ON THE COMPLETE LATTICE OF ESSENTIAL LOCALIZATIONS

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ABSTRACT By a *localization* of category $\mathcal{A}$ we mean a full replete subcategory whose inclusion admits a left–exact left adjoint R ; the localization is *essential* if R itself admits a left adjoint. One has the sets Loc $\mathcal{A} \supset$ Ess $\mathcal{A}$ of localizations and of essential localizations, ordered by inclusion; we study the completeness properties of the latter, comparing them with known results on the former. We show that, when the complete and cocomplete locally–small $\mathcal{A}$ admits either a strong generator or a strong cogenerator, Ess $\mathcal{A}$ is a small complete lattice, suprema in which coincide with those in Loc $\mathcal{A}$. Even when $\mathcal{A}$ is a presheaf category, however, so that infima in Loc $\mathcal{A}$ are just the intersections, the infima in Ess $\mathcal{A}$ (even binary ones) are in general strictly smaller than these.

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1. Introduction

We suppose an inaccessible cardinal ∞ chosen once for all, and call a set *small* if its cardinal is less than ∞. The morphisms of any category $\mathcal{A}$ form a set, and $\mathcal{A}$ is *small* if this set is small, while $\mathcal{A}$ is *locally small* if each hom–set $\mathcal{A}(A,B)$ is small. We use *large* to mean *not small*. Although a category is said to be *complete* when it admits all *small* limits, an ordered set is called a *complete lattice* only when it admits *all* suprema and infima – even large ones if it is a large set. We write Set for the category of *small* sets; similarly, by Grp, Cat, Top, and so on, we mean the categories of *small* groups, categories, topological spaces, and the like.

We refer to [11] for the definitions and properties of *strong epimorphisms* and *strong monomorphisms*. A *strong subobject* is one represented by a strong monomorphism. We call a category *strongly complete* if, besides admitting small limits, it admits arbitrary intersections of strong subobjects. Of course a complete category is strongly complete whenever, as is very commonly the case, every object has but a small set of strong subobjects.

The definition in [11] of strong epimorphism admits an evident generalization to a definition of a *strongly epimorphic family* $(f_k: A_k \longrightarrow B)$ of maps in $\mathcal{A}$, in such a way that, if the coproduct ΣA_k exists, the family is strongly epimorphic if and only if the corresponding map $\Sigma A_k \longrightarrow B$ is so. When $\mathcal{A}$ is finitely complete, it follows as in [11] that the family (f_k) is strongly epimorphic if and only if there is no proper subobject of B through which each f_k factorizes.

By a *generator* [resp. *strong generator*] of $\mathcal{A}$ we mean a *small* set $\mathcal{G}$ of objects of $\mathcal{A}$ such that, for each $A \in \mathcal{A}$, the family $(h: G_h \longrightarrow A)$ of all maps with codomain A and domain in $\mathcal{G}$ is jointly epimorphic [resp. strongly epimorphic].

The full subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be *replete* if every isomorph in $\mathcal{A}$ of an object of $\mathcal{B}$ itself lies in $\mathcal{B}$; throughout this article we abbreviate by using *subcategory* to mean *full replete* subcategory. The subcategory $\mathcal{B}$ of $\mathcal{A}$ is *reflective* if the inclusion $I: \mathcal{B} \to \mathcal{A}$ admits a left adjoint $R: \mathcal{A} \to \mathcal{B}$; the reflective subcategory $\mathcal{B}$ is a *localization* of $\mathcal{A}$ if (for some choice of R, and therefore for any choice) R is left exact; and it is an *essential localization* if R admits a left adjoint. (Although in practice, and in our results below, one never uses the word "localization" unless $\mathcal{A}$ admits finite limits, we can harmlessly avoid circumlocution in these general remarks by taking "left exact" to mean "preserving such finite limits as exist"; then essential localizations are indeed localizations.)

Thus we have the sets $\mathrm{Sub}\,\mathcal{A} \supset \mathrm{Ref}\,\mathcal{A} \supset \mathrm{Loc}\,\mathcal{A} \supset \mathrm{Ess}\,\mathcal{A}$ of all subcategories of $\mathcal{A}$, of reflective subcategories, of localizations, and of essential localizations, each ordered by inclusion. Trivially, $\mathrm{Sub}\,\mathcal{A}$ is a complete lattice, with suprema and infima given by the union $\bigcup \mathcal{B}_k$ and the intersection $\bigcap \mathcal{B}_k$ of subcategories; it is of course large unless $\mathcal{A}$ is (essentially) small, having for instance cardinal 2^{∞} when $\mathcal{A} = \mathrm{Set}$. For the ordered sets $\mathrm{Ref}\,\mathcal{A}$, $\mathrm{Loc}\,\mathcal{A}$, and $\mathrm{Ess}\,\mathcal{A}$, however, we cannot expect reasonable completeness properties unless $\mathcal{A}$ itself has reasonable completeness properties. Then, as we shall see, each of these sets admits small suprema, preserved by the inclusions $\mathrm{Ess}\,\mathcal{A} \subset \mathrm{Loc}\,\mathcal{A} \subset \mathrm{Ref}\,\mathcal{A}$. Accordingly, a proper understanding of the completeness properties of $\mathrm{Ess}\,\mathcal{A}$ requires a brief revision of what is known for $\mathrm{Ref}\,\mathcal{A}$ and for $\mathrm{Loc}\,\mathcal{A}$.

$\mathrm{Ref}\,\mathcal{A}$ has been studied by Kelly in [13]. It is commonly a large set, even though it has just three elements when $\mathcal{A} = \mathrm{Set}$. In fact, for a locally-small and strongly-complete $\mathcal{A}$, the conclusion of [13, Proposition 9] as it stands is that $\mathrm{Ref}\,\mathcal{A}$ cannot be small unless $\mathcal{A}$ has a strong cogenerator; but it follows at once that $\mathrm{Ref}\,\mathcal{A}$ cannot be small unless *every reflective subcategory of* $\mathcal{A}$ *has a strong cogenerator* — and this is such a strong condition that smallness of $\mathrm{Ref}\,\mathcal{A}$ is probably quite exceptional. At any rate, Ref $\mathcal{A}$ is large when $\mathcal{A}$ is the category Grp of (small) groups; for by the Special Adjoint Functor Theorem, Grp has no cogenerator, the functor Grp $\longrightarrow$ Set represented by the coproduct of all small simple groups not being representable by any small group. So Ref $\mathcal{A}$ is large whenever $\mathcal{A}$, like Cat, has Grp as a reflective subcategory. Since the single category $3 = (0 < 1 < 2)$ is dense in Cat, it follows further that Ref $\mathcal{A}$ is large for the presheaf category $\mathcal{A} = [M^{op},Set]$, where M is the finite monoid Cat(3,3).

Large though it commonly is, Ref $\mathcal{A}$ admits *small* suprema whenever $\mathcal{A}$ is strongly complete, the supremum $\vee\mathcal{B}_k$ in Ref $\mathcal{A}$ being the closure in $\mathcal{A}$ of $\bigcup\mathcal{B}_k$ under small limits and all intersections of strong subobjects; see [13, Theorems 14 and 15]. However Ref $\mathcal{A}$ need not admit *arbitrary* suprema, even when $\mathcal{A}$ is as well behaved as the category Top of topological spaces – which is locally small, complete and cocomplete, wellpowered and cowellpowered, and has a generator and a strong cogenerator. For then Ref $\mathcal{A}$ would be a complete lattice, and admit arbitrary infima; yet it is shown in [13, Theorem 7] that, when $\mathcal{A}$ is locally small and strongly complete, an infimum in Ref $\mathcal{A}$, if it exists, must be the intersection $\bigcap\mathcal{B}_k$; while Adámek and Rosický have shown in [1] that intersections of reflective subcategories of Top need not be reflective. (This last is now known to be true even for *binary* intersections, as shown by Trnková, Adámek, and Rosický in [15].)

The study of Loc $\mathcal{A}$ when $\mathcal{A}$ is a presheaf category or a category of modules over a ring is classical, the localizations in these cases corresponding respectively to the Grothendieck topologies and to the Gabriel topologies; and it is nearly as classical when $\mathcal{A}$ is a Grothendieck topos (or an elementary topos for that matter – but this present article is devoted to externally–complete categories $\mathcal{A}$). For a much wider class of such categories, Loc $\mathcal{A}$ has been studied by Borceux and Kelly in [3]. Whenever $\mathcal{A}$ is complete – we no longer need, as we did for Ref $\mathcal{A}$, *strong* completeness – Loc $\mathcal{A}$ admits small suprema; and moreover these are precisely the suprema $\vee\mathcal{B}_k$ in Ref $\mathcal{A}$ of the localizations $\mathcal{B}_k$, although we can now describe $\vee\mathcal{B}_k$ more simply just as the closure in $\mathcal{A}$ of $\bigcup\mathcal{B}_k$ under small limits, without reference to the intersections of strong subobjects; see [3, Theorems 3.1 and 3.3 and Corollary 3.4]. Yet, as [3, Example 5.1] shows, even when the locally–small $\mathcal{A}$ is complete and cocomplete, with a generator and a cogenerator, Loc $\mathcal{A}$ may fail to admit binary infima; in such a case, of course, Loc $\mathcal{A}$ is a large set which fails to admit arbitrary suprema. By [3, Theorem 6.4], however, Loc $\mathcal{A}$ is small whenever the locally–small and finitely–complete $\mathcal{A}$ has a strong generator; thus we have as in [3, Proposition 6.5] the positive result that Loc $\mathcal{A}$ *is a small complete lattice when* $\mathcal{A}$ *is locally small and complete with a strong generator.* Although, for such an $\mathcal{A}$, we have arbitrary infima in Loc $\mathcal{A}$, these need not be the intersections: we exhibit in Example 5.2 below, $\mathcal{A}$ being the dual of a presheaf category, localizations $\mathcal{B}$ and $\mathcal{C}$ whose infimum $\mathcal{B} \wedge \mathcal{C}$ in Loc $\mathcal{A}$ is strictly smaller than $\mathcal{B} \cap \mathcal{C}$, even though $\mathcal{B} \cap \mathcal{C}$ is reflective in $\mathcal{A}$ and is therefore the infimum in Ref $\mathcal{A}$. (No such counter–example was known at the time of writing [3]; the $\mathcal{A}$ of [3, Example 5.2] has no strong generator.)

