Convolution Sum Formula Derivation

Another View:

n=-2	n=-1	n=0	n=1	...	n=k
					$In\ general:$
$x[-2]$	$x[-1]$	$x[0]$	$x[1]$		$x[k]$
$= x[n].\delta[n+2]$	$= x[n].\delta[n+1]$	$= x[n].\delta[n]$	$= x[n].\delta[n-1]$	...	$= x[n].\delta[n-k]$
$= x[-2].\delta[n+2]$	$= x[-1].\delta[n+1]$	$= x[0].\delta[n]$	$= x[1].\delta[n-1]$		$= x[k].\delta[n-k]$
$\downarrow S$	$\downarrow S$	$\downarrow S$	$\downarrow S$		$\downarrow S$
$x[-2].h_{-2}[n]$	$x[-1].h_{-1}[n]$	$x[0].h[n]$	$x[1].h_1[n]$	...	$x[k].h_k[n]$
$= y[-2]$	$= y[-1]$	$= y[0]$	$= y[1]$		$= y[k]$

$$\therefore\ x[n] = sum\ of\ all\ samples$$

$$\therefore\ x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \xrightarrow{S}$$

$$\therefore\ y[n] = sum\ \ of\ \ all\ \ responses$$

$$\therefore\ y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] = \boxed{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

$$as\ S\ is\ LTI\ so\ h_k[n] = h[n-k]$$

As the system is LTI ➔ *$h_k[n] = h[n-k]$* ➔ to characterize/analyze the system *S* we need only:

to know h[n]

Convolution Sum Formula Derivation

$$\therefore x[n] = \sum_{k=-\infty}^{\infty} x[k].\delta[n-k]$$

$$\xrightarrow{S} y[n] = \sum_{k=-\infty}^{\infty} x[k].h[n-k]$$

The independent variable is renamed to *k*

There is a time-reverse

The shift value is the value of *n*

For each shift value *n* there are : multiplications + sum of multiplications

➢ Convolution Sum Computation Algorithm for short finite-domain signals

To compute the convolution sum of x[n] and h[n]:

1- Let the two signals as functions in the independent variable *k* instead of *n*.
So, x[k] instead of x[n]
and h[k] instead of h[n] (just renaming the independent variable will not make any difference)

2- Choose one of the two signals and *time-reverse* it to be either x[-k] or h[-k].

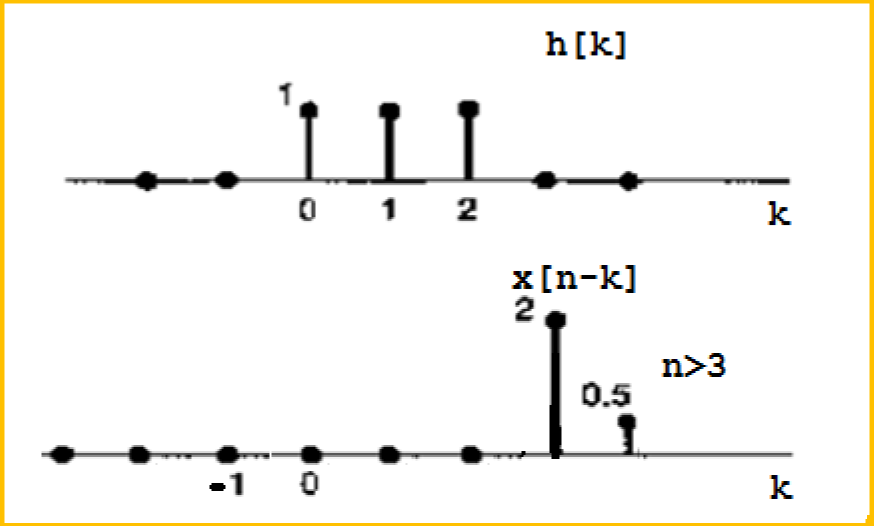
3- Determine the start of the *area of overlapping* ,in terms of shift value (n), between the resulted two signals. (area of overlapping means that this domain that both of the signals have non-zero values at the same time)

