Module 3 Solutions

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1. What is the sample space associated with:

(a) the outcome of tossing a coin three times;

Solution: Let H denote heads and T denote tails. Then the sample space is
$\{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

(b) the number of tails when tossing a coin four times;

Solution: $\{0, 1, 2, 3, 4\}$

2. Suppose that a fair 6-sided die is rolled. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. We then define the following events:

A : The number rolled is even;
B : The number rolled is less than or equal to 3.
C : The number rolled is a 1 or a 2.

Write the events A, B, C as sets, and find $P(B|A)$.

Solutions: $A : \{2, 4, 6\}$; $B : \{1, 2, 3\}$; $C : \{1, 2\}$.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P[\{2,4,6\} \cap \{1,2,3\}]}{P[\{2,4,6\}]} = \frac{P[\{2\}]}{P[\{2,4,6\}]} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

3. What is the probability of getting at least one head when you roll a coin 4 times? [Hint: Think of the complement event.]

Solution: P(no heads in 4 tosses) $= (\frac{1}{2})^4 = \frac{1}{16}$;
hence P(at least one head in 4 tosses) = 1 - P(no heads in 4 tosses) $= \frac{15}{16}$.

4. If events A and B are independent, then is it necessarily the case that $P(A|B) = P(B|A)$?

Solution: No. If A and B are independent, $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$;
where $P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$;
However, $P(A)$ is not necessarily the same as $P(B)$ when A and B are independent.

5. In front of you are 20 M&M's. Out of these 20, 4 are yellow, 5 are brown, 7 are red, and the rest are green.

If you were to pick one M&M at random, what is the probability that it

(a) is green?
(b) is not yellow?
(c) is red or brown?

Solution: (a) P(the M&M is green) $= \frac{4}{20} = \frac{1}{5}$.
(b) P(the M&M is not yellow) = 1 - P(the M&M is yellow) $= 1 - \frac{4}{20} = \frac{4}{5}$.
(c) P(the M&M is red or brown) = P(the M&M is red) + P(the M&M is brown) $= \frac{7}{20} + \frac{5}{20} = \frac{3}{5}$.

6. Let f be a function given by

$$f(x) = \begin{cases} 2x & \text{if} \quad 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Show that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.
(b) X is a random variable with f as its probability density function. What is $P(0.2 < X < 0.5)$?

Solution: (a)

$$\int_{-\infty}^{\infty} 2x\,dx = \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} 2x\,dx + \int_{1}^{\infty} 0\,dx = x^2\Big|_{x=0}^{1} = 1$$

(b)

$$P(0.2 < X < 0.5) = \int_{0.2}^{0.5} 2x\,dx = x^2\Big|_{x=0.2}^{0.5} = 0.21$$

7. Let X be a random variable drawn from a distribution with probability density function given by

$$f(x) = \begin{cases} \frac{3}{4} \text{ if } 0 \le x \le 1 \\ \frac{1}{4} \text{ if } 2 \le x \le 3 \\ 0 \text{ otherwise} \end{cases}$$

Find the expectation and variance of X.

Solution: We know that $\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2$

$$\begin{aligned} \mathbb{E}[X] &= \int_0^1 \frac{3}{4}x\,dx + \int_2^3 \frac{1}{4}x\,dx \\ &= \frac{3}{8}x^2\Big|_{x=0}^{1} + \frac{1}{8}x^2\Big|_{x=2}^{3} \\ &= \frac{3}{8} + \frac{5}{8} \\ &= 1 \end{aligned}$$

$$
\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 \frac{3}{4}x^2\,dx + \int_2^3 \frac{1}{4}x^2\,dx \\
&= \frac{1}{4}x^3\Big|_{x=0}^{1} + \frac{1}{12}x^3\Big|_{x=2}^{3} \\
&= \frac{1}{4} + \frac{19}{12} \\
&= \frac{11}{6}
\end{aligned}
$$

Then $\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{11}{6} - 1^2 = \frac{5}{6}$.

8. X is a random variable which has density function

$$
f(x) = \begin{cases} 2x \text{ if } 0 < x < 1 \\ 0 \text{ otherwise} \end{cases}
$$

(a) Compute the cumulative density function (CDF) of X.

(b) Compute $P(0.2 < X < 0.5)$ using CDF.

Solution: (a)

$$
F_X(x) = \int_0^x 2z\,dz = z^2\big|_0^x = x^2
$$

Hence

$$
F_X(x) = \begin{cases} 0 \text{ if } x < 0 \\ x^2 \text{ if } 0 < x < 1 \\ 1 \text{ if } x > 1 \end{cases}
$$

(b) $P(0.2 < X < 0.5) = F(0.5) - F(0.2) = 0.25^2 - 0.2^2 = 0.21$

9. You make a bet with a generous friend: If you roll a die and get a 5, you win \$200 and if you don't, you can roll a second time. If you get a 5 on the second roll, you win \$100. If not, then you lose. Find the expected amount you will win.

Solution: Let X be the amount of money you win. From the setup, we know that $P(X = 200) = \frac{1}{6}$. In order to get \$100, we need to get something other than 5 on the first roll while getting a 5 on the second roll. Hence $P(X = 100) = \frac{5}{6} \times \frac{1}{6} = \frac{5}{36}$. In all other scenarios, you win nothing. $P(X = 0) = 1 - \frac{1}{6} - \frac{5}{36} = \frac{25}{36}$. Hence the expected amount of money you win is $\mathbb{E}[X] = 0 \times P(X = 0) + 100 \times P(X = 100) + 200 \times P(X = 200) = 100 \times \frac{5}{36} + 200 \times \frac{1}{6} = 47.222$