Shiraz university

Faculty of electrical and computer engineering

Pattern Recognition

Mini-Project #1

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Abstract

This mini-project includes one problem which consists of seven parts.

Part a is designated aimed to recalling an important theorem from probability theory Saying that, the area under the curve of any probability distribution function equals one.

Reviewing the basic algebraic operations mostly important in the future contexts, is the main purpose of part b.

Part c and d are concentrating on the concept of probability of error which is important to classification problems.

Finally, part e and f tend to emphasis classification problems and their possibly solutions based on some criteria (Bayesian minimum error in part e and Bayesian minimum risk in part f), then compare the goodness of the solutions based on the probability of error

In conclude, mini-project one elucidates the basic method of dealing with a pattern recognition problem.

Problem #1

#1. a

We show that the area under the given distribution equals one (as it holds for any probability distribution), by integration over real line

$$\left[\int_{-\infty}^{\infty} \frac{\frac{1}{\pi b}}{1+\left(\frac{x-a_i}{b}\right)^2}\, dx = \left(\frac{1}{\pi b}\right) b * \tan^{-1}\left(\frac{x-a_i}{b}\right)\right]_{-\infty}^{\infty} = \left(\frac{1}{\pi b}\right) * b * \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = 1$$

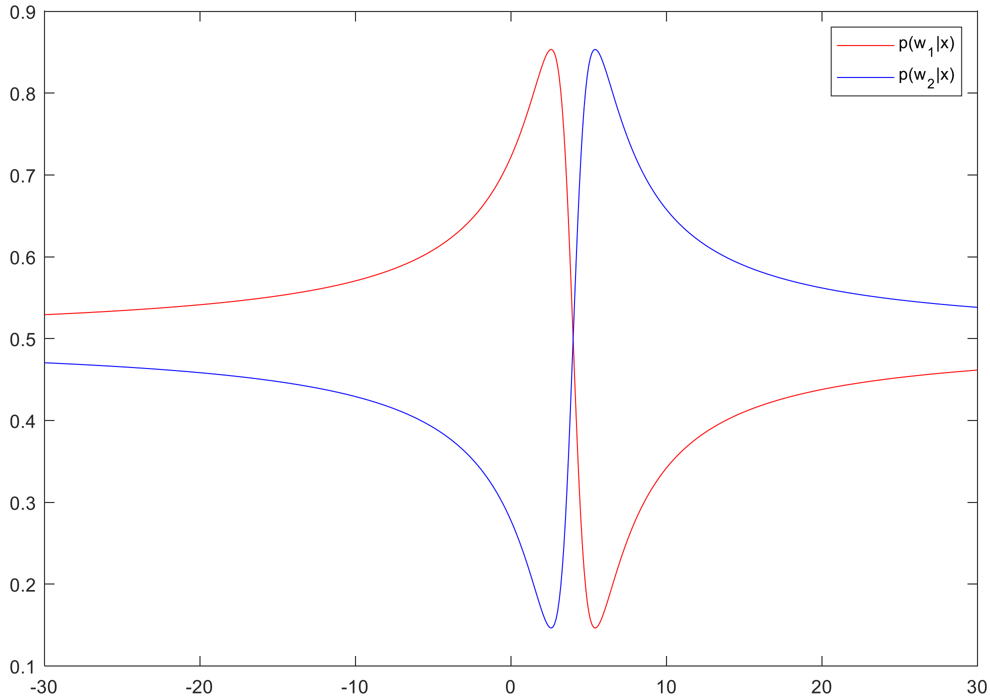
#1. b

$$assumes: p(w_1) = p(w_2) \quad (*)$$

$$p(w_1|x) = p(w_2|x) \Longrightarrow \frac{p(x|w_1)p(w_1)}{p(x)} = \frac{p(x|w_2)p(w_2)}{p(x)} \Longrightarrow p(x|w_1) = p(x|w_2) \Longrightarrow$$

$$\frac{\frac{1}{\pi b}}{1+\left(\frac{x-a_1}{b}\right)^2} = \frac{\frac{1}{\pi b}}{1+\left(\frac{x-a_2}{b}\right)^2} \Longrightarrow 1+\left(\frac{\mathrm{x}-\mathrm{a}_1}{\mathrm{b}}\right)^2 = 1+\left(\frac{\mathrm{x}-\mathrm{a}_2}{\mathrm{b}}\right)^2 \Longrightarrow$$

$$(x-a_1)^2 = (x-a_2)^2 \Longrightarrow \begin{cases} a_1 = a_2 \quad (trivial) \\ \quad x = \dfrac{a_1+a_2}{2} \end{cases}$$



