Solution 1

Solution 1 (a)

Define ℓ_1

The ℓ_1 or $||x||_1$ is defined as:

$$\ell_1 = ||x||_1 = \sum_{i=1}^{d} |x_i|$$

Compute ℓ_1

Let $x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$$\begin{aligned} ||x||_1 &= \sum_{i=1}^{3} |x_i| \\ &= |x_1| + |x_2| + |x_3| \\ &= |1| + |-2| + |3| \\ &= 1 + 2 + 3 \\ &= 6 \end{aligned}$$

$\therefore$ $||x||_1 = 6$

Solution 1

Solution 1 (b)

Define ℓ_2

The ℓ_2 or $||x||_2$ is defined as:

$$\ell_2 = ||x||_2 = \sqrt{\sum_{i=1}^{d} x_i^2}$$

Compute ℓ_2

Let $x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$$\begin{aligned} ||x||_2 &= \sqrt{\sum_{i=1}^{3} x_i^2} \\ &= \sqrt{x_1^2 + x_2^2 + x_3^2} \\ &= \sqrt{1^2 + (-2)^2 + 3^2} \\ &= \sqrt{1 + 4 + 9} \\ &= \sqrt{14} \end{aligned}$$

$\therefore ||x||_2 = \sqrt{14}$

Solution 1

Solution 1 (c)

Define ℓ_∞

The ℓ_∞ or $||x||_\infty$ is defined as:

$$\ell_\infty = ||x||_\infty = \max_i |x_i|$$

Compute ℓ_∞

Let $x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$

$$\begin{aligned} ||x||_\infty &= \max(\{|x_1|, |x_2|, |x_3|\}) \\ &= \max(\{|1|, |-2|, |3|\}) \\ &= \max(\{1, 2, 3\}) \\ &= 3 \end{aligned}$$

$\therefore ||x||_\infty = 3$

Solution 2

Solution 2 (a)

Define ℓ_2 distance

The ℓ_2 distance is defined as:

$$d(x, x')_{\ell_2} = \sqrt{\sum_{i=1}^{n} (x_i - x'_i)^2}$$

Compute ℓ_2 distance

Let $x = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $x' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{aligned} d(x, x')_{\ell_2} &= \sqrt{(x_1 - x'_1)^2 + (x_2 - x'_2)^2 + (x_3 - x'_3)^2 + (x_4 - x'_4)^2} \\ &= \sqrt{((-1) - 1)^2 + (1 - 1)^2 + ((-1) - 1)^2 + (1 - 1)^2} \\ &= \sqrt{(2)^2 + (0)^2 + (2)^2 + (0)^2} \\ &= \sqrt{4 + 0 + 4 + 0} \\ &= \sqrt{8} \end{aligned}$$

$\therefore d(x, x')_{\ell_2} = \sqrt{8}$

Solution 2

Solution 2 (b)

Define ℓ_1 distance

The ℓ_1 distance is defined as:

$$d(x, x')_{\ell_1} = \sum_{i=1}^{n} |x_i - x'_i|$$

Compute ℓ_1 distance

Let $x = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $x' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{aligned} d(x, x')_{\ell_1} &= |x_1 - x'_1| + |x_2 - x'_2| + |x_3 - x'_3| + |x_4 - x'_4| \\ &= |-1-1| + |1-1| + |-1-1| + |1-1| \\ &= |-2| + |0| + |-2| + |0| \\ &= 2 + 0 + 2 + 0 \\ &= 4 \end{aligned}$$

$\therefore d(x, x')_{\ell_1} = 4$

Solution 2

Solution 2 (c)

Define ℓ_∞ distance

The ℓ_∞ distance is defined as:

$$d(x, x')_{\ell_\infty} = \max_i |x_i - x'_i|$$

Compute ℓ_∞ distance

Let $x = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$ and $x' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{aligned} d(x, x')_{\ell_\infty} &= \max\{|x_1 - x'_1|, |x_2 - x'_2|, |x_3 - x'_3|, |x_4 - x'_4|\} \\ &= \max\{|-1-1|, |1-1|, |-1-1|, |1-1|\} \\ &= \max\{|-2|, |0|, |-2|, |0|\} \\ &= \max\{2, 0, 2, 0\} \\ &= 2 \end{aligned}$$

