

Discrete Mathematics

# Predicate Logic

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# Motivation

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- Does this argument make sense?

Suppose that ...

All hummingbirds are richly colored.

No large birds live on honey.

Birds that do not live on honey are dull in color.

Then, hummingbirds are small.



# Predicate

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- A **predicate** is a propositional function over variables
  - once values are assigned to the predicate variables, a predicate becomes a proposition and has a truth value
  - the domain of a predicate variable is a set of values on which the property is asserted
  - $P(x)$  is said to be the value of the propositional function  $P$  at  $x$ .
- Example:
  - Let  $P(x)$  denote “ $x > 0$ ” and the domain be the integers.  
 $P(-3)$  is false.  $P(0)$  is false.  $P(3)$  is true.
  - Let  $Q(x, y)$  denote  $x = y + 3$ .  
 $Q(4, 1)$  is true and  $Q(2, 3)$  is false.

# Examples of Propositional Functions

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- Let “ $x + y = z$ ” be denoted by  $R(x, y, z)$  and  $U$  (for all three variables) be the integers. Find these truth values:

$R(2, -1, 5)$       **Solution: F**

$R(3, 4, 7)$       **Solution: T**

$R(x, 3, z)$       **Solution: Not a proposition**

- Now let “ $x - y = z$ ” be denoted by  $Q(x, y, z)$ , with  $U$  as the integers. Find these truth values:

$Q(2, -1, 3)$       **Solution: T**

$Q(3, 4, 7)$       **Solution: F**

$Q(x, 3, z)$       **Solution: Not a proposition**

# Compound Expressions

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- Connectives from propositional logic carry over to predicate logic.

- If  $P(x)$  denotes “ $x > 0$ ,” find these truth values:

$P(3) \vee P(-1)$       Solution: T

$P(3) \wedge P(-1)$       Solution: F

$P(3) \rightarrow P(-1)$       Solution: F

- Expressions with variables are not propositions and therefore do not have truth values. For example,

$P(3) \wedge P(y)$

$P(x) \rightarrow P(y)$

# Quantification (1/2)

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- We need *quantifiers* to express the meaning of English words including *all* and *some*:
  - “All men are Mortal.”
  - “Some cats do not have fur.”
- A **quantification** expresses the extent to which a predicate is true over a range of elements represented with a variable.
  - A variable is *bound* if a quantifier is used on the variable.
  - A variable is a free variable otherwise

Structure

$\langle \text{Quantifier} \rangle \langle \text{Variables w/ domain condition} \rangle ( \langle \text{Predicate} \rangle )$

# Universal Quantification

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- The universal quantification of  $P(x)$ , denoted as  $\forall x. P(x)$ , is the statement that  $P(x)$  holds for all values of  $x$  in the domain.
  - $\forall$  is called the universal quantifier
  - E.g.,  $\forall x \in \mathbb{R}. x^2 \geq 0$
- $\forall x P(x)$  is read as “For all  $x$ ,  $P(x)$ ” or “For every  $x$ ,  $P(x)$ ”
  - If  $P(x)$  denotes “ $x > 0$ ” and  $U$  is the integers,  $\forall x P(x)$  is false.
  - If  $P(x)$  denotes “ $x > 0$ ” and  $U$  is the positive integers,  $\forall x P(x)$  is true
  - If  $P(x)$  denotes “ $x$  is even” and  $U$  is the integers,  $\forall x P(x)$  is false.

