ATTEMPT ALL QUESTIONS

Question 1: [15 Points]

[a] Determine the properties of each of the following discrete time systems (linearity, causality, memory, time invariance, stability

$$y[n] = x[2n] + 2$$

$$y[n] = x[n-2]\cos\omega_0 n$$

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Linearity	Nonlinear	Linear
Causality	Noncausal	Causal
Memory	Not memoryless	Not memoryless
Time invariance	Time varying	Time varying
Stability	Stable	Stable

[b] The non-zero values of a discrete-time signal are given as $x[0] = 2 + 2j$ and $x[1] = 1$. Decompose x[n] into conjugate symmetric (even) and conjugate anti-symmetric (odd) signals.

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n])$$

$$x_o[n] = \frac{1}{2}(x[n] - x^*[-n])$$

$$x^*[0] = 2 - 2j$$

$$x^*[-1] = 0$$

$$x_e[0] = \frac{1}{2}(2 + 2j + 2 - 2j) = 2$$

$$x_e[1] = \frac{1}{2}(1 + 0) = \frac{1}{2}$$

$$x_e[-1] = \frac{1}{2}(0 + 1) = \frac{1}{2} \qquad x_o[1] = \frac{1}{2}(1 - 0) = \frac{1}{2}$$

$$x_o[0] = \frac{1}{2}(2 + 2j - 2 + 2j) = 2j \qquad x_o[-1] = \frac{1}{2}(0 - 1) = -\frac{1}{2}$$

[c] If a continuous- time sinusoidal signal of frequency is 131 Hz is sampled at a sampling rate of 8000 Hz. What is the discrete-time frequency in rad/sample of the resulting discrete-time signal?

$$\omega = \Omega T_s = \frac{\Omega}{f_s} = \frac{2\pi(131)}{8000} = 0.03275\pi\ rad/sample$$

(And its aliases)

[d] Find the linear and 4-point circular convolutions between x[n] and h[n] given that:

$$x[n] = [\underline{1}, 0, 2, 3]$$

$$h[n] = [\underline{2}, 1, 4]$$

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$$h[n] = [\underline{2}, 1, 4]$$

$$x[n] * h[n] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 4 & 1 & 2 & 0 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 8 \\ 8 \\ 11 \\ 12 \end{bmatrix}$$

$$x[n] \otimes_4 h[n] = \begin{bmatrix} 2 & 0 & 4 & 1 \\ 1 & 2 & 0 & 4 \\ 4 & 1 & 2 & 0 \\ 0 & 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 13 \\ 13 \\ 8 \\ 8 \end{bmatrix}$$

(e) Draw the butterfly diagram used to evaluate the 4-point FFT of the signal $x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$.

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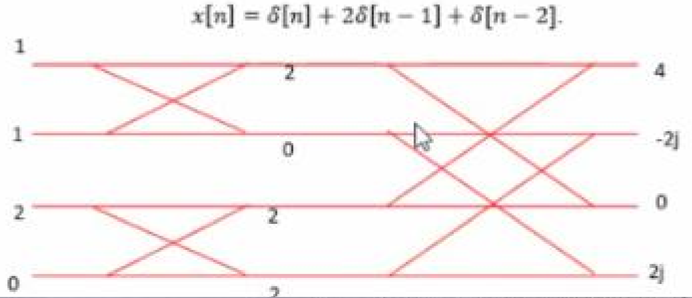

(f) Determine the z-transform (if it exists) and the corresponding ROC. Also determine the DTFT (if it exists) for the signal:

$$x[n] = 2^n u[-n-2] + 0.5^n u[n-1]$$

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$$ROC_1: |z| < 2$$

$$ROC_2: |z| > 0.5$$

$$ROC = ROC_1 \cap ROC_2: 0.5 < |z| < 2$$

Z-transform exists in the ROC $0.5 < |z| < 2$

DTFT exists because the ROC includes the unit circle $|z| = 1$

$$\begin{aligned} x[n] &= 2^n u[-n-2] + 0.5^n u[n-1] \\ &= -2^{-1}(-2^{n+1} u[-(n+1)-1]) + 0.5(0.5^{n-1} u[n-1]) \end{aligned}$$

Z-Transform is:

$$X(z) = \frac{-0.5z}{1 - 2z^{-1}} + \frac{0.5z^{-1}}{1 - 0.5z^{-1}}$$

$$ROC: 0.5 < |z| < 2$$

DTFT is:

$$X(e^{j\omega}) = \frac{-0.5e^{j\omega}}{1 - 2e^{-j\omega}} + \frac{0.5e^{-j\omega}}{1 - 0.5e^{-j\omega}}$$

