consider the signal $e^{jwn}$

$$e^{j(w+2\pi)n} = e^{jwn} \cdot \underbrace{\cancel{e^{j2\pi n}}}_{=1}$$

$$\therefore e^{j(w+2\pi)n} = e^{jwn}$$

let $\phi_k[n] = e^{jkw_0 n} = e^{jk(\frac{2\pi}{N})n}$

then $\phi_k[n] = \phi_{k+N}[n]$

as $\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n} = e^{jk(\frac{2\pi}{N})n} \cdot \underbrace{\cancel{e^{j2\pi n}}}_{=1}$

$$\therefore \phi_{k+N}[n] = \phi_k[n]$$

Then there are only $N$ different $\phi_k[n]$

then $\boxed{x[n] = \sum_{k=\langle N\rangle} a_k e^{jkw_0 n}}$

This is Fourier's Claim

multiply both sides by $e^{-jrw_0 n} = e^{-jr(\frac{2\pi}{N})n}$

$$\therefore x[n] \cdot e^{-jr(\frac{2\pi}{N})n} = \sum_{k=\langle N\rangle} a_k \cdot e^{jk(\frac{2\pi}{N})n} \cdot e^{-jr(\frac{2\pi}{N})n}$$

$$= \sum_{k=\langle N\rangle} a_k e^{j(k-r)(\frac{2\pi}{N})n}$$

summing over $N$ terms we obtain

$$\sum_{n=\langle N\rangle} x[n] e^{-jr(\frac{2\pi}{N})n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(\frac{2\pi}{N})n}$$

interchanging the two summations in right side

$$\therefore \sum_{n=\langle N\rangle} x[n] e^{-jr(\frac{2\pi}{N})n} = \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(\frac{2\pi}{N})n}$$

Recall $\sum_{k=a}^{b} r^k = r^a \cdot \frac{1-r^{(b-a+1)}}{1-r}$ ; $r \neq 1$

$$\therefore \sum_{n=\langle N\rangle} e^{j(k-r)(\frac{2\pi}{N})n} = \sum_{n=N_1}^{N-1+N_1} e^{j(k-r)(\frac{2\pi}{N})n}$$

$$= \sum_{m=0}^{N-1} e^{j(k-r)(\frac{2\pi}{N})(m+N_1)} \quad ; \text{ as } m = n - N_1$$

$$= \sum_{m=0}^{N-1} e^{j(k-r)(\frac{2\pi}{N})m} \cdot e^{j(k-r)(\frac{2\pi}{N})N_1}$$

$$= e^{j(k-r)(\frac{2\pi}{N})N_1} \cdot \sum_{m=0}^{N-1} e^{j(k-r)(\frac{2\pi}{N})m}$$

$$\sum_{m=0}^{N-1} e^{j(k-r)(\frac{2\pi}{N})m} = \frac{1-e^{j(k-r)(\frac{2\pi}{N})(N-1-0+1)}}{1-e^{j(k-r)(\frac{2\pi}{N})}}$$

$$\therefore \sum_{m=0}^{N-1} e^{j(k-r)(\frac{2\pi}{N})m} = \frac{1-e^{j(k-r)(\frac{2\pi}{\cancel{N}})\cancel{N}}}{1-e^{j(k-r)(\frac{2\pi}{N})}}$$

as $k$ is integer and $r$ is integer

$\therefore (k-r)$ is integer $\Rightarrow e^{j(k-r)2\pi} = 1$

$$\therefore \sum_{m=0}^{N-1} e^{j(k-r)(\frac{2\pi}{N})m} = \frac{1-1}{1-e^{j(k-r)(\frac{2\pi}{N})}} = 0 \; ; k \neq r$$

if $k = r \Rightarrow \sum_{n=\langle N\rangle} e^{j(k-r)(\frac{2\pi}{N})n} = \sum_{n=\langle N\rangle} 1 = N$

or if $k = r + pN$ $p$ is integer

$$\sum_{n=\langle N\rangle} e^{j(\cancel{r}+pN-\cancel{r})(\frac{2\pi}{N})n} = \sum_{n=\langle N\rangle} e^{jp\cancel{N}(\frac{2\pi}{\cancel{N}})n} = \sum_{n=\langle N\rangle} 1 = N$$ too

$$\therefore \sum_{n=\langle N\rangle} e^{j(k-r)(\frac{2\pi}{N})n} = \begin{cases} N \; ; & k = r \text{ or } k = r + pN, \; p \text{ is integer} \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$\therefore \sum_{n=\langle N\rangle} x[n] e^{-jr(\frac{2\pi}{N})n} = \sum_{k=\langle N\rangle} a_k \cdot \begin{cases} N \; ; & k = r \text{ or } k = r+pN \\ 0 \; ; & \text{otherwise} \end{cases}$$

$$= N a_r = N a_{r+pN}$$

$$\therefore \boxed{a_r = a_{r+pN}}$$

$$\therefore \boxed{a_k = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(\frac{2\pi}{N})n}}$$

