a) Let $\{x[n]\} = \{1,2,1\}$ and $\{y[n]\} = \{-1,1,2\}$

(i) Evaluate the linear convolution $x[n] * y[n]$.

(ii) Evaluate the 4-point circular convolution $x[n] \otimes y[n]$

(iii) Evaluate the 4-point DFT of $x[n] \otimes y[n]$

Final ~~[illegible]~~ 2021 ①

Q 1

ⓐ $x[n] = \{1, 2, 1\}$ $y[n] = \{-1, 1, 2\}$

	1	2	1
-1	-1	-2	-1
1	1	2	1
2	2	4	2

$h[n] = \{-1, -1, 3, 5, 2\}$

~~[illegible]~~ linear conv

$x[n] = \{1, 2, 1, 0\}$ $y[n] = \{-1, 1, 2, 0\}$

$$\begin{bmatrix} -1 & 0 & 2 & 1 \\ 0 & -1 & 1 & 2 \\ 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 5 \end{bmatrix}$$

1 2 1 0

0 2 1 -1 0 2 1

$x[n] \otimes y[n] = [1, -1, 3, 5]$

$\text{DFT}\{x[n] \underset{4}{\otimes} y[n]\}$

$$\boxed{Z[k] = \sum_{n=0}^{N-1} z[n]\, e^{-j\frac{nk\pi}{N}}}$$

$\boxed{N = 4}$

Direct substitution

$Z[0] = 8$; $Z[1] = -2 + 6j$

$Z[2] = 0$ $Z[3] = -2 - j6$

$\text{DFT}\{x[n] \underset{4}{\otimes} h[n]\} = [8, -2+j6, 0, -2-j6]$

(b) For each of the following signals, determine its z-transform (if it exists) and the corresponding ROC. Also determine the DTFT (if it exists):

(i) $x_1[n] = 0.5^n u[-n] - 0.2^n u[n-1]$

(ii) ~~$x_2[n] = 2^n u[-n-2] + 0.5^n u[n-1]$~~

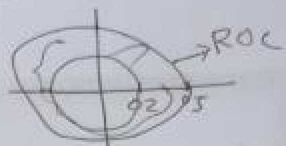
(iii) $x_3[n] = 0.3^n u[n] + 2^n u[-n]$

(2)

(b) $x_1[n] = (0.5)^n u[-n] - (0.2)^n u[n-1]$

$ROC_1\ |z| < 0.5 \qquad ROC_2\ |z| > 0.2$

$ROC\ 0.2 < |z| < 0.5$



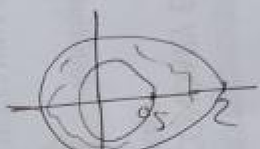
$$X_1(z) = \frac{-z^{-1}(0.5)}{1-0.5z^{-1}} - \frac{(0.2)z^{-1}}{1-0.2z^{-1}}$$

ROC not include unit circle → no DTFT

$x_2[n] = 2^n u[n-2] + 0.5^n u[n-1]$

$ROC_1\ |z| < 2 \qquad ROC_2\ |z| > 0.5$

$ROC \rightarrow 0.5 < |z| < 2$



$$x_2[n] = 2^{-1} 2^{n+1} u[-(n+1)-1] + (0.5)(0.5)^{n-1} u[n-1]$$

$$X_2(z) = (0.5)\frac{-z}{1-2z^{-1}} + \frac{0.5z^{-1}}{1-0.5z^{-1}}$$

ROC includ include unit circle → DTFT ex

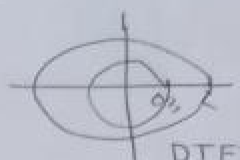
$$X_2(e^{jw}) = \frac{-0.5e^{-jw}}{1-2e^{-jw}} + \frac{0.5e^{-jw}}{1-0.5e^{-jw}}$$

(3)

(iii) $x_3[n] = 0.3^n(n) + 2^n u[-n]$

$ROC_1\ |z| > 0.3 \qquad ROC_2\ |z| < 2$

$ROC \rightarrow 0.3 < |z|\ 2$



$$X_3(z) = \frac{0.3z^{-1}}{1-0.3z^{-1}} + \frac{-z(0.5)}{1-2z^{-1}}$$

DTFT exist

$$X_3(e^{jw}) = \frac{0.3e^{-jw}}{1-0.3e^{-jw}} + \frac{0.5e^{-jw}}{1-\cancel{2z^{-1}}}$$

(iii) $x_3[n] = 0.5^n u[n] + 2^n u[-n]$

(c) For a signal with transfer function given by:

$$X(z) = \frac{2z^{-1}+3}{1-0.03z^{-1}+0.25z^{-2}}$$

Find all the possible ROC's and the corresponding time domain representation of the signal. Which signal has a defined DTFT?

