Solution 1 (a)

Step 1

The loss function is given as:

$$L(w) = w_1^2 + 2w_2^2 + w_3^2 - 2w_3w_4 + w_4^2 + 2w_1 - 4w_2 + 4.$$

Step 2

Calculate partial derivatives of the loss function $L(w)$ with respect to w_1, w_2, w_3, and w_4:

Calculate $\frac{dL}{dw_1}$

$$\frac{dL}{dw_1} = \frac{d}{dw_1}(w_1^2 + 2w_1) = 2w_1 + 2$$

Calculate $\frac{dL}{dw_2}$

$$\frac{dL}{dw_2} = \frac{d}{dw_2}(2w_2^2 - 4w_2) = 4w_2 - 4$$

Calculate $\frac{dL}{dw_3}$

$$\frac{dL}{dw_3} = \frac{d}{dw_3}(w_3^2 - 2w_3w_4) = 2w_3 - 2w_4$$

Calculate $\frac{dL}{dw_4}$

$$\frac{dL}{dw_4} = \frac{d}{dw_4}(-2w_3w_4 + w_4^2) = -2w_3 + 2w_4$$

$\therefore$ The partial derivatives are:

$$\frac{dL}{dw_1} = 2w_1 + 2$$
$$\frac{dL}{dw_2} = 4w_2 - 4$$
$$\frac{dL}{dw_3} = 2w_3 - 2w_4$$
$$\frac{dL}{dw_4} = -2w_3 + 2w_4$$

Solution 1 (b)

Step 1

The gradient $\nabla L(w)$ is defined as:

$$\nabla L(w) = \begin{pmatrix} \frac{dL}{dw_1} \\ \frac{dL}{dw_2} \\ \frac{dL}{dw_3} \\ \frac{dL}{dw_4} \end{pmatrix}$$

Step 2

Substitute values from **Solution 1(a)** into the equation above:

$$\nabla L(w) = \begin{pmatrix} \frac{dL}{dw_1} \\ \frac{dL}{dw_2} \\ \frac{dL}{dw_3} \\ \frac{dL}{dw_4} \end{pmatrix} = \begin{pmatrix} 2w_1 + 2 \\ 4w_2 - 4 \\ 2w_3 - 2w_4 \\ -2w_3 + 2w_4 \end{pmatrix}$$

$\therefore$ The gradient of the loss function is:

$$\nabla L(w) = \begin{pmatrix} 2w_1 + 2 \\ 4w_2 - 4 \\ 2w_3 - 2w_4 \\ -2w_3 + 2w_4 \end{pmatrix}$$

Solution 1 (c)

Step 1

To minimize $l(w)$ using gradient descent we have the following:

$$w_{t+1} = w_t - \eta \nabla L(w_t)$$

Step 2

Substitute given values into the equation above such that $w_1 = w_2 = w_3 = w_4 = 0$ and:

$$t = 0 \quad w_t = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \eta_t = 0.1 \quad \nabla L(w) = \begin{pmatrix} 2w_1 + 2 \\ 4w_2 - 4 \\ 2w_3 - 2w_4 \\ -2w_3 + 2w_4 \end{pmatrix}$$

$$w_{t+1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} - 0.1 \cdot \begin{pmatrix} 2(0) + 2 \\ 4(0) - 4 \\ 2(0) - 2(0) \\ -2(0) + 2(0) \end{pmatrix} = \begin{pmatrix} -0.1 \cdot 2 \\ -0.1 \cdot -4 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0.4 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore$ The next estimate after one step of gradient descent is:

$$w_{t+1} = \begin{pmatrix} -0.2 \\ 0.4 \\ 0 \\ 0 \end{pmatrix}$$

Solution 1 (d)

Step 1

To find the minimum value of $L(w)$, we first set $\nabla L(w) = 0$

$$\nabla L(w) = \begin{pmatrix} 2w_1 + 2 \\ 4w_2 - 4 \\ 2w_3 - 2w_4 \\ -2w_3 + 2w_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

This yeilds the following system of equations:

$$\begin{aligned} 2w_1 + 2 &= 0 \\ 4w_2 - 4 &= 0 \\ 2w_3 - 2w_4 &= 0 \\ -2w_3 + 2w_4 &= 0 \end{aligned}$$

