Wave Math

0x15

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1 Simple 2D Damped Wave Equation

The simplest 2D damped wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y}\right) - \mu\frac{\partial u}{\partial t} \text{ or } D_t^2 u = c^2(D_x^2 u + D_y^2 u) - \mu D_t u$$

For wave magnitude $u(x, y, t)$ (vertical displacement of fluid, etc...) at point (x, y) and time t, wave speed c, and damping coefficient $\mu \geq 0$ (think fluid viscosity).

2 Descretising the Simple 2D Case

Fixing space scales $\Delta x = \Delta y$ and time scale Δt, we let $u_{j,k}^n$ be the numerically calculated value of $u(x_j, y_k, t_n)$ (note that it can be thought $x_j = j\Delta x$ and $y_k = k\Delta y$ as they are in denominations of grid cells, and t_n is the nth timestep so $t_n = n\Delta t$ assuming we're starting from 0 time).

Then for each term applying the limit definition of derivative for forward, backward, and centered difference we get:

First order spatial derivatives:

$$\text{Forward: } \frac{\partial u(x_j, y_k, t_n)}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x_j + \Delta x, y_k, t_n) - u(x_j, y_k, t_n)}{\Delta x} \approx \frac{u_{j+1,k}^n - u_{j,k}^n}{\Delta x}$$

$$\text{Backward: } \frac{\partial u(x_j, y_k, t_n)}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x_j, y_k, t_n) - u(x_j - \Delta x, y_k, t_n)}{\Delta x} \approx \frac{u_{j,k}^n - u_{j-1,k}^n}{\Delta x}$$

$$\text{Centered: } \frac{\partial u(x_j, y_k, t_n)}{\partial x} = \lim_{\Delta x \to 0} \frac{u(x_j + \frac{1}{2}\Delta x, y_k, t_n) - u(x_j - \frac{1}{2}\Delta x, y_k, t_n)}{2\Delta x} \approx \frac{u_{j+1,k}^n - u_{j-1,k}^n}{2\Delta x}$$

And then the same for $\frac{\partial u(x_j, y_k, t_n)}{\partial y}$.

Second order Spatial Derivatives: via taking a Forward difference of a backward derivative (seen below) or vice versa.

$$\frac{\partial^2 u(x_j, y_k, t_n)}{\partial x^2} = \lim_{\Delta x \to 0} \frac{\frac{(u(x_j+\Delta x, y_k, t_n) - u(x_j - \Delta x + \Delta x, y_k, t_n)) - (u(x_j, y_k, t_n) - u(x_j - \Delta x, y_k, t_n))}{\Delta x}}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{u(x_j + \Delta x, y_k, t_n) - 2u(x_j, y_k, t_n) + u(x_j - \Delta x, y_k, t_n)}{\Delta x^2} \approx \frac{u^n_{j+1,k} - 2u^n_{j,k} + u^n_{j-1,k}}{\Delta x^2}$$

And similarly, the same for $\frac{\partial^2 u(x_j, y_k, t_n)}{\partial y^2}$

Time Derivatives (1st and 2nd order):

1st Forward: $\frac{\partial u(x_j, y_k, t_n)}{\partial t} = \lim_{\Delta t \to 0} \frac{u(x_j, y_k, t_n + \Delta t) - u(x_j, y_k, t_n)}{\Delta t} \approx \frac{u^{n+1}_{j,k} - u^n_{j,k}}{\Delta t}$

1st Backward: $\frac{\partial u(x_j, y_k, t_n)}{\partial t} = \lim_{\Delta t \to 0} \frac{u(x_j, y_k, t_n) - u(x_j, y_k, t_n - \Delta t)}{\Delta t} \approx \frac{u^n_{j,k} - u^{n-1}_{j,k}}{\Delta t}$

1st Centered: $\frac{\partial u(x_j, y_k, t_n)}{\partial t} = \lim_{\Delta t \to 0} \frac{u(x_j, y_k, t_n + \frac{1}{2}\Delta t) - u(x_j, y_k, t_n - \frac{1}{2}\Delta t)}{2\Delta t} \approx \frac{u^{n+1}_{j,k} - u^{n-1}_{j,k}}{2\Delta t}$

2st Forward of Backward: $\frac{\partial^2 u(x_j, y_k, t_n)}{\partial t^2} = \lim_{\Delta x \to 0} \frac{u(x_j, y_k, t_n + \Delta t) - 2u(x_j, y_k, t_n) + u(x_j, y_k, t_n - \Delta t)}{\Delta n^2}$
$\approx \frac{u^{n+1}_{j,k} - 2u^n_{j,k} + u^{n-1}_{j,k}}{\Delta t^2}$