c

$$X(z) = \frac{2z^{-1}+3}{1-0.03z^{-1}+0.25z^{-2}}$$

$$\text{poles} = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \qquad a=1,\ b=-0.03,\ c=0.25$$

$$= \frac{0.03 \pm \sqrt{(0.03)^2-4(1)(0.25)}}{2(1)}$$

$$= \frac{0.03 \pm \sqrt{9*10^{-4}-1}}{2} = \frac{0.03 \pm j\sqrt{1-9*10^{-4}}}{2} = \cancel{0.015 \pm}$$

$$= 0.015 \pm j0.49$$

$$|z_{pole}| = \sqrt{(0.015)^2+(0.49)^2} = 0.5$$

ROC $0.5 < |z| < 0.5$

Question 2: **[15 Points]**

a) For the following causal system:

$$H(z) = \frac{1}{1 - 4.8z^{-1} + 3.2z^{-2}}$$

i. Find the locations of the poles of the system. Show why this system is unstable.

ii. Choose a proper value for the constant a such that $g[n] = a^n h[n]$ is stable.

Q2 ③

(ai) $H(z) = \frac{1}{1-4.8z^{-1}+3.2z^{-2}}$ causal → ROC3

$= \frac{1}{(1-4z^{-1})(1-\frac{4}{5}z^{-1})} = \frac{A}{(1-4z^{-1})} + \frac{B}{(1-4/5\, z^{-1})}$

$H(z) = \frac{5/4}{1-4z^{-1}} + \frac{1/4}{1-4/5\, z^{-1}}$ $\left[A = 5/4 \; ; \; B = \frac{-1}{4}\right]$

System is causal → ROC3 ∴ ROC $|z| > 4$

ROC not include unit circle → unstable

(aii) $g[n] = a^n h[n] \Rightarrow ROC[g[n]] = a \cdot ROC\, h[n]$

if $a = \frac{1}{5}$ then $ROC[g[n]]$ $|z| > \frac{4}{5}$

$\boxed{\text{proper } a = \frac{1}{5}}$

[unit circle include]

[so stable]

(b) For a discrete-time system of impulse response:

$$h[n] = \begin{cases} (0.09)^n u[n] & n \text{ is odd} \\ 0 & n \text{ is even} \end{cases}$$

(i) Find the transfer function $H(z)$ of the system and determine whether it is a stable system.

(ii) Find the frequency response of the system.

(iii) If the system is derived from a continuous-time system by the impulse invariance method with sampling rate of 1 Hz. What is the frequency response of the continuous-time system?

(b) $h[n] = \begin{cases} (0.09)^n u[n] & n, \text{odd} \\ 0 & n \to \text{even} \end{cases}$ (i)

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n} = (0.09) z^{-1} + (0.09)^3 z^{-3} + (0.09)^5 z^{-5} \ldots$$

$$= \frac{0.09 z^{-1}}{1-(0.09)^2 z^{-2}} = \frac{A}{1-0.09z^{-1}} + \frac{B}{1+0.09z^{-1}}$$

$$\boxed{A = 0.5 \; ; \; B = -0.5}$$

Write the transfer function ...

b) Draw a realization for the following system using the canonical direct form II:

$$y[n] - y[n-1] + \frac{1}{5}y[n-2] = x[n] - \frac{1}{3}x[n-1]$$

... radians/sec and the bandwidth of

Q3

b

$$y[n] - y[n-1] + \frac{1}{5}y[n-2] = x[n] - \frac{1}{3}x[n-1]$$

$$Y(z) - z^{-1}Y(z) + \frac{1}{5}z^{-2}Y(z) = X(z) - \frac{1}{3}z^{-1}X(z)$$

$$Y(z)\left[1 - z^{-1} + \frac{1}{5}z^{-2}\right] = X(z)\left[1 - \frac{1}{3}\right]$$

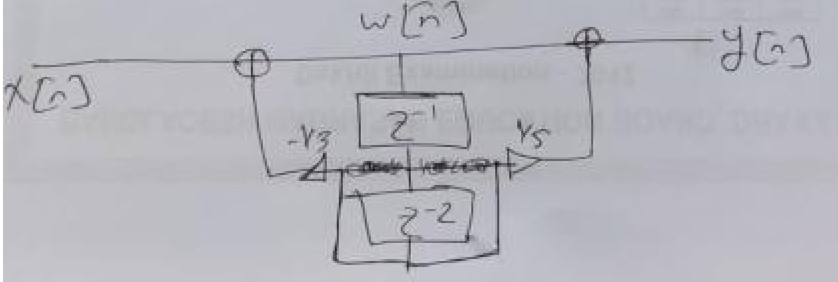
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - \frac{1}{3}z^{-1}}{1 - z^{-1} + \frac{1}{5}z^{-2}}$$

