Fourier Series

- Fourier series and Fourier Transform provide us a set of important tools for signals and systems analysis.
- The main idea is to represent the signals as linear combination of basic signals.
- The basic signal has two properties:-
  1- can be used to represent broad range of signals.
  2- It's response/output is simple to compute.

- Historical Perspective:-
  - The concept of using "trigonometric sums" - that is, sums of harmonically related sines and cosines or periodic complex exponentials - to describe periodic phenomena goes back at least as far as the "Babylonians" → prediction of astronomical events
  - The modern history begins with "Euler" in 1748 with examination of the vibrating string; specifically he noted that if the vertical deflection of the string at any time instant can be represented as linear combination of Normal Modes of this string then {we can do at any subsequent time}
  - in 1753, "Bernoulli" argued on physical grounds all physical motions of string could be represented by linear combinations of normal modes, But he did not mathematically.
  - in 1759, "Lagrange" strongly criticized the use of trigonometric sum in the examination of vibrating string, more specifically signals with corner, he sayed cannot be represented.
  - Around 1800, Fourier presented his ideas and with the claim that harmonically-related sinusoids can be used to describe more and more phenomena like heat propagation, temperature distribution, and he claimed that "any" periodic signal could be represented by such series.

- In 1829 Dirichlet provide precise conditions under which periodic signal could be represented by a Fourier series. Then Fourier extend his claim to aperiodic signals.
- in 1965, The fast Fourier transform FFT was introduced.
- For LTI system, if the input $x(t) = e^{st}$, then the output $y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$

$$\therefore\ y(t) = \int_{-\infty}^{+\infty} h(\tau) . e^{st} . e^{-s\tau}\, d\tau = e^{st} \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau}\, d\tau$$

$$= e^{st} . H(s)$$ { i.e. e^{st} in input appears e^{st} in output too but multiplied by a variable gain }

then e^{st} is called eigenfunction of the LTI systems with eigenvalue is $H(s) = \int_{-\infty}^{+\infty} h(t) e^{-st}\, dt$

then if $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$

$$\therefore\ y(t) = a_1 H_1(s_1) e^{s_1 t} + a_2 H_2(s_2) e^{s_2 t} + a_3 H_3(s_3) e^{s_3 t}$$

Fourier analysis involves restricting $s = j\omega$

Fourier claim that any periodic signal $x(t)$ could be represented as $x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \longrightarrow (1)$

as the set $\phi_k(t) = e^{jk\omega_0 t}$ are harmonically related signals and they are periodic with common period T, which is the fundamental period of $x(t) \Rightarrow$ i.e. $x(t) = x(t+T)$

multiply both sieds of eqn (1) with $e^{-jn\omega_0 t}$

$$\therefore\ x(t)\, e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k . e^{jk\omega_0 t} . e^{-jn\omega_0 t} \quad , \omega_0 = \frac{2\pi}{T}$$

integrating both sides from 0 to $T = \frac{2\pi}{\omega_0}$

$$\therefore \int_0^T x(t)\, e^{-jn\omega_0 t}\, dt = \int_0^T \left\{ \sum_{k=-\infty}^{+\infty} a_k . e^{jk\omega_0 t} . e^{-jn\omega_0 t} \right\} dt$$

$$= \sum_{k=-\infty}^{+\infty} a_k \int_0^T e^{j(k-n)\omega_0 t}\, dt$$

$$\int_0^T e^{j(k-n)\omega_0 t}\, dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j \int_0^T \sin(k-n)\omega_0 t\, dt$$

For $k \neq n$, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic signals with $\omega = (k-n)\omega_0$ when comparing with general form $\cos \omega t$ or $\sin \omega t$

with fundamental period $T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{|(k-n)\omega_0|} = \frac{2\pi}{|\omega_0|} \cdot \frac{1}{|k-n|}$

$= T \cdot \frac{1}{|k-n|}$; with T is the original fundamental period of $x(t)$.

as k is integer and n is integer

$\Rightarrow (k-n)$ is integer $\Rightarrow T_0 = \frac{T}{\text{integer}}$

$\therefore T = \text{integer} \cdot T_0$

$\therefore \int_0^T$ is integrating $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are integrations over integer number of periods of both cos and sin = zeros.

$$\therefore \int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos(k-n)\omega_0 t \, dt + j \int_0^T \sin(k-n)\omega_0 t \, dt$$

$$= \begin{cases} 0 & k \neq n \\ T & k = n \end{cases}$$

as if $k = n$ $\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos(0) \, dt + j \int_0^T \sin(0) \, dt$

$$= \int_0^T 1 \cdot dt + j \cdot 0, \text{ as } \cos 0 = 1, \ \sin 0 = 0$$

$$= t \Big|_0^T = T - 0 = T$$

$$\therefore \int_0^T x(t) e^{-j\omega_0 n t} dt = \sum_{k=-\infty}^{+\infty} a_k \cdot \begin{cases} T & k = n \\ 0 & k \neq n \end{cases}$$

$$= a_n \cdot T$$

$$\therefore a_n = \frac{1}{T} \int_0^T x(t) e^{-j\omega_0 n t} dt$$

it is valid for any integration over any complete period.

$$\therefore \boxed{a_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt}$$ $\Rightarrow$ called analysis equation OR Fourier series coefficients OR spectral coefficients

if $$\boxed{x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}}$$ $\Rightarrow$ called synthesis equation OR Fourier series equation as $a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

$\Rightarrow \quad a_0 = \frac{1}{T}\int_T x(t)\, dt \qquad$ if $k = 0$

= average value of $x(t)$ over one period.

