Discrete Mathematics

Algorithm

Shin Hong

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Problems and Algorithms

- In many domains there are **key general problems** that ask for output with specific properties when a valid input is given

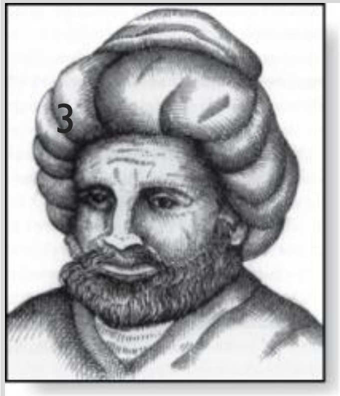
- Generalized solution - algorithm
  - State precisely the general problem by specifying the input and the desired output, using the appropriate structures
  - Specify the steps of a procedure that takes a valid input and produces the desired output.

Algorithms

- An *algorithm* is a finite sequence of precise instructions for performing a computation or for solving a problem.
- **Ex.** Describe an algorithm for finding the maximum value in a finite sequence of integers.

Solution: perform the following steps:

1. Set the temporary maximum equal to the first integer in the sequence.
2. Compare the next integer in the sequence to the temporary maximum.
   - If it is larger than the temporary maximum, set the temporary maximum equal to this integer.
3. Repeat the previous step if there are more integers. If not, stop.
4. When the algorithm terminates, the temporary maximum is the largest integer in the sequence.



Abu Ja'far Mohammed Ibin Musa Al-Khowarizmi (780-850)

Specifying Algorithms in Pseudocode

- Pseudocode is an intermediate step between natural language description and code using a specific programming language
- The form of pseudocode we use is specified in Appendix 3
  - Similar with C++ and Java.
- Programmers can use the description of an algorithm in pseudocode to construct a program in a particular language
- Pseudocode helps us analyze the time required to solve a problem using an algorithm, independent of the actual programming language used to implement algorithm

Properties of Algorithms

- **Input**: An algorithm has input values from a specified set.
- **Output**: From the input values, the algorithm produces the output values from a specified set. The output values are the solution.
- **Correctness**: An algorithm should produce the correct output values for each set of input values.
- **Finiteness**: An algorithm should produce the output after a finite number of steps for any input.
- **Effectiveness**: It must be possible to perform each step of the algorithm correctly and in a finite amount of time.
- **Generality**: The algorithm should work for all problems of the desired form.

Ex. Finding the Maximum Element in a Finite Sequence

- The algorithm in pseudocode:

procedure *max*(a_1, a_2, …., a_n: integers)
  max := a_1
  for i := 2 to n
    if *max* < a_i then *max* := a_i
  return *max* {*max* is the largest element}

- Does this algorithm have all the properties?

The Growth of Functions

- In both computer science and in mathematics, there are many times when we care about how fast a function grows.

- In computer science, we want to understand how quickly an algorithm can solve a problem as the size of the input grows.
  - we can compare the efficiency of two different algorithms for solving the same problem
  - we can also determine whether it is practical to use a particular algorithm as the input grows.

Big-*O* Notation (1/3)