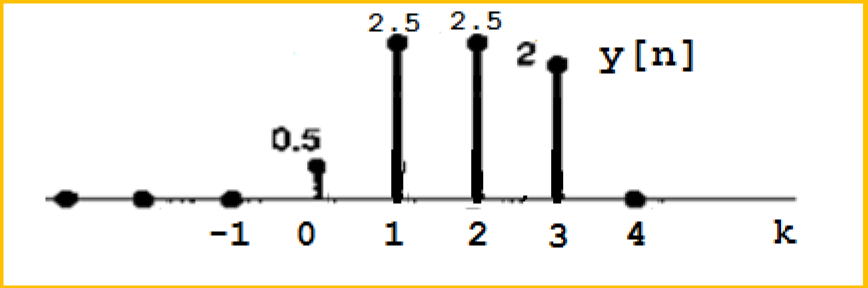
4- Compute the *boundaries* of the overlapping area in terms of shift value (n).
(the first and last point of overlapping)

5- for each shift value of the overlapping area (computed in 4) **compute the output at that time shift by multiplying each point with its corresponding point in the other signal and sum up ALL these multiplications and the result will be the value of the output at that time-shift .**

➢ Convolution Sum : Examples

[1]- Compute the convolution sum of the following input x[n] and system impulse response h[n]:





The answer following the algorithm:

1- let the two signals as functions in k instead of n.

2- let we time-reverse either one of the two signals. Let us choose x[k].

3- if n < 0. there is no overlapping between h[k] and x[n-k] → y[n]=0, n<0

4- the overlapping will start from n=0. and the length of overlapping will be
$(N_x + N_h) - 1 = 2 + 3 - 1 = 4$ points
Starting from n=0 then 1,2,and n=3

5- at n=0 → y[0]= 0x2 + 1x0.5 + 1x0 + 1x0 = 0.5
at n=1 → y[1]= 1x2 + 1x0.5 + 1x0 = 2.5
at n=2 → y[2]= 1x0 + 1x2 + 1x0.5 = 2.5
at n=3 → y[3]= 1x0 + 1x0 + 1x2 + 0x0.5 = 2
for n>3 → there is no overlapping between h[k] and x[n-k] → y[n]= 0; n>3

➢ Convolution Sum Computation Algorithm for lengthy/infinite-domain signals

To compute the convolution sum of x[n] and h[n]:

1- Let the two signals as functions in the independent variable k instead of n.
So, x[k] instead of x[n]
and h[k] instead of h[n] (just renaming the independent variable will not make any difference)

2- Choose one of the two signals and *time-reverse* it to be either x[-k] or h[-k].

3- Determine the *areas of overlapping* that have similar conditions of mathematical treatment, as the shift (n) will slide the time-reversed signal starting from (- ∞) towards (+∞). (area of overlapping means that this domain that both of the signals have non-zero values at the same time) (similar conditions of mathematical treatment means that there is one mathematical formula to compute the sum of multiplication of the two overlapping signals)

4- Compute the *boundaries* (*U*) and (*L*) of each overlapping area. (upper and lower limit of summation)

5- Compute the *mathematical formula* of each overlapping area using the formula:

