Question 1: **[15 Points]**

[a] Determine the properties of each of the following discrete time systems (linearity, causality, memory, time invariance, stability

$$y[n] = x[2n] + 2$$

$$y[n] = x[n-2]\cos\omega_0 n$$

Point one :

Digital Signal Processing

Final 2022

Question 1- Linearity, Causality, memory, time invariance, stability

a) $y[n] = x[2n]+2$ (system equation)

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• $x_1[n] \longrightarrow y_1[n] = x_1[2n]+2$

$x_2[n] \longrightarrow y_2[n] = x_2[2n]+2$

$\because x_3[n] = x_1[n]+x_2[n]$

$\boxed{\therefore y_3[n] = x_3[2n]+2} \neq \boxed{\begin{aligned} y_3[n] &= y_1[n]+y_2[n] \\ &= x_1[2n]+2+x_2[2n]+2 \\ &= x_1[2n]+x_2[2n]+4 \end{aligned}}$

∴ This system is not linear. → (1)

• $y[n] = x[2n]+2$

∵ The output depends on $x[2n]$, which involves evaluating x at twice the time index,

The system is not Causal # → (2)

• $y[n] = x[2n]+2$

$y[n_1] = x[2n_1]+2 \quad y[n_2] = x[2n_2]+2$

∵ To check for memory, consider the output $y[n]$ at some abitrary time n.

The output depends on the value of x at the time index $2n$. So, the system have memory as it only relies on the current input value → (3)

• $y[n] = x[2n] + 2$

To check time invariance, consider two inputs $x_1[n]$, $x_2[n]$ that are time-shifted of each other:

$\therefore x_2[n] = \boxed{x_1[n-n_0]}$

The output will be:

$y_1[n] = x_1[2n] + 2$

$y_2[n] = x_2[2n] + 2 = x_1[2n-2n_0] + 2$

To be system time-invariance $y_2[n]$ should be equal to $y_1[n-n_0]$

$y_2[n] = x_1[2n-2n_0] + 2 \neq y_1[n-n_0]$

$\therefore$ The system is time-Variant # → 4

• $y[n] = x[2n] + 2$

To determine stability, we need to analyze whether the output remains bounded for bounded inputs. Since the given system is not Linear, it is not possible to directly determine its stability using traditional method. $y[n] = x[2n] + 2$, we can observe that the output is always shifted by a constant value of 2 although the input.

$\therefore$ The system can be stable. #

For $y(n) = x(n-2) \cos \omega_0 n$

* Not memory less or memory system
* Causal system as it does not depend on future input.
* not Time invariant or Time Variant
* stable system
* linear system.

Point two :

[b] The non-zero values of a discrete-time signal are given as $x[0] = 2 + 2j$ and $x[1] = 1$. Decompose x[n] into conjugate symmetric (even) and conjugate anti-symmetric (odd) signals.

Q1:(b) $x[0] = 2+2j$ $x[1] = 1$ Conjugate symmetric [even]

anti-Conjugate symmetric [odd]

• Solution:-

1. Conjugate symmetric (even) Component:- $x_e[n] = \frac{1}{2}\left(x[n] + x^*[-n]\right)$

2. Conjugate anti-symmetric (odd) Component:- $x_o[n] = \frac{1}{2}\left(x[n] - x^*[-n]\right)$

∵ $x^*[-n]$ represent the complex conjugate of $x[-n]$

Given:- $x[0] = 2+2j$, $x[1] = 1$

∴ 1. Even Component ($x_e[n]$):

$$x_e[0] = \frac{1}{2}\left(x[0] + x^*[-0]\right) = \frac{1}{2}(2+2j+2-2j) = \boxed{2}$$

$$x_e[1] = \frac{1}{2}\left(x[1] + x^*[-1]\right) = \frac{1}{2}(1+0) = \boxed{\frac{1}{2}}$$

$$x_e[-1] = \frac{1}{2}\left(x[-1] + x^*[1]\right) = \frac{1}{2}(0+1) = \boxed{\frac{1}{2}}$$

