6. Probability Models of Derived RV.

- for PMF

$W = g(X, Y)$ 이라면, $P_W(w) = \sum_{(x,y):\, g(x,y)=w} P_{X,Y}(x,y)$

- for PDF, CDF

if $W = aX \quad (a>0)$

- CDF $F_W(w) = F_X\left(\frac{w}{a}\right)$

$\rightarrow F_W(w) = P[W \le w] = P[aX \le w] = P[X \le \frac{w}{a}] = F_X\left(\frac{w}{a}\right)$

- PDF $f_W(w) = \frac{1}{a} f_X\left(\frac{w}{a}\right)$

$\rightarrow f_W(w) = \frac{dF_W(w)}{dw} = \frac{1}{a} f_X\left(\frac{w}{a}\right)$

Uniform RV

uniform (b, c) RV X

$$f_X(x) = \begin{cases} \frac{1}{c-b}, & b \le x < c \\ 0, & \text{o.w.} \end{cases}$$

$W = aX$ 라면,

$$f_W(w) = \frac{1}{a} f_X\left(\frac{w}{a}\right)$$

$$f_W(w) = \begin{cases} \frac{1}{a}\left(\frac{1}{c-b}\right) \rightarrow \frac{1}{ac-ab}, & b \le \frac{w}{a} < c \Rightarrow ab \le w < ac \\ 0, & \text{o.w.} \end{cases}$$

$\therefore$ $W = aX$ 일 때, W는 uniform (ab, ac) RV 이다.

Exponential RV

$W = aX$ 일 때, $f_W(w) = \frac{1}{a} f_X\left(\frac{w}{a}\right)$

$$f_W(w) = \begin{cases} \frac{1}{a} \lambda e^{-\lambda \frac{w}{a}} = \left(\frac{\lambda}{a}\right) e^{-\left(\frac{\lambda}{a}\right) w}, & \frac{w}{a} \ge 0 \\ 0, & w < 0 \end{cases}$$

Gaussian RV

$W = aX$ 일 때, $f_W(w) = \frac{1}{a} f_X\left(\frac{w}{a}\right)$

$$f_W(w) = \frac{1}{a} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{w}{a} - \mu\right)^2}{2\sigma^2}}$$

$$= \frac{1}{\sqrt{2\pi (a\sigma)^2}} e^{-\frac{(w - a\mu)^2}{2(a\sigma)^2}} \qquad (\mu, \sigma) \rightarrow (a\mu, a\sigma)$$

- $W = X + b$ 이면,

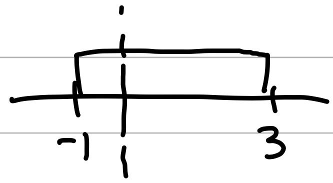
  $F_W(w) = F_X(w-b)$, $f_W(w) = F_X(w-b)$ 이다.

- Uniform $(-1, 3)$ RV X 에 대해서 $W = X^2$ 일 때, W의 CDF와 PDF

$$F_W(w) = P[W \le w] = P[X^2 \le w] = P[-\sqrt{w} \le X \le \sqrt{w}]$$

$$= \int_{-\sqrt{w}}^{\sqrt{w}} f_X(x)\,dx \qquad f_X(x) = \begin{cases} \frac{1}{4}, & -1 \le x \le 3 \\ 0, & \text{o.w.} \end{cases}$$

$\rightarrow 0 \le w \le 9$



$0 \le w \le 1$ 일 때, $F_W(w) = \int_{-\sqrt{w}}^{\sqrt{w}} \frac{1}{4}\,dx = \frac{\sqrt{w}}{2}$

$1 \le w \le 9$ 일 때, $F_W(w) = \int_{-1}^{\sqrt{w}} \frac{1}{4}\,dx = \frac{\sqrt{w}+1}{4}$

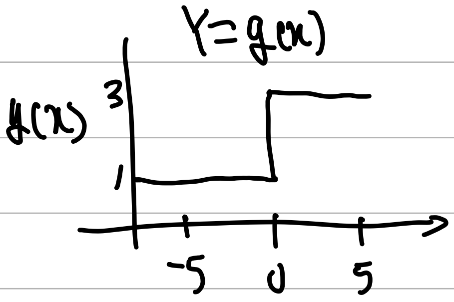
$$\therefore F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{\sqrt{w}}{2} & 0 \le w \le 1 \\ \frac{\sqrt{w}+1}{4} & 1 \le w \le 9 \\ 1 & 9 \le w \end{cases}$$

$\downarrow$

$$f_W(w) = \begin{cases} \frac{1}{4\sqrt{w}} & 0 \le w \le 1 \\ \frac{1}{8\sqrt{w}} & 1 \le w \le 9 \\ 0 & \text{o.w.} \end{cases}$$

Functions Yielding Mixed RV

[1]

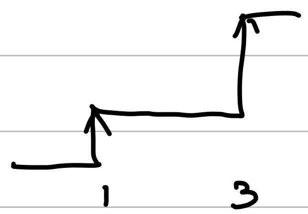


$$g(x) = \begin{cases} 1 & x \le 0 \\ 3 & x > 0 \end{cases}$$

$F_Y(y) = P[Y \le y]$

$y < 1$	$F_Y(y) = 0$
$3 \le y$	$F_Y(y) = 1$
$1 \le y < 3$	$F_Y(1) = P[Y \le 1] = P[X \le 0] = F_X(0)$

$$\therefore F_Y(y) = \begin{cases} 0 & y < 1 \\ F_X(0) & 1 \le y < 3 \\ 1 & 3 \le y \end{cases}$$