= dc component OR constant component.

a_1 and a_{-1} are called coefficients of first harmonic.

a_2 and a_{-2} " " " " second " .

$\vdots \qquad \vdots$

a_k and a_{-1} " " " " k-th " .

Ex 1:-

$$x(t) = \sin \omega_0 t$$

$$\therefore x(t) = \frac{1}{2j}\{e^{j\omega_0 t} - e^{-j\omega_0 t}\}$$

$$= \frac{1}{2j} e^{j(1)\omega_0 t} - \frac{1}{2j} e^{j(-1)\omega_0 t}$$

as $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$

$$\Rightarrow \therefore a_1 = \frac{1}{2j} \quad \text{and} \quad a_{-1} = \frac{-1}{2j}$$

and $a_k = 0 \quad ; \ k \neq +$ or -1

Ex 2:-

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$\therefore x(t) = 1 + \frac{1}{2j}\{e^{j\omega_0 t} - e^{-j\omega_0 t}\} + \cancel{2}\cdot\frac{1}{\cancel{2}}\{e^{j\omega_0 t} + e^{-j\omega_0 t}\} + \frac{1}{2}\{e^{j(2\omega_0 + \frac{\pi}{4})} + e^{-j(2\omega_0 + \frac{\pi}{4})}\}$$

$$\therefore x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j\frac{\pi}{4}}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j\frac{\pi}{4}}\right)e^{-j2\omega_0 t}$$

$\therefore \quad a_0 = 1 =$ dc component

$$a_1 = 1 + \frac{1}{2j} = 1 - \frac{1}{2}j$$

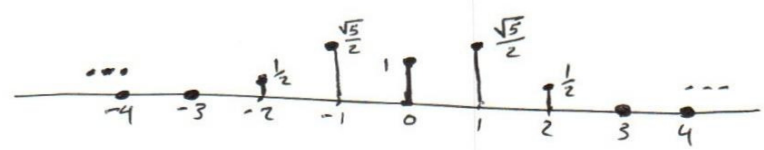
$$a_{-1} = 1 - \frac{1}{2j} = 1 + \frac{1}{2}j$$

$$a_2 = \frac{1}{2}e^{j\frac{\pi}{4}} = \frac{1}{2}\left\{\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right\} = \frac{1}{2}\left\{\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right\} = \frac{\sqrt{2}}{4}(1+j)$$

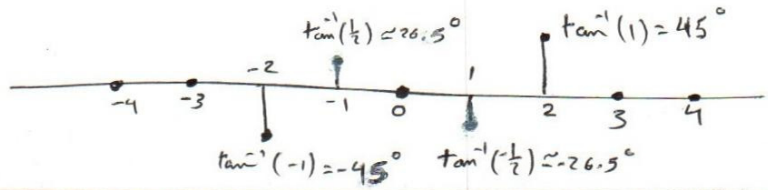
$$a_{-2} = \frac{1}{2}e^{-j\frac{\pi}{4}} = \frac{1}{2}\left\{\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right\} = \frac{1}{2}\left\{\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right\} = \frac{\sqrt{2}}{4}(1-j)$$

$$a_k = 0 \quad , \ |k| > 2$$

magnitudes $|a_k|$



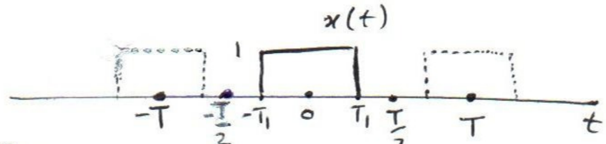
phase $\measuredangle a_k$



Ex 3

$$x(t)=\begin{cases}1 & ; \ |t|<T_1 \\ 0 & ; \ T_1<|t|<\frac{T}{2}\end{cases}$$

as T is the fundamental period of $x(t)$



$$a_0=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt$$

$$=\frac{1}{T}\int_{-T_1}^{T_1} dt=\frac{1}{T}\left\{t\Big|_{-T_1}^{T_1}\right\} \quad \text{as } x(t)=0 \ \ T_1<|t|<\frac{T}{2}$$

$$=\frac{1}{T}\{T_1-(-T_1)\}=\frac{2T_1}{T}$$

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

$$=\frac{1}{T}\int_{-T_1}^{T_1}e^{-jk\omega_0 t}dt=\frac{1}{T}\left\{\frac{e^{-jk\omega_0 T_1}}{-jk\omega_0}-\frac{e^{jk\omega_0 T_1}}{-jk\omega_0}\right\}$$

$$=\frac{1}{jk\omega_0 T}\left\{e^{jk\omega_0 T_1}-e^{-jk\omega_0 T_1}\right\}=\frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T}$$

$$\therefore a_k=\frac{\cancel{2j}\sin(k\omega_0 T_1)}{\cancel{j}k\cancel{\omega_0}\cdot\frac{\cancel{2}\pi}{\cancel{\omega_0}}}=\frac{\sin(k\omega_0 T_1)}{k\pi} \ ; \ k\neq 0$$