∴ 2. odd Component ($x_o[n]$):-

$$x_o[0] = \frac{1}{2}\left(x[0] - x^*[0]\right) = \frac{1}{2}(\cancel{2}+2j-\cancel{2}+2j) = \boxed{2j}$$

$$x_o[-1] = \frac{1}{2}\left(x[-1] - x^*[1]\right) = \frac{1}{2}(0-1) = \boxed{\frac{-1}{2}}$$

$$x_o[1] = \frac{1}{2}\left(x[1] - x^*[-1]\right) = \frac{1}{2}(1-0) = \boxed{\frac{1}{2}}$$

∴ $x_e[n] = 2$ for $n=0$, $x_e[n] = 1/2$ for $n=1$, $x_e[n] = \frac{1}{2}$ for $n=-1$

$x_o[n] = 2j$ for $n=0$, $x_o[n] = 0$ for $n = 1$ #, $x_o[n] = -\frac{1}{2}$ for $n = -1$

[c] If a continuous- time sinusoidal signal of frequency is 131 Hz is sampled at a sampling rate of 8000 Hz. What is the discrete-time frequency in rad/sample of the resulting discrete-time signal?

Q1:- C) frequency = 131 HZ.

Sampling rate = 8000 HZ.

• Solution:- Discrete time frequency in rad/sample

$$= \frac{\text{Analog frequency (HZ)}}{\text{Sampling rate (HZ)}} \times 2\pi$$

• Given:- Analog frequency = 131 HZ

Sampling rate = 8000 HZ

∴ Discrete-time frequency in rad/sample

$$= \frac{131}{8000} \times 2\pi = 0.102887 \text{ rad/sample}$$

[d] Find the linear and 4-point circular convolutions between x[n] and h[n] given that:

$$x[n] = [\underline{1}, 0, 2, 3]$$

$$h[n] = [\underline{2}, 1, 4]$$

8000

Sample #

d) $x[n] = [1, 0, 2, 3]$ $h[n] = [2, 1, 4]$

Linear Convolution

$y[0] = 1*2 = 2$

$y[1] = 1*1 + 2*0 = 1$

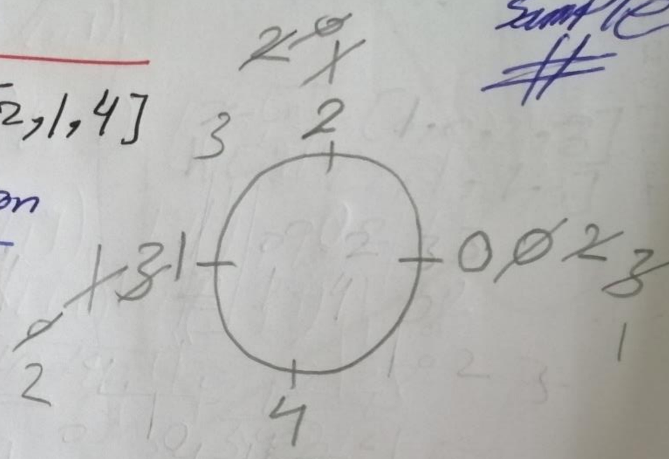
$y[2] = 4*1 + 0*1 + 2*2 = 8$

$y[3] = 4*0 + 1*2 + 3*2 = 2+6 = 8$

$y[4] = 2*4 + 3*1 + 2*0 = 11$

$y[5] = 4*3 = 12$

$\therefore y[n] = [2, 1, 8, 8, 11]$

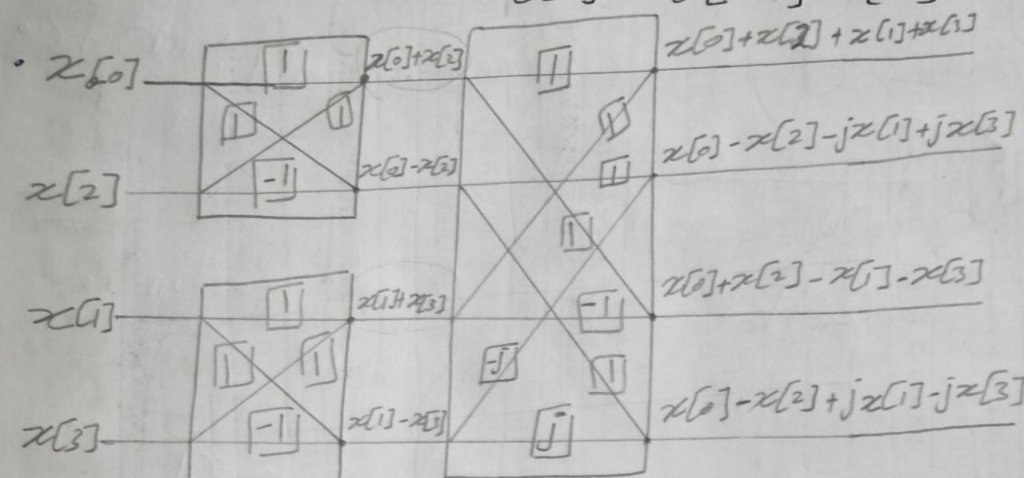


$[13, 13, 8, 8]$

[e] Draw the butterfly diagram used to evaluate the 4-point FFT of the signal $x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2]$.

e) butterfly diagram 4 Point FFT

$x[n] = \delta[n] + 2\delta[n-1] + \delta[n-2].$



$x[0] = \delta[0] + 2\delta[0-1] + \delta[0-2] = 1$

$x[1] = \delta[1] + 2\delta[1-1] + \delta[1-2] = 2$

$x[2] = \delta[2] + 2\delta[2-1] + \delta[2-2] = 1$

$x[3] = \delta[3] + 2\delta[3-1] + \delta[3-2] = 0$

0	0	$x[0]$
1	0	$x[2]$
0	1	$x[1]$
1	1	$x[3]$

$\therefore\ x[0]+x[2] = x[0]+x[2]+x[1]+x[3]$
$= 1+1+2+0 = \boxed{4}$

$x[0]-x[2] = x[0]-x[2]-jx[1]+jx[3]$
$= 1-1-j2+j0 = \boxed{-2j}$

$x[1]+x[3] = x[0]+x[2]-x[1]-x[3]$
$= 1+1-2-0 = \boxed{\text{Zero}}$

$x[1]-x[3] = x[0]-x[2]+jx[1]-jx[3]$
$= 1-1+j2-j\ \text{Zero} = \boxed{2j}$

4 Point FFT

Question 2:

a) Find the impulse response of the following filter assuming it is a causal filter:

$$H(z) = \frac{1 - (\cos\omega_0)z^{-1}}{1 - (2\cos\omega_0)z^{-1} + z^{-2}}$$

Let $\cos\omega_0 = 1$

$$H(Z) = \frac{1 - Z^{-1}}{1 - 2Z^{-1} + Z^{-2}} = \frac{1 - Z^{-1}}{(1 - Z^{-1})(1 - Z^{-1})}$$

$$= \frac{1}{1 - Z^{-1}} \qquad |Z| > 1$$

$$\therefore h[n] = (1)^n u[n]$$

b) For a signal with transfer function given by:

$$X(z) = \frac{2z^{-1} + 3}{1 - 0.3z^{-1} + 0.03z^{-2}}$$

Find all the possible ROC's and the corresponding time domain representation of the signal. Which signal has a defined DTFT?

