Discrete Mathematics

Propositional Logic

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Textbook coverage

- Sec. 1.1 Propositional logic
- Sec. 1.3. Propositional equivalence
- Sec. 1.2 Applications of propositional logic

Logic

- Logic, or a logic system, is a set of rules to specify and derive a certain kind of statements
  - to achieve clarity and correctness in an argument

- A logic system has the syntactic and the semantic aspects
  - syntax: symbolic structure of the statements
  - semantics: a relation between symbolic structures and meaning

Propositional Logic

- A statement in the propositional logic consists one or multiple propositions connected with logical operators

- A proposition is a declarative sentence that is either true or false
  - $1 + 1 = 2$
  - *Vancouver is the capital of Canada*
  - ~~1 + 2 + 3~~
  - ~~x + 1 = 2~~

- A propositional variable is a symbol that represents a propositional statement
  - the value of a propositional variable is either true or false
  - the value is definitive within a statement

Propositional Logic

- An atomic proposition is a propostion that cannot be expressed in term of simpler terms

- A compound proposition is formed with other propositions and logical operators
  - logical operators (connectives): negation, disjunction, conjunction, XOR, implication, etc.
  - E.g., The negation of p for a proposition p, denoted as $\neg p$, is the proposition that is true only when p is false.

- Formal grammar

```
P  :=  A  |  C
A  :=  p  |  q  |  r  |  ...
C  :=  ¬P  |  (P)  |  P ∨ P  |  P ∧ P  |  ...
```

Evaluation

- A propositional statement with propositional variables may have different evaluations (truth values) depending on the values of each propositional variable
  - ex. $p \vee (q \wedge r)$

- An assignment (model or valuation) of a proposional statement is a combination of truth values of the propositional variables
  - e.g. , $\phi_1 = (p\text{: T},\ q\text{: T}, r\text{: T})$ or $[\![\phi_1]\!]_p = \mathrm{T}$, $[\![\phi_1]\!]_q = \mathrm{T}$, $[\![\phi_1]\!]_r = \mathrm{T}$
    $\phi_2 = (p\text{: T},\ q\text{: T}, r\text{: F})$ or $[\![\phi_2]\!]_p = \mathrm{T}$, $[\![\phi_2]\!]_q = \mathrm{T}$, $[\![\phi_2]\!]_r = \mathrm{F}$

Implication (Conditional Statement)

- An implication is a logical connective such that $p \rightarrow q$ evaluates to True when q is true if p is true
  - $p \rightarrow q$ is equivalent with $\neg p \vee q$
  - used to state a condition
    - examples
      - if you do not take midterm, you get F
      - if you are in the Handong campus, you are in Pohang
      - $x < y \rightarrow x < y + 1$
      - $(2 + 3 = 4) \rightarrow (1 + 2 = 4)$

- The converse of $p \rightarrow q$ is $q \rightarrow p$.
- The inverse of $p \rightarrow q$ is $\neg p \rightarrow \neg q$.
- The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$.

Equivalence

- The condition that two propositions p and q evaluate to the same can be expressed as $(p \to q) \land (q \to p)$, or simply $p \leftrightarrow q$
  - have the same truth value for every assignment
  - a statement $p \leftrightarrow q$ refers as p if and only if q (or simply p iff q)

Example

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

- De Morgan's law:

$$\neg(p \wedge q) \leftrightarrow \neg p \vee \neg q$$

$$\neg(p \vee q) \leftrightarrow \neg p \wedge \neg q$$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Propositional Satisfiability

- A proposition p is **satisfiable** if there exists an assignment that makes p true

- A proposition p is **unsatisfiable** if p is not satisfiable
  - A unsatisfiable proposition is called as *contradiction*

- A proposition p is **valid** if p is true for all assignments
  - A valid proposition is called as tautology
  - E.g., if x = y, then x = y
    I just want to live while I am alive - Bon Jovi

Logic Puzzle: Knight or Knaves

- An island has two kinds of inhabitants, *knights*, who always tell the truth, and *knaves*, who always lie.

- You go to the island and meet A and B.
  - A says "B is a knight."
  - B says "The two of us are of opposite types."

  What are the types of A and B?

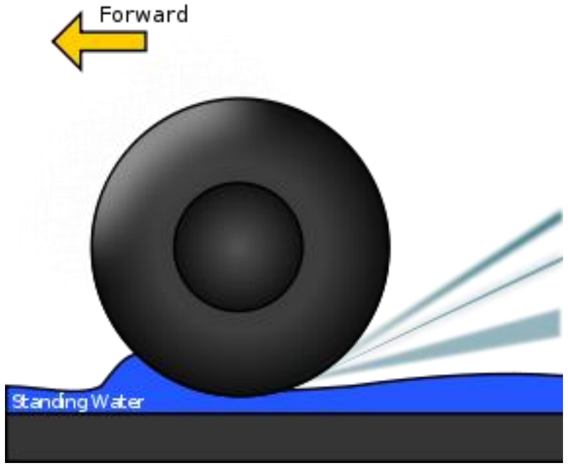
Logic Puzzle: Treasure

- There are 3 trunks only one of which contains a treasure.
- Trunk 1 and Trunk 2 are inscribed with "This trunk is empty" and Trunk 3 is inscribed with "Treasure is in Trunk 2".
- You know that only of the three inscription is true.
- Where's the treasure?

