Discrete Mathematics

Rule of Inference

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How can we know an argument true?

Proof and Inference

- An **argument** is a sequence of statements
- An argument is **valid** iff it is impossible that all preceding statements are true and a final statement is false at the same time
  - a conclusion follows the premises
- An **argument form** is a valid proposition whose structure is $P_1 \land P_2 \ldots \land P_n \to Q$ where P_i and Q are compound propositions
  - **inference rule**
  - E.g.

$$\begin{array}{rl} & p \to q \\ & p \\ \hline \therefore & q \end{array}$$

- An argument is valid if all initial statements are known to be true, and for every non-initial statement, there is an argument form that connects the preceding statements with it
  - Such a sequence of argument is called **proof**

Rules of Inferences

premise$_1$
premise$_2$
...
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conclusion

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Intuitive Examples

p $p \to q$ $\therefore q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
p $\therefore p \vee q$	Addition
$p \wedge q$ $\therefore p$	Simplification
p q $\therefore p \wedge q$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	Resolution

- If there is fire, fire alarm rings.
  There is fire.
  Thus, fire alarm rings
- Fire alarm rings if there's fire.
  There is no fire alarm.
  Thus, there is no fire.
- If one is a man, the one eventually dies.
  If one is a philosoper, the one is a man.
  Thus, a philosoper eventually dies.
- I will take a taxi tonight if it rains.
  Otherwise, I will take a bus tonight.
  Thereby, I will take a taxi or bus tonight.

Example

p $p \to q$ $\therefore q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism
p $\therefore p \vee q$	Addition
$p \wedge q$ $\therefore p$	Simplification
p q $\therefore p \wedge q$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	Resolution

- Premises
  1. $\neg p \wedge q$
  2. $r \to p$
  3. $\neg r \to s$
  4. $s \to t$
- Does t hold upon the premises?
- Inference steps (proof)

1.	$\neg p \wedge q$	Premise 1
2.	$\neg p$	Simplification
3.	$r \to p$	Premise 2
4.	$\neg r$	Modus tollens
5.	$\neg r \to s$	Premise 3
6.	s	Modus ponens
7.	$s \to t$	Premise 4
8.	t	Modus ponens

Quantified Statements

- Valid arguments for quantified statements are a sequence of statements where each statement is either a premise or follows from previous statements by rules of inference
  - rules of inference for propositional logic
  - rules of inference for quantified statements
    - Universal Instantiation (UI)
    - Universal Generalization (UG)
    - Existential Instantiation (EI)
    - Existential Generalization (EG)

Universal Instantiation (UI)

$$\frac{\forall x P(x)}{\therefore P(c)}$$

- c is a specific instance of the domain, or
- c is a variable representing an arbitrary value of the domain

Example:

Our domain consists of all dogs and Bingo is a dog.

“All dogs are cuddly.”

“Therefore, Bingo is cuddly.”
"Therefore, dog *d* is cuddly"

Universal Generalization (UG)

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

Existential Instantiation (EI)

$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

Example:

“There is someone who got an A in the course.”
“Let’s call her *a* and say that *a* got an A”

Existential Generalization (EG)

$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

Example:

“Michelle got an A in the class.”
“Therefore, someone got an A in the class.”

Using Rules of Inference

Construct a valid argument to show that
"John Smith has one wife" is a consequence of the premises:
"Every married man has one wife." "John Smith is a married man."

Solution: Let $M(x)$ denote "x is a married man", and $L(x)$ denote " x has one wife", and let J be an element representing John Smith.

Step	**Reason**
1. $\forall x(M(x) \rightarrow L(x))$	Premise
2. $M(J) \rightarrow L(J)$	UI from (1)
3. $M(J)$	Premise
4. $L(J)$	Modus Ponens using (2) and (3)

Using Rules of Inference

- Construct a valid argument showing that the conclusion:
  "Someone who passed the first exam has not read the book." follows from
  - "A student in this class has not read the book."
  - "Everyone in this class passed the first exam."
- Solution: Let *C*(*x*) denote "*x* is in this class," *B*(*x*) denote " *x* has read the book," and *P*(*x*) denote "*x* passed the first exam."

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \to P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

Step	**Reason**
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	EI from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	UI from (4)
6. $P(a)$	MP from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conj from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	EG from (8)

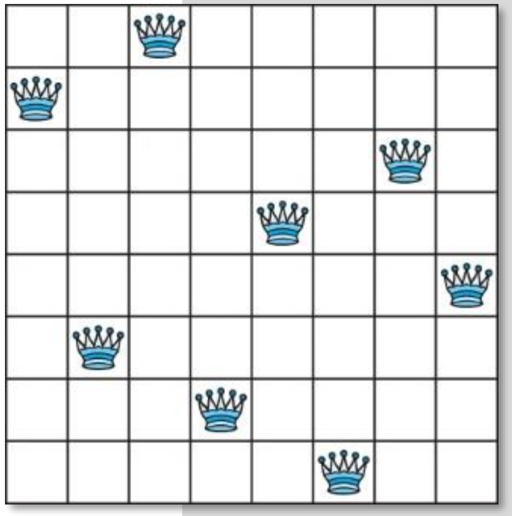
Automatic Theorm Prover

- Propositional logic
  - SAT solvers can determine the satisfiability of any propositional logic formula correctly

- Predicate logic
  - For a limited type of predicate logics, SMT solvers can determine the satisfiability of a predicate logic formula
  - E.g., Quantifier-Free Linear Integer Arithematics
    - $\exists x_1 \in \mathbb{N} \, \exists x_2 \in \mathbb{N} \ldots \exists x_n \in \mathbb{N} \; (P(x_1, x_2, \ldots x_n))$

      where P is a compound predicate formula consist of equality and non-equalities predicates, linear-integer arithematics expressions, and logical connectives

N-Queen Problem in Predicate Logic



- Problem: place N queens on N-by-N grid without any conflict
- Modeling: an integer $p_{i,j}$ is equal to 1 iff a queen is placed at the (i, j) cell; 0 otherwise
- Constraint:

$$\exists p_{1,1} \exists p_{1,2} \dots \exists p_{N,N}$$
$$(Q_{CELL} \land Q_{NUM} \land Q_{ROW} \land Q_{COL} \land Q_{LU} \land Q_{LD} \land Q_{RU} \land Q_{RD})$$

$$Q_{CELL}: \bigwedge_{i=1}^{N} \bigwedge_{j=1}^{N} 0 \leq p_{i,j} \leq 1 \qquad Q_{NUM}: \sum_{i=1}^{N} \sum_{j=1}^{N} p_{i,j} = N$$

$$Q_{ROW}: \bigwedge_{i=1}^{N} \sum_{j=1}^{N} p_{i,j} \leq 1 \qquad Q_{LU}: \bigwedge_{i=1}^{N} \sum_{k=0}^{i-1} p_{i-k,\,1+k} \leq 1 \qquad Q_{RU}: \bigwedge_{i=1}^{N} \sum_{k=0}^{i-1} p_{i-K,\,N-k} \leq 1$$

$$Q_{COL}: \bigwedge_{j=1}^{N} \sum_{i=1}^{N} p_{i,j} \leq 1 \qquad Q_{LD}: \bigwedge_{i=1}^{N} \sum_{k=0}^{N-i} p_{i+k,\,1+k} \leq 1 \qquad Q_{RD}: \bigwedge_{i=1}^{N} \sum_{k=0}^{N-i} p_{i+k,\,N-k} \leq 1$$