Much deeper results on Loc $\mathcal{A}$ were proved in [3] for a special but important class of categories $\mathcal{A}$, namely those locally–presentable $\mathcal{A}$ in which finite limits commute with filtered colimits. Since the writing of [3], the nature of this class has been much clarified by the work of Day and Street reported in [5] and in the very recent [6]; they show the equivalence of the following:

(i) $\mathcal{A}$ is locally presentable and finite limits commute with filtered colimits;

(ii) $\mathcal{A}$ is locally small, cocomplete, and finitely complete, with a strong generator, and finite limits commute with filtered colimits;

(iii) $\mathcal{A}$ is a localization of some locally–finitely–presentable category;

(iv) for some finitely–cocomplete small category C and some Grothendieck topology on C, $\mathcal{A}$ is the subcategory of $[\mathbf{C}^{\mathrm{op}},\mathbf{Set}]$ given by those $F\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{Set}$ which are at once sheaves for the topology and left exact as functors.

Categories satisfying (iii) were called *geometric categories* by Borceux in [2]; let us retain this name. Among the geometric categories are of course all the locally–finitely–presentable categories, such as Grp, Cat, or a presheaf category $[\mathrm{A}^{\mathrm{op}},\mathbf{Set}]$ with A small. Since a Grothendieck topos is a localization of some $[\mathrm{A}^{\mathrm{op}},\mathbf{Set}]$, it too is a geometric category. It is clear from (i) that, if $\mathcal{A}$ is a geometric category, so is the functor category $[\mathrm{K}, \mathcal{A}]$ for any small K and so is the category $\mathcal{A}^{\mathrm{T}}$ of algebras for a finitary monad T on $\mathcal{A}$; note that we call a monad T *finitary* if the functor T is finitary, in the sense that it preserves filtered colimits. It is further the case that a reflective subcategory $\mathcal{B}$ of the geometric $\mathcal{A}$ is geometric if the inclusion $\mathrm{I}\colon \mathcal{B} \to \mathcal{A}$ is finitary, which is to say that $\mathcal{B}$ is closed in $\mathcal{A}$ under filtered colimits; see [3, Examples 6.9(v)]. From this it follows as in [3, Examples 6.9(vi)] that, for a geometric $\mathcal{A}$ and a finitely–complete small K, the category $\mathrm{Lex}[\mathrm{K}, \mathcal{A}]$ of left–exact functors is again geometric. Whether every geometric category is of this latter form for some Grothendieck topos $\mathcal{A}$ remains unknown; we do not see how to deduce this from (iv).

It is proved in [3, Theorem 6.8] that, for a geometric $\mathcal{A}$, the infima in $\mathrm{Loc}\,\mathcal{A}$ are (in contrast to the last example of the penultimate paragraph) precisely the intersections; and that, moreover, these infima are preserved by $\mathcal{B}\vee -$ for each $\mathcal{B}$ in $\mathrm{Loc}\,\mathcal{A}$, so that $(\mathrm{Loc}\,\mathcal{A})^{\mathrm{op}}$ is a *frame* (also called a *complete Heyting algebra*, or a *locale*).

We now turn to $\mathrm{Ess}\,\mathcal{A}$. In Section 2 below we recall the injection Φ from $\mathrm{Ref}\,\mathcal{A}$ to the set of subsets of $\mathrm{mor}\,\mathcal{A}$, sending $\mathcal{B}$ to the set $\mathcal{E}$ of morphisms inverted by the reflexion R, and use it to show that $\mathrm{Ess}\,\mathcal{A}$ is isomorphic to $\mathrm{Ess}(\mathcal{A}^{\mathrm{op}})$. The techniques used in [13] and [3] to study small suprema in $\mathrm{Ref}\,\mathcal{A}$ and in $\mathrm{Loc}\,\mathcal{A}$ involved identifying the images under Φ of Ref $\mathcal{A}$ and of Loc $\mathcal{A}$, and showing these images to be closed under small intersections for any reasonable $\mathcal{A}$; in Section 3 we determine the image under Φ of Ess $\mathcal{A}$, and deduce that Ess $\mathcal{A}$ admits small suprema, which agree with those in Loc $\mathcal{A}$ and in Ref $\mathcal{A}$, when $\mathcal{A}$ is complete and cocomplete. If we further suppose that $\mathcal{A}$ is locally small then, as we have seen, Loc $\mathcal{A}$ [resp. Loc $(\mathcal{A}^{op})$] is small if $\mathcal{A}$ has a strong generator [resp. a strong cogenerator]; so that in either case Ess $\mathcal{A}$ is small, and is therefore a small complete lattice. We give an example, however, of a spatial topos $\mathcal{A}$ for which a countable infimum in Ess $\mathcal{A}$, unlike that in Loc $\mathcal{A}$, is not the intersection.

That counter–example, based on the fact that an intersection of open sets need not be open, tells us nothing about the relation of *binary* infima in Ess $\mathcal{A}$, for a geometric category $\mathcal{A}$, to the intersections. To show that even these can be different, we examine in Section 4 essential localizations of a presheaf category $\mathcal{A} = [A^{op}, \mathrm{Set}]$ in terms of "idempotent ideals" of A, and then produce a concrete counter–example in Section 5, showing at the same time that a binary infimum in Loc $(\mathcal{A}^{op})$, with $\mathcal{A} = [A^{op}, \mathrm{Set}]$, may also differ from the intersection.

2. The basic properties of essential localizations

For a typical reflective subcategory $\mathcal{B}$ of $\mathcal{A}$ with inclusion $I: \mathcal{B} \longrightarrow \mathcal{A}$, we use $\rho: 1 \longrightarrow IR$ for the unit of an adjunction $R \dashv I$. We have no need below to mention the counit $RI \longrightarrow 1$ explicitly; it is of course invertible, and we may always so choose R and ρ that $RI = 1$ and the counit is the identity. Since $IB = B$, we may suppress I where convenient.

Recall that, in the terminology first used in [8], a morphism $f: C \longrightarrow D$ in $\mathcal{A}$ and an object A of $\mathcal{A}$ are said to be *orthogonal* when $\mathcal{A}(f,A): \mathcal{A}(D,A) \longrightarrow \mathcal{A}(C,A)$ is a bijection. Given a subcategory $\mathcal{B}$ of $\mathcal{A}$ we write $\mathcal{B}^{\perp}$ for the set of morphisms orthogonal to every B in $\mathcal{B}$, and given a set $\mathcal{E}$ of morphisms of $\mathcal{A}$ we write $\mathcal{E}^{\perp}$ for the subcategory given by those A orthogonal to every e in $\mathcal{E}$. Let us write Φ for the order-reversing function, from Ref $\mathcal{A}$ to the set $\mathcal{P}$(mor $\mathcal{A}$) of all subsets of the set mor $\mathcal{A}$ of all morphisms of $\mathcal{A}$, which sends $\mathcal{B}$ to $\mathcal{B}^{\perp}$. The following is in [4], but is probably too well known in the folklore to deserve attribution:

Proposition 2.1 *For $\mathcal{B} \in$ Ref $\mathcal{A}$, the set $\Phi(\mathcal{B}) = \mathcal{B}^{\perp}$ coincides with the set $\mathcal{E}$ of those morphisms of $\mathcal{A}$ inverted by the reflexion R: $\mathcal{A} \longrightarrow \mathcal{B}$; moreover the function Φ: Ref $\mathcal{A} \longrightarrow \mathcal{P}$(mor $\mathcal{A}$) is injective, since in fact $\mathcal{B} = \mathcal{E}^{\perp}$.*

Proof $\mathcal{B}^{\perp} = \mathcal{E}$, since to say that $\mathcal{A}$(f,B) is bijective for all B $\in \mathcal{B}$ is equally to say that $\mathcal{B}$(Rf,B) is bijective for all B $\in \mathcal{B}$, or equivalently that Rf is invertible. Since trivially $\mathcal{B} \subset \mathcal{B}^{\perp\perp} = \mathcal{E}^{\perp}$, it remains to show that $\mathcal{E}^{\perp} \subset \mathcal{B}$. Because the unit ρA: A $\longrightarrow$ RA clearly lies in $\mathcal{E}$, it is a coretraction whenever A $\in \mathcal{E}^{\perp}$; but then A, as a retract of an object rA of the reflective $\mathcal{B}$, itself lies in $\mathcal{B}$ by a well-known argument. □

We content ourselves with a clear statement of the following simple result, leaving the reader to supply the easy proofs:

Proposition 2.2 *Let η,ε : S $\dashv$ T : $\mathcal{K} \longrightarrow \mathcal{A}$ be an adjunction in which* T *is fully faithful (or, equivalently, in which ε is invertible). Write $\mathcal{B}$ for the "full replete image" of* T *: that is, the subcategory of $\mathcal{A}$ given by those* A *isomorphic to some* TK *, which are equally those* A *for which ηA : A $\longrightarrow$ TSA is invertible. We have* T = IP *, where* I: $\mathcal{B} \longrightarrow \mathcal{A}$ *is the inclusion and* P: $\mathcal{K} \longrightarrow \mathcal{B}$ *differs from* T *only in that its values are deemed to lie in $\mathcal{B}$. The functor* P *is an equivalence, with equivalence-inverse* SI; *and $\mathcal{B}$ is a* *reflective subcategory of* $\mathcal{A}$ *with reflexion* $R = PS$, *the unit* $\rho: 1 \longrightarrow IR = IPS = TS$ *coinciding with* $\eta: 1 \longrightarrow TS$. *The set* $\mathcal{E} = \mathcal{B}^{\perp}$ *of morphisms inverted by* R *is equally the set of morphisms inverted by* S. *The reflective* $\mathcal{B}$ *is a localization of* $\mathcal{A}$ *if and only if* S *is left exact.*

The next result is again folklore; Kelly recalls learning it from M. Barr in 1976; the authors of [7], unable to find a proof in the literature, gave one in their Lemma 1.3; we too give one, perhaps a few lines shorter, for completeness. The proof applies to adjunctions in any 2–category and uses, besides the triangular equations for the adjunctions, only instances of the 2–categorical equality $X\phi.\theta M = \theta Y.M\phi: MN \longrightarrow XY$ where $\theta: M \longrightarrow X$ and $\phi: N \longrightarrow Y$.