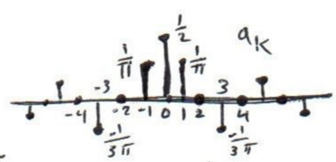
$$\Rightarrow Ta_k=\frac{2\sin(\omega T_1)}{\omega}\Big|_{\omega=k\omega_0}\equiv \text{Samples of envlope } Ta_k$$

$T=4T_1 \Rightarrow a_0=\frac{2T_1}{4T_1}=\frac{1}{2}$

$T_1=\frac{T}{4}=\frac{2\pi}{\omega_0}\cdot\frac{1}{4}=\frac{\pi}{2\omega_0}$

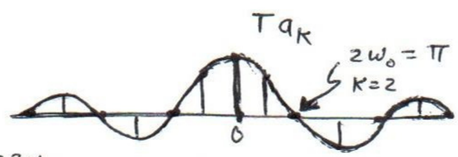
$k\omega_0 T_1=k\cancel{\omega_0}\cdot\frac{\cancel{2}\pi}{\cancel{\omega_0}}\cdot\frac{1}{\cancel{4}_2}=k\cdot\frac{\pi}{2}\Rightarrow$ every 4-points $\equiv$ one cycle $\equiv 2\pi$

$\frac{\sin(k\frac{\pi}{2})}{k\pi}$



$\Rightarrow \omega T_1=\omega\cdot\frac{\pi}{2\omega_0}$

$\therefore \omega T_1=\pi$ if $\omega=2\omega_0$



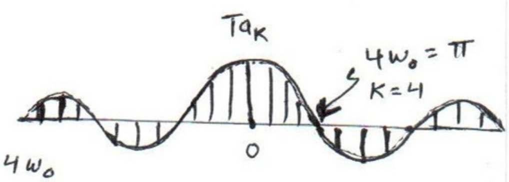
$T=8T_1 \Rightarrow a_0=\frac{2T_1}{8T_1}=\frac{1}{4}$

$T_1=\frac{T}{8}=\frac{2\pi}{\omega_0}\cdot\frac{1}{8}=\frac{\pi}{4\omega_0}$

$k\omega_0 T_1=k\cancel{\omega_0}\cdot\frac{\cancel{2}\pi}{\cancel{\omega_0}}\cdot\frac{1}{\cancel{8}_4}=k\cdot\frac{\pi}{4}\equiv$ every 8-point $\equiv$ one cycle $\equiv 2\pi$

$\Rightarrow \omega T_1=\omega\cdot\frac{\pi}{4\omega_0}$

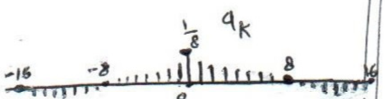
$\therefore \omega T_1=\pi$ if $\omega=4\omega_0$



$T=16T_1 \Rightarrow a_0=\frac{2T_1}{16T_1}=\frac{1}{8}$

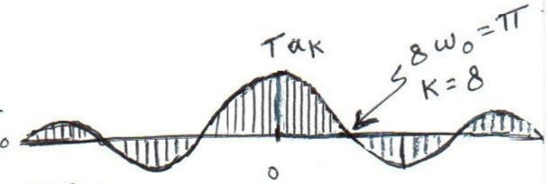
$T_1=\frac{T}{16}=\frac{2\pi}{\omega_0}\cdot\frac{1}{16}=\frac{\pi}{8\omega_0}$

$k\omega_0 T_1=k\cancel{\omega_0}\cdot\frac{\cancel{2}\pi}{\cancel{\omega_0}}\cdot\frac{1}{\cancel{16}_8}=k\cdot\frac{\pi}{8}\equiv$ every 16-point $\equiv$ one cycle $\equiv 2\pi$



$\Rightarrow \omega T_1=\omega\cdot\frac{\pi}{8\omega_0}$

$\therefore \omega T_1=\pi$ if $\omega=8\omega_0$



as T increases $\Rightarrow$ Samples becomes more closer.

- Convergence of Fourier Series

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

Recall: any complex number $z = a + jb = |z| e^{\measuredangle z}$ does NOT go to ∞ unless its magnitude $|z| = \infty$

$\therefore$ as the value $\{x(t)\, e^{-jk\omega_0 t}\}$ is a complex number

$\therefore$ it is not go to ∞ unless $|x(t)| = \infty$

$\therefore$ if $|x(t)| < \infty$, $\forall\, t$

$$\Rightarrow a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt < \infty$$

$\therefore$ ONE ~~E~~enough conditions for Fourier series to converge

is $\boxed{\int_T |x(t)|^2\, dt < \infty} \equiv \{|x(t)| < \infty \ \forall\, t\}$

i.e. $x(t)$ has a finite energy over a period.

