(1) Let $K_\circ$ and $K_\bullet$ be mixed-characteristic local fields, and let $\Sigma'_{K_\square}$ be a subset of $\mathfrak{Primes}$ containing all prime factors of $p_{K_\square}(p_{K_\square} - 1)$ for each $\square \in \{\circ, \bullet\}$. If there exists an isomorphism

$$\alpha_{2,\,\text{pro-}\Sigma'} \colon G_{K_\circ}^{\text{mab, pro-}\Sigma'_{K_\circ}} \xrightarrow{\cong} G_{K_\bullet}^{\text{mab, pro-}\Sigma'_{K_\bullet}}$$

of filtered *profinite groups, then there exists an isomorphism $f \colon K_\circ \to K_\bullet$.*

(2) Let $K_\circ$ and $K_\bullet$ be mixed-characteristic local fields of p-cyclotomic type, and let $\Sigma_{K_\square}$ be a subset of $\mathfrak{Primes}$ containing $p_{K_\square}$ for each $\square \in \{\circ, \bullet\}$. If there exists an isomorphism

$$\alpha_{2,\,\text{pro-}\Sigma} \colon G_{K_\circ}^{\text{mab, pro-}\Sigma_{K_\circ}} \xrightarrow{\cong} G_{K_\bullet}^{\text{mab, pro-}\Sigma_{K_\bullet}}$$

of filtered *profinite groups, then there exists an isomorphism $f \colon K_\circ \to K_\bullet$.*

$\diamond$

Proof. (1) We choose a finite Galois extension $E/\mathbf{Q}_{p_\circ}(= \mathbf{Q}_{p_\bullet})$ that contains both $K_\circ$ and $K_\bullet$. If we denote by $\rho_\circ$ the composition

$$\rho_\circ\colon\ G_{K_\circ}^{\mathrm{ab}} \xrightarrow{\mathrm{Ver}} G_E^{\mathrm{ab}} \twoheadrightarrow G_E^{\mathrm{ab}}(1) \to U_E(1),$$

where the second (resp. third) arrow is the natural surjection (resp. the isomorphism Art_E^{-1}), then $(\rho_\circ, V \coloneqq E^+)$ is a uniformizing representation of $G_{K_\circ}$.

We see that $\rho_\circ$ factors through $G_{K_\circ}^{\mathrm{ab,\,pro\text{-}}p_{K_\circ}}$ (and hence through $G_{K_\circ}^{\mathrm{ab,\,pro\text{-}}\Sigma'_{K_\circ}}$); it follows from Proposition B.9 that $(\rho_\bullet \coloneqq \rho_\circ \circ (\alpha_{1,\,\mathrm{pro}\text{-}p})^{-1}, V)$ is also a uniformizing representation of $G_{K_\bullet}$, where

$$\alpha_{1,\,\mathrm{pro}\text{-}p}\colon G_{K_\circ}^{\mathrm{ab,\,pro\text{-}}p_{K_\circ}} \xrightarrow{\cong} G_{K_\bullet}^{\mathrm{ab,\,pro\text{-}}p_{K_\bullet}}$$

is the isomorphism induced by $\alpha_{2,\,\mathrm{pro}\text{-}\Sigma'}$. Hence there exist an open subgroup $I_\circ$ (resp. $I_\bullet$) of $U_{K_\circ}(1)$ (resp. $U_{K_\bullet}(1)$) and a field homomorphism $\iota_\circ\colon K_\circ \to E$ (resp. $\iota_\bullet\colon K_\bullet \to E$) such that $\alpha_{1,\,\mathrm{pro}\text{-}p}(I_\circ) = I_\bullet$ and the diagram

$$\begin{tikzcd}
 & K_\circ^\times \arrow[rrd, bend left, "\iota_\circ^\times"] & & \\
I_\circ \arrow[ur, "\subseteq"] \arrow[r, "\subseteq"] \arrow[d, "{\cong,\, \alpha_1|_{I_\circ}}"] & U_{K_\circ}(1) \arrow[r, "\mathrm{Art}_{K_\circ}"] \arrow[d, "{\cong,\, \alpha_1|_{U_{K_\circ}(1)}}"] & G_{K_\circ}^{\mathrm{ab,\,pro\text{-}}p_{K_\circ}} \arrow[r, "\rho_\circ"] \arrow[d, "{\cong,\, \alpha_{1,\,\mathrm{pro}\text{-}p}}"] & E^\times \arrow[d, equal] \\
I_\bullet \arrow[dr, "\subseteq"'] \arrow[r, "\subseteq"] & U_{K_\bullet}(1) \arrow[r, "\mathrm{Art}_{K_\bullet}"] & G_{K_\bullet}^{\mathrm{ab,\,pro\text{-}}p_{K_\bullet}} \arrow[r, "\rho_\bullet"] & E^\times \\
 & K_\bullet^\times \arrow[rru, bend right, "\iota_\bullet^\times"'] & &
\end{tikzcd}$$

commutes. In particular, $\iota_\bullet|_{I_\bullet} \circ \alpha_1|_{I_\circ} = \iota_\circ|_{I_\circ}$, and by Lemma 8.1, $\iota_\circ(K_\circ) = \iota_\bullet(K_\bullet)$ in E. Therefore, we have the following field isomorphism:

$$f\colon K_\circ \xrightarrow{\cong,\, \iota_\circ} \iota_\circ(K_\circ) = \iota_\bullet(K_\bullet) \xrightarrow{\cong,\, \iota_\bullet^{-1}} K_\bullet.$$

(2) The proof of (2) follows *mutatis mutandis* from the proof of (1). $\square$

Acknowledgement

The author would like to express his sincere gratitude to Prof. Yuichiro Taguchi for suggesting the topic itself and providing several perspectives through discussions. He also thanks Naganori Yamaguchi for leaving detailed comments and valuable feedback—which eventually led to a substantial improvement regarding the uniqueness assertion of Theorem 2.5—after carefully reading the draft. Furthermore, the author thanks Prof. Akio Tamagawa, Prof. Yuichiro Hoshi, Takahiro Murotani, Kaiji Kondo, Takuya Tanaka and Shota Tsujimura for their useful suggestions and advice.