$\downarrow$

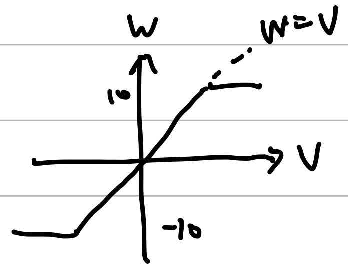
$$f_Y(y) = F_X(0)\,\delta(y-1) + (1 - F_X(0))\,\delta(y-3)$$

or

$$P_Y(y) = \begin{cases} F_X(0) & y = 1 \\ 1 - F_X(0) & y = 3 \\ 0 & \text{o.w.} \end{cases}$$

$\boxed{2}$ V는 가우시안 (0,5)

$$W = g(V) = \begin{cases} -10 & V<-10 \\ V & -10 \le V \le 10 \\ 10 & 10 < V \end{cases}$$



$F_W(w) = P[W \le w]$

$w<-10$: $F_W(w) = 0$

$10 \le w$: $F_W(w) = 1$

$-10 \le w < 10$: $F_W(w) = P[W \le w] = P[V \le w] = F_V(w)$ 이 구간에서 W=V 임.

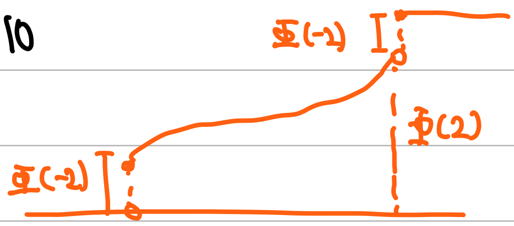
$$\therefore F_W(w) = \begin{cases} 0 & w<-10 \\ F_V(w) & -10 \le w < 10 \\ 1 & 10 \le w \end{cases} = \begin{cases} 0 & w<-10 \\ \Phi(\frac{w}{5}) & -10 \le w < 10 \\ 1 & 10 \le w \end{cases}$$

V가 가우시안 (0,5) 를 따르므로 $F_V(w) = \Phi(\frac{w-0}{5})$ 이다.

$$\therefore f_W(w) = \frac{1}{\sqrt{2\pi 5^2}} e^{-(w-0)^2/2\cdot 5^2}, \quad -10 \le w < 10$$

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} \Phi(-\frac{10}{5})\delta(w+10), & w=-10 \\ \frac{1}{\sqrt{2\pi 5^2}} e^{-(w-0)^2/2\cdot 5^2}, & -10 \le w < 10 \\ (1-\Phi(\frac{10}{5}))\delta(w-10), & w=10 \\ 0, & \text{o.w.} \end{cases}$$

// $\Phi(-2)$



$W = g(X,Y)$ continuous function

$$F_W(w) = P[W \le w] = \iint_{g(x,y)\le w} f_{X,Y}(x,y)\,dx\,dy$$

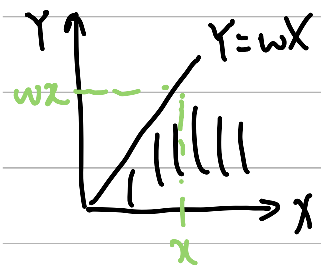
- $W = \max(X,Y)$ 라면

$\{W \le w\} = \{\max(X,Y) \le w\} = \{X \le w\} \cap \{Y \le w\}$

$$F_W(w) = F_{X,Y}(w,w) = \int_{-\infty}^{w}\int_{-\infty}^{w} f_{X,Y}(x,y)\,dx\,dy$$

- $W = Y/X$, $f_{X,Y}(x,y) = \begin{cases} \lambda\mu e^{-(\lambda x + \mu y)}, & x \ge 0,\ y \ge 0 \\ 0 & \text{o.w.} \end{cases}$

$F_W(w) = P[W \le w] = P[\frac{Y}{X} \le w] = P[Y \le wX]$



$$P[Y \le wX] = \int_0^\infty \int_0^{wx} f_{X,Y}(x,y)\,dy\,dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} \int_0^{wx} \mu e^{-\mu y}\,dy\,dx$$

$$= \int_0^\infty \lambda e^{-\lambda x}(1 - e^{-\mu w x})\,dx$$

$$= \int_0^\infty \lambda e^{-\lambda x} - \lambda e^{-(\lambda x + \mu w x)}\,dx$$

$$= \left[-e^{-\lambda x} + \frac{\lambda}{\lambda + \mu w} e^{-(\lambda + \mu w)x}\right]_0^\infty$$

$$= 0 - (-1) + 0 - \left(\frac{\lambda}{\lambda + \mu w}\right)$$

- $W = X + Y$

$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)\, dx = \int_{-\infty}^{\infty} f_{X,Y}(w-y, y)\, dy$$

$\downarrow$ independent

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)\, dx = \int_{-\infty}^{\infty} f_X(w-y) f_Y(y)\, dy$$