Proposition 2.3 *Given adjunctions* $\eta,\varepsilon: S \dashv T: \mathcal{K} \longrightarrow \mathcal{A}$ *and* $\alpha,\beta: U \dashv S: \mathcal{A} \longrightarrow \mathcal{K}$ *in any 2-category, if* ε *is invertible, so is* α. *When we are dealing with the 2-category of categories, therefore,* U *is fully faithful whenever* T *is so.*

Proof We show that the composite γ given by

$$SU \xrightarrow[SU\varepsilon^{-1}]{} SUST \xrightarrow[S\beta T]{} ST \xrightarrow[\varepsilon]{} 1$$

is inverse to $\alpha: 1 \longrightarrow SU$. First, $\gamma\alpha = \varepsilon.S\beta T.SU\varepsilon^{-1}.\alpha = \varepsilon.S\beta T.\alpha ST.\varepsilon^{-1}$, which is the identity since $S\beta.\alpha S$ is an identity. Secondly, $\alpha\gamma = \gamma SU.SU\alpha$ which – since $\varepsilon^{-1}S = S\eta$ – is

$$\varepsilon SU.S\beta TSU.SUS\eta U.SU\alpha = \varepsilon SU.S\eta U.S\beta U.SU\alpha\ ;$$

and this is the identity since $\varepsilon S.S\eta$ and $\beta U.U\alpha$ are identities. $\square$

We remarked in the penultimate paragraph of the Introduction that $\mathrm{Ess}\,\mathcal{A}$ is isomorphic to $\mathrm{Ess}(\mathcal{A}^{\mathrm{op}})$. It is far more convenient, however, to remain within the language of $\mathcal{A}$, speaking of coreflective subcategories of $\mathcal{A}$ rather than reflective subcategories of $\mathcal{A}^{\mathrm{op}}$. To this end we dualize our notation and nomenclature. The subcategory $\mathcal{C}$ of $\mathcal{A}$ is of course *coreflective* if the inclusion $J\colon \mathcal{C} \longrightarrow \mathcal{A}$ has a right adjoint $S\colon \mathcal{A} \longrightarrow \mathcal{C}$; it is then a *colocalization* of $\mathcal{A}$ if S is right exact, and an *essential colocalization* if S has a right adjoint. We write $\mathrm{CoEss}\,\mathcal{A} \subset \mathrm{CoLoc}\,\mathcal{A} \subset \mathrm{CoRef}\,\mathcal{A}$ for the ordered sets of such subcategories, which are of course respectively isomorphic to $\mathrm{Ess}(\mathcal{A}^{\mathrm{op}}) \subset \mathrm{Loc}(\mathcal{A}^{\mathrm{op}}) \subset \mathrm{Ref}(\mathcal{A}^{\mathrm{op}})$. We call a morphism $f\colon C \longrightarrow D$ in $\mathcal{A}$ and an object A of $\mathcal{A}$ *coorthogonal* when $\mathcal{A}(A,f)\colon \mathcal{A}(A,C) \longrightarrow \mathcal{A}(A,D)$ is a bijection. Given a subcategory $\mathcal{C}$ of $\mathcal{A}$ we write $\mathcal{C}^{\top}$ for the set of morphisms coorthogonal to every C in $\mathcal{C}$, and given a set $\mathcal{F}$ of morphisms of $\mathcal{A}$ we write $\mathcal{F}^{\top}$ for the subcategory given by those A coorthogonal to every f in $\mathcal{F}$.

Let us call an ordered pair $(\mathcal{B},\mathcal{C})$ of subcategories of $\mathcal{A}$ an *associated pair* if $\mathcal{B}$ is reflective, $\mathcal{C}$ is coreflective, and $\mathcal{B}^{\perp} = \mathcal{C}^{\top}$.

Theorem 2.4 (a) *Let $(\mathcal{B},\mathcal{C})$ be an associated pair, with* $R \dashv I\colon \mathcal{B} \longrightarrow \mathcal{A}$ *and* $J \dashv S\colon \mathcal{A} \longrightarrow \mathcal{C}$, *where* I *and* J *are the inclusions. Then*

(i) *each of $\mathcal{B}$ and $\mathcal{C}$ is uniquely determined by the other, since we have $\mathcal{C} = \mathcal{B}^{\perp\top}$ and $\mathcal{B} = \mathcal{C}^{\top\perp}$;*

(ii) *the functors* $SI\colon \mathcal{B} \longrightarrow \mathcal{C}$ *and* $RJ\colon \mathcal{C} \longrightarrow \mathcal{B}$ *are mutually inverse equivalences;*

(iii) *$\mathcal{B}$ is an essential localization and $\mathcal{C}$ an essential colocalization.*

(b) *Moreover, every essential localization* $\mathcal{B}$ *forms part of an associated pair* $(\mathcal{B}, \mathcal{C})$ *; and if* $U: \mathcal{B} \longrightarrow \mathcal{A}$ *is any left adjoint of* $R: \mathcal{A} \longrightarrow \mathcal{B}$ *, we can describe* $\mathcal{C} = \mathcal{B}^{\perp\top}$ *alternatively as the full replete image of* U.

(c) $\mathcal{B} \longmapsto \mathcal{B}^{\perp\top}$ *is an order-preserving bijection* Ess $\mathcal{A} \longrightarrow$ CoEss $\mathcal{A}$ *, with inverse* $\mathcal{C} \longmapsto \mathcal{C}^{\top\perp}$.

Proof For (a), let $\mathcal{E}$ be $\mathcal{B}^{\perp} = \mathcal{C}^{\top}$. Then (i) is immediate since Proposition 2.1 gives $\mathcal{B} = \mathcal{E}^{\perp} = \mathcal{C}^{\top\perp}$, while $\mathcal{C} = \mathcal{B}^{\perp\top}$ is just the dual of this. As for (ii), the components of the unit $\rho: 1 \longrightarrow$ IR lie in $\mathcal{C}^{\top}$ since they clearly lie in $\mathcal{B}^{\perp}$; by Proposition 2.1, therefore, they are inverted by S; thus we have an isomorphism $S\rho: S \longrightarrow SIR$. This gives $SIRJ \cong SJ \cong 1$; dually we have $RJSI \cong 1$; whence (ii) follows. Now, since SI is an equivalence and since $S \cong SIR$ by the penultimate sentence, R like S has a left adjoint, proving (iii). Turning to (b), we first observe that U is fully faithful by Proposition 2.3; write $\mathcal{C}$ for its full replete image, so that $U = JP$ where $J: \mathcal{C} \longrightarrow \mathcal{A}$ is the inclusion and $P: \mathcal{B} \longrightarrow \mathcal{C}$ is the functor U seen as taking its values in $\mathcal{C}$. By the dual of Proposition 2.2, $\mathcal{C}$ is coreflective and $\mathcal{C}^{\top}$ is the set of morphisms inverted by R, which by Proposition 2.1 again is $\mathcal{B}^{\perp}$. This proves (b), and now (c) follows trivially. □

We now rationalize our notation, in the following sense. A typical reflective subcategory of $\mathcal{A}$ is still $\mathcal{B}$, with inclusion $I: \mathcal{B} \longrightarrow \mathcal{A}$ and reflexion $R: \mathcal{A} \longrightarrow \mathcal{B}$, the unit being $\rho: 1 \longrightarrow$ IR. When $\mathcal{B}$ is an essential localization, however, the *associated essential colocalization* $\mathcal{B}^{\perp\top}$ will no longer be called $\mathcal{C}$ as in Theorem 2.4, but will henceforth be $\bar{\mathcal{B}}$. This releases $\mathcal{C}$ as a possible name for a second reflective subcategory, perhaps another essential localization. When we deal with general families $(I_k: \mathcal{B}_k \longrightarrow \mathcal{A})$ of subcategories of some type, we of course just add a subscript k throughout as appropriate.

3. Small suprema of essential localizations

We suppose the reader is familiar with the notion of a *factorization system* $(\mathcal{E}, \mathcal{M})$ for a category $\mathcal{A}$, which was introduced in [8] and revised, with more details, in [4] and [3]. Recall that the $\mathcal{M}$ of a factorization system $(\mathcal{E}, \mathcal{M})$ is fully determined by $\mathcal{E}$, being necessarily what was called $\mathcal{E}^{\downarrow}$ in [8] and [3].

Consider the following properties which a set $\mathcal{E}$ of morphisms of $\mathcal{A}$ may possess:

E1. If $fg \in \mathcal{E}$ and $f \in \mathcal{E}$ then $g \in \mathcal{E}$.

