MODULE 3 EXAMPLES

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PROBLEM #: KEY TOPICS FROM PROBLEM

Problem setup and description.

Question

Key notes from readings/lectures needed to answer the question

Solution: written with as much detail as we expect you to give on your homework sets

PROBLEM 1: SAMPLE SPACES

For this problem, consider a standard deck of 52 cards. (A deck of 52 cards has 4 suits: diamonds, hearts, spades, and clubs. There are 13 cards in each suit: Ace, Two–Ten, Jack, Queen, King.) **What is the sample space associated with:**

(a) drawing two cards and recording their sum (assume Ace has a value of 1, and Jack, Queen, King all have a value of 10)

Sample Space: The set of all possible outcomes associated with the event

Solution: {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20}

(b) the number of cards with the suit spades when drawing five cards

Solution: {0, 1, 2, 3, 4, 5}

PROBLEM 2: INDEPENDENT EVENTS

Suppose that a fair 6-sided die is rolled. The probability space is S = {1, 2, 3, 4, 5, 6}. We then define the following events:

A : The number rolled is odd.

B : The number rolled is greater than or equal to 4.

C : The number rolled doesn’t start with the letters “f” or “t”.

(a) Determine the events A, B, C in terms of a set.

Solution: A : {1, 3, 5}; B : {4, 5, 6}; C : {1, 6}.

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(b) Decide whether A and C are independent.

There are two methods we can use to show independence:

Method 1: $P(A \cap C) = P(A) * P(C)$

Method 2: $P(A|C) = \frac{P(A \cap C)}{P(C)} = P(A)$

Solution: Using what we found in part (a), we get

$$P(A) = \frac{\#outcomes\ in\ A}{\#total\ outcomes} = \frac{3}{6} = \frac{1}{2}, P(B) = \frac{1}{2}, P(C) = \frac{1}{3}$$

Let's plug these values in for both methods:

Method 1: $P(A \cap C) = P(\{1,3,5\} \cap \{1,6\}) = P(\{1\}) = \frac{1}{6}$

$P(A) * P(C) = \frac{1}{2} * \frac{1}{3} = \frac{1}{6} = P(A \cap C) \Rightarrow$ A and C are independent

Method 2: $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(\{1\})}{P(\{1,6\})} = \frac{1/6}{1/3} = \frac{1}{2} = P(A) \Rightarrow$ A and C are independent

PROBLEM 3: USING COMPLEMENTS TO FIND PROBABILITIES OF INTEREST

What is the probability of getting at most 5 tails when you toss a coin 6 times? [Hint: think of the complement event.]

Sometimes calculating the complement of an event is easier than calculating the event itself if the main event has many outcomes in its sample space.

Complement: the outcomes in the entire sample space that are not in the event under consideration.

Solution: Let's define the event we're interested in and its complement:

A: getting at most 5 tails in 6 coin tosses: {HHHHHH, THHHHH, HTHHHH, …, TTTTTH, …, HTTTTT}

A^C: getting all tails in 6 tosses: {TTTTTT}

$$P(A) = 1 - P(A^C) = 1 - \mathrm{P}(\text{all tails in 6 tosses}) = 1 - \left(\frac{1}{2}\right)^6 = 1 - \frac{1}{64} = \frac{63}{64}$$

PROBLEM 4: INDEPENDENCE AND CONDITIONAL PROBABILITY

We know that for two events A and B (not necessarily independent), the definition of conditional probability is $P(A|B) = \frac{P(A \cap B)}{P(B)}$

For independent events, we know $P(A \cap B) = P(A) * P(B)$

Thus when finding conditional probability of independent events, we have

$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = P(A)$, and

$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)P(B)}{P(A)} = P(B)$

Also consider if event A = event B, we have $P(A) = P(B)$, so consider the cases in which it holds that

$P(A \cap B) = P(A \cap A) = \mathrm{P(A)} \overset{indep.}{\Longleftrightarrow} \mathrm{P(A)} * \mathrm{P(A)}$

PROBLEM 5: PROBABILITIES FROM A POPULATION

According to the Stanford School of Medicine Blood Center, 44% of Americans have type O blood, 42% have type A blood, 10% have type B blood, and the rest are of type AB.

