

From Forward SDE to Probability Flow ODE

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Derivation of the Probability Flow ODE

We start from a forward stochastic differential equation (SDE):

$$dx_t = f(x_t, t) dt + g(x_t, t) dW_t, \quad (1)$$

where $f(x_t, t)$ is the drift term, $g(x_t, t)$ is the diffusion term, and W_t denotes a standard Wiener process.

The probability density $p(x, t)$ associated with this process satisfies the **Fokker–Planck equation**:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} [f(x, t)p(x, t)] + \frac{1}{2} \frac{\partial^2}{\partial x^2} [g^2(x, t)p(x, t)]. \quad (2)$$

Step 1: Define a deterministic process

We now seek a deterministic process

$$dx_t = v(x_t, t) dt, \quad (3)$$

whose probability density $p(x, t)$ evolves according to the same Fokker–Planck equation (2). For a deterministic ODE, the density evolution is governed by the **continuity equation**:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x} [v(x, t)p(x, t)]. \quad (4)$$

Step 2: Match the two equations

To ensure that the deterministic process reproduces the same marginal distributions, we require the right-hand sides of (2) and (4) to be equal:

$$-\frac{\partial}{\partial x} (vp) = -\frac{\partial}{\partial x} (fp) + \frac{1}{2} \frac{\partial^2}{\partial x^2} (g^2 p). \quad (5)$$

Integrating both sides with respect to x (assuming zero boundary flux) gives:

$$vp = fp - \frac{1}{2} \frac{\partial}{\partial x} (g^2 p). \quad (6)$$

Step 3: Simplify the expression

Expanding the derivative term:

$$\frac{\partial}{\partial x} (g^2 p) = (\partial_x g^2) p + g^2 (\partial_x p), \quad (7)$$

we obtain:

$$vp = fp - \frac{1}{2} [(\partial_x g^2) p + g^2 (\partial_x p)]. \quad (8)$$

Dividing both sides by p yields:

$$v(x, t) = f(x, t) - \frac{1}{2} \frac{\partial}{\partial x} g^2(x, t) - \frac{1}{2} g^2(x, t) \frac{\partial_x p(x, t)}{p(x, t)}. \quad (9)$$

Using the identity $\frac{\partial_x p}{p} = \partial_x \log p$, we can rewrite this as:

$$v(x, t) = f(x, t) - \frac{1}{2} \frac{\partial}{\partial x} g^2(x, t) - \frac{1}{2} g^2(x, t) \frac{\partial}{\partial x} \log p(x, t). \quad (10)$$

Step 4: Final form of the Probability Flow ODE

Thus, the corresponding **Probability Flow ODE (PF-ODE)** is:

$$\boxed{dx_t = \left[f(x_t, t) - \frac{1}{2} \frac{\partial}{\partial x} g^2(x_t, t) - \frac{1}{2} g^2(x_t, t) \frac{\partial}{\partial x} \log p(x_t, t) \right] dt.} \quad (11)$$

Interpretation

The first two terms represent the mean drift correction, while the final term, which involves the score function $\nabla_x \log p_t(x)$, ensures that the deterministic process preserves the same marginal probability flow as the stochastic SDE.