$\therefore$ The Fourier series coefficients of Discrete-time periodic signal is also periodic with same period as $a_k = a_{k+pN}$

## Ex 8

Consider $x[n] = \sin(w_o n)$

by comparing it with general form which is $\sin(wn+\phi)$

$\therefore$ $w = w_o \Rightarrow x[n]$ to be periodic $\frac{w_o}{2\pi}$ should be rational number which is simply $\frac{\text{integer}}{\text{integer}}$

if $\frac{w_o}{2\pi} = \frac{M}{N}$ ; if there is no common factor between M & N

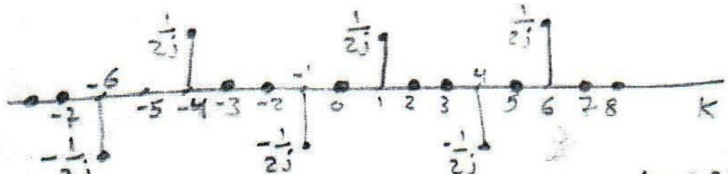
$\Rightarrow$ the fundamental period of $x[n]$ is $N$

$\Rightarrow$ $x[n] = \sin\left(M\left(\frac{2\pi}{N}\right)n\right) = \frac{1}{2j} e^{jM\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-jM\left(\frac{2\pi}{N}\right)n}$

if $M=1, N=5 \Rightarrow x[n] = \frac{1}{2j} e^{j\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j\left(\frac{2\pi}{N}\right)n}$

$$\Rightarrow \begin{cases} a_1 = a_{-4} = a_6 = a_{1+PN} = \frac{1}{2j} \\ a_{-1} = a_{-6} = a_4 = a_{-1+PN} = -\frac{1}{2j} \end{cases} \quad (1)$$

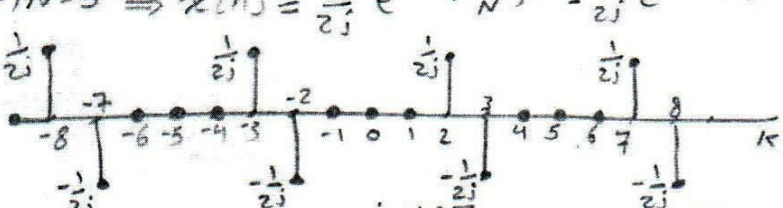
$w_o = \frac{2\pi}{5}$




if $M=2, N=5 \Rightarrow x[n] = \frac{1}{2j} e^{j2\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j2\left(\frac{2\pi}{N}\right)n}$

$$\Rightarrow \begin{cases} a_2 = a_{-3} = a_7 = a_{2+PN} = \frac{1}{2j} \\ a_{-2} = a_{-7} = a_3 = a_{-2+PN} = -\frac{1}{2j} \end{cases} \quad (2)$$

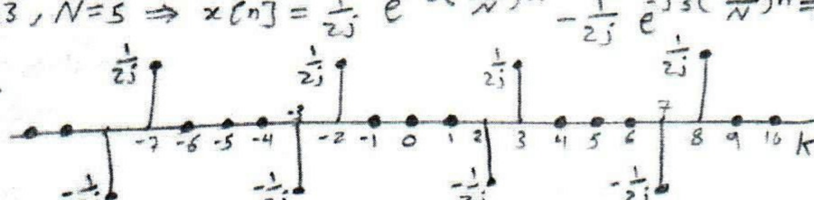
$w_o = \frac{4\pi}{5}$




if $M=3, N=5 \Rightarrow x[n] = \frac{1}{2j} e^{j3\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j3\left(\frac{2\pi}{N}\right)n}$

$$\Rightarrow \begin{cases} a_3 = a_{-2} = a_8 = a_{3+PN} = \frac{1}{2j} \\ a_{-3} = a_2 = a_7 = a_{-3+PN} = -\frac{1}{2j} \end{cases} \quad (3)$$

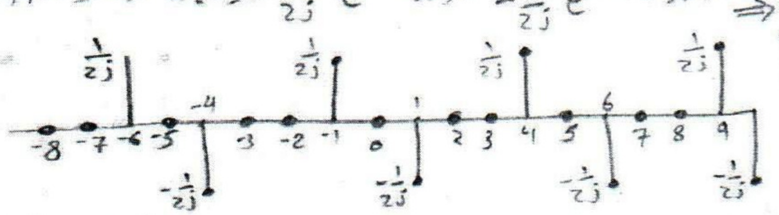
$w_o = \frac{6\pi}{5}$




if $M=4, N=5 \Rightarrow x[n] = \frac{1}{2j} e^{j4\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j4\left(\frac{2\pi}{N}\right)n}$

$$\Rightarrow \begin{cases} a_4 = a_{-1} = a_9 = a_{4+PN} = \frac{1}{2j} \\ a_{-4} = a_1 = a_6 = a_{-4+PN} = -\frac{1}{2j} \end{cases} \quad (4)$$

