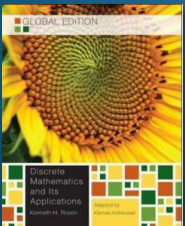


Relations

Chapter 9



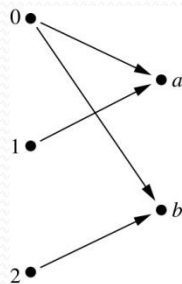
- Taken from the instructor's resource of *Discrete Mathematics and Its Applications*, 7/e
- Edited by Shin Hong hongshin@handong.edu

Binary Relations

Definition: A *binary relation* R from a set A to a set B is a subset $R \subseteq A \times B$.

Example:

- Let $A = \{0,1,2\}$ and $B = \{a,b\}$
- $\{(0, a), (0, b), (1,a) , (2, b)\}$ is a relation from A to B .
- We can represent relations from a set A to a set B graphically or using a table:



R	a	b
0	×	×
1	×	
2		×

Relations are more general than functions. A function is a relation where exactly one element of B is related to each element of A .

Binary Relation on a Set

Definition: A binary relation R on a set A is a subset of $A \times A$ or a relation from A to A .

Example:

- Suppose that $A = \{a, b, c\}$. Then $R = \{(a, a), (a, b), (a, c)\}$ is a relation on A .
- Let $A = \{1, 2, 3, 4\}$. The ordered pairs in the relation $R = \{(a, b) \mid a \text{ divides } b\}$ are $(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3)$, and $(4, 4)$.

Binary Relation on a Set (*cont.*)

Question: How many relations are there on a set A ?

Solution:

Since $A \times A$ has n^2 elements when A has n elements, and a set with m elements has 2^m subsets, there are $2^{|A|^2}$ subsets of $A \times A$.

Therefore, there are $2^{|A|^2}$ relations on a set A .

Binary Relations on a Set (*cont.*)

Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_5 = \{(a,b) \mid a = b + 1\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

Which of these relations contain each of the pairs

$(1,1)$, $(1,2)$, $(2,1)$, $(1,-1)$, and $(2,2)$?

Solution: Checking the conditions that define each relation, we see that the pair $(1,1)$ is in R_1 , R_3 , R_4 , and R_6 ; $(1,2)$ is in R_1 and R_6 ; $(2,1)$ is in R_2 , R_5 , and R_6 ; $(1,-1)$ is in R_2 , R_3 , and R_6 ; $(2,2)$ is in R_1 , R_3 , and R_4 .

Reflexive Relations

R is *reflexive* iff $(a,a) \in R$ for every element $a \in A$.

- R is reflexive if and only if $\forall x \in A ((x,x) \in R)$
- If $A = \emptyset$ then the empty relation is reflexive vacuously.

reflexive relations:

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

non reflexive relations :

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 3 \not> 3),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 3 \neq 3 + 1),$$

$$R_6 = \{(a,b) \mid a + b \leq 3\} \text{ (note that } 4 + 4 \not\leq 3).$$

Symmetric Relations

R is *symmetric* iff $(b,a) \in R$ whenever $(a,b) \in R$ for all $a,b \in A$.

$$- \forall x \in A (\forall y \in A ((x,y) \in R \longrightarrow (y,x) \in R))$$

Example

symmetric:

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\},$$

$$R_6 = \{(a,b) \mid a + b \leq 3\}.$$

not symmetric:

$$R_1 = \{(a,b) \mid a \leq b\} \text{ (note that } 3 \leq 4, \text{ but } 4 \not\leq 3),$$

$$R_2 = \{(a,b) \mid a > b\} \text{ (note that } 4 > 3, \text{ but } 3 \not> 4),$$

$$R_5 = \{(a,b) \mid a = b + 1\} \text{ (note that } 4 = 3 + 1, \text{ but } 3 \neq 4 + 1).$$

Antisymmetric Relations

- R on a set A is **antisymmetric** when $a = b$ if $(a,b) \in R$ and $(b,a) \in R$ for all $a \in A, b \in A$
- **Example**
 - antisymmetric:
 - $R_1 = \{(a,b) \mid a \leq b\},$
 - $R_2 = \{(a,b) \mid a > b\},$
 - $R_4 = \{(a,b) \mid a = b\},$
 - $R_5 = \{(a,b) \mid a = b + 1\}.$
 - not antisymmetric:
 - $R_3 = \{(a,b) \mid a = b \text{ or } a = -b\}$
(note that both $(1,-1)$ and $(-1,1)$ belong to R_3),
 - $R_6 = \{(a,b) \mid a + b \leq 3\}$ (note that both $(1,2)$ and $(2,1)$ belong to R_6).