Step 2

Solve for the critical points (w_1, w_2, w_3, w_4) using the system of equations above:

$$\begin{gathered} 2w_1 + 2 = 0 \to 2w_1 = -2 \to w_1 = -1 \\ 4w_2 - 4 = 0 \to 4_w2 = 4 \to w_2 = 1 \\ 2w_3 - 2w_4 = 0 \to 2w_3 = 2w_4 \to w_3 = w_4 \\ -2w_3 + 2w_4 = 0 \to -2w_3 = -2w_4 \to w_3 = w_4 \end{gathered}$$

It follows that there exists infinitely many critical points in the form:

$$w = \begin{pmatrix} -1 \\ 1 \\ c \\ c \end{pmatrix} \ni c \in \mathbb{R}$$

Step 3

Use Hessian matrix of $L(w)$ to determine if these critical points are minima. In this instance the Hessian matrix of $L(w)$ is defined as:

$$H = \frac{\partial^2 L}{\partial w_j \partial w_k} = \begin{pmatrix} \frac{\partial^2 L}{\partial w_1^2} & \frac{\partial^2 L}{\partial w_1 \partial w_2} & \frac{\partial^2 L}{\partial w_1 \partial w_3} & \frac{\partial^2 L}{\partial w_1 \partial w_4} \\ \frac{\partial^2 L}{\partial w_2 \partial w_1} & \frac{\partial^2 L}{\partial w_2^2} & \frac{\partial^2 L}{\partial w_2 \partial w_3} & \frac{\partial^2 L}{\partial w_2 \partial w_4} \\ \frac{\partial^2 L}{\partial w_3 \partial w_1} & \frac{\partial^2 L}{\partial w_3 \partial w_2} & \frac{\partial^2 L}{\partial w_3^2} & \frac{\partial^2 L}{\partial w_3 \partial w_4} \\ \frac{\partial^2 L}{\partial w_4 \partial w_1} & \frac{\partial^2 L}{\partial w_4 \partial w_2} & \frac{\partial^2 L}{\partial w_4 \partial w_3} & \frac{\partial^2 L}{\partial w_4^2} \end{pmatrix}$$

Calculate each element for the first row:

$$\frac{\partial^2 L}{\partial w_1^2} = \frac{\partial}{\partial w_1}(2w_1 + 2) = 2$$
$$\frac{\partial^2 L}{\partial w_1 \partial w_2} = \frac{\partial}{\partial w_1}(4w_2 - 4) = 0$$
$$\frac{\partial^2 L}{\partial w_1 \partial w_3} = \frac{\partial}{\partial w_1}(2w_3 - 2w_4) = 0$$
$$\frac{\partial^2 L}{\partial w_1 \partial w_4} = \frac{\partial}{\partial w_1}(-2w_3 + 2w_4) = 0$$

Calculate each element for the second row:

$$\frac{\partial^2 L}{\partial w_2 \partial w_1} = \frac{\partial}{\partial w_2}(2w_1 + 2) = 0$$
$$\frac{\partial^2 L}{\partial w_2^2} = \frac{\partial}{\partial w_2}(4w_2 - 4) = 4$$
$$\frac{\partial^2 L}{\partial w_2 \partial w_3} = \frac{\partial}{\partial w_2}(2w_3 - 2w_4) = 0$$
$$\frac{\partial^2 L}{\partial w_2 \partial w_4} = \frac{\partial}{\partial w_2}(-2w_3 + 2w_4) = 0$$

Calculate each element for the third row:

$$\frac{\partial^2 L}{\partial w_3 \partial w_1} = \frac{\partial}{\partial w_3}(2w_1 + 2) = 0$$
$$\frac{\partial^2 L}{\partial w_3 \partial w_2} = \frac{\partial}{\partial w_3}(4w_2 - 4) = 0$$
$$\frac{\partial^2 L}{\partial w_3^2} = \frac{\partial}{\partial w_3}(2w_3 - 2w_4) = 2$$
$$\frac{\partial^2 L}{\partial w_3 \partial w_4} = \frac{\partial}{\partial w_3}(-2w_3 + 2w_4) = -2$$

Calculate each element for the fourth row:

$$\frac{\partial^2 L}{\partial w_4 \partial w_1} = \frac{\partial}{\partial w_4}(2w_1 + 2) = 0$$
$$\frac{\partial^2 L}{\partial w_4 \partial w_2} = \frac{\partial}{\partial w_4}(4w_2 - 4) = 0$$
$$\frac{\partial^2 L}{\partial w_4 \partial w_3} = \frac{\partial}{\partial w_4}(2w_3 - 2w_4) = -2$$
$$\frac{\partial^2 L}{\partial w_4^2} = \frac{\partial}{\partial w_4}(-2w_3 + 2w_4) = 2$$

