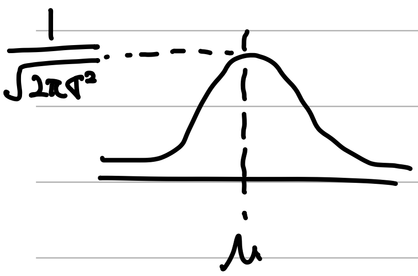
4 - 기말 이후

Gaussian Random Variable

- Gaussian (μ, σ) RV X

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} \qquad \mu\text{는 실수},\ \sigma > 0$$



$E[X] = \mu$, $Var[X] = \sigma^2$, 표준편차 $= \sigma$

Gaussian (μ, σ) RV X에 대해서 $Y = aX + b$이면,
Y는 Gaussian $(a\mu + b, a\sigma)$ RV. $Var[Y] = a^2\sigma^2$ ∴표준편차는 $a\sigma$

Standard Normal RV Z is Gaussian $(0,1)$

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

$$\text{CDF } F_Z(z) = P[Z \le z] = \Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du$$

- Gaussian (μ, σ) RV X 의 CDF

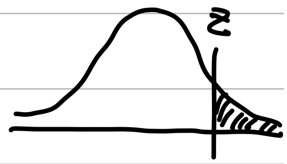
$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

$$P[a < X \le b] = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

- $\Phi(-z) = 1 - \Phi(z)$

- Standard Normal Complementary CDF

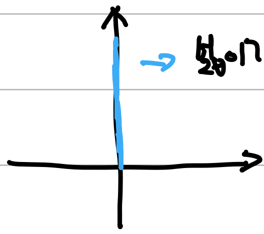
$$Q(z) = P[Z > z] = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} e^{-\frac{u^2}{2}} du = 1 - \Phi(z)$$



Delta function δ (unit impulse)

$$d_\epsilon(x) = \begin{cases} \frac{1}{\epsilon} & -\frac{\epsilon}{2} \le x \le \frac{\epsilon}{2} \\ 0 & \text{o.w.} \end{cases} \rightarrow \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d_\epsilon(x)\, dx = 1$$

$$\delta(x) = \lim_{\epsilon \to 0} d_\epsilon(x) = \begin{cases} \neq 0 \text{ or } \infty, & x = 0 \\ 0, & x \neq 0 \end{cases}$$



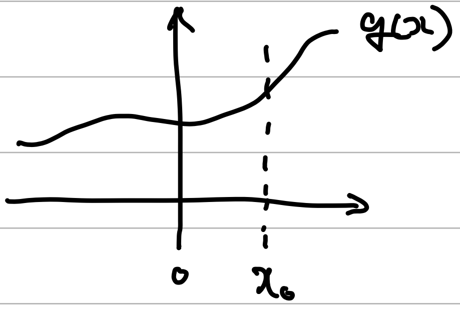
→ 넓이가 항상 1

$$\therefore \int_{-\infty}^{\infty} \delta(x) = 1$$

continuous $g(x)$ 에 대해서

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-x_0)\,dx = \int_{-\infty}^{\infty} g(x_0)\,\delta(x-x_0)\,dx = g(x_0)$$

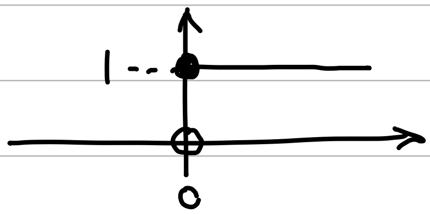
x가 x_0가 아니면 다 0이므로



$$\delta(x-x_0) = \begin{cases} \neq 0 \text{ or } \infty, & x = x_0 \\ 0, & x \neq x_0 \end{cases}$$