E2. Every pullback of an $\mathcal{E}$ is an $\mathcal{E}$.

E3. There is a factorization system $(\mathcal{E}, \mathcal{M})$.

E4. There are factorization systems $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{N}, \mathcal{E})$.

Under mild conditions on $\mathcal{A}$, Cassidy, Hébert, and Kelly identified in [4] the images of $\mathrm{Ref}\,\mathcal{A}$ and of $\mathrm{Loc}\,\mathcal{A}$ under the injection $\Phi\colon \mathrm{Ref}\,\mathcal{A} \longrightarrow \mathcal{P}(\mathrm{mor}\,\mathcal{A})$ of Proposition 2.1. The following extracts their results on this from their Corollaries 3.4 and 4.8; although we use only their $\mathrm{Loc}\,\mathcal{A}$ result, we include their $\mathrm{Ref}\,\mathcal{A}$ result for its inherent interest.

Proposition 3.1 *A set $\mathcal{E}$ of morphisms of the finitely-complete $\mathcal{A}$ is of the form $\mathcal{B}^{\perp}$ for some localization $\mathcal{B}$ of $\mathcal{A}$ if and only if $\mathcal{E}$ satisfies* E1, E2, *and* E3. *If we further suppose that $\mathcal{A}$ admits arbitrary intersections of strong subobjects, $\mathcal{E}$ is of the form $\mathcal{B}^{\perp}$ for some reflective $\mathcal{B}$ if and only if $\mathcal{E}$ satisfies* E1 *and* E3.

We can now identify the image under Φ of $\mathrm{Ess}\,\mathcal{A}$:

Theorem 3.2 *A set $\mathcal{E}$ of morphisms of the finitely-complete and finitely-cocomplete $\mathcal{A}$ is of the form $\mathcal{B}^{\perp}$ for some essential localization $\mathcal{B}$ of $\mathcal{A}$ if and only if $\mathcal{E}$ satisfies* E4.

Proof If $\mathcal{E} = \mathcal{B}^{\perp}$ for some essential localization $\mathcal{B}$, then by Theorem 2.4 we have $\mathcal{E} = \overline{\mathcal{B}}^{\top}$ where $\overline{\mathcal{B}}$ is the associated essential colocalization; so by Proposition 3.1 $\mathcal{E}$ satisfies both E3 and its dual – that is, $\mathcal{E}$ satisfies E4. Suppose conversely that $\mathcal{E}$ satisfies E4 and hence E3. Since $(\mathcal{M}, \mathcal{E})$ is a factorization system, $\mathcal{E}$ satisfies E1 and E2 by [8, Proposition 2.1.1]. Accordingly, by Proposition 3.1, $\mathcal{E} = \mathcal{B}^{\perp}$ for some localization $\mathcal{B}$; dually, however, $\mathcal{E} = \mathcal{C}^{\top}$ for some colocalization $\mathcal{C}$; by Theorem 2.4, therefore, $\mathcal{B}$ is an essential localization. □

Theorem 3.3 *Let $(\mathcal{B}_k)$ be a small family of essential localizations of the complete and cocomplete $\mathcal{A}$, and set $\mathcal{E} = \bigcap \mathcal{E}_k$ where $\mathcal{E}_k = \mathcal{B}_k^{\perp}$. Then $\mathcal{E} = \mathcal{B}^{\perp}$ for an essential localization $\mathcal{B}$ of $\mathcal{A}$, and $\mathcal{B}$ is the supremum of the family $(\mathcal{B}_k)$ not only in* Ess $\mathcal{A}$ *but also in* Loc $\mathcal{A}$ *and in* Ref $\mathcal{A}$. *Explicitly, $\mathcal{B}$ is the closure in $\mathcal{A}$ of $\bigcup \mathcal{B}_k$ under small limits.*

Proof By [3, Theorem 3.1], because $\mathcal{A}$ is complete and the $\mathcal{B}_k$ are localizations, $\mathcal{E}$ satisfies E3, the appropriate $\mathcal{M}$ being constructed explicitly in the proof of that theorem. Using the fact that we also have $\mathcal{E}_k = \overline{\mathcal{B}}_k^{\top}$, and applying [3, Theorem 3.1] now to $\mathcal{A}^{op}$, we see that $\mathcal{E}$ in fact satisfies E4. By Theorem 3.2, therefore, $\mathcal{E} = \mathcal{B}^{\perp}$ for an essential localization $\mathcal{B}$. Since Φ: Ref $\mathcal{A} \to \mathcal{P}(\text{mor } \mathcal{A})$ is an order-reversing injection by Proposition 2.1, it follows at once that $\mathcal{B}$ is the supremum of the $\mathcal{B}_k$ in Ref $\mathcal{A}$, in Loc $\mathcal{A}$, and in Ess $\mathcal{A}$. The explicit description of $\mathcal{B}$ follows from [3, Theorem 3.3]. □

Theorem 3.4 *Suppose that $\mathcal{A}$ is complete and cocomplete. Then* Ess $\mathcal{A}$ *is a complete lattice whenever it is small; and this is surely the case if $\mathcal{A}$ is locally small and has either a strong generator or a strong cogenerator.*

Proof The first assertion is immediate from Theorem 3.3; as for the second, so long as $\mathcal{A}$ is locally small, Loc $\mathcal{A}$ [resp Loc $(\mathcal{A}^{op})$] is small by [3, Proposition 6.5] if $\mathcal{A}$ has a strong generator [resp. strong cogenerator], whence Ess $\mathcal{A}$ is small in either case by Theorem 2.4. □

We recalled in the Introduction the result of [3, Theorem 6.8] that, for a geometric category $\mathcal{A}$, the infima in Loc $\mathcal{A}$ are precisely the intersections. We now show that, even when $\mathcal{A}$ is the category Shv A of sheaves on a compact hausdorff space A, the infima in Ess $\mathcal{A}$ (which of course exist by Theorem 3.4) are in general strictly smaller than those in Loc $\mathcal{A}$ — in contrast to the result of Theorem 3.3 for suprema.

It is classical that a continuous map $i: B \longrightarrow A$ of topological spaces induces a geometric morphism $i^* \dashv i_*: \mathrm{Shv}\, B \longrightarrow \mathrm{Shv}\, A$, and that i_* is fully faithful if i is the inclusion of a subspace. So each subspace B of A gives by Proposition 2.2 a localization $\mathcal{B}$ of the topos $\mathcal{A} = \mathrm{Shv}\, A$, namely the full replete image of i_*. If we identify sheaves on A with local homeomorphsims $p: X \longrightarrow A$, it is very easy to describe i^* and i_*; the former is just the "restriction" functor, sending p to its restriction $p^{-1}(B) \longrightarrow B$, while the latter extends a sheaf $q: Y \longrightarrow B$ on B to one on A by taking the stalk at each point of $A - B$ to be a single point and giving to the result the unique locally–homeomorphic topology. Accordingly $\mathcal{B}$ consists exactly of those sheaves on A whose stalks at each point of $A - B$ are singletons; it is of course equivalent to Shv B.

It is equally classical that, when the subspace B of A is open, i^* itself has a left adjoint $i_!$, necessarily fully faithful by Proposition 2.3; in fact $i_!$ sends a sheaf $q: Y \longrightarrow B$ to one on A by taking the stalk at each point of $A - B$ to be empty – which does produce a local homeomorphism $p: X \longrightarrow A$ when B is open, although not for a general B. Then $\mathcal{B}$ is an essential localization of $\mathcal{A}$; and the associated essential colocalization $\bar{\mathcal{B}}$, which by Theorem 2.4 is the full replete image of $i_!$, consists of those sheaves on A whose stalks at each point of $A - B$ are empty.

Now take for A the (compact) subspace of the reals consisting of 0 and the points $1/n$ for positive integral n; write B_k for the open subspace given by 0 and the $1/n$ for $n \geq k$, where k is again a positive integer; write B for the empty subspace of A; and write C for the subspace $\bigcap B_k = \{0\}$ of A. Taking $\mathcal{A}$ to be Shv A, we have as above the essential localizations $\mathcal{B}_k$ and $\mathcal{B}$ corresponding to the open subspaces B_k and B, along with the associated essential colocalizations $\bar{\mathcal{B}}_k$ and $\bar{\mathcal{B}}$, and we have the localization $\mathcal{C}$ corresponding to the subspace C.

The intersection $\bigcap \bar{\mathcal{B}}_k$ consists of the sheaves on A whose stalks, except that at 0, are all empty; but then the stalk at 0 is necessarily empty too, so that $\bigcap \bar{\mathcal{B}}_k = \bar{\mathcal{B}}$, which is the category (isomorphic to 1) consisting of the empty sheaf alone. It follows that $\bar{\mathcal{B}}$ is the infimum of the $\bar{\mathcal{B}}_k$ in CoEss $\mathcal{A}$; whence, by the isomorphism of Theorem 2.4(c), the infimum in Ess $\mathcal{A}$ of the $\mathcal{B}_k$ is $\mathcal{B}$, which is the category (equivalent to 1) consisting of the sheaves that are isomorphisms $p: X \longrightarrow A$. By [3, Theorem 6.8], however, the infimum of the $\mathcal{B}_k$ in Loc $\mathcal{A}$ is the intersection $\bigcap \mathcal{B}_k$, which is clearly the localization $\mathcal{C}$, equivalent as a category (since the stalk at 0 of an object of $\mathcal{C}$ is arbitrary) to Set.

