Signals and Systems

Lecture # 1

Introduction
and
Complex Numbers Review

Prepared by:

Dr. Mohammed Refaey

Topics of the lecture:

- **Course Outlines.**
- **Introduction.**
- **Review of Complex Numbers.**
  - **Imaginary Number.**
  - **Complex Numbers Representations.**
  - **Complex Conjugate and its Properties.**
  - **Complex Number Magnitude and its Properties.**
  - **Complex Numbers Mathematical Operations**

➢ Course Outline.

- **The Course will be divided into Two parts:**
  - **- Part I : by Dr. Mohammed Refaey**
  - **- Part II : by Dr. Sherif Abdo**

- **The Course Textbooks will be:**
  - **1- "Signals and Systems", 2nd Edition , 1997**
    A.V.Oppenheim & A.S. Willsky (Prentice Hall)
  - **2- "Signals and Systems": 2nd Edition, 2011**
    Edward A. Lee & Pravin Varaiya

- **The Course prerequisites (include but not limited to):**
  The Calculus (especially differentiation and Integration), The Partial Fractions, The Complex Numbers, Trigonometry, and Differential Equations ... etc

- **The Course Total Degrees is 100:**
  - **60 % Final Exam.**
  - **20 % Midterm Exam.**
  - **20 % Two Quizzes and Assignments.**

➢ Introduction

Are you interested to study Signals and Systems?

Do not answer now!

Let us first know some applications of signals and Systems:

1- **Communications. (e.g. the internet carrier signal, the mobile communications...etc)**

2- **Aeronautics علوم الطيران. (Study, design, and manufacturing of air flight-capable machines.)**

3- **Astronautics الملاحة الفضائية . (The science and technology of space flight.)**

4- **Circuit Design. (Design and test electrical circuits.)**

5- **Acoustics الصوتيات. (e.g. how to make the sound clear and effective to the audience)**

6- **Seismology علم الزلازل. (e.g. detecting earthquakes.)**

7- **Biomedical Engineering الهندسة الطبية. (e.g. design of medical imaging devices.)**

8- **Energy Generation and Distribution Systems. (e.g. Microwaves and Heaters.)**

9- **Chemical Process Control. (e.g. Adjusting the mix parameters according to sensors.)**

10- **Speech Processing. (e.g. Speaker Identification, Text to Speech, Speech Recognition.)**

11- **Image Processing. (e.g. Image restoration and enhancement...etc)**

➢ Introduction

Are you involved? No

OK, let us see some examples:

