Joyal 16/5 78.
Lawvere later.

Hardy Fields.

e.g. field $\mathbb{R}(X)$ of rat. functions.

A H-F. is a class of germs at ∞ of functions on $\mathbb{R}$. Assume

Field, i.e.

1) $f \in \mathbb{H}$, $f \sim g \Rightarrow g \in \mathbb{H}$

2) $f, g \in \mathbb{H} \Rightarrow f+g,\ f \cdot g \in \mathbb{H}$

3) $\forall f \in \mathbb{H}$, ~~f(x)~~ either $f \sim 0$ or $f > 0$ eventually
or $f < 0$ — " —

4) $\forall f \neq 0 \quad \exists g : g \cdot f = 1$

[(4) → (3) (?)]

5) $f \in \mathbb{H} \Rightarrow f' \in \mathbb{H}$.

so in particular $\mathbb{H}$ is a differential field.

The main Thm of Hardy is the following:

Given a Hardy field $\mathbb{H}$. Consider the set of all sol'ns of diff eq'ns of type

$$y' \cdot a + y \cdot b = c \qquad \text{with } a, b, c \in \mathbb{H}.$$

Take all solutions of ~~such~~ all such; this is again a Hardy field (containing $\mathbb{H}$), call it $\mathbb{H}^*$

In particular, sol'ns cannot oscillate. So define scale for asymptotic development; $a \in \mathbb{H} \Rightarrow \log a \in \mathbb{H}^*$.

Can iterate this process, and obtain something which is closed under ...

Can we do the algebraic theory of this? Start with a linearly ordered differential field. Adjoin roots. Define an order?

Not possible to extend to second order eq'ns, since $-y'' = y$ has oscillating functions sin, cos as solutions, but $y'' = y$.

So Problem Given lin diff eq'n of higher order $y^{(n)} a_n + y^{(n-1)} a_{n-1}$

Can associate an ordinary alg eq'n.

$$x^n a_n + \cdots$$

(a_n not constant coeff) ~~Since~~ Can compare the coefficients, using Sturm Theorem about number of real roots.

Conjecture: There should be a connection between no. of real roots of this eq'n, and the no. of sol'ns of the diff. eq'n in some linear extension of H.

Notion of linear-differential-real-closed-field

The theory of lin. eq'ns should be done independently of the theory of all diff eq'ns.

Fourman: would $\mathbb{R}^{\infty}_{\infty}$ be a model [for $M = \mathbb{R}$].

Joyal: not a field; and we use excluded middle. Maybe can be extended to local rings.

Lawvere Could ask about the H's: can one define $\mathbb{N}$ inside them. I had the idea that whether or not one can define $\mathbb{N}$ is an indication of nature of

Arndt & Jensen, advocating that a reasonable basis would be: continuum exists but NNO does not.

In connection with topos.

Topos + continuum implies NNO

What category? No Ω, but perhaps cartesian closed

An apparent barrier: Consider R = ring of lin endomorphisms of the line leaving a point fixed; can define ring structure (Gauss) on $R \times R = C$, and can single out the circle S^1 $(= SO(2))$. Using higher order constructions, can define (using cartesian closedness)

$$\mathrm{Hom}_{Ab}(S^1, S^1)$$

is automatically a ring. Usual idea: this is $\mathbb{Z}$. So the discrete infinity has been constructed. However, this $\mathbb{Z}$ cannot be proved (I think) to satisfy any kind of recursion property. Problem: Does it?

(I believe in the ring classifier, get Čebyshef polynomials)

But you should consider R as the given continuum.

Related to philosophical problem:

Does a cat. have NNO if it has such & such properties (hopefully: "no"). Properties line, plane, and integration, $\int$, thought of as an operation on functions.

Does then $\exists$ NNO,
or more specifically: can you deduce from $\int$ that
$y''+y = 0$ has a global solution
(which implies NNO [since sin has an $\mathbb{Z}$ for its zero set].) But not if you only have .. over bounded intervals. So Question:

$$\int \overset{?}{\Longrightarrow} \text{NNO} \qquad \text{hopefully not.}$$

The non-physical counterexamples is connected with the mental assumption that $\mathbb{N}$ exists, not any geometric assumption.

Wraith: There are lot of Nash structure on the circle

Lawvere ..:
$y''+y=0$ should not have global sol'ns.

Axiomatizing Euclidean geom, and its relation to quantities is by no means perfected or finished.

Integration should be taken as a primitive
Postulating an addition $\int$ on line type, having as an axiom,

$$\int_a^{a+d} f = f(a)\cdot d$$

Every set should have a measure.
This kind of general thesis should be tested, but not using NNO.

Reyes: in theory of real closed fields cannot define NNO!
Lawvere: Everything ok if you don't ask for $\int$.

Differentiation should not float by itself, but in opposition to integration.

Tangent vectors should be opposed to cotangent

Dubuc's models should have Λ^1 in it, as Chen's topos.

But unlike the Chen topos ..

One would like

Taking $X = 1$ forces $\Lambda^1 = 1$.

$$\mathrm{Hom}_R(X^D, R) \cong (\Lambda^1)^X$$

or at least for some generating class of objects X.

To make sense, one would have to compute

$$(\Lambda^1)^D \doteq R$$

From gros topos to small topos, in particular, what is $sh(\Lambda^1)$ viewed as a small topos vis a vis $\mathcal{E}/_{\Lambda^1}$ ($\mathcal{E}$ = Chen's topos).

Comparison map?

$$sh(\Lambda^1) \longrightarrow \mathcal{E}/\Lambda^1$$

Also $\mathrm{Hom}_{R^+}(X^D, R)$ (positively homogeneous) play a role in calculus of variations. Is there a

$$\Lambda_1^+ \cong \mathrm{Hom}_{R^+}(X^D, R)$$

Lagrangian function

$$X \xrightarrow{L} \Lambda_+ \qquad L(x, \dot{x})$$

Given manifold X, can construct category $\mathbb{X}$ (Moore category) morphisms generated by path, subject to relation that if you divide a path in two it should be the composite of its parts

Look at functors $\mathbb{X} \xrightarrow{F} (R, +)$ (= additive path functionals)

Problem: to represent ("Riesz"), i.e. write $F(\alpha) \doteq \int_0^{\text{duration } \alpha} \varphi(\alpha(t))$

[for suitable φ, unique]. Real problem: what kind of thing should φ be: In particular $F = \int \varphi(\alpha(t), \dot{\alpha}(t))\, dt$

Sol'ns of diff eq'ns are not global; hence they form a category not a monoid (under composition).

or maybe on even more variables. $\varphi(\alpha, \dot\alpha, \ddot\alpha, \ldots)$

Conjecture would be, that if you define $\dot{X}$ = germs of paths $\dot{X} = \lim_{\varepsilon \to 0} \left(X^{(-\varepsilon,\varepsilon)} \right)$, then φ should be a function defined on $\dot{X}$: F should be defined by something φ which is only infinitesimally additive.