It follows, of course, that the localization $\mathcal{C}$ of $\mathcal{A}$ is not essential. In fact it is easy to exhibit an infinite limit in $\mathcal{A}$ not preserved by the reflexion of $\mathcal{A}$ onto $\mathcal{C}$, or equivalently not preserved by the functor $S: \mathcal{A} \longrightarrow \mathbf{Set}$ given by taking the stalk at 0: namely the intersection in $\mathcal{A}$ of the sheaves $i_k: B_k \longrightarrow A$, seen as subsheaves of $1: A \longrightarrow A$, which is the empty sheaf $i: B \longrightarrow A$; although each $S(i_k) = 1$ while $S(i) = 0$.

In order to give examples where even *binary* infima in Ess $\mathcal{A}$, for a geometric $\mathcal{A}$, differ from those in Loc $\mathcal{A}$, we now turn to the study of essential localizations of presheaf categories.

4. Essential localizations of presheaf categories

Since the earlier parts of what follows apply to categories other than presheaf categories, we give them in a general form. We recall for convenience the following aspects of Propositions 2.3, 2.4, 2.5, 6.2, and 6.3 of [3]:

Proposition 4.1 *For a localization $\mathcal{B}$ of the finitely-complete $\mathcal{A}$, write $\mathcal{E}$ for $\mathcal{B}^{\perp}$ and $\mathcal{E}_m$ for the set of monomorphisms in $\mathcal{E}$. Then*

(i) *if $fg \in \mathcal{E}_m$ and f is monomorphic, we have $f \in \mathcal{E}_m$;*

(ii) $\mathcal{B} = \mathcal{E}_m^{\perp}$;

(iii) *if every morphism f of $\mathcal{A}$ factorizes as a strong epimorphism followed by a monomorphism — the latter then being called the image of f — we have $f \in \mathcal{E}$ if and only if the image of f and the equalizer of the kernel-pair of f both lie in $\mathcal{E}_m$.*

Now suppose further that $\mathcal{A}$ has a strong generator $\mathcal{G}$, and write $\mathcal{T}$ for the set of morphisms in $\mathcal{E}_m$ with codomain in $\mathcal{G}$. Then

(iv) $\mathcal{B} = \mathcal{T}^{\perp}$;

(v) *a monomorphism $f\colon A \to B$ lies in $\mathcal{E}_m$ if and only if, for every $g\colon G \to B$ with $G \in \mathcal{G}$, the pullback g^*f of f along g lies in $\mathcal{T}$.*

The basic result we need is:

Theorem 4.2 *Let the locally-small, complete and cocomplete $\mathcal{A}$ admit a strong generator $\mathcal{G}$ and a cogenerator, and suppose that small products of strong epimorphisms in $\mathcal{A}$ are again strong epimorphisms. Let $\mathcal{B}$ be a localization of $\mathcal{A}$, define $\mathcal{T}$ as in Proposition 4.1, and for* $G \in \mathcal{G}$ *write* $\mathcal{T}(G)$ *for the set of morphisms in $\mathcal{T}$ with codomain* G, *thought of as a set of subobjects of* G. *Then the localization $\mathcal{B}$ is essential if and only if each* $\mathcal{T}(G)$ *has a least element* $t_G \colon T(G) \to G$ *; whereupon* $\mathcal{T}(G)$ *consists of all subobjects of* G *greater than or equal to* t_G.

Proof First observe that, because of its strong generator, $\mathcal{A}$ is wellpowered – see [3, Proposition 6.1]. In consequence, each morphism $f \colon A \to B$ of $\mathcal{A}$ *does* factorize as a strong epimorphism followed by a monomorphism, the image of f being the intersection of those subobjects of B through which f factorizes; the point being that the complete and wellpowered $\mathcal{A}$ admits arbitrary intersections of subobjects. If now the localization $\mathcal{B}$ is essential, the reflexion $R \colon \mathcal{A} \to \mathcal{B}$ preserves all limits and in particular all intersections of subobjects; it follows that $\mathcal{T}(G)$ is closed under intersections, and hence has a least element $t_G \colon T(G) \to G$. For the converse, suppose that each $\mathcal{T}(G)$ does have such a least element. Then, by Proposition 4.1(i), $\mathcal{T}(G)$ in fact consists of all subobjects of G greater than or equal to t_G, whence it is certainly closed under intersections. Now let $(f_k \colon A_k \to B_k)$ be a small family of elements of $\mathcal{E}_m$, and let $f \colon A \to B$ be their product. For $G \in \mathcal{G}$, to give a morphism $g \colon G \to B = \Pi B_k$ is to give its components $g_k \colon G \to B_k$; and since the pullback g^*f as a subobject of G is the intersection $\bigcap g_k^* f_k$, it follows from Proposition 4.1(v) that $\mathcal{E}_m$ is closed under small products. Now it follows easily, from Proposition 4.1(iii) along with the hypothesis that strong epimorphisms are closed under products in $\mathcal{A}$, that $\mathcal{E}$ is closed under small products. We use this to conclude that

$R\colon \mathcal{A} \longrightarrow \mathcal{B}$ preserves small products: the canonical comparison map $h\colon R(\Pi A_k) \longrightarrow \Pi(RA_k)$ is the unique morphism whose composite with the unit $\rho(\Pi A_k)\colon \Pi A_k \longrightarrow R(\Pi A_k)$ is $\Pi(\rho A_k)\colon \Pi A_k \longrightarrow \Pi(RA_k)$; since each $\rho C\colon C \longrightarrow RC$ clearly belongs to the set $\mathcal{E}$ of morphisms inverted by R, and since $\mathcal{E}$ is closed under small products, it follows that h is inverted by R; thus h, being a morphism between objects of $\mathcal{B}$, is itself invertible. Accordingly the left–exact R preserves all small limits. Since $\mathcal{A}$ is locally small, complete, and wellpowered, and has a cogenerator, it follows from Freyd's Special Adjoint Functor Theorem (see [14, Ch.5, §8, Theorem 2]) that R has a left adjoint, and $\mathcal{B}$ is essential. □

Remark 4.3 $\mathrm{Top}^{\mathrm{op}}$, the dual of the category of topological spaces, satisfies the hypotheses of Theorem 4.2; the one–point space 1 is a generator of Top (which has no strong generator), while a strong cogenerator is formed by the chaotic space $2 = \{0,1\}$ and the Sierpinski space 2_s; the strong monomorphisms in Top are the subspace–inclusions, and these are closed under coproducts. Since 2 and 2_s have but a finite number of subobjects in $\mathrm{Top}^{\mathrm{op}}$, it follows from Theorem 4.2 that every localization of $\mathrm{Top}^{\mathrm{op}}$, or equivalently every colocalization of Top, is essential. A very simple analysis of cases shows that there are just three such colocalizations, namely Top itself, the subcategory of discrete spaces, and the subcategory given by the empty space alone. By Theorem 2.4, therefore, the three essential localizations of Top are Top itself, the subcategory of chaotic spaces, and the subcategory of one–element spaces. Theorem 4.2, however, tells us nothing about the set of *all* localizations of Top.

Similar remarks apply to other categories which resemble Top; but we henceforth restrict ourselves to the primary object of this section, by taking $\mathcal{A}$ to be the presheaf category $[\mathbf{A}^{\mathrm{op}},\mathrm{Set}]$ for some small category $\mathbf{A}$. This $\mathcal{A}$ satisfies of course the hypotheses of Theorem 4.2. The strong generator $\mathcal{G}$ that we choose in order to define the $\mathcal{T}$ of Proposition 4.1 and Theorem 4.2 is the set of representable functors $\mathbf{A}(-,A)$ for $A \in \mathbf{A}$;

in fact we abbreviate by treating the fully–faithful Yoneda embedding $\mathbf{A} \longrightarrow [\mathbf{A}^{op},\mathbf{Set}]$ as an inclusion, and writing A for $\mathbf{A}(-,A)$. {Since all monomorphisms and all epimorphisms in $\mathcal{A}$ are strong, there is no difference between a strong generator and a generator, or between a strong cogenerator and a cogenerator.} A cogenerator of $\mathcal{A}$ is given by the presheaves $[\mathcal{A}(A,-),2]$ where 2 is the two–element set $\{0,1\}$ and $[X,Y]$ denotes the power–set Y^X. Finally, products of epimorphisms in $\mathcal{A}$ are again epimorphisms, since this is true in $\mathbf{Set}$ and since limits and colimits in $\mathcal{A} = [\mathbf{A}^{op},\mathbf{Set}]$ are formed pointwise.

By Proposition 4.1(iv), the function sending the localization $\mathcal{B}$ to $\mathcal{T}$ is, like the Φ of Proposition 2.1, an order–reversing injection. In the case of our presheaf category $\mathcal{A}$, it is classical that $\mathcal{T}$ lies in the image of this injection precisely when it is a *Grothendieck topology* on $\mathbf{A}$, in the sense that it satisfies (see, for example [10, Section 0.3] the following three conditions, where $\mathcal{T}(A)$ for $A \in \mathbf{A}$ denotes as in Theorem 4.2 the set of subobjects of A in $\mathcal{A}$ that lie in $\mathcal{T}$, and where f^*u denotes the pullback of u along f:

GT1. For each $A \in \mathbf{A}$ the identity $1_A: A \longrightarrow A$ is in $\mathcal{T}(A)$.

GT2. f^*u is in $\mathcal{T}(B)$ whenever $u: U \longrightarrow A$ is in $\mathcal{T}(A)$ and $f: B \longrightarrow A$ in $\mathbf{A}$.

GT3. A subobject $v: V \longrightarrow A$ of A in $\mathcal{A}$ is in $\mathcal{T}(A)$ if, for some $u: U \longrightarrow A$ in $\mathcal{T}(A)$, we have $f^*v \in \mathcal{T}(B)$ for every $B \in \mathbf{A}$ and every $f: B \longrightarrow A$ that factorizes through u.

