Module 4

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1. Discuss whether the following is a valid assignment of probabilities to a discrete random variable which takes values 2, 4, and 6.

(a) $P(X = 2) = 0.1; P(X = 4) = 0.2; P(X = 6) = 0.4$

(b) $P(X = 2) = -0.1; P(X = 4) = 0.3; P(X = 6) = 0.8$

Solution: (a) This is not valid since the probabilities do not add up to 1.

(b) This is not valid since every individual probability should be in $[0, 1]$.

2. Discuss whether the following are discrete or continuous random variables.

(a) Temperature in La Jolla on a given day.

(b) Number of earthquakes in Taxes in a given year.

(c) The height of a senior in high school.

Solution: (a) This is measured on a continuous scale and doesn't need to be an integer. Therefore it is a continuous random variable.

(b) This will always be an integer, hence is a discrete random variable.

(c) Heights take values on a continuous scale (positive real numbers). Therefore height is a continuous random variable.

3. Jason takes a True or False exam with 20 questions in total. However, he didn't study for the exam at all, so he flips a fair coin to randomly answer each question.

a) What's the probability that he gets exactly 9 question right?

Solution: (a) Let X denote the number of questions he guessed correctly. Then $X \sim Bin(20, 0.5)$.

We are looking for the probability that he gets exactly 9 questions correctly, hence we calculate:

$P(X = 9) = \binom{20}{9} \times 0.5^9 \times 0.5^{20-9} = \binom{20}{9} \times 0.5^{20} = 0.1602$

b) What is the probability that Jason will get a grade of 90% or higher on the exam?

Solution: (b) For him to get a grade of 90% or greater, he needs to get at least $20 \times 0.9 = 18$ questions correctly on the exam. Hence we calculate the probability:

$P(X \geq 18) = P(X = 18) + P(X = 19) + P(X = 20) = \binom{20}{18} \times 0.5^{20} + \binom{20}{19} \times 0.5^{20} + \binom{20}{20} \times 0.5^{20} = 2.0122 \times 10^{-4}$

c) Jason is wondering whether it is likely for him to get 100% on this exam. Could you let him know by providing him with the probability?

Solution: (c) The probability that he gets all questions right is :

$P(X=20) = \binom{20}{20} \times 0.5^{20} = 0.9367 \times 10^{-7}$.

Thus we conclude that it is highly unlikely that he will get all questions correctly simply by guessing.

4. Given that X is a continuous random variable, discuss whether the following statements are true.

(a) $P(X=2)=0$

(b) The value of X must be nonnegative.

Solution: (a) True. If X is a continuous random variable, then the probability that it takes on any one particular value must be zero.

(b) False. A continuous random variable can take negative value.

5. Suppose Z has a standard normal distribution. We know that Z is symmetric. If given $P(Z>a)=0.3$, what is $P(Z>-a)$? Explain.

Solution: Since Z is symmetric and $P(Z>a)=0.3$, we know that $P(Z<-a)=0.3$. Hence $P(Z>-a) = 1-P(Z<-a)=0.7$

6. The percentage of international students in UCSD dorm rooms is around 23.2%. Incoming students are randomly assigned to dorm rooms, where there are 50 students typically on a floor. How many international students do you expect to find on a given floor? With what standard deviation?

Solution: Let X be the number of international students on a floor. Then X has binomial distribution with parameters $n=50; p=0.232$. Then the expected number of international students we expect to find is $\mathbb{E}[X] = np = 50 \times 0.232 = 11.6; SD(X) = \sqrt{np(1-p)} = \sqrt{50 \times 0.232 \times 0.768} = 2.985$

7 Suppose that in a factory, the percentage of defective light bulbs is about 2%. Can we use the Central Limit Theorem to approximate the probability of getting 30 or fewer defective bulbs in a batch of 1000 bulbs? If so, approximate the probability of getting 30 or fewer broken light bulbs.

Solution: Since $np = 20 \geq 10; n(1-p) = 980 \geq 10$, we can approximate $B(1000, 0.02)$ with $N(\mu,\sigma)$, $\mu = np = 20; \sigma = \sqrt{np(1-p)} = \sqrt{1000 \times 0.02 \times 0.98} = 4.427$. Let X be the number of broken light bulbs in the batch. Then

$$P(X<30) = P(\frac{X-20}{4.427} < \frac{30-20}{4.427}) = P(Z<2.259) = 0.988$$

8. If a certain test has its scores approximately normally distributed with a mean of 100 and a standard deviation of 15, what is the IQR of the test scores?

Solution: Since Z has a standard normal distribution, we know that

$$P(Z<0.6745) = 0.75; P(Z<-0.6745) = 0.25.$$

Therefore we have the lower and upper quartiles of the distribution. Then

$$\frac{Q_3-100}{15} = 0.6745; \frac{Q_1-100}{15} = -0.6745$$

Thus $Q_3 - Q_1 = (0.6745 \times 15 + 100) - (-0.6745 \times 15 + 100) = 20.235$.

9. Suppose that a random variable

$$W = \frac{\chi_3^2/3}{\chi_2^2/2}$$

where χ_n^2 is a chi-squared random variable with degrees of freedom n. What distribution does W has? With what degrees of freedom?

Solution: W follows an F distributions with degrees of freedom 3, 2: $F_{3,2}$.