Transitive Relations

- R on a set A is **transitive** if whenever $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$, for all $a,b,c \in A$.
 - $\forall x \forall y \forall z [(x,y) \in R \wedge (y,z) \in R \rightarrow (x,z) \in R]$

- **Examples of transitive relations:**

$$R_1 = \{(a,b) \mid a \leq b\},$$

$$R_2 = \{(a,b) \mid a > b\},$$

$$R_3 = \{(a,b) \mid a = b \text{ or } a = -b\},$$

$$R_4 = \{(a,b) \mid a = b\}.$$

Combining Relations

- Given two relations R_1 and R_2 , we can combine them using basic set operations to form new relations such as $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, and $R_2 - R_1$.
- Example:** Let $A = \{1,2,3\}$ and $B = \{1,2,3,4\}$. The relations $R_1 = \{(1,1), (2,2), (3,3)\}$ and $R_2 = \{(1,1), (1,2), (1,3), (1,4)\}$ can be combined using basic set operations to form new relations:

$$R_1 \cup R_2 = \{(1,1), (1,2), (1,3), (1,4), (2,2), (3,3)\}$$

$$R_1 \cap R_2 = \{(1,1)\} \qquad R_1 - R_2 = \{(2,2), (3,3)\}$$

$$R_2 - R_1 = \{(1,2), (1,3), (1,4)\}$$

Composition

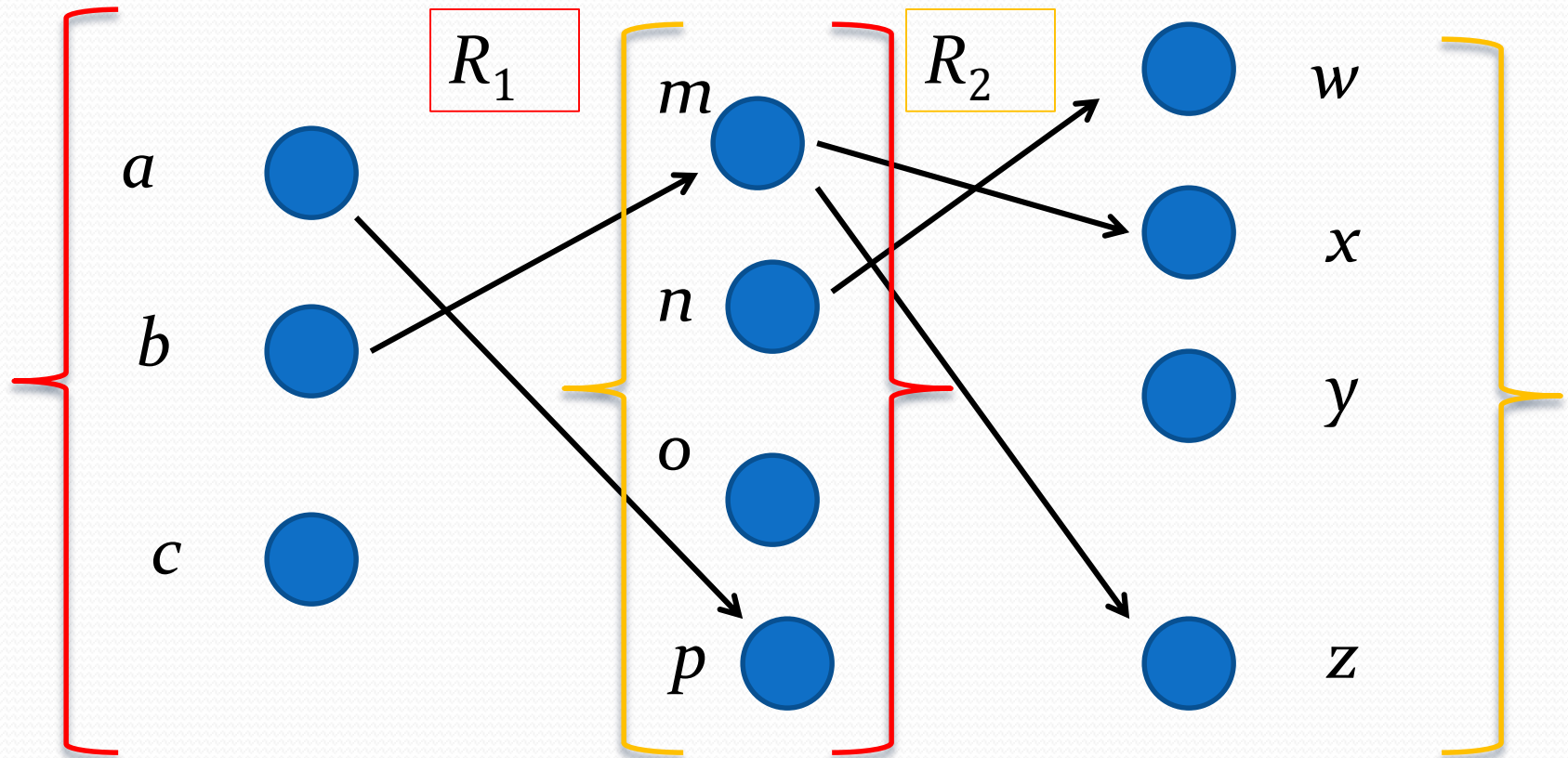
Suppose

- R_1 is a relation from a set A to a set B .
- R_2 is a relation from B to a set C .

The *composition* (or *composite*) of R_2 with R_1 , is a relation from A to C where

- if (x,y) is a member of R_1 and (y,z) is a member of R_2 , then (x,z) is a member of $R_2 \circ R_1$.

Representing the Composition of a Relation



$$R_1 \circ R_2 = \{(b,x), (b,z)\}$$

Powers of a Relation

Definition: Let R be a binary relation on A . Then the powers R^n of the relation R can be defined inductively by:

- Basis Step: $R^1 = R$
- Inductive Step: $R^{n+1} = R^n \circ R$

(see the slides for Section 9.3 for further insights)