$\therefore d(x, x')_{\ell_\infty} = 2$

Solution 3

Solution 3 (a)

Step 1: Define Euclidean distance (ℓ_2)

$$\ell_2 = \|p - q\|_2 = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

Step 2: Compute ℓ_2

Let $p = 1$ and $q = 10$

$$\ell_2 = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$
$$\ell_2 = \sqrt{\sum_{i=1}^{1} (1 - 10)^2}$$
$$\ell_2 = \sqrt{(-9)^2}$$
$$\ell_2 = 9$$

$\therefore \ell_2 = 9$

Solution 2 (b)

Step 1: Define Euclidean distance (ℓ_2)

$$\ell_2 = \|p - q\|_2 = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$

Step 2: Compute ℓ_2

Let $p = \begin{bmatrix} -1 \\ 12 \end{bmatrix}, q = \begin{bmatrix} 6 \\ -12 \end{bmatrix}$

$$\ell_2 = \sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$$
$$\ell_2 = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$$
$$\ell_2 = \sqrt{(-1 - 6)^2 + (12 - (-12))^2}$$
$$\ell_2 = \sqrt{(-7)^2 + (24)^2}$$
$$\ell_2 = \sqrt{625}$$
$$\ell_2 = 25$$

$\therefore \ell_2 = 25$

Solution 2

Solution 2 (c)

Step 1: Define Euclidean distance (ℓ_2)

$$\ell_2 = \|p - q\|_2 = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2}$$

Step 2: Compute ℓ_2

Let $p = \begin{bmatrix} 1 \\ 5 \\ -1 \end{bmatrix}, q = \begin{bmatrix} 5 \\ 2 \\ 11 \end{bmatrix}$

$$\begin{aligned}
\ell_2 &= \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2} \\
\ell_2 &= \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2 + (p_3 - q_3)^2} \\
\ell_2 &= \sqrt{(1 - 5)^2 + (5 - 2)^2 + (-1 - 11)^2} \\
\ell_2 &= \sqrt{(-4)^2 + (3)^2 + (-12)^2} \\
\ell_2 &= \sqrt{169} \\
\ell_2 &= 13
\end{aligned}$$

$\therefore \ell_2 = 13$

Solution 3

Solution 3 (a)

Step 1: Normalize the vector x

Let $x = \begin{bmatrix} 10 \\ 15 \\ 25 \end{bmatrix}$

$$\sum_{i=1}^{3} x_i = x_1 + x_2 + x_3 = 10 + 15 + 25 = 50$$

Now, divide each entry by the total sum:

$$p = \frac{1}{50} \cdot x = \frac{1}{50} \begin{bmatrix} 10 \\ 15 \\ 25 \end{bmatrix} = \begin{bmatrix} 10/50 \\ 15/50 \\ 25/50 \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$$

$\therefore$ the result (p) of scaling vertor x is the following:

$$p = \begin{bmatrix} 0.2 \\ 0.3 \\ 0.5 \end{bmatrix}$$

Solution 3 (b)

Step 1: Define dimension of the probability simplex

The dimension of vector p is 3 and $k = n - 1$ where k is the dimension of the probability simplex

$\therefore$ vector p lies in the probability simplex(Δ_2) for $k = 2$

Solution 4

Step 1: Define probability simplex Δ_2

For a point to be scalable to Δ_2, after scaling it must satisfy:

- All components must be non-negative
- The sum of components must equal 1

Step 2: Give example that violates one of the rules in Step 1

Let $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

The second component of x violates the first rule, all components for a point must be non-negative Δ_2.

$\therefore$ the point $x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ cannot be scaled to lie in Δ_2

Solution 5

Visualizing the Simplex Δ_3 in 2D Projections

Here are the three 2D views of the probability simplex Δ_3. Each plot is a *shadow* of the 3D triangle, viewed along one of the principal axes.