{Recall that to give a subobject $u: U \longrightarrow A = \mathbf{A}(-,A)$ in $\mathcal{A}$ is to give for each $C \in \mathbf{A}$ a subset UC of $\mathbf{A}(C,A)$, in such a way that $gh \in UC$ whenever $g \in UD$ and $h \in \mathbf{A}(C,D)$; then $[U] = \Sigma_{C \in \mathbf{A}} UC$ is the corresponding *sieve* on A, consisting of all the morphisms in $\mathbf{A}$ with codomain A that factorize through u.}

By Theorem 4.2, therefore, to give an *essential* localization $\mathcal{B}$ of the presheaf category $\mathcal{A}$ is equally to give, for each $A \in \mathbf{A}$, a subobject $t_A\colon T(A) \longrightarrow A$ of A in $\mathcal{A}$, in such a way that, if we *define* $\mathcal{T}(A)$ to consist of the subobjects of A greater than or equal to t_A, we have GT1–GT3. To give the subobjects t_A for each A is to give for each $C,A \in \mathbf{A}$ a subset $\mathcal{I}(C,A) = T(A)C$ of $\mathbf{A}(C,A)$ with the property that $gh \in \mathcal{I}(C,A)$ whenever $g \in \mathcal{I}(D,A)$ and $h \in \mathbf{A}(C,D)$; we call such an $\mathcal{I}$ a *right ideal* of $\mathbf{A}$; it is the same thing as a sieve $\mathcal{I}(-,A)$ on A for each $A \in \mathbf{A}$. Of course $f \in \mathbf{A}(C,A)$ factorizes though t_A if and only if $f \in \mathcal{I}$.

With $\mathcal{T}$ so defined in terms of the t_A and thus in terms of the right ideal $\mathcal{I}$, GT1 is trivially satisfied; while GT2 is precisely the condition that ft_B factorizes through t_A for all $f \in \mathbf{A}(B,A)$, or equivalently that $fg \in \mathcal{I}(D,A)$ whenever $g \in \mathcal{I}(D,B)$ – which we express by saying that the right ideal $\mathcal{I}$ of $\mathbf{A}$ is a *two-sided ideal*, or more concisely an *ideal*. It remains to consider what conditions on $\mathcal{I}$ are imposed by GT3.

If $\mathcal{I}$ and $\mathcal{J}$ are ideals of $\mathbf{A}$, so too is the set $\mathcal{I}\mathcal{J}$ of all fg where $f \in \mathcal{I}$ and $g \in \mathcal{J}$. An ideal $\mathcal{I}$ is said to be idempotent if $\mathcal{I}\mathcal{I} = \mathcal{I}$; since trivially $\mathcal{I}\mathcal{I} \subset \mathcal{I}$, idempotence is in fact the assertion that $\mathcal{I} \subset \mathcal{I}\mathcal{I} = \mathcal{I}^2$, or that every $f \in \mathcal{I}$ can be written as $f = gh$ with $g,h \in \mathcal{I}$.

If we have GT3 as stated, we have it in particular when $u\colon U \longrightarrow A$ is taken to be $t_A\colon T(A) \longrightarrow A$; conversely, if we have GT3 when u is t_A, we have it for any $u \in \mathcal{T}(A)$, since an $f\colon B \longrightarrow A$ that factorizes through t_A certainly factorizes through u. What GT3 asserts when u is t_A is that the subobject $v\colon V \longrightarrow A$ is in $\mathcal{T}(A)$ if $f^*v \in \mathcal{T}(B)$ for all $f \in \mathcal{I}(B,A)$. To say that $f^*v \in \mathcal{T}(B)$ is equally to say that ft_B factorizes through v, which in turn is to say that $fg\colon C \longrightarrow A$ factorizes through v for all $C \in \mathbf{A}$ and all $g \in \mathcal{I}(C,B)$. To say that v is in $\mathcal{T}(A)$ is to say, in terms of the corresponding sieve $\{V\}$, that

$\mathcal{I}(-,A) \subset [V]$. Accordingly the import of GT3 is this: given a sieve $[V]$ on A, if $\mathcal{I}^2(-,A) \subset [V]$ then $\mathcal{I}(-,A) \subset [V]$. Since we may always take $[V]$ to be the sieve $\mathcal{I}^2(-,A)$, this is just the assertion that $\mathcal{I}(-,A) \subset \mathcal{I}^2(-,A)$ for each A, or that $\mathcal{I} = \mathcal{I}^2$. Thus:

Theorem 4.4 *There is an order-preserving bijection between essential localizations* $\mathcal{B}$ *of the presheaf category* $\mathcal{A} = [A^{op},Set]$ *and idempotent ideals* $\mathcal{I}$ *of* A. *Given* $\mathcal{B}$, *we find* $\mathcal{I}(-,A) \longrightarrow A$ *as the smallest subobject of* A *in* $\mathcal{A}$ *that lies in* $\mathcal{B}^{\perp}$; *given* $\mathcal{I}$, *we find* $\mathcal{B}$ *as* $\mathcal{T}^{\perp}$ *where* $\mathcal{T}(A)$ *consists of all the subobjects of* A *greater than or equal to* $\mathcal{I}(-,A) \longrightarrow A$.

Remark 4.5 Now suppose that B is a (full replete) subcategory of A, with inclusion $i: B \longrightarrow A$. The latter induces a functor $i^*: \mathcal{A} = [A^{op},Set] \longrightarrow [B^{op},Set]$, which is just composition with i^{op}, and which has left and right adjoints given by the left and right Kan extensions along i^{op}. Because i is fully faithful, so is each of these Kan adjoints. It follows form Section 2 that we have an essential localization $\mathcal{B}$ of $\mathcal{A}$ given by the full replete image of $Ran_{i^{op}} : [B^{op},Set] \longrightarrow [A^{op},Set]$, whose associated essential colocalization $\overline{\mathcal{B}}$ is the full replete image of $Lan_{i^{op}}$.

Theorem 4.6 *The ideal* $\mathcal{I}$ *of* A *corresponding by Theorem 4.4 to the essential localization* $\mathcal{B}$ *of Remark 4.5 consists of those morphisms of* A *which factorize through some object of* B.

Proof $\mathcal{E} = \mathcal{B}^{\perp}$ consists of the morphisms $q: F \longrightarrow G$ in $\mathcal{A}$ inverted by i^*, and therefore of the morphisms for which $qB: FB \longrightarrow GB$ is invertible for each $B \in B$. For $A \in A$, a subobject $u: U \longrightarrow A$ is therefore in $\mathcal{E}$ exactly when UB is all of A(B,A) for each $B \in B$; which is to say that the corresponding sieve [U] on A contains all the morphisms $C \longrightarrow A$ that factorize through some $B \in B$. The result follows. □

Remark 4.7 In the situation of Remark 4.5, different subcategories B, C of A may give rise to the same essential localization $\mathcal{B}$; by Theorems 4.4 and 4.6, this happens precisely when each 1_B for $B \in \mathbf{B}$ factorizes through some $C \in \mathbf{C}$ and each 1_C for $C \in \mathbf{C}$ factorizes through some $B \in \mathbf{B}$ — that is, when each B is a retract of a C and vice versa. Equivalently, we do not change $\mathcal{B}$ if we augment B by adding to its objects all their retracts in A; and then this augmented B is uniquely determined by $\mathcal{B}$, consisting in fact of those $A \in \mathbf{A}$ with $1_A \in \mathcal{I}$.

Remark 4.8 Let us write $\mathbf{A}^*$ for the *Cauchy completion* of a small A, obtained by freely splitting the idempotents of A. Recall that an object of $\mathbf{A}^*$ is an idempotent $e: A \to A$ in A, while a morphism in $\mathbf{A}^*$ from $e : A \to A$ to $e' : A' \to A'$ is an $f \in \mathbf{A}(A,A')$ with $fe = f = e'f$; and that the embedding $\mathbf{A} \to \mathbf{A}^*$ sending A to $1: A \to A$ induces an equivalence $[\mathbf{A}^{*\mathrm{op}},\mathbf{Set}] \to [\mathbf{A}^{\mathrm{op}},\mathbf{Set}] = \mathcal{A}$. Accordingly we can get further essential localizations $\mathcal{B}$ of $\mathcal{A}$ by taking B in Remark 4.5 to be a subcategory not of A but of $\mathbf{A}^*$. The reader will easily verify that the idempotent ideal $\mathcal{I}$ of A corresponding to such a B consists of those morphisms that have the form heg for some $e: B \to B$ in B.

Remark 4.9 It is clear from Theorem 4.2 that *every* localization of $\mathcal{A} = [\mathbf{A}^{\mathrm{op}},\mathbf{Set}]$ is essential if A is such that, for each A, the ordered set of sieves on A satisfies the descending chain condition; and in particular, therefore, if A is finite. Since $\mathbf{A}^*$ is finite when A is so, we may as well, by Remark 4.8, suppose A to be Cauchy complete in the following observation, which is [9, Exercise 9.1.12]. The proof outlined there uses very deep results; there is a quite elementary proof in some unpublished 1977 notes of P.T. Johnstone entitled "Topologies on finite categories"; the proof that follows is shorter still.

Proposition 4.10 *If idempotents split in the finite* A, *every localization of* $\mathcal{A} = [\mathrm{A}^{\mathrm{op}}, \mathrm{Set}]$ *arises as in Remark 4.5 from a subcategory* B *of* A.