The powers of a transitive relation are subsets of the relation. This is established by the following theorem:

Theorem 1: The relation R on a set A is transitive iff $R^n \subseteq R$ for $n = 1, 2, 3, \dots$

(see the text for a proof via mathematical induction)

Representing Relations Using Matrices

- A relation between finite sets can be represented using a zero-one matrix.
- A relation $R \subseteq A \times B$ is represented by the matrix $M_R = [m_{ij}]$, where
$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R, \\ 0 & \text{if } (a_i, b_j) \notin R. \end{cases}$$
- The matrix representing R has a 1 as its (i,j) entry when a_i is related to b_j and a 0 if a_i is not related to b_j .

Example

Example 1: Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a,b) if $a \in A$, $b \in B$, and $a > b$. What is the matrix representing R (assuming the ordering of elements is the same as the increasing numerical order)?

Solution: Because $R = \{(2,1), (3,1), (3,2)\}$, the matrix is

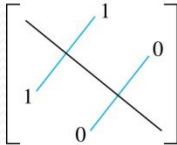
$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

Matrices of Relations on Sets

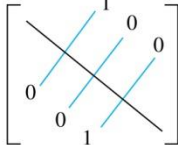
- If R is a reflexive relation, all the elements on the main diagonal of M_R are equal to 1.

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

- R is a symmetric relation, if and only if $m_{ij} = 1$ whenever $m_{ji} = 1$. R is an antisymmetric relation, if and only if $m_{ij} = 0$ or $m_{ji} = 0$ when $i \neq j$.



(a) Symmetric



(b) Antisymmetric

Example of a Relation on a Set

Example 3: Suppose that the relation R on a set is represented by the matrix

$$M_R = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

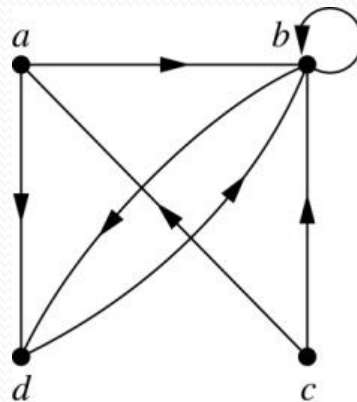
Is R reflexive, symmetric, and/or antisymmetric?