Plugging into the DE: (remember, we want to solve for the current cell at the next time, so $u^{n+1}_{j,k}$. As such I will be using the centered time derivative) Additionally, I will assume the grid cells are perfect squares, so $\Delta x = \Delta y$.

$$\frac{u^{n+1}_{j,k} - 2u^n_{j,k} + u^{n-1}_{j,k}}{\Delta t^2} = c^2 \left(\frac{u^n_{j+1,k} - 2u^n_{j,k} + u^n_{j-1,k}}{\Delta x^2} + \frac{u^n_{j,k+1} - 2u^n_{j,k} + u^n_{j,k-1}}{\Delta y^2} \right) - \mu \frac{u^{n+1}_{j,k} - u^{n-1}_{j,k}}{2\Delta t}$$
$$\frac{u^{n+1}_{j,k} - 2u^n_{j,k} + u^{n-1}_{j,k}}{\Delta t^2} + \mu \frac{u^{n+1}_{j,k} - u^{n-1}_{j,k}}{2\Delta t} = c^2 \left(\frac{u^n_{j+1,k} - 2u^n_{j,k} + u^n_{j-1,k}}{\Delta x^2} + \frac{u^n_{j,k+1} - 2u^n_{j,k} + u^n_{j,k-1}}{\Delta x^2} \right)$$
$$u^{n+1}_{j,k} - 2u^n_{j,k} + u^{n-1}_{j,k} + \frac{\mu\Delta t}{2}(u^{n+1}_{j,k} - u^{n-1}_{j,k}) = \frac{c^2\Delta t^2}{\Delta x^2} \left(u^n_{j+1,k} - 2u^n_{j,k} + u^n_{j-1,k} + u^n_{j,k+1} - 2u^n_{j,k} + u^n_{j,k-1} \right)$$
$$u^{n+1}_{j,k} - 2u^n_{j,k} + u^{n-1}_{j,k} + \frac{\mu\Delta t}{2} u^{n+1}_{j,k} - \frac{\mu\Delta t}{2} u^{n-1}_{j,k} = \frac{c^2\Delta t^2}{\Delta x^2} \left(u^n_{j\pm1,k} + u^n_{j,k\pm1} \right) - \frac{4c^2\Delta t^2}{\Delta x^2} u^n_{j,k}$$
$$u^{n+1}_{j,k} + \frac{\mu\Delta t}{2} u^{n+1}_{j,k} = \frac{c^2\Delta t^2}{\Delta x^2} \left(u^n_{j\pm1,k} + u^n_{j,k\pm1} \right) - \frac{4c^2\Delta t^2}{\Delta x^2} u^n_{j,k} + \frac{\mu\Delta t}{2} u^{n-1}_{j,k} + 2u^n_{j,k} - u^{n-1}_{j,k}$$
$$u^{n+1}_{j,k}(1 + \frac{\mu\Delta t}{2}) = \frac{c^2\Delta t^2}{\Delta x^2} \left(u^n_{j\pm1,k} + u^n_{j,k\pm1} \right) + (2 - \frac{4c^2\Delta t^2}{\Delta x^2}) u^n_{j,k} + (\frac{\mu\Delta t}{2} - 1) u^{n-1}_{j,k}$$
$$u^{n+1}_{j,k} = \frac{c^2\Delta t^2}{\Delta x^2(1 + \frac{\mu\Delta t}{2})} \left(u^n_{j\pm1,k} + u^n_{j,k\pm1} \right) + \frac{2 - 4c^2\Delta t^2/\Delta x^2}{1 + \frac{\mu\Delta t}{2}} u^n_{j,k} + \frac{\frac{\mu\Delta t}{2} - 1}{1 + \frac{\mu\Delta t}{2}} u^{n-1}_{j,k}$$
$$u^{n+1}_{j,k} = \frac{2c^2\Delta t^2}{\Delta x^2(2 + \mu\Delta t)} \left(u^n_{j\pm1,k} + u^n_{j,k\pm1} \right) + \frac{4 - 8c^2\Delta t^2/\Delta x^2}{2 + \mu\Delta t} u^n_{j,k} + \frac{\mu\Delta t - 2}{2 + \mu\Delta t} u^{n-1}_{j,k}$$

3 Expanding to 3D

Similarly to the 2D case, expanding to 3D is

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} + \frac{\partial^2 u}{\partial z}\right) - \mu\frac{\partial u}{\partial t} \text{ or } D_t^2 u = c^2(D_x^2 u + D_y^2 u + D_z^2 u) - \mu D_t u \text{ or } \frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u - \mu\frac{\partial u}{\partial t}$$

