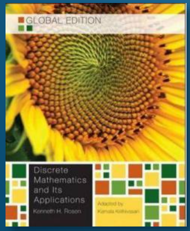
Basic Structures: Sets

Chapter 2



- Taken from the instructor's resource of *Discrete Mathematics and Its Applications,* 7/e
- Edited by Shin Hong hongshin@handong.edu

Sets

- A *set* is an unordered collection of objects.
  - E.g., the students in this class
  - E.g., the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

Defining a Set: Roster Method

- $S = \{a,b,c,d\}$
- Order not important

  $$S = \{a,b,c,d\} = \{b,c,a,d\}$$

- Each distinct object is either a member or not; listing more than once does not change the set.

  $$S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$$

- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear.

  $$S = \{a,b,c,d, \ldots ,z\}$$

Examples

- Set of all vowels in the English alphabet:

  $V = \{a,e,i,o,u\}$

- Set of all odd positive integers less than 10:

  $O = \{1,3,5,7,9\}$

- Set of all positive integers less than 100:

  $S = \{1,2,3,\ldots\ldots..,99\}$

- Set of all integers less than 0:

  $S = \{\ldots., -3,-2,-1\}$

Conventional Notions of Important Sets

N = *natural numbers* = {1,2,3, …}

Z = *integers* = {…,-3,-2,-1,0,1,2,3,…}

Z^{+} = *positive integers* = {1,2,3,…..}

R = set of *real numbers*

R^{+} = set of *positive real numbers*

C = set of *complex numbers*.

Q = set of rational numbers

Defining a Set with Set-Builder Notation

- To specify the properties that all members must satisfy:

  $S = \{x \mid x \text{ is a positive integer less than } 100\}$

  $O = \{x \mid x \text{ is an odd positive integer less than } 10\}$

  $O = \{x \in \mathbf{Z}^+ \mid x \text{ is odd and } x < 10\}$

- A predicate may be used: $S = \{x \mid \mathrm{P}(x)\}$
  - Example: $S = \{x \mid \mathrm{Prime}(x)\}$
  - Example: Positive rational numbers:

$$\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q, \text{ for some positive integers } p, q\}$$

Defining a Set with Interval Notation

$[a,b] = \{x \mid a \leq x \leq b\}$

$[a,b) = \{x \mid a \leq x < b\}$

$(a,b] = \{x \mid a < x \leq b\}$

$(a,b) = \{x \mid a < x < b\}$

E.g., [a,b)

closed interval [a,b]

open interval (a,b)

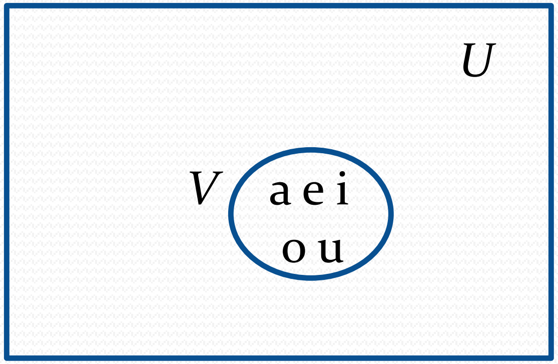
Universal Set and Empty Set

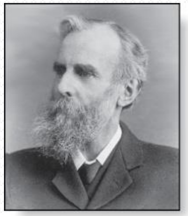
- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.

- The empty set is the set with no elements.

  \- denoted as ∅, or {}

Venn Diagram





John Venn (1834-1923)
Cambridge, UK

Some things to remember

- Sets can be elements of sets.

  $\{\{1,2,3\},\ a, \{b,c\}\}$

  $\{\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}\}$

- The empty set is different from a set containing the empty set.

  $\emptyset \neq \{\emptyset\}$

Set Cardinality

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.

Examples:

1. $|ø| = 0$
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. $|\{1,2,3\}| = 3$
4. $|\{ø\}| = 1$
5. The set of integers is infinite.

Russell's Paradox

- Let R be the set of all sets each of which is not a member of itself. A paradox results from trying to answer the question "Is R a member of itself?"
  - $R = \{S \mid S \notin S\}$

- Related Paradox:
  - Henry is a barber who shaves every man if and only if the man does not shave himself. A paradox results from trying to answer the question "Does Henry shave himself?"