(a) If you were to chose one random American, what is the probability that their blood type is not A?

Solution: P(one American is not of blood type A) = 1 – P(one American IS of blood type A) = 1 – 0.42 = 0.58.

PROBLEM 5: PROBABILITIES FROM A POPULATION

According to the Stanford School of Medicine Blood Center, 44% of Americans have type O blood, 42% have type A blood, 10% have type B blood, and the rest are of type AB.

(b) If I were to randomly choose two potential donors, what is the probability that they both have type O blood?

Solution: P(both Americans are of type O) $= 0.44^2 = 0.1936$.

PROBLEM 5: PROBABILITIES FROM A POPULATION

According to the Stanford School of Medicine Blood Center, 44% of Americans have type O blood, 42% have type A blood, 10% have type B blood, and the rest are of type AB.

(c) What is the probability that one randomly chosen American is of blood type A or AB?

Solution: P(one American has blood type A or AB)

= P(they have type A) + P(they have type AB)

= 0.42 + [1 – (0.44 + 0.42 + 0.10)]

= 0.42 + 0.04 = 0.46

PROBLEM 6: PROBABILITY DENSITY FUNCTIONS FOR CONTINUOUS RANDOM VARIABLES

Let f be a function given by $f(y) = 4y^3$ when $0 \leq y \leq 1$, and $f(y) = 0$ otherwise.

(a) Show that $\int_{-\infty}^{\infty} f(y)dy = 1$

Solution: $\int_{-\infty}^{\infty} f(y)dy = \int_{-\infty}^{0} 0dy + \int_{0}^{1} 4y^3 dy + \int_{1}^{\infty} 0dy$

$$= y^4|_{y=0}^{1} = 1$$

(b) Consider a random variable Y whose density function is $f(y)$. What is $P\left(0 \leq Y \leq \frac{1}{2}\right)$?

Solution: $P\left(0 \leq Y \leq \frac{1}{2}\right) = \int_{0}^{1/2} 4y^3 dy = y^4|_{y=0}^{1/2} = \frac{1}{16}$

PROBLEM 7: EXPECTATION AND VARIANCE OF CONTINUOUS RANDOM VARIABLES

Suppose that a random variable X has the density function

$$f(x) = \begin{cases} 81x^{-4} \; if \; x \geq 3 \\ 0 \quad otherwise \end{cases}$$

(a) Find the expectation of X.

$$\mathbb{E}[X] = \int_{-\infty}^{\infty} xf(x)dx$$

Solution: $\mathbb{E}[X] = \int_{3}^{\infty} x(81x^{-4})dx$

$$= \int_{3}^{\infty} 81x^{-3}dx$$

$$= -40.5x^{-2}|_{x=3}^{\infty} = 4.5$$

(b) Find the variance of X.

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2$$

$$\mathbb{E}[X^2] = \int_{-\infty}^{\infty} x^2 f(x)dx$$

Solution: $\mathbb{E}[X^2] = \int_{3}^{\infty} x^2(81x^{-4})dx$

$$= \int_{3}^{\infty} 81x^{-2}dx$$

$$= -81x^{-1}|_{x=3}^{\infty} = 27$$

Thus $Var(X) = 27 - 4.5^2 = 6.75$

PROBLEM 8: CUMULATIVE DENSITY FUNCTIONS

Consider a random variable Y whose density function is $f_Y(y) = 4y^3$ when $0 \leq y \leq 1$.

(a) Find the CDF of Y.