$w_o = \frac{8\pi}{5}$



if $M=5, N=5 \Rightarrow x[n] = \sin(2\pi n) = 0 \Rightarrow$ No Fourier series coefficients

if $M=6, N=5 \Rightarrow x[n] = \frac{1}{2j} e^{j6\left(\frac{2\pi}{N}\right)n} - \frac{1}{2j} e^{-j6\left(\frac{2\pi}{N}\right)n}$

$$\Rightarrow \begin{cases} a_6 = a_1 = a_{-4} = \frac{1}{2j} \\ a_{-6} = a_{-1} = a_4 = -\frac{1}{2j} \end{cases} \quad (5)$$

$=$ if $M=1, N=5 \Rightarrow w_o = \frac{2\pi}{5} = \frac{12\pi}{5}$ = Same coefficients in (1)

**Note** $\frac{12\pi}{5} = \frac{10\pi}{5} + \frac{2\pi}{5} = 2\pi + \left(\frac{2\pi}{5}\right)$

which is correct as $w_o$ & $w_o + 2\pi$ are the same frequencies for discrete-time signals $\Rightarrow$ Recall slides # 4.

**Note** However $a_k$ is periodic as $a_k = a_{k+PN}$ ; $P$ is integer only ONE period is used in synthesis equation.

**Note** $a_{-k} = a_k^*$ as $x[n]$ is real for any $M$

**Ex 9** if $x[n] = \begin{cases} 1 & ; |n| \le N_1 \\ 0 & ; N_1 < |n| < \frac{N}{2} \end{cases}$ is Periodic with Period $N$

Find $a_k$ if any?

(Solution)

$x[n] \Longrightarrow$


as $x[n]$ is periodic with period $N \Rightarrow \therefore$ has Fourier series coefficients

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]\, e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=-\frac{N}{2}}^{\frac{N}{2}} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk\left(\frac{2\pi}{N}\right)n}$$

Recall $\sum_{k=a}^{b} r^k = r^a \cdot \frac{1-r^{b-a+1}}{1-r}$ ; $r \ne 1$

if $k \ne 0, \pm N, \pm 2N, \dots$

$$\Rightarrow \therefore a_k = \frac{1}{N}\cdot\left\{ e^{-jk\left(\frac{2\pi}{N}\right)(-N_1)} \cdot \frac{1 - e^{-jk\left(\frac{2\pi}{N}\right)(N_1-(-N_1)+1)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}} \right\}$$

$$\therefore a_k = \frac{1}{N}\, e^{jk\left(\frac{2\pi}{N}\right)N_1} \cdot \left[\frac{1 - e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}}\right]$$

$$= \frac{1}{N}\left[\frac{e^{jk\left(\frac{2\pi}{N}\right)N_1} - e^{-jk\left(\frac{2\pi}{N}\right)(2N_1+1-N_1)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}}\right]$$

$$= \frac{1}{N}\left[\frac{e^{jk\left(\frac{2\pi}{N}\right)N_1} - e^{-jk\left(\frac{2\pi}{N}\right)(N_1+1)}}{1 - e^{-jk\left(\frac{2\pi}{N}\right)}}\right]$$

take $e^{-jk\left(\frac{2\pi}{N}\right)\left(\frac{1}{2}\right)}$ as a common factor from numerator and denomenator

$$\therefore a_k = \frac{1}{N}\cdot\frac{\cancel{e^{-jk\left(\frac{2\pi}{N}\right)\left(\frac{1}{2}\right)}}\left[e^{jk\left(\frac{2\pi}{N}\right)\left(N_1+\frac{1}{2}\right)} - e^{-jk\left(\frac{2\pi}{N}\right)\left(N_1+\frac{1}{2}\right)}\right]}{\cancel{e^{-jk\left(\frac{2\pi}{N}\right)\left(\frac{1}{2}\right)}}\left[e^{jk\left(\frac{2\pi}{N}\right)\left(\frac{1}{2}\right)} - e^{-jk\left(\frac{2\pi}{N}\right)\left(\frac{1}{2}\right)}\right]}$$

$$\therefore a_k = \frac{1}{N}\cdot\left.\frac{\cancel{2j}\sin\left(k\left(\frac{2\pi}{N}\right)\left(N_1+\frac{1}{2}\right)\right)}{\cancel{2j}\sin\left(k\left(\frac{2\pi}{2N}\right)\right)}\right]$$

$$\therefore \boxed{a_k = \frac{1}{N}\cdot\frac{\sin\left[k\left(\frac{2\pi}{N}\right)\left(N_1+\frac{1}{2}\right)\right]}{\sin\left[k\left(\frac{\pi}{N}\right)\right]}}$$ ; $k \ne 0, \pm N, \pm 2N, \dots, pN$