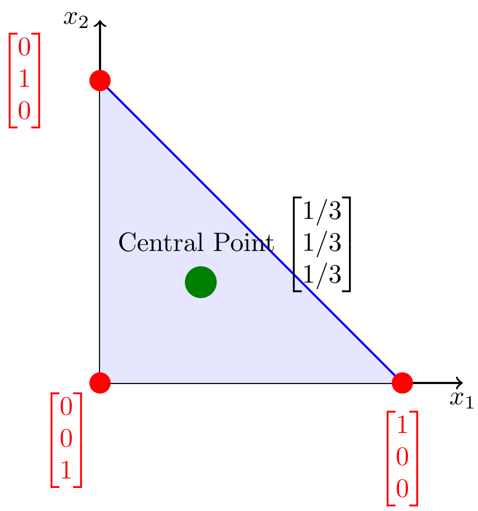

Figure 1: View 1: Projection onto the x_1-x_2 plane.

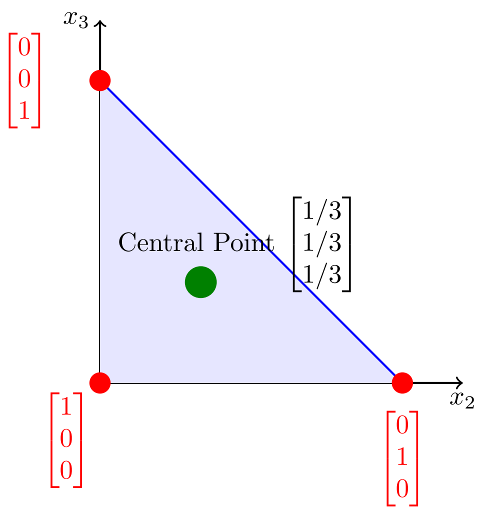

Figure 2: View 2: Projection onto the x_2-x_3 plane.

Solution 5

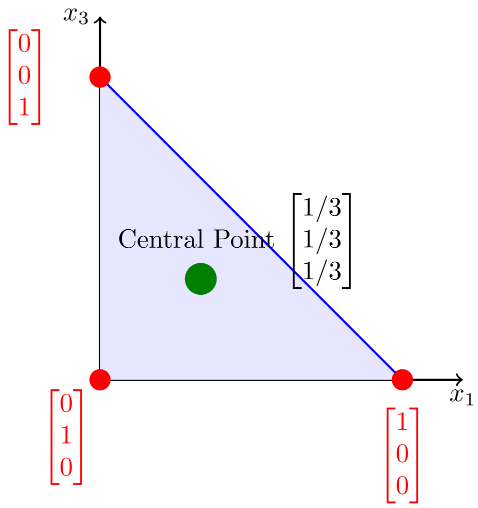

Figure 3: View 3: Projection onto the x_1-x_3 plane.

Solution 6

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6 (a): ℓ_1 for p and q

The ℓ_1 distance between two vectors $p, q \in \mathbb{R}^n$ is given by:

$$\|p - q\|_1 = \sum_{i=1}^{n} |p_i - q_i|$$

Let $p = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/8 \\ 1/8 \end{bmatrix}$ and $q = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

$$\begin{aligned}
\|p - q\|_1 &= \left|\frac{1}{2} - \frac{1}{4}\right| + \left|\frac{1}{4} - \frac{1}{4}\right| + \left|\frac{1}{8} - \frac{1}{4}\right| + \left|\frac{1}{8} - \frac{1}{4}\right| \\
&= \left|\frac{2}{4} - \frac{1}{4}\right| + |0| + \left|\frac{1}{8} - \frac{2}{8}\right| + \left|\frac{1}{8} - \frac{2}{8}\right| \\
&= \frac{1}{4} + 0 + \left|-\frac{1}{8}\right| + \left|-\frac{1}{8}\right| \\
&= \frac{1}{4} + \frac{1}{8} + \frac{1}{8} = \frac{2}{8} + \frac{1}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}
\end{aligned}$$