Solution: Because all the diagonal elements are equal to 1, R is reflexive. Because M_R is symmetric, R is symmetric and not antisymmetric because both $m_{1,2}$ and $m_{2,1}$ are 1.

Representing Relations Using Digraphs

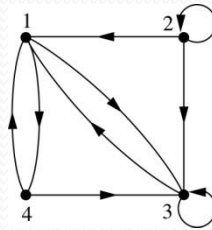
- A *directed graph*, or *digraph*, consists of a set V of *vertices* (or *nodes*) together with a set E of ordered pairs of elements of V called *edges* (or *arcs*). The vertex a is called the *initial vertex* of the edge (a,b) , and the vertex b is called the *terminal vertex* of this edge.
 - An edge of the form (a,a) is called a *loop*.

Example 7: A drawing of the directed graph with vertices a , b , c , and d , and edges (a, b) , (a, d) , (b, b) , (b, d) , (c, a) , (c, b) , and (d, b) is shown here.



Examples of Digraphs Representing Relations

Example 8: What are the ordered pairs in the relation represented by this directed graph?



Solution: The ordered pairs in the relation are $(1, 3)$, $(1, 4)$, $(2, 1)$, $(2, 2)$, $(2, 3)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, and $(4, 3)$

Determining which Properties a Relation has from its Digraph

- *Reflexivity*: A loop must be present at all vertices in the graph.
- *Symmetry*: If (x,y) is an edge, then so is (y,x) .
- *Antisymmetry*: If (x,y) with $x \neq y$ is an edge, then (y,x) is not an edge.
- *Transitivity*: If (x,y) and (y,z) are edges, then so is (x,z) .

Equivalence Relations

- A relation on a set A is called an *equivalence relation* if it is reflexive, symmetric, and transitive.

Definition 2: Two elements a , and b that are related by an equivalence relation are called *equivalent*. The notation $a \sim b$ is often used to denote that a and b are equivalent elements with respect to a particular equivalence relation.

Example

Suppose that R is the relation on the set of strings of English letters such that $(a, b) \in R$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x . Is R an equivalence relation?

Solution

- *Reflexivity*: as $l(a) = l(a)$, it follows $(a, a) \in R$ for every string a .
- *Symmetry*: $(b, a) \in R$ if $(a, b) \in R$ because $l(a) = l(b)$, $l(b) = l(a)$
- *Transitivity*: $(a, c) \in R$ if $(a, b) \in R \wedge (b, c) \in R$ because, if $l(a) = l(b)$ and $l(b) = l(c)$ then $l(a) = l(c)$

Congruence Modulo m

Example: Let m be an integer with $m > 1$. Show that the relation
$$R = \{(a,b) \mid a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.

Solution: Recall that $a \equiv b \pmod{m}$ if and only if m divides $a - b$.

- *Reflexivity:* $a \equiv a \pmod{m}$ since $a - a = 0$ is divisible by m since $0 = 0 \cdot m$.
- *Symmetry:* Suppose that $a \equiv b \pmod{m}$. Then $a - b$ is divisible by m , and so $a - b = km$, where k is an integer. It follows that $b - a = (-k)m$, so $b \equiv a \pmod{m}$.
- *Transitivity:* Suppose that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Then m divides both $a - b$ and $b - c$. Hence, there are integers k and l with $a - b = km$ and $b - c = lm$. We obtain by adding the equations:
$$a - c = (a - b) + (b - c) = km + lm = (k + l)m.$$

Therefore, $a \equiv c \pmod{m}$.

Divides

Example: Show that the “divides” relation on the set of positive integers is not an equivalence relation.

Solution: The properties of reflexivity, and transitivity do hold, but there relation is not transitive. Hence, “divides” is not an equivalence relation.

- *Reflexivity:* $a \mid a$ for all a .
- *Not Symmetric:* For example, $2 \mid 4$, but $4 \nmid 2$. Hence, the relation is not symmetric.
- *Transitivity:* Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.

Equivalence Classes

Definition 3: Let R be an equivalence relation on a set A . The set of all elements that are related to an element a of A is called the *equivalence class* of a . The equivalence class of a with respect to R is denoted by $[a]_R$.

When only one relation is under consideration, we can write $[a]$, without the subscript R , for this equivalence class.

Note that $[a]_R = \{s \mid (a, s) \in R\}$.