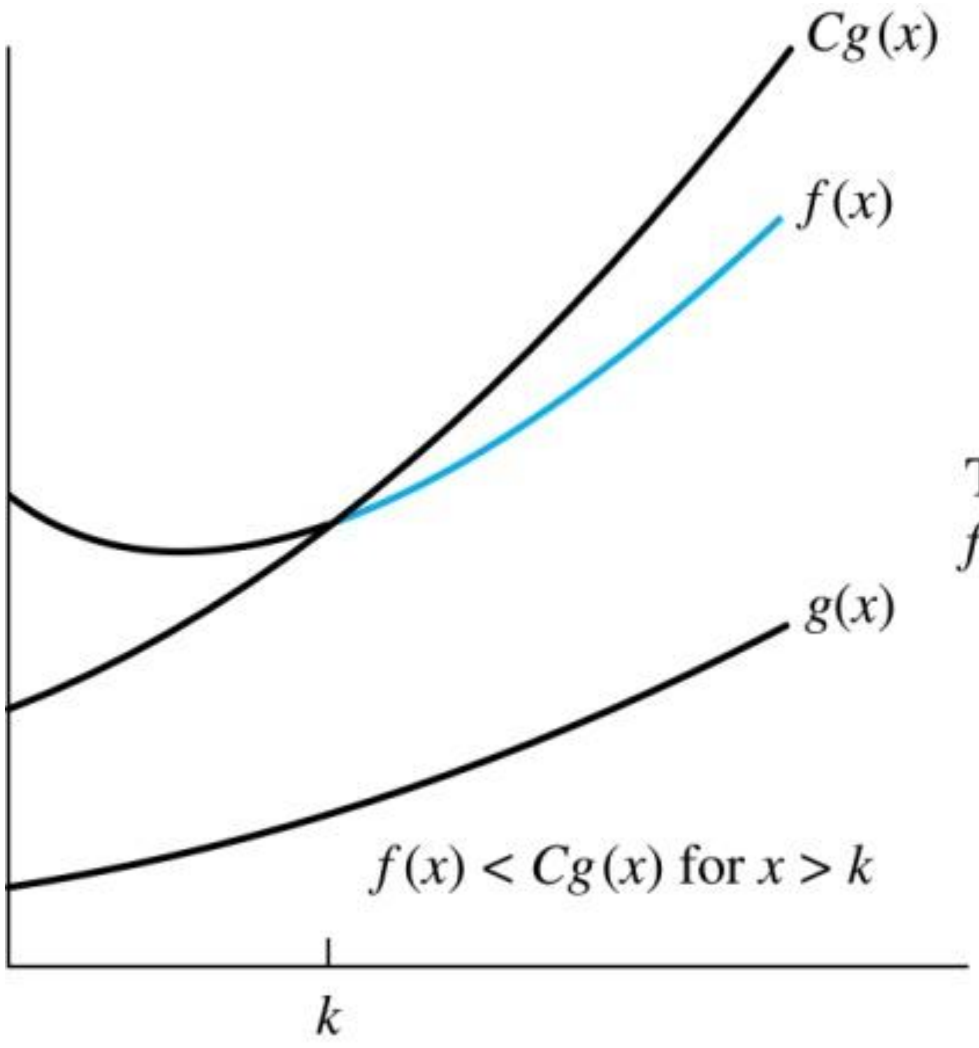
- Let *f* and *g* be functions from the set of real numbers to the set of real numbers. We say that *f*(*x*) is *O*(*g*(*x*)) if there are constants *C* and *k* such that

$$|f(x)| \leq C|g(x)|$$

whenever $x > k$. (illustration on next slide)

- This is read as "*f*(*x*) is big-*O* of *g*(*x*)" or "*g* asymptotically dominates *f*."
- The constants C and k are called *witnesses* to the relationship *f*(*x*) is *O*(*g*(*x*)). Only one pair of witnesses is needed.

Big-O Notation (2/3)



$f(x)$ is $O(g(x))$

The part of the graph of $f(x)$ that satisfies $f(x) < Cg(x)$ is shown in color.

Big-O Notation (3/3)

- If one pair of witnesses is found, then there are infinitely many pairs
  - We can always make the *k* or the *C* larger and still maintain the inequality $|f(x)| \leq C|g(x)|$
  - Any pair *C*' and *k*′ where *C* < *C*′ and *k* < *k*' is also a pair of witnesses since $|f(x)| \leq C|g(x) \leq C'|g(x)|$ whenever *x* > *k*′ > *k*.

- Usually, we will drop the absolute value sign since we will always deal with functions that take on positive values.