$p$ is integer

if $k = 0, \pm N, \pm 2N, \dots$

$$\Rightarrow a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jpN\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jpn(2\pi)} = \frac{1}{N}\cdot\sum_{n=-N_1} 1$$

$$\therefore a_k = \frac{1}{N}\cdot(2N_1+1) = \boxed{\frac{2N_1+1}{N}}$$ ; $k = 0, \pm N, \pm 2N, \dots, pN$

$p$ is integer

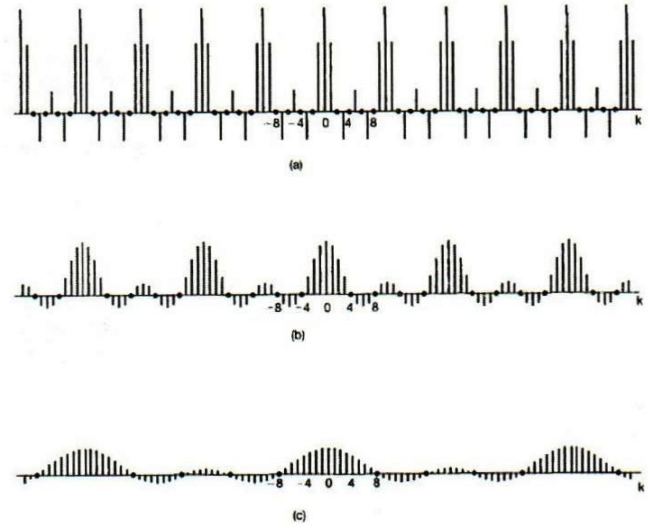
Follow Ex9

plot of $a_k$
for $2N_1+1=5$
and different $N$

$N=10 \Rightarrow$

$N=20 \Rightarrow$

$N=40 \Rightarrow$




## - Convergence of discrete-time Fourier series :-

as $x[n]$ is a sequence of finite-amplitude samples

$$\Rightarrow \quad a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk(\frac{2\pi}{N})n}$$

$$= \text{finite sum of finite values}$$

$$\Rightarrow \quad a_k < \infty \quad ; \quad \forall k$$

$\Rightarrow$ No convergence issues with discrete-time Fourier series

## properties of discrete-time Fourier Series

- They are as the properties of continuous-time Fourier series

Except ① First difference :- if $x[n] \xleftrightarrow{FS} a_k$

and $y[n] = x[n] - x[n-1] \xleftrightarrow{FS} b_k$

From time-shift property $\Rightarrow x[n-1] \xleftrightarrow{FS} e^{jk(\frac{2\pi}{N})} . a_k$

$\Rightarrow$ From linearity property $\Rightarrow x[n] - x[n-1] \xleftrightarrow{FS} a_k - e^{jk(\frac{2\pi}{N})} . a_k$

$$\therefore \quad y[n] = x[n] - x[n-1] \xleftrightarrow{FS} \left(1 - e^{jk(\frac{2\pi}{N})}\right) . a_k$$

this is parallel to differentiation property in continuous-time Fourier Series

② <u>Multiplication property</u>, if $x[n] \xleftrightarrow{FS} a_k$
$y[n] \xleftrightarrow{FS} b_k$

$$\therefore w[n] = x[n] \cdot y[n] \xleftrightarrow{FS} c_k$$

as $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(\frac{2\pi}{N})n} = \sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n}$

and $y[n] = \sum_{m=\langle N\rangle} b_m e^{jm(\frac{2\pi}{N})n} = \sum_{m=0}^{N-1} b_m e^{jm(\frac{2\pi}{N})n}$

$$\therefore x[n] \cdot y[n] = \left\{ \sum_{k=0}^{N-1} a_k e^{jk(\frac{2\pi}{N})n} \right\} \left\{ \sum_{m=0}^{N-1} b_m e^{jm(\frac{2\pi}{N})n} \right\}$$

$$= \sum_{k=0}^{N-1} \sum_{m=0}^{N-1} a_k b_m e^{j(k+m)(\frac{2\pi}{N})n}$$

let $p = k+m \Rightarrow m = p-k$

$$\therefore x[n] \cdot y[n] = \sum_{k=0}^{N-1} \sum_{p=k}^{k+N-1} a_k b_{p-k} e^{jp(\frac{2\pi}{N})n}$$

remember $b_k$ is periodic with period $N$ as $b_k = b_{k+lN}$
$l$ is integer
and $e^{jp(\frac{2\pi}{N})n}$ is also periodic with period $N$ as $w = p(\frac{2\pi}{N})$
and $\frac{w}{2\pi} = p(\frac{2\pi}{N}) \cdot \frac{1}{2\pi} = \frac{p}{N}$ = Rational Number
as $p$ is integer.