Bertrand Russell (1872-1970)
Cambridge, UK
Nobel Prize Winner

Set Equality

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$
- We write *A* = *B* if *A* and *B* are equal sets.

$$\{1,3,5\} = \{3, 5, 1\}$$

$$\{1,5,5,5,3,3,1\} = \{1,3,5\}$$

Another look at Equality of Sets

- Recall that two sets A and B are *equal*, denoted by $A = B$, iff
$$\forall x(x \in A \leftrightarrow x \in B)$$

- Using logical equivalences we have that $A = B$ iff
$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

- This is equivalent to

$A \subseteq B$ and $B \subseteq A$

Subsets

Definition: The set A is a *subset* of B, if and only if every element of A is also an element of B.

- The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
- $A \subseteq B$ holds if and only if $\forall x(x \in A \to x \in B)$ is true.
  1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$,for every set S.
  2. Because $a \in S \to a \in S$, $S \subseteq S$, for every set S.

Proper Subsets

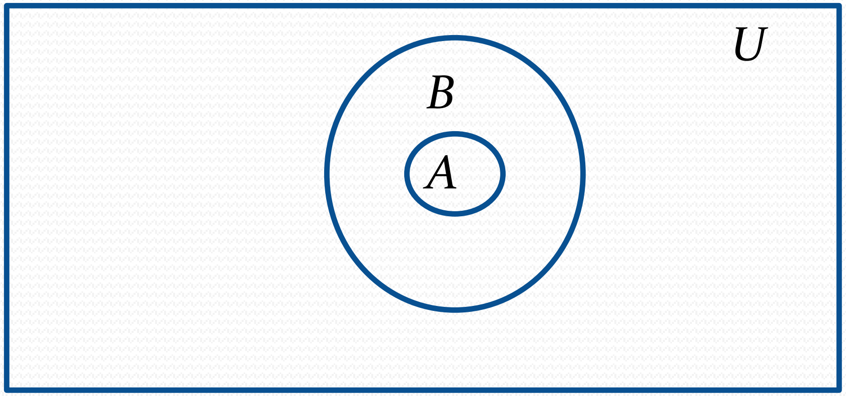
Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$ or $A \subsetneq B$

If $A \subset B$, then

$$\forall x (x \in A \rightarrow x \in B) \land \exists x (x \in B \land x \notin A)$$

is true.

Venn Diagram



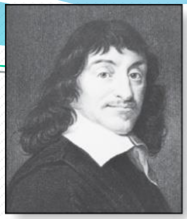
Power Sets

Definition: The set of all subsets of a set *A*, denoted P(*A*), is called the *power set* of *A*.

Example: If A = {a,b} then

$$\mathcal{P}(A) = \{\emptyset, \{a\},\{b\},\{a,b\}\}$$

- If a set has *n* elements, then the cardinality of the power set is 2^n. (In Chapters 5 and 6, we will discuss different ways to show this.)



René Descartes
(1596-1650)

Cartesian Product

Definition: The *Cartesian Product* of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a, b) where $a \in A$ and $b \in B$.

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

Example:

$A = \{a,b\}$ $B = \{1,2,3\}$

$A \times B = \{(a,1),(a,2),(a,3), (b,1),(b,2),(b,3)\}$

- **c.f. Relation**
  - A subset R of the Cartesian product $A \times B$ is called a *relation* from the set A to the set B.
  - Will be covered in depth in Chapter 9.

Cartesian Product

Definition: The cartesian products of the sets $A_1, A_2, \ldots\ldots, A_n$, denoted by $A_1 \times A_2 \times \ldots\ldots \times A_n$, is the set of ordered n-tuples $(a_1, a_2, \ldots\ldots, a_n)$ where a_i belongs to A_i for $i = 1, \ldots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \ldots n\}$$

Example: What is $A \times B \times C$ where $A = \{0,1\}$, $B = \{1,2\}$ and $C = \{0,1,2\}$

Solution: $A \times B \times C = \{(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,1,2)\}$

Tuples

- The *n-tuple* $(a_1, a_2, \ldots\ldots, a_n)$ is the ordered collection of objects, which has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
  - Two n-tuples are equal if and only if their corresponding elements are equal.
- 2-tuples are called *ordered pairs*.
  - The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