- If $b \in [a]_R$, then b is called a representative of this equivalence class. Any element of a class can be used as a representative of the class.
- The equivalence classes of the relation congruence modulo m are called the *congruence classes modulo m* . The congruence class of an integer a modulo m is denoted by $[a]_m$, so $[a]_m = \{\dots, a-2m, a-m, a, a+m, a+2m, \dots\}$. For example,

$$[0]_4 = \{\dots, -8, -4, 0, 4, 8, \dots\}$$

$$[1]_4 = \{\dots, -7, -3, 1, 5, 9, \dots\}$$

$$[2]_4 = \{\dots, -6, -2, 2, 6, 10, \dots\}$$

$$[3]_4 = \{\dots, -5, -1, 3, 7, 11, \dots\}$$

Equivalence Classes and Partitions

Theorem 1: let R be an equivalence relation on a set A . These statements for elements a and b of A are equivalent:

- (i) aRb
- (ii) $[a] = [b]$
- (iii) $[a] \cap [b] = \emptyset$

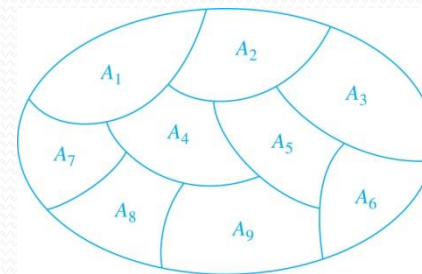
Proof: We show that (i) implies (ii). Assume that aRb . Now suppose that $c \in [a]$. Then aRc . Because aRb and R is symmetric, bRa . Because R is transitive and bRa and aRc , it follows that bRc . Hence, $c \in [b]$. Therefore, $[a] \subseteq [b]$. A similar argument (omitted here) shows that $[b] \subseteq [a]$. Since $[a] \subseteq [b]$ and $[b] \subseteq [a]$, we have shown that $[a] = [b]$.

(see text for proof that (ii) implies (iii) and (iii) implies (i))

Partition of a Set

Definition: A *partition* of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i , where $i \in I$ (where I is an index set), forms a partition of S if and only if

- $A_i \neq \emptyset$ for $i \in I$,
- $A_i \cap A_j = \emptyset$ when $i \neq j$,
- and $\bigcup_{i \in I} A_i = S$.



A Partition of a Set

An Equivalence Relation Partitions a Set

- Let R be an equivalence relation on a set A . The union of all the equivalence classes of R is all of A , since an element a of A is in its own equivalence class $[a]_R$. In other words,

$$\bigcup_{a \in A} [a]_R = A.$$

- From Theorem 1, it follows that these equivalence classes are either equal or disjoint, so $[a]_R \cap [b]_R = \emptyset$ when $[a]_R \neq [b]_R$.
- Therefore, the equivalence classes form a partition of A , because they split A into disjoint subsets.

An Equivalence Relation Partitions a Set

Theorem 2: Let R be an equivalence relation on a set S . Then the equivalence classes of R form a partition of S . Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S , there is an equivalence relation R that has the sets A_i , $i \in I$, as its equivalence classes.

Proof: We have already shown the first part of the theorem.

For the second part, assume that $\{A_i \mid i \in I\}$ is a partition of S . Let R be the relation on S consisting of the pairs (x, y) where x and y belong to the same subset A_i in the partition. We must show that R satisfies the properties of an equivalence relation.

- *Reflexivity:* For every $a \in S$, $(a, a) \in R$, because a is in the same subset as itself.
- *Symmetry:* If $(a, b) \in R$, then b and a are in the same subset of the partition, so $(b, a) \in R$.
- *Transitivity:* If $(a, b) \in R$ and $(b, c) \in R$, then a and b are in the same subset of the partition, as are b and c . Since the subsets are disjoint and b belongs to both, the two subsets of the partition must be identical. Therefore, $(a, c) \in R$ since a and c belong to the same subset of the partition.

Partial Orderings

Definition 1: A relation R on a set S is called a *partial ordering*, or *partial order*, if it is reflexive, antisymmetric, and transitive. A set together with a partial ordering R is called a *partially ordered set*, or *poset*, and is denoted by (S, R) . Members of S are called *elements* of the poset.

Partial Orderings (*continued*)

Example 1: Show that the “greater than or equal” relation (\geq) is a partial ordering on the set of integers.

