Equality in hyperdoctrines and comprehension schema as an adjoint functor

F. William Lawvere

0. The notion of hyperdoctrine was introduced (*Adjointness in Foundations*, to appear in Dialectica) in an initial study of systems of categories connected by specific kinds of adjoints of a kind that arise in formal logic, proof theory, sheaf theory, and group-representation theory. It appears that abstract structures of this kind are also intimately related to Gödel's proof of the consistency of number theory (Dialectica 1958) and to Läuchli's complete semantics for intuitionistic logic (to appear in Proceedings of the Buffalo Conference on Intuitionism and Proof Theory), although the precise relationship is yet to be worked out. Since then the author has noticed that yet another "logical operation", namely that which assigns to every formula φ its "extension" $\{x: \varphi(x)\}$ is characterized by adjointness, and that the "same" adjoint in a different hyperdoctrine leads to the notion of fibered category (or in particular the covering groupoid of a permutation group). The second part of this article is devoted to a preliminary discussion of this sort of adjoint, which we call tentatively the Comprehension Schema. The first part of the article concerns two kinds of identities which a hyperdoctrine may satisfy, and which lead in particular to a more or less satisfactory theory of the attribute "equality". One of these kinds of identities is formally similar to, and reduces in particular to, the Frobenius reciprocity formula for permutation representations of groups. Actually our definition of "equality" is *not* satisfactory when these identities do not hold, though from examples one surmises that a satisfactory theory could be developed by introducing still more structure into the already rather rich notion of hyperdoctrine.

We recall the basic ingredients of a hyperdoctrine: there is to be a category $\mathbf{T}$ of "types", whose morphisms are called "terms", and which is assumed to be cartesian closed. For each type X there is a cartesian closed category $P(X)$ of "attributes of type X", whose morphisms are called "deductions over X", and for every term $f: X \to Y$ there is a functor $f \cdot (\ \ ) P(Y) \to P(X)$ called "substitution of f in ( )" for which it is assumed that $f \cdot (g \cdot \varphi) = (fg) \cdot \varphi$ for $g: Y \to Z$ a term and φ an attribute of type Z (or a deduction over Z). Actually we should in principle only give natural isomorphisms $f \cdot (g \cdot (\ \ )) \cong (fg) \cdot (\ \ )$ and assume that these are coherent, but actual equality holds in the examples which we consider here. Finally there are given, for each term $f: X \to Y$, two functors $(\ \ )\Sigma f$ and $(\ \ )\Pi f$ respectively left and right adjoint to substitution, called "existential, respectively universal, quantification along f". By general properties of adjoints we have then canonical natural isomorphisms

$$(\Phi\Sigma f)\Sigma g \cong \varphi\Sigma(fg), \qquad (\varphi\Pi f)\Pi g \cong \varphi\Pi(fg)$$

for any attribute φ of type X.

All the adjointness relations involved in a hyperdoctrine are supposed to involve given front and back adjunction maps, so that the theory of hyperdoctrines is a purely equational calculus. Nevertheless, we shall mostly use only the hom-set bijections induced by the adjunction morphisms, and in fact we will indicate these bijections in the manner usually used for rules of inference. Thus the cartesian closed structure of **T**, for example, involves three adjoints: First there is the terminal object 1, right adjoint to $\mathbf{T} \to \mathbf{1}$, whose characteristic property is

$$\frac{X \to 1 \text{ (in } \mathbf{T})}{\circ \quad \text{(in } \mathbf{1})}$$

where the horizontal line indicates the canonical bijection of the morphisms of the sort above the line with those of sort below the line, and the dot denotes the unique morphism of the category **1**. Secondly there is the cartesian product, right adjoint to the diagonal functor $\mathbf{T} \to \mathbf{T} \times \mathbf{T}$, whose adjunction morphisms are the diagonal $X\delta: X \to X \times X$ and the projections $\langle Y_1, Y_2\rangle\pi_i: Y_1 \times Y_2 \to Y_i$, and whose characteristic property is expressed by the bijection

$$\frac{X \to Y_1 \times Y_2}{X \to Y_1,\ X \to Y_2}$$

where the ordered pair below the line may be thought of as a morphism in $\mathbf{T} \times \mathbf{T}$. Finally, for each type A, we have the right adjoint to $A \times (\ \ )$, called exponentation by A, whose adjunction natural transformations λ_A and ϵ_A can be "deduced" from the basic bijection

$$\frac{X \to Y^A}{A \times X \to Y}$$

by setting $Y = A \times X$ and considering the identity term below the line, respectively by setting $X = Y^A$ and considering the identity term above the line.