CDF: $F_Y(y) = P(Y \leq y) = \int_{-\infty}^{y} f(z)dz$

Solution: $F_Y(y) = \int_0^y 4z^3 dz = z^4|_{z=0}^{y} = y^4$

(b) Compute $P\left(0 \leq Y \leq \frac{1}{2}\right)$.

Given a CDF of a random variable Y: $P(a \leq Y \leq b) = F_Y(b) - F_Y(a)$

Solution: $P\left(0 \leq Y \leq \frac{1}{2}\right) = F_Y\left(\frac{1}{2}\right) - F_Y(0) = \left(\frac{1}{2}\right)^4 - (0)^4 = \frac{1}{16}$

PROBLEM 9: EXPECTATION FOR DISCRETE RANDOM VARIABLES

Suppose that the probability that you receive 0 calls in a day is 0.1, 1 call is 0.3, 2 calls is 0.4, and 3 calls is 0.2.

(a) What is your expected number of calls received per day?

For discrete random variables: $\mathbb{E}[X] = \sum_{k \in all\ values\ of\ x} k p_k(x)$

Solution: $\mathbb{E}[X] = (0 * 0.1) + (1 * 0.3) + (2 * 0.4) + (3 * 0.2) = 1.7$

(b) What is the standard deviation of number of calls received per day?

$Var(X) = \mathbb{E}[(X - \mathbb{E}[X])^2] = \mathbb{E}[X^2] - \mathbb{E}[X]^2, \sigma_X = \sqrt{Var(X)}$

Solution: $Var(X) = (0 - 1.7)^2 * 0.1 + (1 - 1.7)^2 * 0.3 + (2 - 1.7)^2 * 0.4 + (3 - 1.7)^2 * 0.2 = 0.81$, thus standard deviation $= \sqrt{0.81} = 0.9$

PROBLEM 10: LINEARITY OF EXPECTATION, INDEPENDENT RANDOM VARIABLES

Suppose X and Y are independent random variables. We know that $\mathbb{E}[X] = 100,\ Var(X) = 144, \mathbb{E}[Y] = 150, Var(Y) = 169$.

What is the mean and standard deviation of:

(a) $\frac{X+Y}{2}$, **(b)** $X - 5$, **(c)** $\frac{1}{4}X - Y$

Linearity of expectation: For any two random variables (not necessariliy independent), we have:

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$$

For independent random variables, we have:

$$Var(X + Y) = Var(X) + Var(Y)$$

$$Var(aX + b) = a^2 Var(X)$$

(a) $\frac{X+Y}{2}$

Solution: $\mathbb{E}\left[\frac{X+Y}{2}\right] = \frac{1}{2}(\mathbb{E}[X] + \mathbb{E}[Y])$

$= \frac{1}{2}(100 + 150) = 125$

$Var\left(\frac{X+Y}{2}\right) = (\frac{1}{2})^2[Var(X) + Var(Y)]$

$= \frac{1}{4}(144 + 169) = \frac{313}{4}$

Thus standard deviation $= \sqrt{\frac{313}{4}}$

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$Var(X + Y) = Var(X) + Var(Y)$

$Var(aX + b) = a^2 Var(X)$

(b) X – 5

Solution: $\mathbb{E}[X - 5] = \mathbb{E}[X] - 5$

$= 100 - 5 = 95$

$Var(X - 5) = Var(X) = 144$

Thus standard deviation $= \sqrt{144} = 12$

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(c) $\frac{1}{4}X - Y$

Solution: $\mathbb{E}\left[\frac{1}{4}X - Y\right] = \frac{1}{4}\mathbb{E}[X] - \mathbb{E}[Y]$

$= \frac{1}{4}(100) - 150 = -125$

$Var\left(\frac{1}{4}X - Y\right) = (\frac{1}{4})^2 Var(X) + (-1)^2 Var(Y)$

$= \frac{1}{16}(144) + 169 = 178$

Thus standard deviation $= \sqrt{178}$