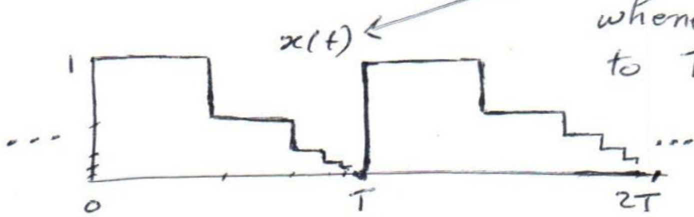
Dirichlet conditions:- are as follow:-

① $\int_T |x(t)|\, dt < \infty \equiv$ absolutely integrable over any period.

$\equiv$ Counter example $x(t) = \frac{1}{t}$; $0 < t \le 1$

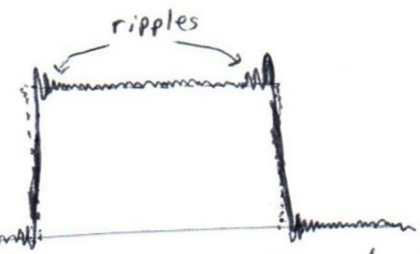
② $x(t)$ has a finite number of maxima and minima during any single period $\equiv$ Counter example $x(t) = \sin\left(\frac{2\pi}{t}\right)$, $0 < t \le 1$ with period $= 1$

③ $x(t)$ has a finite number of discontinuities over any finite interval of time $\equiv$ Counter example the value of $x(t)$ decreases by a factor of 2 whenever the distance from t to T decreases by a factor of 2



- Gibbs phenomenon :-

- An American physicist, Albert Michelson, tested the convergence of Fourier series through an experiment
- He uses the approximation formula $x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}$ in a trial to test the position of discontinuity of the square wave.
- He found that around the position of discontinuity there are ripples that do not vanish as he increases N instead the ripples are compressed but its amplitude does not decreases.



- it is called Gibbs phenomenon as the mathematical physicist "Josiah Gibbs" is who explained it mathmatically. He found the height of ripples = 0.09% of the height of discontinuity.

Properties of Continuous-time Fourier series

① Linearity :-

$$x(t) \xleftrightarrow{FS} a_k$$
$$y(t) \xleftrightarrow{FS} b_k$$

let $w(t) = m\,x(t) + n\,y(t) \xleftrightarrow{FS} C_k$

$$C_k = \frac{1}{T}\int_T w(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T \{m\,x(t) + n\,y(t)\}\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T m\,x(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int_T n\,y(t)\, e^{-jk\omega_0 t}\, dt$$

$$= m\left\{\frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt\right\} + n\left\{\frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt\right\}$$

$$\therefore \boxed{C_k = m\,a_k + n\,b_k}$$

② Time-shifting :- if $x(t) \xleftrightarrow{FS} a_k$

and $y(t) = x(t-t_0) \xleftrightarrow{FS} b_k$

$$\therefore \quad b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t}\, dt$$

let $\tau = t - t_0 \Rightarrow d\tau = dt \Rightarrow t = \tau + t_0$

$$\therefore \quad b_k = \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0(\tau - t)}\, dt$$

$$= \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0 t}\, e^{-jk\omega_0 \tau}\, d\tau$$

$$= e^{-jk\omega_0 t} \left\{ \frac{1}{T}\int_T x(\tau)\, e^{-jk\omega_0\tau}\, d\tau \right\}$$ as t is constant with respect to τ

$$\therefore \quad b_k = e^{-jk\omega_0 t} \cdot a_k$$

$$\therefore \quad \boxed{x(t-t_0) \xleftrightarrow{FS} e^{-jk\omega_0 t}\, a_k}$$

{Note magnitude does not changed just the phase that is changed}

③ Time-Reversal :- if $x(t) \xleftrightarrow{FS} a_k$

and $y(t) = x(-t) \xleftrightarrow{FS} b_k$

$$\therefore \quad b_k = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T}\int_T x(-t)\, e^{-jk\omega_0 t}\, dt$$

let $\tau = -t \Rightarrow d\tau = -dt$ and $t = -\tau$

$$\therefore \quad b_k = \frac{1}{T}\int_T x(\tau)\, e^{+jk\omega_0\tau}\, (-d\tau)$$ {with boundaries of integration is still w.r.t t}

$$= \frac{1}{T}\int_T x(\tau)\, e^{jk\omega_0\tau}\, d\tau$$ {with boundaries of integration is replaced with boundaries of τ and $\tau = -t$}

as if t is from $-\frac{T}{2}$ to $\frac{T}{2}$

τ will be from $\frac{T}{2}$ to $-\frac{T}{2}$

and by making τ to range from $-\frac{T}{2}$ to $\frac{T}{2}$

You will multiply by (-1) which will be cancelled with the $-ve$ associated with $(-d\tau)$

$$\therefore \quad \boxed{b_k = a_{-k} \xleftrightarrow{FS} x(-t)}$$

{i.e. time-reversal make time-reversal in the coefficients too.}

Another Proof if $x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \Rightarrow x(-t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{-jk\omega_0 t}$