In the cartesian closed category $P(X)$ of attributes of type X, we call the terminal object 1_X the "identically true attribute of type X" (deductions over X with domain 1_X will sometimes be called "proofs over X"), and we denote product and exponentation as conjunction and implication, respectively. Thus the "evaluation" natural transformation ϵ could instead be called "modus ponens", and the adjointness relations become bijections of deductions over X as follows.

$$\frac{\varphi \to \psi_1 \wedge \psi_2}{\varphi \to \psi_1,\ \varphi \to \psi_2}, \qquad \frac{\varphi \to (\alpha \Rightarrow \psi)}{\alpha \wedge \varphi \to \psi}.$$

Finally the adjointness property for existential (and dually for universal) quantification along $f: X \to Y$ is expressed by the bijection

$$\frac{\varphi\Sigma f \to \psi}{\varphi \to f \cdot \psi}$$

between deductions over Y above and deductions over X below for each attribute φ of type X and attribute ψ of type Y. Here we have not bothered to give names to the adjunction transformations. This neglect, and our use of the "rule of inference" notation, indicates in particular that we are ignoring coherence questions; that is, in our assertions below in which we assert the existence of a canonical natural deduction $\varphi_1 \to \varphi_2$, we have not verified that there do not exist several such. Lambek, in the Proceedings of the Batelle Conference on Categorical Algebra and Homology Theory, has made a healthy start on the coherence problem by establishing Cut-Elimination for certain categories closely related to cartesian closed categories. In the same place, Gray, by introducing the appropriate notion of 2-dimensional adjointness, has shown that all the features of a hyperdoctrine, including our comprehension schema, can be obtained by defining a type to be an arbitrary category and an attribute of type **B** to be any fibration over **B**.

As pointed out in our Dialectica article, terms corresponding to all higher-type primitive recursive functions can be guaranteed by assuming a left adjoint to the forgetful functor $\tilde{T} \to \mathbf{T}$ (the domain being the usual category whose objects are endo-terms). However we have not here included this adjoint in our general definition as it plays no role in this paper.

We mention now some examples of hyperdoctrines. Given any theory (several sorted, institutionistic or classical) formulated in the language of finite types, define $\mathbf{T}$ to have as objects all type symbols $1, V_0, V_1, V_2, \ldots$ (one V_i for each sort), $V_i \times V_j, V_i^{V_j}, \ (V_i \times V_j)^{V_k} \times V_e^{(V_i^{V_n})}$, (i.e. all expressions obtained by closing the V_i with respect to product and exponentation) and as morphisms suitable equivalence classes of (tuples of) terms from the theory. The adjunction equations force certain identifications of terms, and additional identifications may be forced by axioms of the theory if there are terms provided by the theory in addition to those guaranteed by the requirement that $\mathbf{T}$ be cartesian closed (for instance, in higher-order number theory, the recursion-adjoint F of the preceding paragraph exists, and the natural numbers $1 \xrightarrow{n} 1F$, the successor map, $1F \xrightarrow{\sigma} 1F$, etc. are such additional terms, while the distributive law is such an additional identification). As objects in $P(X)$ take all formulas of the theory whose free variables correspond to the type X. For deductions over X, one may take provable entailments (so that the category $P(X)$ reduces to a preordered set) or one may take suitable "homotopy classes" of deductions in the usual sense. One can write down an inductive definition of the "homotopy" relation, but the author does not understand well what results (some light is shed on this question by the work of Läuchli and Lambek cited above). Thus, although such syntactically presented hyperdoctrines are quite important, it is fortunate for the intuition that there are also semantically-defined examples, as below.

There are two basic examples in which $\mathbf{T} = \mathscr{S}$ the category of all (small) sets and mappings. One has $P(X) = 2^X$ = the partially-ordered set of all propositional functions defined on X; if we confuse propositional functions with the corresponding subsets, we then must have that $\varphi_1 \wedge \varphi_2 = \varphi_1 \cap \varphi_2$ and that $\varphi\Sigma f$ is the direct image of φ along f (understanding that substitution is defined by composition, so that, under the confusion, $f \cdot \psi$ is the inverse image by f of ψ). Every model of a higher-order theory induces a morphism from the corresponding hyperdoctrine to this set-hyperdoctrine, and conversely. The other example has $P(X) = \mathscr{S}^X$, so that an attribute φ of type X is any family $x \cdot \varphi$ of sets indexed by $x \in X$ and a deduction $\varphi_1 \xrightarrow{d} \varphi_2$ over X is any family $x \cdot \varphi_1 \xrightarrow{x \cdot d} x \cdot \varphi_2$ of mappings. Thus $P(1) = \mathscr{S}$ is the "category of truth-values" for this hyperdoctrine. The relations

$$x \cdot (\alpha \Rightarrow \psi) = (x \cdot \psi)^{(x \cdot \alpha)}, \qquad y \cdot (\varphi \Pi f) = \prod_{xf=y} x \cdot \varphi,$$

$$y \cdot (\varphi \Sigma f) = \sum_{xf=y} x \cdot \varphi \text{ (disjoint sum)}$$

follow (from the definition of substitution as composition). By our general definition of "proof over X" it follows that the proofs (over 1) of $x \cdot \varphi$ for $1 \xrightarrow{x} X$ are precisely the elements of the set $x \cdot \varphi$. Thus, this hyperdoctrine may be viewed as a kind of set-theoretical surrogate of proof theory (honest proof theory would presumably also yield a hyperdoctrine with nontrivial $P(X)$, but a syntactically-presented one). For example, by the above equations, a proof over X of $\alpha \Rightarrow \psi$ is a function which, for each $1 \xrightarrow{x} X$ assigns to every proof that $x \cdot \alpha$ a corresponding proof that $x \cdot \psi$, while a proof over Y of $\varphi\Sigma f$ is a function assigning to every $1 \xrightarrow{y} Y$ an ordered pair consisting of an x such that $xf = y$ and a proof that x has the attribute φ.