Q2:- b) $X(Z) = \frac{2Z^{-1}+3}{1-0.4Z^{-1}+0.03Z^{-2}}$

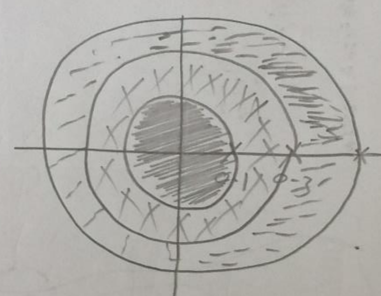
تحليل $\frac{2Z^{-1}+3}{1-0.4Z^{-1}+0.03Z^{-2}} = \frac{2Z^{-1}+3}{(1-0.3Z^{-1})(1-0.1Z^{-1})}$

$= \frac{A}{1-0.3Z^{-1}} + \frac{B}{1-0.1Z^{-1}}$ (Partial fraction)

$A = \frac{2Z^{-1}+3}{1-0.1Z^{-1}} = \frac{\frac{2}{0.3}+3}{1-\frac{0.1}{0.3}} = \frac{2.9}{0.2} = \frac{29}{2}$

$B = \frac{2Z^{-1}+3}{1-0.3Z^{-1}} = \frac{\frac{2}{0.1}+3}{1-\frac{0.3}{0.1}} = \frac{2.3}{-0.2} = -\frac{23}{2}$

$\frac{\frac{29}{2}}{1-0.3Z^{-1}} - \frac{\frac{23}{2}}{1-0.1Z^{-1}}$



ROC1:- $|Z| < 0.1$

$x[n] = -\frac{29}{2}(0.3)^n u[-n-1] + \frac{23}{2}(0.1)^n u[-n-1]$ Unstable

ROC2:- $0.1 < |Z| < 0.3$ Unstable

$x[n] = -\frac{29}{2}(0.3)^n u[-n-1] - \frac{23}{2}(0.1)^n u[n]$

ROC 3:- $|Z| > 0.3$ Stable

$x[n] = \frac{29}{2}(0.3)^n u[n] - \frac{23}{2}(0.1)^n u[n]$

DTFT

c) Consider a digital filter where the z-transform of the impulse response is

$$H(z) = \frac{z^2 - 1}{z^2 + \frac{49}{64}}$$

Plot the poles and zeros of the filter. Find the ROC of the stable realization of the filter. Sketch the frequency response and determine the type of the filter

Q2 (c)

$$H(z) = \frac{z^2 - 1}{z^2 + \frac{49}{64}}$$

Sol

Zeros $z^2 - 1 = 0$ $\quad z^2 = 1$

$|z| = \pm 1$

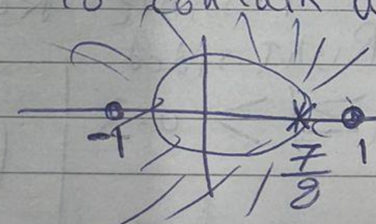
Poles $z^2 + \frac{49}{64} = 0$

$z^2 = -\frac{49}{64}$

$|z| = \sqrt{-\frac{49}{64}} = \frac{7}{8}$

RoC $z > \frac{7}{8}$

To contain unit circle



$$H(e^{jw}) = \frac{e^{2jw} - 1}{e^{2jw} + \frac{49}{64}}$$

Question 3: **[15 Points]**

a) A discrete-time system is given as

$$y[n] = ay[n-1] + x[n] - ax[n-1],$$

where a is a real scalar constant. Find:

i) The impulse response of the system.
ii) The range of values of a for which the system is BIBO stable.

$$Y(Z) = a\,Y(Z)\,Z^{-1} + X(Z) - a\,X(Z)\,Z^{-1}$$

$$Y(Z) - a\,Y(Z)\,Z^{-1} = X(Z) - a\,X(Z)\,Z^{-1}$$

$$Y(Z)\left[1 - aZ^{-1}\right] = X(Z)\left[1 - aZ^{-1}\right]$$

$$\frac{Y(Z)}{X(Z)} = \frac{1 - aZ^{-1}}{1 - aZ^{-1}} = 1 = \delta[n]$$

any value for $\boxed{a}$ will make the system stable.