$$\therefore x[n] \cdot y[n] = \sum_{k=0}^{N-1} \sum_{p=0}^{N-1} a_k b_{p-k} e^{jp(\frac{2\pi}{N})n}$$

$$= \sum_{p=0}^{N-1} \left\{ \sum_{k=0}^{N-1} a_k b_{p-k} \right\} e^{jp(\frac{2\pi}{N})n}$$

$$= \sum_{p=0}^{N-1} c_p e^{jp(\frac{2\pi}{N})n} \quad ; \quad c_p = \sum_{k=0}^{N-1} a_k b_{p-k}$$

by renaming $p$ by $k$ and $k$ by $l$

$$\therefore x[n] \cdot y[n] = \sum_{k=0}^{N-1} c_k e^{jk(\frac{2\pi}{N})n} \quad ; \quad c_k = \sum_{l=0}^{N-1} a_l b_{k-l}$$

$$\boxed{\therefore x[n] \cdot y[n] \xleftrightarrow{FS} c_k = \sum_{l=\langle N\rangle} a_l b_{k-l}}$$

## (3) Parseval Relation :-

From multiplication property and analysis equation

$$\therefore \quad c_k = \frac{1}{N} \sum_{k=\langle N \rangle} \{x[n] \cdot y[n]\} \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

$$\therefore N \cdot \left\{ \sum_{\ell=\langle N \rangle} a_\ell \, b_{k-\ell} \right\} = \sum_{n=\langle N \rangle} \{x[n]\, y[n]\} \, e^{-jk\left(\frac{2\pi}{N}\right)n}$$

if $k=0 \Rightarrow N \cdot \sum_{\ell=\langle N \rangle} a_\ell \, b_{-\ell} = \sum_{n=\langle N \rangle} x[n]\, y[n]$

if $y[n] = x^*[n] \xleftrightarrow{FS} b_\ell = a^*_{-\ell} \Rightarrow b_{-\ell} = a^*_\ell$

$$\therefore \quad N \cdot \sum_{\ell=\langle N \rangle} a_\ell \cdot a^*_\ell = \sum_{n=\langle N \rangle} x[n] \cdot x^*[n]$$

as $z \cdot z^* = |z|^2$

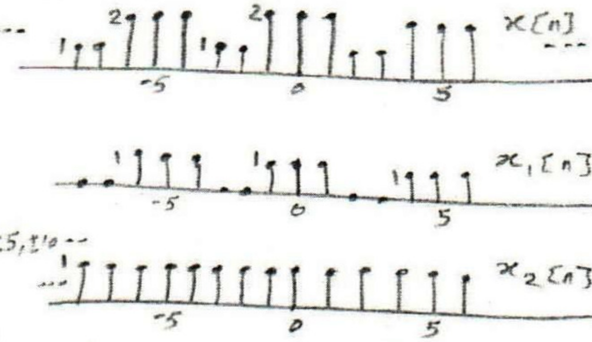
$$\therefore \quad N \sum_{\ell=\langle N \rangle} |a_\ell|^2 = \sum_{n=\langle N \rangle} |x[n]|^2$$

$$\therefore \quad \frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

= average power in one period

$\therefore$ The total average power of $x[n] = \sum_{k=\langle N \rangle} |a_k|^2$

**Ex 10** if $x[n] \Rightarrow$ is periodic with period $N=5$




$x[n] = x_1[n] + x_2[n]$

From Ex9 with $N_1 = 1$ and $N = 5$

$$\therefore x_1[n] \xleftrightarrow{FS} b_k = \begin{cases} \frac{1}{5} \cdot \dfrac{\sin\left(k\left(\frac{2\pi}{5}\right)\left(\frac{3}{2}\right)\right)}{\sin\left(k\left(\frac{\pi}{5}\right)\right)} & ; k \neq 0, \pm 5, \pm 10 \cdots \\ \frac{3}{5} & ; k = 0, \pm 5, \pm 10, \cdots \end{cases}$$

$\because x_2[n]$ is constant $x_2[n] = 1 \Rightarrow$ has only $c_0 = \frac{1}{5} \sum_{n=\langle 5 \rangle} x_2[n] \cdot 1 = \frac{1}{5} \sum_{n=0}^{4} 1 = \frac{5}{5} = 1$

$$\therefore \quad c_k = \begin{cases} 1 & ; k = 0, \pm 5, \pm 10 \cdots \\ 0 & ; k \neq 0, \pm 5, \pm 10 \cdots \end{cases}$$

as $x[n] = x_1[n] + x_2[n] \xleftrightarrow{FS} a_k = b_k + c_k$ {Linearity property}

$$\therefore \boxed{a_k = \begin{cases} \frac{1}{5} \cdot \dfrac{\sin\left(k\left(\frac{3\pi}{5}\right)\right)}{\sin\left(k\left(\frac{\pi}{5}\right)\right)} & ; k \neq 0, \pm 5, \pm 10, \cdots \\ \frac{8}{5} & ; k = 0, \pm 5, \pm 10, \cdots \end{cases}}$$