The functor $\mathscr{S} \to \mathbf{2}$ taking the empty set to 0 and every other set to 1 induces a functor from the "proof" hyperdoctrine on $\mathbf{T}$ = sets to the "propositional-function" hyperdoctrine on $\mathbf{T}$ = sets which commutes with all the mentioned logical operations. The fact that it commutes with universal quantification is equivalent to the axiom of choice, or in the language of proofs, to a strong form of ω-completeness.

We will consider three examples in which types are small categories and terms are all functors between them. Here of course exponentiation of types must be the usual functor-category construction. One has $P(\mathbf{B}) = \mathbf{2}^{\mathbf{B}}$ = the category of all functors from $\mathbf{B}$ into the arrow category $\cong$ the Brouwerian lattice of all sets φ of objects of $\mathbf{B}$ with the property that if $B \xrightarrow{b} B'$ in $\mathbf{B}$ and $B \in \varphi$ then $B' \in \varphi$; we leave as an exercise the computation of implication and quantification. The second example has $P(\mathbf{B}) = \mathscr{S}^{\mathbf{B}}$. Hence one has

$$\alpha \Rightarrow \psi : B \rightsquigarrow \text{nat}\,(H^B \times \alpha, \psi) = \mathscr{S}^{\mathbf{B}}(H^B \wedge \alpha, \psi),$$

$$\varphi\Sigma f : C \rightsquigarrow \varinjlim\,[(f, C) \to \mathbf{B} \xrightarrow{\varphi} \mathscr{S}]$$

for $\mathbf{B} \xrightarrow{f} \mathbf{C}$ a functor and $\alpha, \varphi, \psi : \mathbf{B} \to \mathscr{S}$. The third example also has $P(\mathbf{B}) = \mathscr{S}^{\mathbf{B}}$, but we restrict the category $\mathbf{T}$ of types to consist only of those $\mathbf{B}$ which are groupoids i.e. categories in which all morphisms are isomorphisms. $\mathbf{T}$ is still cartesian closed since in fact $\mathbf{B}^{\mathbf{A}}$ is a groupoid for any category $\mathbf{A}$ if $\mathbf{B}$ is. If $\mathbf{B}$ and $\mathbf{C}$ have one object (i.e. are groups) then $P(\mathbf{B})$ is the category of all permutation representations of $\mathbf{B}$ and $\varphi\Sigma f$ is the so-called induced representation of $\mathbf{C}$. (Actually, there are two induced representations, the other being $\varphi\Pi f$, calculated roughly as the fixed point set of $\varphi^{\mathbf{C}}$ rather than the orbit set of $\varphi \times \mathbf{C}$. If f is of finite index the analogous constructions for *linear* representations yield isomorphic results, which is perhaps why there seems to be no established name for "universal quantification" in representation theory.)

Since we have not taken recursion as part of the definition, hyperdoctrines are also obtained if in the last five examples we replace small set, category, functor, etc. by finite set, category, functor.

Finally we remark that although our discussion below of comprehension hinges on the operation Σ, there is at least one structure, namely with types = Kelly spaces and attributes = set-valued sheaves in which all features of hyperdoctrines except Σ exist ($f \cdot (\;\;)$ is only exact, not continuous in general) but in which there is clearly a kind of "extension", namely the espace etalé.

1. We define, for each type X, an attribute of type $X \times X$ as follows

$$\Theta_X = 1_X\Sigma(X\delta).$$

The adjunction then provides a canonical deduction $1_X \to (X\delta) \cdot \Theta_X$ which we interpret to mean that "reflexivity" holds for "equality" so defined. We wish to investigate what other expected properties of equality hold, and more generally to study the interaction of existential quantification of attributes and cartesian products of types.

There are other expected properties of equality which we have not investigated; for example, considering the projections p, π_1, π_2, and the evaluation adjunction ϵ in

$$\begin{array}{ccc} X \times Y^X \times Y^X & \xrightarrow{p} & Y^X \times Y^X \\ \downarrow{\scriptstyle \pi_i} & & \\ X \times Y^X & \xrightarrow{\epsilon} & Y \end{array}$$

one might expect $\Theta_{Y^X} = ((\pi_1\epsilon)\Theta_Y(\pi_2\epsilon))\Pi p$ to hold. The intuitive interpretation of this equation, $f_1 = f_2 \Leftrightarrow \forall x[xf_1 = xf_2]$ does not quite reflect it adequately, for it does not necessarily mean that 1 is a generator for $\mathbf{T}$; for example, the equation holds in the hyperdoctrine derived from a higher-order theory, even though there may be no morphisms $x: 1 \to X$ in $\mathbf{T}$. However for what we are able to prove in this paper neither exponentiation of types nor universal quantification of attributes plays any role. Thus we only assume that we work in an arbitrary eed (elementary existential doctrine, defined like a hyperdoctrine except that Y^X and $\varphi\Pi f$ are not necessarily assumed to exist).

Reasonable relationships in an eed between products and equality as we have defined it turn partly on implication being strictly preserved by substitution.

PROPOSITION (SUBSTITUTIVITY OF EQUALITY). *In any eed in which, for every term $f: X \to Y$ and any two attributes α, ψ of type Y, the canonical deduction*

$$f \cdot (\alpha \Rightarrow \psi) \to f \cdot \alpha \Rightarrow f \cdot \psi$$

over X is an isomorphism, one also has, for any attribute φ of type X, a canonical deduction

$$\Theta_X \to \pi_1 \cdot \varphi \Rightarrow \pi_2 \cdot \varphi$$

over $X \times X$.