Using the Definition of Big-*O* Notation

Example: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$

Solution: Since when $x > 1$, $x < x^2$ and $1 < x^2$

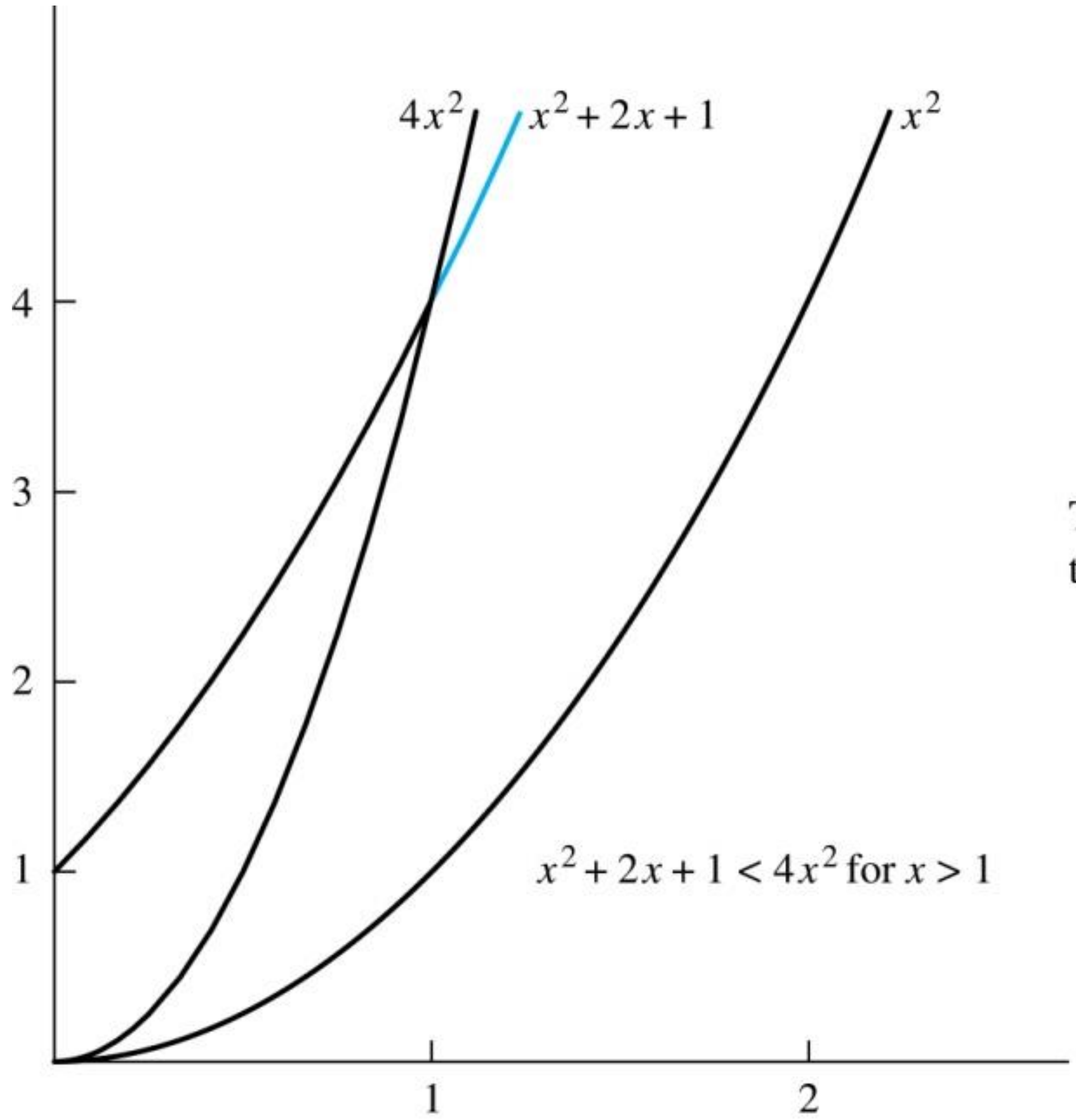
$$0 \leq x^2 + 2x + 1 \leq x^2 + 2x^2 + x^2 = 4x^2$$