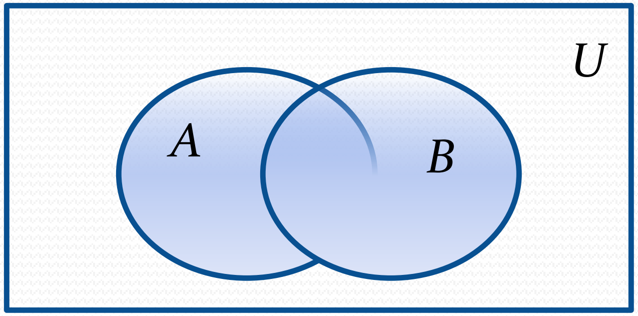
Union

- **Definition**: Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set:

$$\{x | x \in A \vee x \in B\}$$

- **Example**: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Venn Diagram for $A \cup B$



Solution: $\{1,2,3,4,5\}$

Intersection

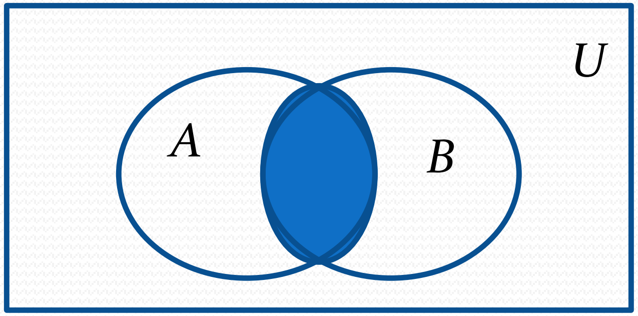
- **Definition**: The *intersection* of sets A and B, denoted by $A \cap B$, is $\{x | x \in A \wedge x \in B\}$
- Note if the intersection is empty, then A and B are said to be *disjoint*.
- **Example**: What is? $\{1,2,3\} \cap \{3,4,5\}$?

  Solution: $\{3\}$
- **Example:** What is?

  $\{1,2,3\} \cap \{4,5,6\}$?

  Solution: $\emptyset$

Venn Diagram for $A \cap B$



Complement

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set U - A

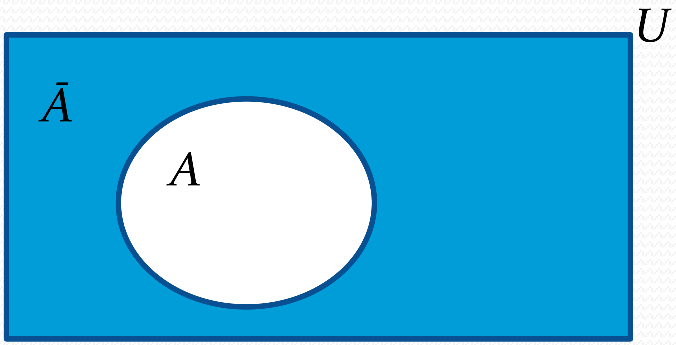
$$\bar{A} = \{x \in U \mid x \notin A\}$$

(The complement of A is sometimes denoted by A^c .)

Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \mid x \leq 70\}$

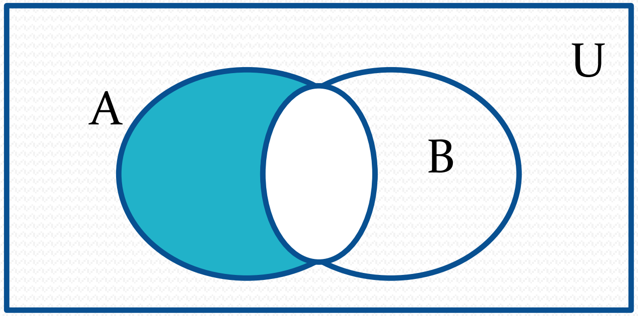
Venn Diagram for Complement



Difference

- **Definition**: Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$

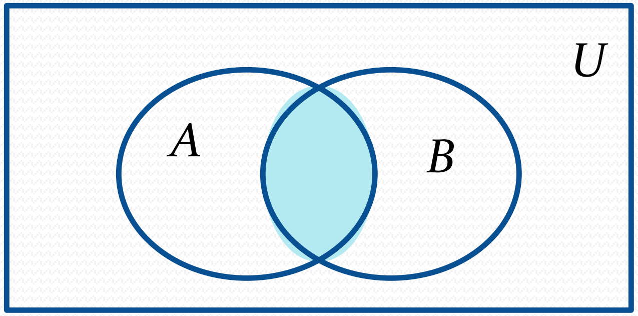


Venn Diagram for $A - B$

The Cardinality of the Union of Two Sets

- Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$



Venn Diagram for A, B, $A \cap B$, $A \cup B$

- **Example**: Let A be the math majors in your class and B be the CS majors. To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.
- We will return to this principle in Chapter 6 and Chapter 8 where we will derive a formula for the cardinality of the union of n sets, where n is a positive integer.