$\therefore \ell_1 = \frac{1}{2}$

Solution 6

---

6 (b): ℓ_1 for q and r

The ℓ_1 distance between two vectors $q, r \in \mathbb{R}^n$ is given by:

$$\|q-r\|_1 = \sum_{i=1}^{n} |q_i - r_i|$$

Let $q = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$ and $r = \begin{bmatrix} 1/2 \\ 0 \\ 1/4 \\ 1/4 \end{bmatrix}$

$$\begin{aligned}
\|q-r\|_1 &= \sum_{i=1}^{4} |q_i - r_i| \\
&= \left|\frac{1}{4} - \frac{1}{2}\right| + \left|\frac{1}{4} - 0\right| + \left|\frac{1}{4} - \frac{1}{4}\right| + \left|\frac{1}{4} - \frac{1}{4}\right| \\
&= \left|\frac{1}{4} - \frac{2}{4}\right| + \left|\frac{1}{4}\right| + |0| + |0| \\
&= \left|-\frac{1}{4}\right| + \frac{1}{4} + 0 + 0 \\
&= \frac{1}{4} + \frac{1}{4} \\
&= \frac{1}{2}
\end{aligned}$$

$\therefore \ell_1 = \frac{1}{2}$

Solution 6

---

6 (c): KL divergence $K(p, q)$

The Kullback-Leibler (KL) divergence from a distribution p to a distribution q is defined as:

$$K(p,q) = \sum_i p_i \ln \frac{p_i}{q_i}$$

Let $p = \begin{bmatrix} 1/2 \\ 1/4 \\ 1/8 \\ 1/8 \end{bmatrix}$ and $q = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$

$$\begin{aligned}
K(p,q) &= \sum_{i=1}^{4} p_i \ln\left(\frac{p_i}{q_i}\right) \\
&= p_1 \ln\left(\frac{p_1}{q_1}\right) + p_2 \ln\left(\frac{p_2}{q_2}\right) + p_3 \ln\left(\frac{p_3}{q_3}\right) + p_4 \ln\left(\frac{p_4}{q_4}\right) \\
&= \frac{1}{2} \ln\left(\frac{1/2}{1/4}\right) + \frac{1}{4} \ln\left(\frac{1/4}{1/4}\right) + \frac{1}{8} \ln\left(\frac{1/8}{1/4}\right) + \frac{1}{8} \ln\left(\frac{1/8}{1/4}\right) \\
&= \frac{1}{2} \ln(2) + \frac{1}{4} \ln(1) + \frac{1}{8} \ln\left(\frac{1}{2}\right) + \frac{1}{8} \ln\left(\frac{1}{2}\right) \\
&= \frac{1}{2} \ln(2) + \frac{1}{4}(0) - \frac{1}{8} \ln(2) - \frac{1}{8} \ln(2) \\
&= \frac{1}{2} \ln(2) - \frac{2}{8} \ln(2) \\
&= \left(\frac{1}{2} - \frac{1}{4}\right) \ln(2) \\
&= \frac{1}{4} \ln(2)
\end{aligned}$$

$\therefore K(p,q) = \frac{1}{4} \ln(2)$

Solution 6

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6 (d): KL divergence $K(q, r)$

The Kullback-Leibler (KL) divergence from a distribution q to a distribution r is defined as:

$$K(q, r) = \sum_i q_i \ln \frac{q_i}{r_i}$$

Let $q = \begin{bmatrix} 1/4 \\ 1/4 \\ 1/4 \\ 1/4 \end{bmatrix}$ and $r = \begin{bmatrix} 1/2 \\ 0 \\ 1/4 \\ 1/4 \end{bmatrix}$

Looking at the second component ($i = 2$). Here, $q_2 = \frac{1}{4} > 0$ while $r_2 = 0$. The corresponding term in the KL divergence sum, $q_2 \ln \left(\frac{q_2}{r_2}\right)$, involves division by zero.

Hence, the divergence will be infinite.

$\therefore K(q, r) = \infty$