# Existential Quantification

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- The existential quantification of  $P(x)$ , denoted as  $\exists x. P(x)$ , is the statement that  $P(x)$  holds for a value of  $x$  in the domain.
  - $\exists$  is called the existential quantifier
  - E.g.,  $\exists x \in \mathbb{R}. x^2 = 1$
- $\exists x P(x)$  is read as “For some  $x$ ,  $P(x)$ ”, or as “There is an  $x$  such that  $P(x)$ ,” or “For at least one  $x$ ,  $P(x)$ .”
  - If  $P(x)$  denotes “ $x > 0$ ” and  $U$  is the integers,  $\exists x P(x)$  is true  
It is also true if  $U$  is the positive integers
  - If  $P(x)$  denotes “ $x < 0$ ” and  $U$  is the positive integers,  $\exists x P(x)$  is false
  - If  $P(x)$  denotes “ $x$  is even” and  $U$  is the integers,  $\exists x P(x)$  is true



# Uniqueness Quantifier

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- $\exists!x P(x)$  means that  $P(x)$  is true for one and only one  $x$  in the universe of discourse.
- This is commonly expressed in English in the following equivalent ways:
  - “There is a unique  $x$  such that  $P(x)$ .”
  - “There is one and only one  $x$  such that  $P(x)$ ”
- Examples:
  1. If  $P(x)$  denotes “ $x + 1 = 0$ ” and  $U$  is the integers, then  $\exists!x P(x)$  is true.
  2. But if  $P(x)$  denotes “ $x > 0$ ,” then  $\exists!x P(x)$  is false.
- The uniqueness quantifier is not really needed as the restriction that there is a unique  $x$  such that  $P(x)$  can be expressed as:

$$\exists x (P(x) \wedge \forall y (P(y) \rightarrow y=x))$$

# Properties of Quantifiers

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- The truth value of  $\exists x P(x)$  and  $\forall x P(x)$  depend on both the propositional function  $P(x)$  and on the domain  $U$ .
- Examples
  1. If  $U$  is the positive integers and  $P(x)$  is the statement “ $x < 2$ ”,  $\exists x P(x)$  is true, and  $\forall x P(x)$  is false.
  2. If  $U$  is the negative integers and  $P(x)$  is the statement “ $x < 2$ ”, both  $\exists x P(x)$  and  $\forall x P(x)$  are true.
  3. If  $U$  consists of 3, 4, and 5, and  $P(x)$  is the statement “ $x > 2$ ”, both  $\exists x P(x)$  and  $\forall x P(x)$  are true.  
But if  $P(x)$  is the statement “ $x < 2$ ”, both  $\exists x P(x)$  and  $\forall x P(x)$  are false.

# Thinking about Quantifiers

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- When the domain of discourse is finite, we can think of quantification as looping through the elements of the domain.
- To evaluate  $\forall x P(x)$  loop through all  $x$  in the domain.
  - If at every step  $P(x)$  is true, then  $\forall x P(x)$  is true.
  - If at a step  $P(x)$  is false, then  $\forall x P(x)$  is false and the loop terminates.
- To evaluate  $\exists x P(x)$  loop through all  $x$  in the domain.
  - If at some step,  $P(x)$  is true, then  $\exists x P(x)$  is true and the loop terminates.
  - If the loop ends without finding an  $x$  for which  $P(x)$  is true, then  $\exists x P(x)$  is false.
- Even if the domains are infinite, we can still think of the quantifiers this fashion, but the loops will not terminate in some cases.

# Precedence of Quantifiers

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- The quantifiers  $\forall$  and  $\exists$  have higher precedence than all the logical operators.

- E.g.,  $\forall x P(x) \vee Q(x)$  means  $(\forall x P(x)) \vee Q(x)$   
 $\forall x (P(x) \vee Q(x))$  ?