- *Reflexivity:* $a \geq a$ for every integer a .
- *Antisymmetry:* If $a \geq b$ and $b \geq a$, then $a = b$.
- *Transitivity:* If $a \geq b$ and $b \geq c$, then $a \geq c$.

These properties all follow from the order axioms for the integers.
(See Appendix 1).

Partial Orderings (*continued*)

Example 2: Show that the divisibility relation ($|$) is a partial ordering on the set of integers.

- *Reflexivity:* $a | a$ for all integers a . (see Example 9 in Section 9.1)
- *Antisymmetry:* If a and b are positive integers with $a | b$ and $b | a$, then $a = b$. (see Example 12 in Section 9.1)
- *Transitivity:* Suppose that a divides b and b divides c . Then there are positive integers k and l such that $b = ak$ and $c = bl$. Hence, $c = a(kl)$, so a divides c . Therefore, the relation is transitive.
- $(\mathbb{Z}^+, |)$ is a poset.

Partial Orderings (*continued*)

Example 3: Show that the inclusion relation (\subseteq) is a partial ordering on the power set of a set S .

- *Reflexivity:* $A \subseteq A$ whenever A is a subset of S .
- *Antisymmetry:* If A and B are positive integers with $A \subseteq B$ and $B \subseteq A$, then $A = B$.
- *Transitivity:* If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

The properties all follow from the definition of set inclusion.

Comparability

Definition 2: The elements a and b of a poset (S, \preceq) are *comparable* if either $a \preceq b$ or $b \preceq a$. When a and b are elements of S so that neither $a \preceq b$ nor $b \preceq a$, then a and b are called *incomparable*.

The symbol \preceq is used to denote the relation in any poset.

Definition 3: If (S, \preceq) is a poset and every two elements of S are comparable, S is called a *totally ordered* or *linearly ordered set*, and \preceq is called a *total order* or a *linear order*. A totally ordered set is also called a *chain*.

Definition 4: (S, \preceq) is a well-ordered set if it is a poset such that \preceq is a total ordering and every nonempty subset of S has a least element.

Lexicographic Order

Definition: Given two posets (A_1, \preceq_1) and (A_2, \preceq_2) , the *lexicographic ordering* on $A_1 \times A_2$ is defined by specifying that (a_1, a_2) is less than (b_1, b_2) , that is,

$$(a_1, a_2) < (b_1, b_2),$$

either if $a_1 <_1 b_1$ or if $a_1 = b_1$ and $a_2 <_2 b_2$.

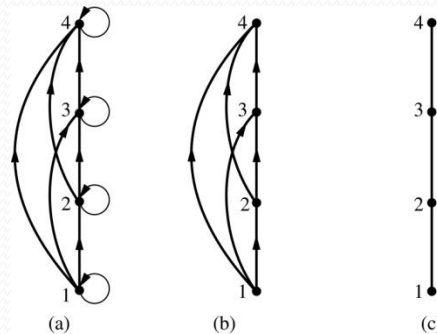
- This definition can be easily extended to a lexicographic ordering on strings (*see text*).

Example: Consider strings of lowercase English letters. A lexicographic ordering can be defined using the ordering of the letters in the alphabet. This is the same ordering as that used in dictionaries.

- *discreet* < *discrete*, because these strings differ in the seventh position and $e < t$.
- *discreet* < *discreetness*, because the first eight letters agree, but the second string is longer.

Hasse Diagrams

Definition: A *Hasse diagram* is a visual representation of a partial ordering that leaves out edges that must be present because of the reflexive and transitive properties.



A partial ordering is shown in (a) of the figure above. The loops due to the reflexive property are deleted in (b). The edges that must be present due to the transitive property are deleted in (c). The Hasse diagram for the partial ordering (a), is depicted in (c).

Procedure for Constructing a Hasse Diagram

- To represent a finite poset (S, \preceq) using a Hasse diagram, start with the directed graph of the relation:
 - Remove the loops (a, a) present at every vertex due to the reflexive property.
 - Remove all edges (x, y) for which there is an element $z \in S$ such that $x \prec z$ and $z \prec y$. These are the edges that must be present due to the transitive property.
 - Arrange each edge so that its initial vertex is below the terminal vertex. Remove all the arrows, because all edges point upwards toward their terminal vertex.