Question 2: **[15 Points]**

a) Find the impulse response of the following filter assuming it is a causal filter:

$$H(z) = \frac{1-(\cos\omega_0)z^{-1}}{1-(2\cos\omega_0)z^{-1}+z^{-2}}$$

b) For a signal with transfer function given by:

$$X(z) = \frac{2z^{-1}+3}{1-0.3z^{-1}+0.03z^{-2}}$$

Find all the possible ROC's and the corresponding time domain representation of the signal. Which signal has a defined DTFT?

$$= \frac{\quad}{1-e^{j2\omega_0}} = 0.5$$

Causal realization:

$$h[n] = 0.5e^{j\omega_0 n}u[n] + 0.5e^{-j\omega_0 n}u[n] = \cos\omega_0 n\, u[n]$$

a)

b) For a signal with transfer function given by:

$$\frac{2z^{-1}+3}{1-0.4z^{-1}+0.03z^{-1}} = \frac{2z^{-1}+3}{(1-0.3z^{-1})(1-0.1z^{-1})}$$

$$= \frac{A}{1-0.3z^{-1}} + \frac{B}{1-0.1z^{-1}}$$

$$A = \left.\frac{2z^{-1}+3}{1-0.1z^{-1}}\right|_{z=0.3} = \frac{\frac{2}{0.3}+3}{1-\frac{0.1}{0.3}} = \frac{2.9}{0.2} = \frac{29}{2}$$

$$B = \left.\frac{2z^{-1}+3}{1-0.3z^{-1}}\right|_{z=0.1} = \frac{\frac{2}{0.1}+3}{1-\frac{0.3}{0.1}} = \frac{2.3}{-0.2} = -\frac{23}{2}$$

b)

c) Consider a digital filter where the z-transform of the impulse response is

$$H(z) = \frac{z^2-1}{z^2+\frac{49}{64}}$$

Plot the poles and zeros of the filter. Find the ROC of the stable realization of the filter. Sketch the frequency response and determine the type of the filter

$$\textit{Zeros at } z = 1 \textit{ and } z = -1$$

$$\textit{poles at } z = j\frac{7}{8} \textit{ and } z = -j\frac{7}{8}$$

$$\textit{Stable realization is when } |z| = 1 \textit{ which lies on the ROC: } |z| > \frac{7}{8}$$

$$H(e^{j\omega}) = \frac{e^{j2\omega} - 1}{e^{j2\omega} + \frac{49}{64}} = \frac{e^{j\omega}\left(e^{j\omega} - e^{-j\omega}\right)}{e^{j\omega}\left(e^{j\omega} + \frac{49}{64}e^{-j\omega}\right)} = \frac{2j\sin\omega}{e^{j\omega} + \frac{49}{64}e^{-j\omega}}$$

$$\left|H(e^{j\omega})\right|^2 = \frac{4\sin^2\omega}{\left(\frac{113}{64}\cos\omega\right)^2 + \left(\frac{15}{64}\sin\omega\right)^2} = \frac{4\sin^2\omega}{\left(\left(\frac{113}{64}\right)^2 - \left(\frac{15}{64}\right)^2\right)\cos^2\omega + \left(\frac{15}{64}\right)^2}$$

Question 3: **[15 Points]**

a) A discrete-time system is given as

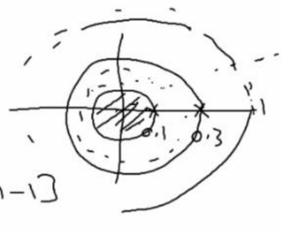
$$y[n] = ay[n-1] + x[n] - ax[n-1],$$

where a is a real scalar constant. Find:

i) The impulse response of the system.

ii) The range of values of a for which the system is BIBO stable.

$$\frac{29/2}{1 - 0.3z^{-1}} - \frac{23/2}{1 - 0.1z^{-1}}$$



unstable ROC1: $|z| < 0.1$

$$x[n] = \frac{-29}{2}\, 0.3^n u[-n-1] + \frac{23}{2}\, 0.1^n u[-n-1]$$

unstable ROC2: $0.1 < |z| < 0.3$

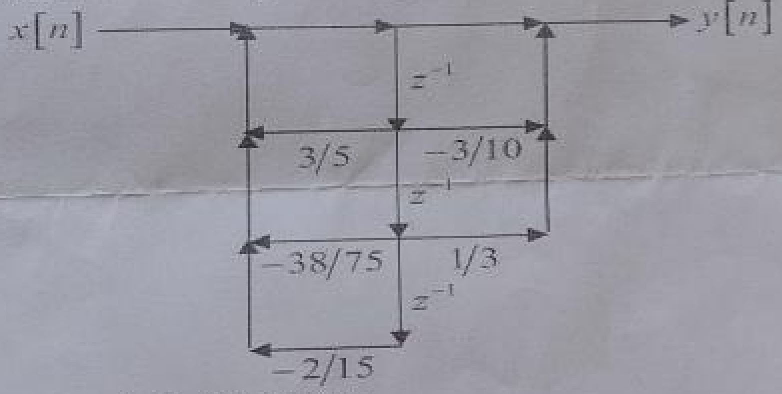
$$x[n] = \frac{-29}{2}\, 0.3^n u[-n-1] - \frac{23}{2}\, 0.1^n u[n]$$

DTFT e

Stable ✓ROC3 $|z| > 0.3$

$$x[n] = \frac{29}{2}\, 0.3^n u[n] - \frac{23}{2}\, 0.1^n u[n]$$

b) Consider the filter defined by the following block diagram:



Write the transfer function of the filter.