Symmetric Difference (*optional*)

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set

$$(A - B) \cup (B - A)$$

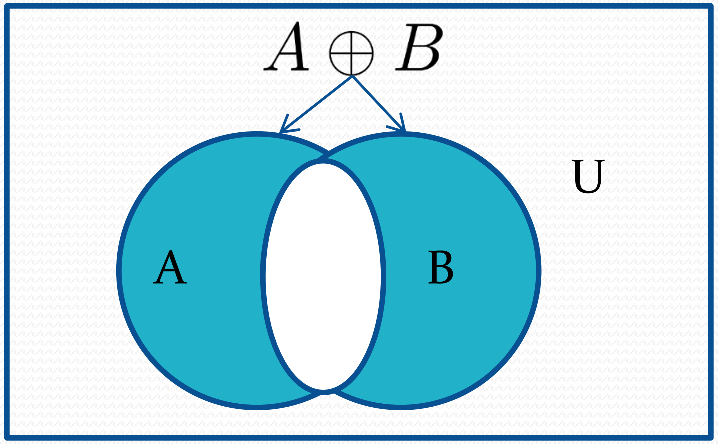
Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is:

- **Solution**: $\{1,2,3,6,7,8\}$



Venn Diagram

Set Identities

- Identity laws

$$A \cup \emptyset = A \qquad A \cap U = A$$

- Domination laws

$$A \cup U = U \qquad A \cap \emptyset = \emptyset$$

- Idempotent laws

$$A \cup A = A \qquad A \cap A = A$$

- Complementation law

$$\overline{(\overline{A})} = A$$

Continued on next slide →

Set Identities

- Commutative laws

$$A \cup B = B \cup A \qquad A \cap B = B \cap A$$

- Associative laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

- Distributive laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Continued on next slide →

Set Identities

- De Morgan's laws

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \qquad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- Absorption laws

$$A \cup (A \cap B) = A \qquad A \cap (A \cup B) = A$$

- Complement laws

$$A \cup \overline{A} = U \qquad A \cap \overline{A} = \emptyset$$

Proving Set Identities

- Different ways to prove set identities:
  1. Prove that each set (side of the identity) is a subset of the other.
  2. Use set builder notation and propositional logic.
  3. Membership Tables: Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Proof of Second De Morgan Law

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

Lemma 1 $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and

Lemma 2 $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

Continued on next slide →

Proof of Second De Morgan Law

Lemma 1: $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

1. $\forall x\ (\neg(x \in \overline{A \cap B}) \vee x \in (\bar{A} \cup \bar{B}))$ By def.
2. $\forall x\ (\neg(x \in \overline{A \cap B}) \vee x \in \bar{A}\ \vee x \in \bar{B})$ By def. $\cup$
3. $\forall x\ ((x \in A \wedge x \in B) \vee x \in \bar{A}\ \vee x \in \bar{B})$ By def. comp.
4. $\forall x\ ((x \in A \wedge x \in B \wedge x \in \bar{A}) \vee (x \in A \wedge x \in \bar{B} \wedge x \in \bar{A}))$ By distributive law
5. $\forall x$ T By negation laws

Set-Builder Notation: Second De Morgan Law

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
 & = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
 & = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
 & = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
 & & & \text{for Prop Logic} \\
 & = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
 & = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
 & = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
 & = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

Membership Table

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Solution:

A	**B**	**C**	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

Generalized Unions and Intersections

- Let A_1, A_2 ,…, A_n be an indexed collection of sets.
  We define:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n \qquad \bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

These are well defined, since union and intersection are associative.

- For i = 1,2,…, let A_i = {i, i + 1, i + 2, ….}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$

Representing Sets in Computer

- List of elements
- Sorted list of elements
- Bit string (bit vector)
  - For a universal set U={a1,a2,a3,…,a_n} , a set is represented by a vector *v* with *n* Boolean elements such that *v*'s *i*-th element is true iff a_i is a member of the set.
  - Example: U={1,2,3,4,…,10}, S1={1,3,5,7,9}, S2={1,2,3,4,5}

    v_{S_1} = <1010101010>

    v_{S_2} = <1111100000>