let $m = -k \Rightarrow x(-t) = \sum_{m=+\infty}^{-\infty} a_{-m}\, e^{jm\omega_0 t} = \sum_{m=-\infty}^{+\infty} a_{-m}\, e^{jm\omega_0 t}$

$$\therefore \quad \boxed{x(-t) \xleftrightarrow{FS} b_k = a_{-k}}$$

- As a consequence of time-reversal property :-

(A) if $\boxed{x(t) \text{ is even}}$ $\Rightarrow$ $x(t) = x(-t)$

$\therefore$ $\boxed{a_K = a_{-K}}$ i.e. coefficients are even too

(B) if $\boxed{x(t) \text{ is odd}}$ $\Rightarrow$ $x(t) = -x(-t)$

$\therefore$ $\boxed{a_K = -a_{-K}}$ i.e. coefficients are odd too

OR $a_{-K} = -a_K$

(4) Time-scaling :- if $x(t) = \sum_{K=-\infty}^{+\infty} a_K e^{jK(\omega_0)t}$

then $x(\alpha t) = \sum_{K=-\infty}^{+\infty} a_K e^{jK(\alpha \omega_0)t}$

which means as the fundamental period becomes $\left(\frac{T}{\alpha}\right)$ then the fundamental frequency becomes $(\alpha \omega_0)$ but the coefficients of Fourier series Not changed

$\therefore$ $\boxed{x(\alpha t) \xleftrightarrow{FS} a_K}$

(5) Multiplication :- if $x(t) \xleftrightarrow{FS} a_K$

$y(t) \xleftrightarrow{FS} b_K$

then $x(t).y(t) \xleftrightarrow{FS} C_K$

$$x(t).y(t) = \left\{ \sum_{K=-\infty}^{+\infty} a_K e^{jK\omega_0 t} \right\} \left\{ \sum_{m=-\infty}^{+\infty} b_m e^{jm\omega_0 t} \right\}$$

$$\therefore \quad C_K = \frac{1}{T} \int_T x(t).y(t)\, e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_T \left\{ \sum_{m=-\infty}^{+\infty} a_m e^{jm\omega_0 t} \right\} \left\{ \sum_{\ell=-\infty}^{+\infty} b_\ell e^{j\ell\omega_0 t} \right\} e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \sum_{m=-\infty}^{+\infty} a_m \sum_{\ell=-\infty}^{+\infty} b_\ell \int_T e^{j(m+\ell-k)\omega_0 t}\, dt$$

remember $\int_T e^{j(m+\ell-k)\omega_0 t}\, dt = \begin{cases} T \,; & K = m+\ell \\ 0 \,; & \text{otherwise} \end{cases}$

as $e^{j\theta} = \cos\theta + j\sin\theta$

and integrating the cos or sin over one period = zero

$$\therefore \quad C_K = \sum_{m=-\infty}^{+\infty} a_m \sum_{\ell=-\infty}^{+\infty} b_\ell \cdot \frac{1}{T} \cdot \begin{cases} T & \text{if } K = m+\ell \text{ or } \ell = K-m \\ 0 & \text{otherwise} \end{cases}$$

$$\therefore \quad \boxed{C_K = \sum_{m=-\infty}^{+\infty} a_m b_{K-m}} \equiv C(n) = \sum_{K=-\infty}^{+\infty} a(K)\, b(n-K)$$

i.e. $C_K \equiv$ discrete-time convolution between a_K and b_K

⑥ Conjugation :- if $x(t) \xleftrightarrow{FS} a_k$

$$x^*(t) \xleftrightarrow{FS} b_k$$

$$\therefore \quad b_k = \frac{1}{T}\int_T x^*(t)\, e^{-jk\omega_0 t}\, dt$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \longrightarrow \text{I}$$

$$\therefore \quad x^*(t) = \left\{ \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \right\}^*$$

Recall $\{z_1 + z_2\}^* = z_1^* + z_2^*$

$$\therefore \quad x^*(t) = \sum_{k=-\infty}^{+\infty} \left\{ a_k\, e^{jk\omega_0 t} \right\}^*$$

Recall $\{z_1 z_2\}^* = z_1^* z_2^*$

$$\therefore \quad x^*(t) = \sum_{k=-\infty}^{+\infty} a_k^*\, e^{-jk\omega_0 t}$$

let $m = -k \Rightarrow x^*(t) = \sum_{m=+\infty}^{-\infty} a_{-m}^*\, e^{jm\omega_0 t}$

by comparing to equation # I $\Rightarrow$ $\boxed{b_k = a_{-k}^*}$

if $\boxed{x(t) \text{ is real}} \Rightarrow x^*(t) = x(t) \Rightarrow a_k = a_{-k}^*$ or $\boxed{a_{-k} = a_k^*}$

i.e. Conjugate symmetric and $\boxed{|a_k| = |a_{-k}|}$ and a_0 is real

if $\boxed{x(t) \text{ is real and even}}$ i.e. $a_k = a_{-k} \Rightarrow a_k = a_k^*$

i.e. $\boxed{\text{Fourier series coefficients are real and even}}$

if $\boxed{x(t) \text{ is real and odd}} \Rightarrow \boxed{\text{Fourier series coefficients are pure imaginary and odd}}$