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$$H(z) = \frac{1 - \frac{3}{10}z^{-1} + \frac{1}{3}z^{-2}}{1 - \frac{3}{5}z^{-1} + \frac{38}{75}z^{-2} + \frac{2}{15}z^{-3}}$$

Question 4:

a) For the following analog filter:

$$H(s) = \frac{10(s+1)}{s^2 - 6s + 10}$$

Find the transfer function of the corresponding digital filter using impulse invariance method with sampling period $T_s = 0.001\ s$

b) Write the transfer function of the second order digital Butterworth filter with normalized cutoff frequency $\omega_c = 0.4\pi\ rad$ if bilinear transformation is used to convert the analog filter to digital and assuming T=1 sec.

$$s^2 - 6s + 10 = 0$$

poles: $s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{6 \pm \sqrt{36 - 40}}{2} = \frac{6 \pm j2}{2} = 3 \pm j1$

$$H(s) = \frac{10(s+1)}{s^2 - 6s + 10} = \frac{10(s+1)}{(s-3-j)(s-3+j)}$$

$$= \frac{A}{s-3-j} + \frac{B}{s-3+j}$$

$$A = \frac{10(s+1)}{s-3+j}\bigg|_{s=3+j} = \frac{10(3+j+1)}{3+j-3+j} = \frac{40+10j}{2j} = 5 - j20$$

$T = 0.01$

Impulse invariance $\quad H(s) = \dfrac{10/3}{s+5} - \dfrac{1/3}{s+2}$

Transform the poles: $s=-5 \rightsquigarrow z = e^{-5T} = e^{-0.05}$

$s=-2 \rightsquigarrow z = e^{-2T} = e^{-0.02}$

$$H(z) = T\left[\frac{10/3}{1-e^{-0.05}z^{-1}} - \frac{1/3}{1-e^{-0.02}z^{-1}}\right]$$

$$= \frac{0.1/3}{1-e^{-0.05}z^{-1}} - \frac{0.01/3}{1-e^{-0.02}z^{-1}}$$

$$H(s) = \frac{3s+5}{s^2+7s+10} = \frac{3s+5}{(s+5)(s+2)}$$

$$= \frac{A}{s+5} + \frac{B}{s+2} = \frac{10/3}{s+5} - \frac{1/3}{s+2}$$

$$A = \left.\frac{3s+5}{s+2}\right|_{s=-5} = \frac{-10}{-3} = \frac{10}{3}$$

$$B = \left.\frac{3s+5}{s+5}\right|_{s=-2} = \frac{-1}{3}$$

Bilinear Trasformation

$$H(s) = \frac{3s + 5}{s^2 + 7s + 10}$$

just set $s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} = \frac{200(1 - z^{-1})}{1 + z^{-1}}$

$$H(z) = \frac{\frac{600(1 - z^{-1})}{1 + z^{-1}} + 5}{\frac{40000(1 - z^{-1})^2}{(1 + z^{-1})^2} + \frac{1400(1 - z^{-1})}{1 + z^{-1}} + 10}$$

$$\vdots$$

$$H(s) = \frac{\Omega_c^2}{s^2 + \sqrt{2}\,\Omega_c s + \Omega_c^2}$$

$\omega_c = 0.4\pi\,\text{rad}$

Bilinear

$$\Omega_c = \frac{2}{T} \tan \frac{\omega_c}{2}$$

$$\Omega_c = \frac{2}{T} \tan \frac{0.4\pi}{2} =$$

$$y[n] = a\,y[n-1] + x[n] - a\,x[n-1]$$

Take z-transform:

$$Y(z) = a z^{-1} Y(z) + X(z) - a z^{-1} X(z)$$

$$Y(z)\left[1 - a z^{-1}\right] = X(z)\left[1 - a z^{-1}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - a z^{-1}}{1 - a z^{-1}} = 1 \qquad a z^{-1} \neq 1 \qquad z \neq a$$

$h[n] = \delta[n] \longrightarrow$ impulse response

valid when $z \neq a$

ROC: $\mathbb{C} - \{a\}$

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