Proof Let the localization $\mathcal{B}$ of $\mathcal{A}$, which is essential by Remark 4.9, correspond as in Theorem 4.4 to the idempotent ideal $\mathcal{I}$ of A. Define the subcategory B of A to consist of those $A \in \mathrm{A}$ for which $1_A \in \mathcal{I}$, and write $\mathcal{J}$ for the idempotent ideal of A consisting of those morphisms which factorize through some $B \in \mathrm{B}$. Clearly $\mathcal{J} \subset \mathcal{I}$, and we are to prove that $\mathcal{I} \subset \mathcal{J}$. First, since $\mathcal{I} = \mathcal{I}^2 = \mathcal{I}^3 = \dots$, any f in $\mathcal{I}$ has the form

$$A_0 \xrightarrow[f_1]{} A_1 \xrightarrow[f_2]{} A_2 \longrightarrow \cdots \longrightarrow A_{n-1} \xrightarrow[f_n]{} A_n$$

with each $f_i \in \mathcal{I}$ and n arbitrarily large. Because A is finite, the A_i here with $0 < i < n$ cannot be all different when n is large enough; accordingly f has the form htg for some *endomorphism* $t: E \longrightarrow E$, with $h,t,g \in \mathcal{I}$. Now apply the same argument with t in place of f; we get $t = h_1 t_1 g_1$ where t_1 is an endomorphism and $h_1, t_1, g_1 \in \mathcal{I}$. Continue thus, with $t_1 = h_2 t_2 g_2$, and so on. Because A is finite, we must have $t_r = t_{r+m}$ for some r and some $m > 0$. It suffices, of course, to show that $t_r \in \mathcal{I}$. Writing s for the endomorphism t_r, we have $s = ysx$ where $y,s,x \in \mathcal{I}$. Thus $s = y^k s x^k$ for all $k \geq 1$. For k large enough, we must since A is finite have $y^k = y^{2k}$; so $y^k = e$ is an idempotent belonging to $\mathcal{I}$, and it suffices to show that $e \in \mathcal{J}$. Let $e: A \longrightarrow A$ split as $e = ip$, where $i: B \longrightarrow A$ and $p: A \longrightarrow B$ satisfy $pi = 1_B$. Since $p = pip = pe$, we have $p \in \mathcal{I}$; so $1_B = pi$ lies in $\mathcal{I}$, and $B \in \mathrm{B}$. This completes the proof.

5. PARTICULAR EXAMPLES IN PRESHEAF CATEGORIES

Because of the need to relate our considerations in Section 4 to classical Grothendieck topologies, we felt constrained to write a typical presheaf category $\mathcal{A}$ as

$[A^{op}, Set]$. For our examples below, however, it is much more convenient to avoid a superfluous dualization by taking $\mathcal{A}$ to be $[A, Set]$, which we do henceforth. Note that, because the notion of an ideal $\mathcal{I}$ of A is self–dual, Theorem 4.4 and 4.6 are totally insensitive to this change. We abbreviate further by writing $\mathcal{S}$ for Set, writing $\mathcal{S}^A$ for $[A, Set]$ whenever convenient, and writing $\mathcal{S}^q: \mathcal{S}^B \longrightarrow \mathcal{S}^A$ for the functor q^* induced by $q: A \longrightarrow B$.

Example 5.1 For our primary example we take for A the category freely generated by the graph with two objects P and Q and two arrows $f: P \longrightarrow Q$ and $g: Q \longrightarrow P$. An object F of $\mathcal{A} = \mathcal{S}^A$ is accordingly given by two sets $X = FP$ and $Y = FQ$, along with two functions

$$\phi = Ff: X \longrightarrow Y \quad , \quad \psi = Fg: Y \longrightarrow X. \tag{5.1}$$

Taking for B and C respectively the full subcategories of A whose object–sets are $\{P\}$ and $\{Q\}$, we get as in Remark 4.5 essential localizations $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A}$, with associated essential colocalizations $\overline{\mathcal{B}}$ and $\overline{\mathcal{C}}$. Write N for the monoid of natural numbers, seen as a category with a single object $*$ and with generator $e: * \longrightarrow *$, so that a typical morphism is (not n but) e^n. Each of B and C is a monoid isomorphic to N, the respective generators being gf and fg. Instead of taking i to be as in Remark 4.5 the inclusion $B \longrightarrow A$, and j to be the corresponding inclusion $C \longrightarrow A$, we find it more convenient to invoke the isomorphisms above, and to define i and j to be the fully–faithful functors $N \longrightarrow A$ given by $i(*) = P$, $i(e) = gf$ and by $j(*) = Q$, $j(e) = fg$. It is of course still the case that $\mathcal{B}$ and $\mathcal{C}$ are the full replete images of Ran_i and Ran_j, while $\overline{\mathcal{B}}$ and $\overline{\mathcal{C}}$ are the full replete images of Lan_i and Lan_j.

It is immediate that each of B and C is both reflective and coreflective in A. There is in addition a relation between the two reflexions and the two coreflexions, which is best expressed by speaking instead of the adjoints to i and to j, as follows. The category A being free on the appropriate graph, we can define functors $r,s\colon A \to N$, necessarily sending P and Q to $\star$, by

$$r(f) = e\,,\ r(g) = 1\ ;\ s(f) = 1,\ s(g) = e. \tag{5.2}$$

It is at once clear that we have a cycle of adjunctions

$$r \dashv i \dashv s \dashv j \dashv r, \tag{5.3}$$

with $ri = si = rj = sj = 1$.

The adjunctions (5.3) induce adjunctions

$$\mathcal{S}^{r} \dashv \mathcal{S}^{j} \dashv \mathcal{S}^{s} \dashv \mathcal{S}^{i} \dashv \mathcal{S}^{r}; \tag{5.4}$$

from which it follows that we can take

$$\mathrm{Ran}_i = \mathrm{Lan}_j = \mathcal{S}^{r}\ ,\ \mathrm{Ran}_j = \mathrm{Lan}_i = \mathcal{S}^{s}. \tag{5.5}$$

The functor $\mathcal{S}^{r}\colon \mathcal{S}^{N} \to \mathcal{S}^{A}$ sends an object $\varepsilon\colon W \to W$ of $\mathcal{S}^{N}$ to the object (5.1) of $\mathcal{A}$ given by $\phi = \varepsilon\colon W \to W$ and $\psi = 1\colon W \to W$. It is easy to see that a general object (5.1) of $\mathcal{A}$ is isomorphic to one of this form, and hence belongs to the full replete image of $\mathcal{S}^{r}$, if and only if ψ is invertible. Thus, using (ϕ,ψ) for a typical object of $\mathcal{A}$ as in (5.1), we have

$$\mathcal{B} = \overline{\mathcal{C}} = \{(\phi,\psi)\,|\,\psi \text{ invertible}\}. \tag{5.6}$$

Exactly similar considerations applied to $\mathcal{S}^s$ give

$$\mathcal{C} = \overline{\mathcal{B}} = \{(\phi,\psi) \mid \phi \text{ invertible}\}. \tag{5.7}$$

Writing henceforth $\mathcal{D}$ for $\mathcal{B} \cap \mathcal{C}$, we accordingly have

$$\mathcal{D} = \mathcal{B} \cap \mathcal{C} = \overline{\mathcal{B}} \cap \overline{\mathcal{C}} = \{(\phi,\psi) \mid \phi \text{ and } \psi \text{ invertible}\}. \tag{5.8}$$

We now use Theorem 4.4 to verify that $\mathcal{D} = \mathcal{B} \cap \mathcal{C}$, although by [3, Theorem 6.8] it is a localization of $\mathcal{A}$, is not an essential one. Let $\mathcal{I}$ and $\mathcal{J}$ be the idempotent ideals of A corresponding by Theorem 4.4 to the essential localizations $\mathcal{B}$ and $\mathcal{C}$ respectively. By Theorem 4.6, $\mathcal{I}$ consists of all the morphisms of A that factorize through P; that is, all the morphisms of A except 1_Q. Similarly $\mathcal{J}$ consists of all the morphisms of A except 1_P. So the ideal $\mathcal{I} \cap \mathcal{J}$ of A consists of all the words in f and g of length > 0. Let $\mathcal{K}$ be an idempotent ideal contained in $\mathcal{I} \cap \mathcal{J}$. If $\mathcal{K}$ were not empty, it would contain a word in f and g of minimal length; but then it could not be idempotent. So the idempotent ideal $\mathcal{K}$ corresponding to the infimum $\mathcal{B} \wedge \mathcal{C}$ in Ess $\mathcal{A}$ is empty. By Theorem 4.4, therefore, the corresponding $\mathcal{T}$(A) consists of *all* the subobjects of A in $\mathcal{A}$, whence $\mathcal{B} \wedge \mathcal{C}$ consists of the terminal object 1 of $\mathcal{A}$ and its isomorphs.

It follows from Theorem 2.4 that the infimum $\overline{\mathcal{B}} \wedge \overline{\mathcal{C}}$ in CoEss $\mathcal{A}$ is $\overline{\mathcal{B} \wedge \mathcal{C}}$; it is therefore the subcategory of $\mathcal{A}$ consisting of the initial object 0 alone. Thus $\overline{\mathcal{B}} \wedge \overline{\mathcal{C}}$, too, differs from $\overline{\mathcal{B}} \cap \overline{\mathcal{C}} = \mathcal{D}$. In fact $\overline{\mathcal{B}} \cap \overline{\mathcal{C}}$, although a coreflective subcategory of $\mathcal{A}$, is not even a colocalization; this is the example we promised in the Introduction that goes beyond [3, Example 5.2]. The following considerations not only establish this, but provide an alternative proof that $\mathcal{B} \cap \mathcal{C}$ is not an essential localization by showing that the reflexion fails to preserve infinite products.

