Discrete Mathematics

Sequence

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Sequences

- A *sequence* of a set *S* is a function from the set of non-negative (or positive) integers to *S*
    - a_n, a term of the sequence, denotes the image of n.
    - ex. consider the sequence $\{a_n\}$ where $a_n = \frac{1}{n}$

$$\{a_n\} = \{a_1, a_2, a_3, \ldots\} \qquad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots$$

Strings

Definition: A *string* is a finite sequence from a finite set (an alphabet)

- Sequences of characters or bits are important in computer science
- The *empty string* is represented by λ.
- The string *abcde* has *length* 5.

Arithmetic Progression

Definition: A *arithmetic progression* is a sequence of the form:

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the *initial term a* and the *common difference d* are real numbers.

Examples:

1. Let $a = -1$ and $d = 4$:

$$\{s_n\} = \{s_0, s_1, s_2, s_3, s_4, \ldots\} = \{-1, 3, 7, 11, 15, \ldots\}$$

2. Let $a = 7$ and $d = -3$:

$$\{t_n\} = \{t_0, t_1, t_2, t_3, t_4, \ldots\} = \{7, 4, 1, -2, -5, \ldots\}$$

3. Let $a = 1$ and $d = 2$:

$$\{u_n\} = \{u_0, u_1, u_2, u_3, u_4, \ldots\} = \{1, 3, 5, 7, 9, \ldots\}$$

Geometric Progression

Definition. A *geometric progression* is a sequence of the form:

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

Examples:

1. Let $a = 1$ and $r = -1$. Then:

$$\{b_n\} = \{b_0, b_1, b_2, b_3, b_4, \ldots\} = \{1, -1, 1, -1, 1, \ldots\}$$

2. Let $a = 2$ and $r = 5$. Then:

$$\{c_n\} = \{c_0, c_1, c_2, c_3, c_4, \ldots\} = \{2, 10, 50, 250, 1250, \ldots\}$$

3. Let $a = 6$ and $r = 1/3$. Then:

$$\{d_n\} = \{d_0, d_1, d_2, d_3, d_4, \ldots\} = \{6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots\}$$

Recurrence Relations

- A *recurrence relation* for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms (e.g., a_0, a_1, a_{n-1}) for all non-negative integers n

- A sequence is called a *solution* of a recurrence relation if its terms satisfy the recurrence relation.

- The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

Fibonacci Sequence

Definition: Define the *Fibonacci sequence*, $f_0, f_1, f_2, \ldots$, by:
- Initial Conditions: $f_0 = 0, f_1 = 1$
- Recurrence Relation: $f_n = f_{n-1} + f_{n-2}$

Example: Find f_2, f_3, f_4, f_5 and f_6

Answer:

$$f_2 = f_1 + f_0 = 1 + 0 = 1,$$
$$f_3 = f_2 + f_1 = 1 + 1 = 2,$$
$$f_4 = f_3 + f_2 = 2 + 1 = 3,$$
$$f_5 = f_4 + f_3 = 3 + 2 = 5,$$
$$f_6 = f_5 + f_4 = 5 + 3 = 8$$

Questions about Recurrence Relations

- **Example.** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for $n = 1,2,3,4,\ldots.$ and suppose that $a_0 = 2$. What are a_1 , a_2 and a_3?

- **Solution**: We see from the recurrence relation that

$$a_1 = a_0 + 3 = 2 + 3 = 5$$
$$a_2 = 5 + 3 = 8$$
$$a_3 = 8 + 3 = 11$$

Solving Recurrence Relations

- Finding a formula for a *i*-th term of a sequence generated by a recurrence relation is called *solving the recurrence relation*
    - such a formula is called a *closed formula*.

- Various methods for solving recurrence relations will be covered in Chapter 8 where recurrence relations will be studied in greater depth.

Ex. Financial Application

- Suppose that a person deposits \$10,000 in a savings account at a bank yielding 11% per year with interest compounded annually

- How much will be in the account after 30 years?

  Let P_n denote the amount in the account after 30 years.
  P_n satisfies the following recurrence relation:

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\,P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Financial Application

$$P_n = P_{n-1} + 0.11P_{n-1} = (1.11)\, P_{n-1}$$

with the initial condition $P_0 = 10{,}000$

Solution: Forward Substitution

$P_1 = (1.11)P_0$

$P_2 = (1.11)P_1 = (1.11)^2P_0$

$P_3 = (1.11)P_2 = (1.11)^3P_0$

$\vdots$

$P_n = (1.11)P_{n-1} = (1.11)^n P_0 = (1.11)^n\, 10{,}000$

$P_n = (1.11)^n\, 10{,}000$ (Can prove by induction, covered in Chapter 5)

$P_{30} = (1.11)^{30}\, 10{,}000 = \$228{,}992.97$

Useful Sequences & Useful Summation Formulae

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...
f_n	1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Geometric Series: We just proved this

Later we will prove some of these by induction.

Proof in text (requires calculus)