Substituting all the partial second derivatives calculated above yeilds the following Hessian matrix:

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & -2 & 2 \end{pmatrix}$$

Step 4

Apply $det(H - \lambda I) = 0$ and calculate eigenvalues of H.

$$\lambda_1 - 2 = 0 \rightarrow \lambda_1 = 2 \quad \text{and} \quad \lambda_2 - 4 = 0 \rightarrow \lambda_2 = 4$$

To solve for λ_3 and λ_4 we do the following:

$$det \begin{pmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 2 \end{pmatrix} = 0 \rightarrow (\lambda - 2)^2 - 4 = 0 \rightarrow \lambda(\lambda - 4) = 0$$

It follows that $\lambda_1 = 2$, $\lambda_2 = 4$, $\lambda_3 = 0$, and $\lambda_4 = 4$.

Hence, the critical points, $\begin{pmatrix} -1 \\ 1 \\ c \\ c \end{pmatrix}$ $\forall c \in \mathbb{R}$, are minima since the matrix is positive semi-definite.

Step 5

Calculate the minimum value of $L(w)$ by substituting $w = \begin{pmatrix} -1 \\ 1 \\ c \\ c \end{pmatrix}$ into the loss function:

$$\begin{aligned} L(w) &= w_1^2 + 2w_2^2 + w_3^2 - 2w_3w_4 + w_4^2 + 2w_1 - 4w_2 + 4 \\ &= (-1)^2 + 2(1)^2 + c^2 - 2cc + c^2 + 2(-1) - 4(1) + 4 \\ &= 1 + 2 + c^2 - 2c^2 + c^2 - 2 - 4 + 4 \\ &= 3 + 0c^2 - 2 \\ &= 1 \end{aligned}$$

$\therefore$ The minimum value of $L(w)$ is 1.

Solution 1 (e)

Step 1

From **Solution 1 (d)**, we found that the critical points of $L(w)$ are of the form:

$$w = \begin{pmatrix} -1 \\ 1 \\ c \\ c \end{pmatrix} \forall c \in \mathbb{R}$$

Since there are infinitely many real numbers and c can be any real number that yeilds the same minimum value of $L(w) = 1$, there is not a unique solution at which this minimum is realized.

$\therefore$ there is not a unique solution at which the minimum is realized. The minimum occurs at any point of the form:

$$w = \begin{pmatrix} -1 \\ 1 \\ c \\ c \end{pmatrix} \forall c \in \mathbb{R}$$

Solution 2 (a)

Step 1

Given:

$$L(w) = \sum_{i=1}^{n}(w \cdot x^{(i)}) + \frac{1}{2}c||w||^2$$

Step 2

We have the following identities:

$$w \cdot x^{(i)} = \sum_{j=1}^{d} w_j x_j^{(i)} \quad \text{and} \quad ||w||^2 = \sum_{j=1}^{d} w_j^2$$

Step 3

Substitute identies above into $L(w)$ and find partial derivative with respect to w_j:

$$L(w) = \sum_{i=1}^{n}(w \cdot x^{(i)}) + \frac{1}{2}c||w||^2 = \sum_{i=1}^{n}\left(\sum_{k=1}^{d} w_k x_k^{(i)}\right) + \frac{1}{2}c\sum_{k=1}^{d} w_k^2$$

$$\begin{aligned}
\frac{dL}{dw_j} &= \frac{d}{dw_j}\left[\sum_{i=1}^{n}\left(\sum_{k=1}^{d} w_k x_k^{(i)}\right) + \frac{1}{2}c\sum_{k=1}^{d} w_k^2\right] \\
&= \sum_{i=1}^{n}\frac{d}{dw_j}\left[w_j x_j^{(i)} + \sum_{k \neq j} w_k x_k^{(i)}\right] + \frac{1}{2}c\frac{d}{dw_j}\left[w_j^2 + \sum_{k \neq j} w_k^2\right] \\
&= \sum_{i=1}^{n} x_j^{(i)} + c w_j
\end{aligned}$$

$\therefore$ The partial derivative $\frac{dL}{dw_j}$ is:

$$\frac{dL}{dw_j} = \sum_{i=1}^{n} x_j^{(i)} + c w_j$$

Solution 2 (b)