PROOF. The identity deduction $\varphi \to \varphi$ yields a canonical

$$1_X \to \varphi \Rightarrow \varphi = (\delta\pi_i)\varphi \Rightarrow (\delta\pi_2) \cdot \varphi \xleftarrow{\sim} \delta \cdot (\pi_1 \cdot \varphi \Rightarrow \pi_2 \cdot \varphi)$$

which by the adjointness of existential quantification along the diagonal used to define equality yields the conclusion: Thus in fact we only used the assumption for the case $f = \delta$.

DEFINITION-THEOREM. *In any eed, the following are equivalent:*

(1) *Frobenius Reciprocity holds.*

(2) *For any $f: X \to Y$, α, ψ in $P(Y)$ $\quad f \cdot (\alpha \Rightarrow \psi) \xrightarrow{\approx} f \cdot \alpha \Rightarrow f \cdot \psi$.*

(3) *For any $f: X \to Y$, $\varphi \in P(X)$, $\alpha \in P(Y)$ $\quad ((f \cdot \alpha) \wedge \varphi)\Sigma f \xrightarrow{\approx} \alpha \wedge (\varphi\Sigma f)$.*

PROOF. The second condition means that the diagram of functors

$$\begin{tikzcd}
P(Y) \arrow[r, "\alpha \Rightarrow (\ )"] \arrow[d, "f\cdot(\ )"'] & P(Y) \arrow[d, "f\cdot(\ )"] \\
P(X) \arrow[r, "(f\cdot\alpha) \Rightarrow (\ )"'] & P(X)
\end{tikzcd}$$

commutes up to canonical natural equivalence. Hence replacing each functor by its left adjoint also yields a diagram which commutes up to canonical natural equivalence:

$$\begin{tikzcd}
P(Y) & P(Y) \arrow[l, "\alpha\wedge(\ )"'] \\
P(X) \arrow[u, "(\ )\Sigma f"] & P(X) \arrow[l, "(f\cdot\alpha)\wedge(\ )"] \arrow[u, "(\ )\Sigma f"']
\end{tikzcd}$$

But the latter is just the third condition. Conversely if the third condition holds, we can replace the functors in the latter diagram by their right adjoints, yielding the second condition.

It is clear that Frobenius Reciprocity holds in both the 2-valued and set-valued hyperdoctrines with sets as types. However it does not hold in the set-valued hyperdoctrines with small categories as types. We provide a

COUNTEREXAMPLE. Let $f:\mathbf{1} \to \mathbf{2}$ be 1 and consider $\alpha, \psi:\mathbf{2} \to \mathscr{S}$ represented as $A \xrightarrow{\alpha} B$, $U \xrightarrow{\psi} V$ in $\mathscr{S}$ Then in general $f\cdot(\alpha \Rightarrow \psi) \to f\cdot \alpha \Rightarrow f\cdot \psi$ is not an isomorphism in $\mathscr{S}$

PROOF. $f\cdot\zeta$ is just the value of ζ at 1 for any $\zeta \in \mathscr{S}^{\mathbf{2}}$. We do have

$$(\alpha \Rightarrow \psi)_0 = U^A = \alpha_0 \Rightarrow \psi_0$$

but, since $H^1 = 1_2$,

$$(\alpha \Rightarrow \psi)_1 = \mathscr{S}^{\mathbf{2}}(\alpha, \psi) = U^A \underset{V^A}{\times} V^B$$

while

$$a_1 \Rightarrow \psi_1 = V^B.$$

Nevertheless, group theory is simpler than category theory.

PROPOSITION. *In the groupoid-permutation hyperdoctrine, Frobenius Reciprocity holds.*

PROOF. We need only show that substitution preserves implication. But in fact we have for any *groupoid* $\mathbf{C}$ and object $c \in \mathbf{C}$ and any two functors $\alpha, \psi:\mathbf{C} \to \mathscr{S}$ that

$$\mathscr{S}^{\mathbf{C}}(H^C \times \alpha, \psi) \to (C\psi)^{(C\alpha)}$$

defined by evaluating a natural transformation at the identity in $(C)H^C = \mathbf{C}(C, C)$, is a bijection. The inverse sends any mapping: $g:C\alpha \to C\psi$ into the natural transformation $\bar{g}$ for which

$$D\bar{g}:\langle u, x\rangle \to x(u^{-1}\alpha)g(u\psi)$$

for any $C \xrightarrow{u} D$ in DH^C and $x \in D\alpha$. This actually shows that for $\mathbf{1} \xrightarrow{C} \mathbf{C}$, $C\cdot(\ ):\mathscr{S}^{\mathbf{C}} \to \mathscr{S}^{\mathbf{1}} = \mathscr{S}$ preserves implication, that implication is defined objectwise. ($C \rightsquigarrow (C\psi)^{(C\alpha)}$ becomes a functor by means of $u \rightsquigarrow (u\psi)^{(u^{-1}\alpha)}$). Thus for any $f:\mathbf{B} \to \mathbf{C}$ the sets involved in an implication-representation are preserved, and it is clear that the action is also preserved.