#

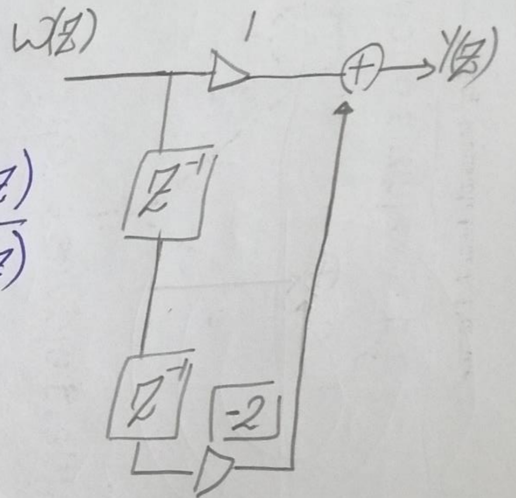
c) Draw a realization for the following system using the canonical direct form II and parallel form:

$$H(z) = \frac{1-2z^{-2}}{1-1.2z^{-1}+0.32z^{-2}}$$

c) $H(Z) = \frac{1-2Z^{-2}}{1-1.2Z^{-1}+0.32Z^{-2}}$

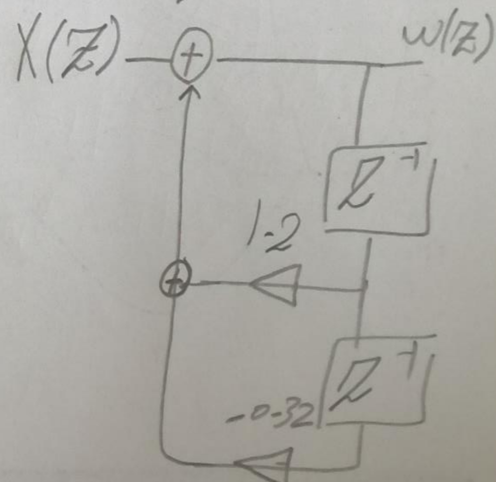
• ∴ $H(Z) = \frac{b_0 + b_1 Z^{-1} + \cdots}{1 + a_1 Z^{-1} + \cdots} = \frac{Y(Z)}{X(Z)}$

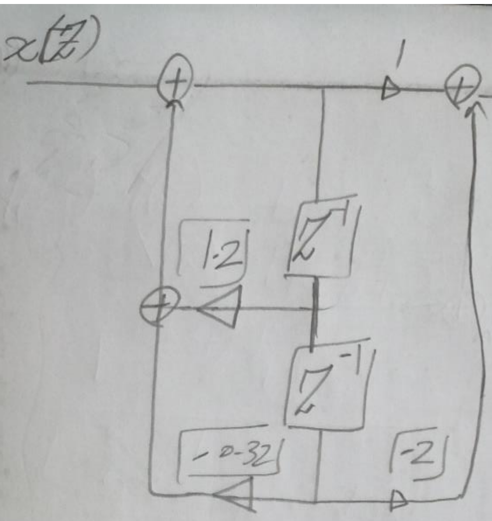
$$\boxed{\frac{Y(Z)}{W(Z)}} \cdot \boxed{\frac{W(Z)}{X(Z)}}$$



∴ $\frac{Y(Z)}{W(Z)} = 1-2Z^{-2}$

∴ $\frac{W(Z)}{X(Z)} = 1-1.2Z^{-1}+0.32Z^{-2}$





Direct form II

by Parallel form:-

$$H(Z) = \frac{1-2Z^{-2}}{1-1.2Z^{-1}+0.32Z^{-2}} \quad \text{by (Partial fraction)}$$