Step 1

The gradient $\nabla L(w)$ is defined as:

$$\nabla L(w) = \begin{pmatrix} \frac{dL}{dw_1} \\ \frac{dL}{dw_2} \\ \vdots \\ \frac{dL}{dw_d} \end{pmatrix}$$

Step 2

Substitute result from **Solution 2 (a)**.

$$\nabla L(w) = \begin{pmatrix} \frac{dL}{dw_1} \\ \frac{dL}{dw_2} \\ \vdots \\ \frac{dL}{dw_d} \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_1^{(i)} + cw_1 \\ \sum_{i=1}^n x_2^{(i)} + cw_2 \\ \vdots \\ \sum_{i=1}^n x_d^{(i)} + cw_d \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n x_1^{(i)} \\ \sum_{i=1}^n x_2^{(i)} \\ \vdots \\ \sum_{i=1}^n x_d^{(i)} \end{pmatrix} + c \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} x_1^{(i)} \\ x_2^{(i)} \\ \vdots \\ x_d^{(i)} \end{pmatrix} + cw = \sum_{i=1}^{n} x^{(i)} + cw$$

$\therefore$ The gradient of the loss function is:

$$\nabla L(w) = \sum_{i=1}^{n} x^{(i)} + cw$$

Solution 2 (c)

Step 1

Set the gradient equal to zero and solve for w to find value that minimizes $L(w)$:

$$\nabla L(w) = \sum_{i=1}^{n} x^{(i)} + cw = 0$$

$$\begin{aligned}\sum_{i=1}^{n} x^{(i)} + cw = 0 \\ cw &= -\sum_{i=1}^{n} x^{(i)} \\ w &= -\frac{1}{c}\sum_{i=1}^{n} x^{(i)}\end{aligned}$$

Step 2

Check the Hessian matrix to verify the critical point w is a minimum. The Hessian of $L(w)$ is:

$$H = \begin{pmatrix} \frac{\partial^2 L}{\partial w_1^2} & \frac{\partial^2 L}{\partial w_1 \partial w_2} & \cdots & \frac{\partial^2 L}{\partial w_1 \partial w_d} \\ \frac{\partial^2 L}{\partial w_2 \partial w_1} & \frac{\partial^2 L}{\partial w_2^2} & \cdots & \frac{\partial^2 L}{\partial w_2 \partial w_d} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial w_d \partial w_1} & \frac{\partial^2 L}{\partial w_d \partial w_2} & \cdots & \frac{\partial^2 L}{\partial w_d^2} \end{pmatrix} = \begin{pmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{pmatrix}$$

Since $c > 0$, all eigenvalues of H are positive, which confirms that our critical point is indeed a minimum.

$\therefore$ The value of w that minimizes $L(w)$ is:

$$w = -\frac{1}{c}\sum_{i=1}^{n} x^{(i)}$$

Solution 3 (a)

Step 1

The ridge regression loss function is defined as:

$$L(w) = \sum_{i=1}^{n} (y^{(i)} - w \cdot x^{(i)})^2 + \lambda ||w||^2$$

where $(x^{(1)}, y^{(1)}), ..., (x^{(n)}, y^{(n)}) \in \mathbb{R}^d \times \mathbb{R}$ are the data points, $w \in \mathbb{R}^d$, and $\lambda > 0$ is the regularization parameter.

The gradient $\nabla L(w)$ is defined as:

$$\nabla L(w) = \begin{pmatrix} \frac{dL}{dw_1} \\ \frac{dL}{dw_2} \\ \vdots \\ \frac{dL}{dw_d} \end{pmatrix}$$

Step 2

Calculate partial derivative with resect to w_j:

$$\begin{aligned}
\frac{\partial L}{\partial w_j} &= \frac{\partial}{\partial w_j} \left[\sum_{i=1}^{n} (y^{(i)} - w \cdot x^{(i)})^2 + \lambda ||w||^2 \right] \\
&= \frac{\partial}{\partial w_j} \left[\sum_{i=1}^{n} (y^{(i)} - w \cdot x^{(i)})^2 \right] + \frac{\partial}{\partial w_j} \left[\lambda ||w||^2 \right] \\
&= \sum_{i=1}^{n} 2(y^{(i)} - w \cdot x^{(i)}) \frac{\partial}{\partial w_j} \left[y^{(i)} - w \cdot x^{(i)} \right] + \lambda \frac{\partial}{\partial w_j} \left[w_j^2 + \sum_{k \neq j} w_k^2 \right] \\
&= \sum_{i=1}^{n} 2(y^{(i)} - w \cdot x^{(i)})(-x_j^{(i)}) + 2\lambda w_j \\
&= -2 \sum_{i=1}^{n} (y^{(i)} - w \cdot x^{(i)}) x_j^{(i)} + 2\lambda w_j
\end{aligned}$$