In order to prove the theorems we are aiming at in this section, we need to consider another condition, which first came to the writer's attention in unpublished work of Jon Beck on Descent Theory but which was surely considered earlier in topology.

DEFINITION. An eed satisfies the Beck condition iff for every diagram

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "y"'] & Y' \end{tikzcd}$$

of types and terms which is a *meet* (pullback, fibered product) diagram and for any attribute ψ of type Y, the canonical deduction $(f \cdot \psi)\Sigma x \rightarrow f' \cdot (\psi\Sigma y)$ (induced by the identity deduction on $\psi\Sigma y$) is an isomorphism. (We should require the same for Π if it exists.) Since we have not assumed that $\mathbf{T}$ has meets in general, we are led to ask

QUESTION. What is the form of the diagrams which must be meet diagrams in any category with products? Only two forms enter into our theorems; we do not know whether there are essentially different forms.

PROPOSITION. *For any morphism (term)* $f: X \rightarrow Y$

(a)
$$\begin{tikzcd} X \arrow[r, "\langle X, f\rangle"] \arrow[d, "f"'] & X \times Y \arrow[d, "f \times Y"] \\ Y \arrow[r, "Y\delta"'] & Y \times Y \end{tikzcd}$$

is a meet diagram.

PROOF. Clear.

Case (a) of the Beck condition enables us to settle the following, which may have puzzled some readers. Our notion of quantification along an arbitrary term seems a considerable generalization of the usual quantification with respect to a variable x, which corresponds to the case when the term f quantified along is a projection $\pi_Y: X \times Y \rightarrow Y$. The greater generality was used in defining equality, since there we quantified along a diagonal term, which is not reducible to quantification along a projection. But perhaps that is the only essential case gained by the generalization: that is, perhaps the general case of $\varphi\Sigma f$ can be expressed in terms of Θ_Y and $(\quad)\Sigma\pi_Y$. In fact, that is true in the basic set-propositional function hyperdoctrine where $y \cdot (\varphi\Sigma f) = 1$ iff $\exists x[xf = y \wedge x \cdot \varphi = 1]$. More generally, this relation (suitably translated into our variable-free language) holds in many eeds, as asserted below. First we introduce a slight abbreviation of notation: if $f_i: X \rightarrow Y$, $i = 1, 2$, denote by $f_1 \Theta f_2 = \langle f_1, f_2\rangle \cdot \Theta_Y$ the attribute of type X obtained by substituting $\langle f_1. f_2\rangle: X \rightarrow Y \times Y$ into the equality attribute of type $Y \times Y$. Then

THEOREM. *In any eed in which Frobenius Reciprocity and case* (a) *of the Beck condition holds,*

$$\varphi\Sigma f \xrightarrow{\approx} (\pi_X \cdot \varphi \wedge (\pi_X f \Theta \pi_Y))\Sigma\pi_Y.$$

PROOF. We show first that Frobenius Reciprocity implies

$$\varphi\Sigma f \xrightarrow{\approx} (\pi_X \cdot \varphi \wedge (1_X\Sigma\langle X,f\rangle))\Sigma\pi_Y.$$

Indeed,

$$\varphi\Sigma f = \varphi\Sigma(\langle X,f\rangle\pi_Y) \cong (\varphi\Sigma\langle X,f\rangle)\Sigma\pi_Y$$

so we are reduced to showing that

$$\varphi\Sigma\langle X,f\rangle \xrightarrow{\approx} \pi_X \cdot \varphi \wedge (1_X\Sigma\langle X,f\rangle),$$

which is equivalent to

$$(\langle X,f\rangle \cdot (\pi_X \cdot \varphi) \wedge 1_X)\Sigma\langle X,f\rangle \xrightarrow{\approx} \pi_X \cdot \varphi \wedge (1_X\Sigma\,\langle X,f\rangle);$$

but the latter follows from our statement of Frobenius Reciprocity by making the substitutions $a \rightsquigarrow \pi_X \cdot \varphi$, $\varphi \rightsquigarrow 1_X, f \rightsquigarrow \langle X,f\rangle$.

To complete the proof we show that Beck's condition applied to diagrams of form (a) yields a canonical isomorphism $1_X\Sigma\langle X,f\rangle \xrightarrow{\approx} \pi_X f\Theta\pi_Y$ (note that both of the expressions intuitively express the attribute of type $X \times Y$ which corresponds to the graph of f). In fact Beck (a) is explicitly $(f \cdot \psi)\Sigma\langle X,f\rangle \xrightarrow{\sim} (f \times Y) \cdot (\psi\Sigma(Y\delta))$; noting that $f \times Y = \langle \pi_X f, \pi_Y\rangle$ and that $f \cdot 1_Y \xrightarrow{\sim} 1_X$, the stated isomorphism follows by setting $\psi = 1_Y$ and using our definition of equality.

PROPOSITION. *For any type A and term $f{:}X \to Y$, the following is a meet diagram*

(b)
$$\begin{tikzcd} A \times X \arrow[r, "A\times f"] \arrow[d] & A \times Y \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

PROOF. Special case of the following, whose proof is clear.