- Can take $C = 4$ and $k = 1$ as witnesses to show that $f(x)$ is $O(x^2)$

- Alternatively, when $x > 2$, we have $2x \leq x^2$ and $1 < x^2$. Hence, when $x > 2$.
  - Can take $C = 3$ and $k = 2$ as witnesses instead.

$$0 \leq x^2 + 2x + 1 \leq x^2 + x^2 + x^2 = 3x^2$$

Illustration of Big-O Notation



$$f(x) = x^2 + 2x + 1$$

is $O(x^2)$

The part of the graph of $f(x) = x^2 + 2x + 1$ that satisfies $f(x) < 4x^2$ is shown in blue.

Big-*O* Notation

- When both $f(x) = x^2 + 2x + 1$ and $g(x) = x^2$ are such that $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$, two functions are of the same order

- If $f(x)$ is $O(g(x))$ and *h(x)* is larger than *g(x)* for all positive real numbers, $f(x)$ is $O\big(h(x)\big)$

- If $|f(x)| \le C|g(x)|$ for $k < x$ and if $|g(x)| < |h(x)|$ for all x, $|f(x)| \le C|h(x)|$ if $k < x$. Hence, $f(x)$ is $O(h(x))$

- For many applications, the goal is to select the function g(*x*) in *O(g(x))* as small (tight) as possible (up to multiplication by a constant, of course)

Using the Definition of Big-O Notation

- **Example**: Show that $7x^2$ is $O(x^3)$.
- **Solution**: When $x > 7$, $7x^2 < x^3$. Take $C = 1$ and $k = 7$ as witnesses to establish that $7x^2$ is $O(x^3)$

- **Example**: Show that n^2 is not $O(n)$
- **Solution**: Suppose there are constants C and k for which $n^2 \leq Cn$, whenever $n > k$. Then (by dividing both sides of $n^2 \leq Cn$) by n, then $n \leq C$ must hold for all $n > k$. A contradiction!

Big-*O* Estimates for Polynomials

Example: Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_o$ where $a_0, a_1, \ldots, a_n$ are real numbers with $a_n \neq 0$. Then $f(x)$ is $O(x^n)$.

Proof: $|f(\mathrm{x})| = |a_n x^n + a_{n-1}\, x^{n-1} + \cdots + a_1 x^1 \; + a_1|$

$$\leq |a_n| x^n + |a_{n-1}|\, x^{n-1} + \cdots + |a_1| x^1 + |a_1|$$

$$= x^n \,(|a_n| + |a_{n-1}| \,/x + \cdots + |a_1|/x^{n-1} + |a_1|/\, x^n)$$

$$\leq x^n \,(|a_n| + |a_{n-1}| + \cdots + |a_1| + |a_1|)$$

- Take $C = |a_n| + |a_{n-1}| + \cdots + |a_1| + |a_1|$ and $k = 1$. Then $f(x)$ is $O(x^n)$.
- The leading term $a_n x^n$ of a polynomial dominates its growth.

Big-Theta Notation

- **Definition**: Let *f* and *g* be functions from the set of integers (or real numbers) to the set of real numbers. The function if $f(x)\ is\ \Theta(g(x))$, $f(x)\ is\ O(g(x))$ and $g(x)\ is\ O(f(x))$ (i.e., $f(x)\ is\ \Omega(g(x))$)
  - We say that "f is big-Theta of *g*(*x*)" and also that "*f*(*x*) is of *order* *g*(*x*)" and also that "*f*(*x*) and *g*(*x*) are of the *same order*."

