Calculating Functions with Taylor Polynomials

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Contents

1 Trigonometric Functions

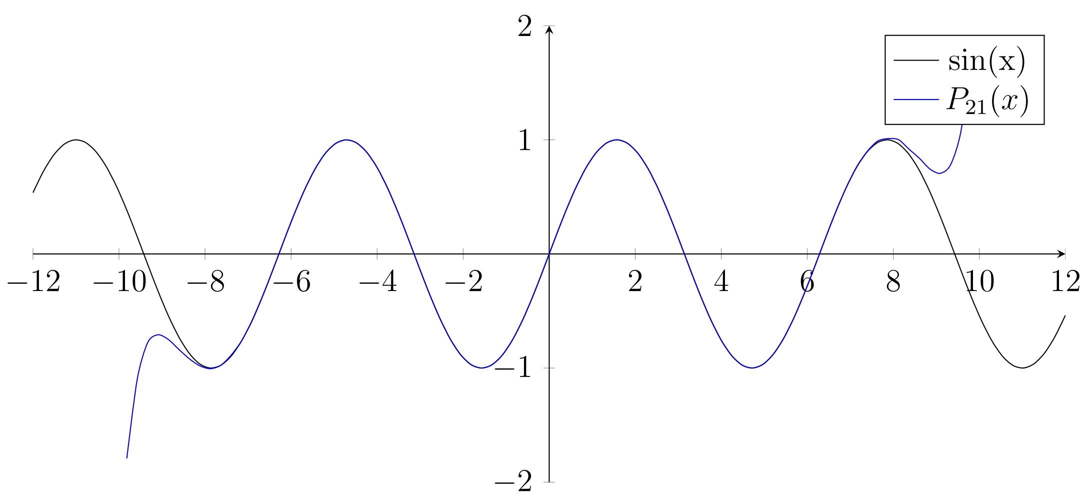
1.1 Sine and Cosine

We can use the Maclaurin Series for sine and cosine and calculate several terms of the series using a for loop. The series are shown below.

$$\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

With just 11 terms (21st degree Taylor polynomial), the sums above will be extremely accurate for small inputs. However, the margin of error increase quickly as we use larger inputs.



For better accuracy with larger inputs, we need to transform all inputs to be in the range $0 \leq x < 2\pi$. Because sine and cosine have a period of 2π, the below is true.

$$x_m = x \bmod 2\pi$$

$$\sin(x) = \sin(x_m)$$

$$\cos(x) = \cos(x_m)$$

Instead of calculating $\sin(x)$ and $\cos(x)$ directly, we can calculate $\sin(x_m)$ and $\cos(x_m)$.

1.2 Inverse Tangent

It is possible to construct an infinite series that converges to $\arctan(x)$ and use it to accurately calculate values of $\arctan(x)$. First take the derivative and then rewrite.

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$$

Using the formula for the sum of an infinite geometric series gives us the following.

$$\frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + ... + (-x^2)^n + ...$$

$$\begin{aligned}\arctan(x) &= \int \left(1 - x^2 + x^4 - x^6 + ... + (-x^2)^n + ...\right) dx \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + ... = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

Now, we can use a for loop to calculate $arctan(x)$ for values between 0 and 1. This method is very accurate with 25 or more terms.

1.3 Other Trig Functions

Since we can already calculate $sin(x)$ and $cos(x)$, it is easy to calculate other trig functions using basic trig identities.

$$\tan(x) = \frac{sin(x)}{cos(x)} \qquad \cot(x) = \frac{cos(x)}{sin(x)} \qquad \csc(x) = \frac{1}{sin(x)} \qquad \sec(x) = \frac{1}{cos(x)}$$

2 Exponential

2.1 Natural Exponential

The Maclaurin series for e^x is shown below. Using this formula with around 30 to 40 terms, it is simple to accurately calculate e^x for all $x > 0$.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

2.2 Other Bases

To calculate b^x, we can use exponent rules to rewrite it using base e.

$$b^x = c^{b\log_c(x)} \quad \Rightarrow \quad b^x = e^{b\ln(x)}$$

3 Logarithms

3.1 Natural Log

The Taylor series for $\ln(x)$ centered around $x = 0$ is shown below.

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

Because the range of convergence of this function is $0 < x \leq 2$, it cannot be used to calculate values outside of that range. In order to calculate any value greater than 0, we need a way to transform the inputs to be inside the interval of convergence. The below is true because of log rules.

$$\ln(ab) = \ln(a) + \ln(b)$$
$$\ln(a^b) = b\ln(a)$$

We can rewrite the ln function using this rule.

$$\ln(x) = \ln\left(\frac{2^b x}{2^b}\right) = \ln\left(\frac{x}{2^b}\right) - \ln\left(2^b\right) = \ln\left(\frac{x}{2^b}\right) - b\ln(2)$$

Where b is the smallest positive integer such that $2^b > x$. We can easily find b in code by starting with a variable equal 1, and doubling its value until it is greater than x. b is the number of times the loop was run.

3.2 Other Bases

Since we can calculate $ln(x)$ we can calculate logarithms for all bases using the change of base rule.

$$\log_b(x) = \frac{\log_d(x)}{\log_d(b)} \quad \Rightarrow \quad \log_b(x) = \frac{\ln(x)}{\ln(b)}$$