PROPOSITION. *For any pair of terms $f_i{:}X_i \to Y_i$, $i = 1, 2$ the following is a meet diagram*

(c)
$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "X_1\times f_2"] \arrow[d, "f_1\times X_2"'] & X_1 \times Y_2 \arrow[d, "f_1\times Y_2"] \\ Y_1 \times X_2 \arrow[r, "Y_1\times f_2"'] & Y_1 \times Y_2 \end{tikzcd}$$

Our other theorem concerning the interaction of products and quantifications will have a corollary concerning equality of vectors, and will be based on Beck's condition applied to diagrams of form (b). The theorem itself states in effect that, though conjunction and existential quantification do not usually commute, they do in a certain sense if the quantified variables are "independent" of each other inside the matrix. First, to make the notation more readable, we define the functor

$$P(X_1) \times P(X_2) \xrightarrow{\otimes} P(X_1 \times X_2)$$

by $\varphi_1 \otimes \varphi_2 = \pi_1 \cdot \varphi_1 \wedge \pi_2 \cdot \varphi_2$, the conjunction being of course the product in $P(X_1 \times X_2)$.

THEOREM. *In an eed in which Frobenius Reciprocity and Beck* (b) *hold, one has for any term $f{:}X \to Y$ and type A, and for any attributes φ and α of types X and A respectively, a canonical natural isomorphism $(\alpha \otimes \varphi)\Sigma(A \times f) \xrightarrow{\approx} \alpha \otimes (\varphi\Sigma f)$ of attributes of type $A \times Y$.*

PROOF. Let $\pi_X{:}A \times X \to X$, $\pi_Y{:}A \times Y \to Y$ denote the projections. Then Beck (b) yields explicitly

$$(\pi_X \cdot \varphi)\Sigma(A \times f) \xrightarrow{\approx} \pi_Y \cdot (\varphi\Sigma f).$$

Thus

$$\begin{aligned} \alpha \otimes (\varphi\Sigma f) &\xleftarrow{\sim} \pi_A \cdot \alpha \wedge (\pi_X \cdot \varphi)\Sigma(A \times f) \\ &\xleftarrow{\sim} ((A \times f) \cdot (\pi_A \cdot \alpha) \wedge \pi_X \cdot \varphi)\Sigma(A \times f) \quad \text{by Frobenius} \\ &= (\alpha \otimes \varphi)\Sigma(A \times f) \end{aligned}$$

since $(A \times f)\pi_A = \pi_A$.

COROLLARY. *If $g{:}B \to A$, $f{:}X \to Y$ are any two terms, β and φ attributes of types B and X respectively, then under the hypotheses of the foregoing theorem, one has a canonical natural isomorphism*

$$(\beta \otimes \varphi)\Sigma(g \times f) \xrightarrow{\sim} (\beta\Sigma g) \otimes (\varphi\Sigma f).$$

PROOF. Set $\alpha = \beta\Sigma g$ in the foregoing theorem, use also the symmetrized form

$$(\beta \otimes \varphi)\Sigma(g \times B) \xrightarrow{\approx} (\beta\Sigma g) \otimes \varphi$$

of the theorem and the fact that $g \times f = (g \times B)(A \times f)$.

COROLLARY. *Under the hypotheses of the theorem, one has for any two types X_1, X_2 an isomorphism*

$$\Theta_{X_1 \times X_2} \xrightarrow{\approx} \theta \cdot (\Theta_{X_1} \otimes \Theta_{X_2})$$

where θ is the term "middle four exchange isomorphism":

$$\theta{:}(X_1 \times X_2)^2 \xrightarrow{\approx} X_1^2 \times X_2^2.$$

Thus our culminating result states that two ordered pairs are equal iff their first components are equal and their second components are equal.

PROOF. Setting $\beta = 1_{X_1}$, $\varphi = 1_{X_2}$, $g = X_1\delta$, $f = X_2\delta$ in the previous corollary, one obtains

$$(1_{X_1} \otimes 1_{X_2})\Sigma(X_1\delta \times X_2\delta) \xrightarrow{\sim} \Theta_{X_1} \otimes \Theta_{X_2}.$$

But $1_{X_1} \otimes 1_{X_2} = 1_{X_1 \times X_2}$, since both conjuncts are $\pi_i \cdot 1_{X_i} = 1_{X_1 \times X_2}$. Finally

$$(X_1 \times X_2)\delta = (X_1\delta \times X_2\delta)\theta^{-1}$$

but since θ is an isomorphism

$$\xi\Sigma\theta^{-1} \xrightarrow{\sim} \theta \cdot \xi,$$

so the statement follows.

Even these meager theorems apparently do not hold in the doctrines whose attributes are set-valued functors on small categories or groupoids. *Counterexample* (albeit to the hypotheses, not the conclusion, of the theorems). Let $\mathbf{G}$ be the groupoid

$$A \underset{f^{-1}}{\overset{f}{\rightleftarrows}} B$$

with only four morphisms and consider the two constant endofunctors ∂_A, ∂_B of $\mathbf{G}$. Then

$$\begin{tikzcd} \mathbf{O} \arrow[r] \arrow[d] & \mathbf{G} \arrow[d, "\partial_B"] \\ \mathbf{G} \arrow[r, "\partial_A"'] & \mathbf{G} \end{tikzcd}$$

is a meet diagram (where $\mathbf{O}$ is the empty category) and yet the Beck condition applied to this diagram does not hold at any nonempty attribute φ.