#1. c

$$P(error) = p(w_1)p(e|w_1) + p(w_2)p(e|w_2) \Longrightarrow$$

$$p(error) = \ p(w_1)\int_{th}^{\infty} p(x|w_1)dx + p(w_2)\int_{-\infty}^{th} p(x|w_2)dx \Longrightarrow$$

$$p(error) = \left(\frac{1}{2\pi b}\right) * b(\ [\tan^{-1}\frac{x-a_1}{b}]_{th}^{\infty} + [\ \tan^{-1}\frac{x-a_2}{b}]_{-\infty}^{th}) \Longrightarrow$$

$$p(error) = \left(\frac{1}{2\pi}\right)[\frac{\pi}{2} - \tan^{-1}\frac{th-a_1}{b} + tan^{-1}\frac{th-a_2}{b} + \frac{\pi}{2}] \quad (1)$$

Using derivation in order to find extrema:

$$\frac{\partial P(error)}{\partial th} = 0 \Longrightarrow \frac{1}{b} * \left(\frac{1}{1+\left(\frac{th-a_2}{b}\right)^2}\right) - \frac{1}{b} * \left(\frac{1}{1+\left(\frac{th-a_1}{b}\right)^2}\right) = 0 \Longrightarrow$$

$$th = \frac{a_1+a_2}{2} \quad or \quad th \to \infty$$

Substituting in (1):

$$p(error_{\min}\ \ ) = \left(\frac{1}{2\pi}\right) * [\pi - \tan^{-1}\frac{\frac{a_1+a_2}{2}-a_1}{b} + \tan^{-1}\frac{\frac{a_1+a_2}{2}-a_2}{b}] =$$

$$\left(\frac{1}{2\pi}\right) * [\pi - \tan^{-1}\frac{a_2-a_1}{2b} + \tan^{-1}\frac{a_1-a_2}{2b}] \Longrightarrow$$

$$p(error_{\min}\ \ ) = \left(\frac{1}{2\pi}\right) * \left(\pi - 2\tan^{-1}\frac{a_2-a_1}{2b}\right) = \frac{1}{2} - \left(\frac{1}{\pi}\right)\tan^{-1}\left|\frac{a_2-a_1}{2b}\right|$$

$$(**)\ \tan^{-1}(-x) = -\tan^{-1}(x)$$

#1. d

According to equation (1) in PART (#1. C), we have two cases to consider:

1) $if\ a_1 = a_2$:

   In this case, two distributions are identical. so, it is not possible to distinguish between two classes. thus, assuming equal priori, the maximum error (and error) equals ½

2) $If\ th \rightarrow \infty$:

   In this case, classifier always decides in favor of one of two classes. again, the maximum error (and error) equals $1/2$

#1. e

Bayesian minimum error classifier:

$$p(w_1|x) > p(w_2|x) \Longrightarrow choose\ w1:$$

$$p(w_1|x) > p(w_2|x) \Longrightarrow \frac{p(x|w_1)p(w_1)}{p(x)} > \frac{p(x|w_2)p(w_2)}{p(x)}$$

$$\frac{\frac{1}{\pi b}}{1+\left(\frac{x-a_1}{b}\right)^2} > \frac{\frac{1}{\pi b}}{1+\left(\frac{x-a_2}{b}\right)^2} \Longrightarrow x^2 - 2a_1x + a_1^2 < x^2 - 2a_2x + a_2^2$$