# Translating from English to Predicate Logic (1/2)

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- Ex. “Every student in this class has taken a course in Java.”
- Solution
  - If  $U$  is all students in this class, define a propositional function  $J(x)$  denoting “ $x$  has taken a course in Java” and translate as  $\forall x J(x)$
  - If  $U$  is all people, define a propositional function  $S(x)$  denoting “ $x$  is a student in this class” and translate as  $\forall x (S(x) \rightarrow J(x))$

# Translating from English to Predicate Logic (2/2)

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- Let the domain  $U$  as all people
- Denote “ $x$  is a man” as  $Man(x)$  and “ $x$  is mortal” as  $Mortal(x)$
- Premises:  $\forall x (Man(x) \rightarrow Mortal(x))$   
 $Man(Socrates)$
- Conclusion:  $Mortal(Socrates)$

# Equivalences in Predicate Logic

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- Predicate statements are *logically equivalent* if and only if they have the same truth value
  - Denoted as  $S \equiv T$
  - E.g.,  $\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$
- De Morgan's Law
  - $\neg \forall x P(x) \equiv \exists x \neg P(x)$
  - $\neg \exists x P(x) \equiv \forall x \neg P(x)$
  - E.g.,
    - $\forall x J(x)$ : “Every student in your class has taken a course in Java.”
    - $\exists x J(x)$ : “There is a student in this class who has taken a course in Java.”

# Nested Quantifiers

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- Nested quantifiers have one quantifier within the scope of another quantifier
  - E.g.,  $\forall x(\exists y(x + y = 0))$
  - E.g., *every man has exactly one wife*
  - E.g., every real number except zero has a multiplication inverse
- Depending on quantification orders, the statements containing the same predicate may have different truth values
  - E.g.,  $\forall x(\exists y(x + y = 0))$ ,  $\exists y(\forall x(x + y = 0))$  for that  $x$  and  $y$  are integers



# Questions on Order of Quantifiers

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**Example 1:** Let  $U$  be the real numbers,

Define  $P(x, y)$  as  $x \times y = 0$

What is the truth value of the following:

- |                                  |                      |
|----------------------------------|----------------------|
| 1. $\forall x \forall y P(x, y)$ | <b>Answer:</b> False |
| 2. $\forall x \exists y P(x, y)$ | <b>Answer:</b> True  |
| 3. $\exists x \forall y P(x, y)$ | <b>Answer:</b> True  |
| 4. $\exists x \exists y P(x, y)$ | <b>Answer:</b> True  |

# Negating Nested Quantifiers

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- E.g.,  $\forall w \neg \forall a \exists f (P(w, f) \wedge Q(f, a)) \equiv \forall w \exists a \neg \exists f (P(w, f) \wedge Q(f, a))$   
 $\equiv \forall w \exists a \forall f \neg (P(w, f) \wedge Q(f, a))$   
 $\equiv \forall w \exists a \forall f (\neg P(w, f) \vee \neg Q(f, a)).$



# Exercises 1.5.33

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- Rewrite each of these statements so that negations appear only within predicates

(b)  $\neg \forall y \exists x P(x, y)$

(d)  $\neg (\exists x \exists y \neg P(x, y) \wedge \forall x \forall y Q(x, y))$

# Revisiting Motivating Example

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All hummingbirds are richly colored.

$$\forall x \in \text{Bird} (\text{Hummingbird}(x) \rightarrow \text{RichlyColored}(x))$$

No large birds live on honey.

$$\begin{aligned} &\neg \exists x \in \text{Bird} (\text{Large}(x) \wedge \text{Honey}(x)) \\ &\forall x \in \text{Bird} (\neg \text{Large}(x) \vee \neg \text{Honey}(x)) \end{aligned}$$

Birds that do not live on honey are dull in color.

$$\forall x \in \text{Bird} (\neg \text{Honey}(x) \rightarrow \neg \text{RichlyColored}(x))$$

Then, hummingbirds are small.

$$\forall x \in \text{Bird}.$$

$$(\text{Hummingbird}(x) \rightarrow \text{RichlyColored}(x)) \wedge$$

$$(\neg \text{Large}(x) \vee \neg \text{Honey}(x)) \wedge$$

$$(\neg \text{Honey} \rightarrow \neg \text{RichlyColored}(x))$$

$$(\text{Honey}(x) \rightarrow \neg \text{Large}(x))$$

$$(\text{RichlyColored}(x) \rightarrow \text{Honey}(x))$$