- $f(x)$ is $\Theta(g(x))$ if and only if there exists constants C_1, C_2 and *k* such that $C_1 g(x) < f(x) < C_2\, g(x)$ if $k < x$

Big Theta Notation

- **Example**: Show that the sum of the first n positive integers is $\Theta(n^2)$
- **Solution**: Let $f(n) = 1 + 2 + \cdots + n$
  - We have already shown that $f(n)$ is $O(n^2)$.
  - To show that $f(n)$ is $\Omega(n^2)$, we need a positive constant C such that $f(n) > Cn^2$ for sufficiently large n.
  - Summing only the terms greater than $n/2$ we obtain the inequality

$$\begin{aligned} 1 + 2 + \cdots + n &\geq \lceil n/2 \rceil + (\lceil n/2 \rceil + 1) + \cdots + n \\ &\geq \lceil n/2 \rceil + \lceil n/2 \rceil + \cdots + \lceil n/2 \rceil \\ &= (\mathrm{n} - \lceil n/2 \rceil + 1)\lceil n/2 \rceil \\ &\geq (\mathrm{n}/2)(\mathrm{n}/2) = n^2/4 \end{aligned}$$

  - Taking $C = ¼$, $f(n) > Cn^2$ for all positive integers n.
    Hence, $f(n)$ is $\Omega(n^2)$, and we can conclude that $f(n)$ is $\Theta(n^2)$

Big-Theta Notation

- **Example**: Show that $f(x) = 3x^2 + 8x\log x$ is $\Theta(x^2)$
- **Solution**:
  - $3x^2 + 8x \log x \le 11x^2$ for $x > 1$, since $0 \le 8x \log x \le 8x^2$,

    Hence, $3x^2 + 8x \log x$ is $O(x^2)$.
  - x^2 is clearly $O(3x^2 + 8x \log x)$, hence, $3x^2 + 8x \log x$ is $\Theta(x^2)$.

The Complexity of Algorithms (1/2)

- Given an algorithm, how efficient is this algorithm for solving a problem given input of a particular size?
  - Time complexity - How much time does this algorithm use to solve a problem?
  - Space complexity - How much computer memory does this algorithm use to solve a problem?

- To analyze the time complexity of algorithms, we determine the number of operations, such as comparisons and arithmetic operations (addition, multiplication, etc.)

- We will focus on the *worst-case time* complexity of an algorithm. This provides an upper bound on the number of operations an algorithm uses to solve a problem with input of a particular size.

The Complexity of Algorithms (2/2)

- We will measure time complexity in terms of the number of operations an algorithm uses and we will use big-*O* and big-Theta notation to estimate the time complexity.

- We can use this analysis to see whether it is practical to use this algorithm to solve problems with input of a particular size. We can also compare the efficiency of different algorithms for solving the same problem.

- We ignore implementation details (including the data structures used and both the hardware and software platforms) because it is extremely complicated to consider them.

Complexity Analysis of Algorithms

- **Example**: Describe the time complexity of the algorithm for finding the maximum element in a finite sequence.

```
procedure max(a_1, a_2, …., a_n: integers)
   max := a_1
   for i := 2 to n
      if max < a_i then max := a_i
   return max  {max is the largest element}
```

Solution: Count the number of comparisons.

- The $max < a_i$ comparison is made $n - 2$ times.
- Each time i is incremented, a test is made to see if $i \leq n$.
- One last comparison determines that $i > n$.
- Exactly $2(n - 1) + 1 = 2n - 1$ comparisons are made.

Hence, the time complexity of the algorithm is $\Theta(n)$.

Worst-Case Complexity of Linear Search

Ex. Determine the time complexity of the linear search algorithm.

```
procedure linear search(x : integer,  a_1, a_2, …,a_n: distinct integers)
i := 1
while (i ≤ n and x ≠ a_i)
    i := i + 1
if i ≤ n then location := i
else location := 0
return location
```

Solution: Count the number of comparisons.

- At each step two comparisons are made; $i \leq n$ and $x \neq a_i$.
- To end the loop, one comparison $i \leq n$ is made.
- After the loop, one more $i \leq n$ comparison is made.

If $x = a_i$, $2i + 1$ comparisons are used. If x is not on the list, $2n + 1$ comparisons are made and then an additional comparison is used to exit the loop. So, in the worst case $2n + 2$ comparisons are made. Hence, the complexity is $\Theta(n)$.