$$\boxed{y[n] = \sum_{k=L}^{U} x[k]h[n-k]_{(if\ \ you\ reversed\ h)}}$$

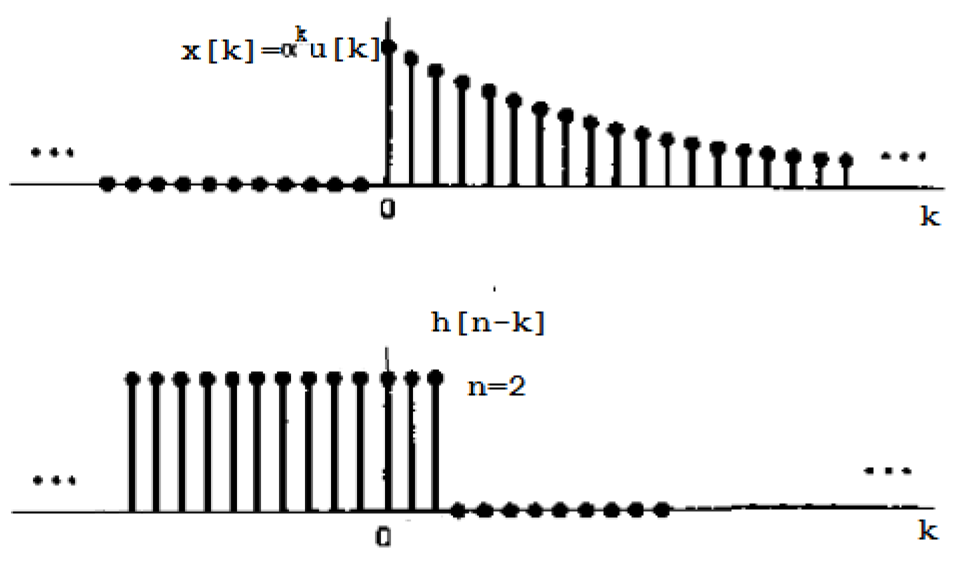
OR

$$\boxed{y[n] = \sum_{k=L}^{U} h[k]x[n-k]_{(if\ \ you\ reversed\ x)}}$$

6- Repeat steps 4 and 5 for each overlapping area.

➢ Convolution Sum : Examples

[2]- Compute the convolution sum of the following input x[n] and system impulse response h[n], with 0<α<1: $x[n]=\alpha^n u[n]$, $h[n]=u[n]$



The answer following the algorithm:

1- let the two signals as functions in k instead of n.

2- let we time-reverse either one of the two signals. Let us choose h[k].

3- if n < 0. there is no overlapping between x[k] and h[n-k] → y[n]=0, n<0

4- the overlapping will start from n=0. and slightly and gradually the signal h[n-k] will slides under x[k] as n becomes larger and larger.
at n=0 → there is overlapping from (0) to (0)
at n=1 → there is overlapping from (0) to (1)
at n=2 → there is overlapping from (0) to (2)
And so on … then the boundaries of overlapping are: L=0 (as the lower limit is fixed at 0)
U=n (as the upper limit is equal to n)

5- for n ≥ 0: $$y[n]=\sum_{k=L}^{U} x[k]h[n-k]$$

$$\therefore\ y[n]=\sum_{k=0}^{n}\alpha^k u[k]u[n-k]$$

$$\therefore\ y[n]=\sum_{k=0}^{n}\alpha^k$$

$$\therefore\ y[n]=\frac{1-\alpha^{n+1}}{1-\alpha};n\geq 0$$

$$Recall \Rightarrow \sum_{a}^{b} r^k = r^a\frac{(1-r^{b-a+1})}{1-r}, r\neq 1$$

➢ Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n] = \begin{cases} 1 & ; \ 0 \le n \le 4 \\ 0 & ; otherwise \end{cases}$ $h[n] = \begin{cases} \alpha^n & ; \ 0 \le n \le 6 \\ 0 & ; otherwise \end{cases} ; \alpha > 1$

The answer following the algorithm:

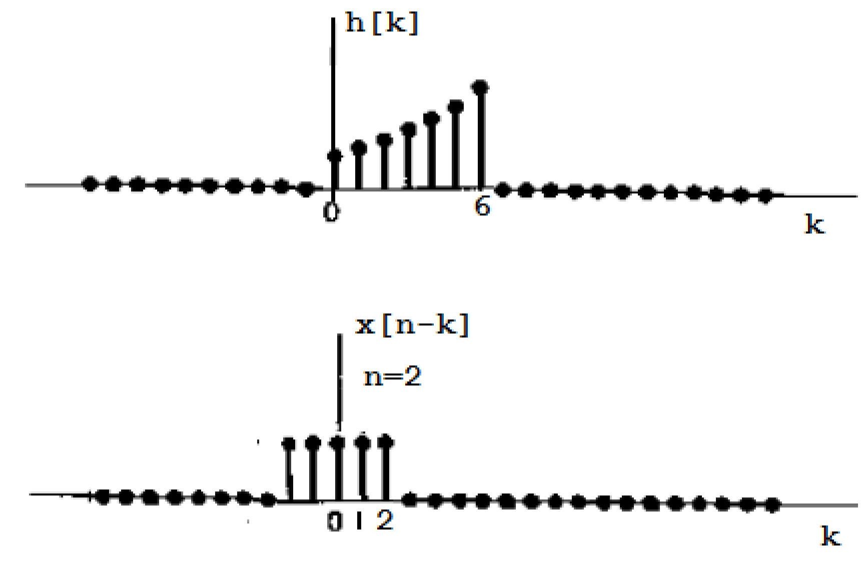
1- let the two signals as functions in k instead of n.

2- let we time-reverse either one of the two signals. Let us choose x[k].

3- if n < 0. there is no overlapping between h[k] and x[n-k] → y[n]=0, n<0

4- the overlapping will start from n=0. and slightly and gradually the signal x[n-k] will slides under h[k] as 0 ≤ n ≤ 3 . (partial overlapping)
at n=0 → there is overlapping from (0) to (0)
at n=1 → there is overlapping from (0) to (1)
at n=2 → there is overlapping from (0) to (2)
And so on … then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)
U=n (as the upper limit is equal to n)

5- for 0 ≤ n ≤3 : $y[n] = \sum_{k=L}^{U} h[k]x[n-k]$



$$y[n] = \sum_{k=0}^{n} \alpha^k .1 = \sum_{k=0}^{n} \alpha^k$$

$$\therefore \ y[n] = \frac{1-\alpha^{n+1}}{1-\alpha} \quad ; \quad 0 \le n \le 3$$

6- as n=4 there is total overlapping:
So, repeat steps 4 and 5 for this region too.

See next page

➢ Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n]=\begin{cases}1 & ; \ 0\le n\le 4\\ 0 & ; otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n & ; \ 0\le n\le 6\\ 0 & ; otherwise\end{cases}; \alpha>1$

(continued)

4'- the total overlapping will start from n=4 as slightly and gradually the signal x[n-k] will slides under h[k] as 4 ≤ n ≤ 6 . (total overlapping)

at n=4 → there is overlapping from (0) to (4)

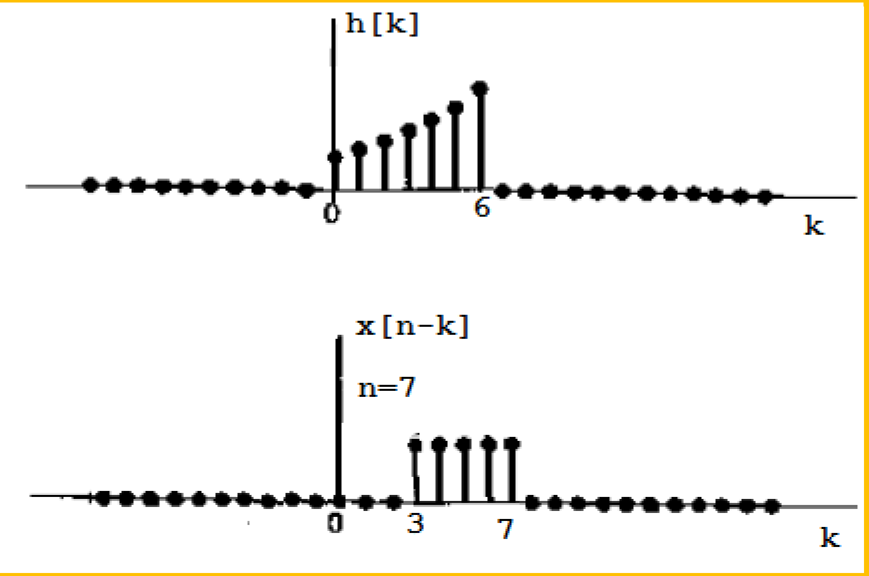
at n=5 → there is overlapping from (1) to (5)

at n=6 → there is overlapping from (2) to (6)

then the overlapping boundaries of this area are:

L=n-4 (as the lower limit is less n by 4)

U=n (as the upper limit is equal to n)



5'- for 4 ≤ n ≤ 6 : $y[n]=\sum_{k=L}^{U} h[k]x[n-k]$

$$y[n]=\sum_{k=n-4}^{n}\alpha^k .1=\sum_{k=n-4}^{n}\alpha^k$$

$$\therefore \ y[n]=\frac{\alpha^{n-4}(1-\alpha^5)}{1-\alpha}$$

$$\therefore \ y[n]=\frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} \quad ; \quad 4\le n\le 6$$

6'- as n>6 the signal x[n-k] will slide out of h[k] (partial overlapping again):
So, repeat steps 4 and 5 for this region too.

See next page

➢ Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n]=\begin{cases}1 & ; \ 0\le n\le 4\\ 0 & ; otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n & ; \ 0\le n\le 6\\ 0 & ; otherwise\end{cases} ; \alpha>1$

(continued)

4”- The second partial overlapping will start from n=7 as slightly and gradually the signal x[n-k] will slides out of h[k] as 7 ≤ n ≤ 10 . (partial overlapping)

at n=7 → there is overlapping from (3) to (6)

at n=8 → there is overlapping from (4) to (6)

at n=9 → there is overlapping from (5) to (6)

then the overlapping boundaries of this area are:

L=n-4 (as the lower limit is less n by 4)

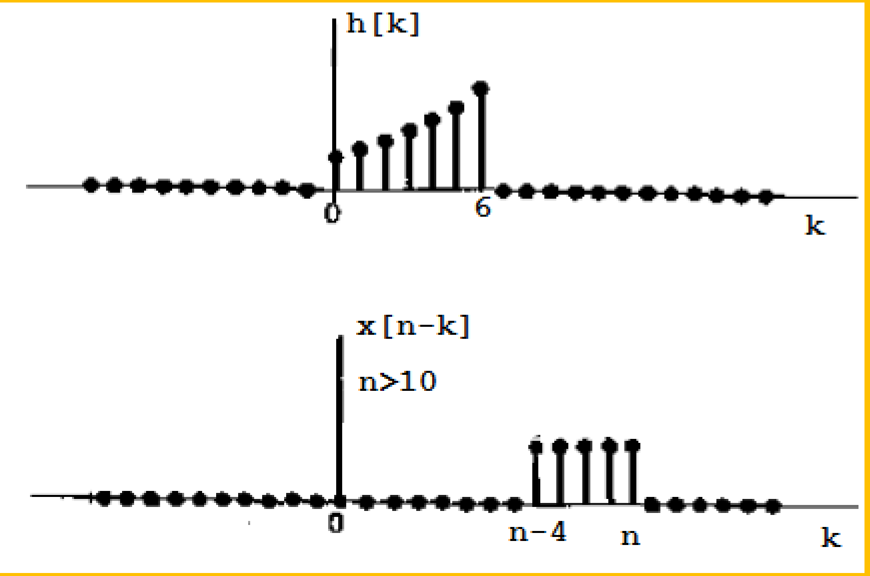
U=6 (as the upper limit is fixed to 6)

5”- for 7 ≤ n ≤ 10 : $y[n]=\sum_{k=L}^{U} h[k]x[n-k]$

$$y[n]=\sum_{k=n-4}^{6}\alpha^k .1=\sum_{k=n-4}^{6}\alpha^k$$

$$\therefore\ y[n]=\frac{\alpha^{n-4}(1-\alpha^{6-n+4+1})}{1-\alpha}=\frac{\alpha^{n-4}(1-\alpha^{11-n})}{1-\alpha}$$

$$\therefore\ y[n]=\frac{\alpha^{n-4}-\alpha^7}{1-\alpha} \quad ; \quad 7\le n\le 10$$



6”- as n>10 there is no overlapping between the signal x[n-k] and h[k] → y[n]=0; n>10.