$a_k = -a_{-k} \Rightarrow a_k^* = -a_k$ and $a_{-k}^* = -a_{-k} \Rightarrow a_k$ pure imaginary & odd ($a_0 = 0$)

⑦ Parseval's Relation :- The average in one Period of Periodic signal

$$x(t) \equiv \frac{1}{T}\int_T |x(t)|^2 dt$$

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T \left| \sum_{k=-\infty}^{+\infty} a_k\, e^{jk\omega_0 t} \right|^2 dt$$

$$= \sum_{k=-\infty}^{+\infty} |a_k|^2 \cdot \left[\frac{1}{T}\int_T |e^{jk\omega_0 t}|^2 dt \right.$$

as $|A \cdot B| = |A| \cdot |B|$

$$\therefore \quad \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2 \left[\frac{1}{T}\int_T dt \right] \quad \text{as } |e^{j\theta}|^2 = 1$$

$$= \sum_{k=-\infty}^{+\infty} |a_k|^2 \left[\frac{1}{\cancel{T}} \cdot \cancel{T} \right]$$

$$\therefore \quad \frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

as $|a_k|$ is the average power in kth harmonic

$\Rightarrow$ The total average power in a periodic signal equals the sum of the average powers in all of its harmonic components

⑧ Periodic convolution :- if $x(\tau) \xleftrightarrow{FS} a_K$
and $y(\tau) \xleftrightarrow{FS} b_K$

$$\therefore x(\tau) = \sum_m a_m e^{jm\omega_0\tau}$$

$$\text{and } y(\tau) = \sum_n b_n e^{jn\omega_0\tau}$$

$$\therefore y(t-\tau) = \sum_n b_n e^{jn\omega_0(t-\tau)} = \sum_n b_n e^{jn\omega_0 t} e^{-jn\omega_0\tau}$$

$$\therefore x(\tau)\cdot y(t-\tau) = \left\{\sum_m a_m e^{jm\omega_0\tau}\right\}\left\{\sum_n b_n e^{jn\omega_0 t} e^{-jn\omega_0\tau}\right\}$$

$$= \sum_m a_m \sum_n b_n e^{jn\omega_0 t} \cdot e^{j(m-n)\omega_0\tau}$$

$$\therefore \int_T x(\tau) y(t-\tau)\, d\tau = \int_T \sum_m a_m \sum_n b_n e^{jn\omega_0 t} e^{j(m-n)\omega_0\tau}\, d\tau$$

$$= \sum_m a_m \sum_n b_n e^{jn\omega_0 t} \int_T e^{j(m-n)\omega_0\tau}\, d\tau$$

$$= \sum_m a_m \sum_n b_n e^{jn\omega_0 t} \cdot \begin{cases} T & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

$$= T \sum_m a_m b_m e^{jm\omega_0 t}$$

$$\therefore \int_T x(\tau) y(t-\tau)\, d\tau = T\sum_m a_m b_m e^{jm\omega_0 t} \xleftrightarrow{FS} C_K$$

$$\therefore C_K = \frac{1}{T}\int_T \left\{T\sum_m a_m b_m e^{jm\omega_0 t}\right\} e^{-jK\omega_0 t}\, dt$$

$$= \frac{1}{\cancel{T}} \cdot \cancel{T} \sum_m a_m b_m \cdot \int_T e^{j(m-K)\omega_0 t}\, dt$$

$$= \sum_m a_m b_m \cdot \begin{cases} T & \text{if } m = K \\ 0 & \text{if } m \neq K \end{cases}$$

$$\therefore \boxed{C_K = T\, a_K\, b_K}$$

⑨ Differentiation property :- if $x(t) \xleftrightarrow{FS} a_K$

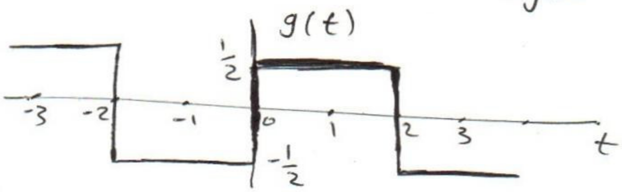
$$\therefore x(t) = \sum_K a_K e^{jK\omega_0 t}$$

$$\therefore \frac{d}{dt} x(t) = \frac{d}{dt}\left\{\sum_K a_K e^{jK\omega_0 t}\right\} = \sum_K a_K \cdot jK\omega_0 \cdot e^{jK\omega_0 t}$$

$$\Rightarrow \frac{d}{dt} x(t) = \sum_K \{jK\omega_0 a_K\} \cdot e^{jK\omega_0 t}$$

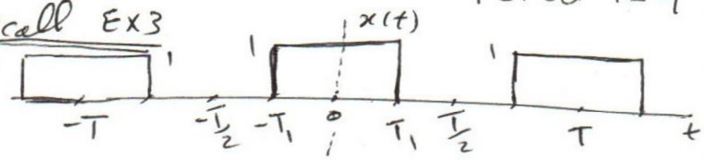
$$\therefore \boxed{\frac{d}{dt} x(t) \xleftrightarrow{FS} jK\omega_0 \cdot a_K}$$