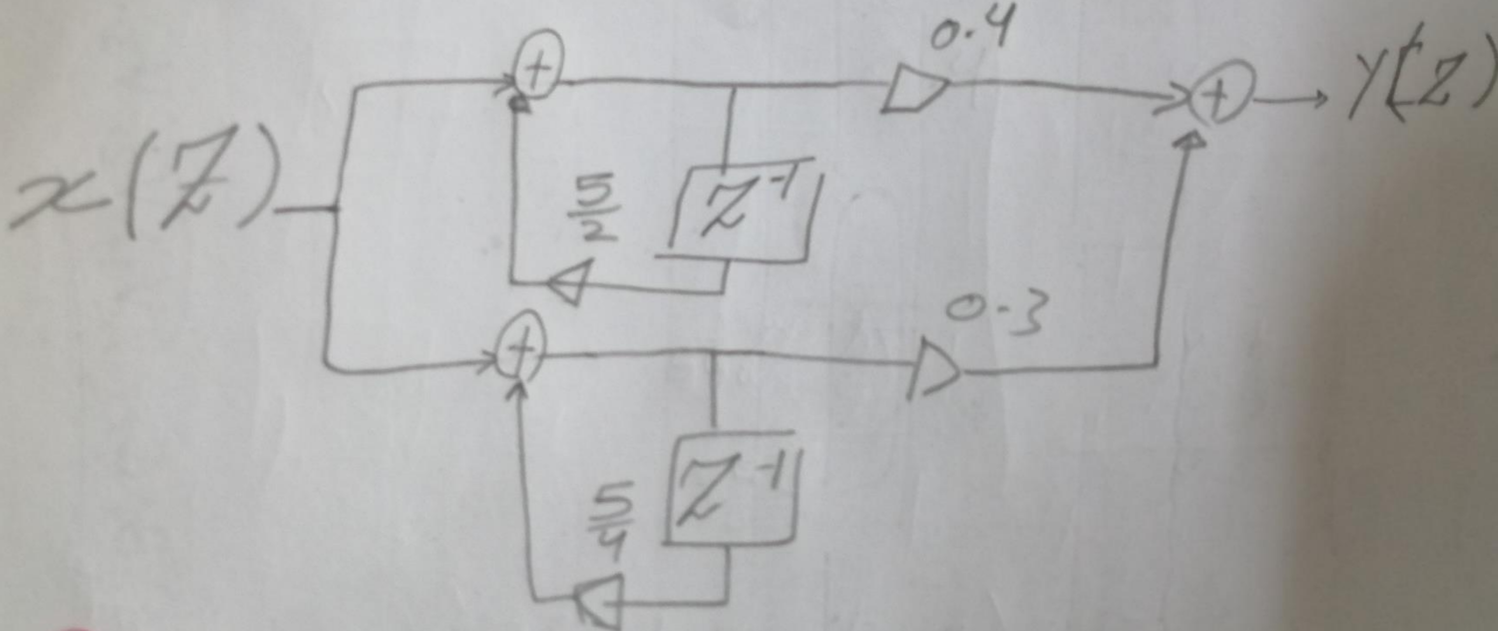
$$= \frac{1-2Z^{-2}}{\left(1-\frac{5}{2}Z^{-1}\right)\left(1-\frac{5}{4}Z^{-1}\right)} = \frac{A}{1-\frac{5}{2}\boxed{Z^{-1}}} + \frac{B}{1-\frac{5}{4}\boxed{Z^{-1}}}$$

$\rightarrow \frac{2}{5}$ $\rightarrow \frac{4}{5}$

$$A = \frac{1-2Z^{-1}}{1-\frac{5}{4}Z^{-1}} = \frac{1-\frac{2}{2.5}}{1-\left(\frac{5}{4}\right)/2.5} = \frac{2}{5} = \boxed{0.4}$$

$$B = \frac{1-2Z^{-1}}{1-\frac{5}{2}Z^{-1}} = \frac{1-2\cdot\frac{4}{5}}{1-\frac{5}{2}\cdot\frac{4}{5}} = \frac{3}{5} \; \boxed{0.3}$$

$$\therefore H(Z) = \frac{0.4}{1 - \frac{5}{2} Z^{-1}} + \frac{0.3}{1 - \frac{5}{4} Z^{-1}}$$

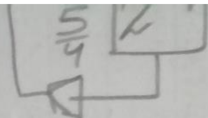


Question 4:

a) For the following analog filter:

$$H(s) = \frac{10(s+1)}{s^2 - 6s + 10}$$

Find the transfer function of the corresponding digital filter using impulse invariance method with sampling period $T_s = 0.001\ s$



Q4:-

a) $H(s) = \dfrac{10(s+1)}{s^2 - 6s + 10}$.