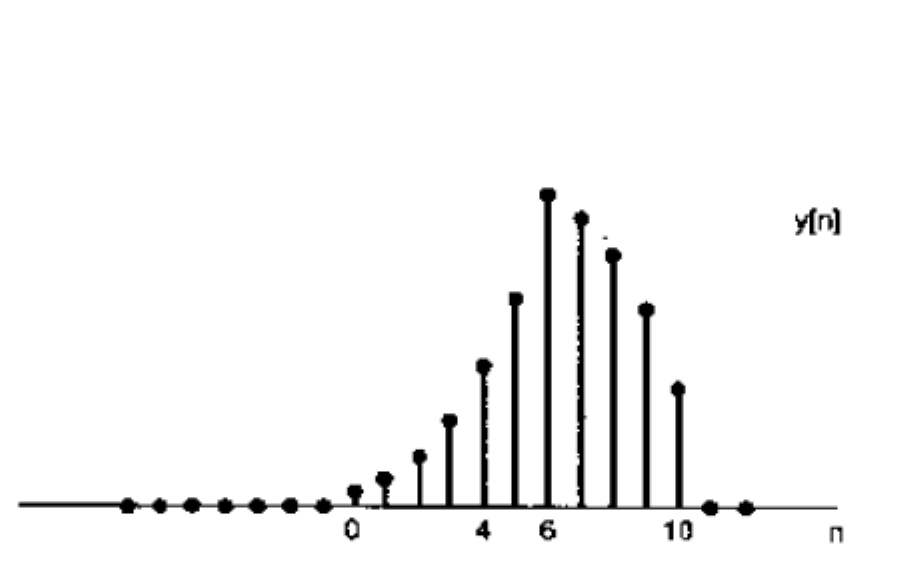
➢ Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n]=\begin{cases}1 & ; 0\le n\le 4\\ 0 & ; otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n & ; 0\le n\le 6\\ 0 & ; otherwise\end{cases};\alpha>1$

(continued)

Then collecting results of all areas gives us:

$$\therefore\ y[n] = \begin{cases} 0 & ; \quad n<0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & ; \quad 0\le n\le 3 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} & ; \quad 4\le n\le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha} & ; \quad 7\le n\le 10 \\ 0 & ; \quad n>10 \end{cases}$$



➢ Convolution Sum : Examples

[4]- Have fun with this applet in this web page { http://www.jhu.edu/~signals/discreteconv2/index.html }

Please wait the Web page to be downloaded

You should be connected to the INTERNET and configured the liveWeb Add in

Visit this site for help https://docs.google.com/a/fci-cu.edu.eg/document/d/1tOrwbl71BR42ZV561i1DKcxs0-tMETaXtj5JzikBZpU/edit

Also if the applet not working visit: htwww.java.com/en/download/help/java_blocked.xtps://ml

Please wait the Web page to be downloaded
http://www.jhu.edu/~signals/discreteconv2/index.html

You should be connected to the INTERNET
and
configured the liveWeb Add in

Signals and Systems

Lectures # 10 & # 11

Continuous-time LTI Systems

(Convolution Integral)

Prepared by:

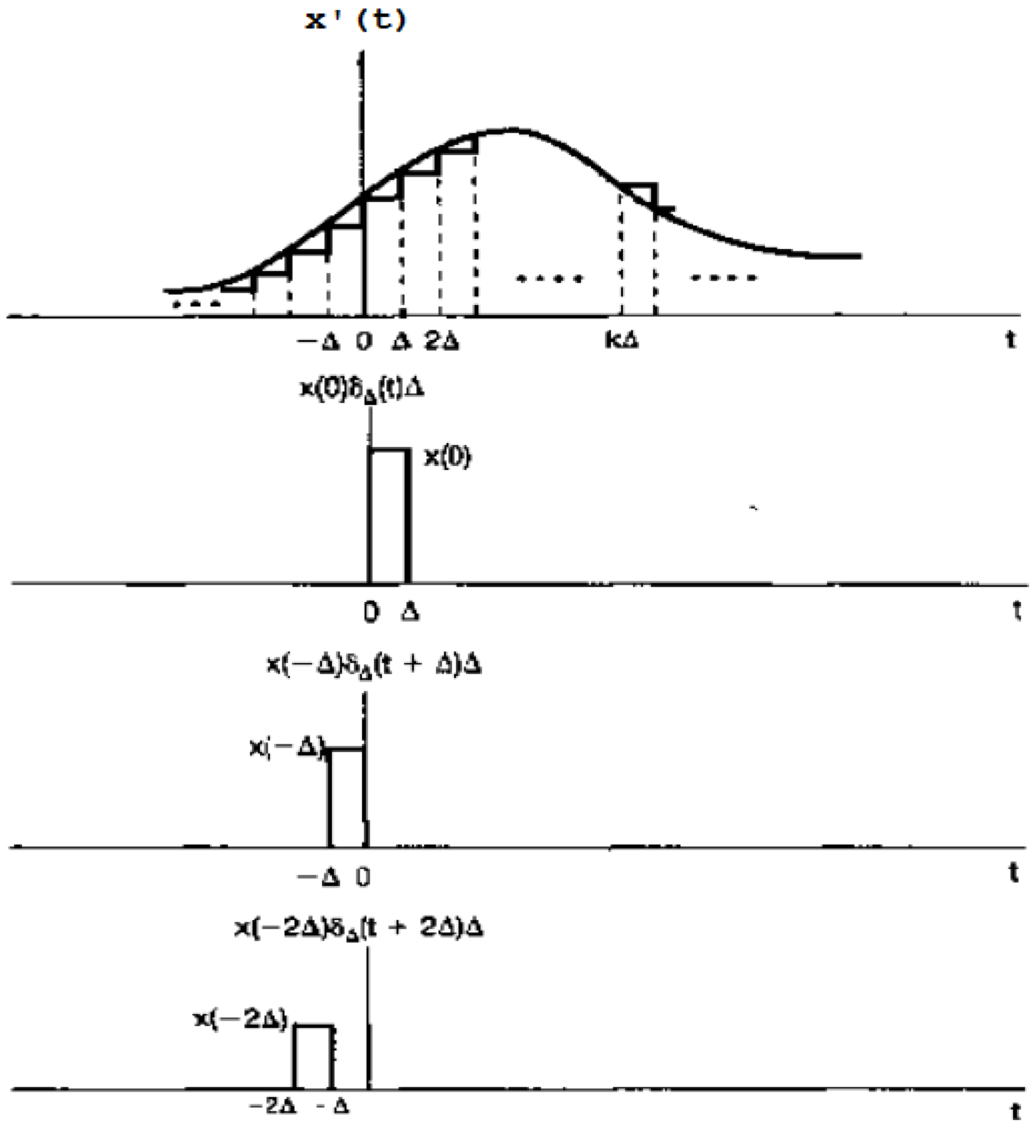
Dr. Mohammed Refaey

Topics of the lecture:

- **Convolution Integral Formula Derivation**
- **Convolution Integral Computation Algorithm**
- **Examples.**

➢ Convolution Integral Formula Derivation for LTI Systems

Then the pulse approximation x'(t) of x(t) will be as in figure →



$$Recall: \quad \delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$then \quad \delta_\Delta(t).\Delta = \begin{cases} 1 & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\Rightarrow \quad x'(t).\delta_\Delta(t).\Delta = \begin{cases} x'(0) & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t).\Delta \; = \; x'(0).\delta_\Delta(t).\Delta$$

$$\Rightarrow \quad x'(t).\delta_\Delta(t+\Delta).\Delta = \begin{cases} x'(-\Delta) & ; \quad -\Delta < t < 0 \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t+\Delta).\Delta \; = \; x'(-\Delta).\delta_\Delta(t+\Delta).\Delta$$

$$\Rightarrow \quad x'(t).\delta_\Delta(t+2\Delta).\Delta = \begin{cases} x'(-2\Delta) & ; \quad -2\Delta < t < -\Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t+2\Delta).\Delta \; = \; x'(-2\Delta).\delta_\Delta(t+2\Delta).\Delta$$