Ex 4:- if $g(t)$ is as shown in Figure



$g(t)$ is periodic with period $T=4$

Recall Ex3



if $T=4$ and $T_1=1$

$\therefore\ g(t) = x(t-1) - \frac{1}{2}$

Recall $x(t) \xleftrightarrow{FS} a_k = \begin{cases} \frac{\sin(k\omega_0 T_1)}{k\pi} & ; k \neq 0 \\ \frac{2T_1}{T} & ; k=0 \end{cases}$

$$\therefore\ \boxed{a_k = \begin{cases} \frac{2}{4} = \frac{1}{2} & ; k=0 \\ \frac{\sin(k\omega_0)}{k\pi} & ; k \neq 0 \end{cases}}$$

using FS properties

$x(t) \xleftrightarrow{FS} a_k$

$\therefore\ x(t-1) \xleftrightarrow{FS} e^{-jk\omega_0} a_k = c_k$

$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$

$\therefore\ x(t-1) \xleftrightarrow{FS} \boxed{e^{-jk\frac{\pi}{2}} \cdot a_k = c_k}$

if $g(t) = -\frac{1}{2} \xleftrightarrow{FS} d_k$

$\therefore\ d_k = \frac{1}{4}\int_T (-\tfrac{1}{2}) e^{-jk\omega_0 t}\, dt$

$= -\frac{1}{8} \cdot \begin{cases} 4 & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$

$$\therefore\ \boxed{d_k = \begin{cases} -\frac{1}{2} & k=0 \\ 0 & k \neq 0 \end{cases}}$$

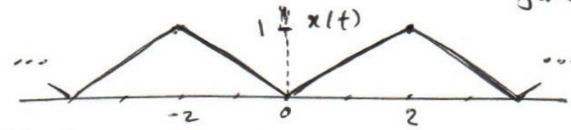
$\therefore\ g(t) \xleftrightarrow{FS} \boxed{b_k = c_k + d_k}$

$\therefore\ b_k = a_k \cdot e^{-jk\frac{\pi}{2}} + d_k$

$= \begin{cases} 0 + \frac{\sin(k\omega_0)}{k\pi} \cdot e^{-jk\frac{\pi}{2}} & ; k \neq 0 \\ -\frac{1}{2} + \frac{1}{2} & ; k=0 \end{cases}$

$$\therefore\ \boxed{b_k = \begin{cases} \frac{\sin(k\cdot\frac{\pi}{2})}{k\pi} \cdot e^{-jk\frac{\pi}{2}} & ; k \neq 0 \\ 0 & ; k=0 \end{cases}}$$

Ex 5:- if $x(t)$ is as shown in Figure



is periodic with period $T=4$

Note:- $g(t) = \frac{d}{dt} x(t)$
of Ex4

if $g(t) \xleftrightarrow{FS} b_k$

and $x(t) \xleftrightarrow{FS} e_k$

using FS properties

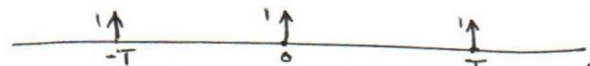
$\therefore\ b_k = jk\omega_0 \cdot e_k = \frac{jk\pi}{2} \cdot e_k$

$\therefore\ e_k = \frac{2 b_k}{jk\pi} = \frac{2\sin(k\frac{\pi}{2}) \cdot e^{-jk\frac{\pi}{2}}}{j\, k\pi \cdot k\pi}$; $k \neq 0$

for $k=0 \Rightarrow e_k$ = average of one period of $x(t)$

$$\therefore\ \boxed{e_k = \begin{cases} \frac{2\sin(k\frac{\pi}{2})}{j(k\pi)^2} \cdot e^{-jk\frac{\pi}{2}} & ; k \neq 0 \\ \frac{1}{2} & ; k=0 \end{cases}}$$

Ex 6:- $x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT)$

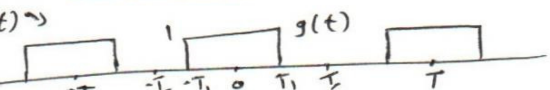


$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$

$= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t)\, e^{-jk(\frac{2\pi}{T})t}\, dt$

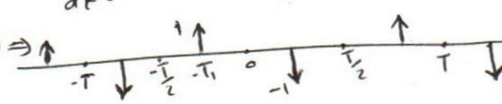
$$\therefore\ \boxed{a_k = \frac{1}{T}}$$

if $g(t) \Rightarrow$



and if $q(t) = \frac{d}{dt} g(t)$

$\therefore\ q(t) \Rightarrow$



$\therefore\ q(t) = x(t+T_1) - x(t-T_1)$

if $q(t) \xleftrightarrow{FS} b_k$

$$\therefore\ \boxed{b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k}$$

$\therefore\ b_k = a_k \left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1} \right]$

$$\therefore\ \boxed{b_k = \frac{1}{T} \cdot 2j\sin(k\omega_0 T_1)}$$

as $q(t) = \frac{d}{dt} g(t)$

$\therefore\ b_k = jk\omega_0 C_k$ as $g(t) \xleftrightarrow{FS} c_k$

$\therefore\ C_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T}$; $k \neq 0$