Step 3

Substitute into the gradient $\nabla L(w)$ equation from **Step 1**:

$$\begin{aligned}
\nabla L(w) &= \begin{pmatrix} \frac{\partial L}{\partial w_1} \\ \frac{\partial L}{\partial w_2} \\ \vdots \\ \frac{\partial L}{\partial w_d} \end{pmatrix} \\
&= \begin{pmatrix} -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x_1^{(i)} + 2\lambda w_1 \\ -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x_2^{(i)} + 2\lambda w_2 \\ \vdots \\ -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x_d^{(i)} + 2\lambda w_d \end{pmatrix} \\
&= \begin{pmatrix} -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x_1^{(i)} \\ -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x_2^{(i)} \\ \vdots \\ -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x_d^{(i)} \end{pmatrix} + 2\lambda \cdot \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_d \end{pmatrix} \\
&= -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x^{(i)} + 2\lambda w
\end{aligned}$$

$\therefore$ The gradient of the ridge regression loss function is:

$$\nabla L(w) = -2\sum_{i=1}^{n}(y^{(i)} - w \cdot x^{(i)})x^{(i)} + 2\lambda w$$

Solution 3 (b)

Step 1

The update rule for gradient descent is defined as:

$$w_{t+1} = w_t - \eta \nabla L(w_t)$$

where $\eta > 0$ is the step size.

Step 2

Substitute $\nabla L(w)$ from **Solution 3(a)** into the equation above.

$$\begin{aligned}
w_{t+1} &= w_t - \eta \nabla L(w_t) \\
&= w_t - \eta \left(-2 \sum_{i=1}^{n} (y^{(i)} - w_t \cdot x^{(i)}) x^{(i)} + 2\lambda w_t \right) \\
&= w_t + 2\eta \sum_{i=1}^{n} (y^{(i)} - w_t \cdot x^{(i)}) x^{(i)} - 2\eta\lambda w_t \\
&= (1 - 2\eta\lambda) w_t + 2\eta \sum_{i=1}^{n} (y^{(i)} - w_t \cdot x^{(i)}) x^{(i)}
\end{aligned}$$

$\therefore$ The update step for gradient descent is:

$$w_{t+1} = (1 - 2\eta\lambda) w_t + 2\eta \sum_{i=1}^{n} (y^{(i)} - w_t \cdot x^{(i)}) x^{(i)}$$

Solution 3 (c)

In stochastic gradient descent (SGD), instead of using the full gradient, we use the gradient computed from a single randomly selected data point at each iteration.

For a single data point $(x^{(i)}, y^{(i)})$, the loss function is:

$$L_i(w) = (y^{(i)} - w \cdot x^{(i)})^2 + \frac{\lambda}{n}||w||^2$$

Note: the regularization term must be scaled by the number of itterations $\frac{1}{n}$ *since we don't want to add the full regularization term when considering a single point.*

It follows that gradient with respect to w for the single point loss function would be:

$$\nabla L_i(w) = -2(y^{(i)} - w \cdot x^{(i)})x^{(i)} + \frac{2\lambda}{n}w$$

$\therefore$ the stochastic gradient descent algorithm with a fixed step size η can be written as:

Stochastic Gradient Descent for Ridge Regression

1. Initialize $w_0 = 0$ (or randomly)
2. Set $t = 0$
3. While not converged:
   (a) Randomly select a data point index $i \in \{1, 2, \ldots, n\}$
   (b) Compute the gradient for the random point selected: $g_t = -2(y^{(i)} - w_t \cdot x^{(i)})x^{(i)} + \frac{2\lambda}{n}w_t$
   (c) Update: $w_{t+1} = w_t - \eta g_t$
   (d) $t = t + 1$
4. Return w_t

Solution 4 (a)

Rules to determine convexity of $f(x)$:

- If $f''(x) > 0$ for all x in the domain, then f is convex.
- If $f''(x) < 0$ for all x in the domain, then f is concave.
- If $f''(x) = 0$ for all x in the domain, then f is both convex and concave.
- If $f''(x)$ changes sign over the domain, then f is neither convex nor concave.

Let $f(x) = x^2$, then the first and second derivatives are:

$$f'(x) = 2x$$
$$f''(x) = 2$$

Hence, $f''(x) = 2 > 0 \; \forall \; x \in \mathbb{R}$.