**Ex 11** Determine $x[n]$ if :-

1- $x[n]$ is periodic with period $N=6$

2- $\sum_{n=0}^{5} x[n] = 2$

3- $\sum_{n=2}^{7} (-1)^n \cdot x[n] = 1$

4 - $x[n]$ has the minimum average power among all of its possibilities

Solution

$$x[n] \xleftrightarrow{FS} a_k$$

$\boxed{\text{From Fact \#1}}$ $\quad N=6 \Rightarrow \omega_0 = \frac{2\pi}{6} = \boxed{\frac{\pi}{3}}$

$\boxed{\text{From Fact \#2}}$ $\quad$ as $a_0 = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] = \frac{1}{6} \sum_{n=0}^{5} x[n] = \frac{1}{6} \cdot (2)$

$$\Rightarrow a_0 = \frac{2}{6} = \boxed{\frac{1}{3}}$$

as $(-1)^n = \cos(\pi n) = e^{j\pi n} = e^{-j\pi n}$ as $\cos(\pi n) = \cos(-\pi n)$

$$\therefore (-1)^n = e^{-j\pi n} = e^{-j\frac{6\pi}{6}n} = e^{-j3(\frac{2\pi}{6})n}$$

$\therefore$ $\boxed{\text{From Fact \#3}}$ $\quad \sum_{n=2}^{7} (-1)^n x[n] = \sum_{n=\langle 6\rangle} x[n] e^{-j3(\frac{2\pi}{6})n} = 1$

$$\because a_3 = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{j3(\frac{2\pi}{N})n}$$

$$\Rightarrow 6a_3 = \sum_{n=\langle 6\rangle} x[n] e^{j3(\frac{2\pi}{6})n} \Rightarrow 6a_3 = 1 \Rightarrow \boxed{a_3 = \frac{1}{6}}$$

as $N=6 \Rightarrow x[n]$ has $a_0, a_1, a_2, a_3, a_4, a_5$

as the average power $\boxed{\text{from parseval's relation}}$

$$\text{is } \frac{1}{N} \sum_{n=\langle N\rangle} x[n] = \sum_{K=\langle N\rangle} |a_K|^2$$

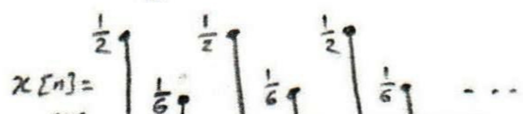
$\boxed{\text{From Fact \#4}}$ as $x[n]$ should to has minimum power then the remaining coefficients should be = zero

$$\therefore a_0 = \frac{1}{3},\ a_1 = 0,\ a_2 = 0,\ a_3 = \frac{1}{6},\ a_4 = 0,\ a_5 = 0$$

$$\Rightarrow x[n] = \sum_{K=\langle N\rangle} a_K e^{jK(\frac{2\pi}{N})n} = a_0 + 0 + 0 + \frac{1}{6} e^{j3(\frac{2\pi}{6})n} + 0 + 0$$

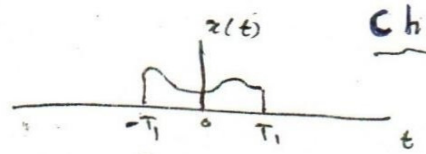
$$\therefore x[n] = \frac{1}{3} + \frac{1}{6} e^{j\pi n}$$

$$\therefore \boxed{x[n] = \frac{1}{3} + \frac{1}{6}\cos(\pi n)} \Rightarrow$$

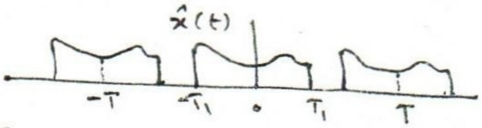


# Chapter #4

let $x(t) \Rightarrow$




and $\hat{x}(t)$ as in figure $\Rightarrow$




$\equiv$ Periodic with Period $T$

using Fourier series $\therefore$ $\hat{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

$$\therefore a_k = \frac{1}{T} \int_T \hat{x}(t) e^{-jk\omega_0 t} dt$$

let the integration be carried over the interval $-\frac{T}{2} < t < \frac{T}{2}$

$$\therefore a_k = \frac{1}{T} \int_{-T/2}^{T/2} \hat{x}(t) e^{-jk\omega_0 t} dt$$

$$\therefore a_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-jk\omega_0 t} dt$$

as $x(t) = \hat{x}(t)$ over $-\frac{T}{2} < t < \frac{T}{2}$

$$\therefore a_k = \frac{1}{T} \int_{-\infty}^{+\infty} x(t) e^{-jk\omega_0 t} dt$$