$$\Rightarrow \frac{Y(z)}{\cancel{X(z)}\; W(z)} \cdot \frac{W(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{3}z^{-2}} \cdot \left(1 + \frac{1}{3}z^{-1}\right)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - z^{-1} + \frac{1}{3}z^{-2}} \quad \therefore x[n] = w[n] - w[n-1] + \frac{1}{3}w[n-2]$$

$$\boxed{\therefore w[n] = x[n] + w[n-1] - \frac{1}{3}w[n-2]}$$

$$\cancel{W(z)} \quad \frac{Y(z)}{W(z)} = 1 + \frac{1}{5}z^{-1} \Rightarrow \boxed{y[n] = w[n] + \frac{1}{5}w[n-1]}$$



$$y[n] - y[n-1] + \frac{1}{5}y[n-2] = x[n] - \frac{1}{3}x[\ldots]$$

c) If the bandwidth of the signal $x_1(t)$ is 200π radians/sec and the bandwidth of the signal $x_2(t)$ is 500π radians/sec, find the minimum sampling frequency needed for:

i. $x_1(t) + x_2(t)$

ii. $x_1(t) * x_2(t)$

iii. $x_1(t)x_2(t)$

[15 Points]

c

$x_1(t) = 200\pi \quad x_2(500\pi)$

(i) $x_1(t) + x_2(t) \Rightarrow \max(200\pi, 500\pi) \times 2 = 1000\pi$

(ii) $x_1(t) * x_2(t) \Rightarrow \min(200\pi, 500\pi) \times 2 = 400\pi$

(i) $x_1(t) \cdot x_2(t) \Rightarrow (200\pi + 500\pi) \times 2 = 1400\pi$

a

[15 Points]

Question 4:

a) A Butterworth low-pass filter with frequency response $H(e^{j\omega})$ is to be designed to meet the following specifications:

$\delta_1 = 0.02 \quad 0.98 \le |H(e^{j\omega})| \le 1.02 \quad for\ |\omega| \le \frac{\pi}{6} \qquad \Omega_p = \frac{\pi}{6}$

$\delta_2 = 0.02 \quad |H(e^{j\omega})| \le 0.02 \quad for\ \frac{2\pi}{3} \le |\omega| \le \pi \qquad \Omega_s = \pi$

If the impulse invariance transformation is used in this design with sampling period T=1, what are the poles of the filter?

(a) $0.98 < |H(j\Omega_p)| < 1.02$

$\therefore |H(j\Omega_p)|^2 = (0.98)^2 \qquad |H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}$

$\therefore (0.98)^2 = \frac{1}{1+\left(\frac{\Omega_p}{\Omega_c}\right)^{2n}} \Rightarrow \left(\frac{\Omega_p}{\Omega_c}\right)^{2n} = \frac{1}{(0.98)^2} - 1$

$= \qquad$ — (i)

$\therefore \left(\frac{\Omega_s}{\Omega_c}\right)^{2n} = \frac{1}{(0.02)^2} - 1 \quad = \qquad$ — (ii)

$\Rightarrow \left(\frac{\Omega_p}{\Omega_s}\right)^{2n} = \frac{0/0\cancel{4}}{\cancel{1}/99} = \frac{\frac{1}{(0.98)^2} - 1}{\frac{1}{(0.02)^2} - 1} =$

$\cancel{j(n+2k-1)}\pi$

period T=1, what are the ...

(b) A second-order analog filter with two poles at $s = -3 \mp j$, two zeros at $s = 1 \mp 2j$ and a DC gain of 1. The filter is to be transformed to digital via the bilinear transformation method with sampling period of 1. Find the transfer function $H(z)$ of the digital filter.

$\cancel{S = \Omega_c e^{j\frac{(N+2k-1)\pi}{2N}}}$

Gain = 1   poles $-3+j$ zeros $1+2j$

$$H(s) = \frac{K(s-1-2j)(s-1+2j)}{(s+3-j)(s+3+j)} \qquad \boxed{j^2 = 1}$$

$H(0) = 1$

$$= \frac{K(5)}{10}$$

$\therefore K = 2$

$$= \frac{K[(s-1)^2 + (4j)^2]}{(s+3)^2 + j^2} = \frac{K[(s-1)^2 + 4]}{(s+3)^2 + 1}$$

$$= \frac{K[s^2 - 2s + 5]}{s^2 + 6s + 10}$$

$$\therefore H(s) = \frac{2s^2 - 4s + 10}{s^2 + 6s + 10}$$

$$s = \frac{2}{T_s} \cdot \frac{1 - z^{-1}}{1 + z^{-1}}$$