C_0 = average value of one period $= \frac{2T_1}{T}$

$$\therefore\ \boxed{c_k = \begin{cases} \frac{\sin(k\omega_0 T_1)}{k\pi} & ; k \neq 0 \\ \frac{2T_1}{T} & ; k=0 \end{cases}}$$

Ex 7 :- given the following facts about a signal $x(t)$

1- $x(t)$ is a real signal

2- $x(t)$ is periodic with period $T=4$ and it has Fourier series coefficients a_k

3- $a_k = 0$ for $|k| > 1$

4- The signal with Fourier coefficients $b_k = e^{-jk\frac{\pi}{2}} \cdot a_{-k}$ is odd.

5- $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$

Determine $x(t)$ within a sign factor ?

Solution

$\boxed{\text{Fact \# 3}}$ The non-zero coefficients are :- a_0, a_1, and a_{-1}

$\therefore x(t) = a_0 + a_1 e^{j \cdot 1 \cdot w_0} + a_{-1} e^{j \cdot (-1) \cdot w_0}$

$\boxed{\text{Fact \# 2}}$ $T = 4 \Rightarrow w_0 = \frac{2\pi}{4} = \frac{\pi}{2}$

$\therefore x(t) = a_0 + a_1 e^{j\frac{\pi}{2}} + a_{-1} e^{-j\frac{\pi}{2}}$

$\boxed{\text{Fact \# 1}}$ as $x(t)$ is real $\Rightarrow a_{-k} = a_k^*$

$\therefore a_{-1} = a_1^*$ and a_0 is real

$\therefore x(t) = a_0 + a_1 e^{j\frac{\pi}{2}} + a_1^* e^{-j\frac{\pi}{2}}$

$\boxed{\text{Fact \# 4}}$ - From time-reversal property

if $x(t) \xleftrightarrow{FS} a_k$

then $x(-t) \xleftrightarrow{FS} a_{-k}$

- from time-shifting property

$x(t-t_0) \xleftrightarrow{FS} e^{-jkw_0} a_k$

$\therefore x(-(t-1)) \xleftrightarrow{FS} b_k = e^{-jk\frac{\pi}{2}} \cdot a_{-k}$

as $y_1(t) = x(-t)$ time-reversal

then $y(t) = y_1(t-1)$ time-shift

$= x(-(t-1)) = x(-t+1)$

$\therefore x(-t+1) \xleftrightarrow{FS} e^{-jk\frac{\pi}{2}} \cdot a_{-k}$

- From Fact # 4 $y(t) = x(-t+1)$ is odd

- From Fact # 1 as $x(t)$ is real

$\Rightarrow x(-t+1)$ is also real.

$\therefore x(-t+1)$ is real and odd

$\therefore b_k$ is pure-imaginary and odd

$\Rightarrow b_0 = 0$

and $b_{-1} = -b_1$

$\boxed{\text{Fact \# 5}}$ average power over one period of $x(t) = \frac{1}{2}$

- As the time-reversal and time-shift cannot change the average power

$\therefore$ the average power of $x(t)$

$=$ the average power of $x(-t+1)$

then Fact # 5 holds for $x(-t+1)$ too

$\therefore \frac{1}{4}\int_4 |x(-t+1)|^2 dt = \frac{1}{2}$

- From parseval's relation

$\therefore \sum_k |b_k|^2 = \frac{1}{2}$

as $b_0 = 0$

$\therefore |b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$

as b_k is odd $\Rightarrow b_1 = -b_{-1}$

$\therefore |b_1|^2 + |-b_1|^2 = 2|b_1|^2 = \frac{1}{2}$

$\therefore |b_1|^2 = \frac{1}{4} \Rightarrow |b_1| = \frac{1}{2}$

as b_k is pure imaginary {conjugation property}

$\therefore b_1$ is either $\frac{1}{2}j$ or $-\frac{1}{2}j$

$\because b_k = e^{-jk\frac{\pi}{2}} \cdot a_{-k} \Rightarrow b_0 = a_0 = 0$ {Fact # 4}

if $k=-1 \Rightarrow b_{-1} = e^{-j\frac{\pi}{2}} a_1 \Rightarrow a_1 = -b_1 \cdot e^{-j\frac{\pi}{2}}$

$\Rightarrow a_1 = (-b_1) \cdot (-j) = j b_1$

Case 1 :- $b_1 = \frac{1}{2}j \Rightarrow a_1 = -\frac{1}{2}$

$\Rightarrow x(t) = \frac{-1}{2}\{e^{j\frac{\pi}{2}} + e^{-j\frac{\pi}{2}}\} = -\cos(\frac{\pi}{2})$

Case 2 :- $b_1 = -\frac{1}{2}j \Rightarrow a_1 = \frac{1}{2}$

$\Rightarrow x(t) = \frac{1}{2}\{e^{j\frac{\pi}{2}} + e^{-j\frac{\pi}{2}}\} = \cos(\frac{\pi}{2})$