→And so on, then:

$$\therefore x'(t) = sum\ of\ all\ pulses = \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

➢ Convolution Integral Formula Derivation for LTI Systems

$$x'(t) = sum \quad of \quad all \quad pulses$$

$$= \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\because \quad x'(t) \xrightarrow{S} y'(t)$$

$$let \quad \delta_\Delta(t) \xrightarrow{S} h'(t)$$

$$as \quad S \quad LTI \quad system:$$

$$\therefore \delta_\Delta(t-k\Delta) \xrightarrow{S} h'(t-k\Delta) \; (LTI \equiv same \;\; shift)$$

$$\therefore x'(t) = \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\xrightarrow{S} y'(t) = \sum_{k=-\infty}^{\infty} x'(k\Delta)h'(t-k\Delta)\Delta$$

$$\because \quad x(t) = \lim_{\Delta\to 0} \quad x'(t)$$

$$\therefore \quad y(t) = \lim_{\Delta\to 0} \quad y'(t)$$

$$and \quad h(t) = \lim_{\Delta\to 0} \quad h'(t)$$

$$then \quad as \quad \Delta \to 0$$

$$x(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

$$\xrightarrow{S} y(t) = \lim_{\Delta\to 0} \sum_{k=-\infty}^{\infty} x'(k\Delta)h'(t-k\Delta)\Delta$$

$as\ \Delta \to 0$ *the summation tends to be integration*

$$\sum_{-\infty}^{\infty} \quad \to \quad \int_{-\infty}^{\infty}$$

$$x'(k\Delta) \quad \to \quad x(k\Delta)$$

$$h'(t-k\Delta) \to h(t-k\Delta)$$

$$let \quad k.\Delta = \tau \qquad \therefore \Delta.dk = d\tau$$

as dk *is the step of summation* $\to$ *then* $dk = 1$

$$\therefore \Delta.1 = d\tau \qquad \to \qquad \Delta = d\tau$$

$$\boxed{\therefore \; y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

called Convolution Integral Formula

➢ Convolution Integral Formula Derivation for LTI Systems

$$\therefore\ y(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau)\,d\tau$$

The independent variable is renamed to ***τ***

There is a time-reverse

The shift value is the value of ***t***

For each shift value ***t*** there are : multiplications + integral of multiplications

➢ Convolution Integral Computation Algorithm for LTI

To compute the convolution integral of x(t) and h(t):

1- Let the two signals as functions in the independent variable τ instead of t.

So,   x(τ) instead of x(t)

and   h(τ) instead of h(t)   (just renaming the independent variable will not make any difference)

2- Choose one of the two signals and *time-reverse* it to be either x(-τ) or h(-τ).

3- Determine the *areas of overlapping* that have similar conditions of mathematical treatment, as the shift (t) will slide the time-reversed signal starting from (- ∞) towards (+∞). (area of overlapping means that this domain that both of the signals have non-zero values at the same time) (similar conditions of mathematical treatment means that there is one mathematical formula to compute the integration of multiplication of the two overlapping signals)

4- Compute the *boundaries* (*U*) and (*L*) of each overlapping area. (upper and lower limit of integration)

5- Compute the *mathematical formula* of each overlapping area using the formula:

$$y(t) = \int_{L}^{U} x(\tau)h(t-\tau)d\tau \;{\scriptstyle (if\ \ you\ reversed\ h)}$$

OR

$$y(t) = \int_{L}^{U} h(\tau)x(t-\tau)d\tau \;{\scriptstyle (if\ \ you\ reversed\ x)}$$

6- Repeat steps 4 and 5 for each overlapping area.