as $x(t) = 0$ outside the interval $-\frac{T}{2} < t < \frac{T}{2}$

let $$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt}$$

$$\therefore a_k = \frac{1}{T} X(jk\omega_0)$$

$$\therefore \hat{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t}$$

as $\omega_0 = \frac{2\pi}{T} \Rightarrow \frac{\omega_0}{2\pi} = \frac{1}{T}$

$$\therefore \hat{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0$$

as $T \to \infty$ then $\hat{x}(t) \to x(t)$

$k\omega_0 \to 0$ or $\omega_0 \to d\omega$

$\Sigma \to \int$

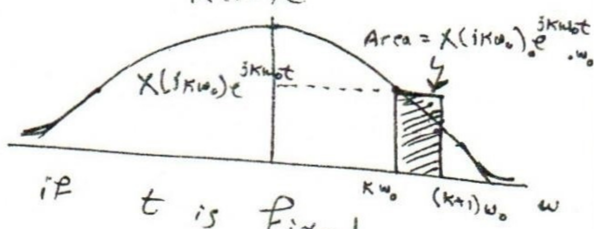
then as $T \to \infty$ $$\boxed{x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega}$$

synthesis eqn $\rightsquigarrow$ inverse Fourier Transform

and $$\boxed{X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt}$$

$\rightsquigarrow$ Fourier Transform or Analysis eqn

$X(j\omega)e^{j\omega t}$




if $t$ is fixed

as $\omega_0 \to 0$ the $\Sigma \to \int$

$\omega_0 \to d\omega$

$$a_k = \frac{1}{T} X(j\omega) \Big|_{\omega = k\omega_0}$$

= coefficients of the Periodic signal $\hat{x}(t)$

$X(j\omega) \equiv$ Fourier Transform of $x(t)$

$\equiv$ Fourier Transform of one Period of $\hat{x}(t)$

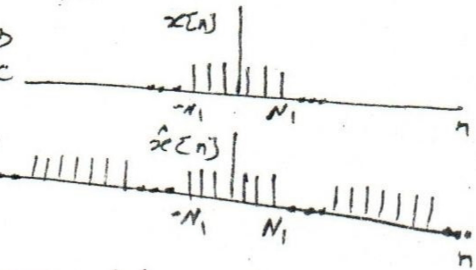
then we can say

The Fourier coefficients of any Periodic signal equals to samples of Fourier Transform of one Period of itself.

# Chapter #5

let $x[n] \Rightarrow$ Aperiodic

and $\hat{x}[n] \Rightarrow$ periodic




Fourier series representation of $\hat{x}[n]$ is

$$\hat{x}[n] = \sum_{K=\langle N \rangle} a_K \, e^{jK(\frac{2\pi}{N})n}$$

and

$$a_K = \frac{1}{N} \sum_{n=\langle N \rangle} \hat{x}[n] \, e^{-jK(\frac{2\pi}{N})n}$$

as $\hat{x}[n] = \hat{x}[n]$ over a period that includes the interval $-N_1 \leq n \leq N_1$ then

$$a_K = \frac{1}{N} \sum_{n=-N_1}^{N_1} x[n] \, e^{-jK(\frac{2\pi}{N})n}$$

as $x[n] = 0$ outside the interval $N_1 \leq n \leq N_1$

then

$$a_K = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] \, e^{-jK(\frac{2\pi}{N})n}$$

if

$$\boxed{X(e^{jw}) = \sum_{n=-\infty}^{+\infty} x[n] \, e^{-jwn}}$$

then

$$a_K = \frac{1}{N} X(e^{jKw_0}) \qquad ; \; w_0 = \frac{2\pi}{N}$$

then

$$\hat{x}[n] = \sum_{K=\langle N \rangle} \frac{1}{N} X(e^{jKw_0}) \, e^{jKw_0 n}$$

as $w_0 = \frac{2\pi}{N} \Rightarrow \frac{w_0}{2\pi} = \frac{1}{N}$

then

$$\hat{x}[n] = \sum_{K=\langle N \rangle} \frac{1}{2\pi} X(e^{jKw_0}) \, e^{jKw_0 n} \cdot w_0$$

as $N$ increases $w_0$ decreases as $N \propto \frac{1}{w_0}$

as $N \to \infty \Rightarrow w_0 \to 0$ or equivalently $w_0 \to dw$

then as $N \to \infty$ :

$\hat{x}[n] \to x[n]$

$w_0 \to dw$

$\sum \to \int$

$K = \langle N \rangle \to w = \langle 2\pi \rangle$

as $(K w_0) = \frac{2\pi}{N} \cdot K$

$$X(e^{j(w+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n] \, e^{-j(w+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] \, e^{-jwn} \cdot \underbrace{e^{-j2\pi n}}_{=1}$$

$$\therefore X(e^{j(w+2\pi)}) = X(e^{jw})$$

$\equiv$ Periodic every $2\pi$

also $e^{j(w+2\pi)n} = e^{jwn} \cdot \underbrace{e^{j2\pi n}}_{=1}$

$$\therefore e^{j(w+2\pi)n} = e^{jwn}$$

$\equiv$ Periodic every $2\pi$

then as $N \to \infty$

$$\boxed{x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{jw}) \, e^{jwn} \, dw}$$