PROOF. Obviously $((\mathbf{O} \to \mathbf{G}) \cdot \varphi)\Sigma(\mathbf{O} \to \mathbf{G}) = 0$ and yet, since

$$\begin{tikzcd}[column sep=tiny] \mathscr{S}^{\mathbf{G}}(\partial_B \cdot \phi, \psi) \arrow[rr, "\approx"] \arrow[dr, "\approx"'] & & \mathscr{S}^{\mathbf{G}}(\phi, \partial_A \cdot \psi) \arrow[dl, "\approx"] \\ & \mathscr{S}(B \cdot \varphi, A \cdot \psi), & \end{tikzcd}$$

we have that $\partial_B \cdot (\varphi\Sigma\partial_A) = \partial_B \cdot (\partial_B \cdot \varphi) = \partial_B^2 \cdot \varphi = \partial_B \cdot \varphi \neq 0$.

This should not be taken as indicative of a lack of vitality of $\mathscr{S}^{\mathbf{B}}$, $\mathbf{B} \in \text{Cat}$ as a hyperdoctrine, or even of a lack of a satisfactory theory of equality for it. Rather, it indicates that we have probably been too naive in defining equality in a manner too closely suggested by the classical conception. Equality should be the "graph" of the identity term. But present categorical conceptions indicate that, in the context of set-valued attributes, the graph of a functor $f\colon \mathbf{B} \to \mathbf{C}$ should be, not $1_{\mathbf{B}}\Sigma\langle \mathbf{B}, f\rangle$, but rather the corresponding "profunctor", a binary attribute of *mixed* variance in $P(\mathbf{B}^{\mathrm{op}} \times \mathbf{C})$. Thus in particular "equality" should be the functor $\hom_{\mathbf{B}}$ (rather than the rather uninformative attribute $\Theta_{\mathbf{B}}$ in $P(\mathbf{B} \times \mathbf{B})$, given by our present definition). The term which would take the place of δ in such a more enlightened theory of equality would then be the forgetful functor

$$\tilde{\mathbf{B}} \to \mathbf{B}^{\mathrm{op}} \times \mathbf{B}$$

from the "twisted morphism category", as follows from the "extensional" considerations of the following section. Of course to abstract from this example would require at least the addition of a functor $\mathbf{T} \xrightarrow{\text{op}} \mathbf{T}$ to the structure of an eed.

2. In any elementary existential doctrine we have a functor

$$(\mathbf{T}, B) \xrightarrow{1(\ )\Sigma(\ )} P(B)$$

for each type B, defined on objects by

$$\begin{tikzcd} E \arrow[d, "p"'] \\ B \end{tikzcd} \rightsquigarrow 1_E \Sigma p$$

The morphisms in the category $(\mathbf{T}, B)$ are of course arbitrary commutative triangles

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B & \end{tikzcd}$$

of terms, and it is easy to verify that the above definition can be canonically extended to these morphisms to become a functor. For example, in the hyperdoctrine with $\mathbf{T} = \mathscr{S}$, $P(X) = 2^X$, our functor

$$(\mathscr{S}, B) \rightarrow 2^B$$

assigns to any mapping p with codomain B the propositional function $\bar{p}$ defined on B such that $b\bar{p} = 1$ iff $b \in$ image (p) or in the example $P(X) = \mathscr{S}^X$, our functor takes $p: E \rightarrow B$ into the family E_b, $b \in B$ of sets in which E_b is the fiber of p over b.

When the functor defined in the previous paragraph is equipped with a right adjoint

$$P(B) \rightarrow (\mathbf{T}, B)$$

we say that the eed satisfies the *Comprehension Schema* and denote the adjoint by

$$\psi \rightsquigarrow \begin{tikzcd} \{B : \psi\} \arrow[d, "p_\psi"] \\ B \end{tikzcd}$$

The new rule of inference is then expressed by the adjointness bijection

$$\frac{\begin{tikzcd} E \arrow[rr] \arrow[dr, "p"'] & & \{B : \psi\} \arrow[dl] \\ & B & \end{tikzcd}}{1_E \Sigma p \rightarrow \psi}$$

between terms $E \xrightarrow{f} \{B:\psi\}$ for which $fp_\psi = p$ and deductions $1_E \Sigma p \to \psi$ over B. We may call $p_\psi:\{B:\psi\} \to B$ the "extension" of ψ, justified by the fact that in the hyperdoctrine 2^X, $X \in \mathscr{S}$, p_ψ reduces to the inclusion $\tilde{p}_\psi$ of that part of B whose characteristic function is the propositional function ψ. For since $\tilde{p}_\psi$ is then monomorphic, there is for any p at most one f such that $f\tilde{p}_\psi = p$; there is such an f iff the image of p is contained in the part of B in question, which holds iff there is a "deduction" $1_E\Sigma p \to \psi$ in 2^B; hence $p_\psi \approx \tilde{p}_\psi$ in $(\mathscr{S}, B)$.

Similarly the set-valued hyperdoctrine on $\mathbf{T} = \mathscr{S}$ satisfies the comprehension schema; for a family ψ of sets indexed by the elements of B, $\{B:\psi\} = \sum_{b\in B} b \cdot \psi$, the disjoint sum, with p_ψ the obvious projection. Thus in this case the Comprehension Schema is more nearly the Replacement Schema.