Unit Step function

$$u(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$$



$$\int_{-\infty}^{x} \delta(v)\,dv = u(x)$$

왜? x가 0보다 작으면 적분값 0

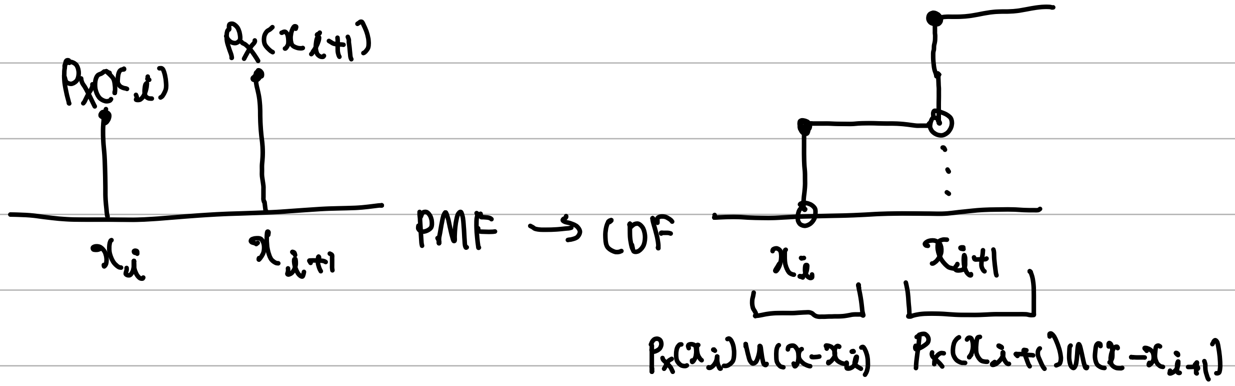
x가 0보다 크면 적분값 1

↓ 미분

$$\delta(x) = \frac{d\,u(x)}{dx}$$

Discrete RV with $\delta(x)$, $u(x)$

$$F_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, u(x - x_i)$$



$F_X(x)$를 미분 $\quad f_X(x) = \sum_{x_i \in S_X} P_X(x_i)\, \delta(x - x_i)$

ex) $\quad P_Y(y) = \begin{cases} \frac{1}{3} & y = 1, 2, 3 \\ 0 & \text{o.w.} \end{cases} \qquad F_Y(y) = \begin{cases} 0 & y < 1 \\ \frac{1}{3} & 1 \le y < 2 \\ \frac{2}{3} & 2 \le y < 3 \\ 1 & y \ge 3 \end{cases}$

$$F_Y(y) = \frac{1}{3}u(y-1) + \frac{1}{3}u(y-2) + \frac{1}{3}u(y-3)$$

$$\downarrow$$

$$f_Y(y) = \frac{1}{3}\delta(y-1) + \frac{1}{3}\delta(y-2) + \frac{1}{3}\delta(y-3)$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_{-\infty}^{\infty} y\left(\frac{1}{3}\delta(y-1) + \frac{1}{3}\delta(y-2) + \frac{1}{3}\delta(y-3)\right) dy$$

$$= \int_{-\infty}^{\infty} \frac{y}{3}\delta(y-1) + \frac{y}{3}\delta(y-2) + \frac{y}{3}\delta(y-3)\, dy = \frac{1}{3} + \frac{2}{3} + 1 = 2$$

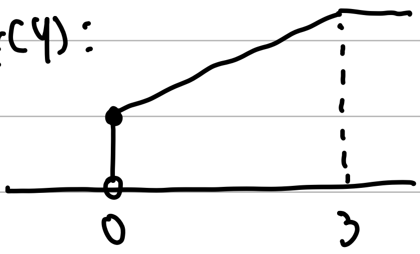
- $P[X = x_0] = q$
- $P_X(x_0) = q$
- $F_X(x_0^+) - F_X(x_0^-) = q$
- $f_X(x_0) = q \times \delta(x - x_0)$

Example 4.17

$P[Y=0] = P_Y(0) = \frac{1}{3}$ discrete

$P[0 < Y \le 3] = \frac{1}{3}$ uniform continuous

$F_Y(y)$:



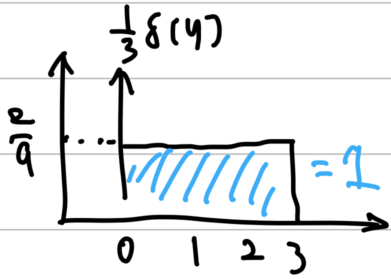
$= \left(\frac{1}{3}\right)(1) + \left(\frac{2}{3}\right)\left(\frac{y}{3}\right) = \frac{1}{3} + \frac{2}{9}y$

전화 안받을 확률 그때 $Y \le y$의 확률

$$F_Y(y) = \begin{cases} 0 & , y < 0 \\ \frac{1}{3} + \frac{2}{9}y & , 0 \le y < 3 \\ 1 & , y \ge 3 \end{cases}$$

↓ 미분

$$f_Y(y) = \begin{cases} \frac{1}{3}\delta(y) + \frac{2}{9} & , 0 \le y \le 3 \\ 0 & , \text{o.w.} \end{cases}$$



$$E[Y] = \underbrace{\int_{-\infty}^{\infty} y \cdot \frac{1}{3}\delta(y)\,dy}_{} + \int_0^3 y \cdot \frac{2}{9}\,dy = 0 + \frac{2}{9}\left[\frac{1}{2}y^2\right]_0^3 = \frac{2}{9} \cdot \frac{9}{2} = 1$$

$\frac{1}{3}y$에 $y=0$ 대입, 따라서 0.