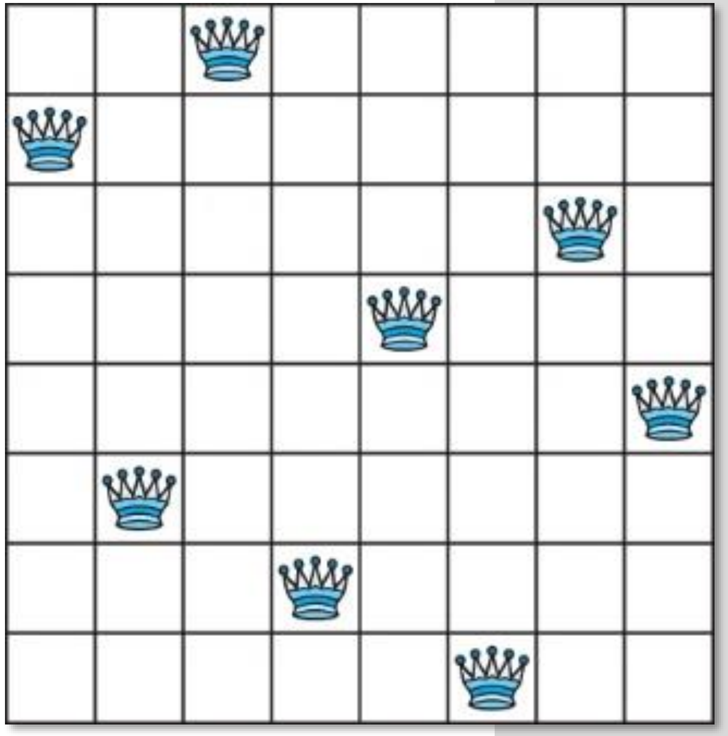
System Specification

- Logic-based languages (formal languages) are powerful tools for specifying and analyzing software requirements rigorously

- E.g., Lufthansa A320 Airbus accident at Warsaw in 1993
  - Specification: Turn on reverse thrust when the airplane is running on runway for landing
  - System Design (adopted)
    - Set REVERSE_THRUST as ON iff
      MODE = LANDING and ALTITUDE = 0
    - Set MODE as LAND iff
      VELOCITY > 0 and LANDING_GEAR_ANG > 0



Application: N-Queen Problem



- Problem
  - Place N Queens on a NxN grid, while not placing two Queens on the same vertical, horizontal or diagonal line
- Modeling
  - Proposition $p_{i,j}$ indicates whether a Queen is placed at the i-row and at the j-th column

$$Q_1 = \bigwedge_{i=1..n} \bigvee_{j=1..n} p_{i,j}$$

$$Q_2 = \bigwedge_{i=1..n} \bigwedge_{j=1..n-1} \bigwedge_{k=j+1..n} \neg(p_{i,j} \wedge p_{i,k})$$

$$Q_3 = \bigwedge_{i=1..n} \bigwedge_{j=1..n-1} \bigwedge_{k=j+1..n} \neg(p_{j,i} \wedge p_{k,i})$$

$$Q_4 = \bigwedge_{i=1..n} \bigwedge_{j=1..n} \bigwedge_{k=1..i+j-1} \neg(p_{i,j} \wedge p_{k,i+j-k})$$

$$Q_5 = \bigwedge_{i=1..n} \bigwedge_{j=1..n} \bigwedge_{k=1..n-i+j-1} \neg(p_{i,j} \wedge p_{k,n-i+j})$$

$$Q_1 \wedge Q_2 \wedge Q_3 \wedge Q_4 \wedge Q_5$$

Application: Sudoku Puzzle

- A Sudoku puzzle is represented as a 9x9 grid with nine 3x3 subgrids called subgrids
  - each cell has a number in 1 to 9
- The puzzle is solved by assigning a number to each cell so that every row, every column, and every of a block contains each of the 9 numbers.
- Modeling
  - $p(i, j, n)$ holds when row i and column j has n

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$$

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$$

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r + i, 3s + j, n)$$

$$\bigwedge_{i=1}^{9} \bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigwedge_{m=1}^{9} (p(i,j,n) \wedge n \neq m) \rightarrow \neg p(i,j,m)$$