With the discretization of $\frac{\partial^2 u}{dz^2}$ following similarly to that of x and y (and of course $u(x_j, y_k, z_\ell, t_n) = u^n_{j,k,\ell}$). And similarly the assumption $\Delta x = \Delta y = \Delta z$.

$$\frac{u^{n+1}_{j,k,\ell} - 2u^n_{j,k,\ell} + u^{n-1}_{j,k,\ell}}{\Delta t^2} = c^2\left(\frac{u^n_{j+1,k,\ell} - 2u^n_{j,k,\ell} + u^n_{j-1,k,\ell}}{\Delta x^2} + \frac{u^n_{j,k+1,\ell} - 2u^n_{j,k,\ell} + u^n_{j,k-1,\ell}}{\Delta y^2} + \frac{u^n_{j,k,\ell+1} - 2u^n_{j,k,\ell} + u^n_{j,k,\ell-1}}{\Delta z^2}\right) - \mu\frac{u^{n+1}_{j,k,\ell} - u^{n-1}_{j,k,\ell}}{2\Delta t}$$

$$\frac{u^{n+1}_{j,k,\ell} - 2u^n_{j,k,\ell} + u^{n-1}_{j,k,\ell}}{\Delta t^2} + \mu\frac{u^{n+1}_{j,k,\ell} - u^{n-1}_{j,k,\ell}}{2\Delta t} = c^2\left(\frac{u^n_{j+1,k,\ell} - 2u^n_{j,k,\ell} + u^n_{j-1,k,\ell}}{\Delta x^2} + \frac{u^n_{j,k+1,\ell} - 2u^n_{j,k,\ell} + u^n_{j,k-1,\ell}}{\Delta x^2} + \frac{u^n_{j,k,\ell+1} - 2u^n_{j,k,\ell} + u^n_{j,k,\ell-1}}{\Delta x^2}\right)$$

$$u^{n+1}_{j,k,\ell} - 2u^n_{j,k,\ell} + u^{n-1}_{j,k,\ell} + \frac{\mu\Delta t}{2}(u^{n+1}_{j,k,\ell} - u^{n-1}_{j,k,\ell}) = \frac{c^2}{\Delta x^2}\left(u^n_{j+1,k,\ell} - 2u^n_{j,k,\ell} + u^n_{j-1,k,\ell} + u^n_{j,k+1,\ell} - 2u^n_{j,k,\ell} + u^n_{j,k-1,\ell} + u^n_{j,k,\ell+1} - 2u^n_{j,k,\ell} + u^n_{j,k,\ell-1}\right)$$

$$u^{n+1}_{j,k,\ell} + \frac{\mu\Delta t}{2}u^{n+1}_{j,k,\ell} - 2u^n_{j,k,\ell} + u^{n-1}_{j,k,\ell} - \frac{\mu\Delta t}{2}u^{n-1}_{j,k,\ell} = \frac{c^2}{\Delta x^2}\left(u^n_{j\pm1,k,\ell} + u^n_{j,k\pm1,\ell} + u^n_{j,k,\ell\pm1} - 6u^n_{j,k,\ell}\right)$$

$$u^{n+1}_{j,k,\ell}(1 + \frac{\mu\Delta t}{2}) = \frac{c^2}{\Delta x^2}\left(u^n_{j\pm1,k,\ell} + u^n_{j,k\pm1,\ell} + u^n_{j,k,\ell\pm1}\right) - \frac{6c^2}{\Delta x^2}u^n_{j,k,\ell} + 2u^n_{j,k,\ell} - u^{n-1}_{j,k,\ell} + \frac{\mu\Delta t}{2}u^{n-1}_{j,k,\ell}$$

$$u^{n+1}_{j,k,\ell} = \frac{c^2}{\Delta x^2(1 + \frac{\mu\Delta t}{2})}\left(u^n_{j\pm1,k,\ell} + u^n_{j,k\pm1,\ell} + u^n_{j,k,\ell\pm1}\right) + \frac{2 - 6c^2/\Delta x^2}{1 + \frac{\mu\Delta t}{2}}u^n_{j,k,\ell} + \frac{\frac{\mu\Delta t}{2} - 1}{1 + \frac{\mu\Delta t}{2}}u^{n-1}_{j,k,\ell}$$

$$u^{n+1}_{j,k,\ell} = \frac{2c^2}{\Delta x^2(2 + \mu\Delta t)}\left(u^n_{j\pm1,k,\ell} + u^n_{j,k\pm1,\ell} + u^n_{j,k,\ell\pm1}\right) + \frac{4 - 12c^2/\Delta x^2}{2 + \mu\Delta t}u^n_{j,k,\ell} + \frac{\mu\Delta t - 2}{2 + \mu\Delta t}u^{n-1}_{j,k,\ell}$$