$\hookrightarrow$ Synthesis eqn of DTFT

Analysis eqn of DTFT

**we can say that :**

① $a_K$ which are Fourier series coefficients of periodic signal $\hat{x}[n]$ are equally spaced samples of the Fourier Transform of aperiodic signal $x[n]$

as $a_K = \frac{1}{N} X(e^{jKw_0})$

② Fourier Transform of aperiodic signal $X(e^{jw})$ is periodic in $w$ with period $= 2\pi$

③ $x[n]$ can be reconstructed only from one period of $X(e^{jw})$ as the integration carried out over one interval of $2\pi$

④ low frequencies lies at $0, 2\pi, 4\pi, \ldots$
High frequencies lies at $-\pi, \pi, 3\pi, 5\pi, \ldots$

slides #4 From chapter #1

⇒ Consider the signal $e^{j\omega n}$

$$e^{j(\omega+2\pi)n} = e^{j\omega n}\cdot \underbrace{e^{j2\pi n}}_{=1}$$

$$\therefore\ e^{j(\omega+2\pi)n} = e^{j\omega n}$$

let $\phi_k[n] = e^{jk\omega_0 n} = e^{jk(\frac{2\pi}{N})n}$

then $\phi_k[n] = \phi_{k+N}[n]$

as $\phi_{k+N}[n] = e^{j(k+N)(\frac{2\pi}{N})n} = e^{jk\frac{2\pi}{N}n}\,\underbrace{e^{j2\pi n}}_{=1}$

$$\therefore\ \phi_{k+N}[n] = \phi_k[n]$$

Then there are only $N$ different $\phi_k[n]$

then $\boxed{x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n}}$

this is Fourier's claim

Multiply both sides by $e^{-jr\omega_0 n} = e^{-jr(\frac{2\pi}{N})n}$

$$\therefore\ x[n]e^{-jr(\frac{2\pi}{N})n} = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} e^{-jr\omega_0 n}$$

$$= \sum_{k=\langle N\rangle} a_k e^{j(k-r)(\frac{2\pi}{N})n}$$

Summing over $N$ terms we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(\frac{2\pi}{N})n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(\frac{2\pi}{N})n}$$

interchanging the two summations in RHS

$$\therefore\ \sum_{n=\langle N\rangle} x[n]e^{-jr(\frac{2\pi}{N})n} = \sum_{k=\langle N\rangle}\sum_{n=\langle N\rangle} a_k e^{j(k-r)(\frac{2\pi}{N})n}$$

$$= \sum_{k=\langle N\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(\frac{2\pi}{N})n}$$

$$\sum_{n=\langle N\rangle} e^{jk(\frac{2\pi}{N})n} = \begin{cases} N & k=0,\pm N,\pm 2N,\ldots \\ 0 & o.w.\end{cases}$$

as $\sum_{k=a}^{b} r^k = r^a\cdot\frac{1-r^{b-a+1}}{1-r}$

$$\sum_{n=0}^{N-1} e^{jk(\frac{2\pi}{N})n} = \frac{1-e^{jk(\frac{2\pi}{N})(\cancel{N-1-0+1})\,N}}{1-e^{jk(\frac{2\pi}{N})}}$$

$$= \frac{1-e^{jk2\pi}}{1-e^{jk(\frac{2\pi}{N})}} = \frac{1-1}{1-e^{jk(\frac{2\pi}{N})}} = 0$$

if $k=Np$ $\sum_{n=0}^{N-1} e^{j2\pi np} = N$ ; $p$ integer

if we choose another period for example from $N_1$ to $N-1+N_1$

$$\sum_{n=N_1}^{N-1+N_1} e^{jk(\frac{2\pi}{N})n}$$

let $m = n - N_1$

then $\sum_{m=0}^{N-1} e^{jk(\frac{2\pi}{N})m}\cdot e^{+jk(\frac{2\pi}{N})N_1}$

$$= e^{+jk(\frac{2\pi}{N})N_1}\sum_{n=0}^{N-1} e^{jk(\frac{2\pi}{N})n}$$

if $k=pN \Rightarrow \sum_{n=N_1}^{N-1+N_1} e^{jN(\frac{2\pi}{N})np} = N$ $p$ is integer, $n$ is integer

as $\sum_{n=N_1}^{N-1+N_1} 1 = N$

if $k\neq pN \Rightarrow \sum_{n=N_1}^{N-1+N_1} e^{jk(\frac{2\pi}{N})n} = e^{jk\frac{2\pi}{N}N_1}\cdot 0$

$$= 0$$

then

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(\frac{2\pi}{N})n} = \sum_{k=\langle N\rangle} a_k \begin{cases} N & k-r=pN,\ p \text{ integer} \\ 0 & o.w.\end{cases}$$

$$= Na_r = Na_{r+pN}$$

then $a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jr(\frac{2\pi}{N})n}$

or $\boxed{a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(\frac{2\pi}{N})n}}$

$$a_k = a_{k+pN}$$

i.e. the Fourier series coefficients of Discrete-time periodic signal is also periodic