Example 5.2 We retain the notation of Example 5.1, of which this is but a continuation. The subcategory $\mathcal{D}$ of $\mathcal{A}$ given by (5.8) may be identified with $\mathcal{S}^{D}$, where D is the category generated by the graph with two objects P and Q and four arrows $\bar{f}, \bar{g}': P \longrightarrow Q$ and $\bar{g}, \bar{f}': Q \longrightarrow P$, subject to the conditions that $\bar{f}$ and $\bar{f}'$ be mutually inverse and that $\bar{g}$ and $\bar{g}'$ be mutually inverse. Write $p: A \longrightarrow D$ for the functor sending P, Q, f, g to P, Q, $\bar{f}$, $\bar{g}$; then we may identify the inclusion $\mathcal{D} \longrightarrow \mathcal{A}$ with $\mathcal{S}^{p}: \mathcal{S}^{D} \longrightarrow \mathcal{S}^{A}$. This of course has left and right adjoints Lan_p and Ran_p, exhibiting $\mathcal{D}$ as both a reflective subcategory of $\mathcal{A}$ (known by [3, Theorem 6.8] to be in fact a localization) and as a coreflective subcategory of $\mathcal{A}$ (which latter we had no *a priori* reason to suppose). We re-verify that the localization $\mathcal{D}$ is not essential by observing that Lan_p fails to preserve infinite products, and we verify that $\mathcal{D}$ is not a colocalization by observing that Ran_p fails to preserve epimorphisms.

We simplify the calculations by arguing somewhat indirectly. Write Z for the infinite cyclic group on the generator $\bar{e}$, seen as a category with one object $*$, and write $q: N \longrightarrow Z$ for the functor sending e to $\bar{e}$. There are obviously unique functors $\bar{i}, \bar{r}, \bar{j}, \bar{s}$ rendering commutative

$$\begin{tikzcd}
N \arrow[r, "i"] \arrow[d, "q"'] & A \arrow[r, "r"] \arrow[d, "p"] & N \arrow[r, "j"] \arrow[d, "q"] & A \arrow[r, "s"] \arrow[d, "p"] & N \arrow[d, "q"] \\
Z \arrow[r, "\bar{i}"'] & D \arrow[r, "\bar{r}"'] & Z \arrow[r, "\bar{j}"'] & D \arrow[r, "\bar{s}"'] & Z ,
\end{tikzcd} \tag{5.9}$$

and they are clearly equivalences. Recall from, say, [12, Theorem 4.47] that $\mathrm{Lan}_p \mathrm{Lan}_j \equiv \mathrm{Lan}_{pj} = \mathrm{Lan}_{\bar{j}q} \equiv \mathrm{Lan}_{\bar{j}} \mathrm{Lan}_q$. Here $\mathrm{Lan}_{\bar{j}}$ like $\bar{j}$ is an equivalence, while Lan_j, being $\mathcal{S}^{r}$ by (5.5), preserves all limits; so if Lan_p preserves all products, so does Lan_q. In the same way we have $\mathrm{Ran}_p \mathrm{Ran}_i \equiv \mathrm{Ran}_{\bar{i}} \mathrm{Ran}_q$, where $\mathrm{Ran}_{\bar{i}}$ is an equivalence and Ran_i, which by (5.5) is again $\mathcal{S}^{\Gamma}$, preserves all colimits; so if Ran_p preserves epimorphisms, so does Ran_q. We are thus reduced to proving that Lan_q does not preserve products, nor Ran_q epimorphisms; and we do so by calculating these Kan adjoints explicitly.

We can identify an object F of $\mathcal{S}^{\mathbf{N}}$ with a set W along with an endofunction $\varepsilon\colon W \longrightarrow W$, the object being in $\mathcal{S}^{\mathbf{Z}}$ when ε is invertible. Let us write Ran_q and Lan_q of $\varepsilon\colon W \longrightarrow W$ as $\varepsilon'\colon W' \longrightarrow W'$ and $\varepsilon''\colon W'' \longrightarrow W''$ respectively. Using the classical formulas for Kan extensions in terms of limits and colimits – see [14, Chapter X, Theorem 1] or [12, Section 4.2] – we see that W' is given as the limit, and W'' as the colimit, of the doubly-infinite sequence

$$\cdots \xrightarrow[\varepsilon]{} W \xrightarrow[\varepsilon]{} W \xrightarrow[\varepsilon]{} W \xrightarrow[\varepsilon]{} W \xrightarrow[\varepsilon]{} \cdots .$$

An element of W', then, is a family $w = (w_n \mid n \in \mathbf{Z})$ of elements of W which satisfy $\varepsilon w_n = w_{n+1}$; and it is easy to see that the isomorphism $\varepsilon'\colon W' \longrightarrow W'$ is given by $(\varepsilon' w)_n = w_{n+1}$. On the other hand W'' is the quotient of $\mathbf{Z} \times W$ by the relation $(n+1,w) \sim (n,\varepsilon w)$, and ε'' sends $[(n,w)]$ to $[(n+1,w)]$.

Now take $\varepsilon\colon W \longrightarrow W$ to be $\sigma\colon \mathbf{N} \longrightarrow \mathbf{N}$, where σ is the successor function. It is clear that W' is empty. On the other hand Ran_q takes the terminal object $1\colon 1 \longrightarrow 1$ of $\mathcal{S}^{\mathbf{N}}$, which in fact lies in $\mathcal{S}^{\mathbf{N}}$, to itself. So Ran_q does not preserve the epimorphism $(\mathbf{N},\sigma) \longrightarrow 1$, where henceforth (W,ε) is used for $\varepsilon\colon W \longrightarrow W$. Not only does this confirm that $\mathcal{D}$ is not a colocalization of $\mathcal{A}$; it enables us to conclude that the only colocalization $\mathcal{F}$ contained in $\mathcal{D}$ is $\{0\}$; for the coreflexion of $\mathcal{D}$ onto $\mathcal{F}$ must invert $0 \longrightarrow 1$. Thus the infimum of $\overline{\mathcal{B}}$ and $\overline{\mathcal{C}}$ in $\mathrm{CoLoc}\,\mathcal{A}$ is their intersection $\{0\}$ in $\mathrm{CoEss}\,\mathcal{A}$.

Again with $(W,\varepsilon) = (N,\sigma)$, consider (W'',ε''). An element of $Z \times N$ has, under the relation $\sim$ above, a normal form $(n,0)$; so that W'' may be identified with Z, and ε'' with $\sigma\colon Z \longrightarrow Z$ sending n to $n+1$. Next, take for (W,ε) the N–th power (N^N,σ^N) of (N,σ) in $\mathcal{S}^N$. An element of $Z \times N^N$ is a pair (n,w) where w is a sequence $(w_n \mid n \in N)$ of elements of N; and $\sigma^N(w)$ being the sequence $(w_n{+}1 \mid n \in N)$, such an element has under $\sim$ a normal form $(0,w)$, where w is now a sequence $(w_n \mid n \in N)$ of elements not of N but of Z, subject however to the condition that the sequence be bounded below in Z. To accord with this, the normal form above for an element of $Z \times N$ could have been written $(0,n)$ rather than $(n,0)$, with the understanding that n here may be negative. Abbreviating Lan_q to L, we see that the canonical comparison map $L((N,\sigma)^N) \longrightarrow (L(N,\sigma))^N$ is the inclusion into Z^N of those sequences $w \in Z^N$ which are bounded below. Since this is not an isomorphism, Lan_q does not preserve infinite products.

Example 5.3 The authors spent some little time looking for an example, of the same general kind as that above, where $\mathcal{B} \cap \mathcal{C}$ *is* the infimum of $\mathcal{B}$ and $\mathcal{C}$ in Ess $\mathcal{A}$, but $\bar{\mathcal{B}} \cap \bar{\mathcal{C}}$ is *not* the infimum of $\bar{\mathcal{B}}$ and $\bar{\mathcal{C}}$ in CoEss $\mathcal{A}$. They did not succeed in the time available; but the following example may be worth noting, in that $\bar{\mathcal{B}} = \mathcal{B}$ and $\bar{\mathcal{C}} = \mathcal{C}$.

This time we again take A to be generated by the graph with two objects P and Q and with two arrows $f\colon P \longrightarrow Q$ and $g\colon Q \longrightarrow P$, but now subject to the relations $fgf = f$ and $gfg = g$. The full subcategories B and C of A again give rise, as in Remark 4.5, to essential localizations $\mathcal{B}$ and $\mathcal{C}$ of $\mathcal{A} = \mathcal{S}^A$. It turns out that both $\mathcal{B}$ and $\bar{\mathcal{B}}$ consist of those objects $(\phi\colon X \longrightarrow Y, \psi\colon Y \longrightarrow X)$ of $\mathcal{A}$ for which $\psi\phi = 1$, while both $\mathcal{C}$ and $\bar{\mathcal{C}}$ consist of those with $\phi\psi = 1$; so that $\mathcal{D} = \mathcal{B} \cap \mathcal{C}$ consists of those for which ϕ and ψ are mutually inverse. Since $\mathcal{B} \cap \mathcal{C}$ is known to be a localization of $\mathcal{A}$ by [3, Theorem 6.8], and since (A here being finite) all localizations of $\mathcal{A}$ are essential by Remark 4.9, $\mathcal{B} \cap \mathcal{C}$ is necessarily the infimum of $\mathcal{B}$ and $\mathcal{C}$ in Ess $\mathcal{A}$. Once again $\mathcal{D}$ has the form $\mathcal{S}^D$

for a suitable D, and is hence not only reflective in $\mathcal{A}$ but also coreflective. In fact, D being equivalent to 1, the category $\mathcal{S}^{\mathrm{D}}$ is equivalent to $\mathcal{S}$, and the inclusion $\mathcal{D} \longrightarrow \mathcal{A}$ is in effect the diagonal functor $\Delta: \mathcal{S} \longrightarrow \mathcal{S}^{\mathrm{A}}$. Its left and right adjoints are lim and colim; and for this A, these coincide.

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