$\cdot\ H(s) = \dfrac{10s + 10}{s^2 - 6s + 10}$

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لازم بالقانون العام

$s = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$= \dfrac{6 \pm \sqrt{36 - 40}}{2}$

$= \dfrac{6 \pm 2j}{2} = \boxed{3 \pm j}$

$\therefore H(s) = \dfrac{10(s+1)}{(s-3-j)(s-3+j)} = \dfrac{A}{s-3-j} + \dfrac{B}{s-3+j}$

$\therefore A = \left.\dfrac{10(s+1)}{s-3+j}\right|_{s=3+j} = \dfrac{10(3+j+1)}{\cancel{3}+j-\cancel{3}+j} = \dfrac{40+10j}{2j}$

$= \dfrac{20+5j}{j}$

$= -20j + 5$

$$B = \frac{10(s+1)}{s-3-j}\Bigg|_{s=3-j} = \frac{10(3-j+1)}{\cancel{3}-j-\cancel{3}-j} = \frac{40-10j}{-2j}$$

$$= \frac{\overset{20}{\cancel{40}}}{-\cancel{2}j} + \boxed{\frac{-10j}{-2j}}_{5}$$

$$= \frac{-20}{j} + 5$$

$$= 20j + 5$$

$$\therefore H(s) = \frac{-20j+5}{s-3-j} + \frac{20j+5}{s-3+j}$$

$$\therefore H(Z) = \frac{(-20j+5)T_s}{1-e^{(3-j)T_s}Z^{-1}} + \frac{(20j+5)T_s}{1-e^{(3+j)T_s}Z^{-1}}$$

$$= \frac{(-20j+5)(0.001)}{1-e^{(3-j)(0.001)}Z^{-1}} + \frac{(20j+5)(0.001)}{1-e^{(3+j)(0.001)}Z^{-1}}$$

$$= \frac{-0.02j+0.005}{1-e^{(0.003-0.001j)}Z^{-1}} + \frac{0.02j+0.005}{1-e^{(0.003-0.001j)}Z^{-1}}$$

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b) Write the transfer function of the second order digital Butterworth filter with normalized cutoff frequency $\omega_c = 0.4\pi\ rad$ if bilinear transformation is used to convert the analog filter to digital and assuming T=1 sec.

دي مش حل السؤال دي سؤال شبهه

Example:- Write the Transfer of the butterworth filter of order 3 and Cutoff frequency 1000 HZ

Solution:- $n = 3$, $\Omega_c = 2\pi(1000)$

$= 2000\pi$ rad/sec

$$S_k = \Omega_c e^{j\frac{\pi}{2n}(n+2k-1)} \quad , k = 1, 2, 3$$

$$S_1 = \Omega_c e^{j\frac{\pi}{6}(3+2-1)} = \Omega_c e^{j\frac{2\pi}{3}}$$

$$S_3 = \Omega_c e^{-j\frac{2\pi}{3}}$$

$$S_2 = -\Omega_c$$

$$(s-S_1)(s-S_3) = \left(s-\Omega_c e^{j\frac{2\pi}{3}}\right)\cdot\left(s-\Omega_c e^{-j\frac{2\pi}{3}}\right)$$

JULY 17

$$= \left(S^2 + 2\Omega_c \cos\frac{2\pi}{3} S + \Omega_c^2\right) \#$$

$$= \left(S^2 + \Omega_c S + \Omega_c^2\right)$$

$$H(s) = \frac{\Omega_c^3}{(s-s_1)(s-s_2)(s-s_3)} = \frac{\Omega_c^3}{(s^2+\Omega_c s+\Omega_c^2)\cdot(s+\Omega_c)}$$

. In Case of using Bi-Linear Transformation:

ω_s, ω_p should be Transformed to Ω_s, Ω_p using

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the equation:

$$\Omega = \frac{2}{T_s} \tan\frac{\omega}{2}$$

. In Case of impulse invariance

$$\Omega = \frac{\omega}{T_s}$$

Digital filter Design using