$\therefore$ the function $f(x) = x^2$ is convex.

Solution 4 (b)

Rules to determine convexity of $f(x)$:

- If $f''(x) > 0$ for all x in the domain, then f is convex.
- If $f''(x) < 0$ for all x in the domain, then f is concave.
- If $f''(x) = 0$ for all x in the domain, then f is both convex and concave.
- If $f''(x)$ changes sign over the domain, then f is neither convex nor concave.

Let $f(x) = -x^2$, then the first and second derivatives are:

$$f'(x) = -2x$$
$$f''(x) = -2$$

Hence, $f''(x) = -2 < 0 \ \forall \ x \in \mathbb{R}$.

$\therefore$ the function $f(x) = -x^2$ is concave.

Solution 4 (c)

Rules to determine convexity of $f(x)$:

- If $f''(x) > 0$ for all x in the domain, then f is convex.
- If $f''(x) < 0$ for all x in the domain, then f is concave.
- If $f''(x) = 0$ for all x in the domain, then f is both convex and concave.
- If $f''(x)$ changes sign over the domain, then f is neither convex nor concave.

Let $f(x) = x^2 - 2x + 1$, then the first and second derivatives are:

$$f'(x) = 2x - 2$$
$$f''(x) = 2$$

Hence, $f''(x) = 2 > 0 \ \forall \ x \in \mathbb{R}$.

$\therefore$ the function $f(x) = x^2 - 2x + 1$ is convex.

Solution 4 (d)

Rules to determine convexity of $f(x)$:

- If $f''(x) > 0$ for all x in the domain, then f is convex.
- If $f''(x) < 0$ for all x in the domain, then f is concave.
- If $f''(x) = 0$ for all x in the domain, then f is both convex and concave.
- If $f''(x)$ changes sign over the domain, then f is neither convex nor concave.

Let $f(x) = x$, then the first and second derivatives are:

$$f'(x) = 1$$
$$f''(x) = 0$$

Hence, $f''(x) = 0 \; \forall \; x \in \mathbb{R}$.

$\therefore$ the function $f(x) = x$ is both convex and concave.

Solution 4 (e)

Rules to determine convexity of $f(x)$:

- If $f''(x) > 0$ for all x in the domain, then f is convex.
- If $f''(x) < 0$ for all x in the domain, then f is concave.
- If $f''(x) = 0$ for all x in the domain, then f is both convex and concave.
- If $f''(x)$ changes sign over the domain, then f is neither convex nor concave.

Let $f(x) = x^3$, then the first and second derivatives are:

$$f'(x) = 3x^2$$
$$f''(x) = 6x$$

Hence, $f''(x) = 6x$ which changes sign:

- When $x > 0 \rightarrow f''(x) > 0$
- When $x < 0 \rightarrow f''(x) < 0$
- When $x = 0 \rightarrow f''(x) = 0$

$\therefore$ the function $f(x) = x^3$ is neither convex nor concave.

Solution 4 (f)

Rules to determine convexity of $f(x)$:

- If $f''(x) > 0$ for all x in the domain, then f is convex.
- If $f''(x) < 0$ for all x in the domain, then f is concave.
- If $f''(x) = 0$ for all x in the domain, then f is both convex and concave.
- If $f''(x)$ changes sign over the domain, then f is neither convex nor concave.

Let $f(x) = x^4$, then the first and second derivatives are:

$$f'(x) = 4x^3$$
$$f''(x) = 12x^2$$

Hence, $f''(x) = 12x^2 \geq 0 \ \forall \ x \in \mathbb{R}$.

$\therefore$ the function $f(x) = x^4$ is convex.

Solution 4 (g)

Rules to determine convexity of $f(x)$:

- If $f''(x) > 0$ for all x in the domain, then f is convex.
- If $f''(x) < 0$ for all x in the domain, then f is concave.
- If $f''(x) = 0$ for all x in the domain, then f is both convex and concave.
- If $f''(x)$ changes sign over the domain, then f is neither convex nor concave.

Let $f(x) = \ln x$, then the first and second derivatives are:

Note that the domain of $f(x) = \ln x$ is $(0, \infty)$, i.e., $x > 0$.

$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2}$$

Hence, $f''(x) = -\frac{1}{x^2} < 0 \; \forall \; x$ in the domain.

$\therefore$ the function $f(x) = \ln x$ is concave.