References

[1] V. Abrashkin. “Modified Proof of a Local Analogue of the Grothendieck Conjecture”. In: *Journal de Théorie des Nombres de Bordeaux* 22.1 (2010), pp. 1–50.

[2] V. A. Abrashkin. “On a Local Analogue of the Grothendieck Conjecture”. In: *Int. J. Math.* 11.02 (2000), pp. 133–175.

[3] J. Ax. “Zeros of Polynomials over Local Fields—The Galois Action”. In: *Journal of Algebra* 15.3 (1970), pp. 417–428.

[4] I. Fesenko and S. Vostokov. *Local Fields and Their Extensions*. Trans. by I. Fesenko. Vol. 121. Translations of Mathematical Monographs. American Mathematical Society, 1993.

[5] J.-M. Fontaine and Y. Ouyang. *Theory of p-adic Galois Representations*. Available at `https://www.imo.universite-paris-saclay.fr/~fontaine/galoisrep.pdf`, retrieved June 19, 2023.

[6] A. Grothendieck. "Letter to G. Faltings (Translation into English)". In: *Geometric Galois Actions: Volume 1: Around Grothendieck's Esquisse d'un Programme*. Ed. by L. Schneps and P. Lochak. Vol. 1. London Mathematical Society Lecture Note Series. Cambridge University Press, 1997, pp. 285–293.

[7] Y. Hoshi. "Introduction to Mono-anabelian Geometry". In: *Publications mathématiques de Besançon. Algèbre et théorie des nombres* (2022), pp. 5–44.

[8] Y. Hoshi. "On the Grothendieck Conjecture for Affine Hyperbolic Curves Over Kummer-Faithful Fields". In: *Kyushu Journal of Mathematics* 71.1 (2017), pp. 1–29.

[9] K. Iwasawa. *Local Class Field Theory*. Oxford University Press, 1986.

[10] N. Koblitz. *p-Adic Numbers, p-Adic Analysis, and Zeta-Functions*. Vol. 58. Graduate Texts in Mathematics. Springer New York, 1984.

[11] S. Ladkani. "On nilpotency problems in metabelian extensions of local fields". Master's thesis. University of Bonn.

[12] J. S. Milne. *Algebraic Number Theory (v3.08)*. Available at `https://www.jmilne.org/math/CourseNotes/ANT.pdf`. 2020.

[13] J. S. Milne. *Class Field Theory (v4.03)*. Available at `https://www.jmilne.org/math/CourseNotes/CFT.pdf`. 2020.

[14] S. Mochizuki. "A Version of the Grothendieck Conjecture for p-adic Local Fields". In: *International Journal of Mathematics* 08.04 (1997), pp. 499–506.

[15] S. Mochizuki. "Topics in Absolute Anabelian Geometry III: Global Reconstruction Algorithms". In: *Journal of Mathematical Sciences, the University of Tokyo* 22.4 (2015), pp. 939–1156.

[16] J. Neukirch. *Algebraic Number Theory*. Vol. 322. Grundlehren Der Mathematischen Wissenschaften. Springer Berlin Heidelberg, 1999.

[17] J. Neukirch, A. Schmidt, and K. Wingberg. *Cohomology of Number Fields*. Vol. 323. Grundlehren Der Mathematischen Wissenschaften. Springer Berlin Heidelberg, 2008.

[18] M. Saïdi and A. Tamagawa. "The m-step solvable anabelian geometry of number fields". In: *Journal für die reine und angewandte Mathematik (Crelles Journal)* 2022.789 (2022), pp. 153–186.

[19] J.-P. Serre. *Abelian ℓ-Adic Representations and Elliptic Curves*. W. A. Benjamin, Inc., 1968.

[20] J.-P. Serre. "Local Class Field Theory". In: *Algebraic Number Theory*. Ed. by J.W.S. Cassels and A. Frölich. London Mathematical Society, 1976.

[21] J.-P. Serre. *Local Fields*. Vol. 67. Graduate Texts in Mathematics. Springer New York, 1979.

[22] J. T. Tate. "p-Divisible Groups". In: *Proceedings of a Conference on Local Fields*. Ed. by T. A. Springer. Springer Berlin Heidelberg, 1967, pp. 158–183.

[23] K. Uchida. "Isomorphisms of Galois Groups". In: *Journal of the Mathematical Society of Japan* 28.4 (1976), pp. 617–620.

[24] C. A. Weibel. *An Introduction to Homological Algebra*. Cambridge University Press, 1994.

[25] S. Yamagata. "A Counterexample for the Local Analogy of a Theorem by Iwasawa and Uchida". In: *Proceedings of the Japan Academy* 52.6 (1976), pp. 276–278.

[26] T. Yoshida. "Local Class Field Theory via Lubin-Tate Theory". In: *Annales de la Faculté des sciences de Toulouse: Mathématiques* 17.2 (2008), pp. 411–438.