Discrete Mathematics

# Rule of Inference

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How can we know an argument true?

# Proof and Inference



- An **argument** is a sequence of statements

- An argument is **valid** iff it is impossible that all preceding statements are true and a final statement is false at the same time
  - a conclusion follows the premises

- An **argument form** is a valid proposition whose structure is $P_1 \land P_2 \ldots \land P_n \rightarrow Q$ where $P_i$ and $Q$ are compound propositions
  - **Inference rule**

- An argument is valid if all initial statements are known to be true, and for every non-initial statement, there is an argument form that connects the preceding statements with it
  - Such a sequence of argument is called **proof**

$$\begin{array}{rl} & p \rightarrow q \\ & p \\ \hline \therefore & q \end{array}$$

| *Rule of Inference* | *Tautology* | *Name* |
|---|---|---|
| $p$<br>$p \to q$<br>$\therefore\ q$ | $(p \land (p \to q)) \to q$ | Modus ponens |
| $\neg q$<br>$p \to q$<br>$\therefore\ \neg p$ | $(\neg q \land (p \to q)) \to \neg p$ | Modus tollens |
| $p \to q$<br>$q \to r$<br>$\therefore\ p \to r$ | $((p \to q) \land (q \to r)) \to (p \to r)$ | Hypothetical syllogism |
| $p \lor q$<br>$\neg p$<br>$\therefore\ q$ | $((p \lor q) \land \neg p) \to q$ | Disjunctive syllogism |
| $p$<br>$\therefore\ p \lor q$ | $p \to (p \lor q)$ | Addition |
| $p \land q$<br>$\therefore\ p$ | $(p \land q) \to p$ | Simplification |
| $p$<br>$q$<br>$\therefore\ p \land q$ | $((p) \land (q)) \to (p \land q)$ | Conjunction |
| $p \lor q$<br>$\neg p \lor r$<br>$\therefore\ q \lor r$ | $((p \lor q) \land (\neg p \lor r)) \to (q \lor r)$ | Resolution |

# Rules of Inferences

premise$_1$
premise$_2$
…
------------------------
conclusion

# Intuitive Examples



| | |
|---|---|
| $p$<br>$p \to q$<br>$\therefore q$ | Modus ponens |
| $\neg q$<br>$p \to q$<br>$\therefore \neg p$ | Modus tollens |
| $p \to q$<br>$q \to r$<br>$\therefore p \to r$ | Hypothetical syllogism |
| $p \vee q$<br>$\neg p$<br>$\therefore q$ | Disjunctive syllogism |
| $p$<br>$\therefore p \vee q$ | Addition |
| $p \wedge q$<br>$\therefore p$ | Simplification |
| $p$<br>$q$<br>$\therefore p \wedge q$ | Conjunction |
| $p \vee q$<br>$\neg p \vee r$<br>$\therefore q \vee r$ | Resolution |

- If there is fire, fire alarm rings.
  There is fire.
  Thus, fire alarm rings
- Fire alarm rings if there's fire.
  There is no fire alarm.
  Thus, there is no fire.
- If one is a man, the one eventually dies.
  If one is a philosoper, the one is a man.
  Thus, a philosoper eventually dies.
- I will take a taxi tonight if it rains.
  Otherwise, I will take a bus tonight.
  Thereby, I will take a taxi or bus tonight.

# Quantified Statements



- Valid arguments for quantified statements are a sequence of statements where each statement is either a premise or follows from previous statements by rules of inference
  - rules of inference for propositional logic
  - rules of inference for quantified statements
    - Universal Instantiation (UI)
    - Universal Generalization (UG)
    - Existential Instantiation (EI)
    - Existential Generalization (EG)

# Universal Instantiation (UI)



$$\frac{\forall x P(x)}{\therefore P(c)}$$

- $c$ is a specific instance of the domain, or
- $c$ is a variable representing an arbitrary value of the domain

**Example**:

Our domain consists of all dogs and Bingo is a dog.

"All dogs are cuddly."

"Therefore,  Bingo is cuddly."
"Therefore, dog *d* is cuddly"

# Universal Generalization (UG)



$$\frac{P(c) \text{ for an arbitrary } c}{\therefore \forall x P(x)}$$

Used often implicitly in Mathematical Proofs.

# Existential Instantiation (EI)



$$\frac{\exists x P(x)}{\therefore P(c) \text{ for some element } c}$$

**Example**:

"There is someone who got an A in the course."
"Let's call her *a* and say that *a* got an A"

# Existential Generalization (EG)



$$\frac{P(c) \text{ for some element } c}{\therefore \exists x P(x)}$$

**Example**:

"Michelle got an A in the class."
"Therefore, someone got an A in the class."

# Using Rules of Inference



Construct a valid argument to show that
"John Smith has one wife" is a consequence of the premises:
"Every married man has one wife." "John Smith is a married man."

Solution: Let $M(x)$ denote "$x$ is a man", and $L(x)$ denote " $x$ has one wife", and let $J$ be an element representing John Smith.

| **Step** | **Reason** |
|---|---|
| 1. $\forall x(M(x) \to L(x))$ | Premise |
| 2. $M(J) \to L(J)$ | UI from (1) |
| 3. $M(J)$ | Premise |
| 4. $L(J)$ | Modus Ponens using (2) and (3) |

# Using Rules of Inference



- Construct a valid argument showing that the conclusion:
  "Someone who passed the first exam has not read the book." follows from
  - "A student in this class has not read the book."
  - "Everyone in this class passed the first exam."
- Solution: Let *C*(*x*) denote "*x* is in this class," *B*(*x*) denote " *x* has read the book," and *P*(*x*) denote "*x* passed the first exam."

$$\begin{array}{l} \exists x(C(x) \land \neg B(x)) \\ \forall x(C(x) \to P(x)) \\ \hline \therefore\ \exists x(P(x) \land \neg B(x)) \end{array}$$

| **Step** | **Reason** |
|---|---|
| 1. $\exists x(C(x) \land \neg B(x))$ | Premise |
| 2. $C(a) \land \neg B(a)$ | EI from (1) |
| 3. $C(a)$ | Simplification from (2) |
| 4. $\forall x(C(x) \to P(x))$ | Premise |
| 5. $C(a) \to P(a)$ | UI from (4) |
| 6. $P(a)$ | MP from (3) and (5) |
| 7. $\neg B(a)$ | Simplification from (2) |
| 8. $P(a) \land \neg B(a)$ | Conj from (6) and (7) |
| 9. $\exists x(P(x) \land \neg B(x))$ | EG from (8) |

# Proof



- A **theorem** is an important proposition that can be shown true
  - a theorem (or fact) is a proposition that is true
  - a lemma is a less important proposition that is true (usullay a part of a theorem)
  - a corollary is a theorem directly established from a main theorem
- A **proof** is a valid argument that establishes the truth of a theorem
  - a proof can include axioms (postulates) which are statement known, assumed, or believed to be true
  - a proof can include proven theorems
  - a proof can include premises
  - a proof include a conclusion from valid assertions by a valid inference rule
- A **conjecture** is a statement proposed to be true (yet) without a proof