NM05

Newton's Method

- 반드시 미분가능해야함. initial guess 1개만 필요.

- tangent line 식 유도 시험

$y - f(x_0) = f'(x_0)(x - x_0)$ $y=0$, $x=x_1$ 을 지나게 만들기

$-f(x_0) = f'(x_0)(x_1 - x_0)$

↓ 정리

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \quad \cdots \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0$$

- Convergence of Newton's method

$$\lim_{n\to\infty} \frac{|e_{n+1}|}{|e_n|^k} = C$$

0이거나 양수면 수렴.

k : 얼마나 빨리 수렴하는지

C : 수렴하는지

- 유도

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$e_{n+1} = \alpha - x_{n+1} = \alpha - x_n - \frac{f(x_n)}{f'(x_n)}$$

x_n 근처에서 f를 테일러로 전개

$$f(\alpha) = f(x_n) + (\alpha - x_n) f'(x_n) + \frac{(\alpha - x_n)^2}{2} f''(\xi_n) \quad (\alpha < \xi_n < x_n)$$

알고있는 $f(\alpha) = 0$, $f'(\alpha) \neq 0$ 활용. (양변을 $f'(x_n)$로 나눈다)

$$0=\underbrace{\frac{f(x_n)}{f'(x_n)}+\alpha-x_n}_{e_{n+1}}+\frac{(\alpha-x_n)^2}{2f'(x_n)}f''(\xi_n)$$

$$e_{n+1}=-\frac{(\underbrace{\alpha-x_n}_{e_n})^2}{2f'(\xi_n)}f''(\xi_n)=-e_n^2\frac{f''(\xi_n)}{2f'(\xi_n)}$$

$$c=\left|\frac{f''(\xi_n)}{2f'(\xi_n)}\right|$$

$k=2$

- Convergence of secant method.

$$x_{n+1}=x_n-f(x_n)\left(\frac{x_n-x_{n-1}}{f(x_n)-f(x_{n-1})}\right)$$

$\vdots$

$$k=\lim_{n\to\infty}\frac{x_{n+1}}{x_n}=\frac{1+\sqrt{5}}{2}\approx 1.62$$

$\boxed{\text{Nonlinear system}}$

$$\begin{cases} f_1(x_1,x_2)=0 \\ f_2(x_1,x_2)=0 \\ \text{initial estimate } (x_1^{(0)},x_2^{(0)}) \end{cases}$$

tangent plane $y=f_1(x_1,x_2)$ at $(x_1^{(0)},x_2^{(0)})$

$\downarrow$

$$y-f_1(x_1^{(0)},x_2^{(0)})=\frac{\partial}{\partial x_1}\left(f_1(x_1^{(0)},x_2^{(0)})\right)(x_1-x_1^{(0)})+\frac{\partial}{\partial x_2}\left(f_1(x_1^{(0)},x_2^{(0)})\right)(x_2-x_2^{(0)})$$

tangent plane $y = f_2(x_1, x_2)$ at $(x_1^{(0)}, x_2^{(0)})$

$$\downarrow$$

$$y - f_2(x_1^{(0)}, x_2^{(0)}) = \frac{\partial}{\partial x_1}\left(f_2(x_1^{(0)}, x_2^{(0)})\right)(x_1 - x_1^{(0)}) + \frac{\partial}{\partial x_2}\left(f_2(x_1^{(0)}, x_2^{(0)})\right)(x_2 - x_2^{(0)})$$

두 식을 $y=0$ 일때 가우스 소거법으로 정리하면...

$\Delta x_1^{(0)} = x_1 - x_1^{(0)}$, $\Delta x_2^{(0)} = x_2 - x_2^{(0)}$

$\therefore x_1^{(1)} = x_1^{(0)} + \Delta x_1^{(0)}$, $x_2^{(1)} = x_2^{(0)} + \Delta x_2^{(0)}$

general form

부호 유의

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \Delta x_1^{(i)} \\ \vdots \\ \Delta x_n^{(i)} \end{bmatrix} = - \begin{bmatrix} f_1 \\ \vdots \\ f_2 \end{bmatrix}$$

$\uparrow$

jacobian matrix (미분의 재응집) non-singular 이어야함.

— distance metrics 수식 암기!

L_1-norm $\sum_{i=1}^{n} |p_i - q_i|$

L_2-norm $\sqrt{\sum_{i=1}^{n} (p_i - q_i)^2}$

L_∞-norm $\max_i |p_i - q_i|$

L_p-norm $\left(\sum_{i=1}^{n} |p_i - q_i|^p\right)^{\frac{1}{p}}$

시험

예제

$$f_1(x_1, x_2) = x_1^3 + 3x_2^2 - 21 = 0$$

$$f_2(x_1, x_2) = x_1^2 + 2x_2 + 2 = 0$$

$$J = \begin{bmatrix} 3x_1^2 & 6x_2 \\ 2x_1 & 2 \end{bmatrix}$$

(1, -1)에서 $\begin{bmatrix} 3 & -6 \\ 2 & 2 \end{bmatrix}$

$$\begin{bmatrix} 3 & -6 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} \Delta x_1^{(0)} \\ \Delta x_2^{(0)} \end{bmatrix} = -\begin{bmatrix} 1^3 + 3 - 21 = -17 \\ 1^2 - 2 + 2 = 1 \end{bmatrix}$$