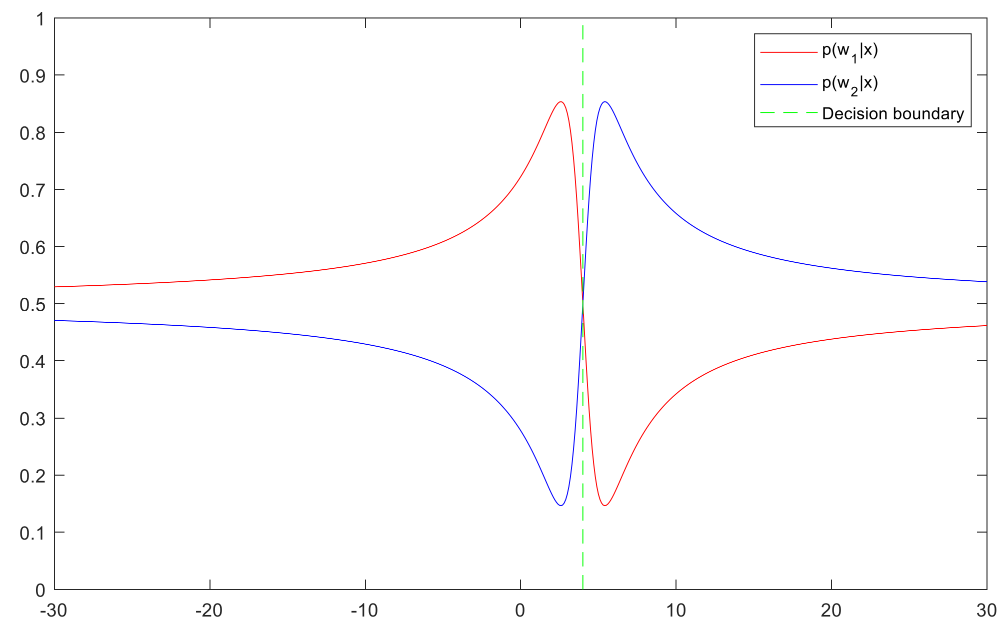
$$\Longrightarrow 2(a_2 - a_1)x < a_2^2 - a_1^2 \Longrightarrow x < \frac{a_1 + a_2}{2}$$

$$Decision\ rule = \begin{cases} w_1, & x < \dfrac{a_1 + a_2}{2} \\ w_2, & o.t.w \end{cases}$$

** P(error) can be derived as same as (#1. C)

For $a_1 = 3, a_2 = 5, b = 1$ we have:

$$p(error) = \frac{1}{2} - \frac{1}{\pi} * \tan^{-1}\frac{5-3}{2} = \frac{1}{4}$$



#1. f

following Bayesian minimum risk:

$$\frac{p(x|w_1)}{p(x|w_2)} > \frac{\lambda_{12}-\lambda_{22}}{\lambda_{21}-\lambda_{11}} . \frac{p(w_2)}{p(w_1)} \Longrightarrow \frac{p(x|w_1)}{p(x|w_2)} > \frac{1}{2} \Longrightarrow choose\ w_1$$

We have:

$$\frac{\dfrac{\frac{1}{\pi b}}{1+\left(\frac{x-a_1}{b}\right)^2}}{\dfrac{\frac{1}{\pi b}}{1+\left(\frac{x-a_2}{b}\right)^2}} > \frac{1}{2} \Longrightarrow \frac{b^2+(x-a_2)^2}{b^2+(x-a_1)^2} > \frac{1}{2} \Longrightarrow$$

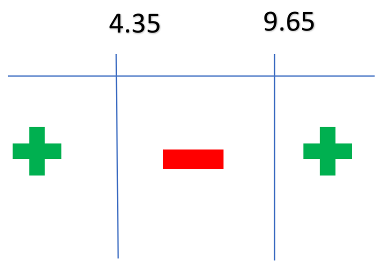
$$x^2 + (2a_1 - 4a_2)x > a_1^2 - b^2 - 2a_2^2$$

Substituting for $a_1 = 3, a_2 = 5, b = 1$ the decision rule is:

$$if\ x^2 - 14x + 42 > 0 \Longrightarrow choose\ w_1$$
$$otherwise\ choose\ w_2$$

Or equivalently

$$x^2 - 14x + 42 = 0 \Longrightarrow x = \frac{14 \pm \sqrt{28}}{2} \approx 9.65{,}4.35$$