Given $f_i:X \to Y$, $i = 1, 2$ one would expect that the extension $\{X:f_1\Theta f_2\}$ of the attribute of type X expressing that f_1 and f_2 are equal should in fact give the equalizer in the category T of f_1, f_2. This is true under certain conditions.

THEOREM. *Suppose that in a given eed in which the Comprehension Schema holds, we have further the following conditions for any two terms $h_i:E \to Y$:*

(i) *There is at most one proof $1_E \to h_1\Theta h_2$*
(ii) *If there is such a proof, then $h_1 = h_2$.*

Then if $f_i:X \to Y$ are any two given terms and we set $\varphi = f_1\Theta f_2$, it follows that

$$\{X:\varphi\} \xrightarrow[p_\varphi]{} X \overset{f_1}{\underset{f_2}{\rightrightarrows}} Y$$

is an equalizer diagram.

NOTE. It would be too restrictive to replace (i) by the assumption that all attributes have at most one proof. An equality statement tends to be a very special sort of attribute; consider for example $P(X) = \mathscr{S}^X$, $X \in \mathscr{S}$, where (i) holds but most attributes have many distinct proofs. Condition (ii) seems difficult to guarantee by other kinds of assumptions.

PROOF. Consider any "test" term $E \xrightarrow{p} X$ as an object in $(\mathbf{T}, X)$. We must show that there is at most one term $E \to \{X:f_1\Theta f_2\}$ which when composed with p_φ gives p, and that there is such a term iff $pf_1 = pf_2$. By adjointness

$$\begin{array}{c}
\begin{tikzcd} E \arrow[r] \arrow[dr, "p"'] & \{X:f_1\Theta f_2\} \arrow[d, "p_\varphi"] \\ & X \end{tikzcd} \\
\hline
1_E\Sigma p \to f_1\Theta f_2 \\
\hline
1_E \to p\cdot(f_1\Theta f_2)
\end{array}$$

But $p \cdot (f_1\Theta f_2) = p\langle f_1, f_2\rangle \cdot \Theta = \langle pf_1, pf_2\rangle \cdot \Theta = pf_1\Theta pf_2$ so the result follows by setting $h_i = pf_i$.

The notation of "extension" surely belongs to logic, yet its own extension is considerably broader than the case traditionally considered by logicians. For example

THEOREM. *The hyperdoctrine with* $\mathbf{T} = \mathrm{Cat}$, $P(\mathbf{B}) = \mathscr{S}^{\mathbf{B}}$ *satisfies the Comprehension Schema. Indeed, if* $\varphi : \mathbf{B} \to \mathscr{S}$ *is any functor, its extension* $p_\varphi : \{\mathbf{B} : \varphi\} \to \mathbf{B}$ *is the op-fibration with discrete fibers associated to* φ.

PROOF. We need only show that the op-fibration $\tilde{\varphi} \to \mathbf{B}$ in question, has the required universal property. Recall that $\tilde{\varphi}$ has as objects pairs $\langle B, x\rangle$ with $1 \xrightarrow{x} B\varphi$ in $\mathscr{S}$, and as morphisms $\langle B, x\rangle \to \langle B', x'\rangle$ the morphisms $B \to B'$ in $\mathbf{B}$ which under the action φ take $x \rightsquigarrow x'$. For any $p : \mathbf{E} \to \mathbf{B}$ one has clearly that the commuting diagrams

$$\begin{tikzcd} \mathbf{E} \arrow[rr] \arrow[dr, "p"'] & & \tilde{\varphi} \arrow[dl] \\ & \mathbf{B} & \end{tikzcd}$$

correspond to the elements of proj lim $(p \cdot \varphi)$. But on the other hand for deductions (i.e. natural transformations) one has

$$\frac{1_{\mathbf{E}}\Sigma p \to \varphi}{1_{\mathbf{E}} \to p \cdot \varphi}$$

and the deductions of the sort below the line also correspond canonically to the elements of proj lim $(p \cdot \varphi)$ since the terminal object represents the inverse limit functor on $\mathscr{S}^{\mathbf{E}}$. Thus

$$\frac{\begin{tikzcd} \mathbf{E} \arrow[rr] \arrow[dr, "p"'] & & \tilde{\varphi} \arrow[dl] \\ & \mathbf{B} & \end{tikzcd}}{1_{\mathbf{E}}\Sigma p \to \varphi}$$

canonically for all $\mathbf{E}, p$, and hence

$$\tilde{\varphi} \approx \{\mathbf{B} : \varphi\}.$$

It is clear that if $\varphi : \mathbf{B} \to \mathscr{S}$ is a functor whose domain $\mathbf{B}$ is a groupoid, then the corresponding cofibered category $\tilde{\varphi}$ is also a groupoid; it is in fact the "covering groupoid" used by Higgins in his proof of the subgroup theorem and, in a measure-theoretic context, by Mackey in his theory of virtual subgroups. Thus

COROLLARY. *The hyperdoctrine with* $\mathbf{T} =$ *groupoids,* $P(\mathbf{B}) = \mathscr{S}^{\mathbf{B}}$ *satisfies the Comprehension Schema, with* $\{\mathbf{B} : \varphi\} =$ *the covering groupoid of* φ *for any permutation representation* φ *of the groupoid* $\mathbf{B}$.

CUNY GRADUATE CENTER
FORSCHUNGSINSTITUT FÜR MATHEMATIK
ETH ZÜRICH, SWITZERLAND