$$if\ 4.35 < x < 9.65 \Longrightarrow choose\ w_2$$
$$otherwise\ choose\ w_1$$

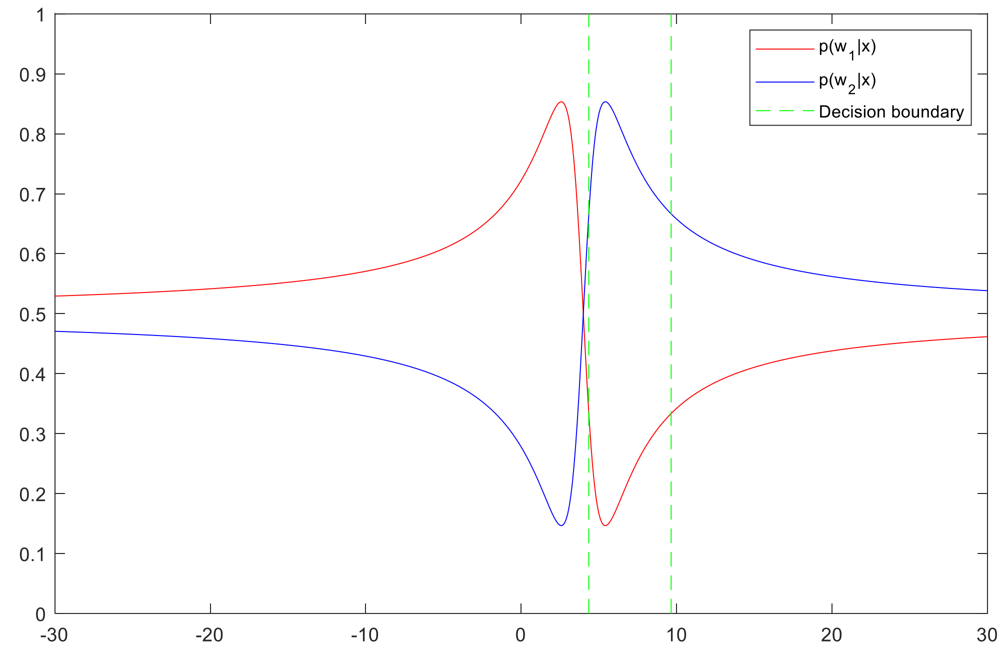
Therefore, p(error) can be computed as follows:

$$p(error) = p(w_1)p(e|w_1) + p(w_2)p(e|w_2) \Longrightarrow$$

$$p(error) = \frac{1}{2} * \int_{4.35}^{9.65} p(x|w_1)dx + \frac{1}{2} * \int_{-infty}^{4.35} p(x|w_2)dx + \frac{1}{2} * \int_{9.65}^{\infty} p(x|w_2)dx$$

$$p(error) = \left(\frac{1}{2\pi}\right) * [\tan^{-1}(x-3)|_{4.35}^{9.65} + \tan^{-1}(x-5)|_{-\infty}^{4.35} + \tan^{-1}(x-5)|_{9.65}^{\infty}]$$

$$\Longrightarrow p(error) \approx 0.27$$



as above graph illustrated, in the limit of infinity both distributions are approximately equal so, it makes hard to distinguish between two classes!

In such situation, our Bayesian minimum risk classifier always decides in favor of class w1 because the risk of wrong decision when the true class is w2 is less than wrong decision when the true class is w1.

The two classifiers proposed in part 1.e and part 1.f shine upon two different situations (equal \ not equal loss per actions) so, actually they cannot be compared based on the probability of error. But, while in the case of 1.e The real line is divided equally between two decision regions, in the case of 1.f, nearly the whole of the real line belongs to w1 decision region. This should not be surprised because in the last case, the cost of choosing w2 is generally higher than choosing it's alternative

Appendix

Scripts

MATLAB Script used to define probability density functions:

```
x = (-30:0.1:30);
p_x_w1 =  (1/pi) * 1./(1+(x-3).^2); %likelihood of x while state of nature is w1
p_x_w2 =  (1/pi) * 1./(1+(x-5).^2); %likelihood of x while state of nature is w2

p_w1 = 1/2; %priori probability of class w1
p_w2 = 1/2; %priori probability of class w2

p_x = p_w1 .* p_x_w1 + p_w2 .* p_x_w2; %evidence

p_w1_x = (p_x_w1 .* p_w1) ./ p_x ; %posteriori probability of class w1
p_w2_x = (p_x_w2 .* p_w2) ./ p_x; %posteriori probability of class w2
```

MATLAB script used to generate graph in 1.b:

```
figure()
xlim([-30,30]);
ylim([0,1]);
plot(x,p_w1_x,'r');

hold on
plot(x,p_w2_x,'b');
legend('p(w_1|x)','p(w_2|x)')
```

MATLAB script used to generate graph in 1.e:

```
figure()
xlim([-30,30]);
ylim([0,1]);
plot(x,p_w1_x,'r');
hold on
plot(x,p_w2_x,'b');
th = (3 + 5)/2;
plot([th,th],[0,1],'g--');
legend('p(w_1|x)','p(w_2|x)','Decision boundary')
```

MATLAB script used to generate graph in 1.f:

```
figure()
xlim([-30,30]);
ylim([0,1]);
plot(x,p_w1_x,'r');
hold on
plot(x,p_w2_x,'b');
th1 = 4.35;
th2 = 9.65;
plot([th1,th1],[0,1],'g--');
plot([th2,th2],[0,1],'g--');
legend('p(w_1|x)','p(w_2|x)','Decision boundary')
```