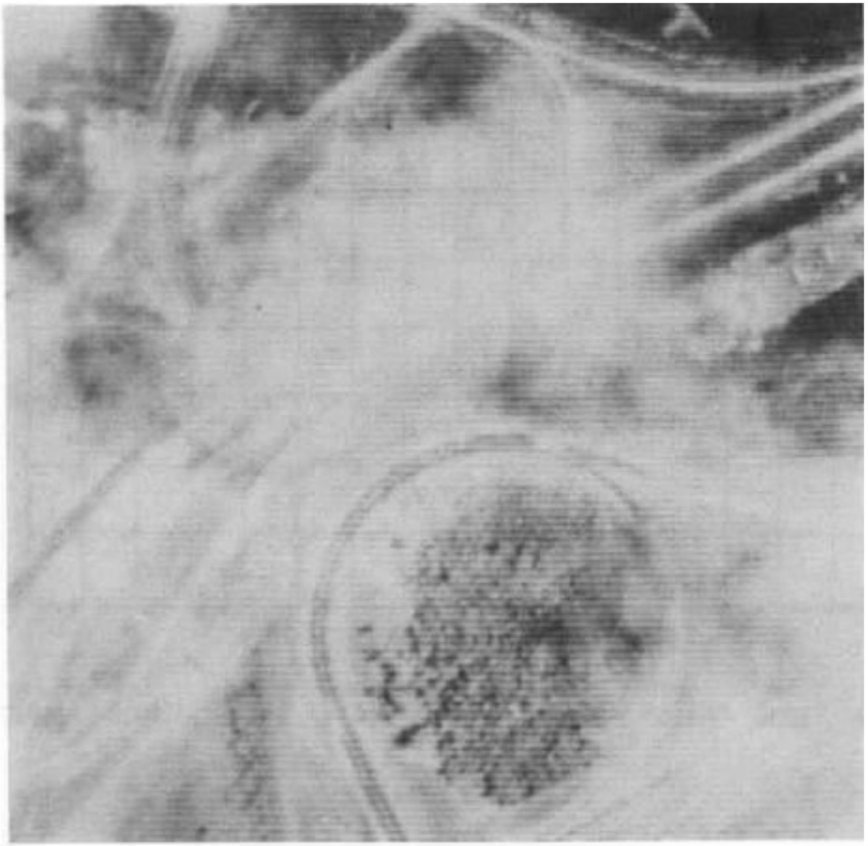
Aeronautics example
صناعة

Astronautics example
قيادة

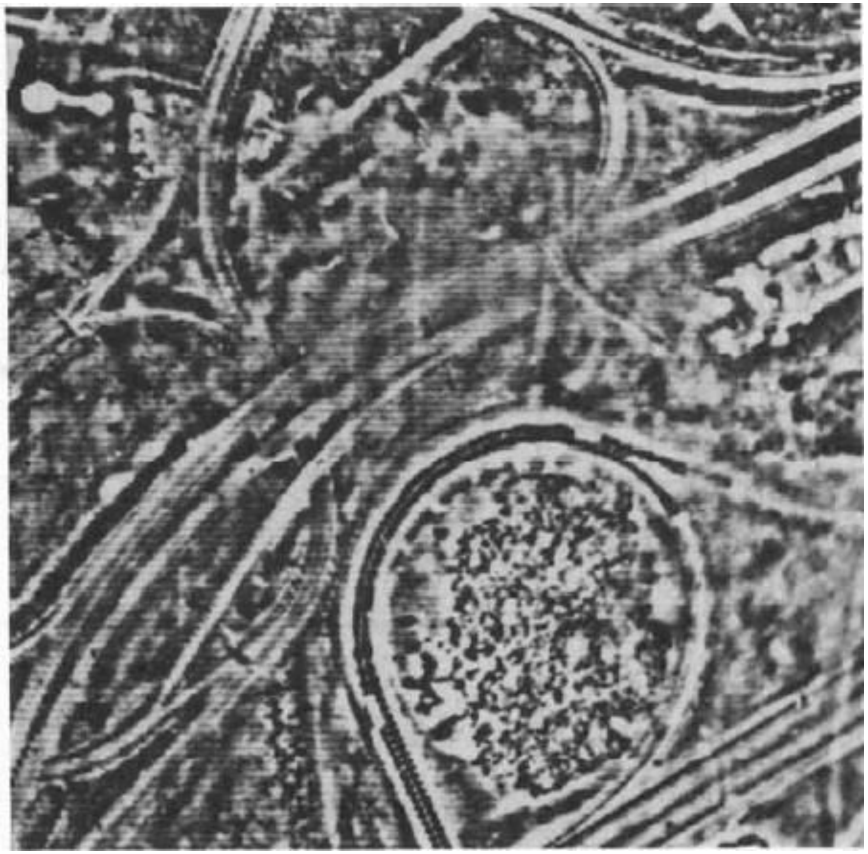
➢ Introduction

Are you involved? No

OK, let us see more examples:



Satellite image Before Enhancement

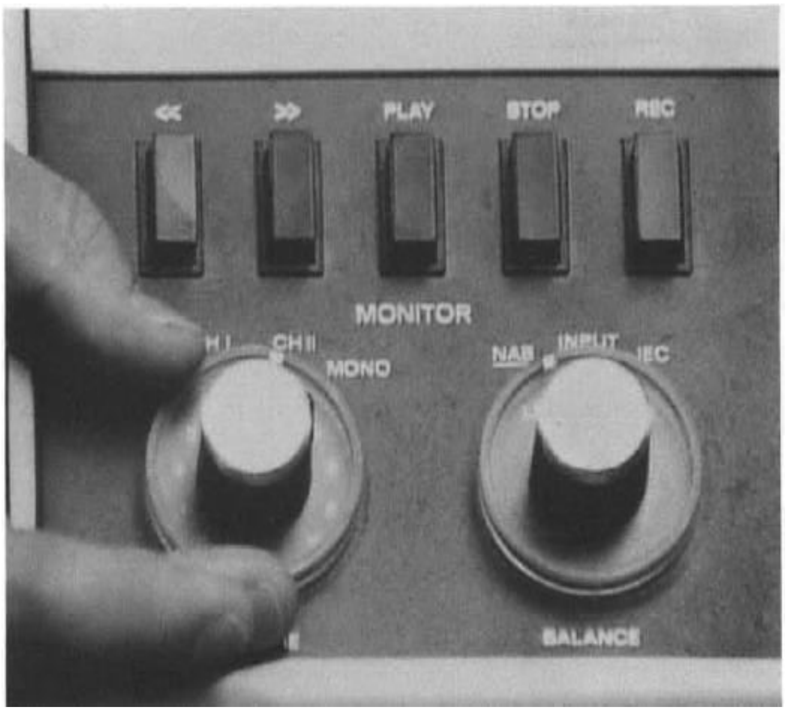


After Enhancement

➢ Introduction

Are you involved? No

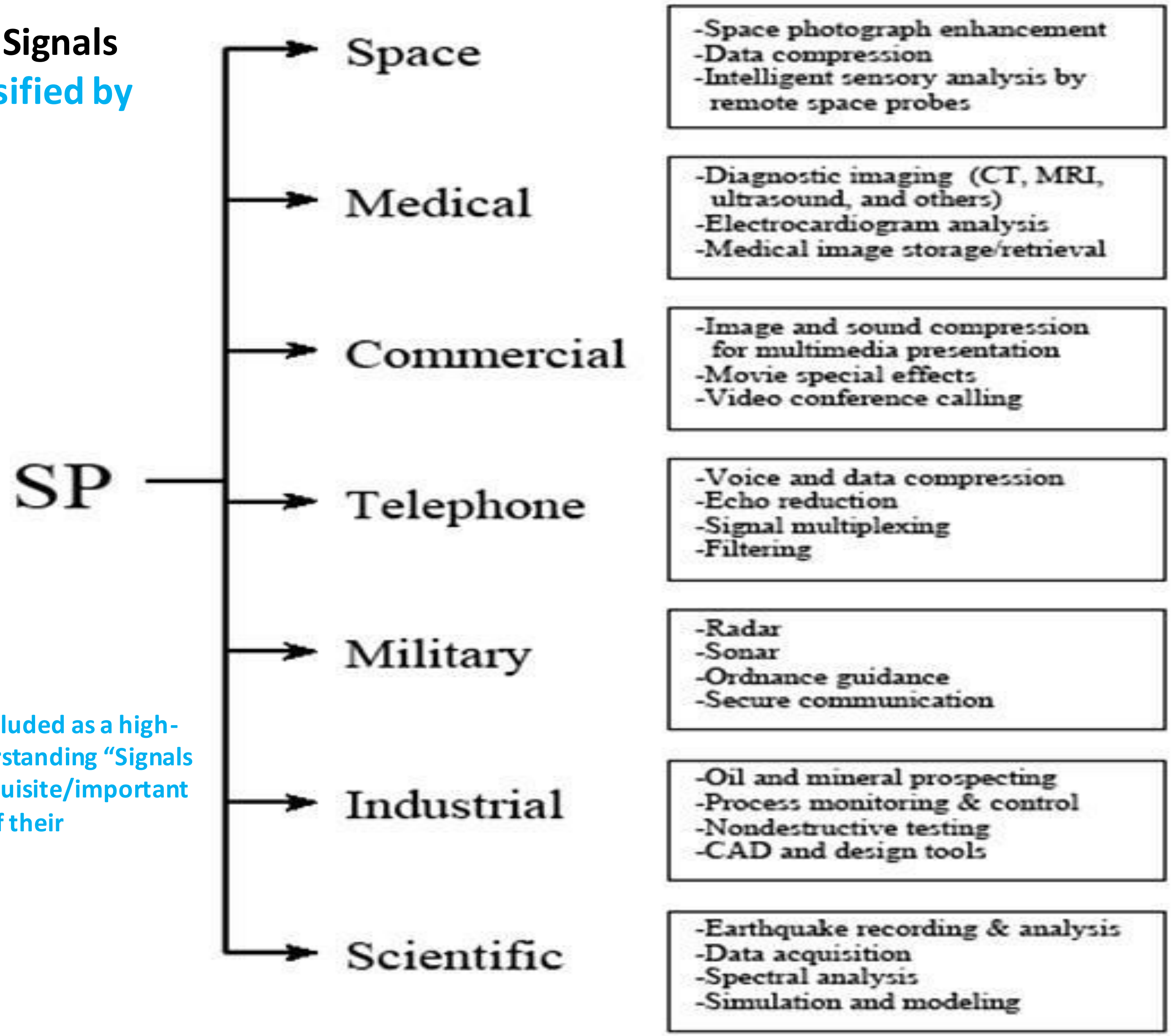
OK, let us see more examples:



Recording Enhancement Process

➢ Introduction

Applications of Signals Processing classified by field:



Note: it may not be included as a high-end product but understanding "Signals and Systems" is prerequisite/important to development/use of their products/machines.

Signals and systems Definition

Now let us know what is the signal? And what is the system?

The Signal:

It is a mean to convey information **(or *an abstraction of any measurable quantity*) that is usually have some form of variations. The contained information point to the behavior or nature of some phenomena.**

Mathematically:

The signal is a function of one or more independent variable **that maps a domain, often time or space, into a range, often a physical measure such as air pressure or light intensity. We usually called the independent variable time even if it is not actually time.**

The System:

It is a tool that transform a signal to get another signal or process a signal to obtain a desired behavior or for extracting a piece of information.

Mathematically:

***A system is a function that maps signals from its domain—its input signals—into a signal in its range—its output signals*. The domain and the range are both sets of signals; we call a set of signals a signal space. Thus, systems are functions whose domains and ranges are signal spaces.**

➢ Signals and systems Definition

Examples:

Sequence of bits stored on a memory	Audio Player	High Quality Stereo Sound
Spoken Words	Interactive Voice Response (IVR)	Sequence of commands to a computer/machine
Sound	Ultrasound Imaging System	Images
Driver Actions	Automobile (Car, Train, Airplane)	Increased/Decrease Automobile's speed, and/or direction
Images	Image Processing System	Recognized Objects/ Detecting manufacturing errors

➢ Motivations

What are the types of problems that signals and systems techniques try to answer?

1- *System Characterization* in detail to understand how it will respond to different inputs. (e.g. aircraft/ electrical circuit)

2- *System design* to react to inputs in a specific way. This usually involves a signal enhancement or restoration. (e.g. air traffic control tower in the airport to avoid the loud noise around when communicating with a pilot)

3- *Extracting specific pieces of information*. (e.g. electrocardiogram estimation of heart rates)

4- *Design of Signals with particular properties*. (e.g. the carrier signal in long distance communications)

5- *Modification and control* the characteristics *of a given system*. (e.g. chemical process control through sensors)

➢Review of Complex Numbers.

Imaginary Number:

An Imaginary Number, when squared, gives a negative result

- **Unit Imaginary Number:**

  Each numbering system should have a unit to can be counted !!!
  The "unit" Imaginary Number (the equivalent of 1 for Real Numbers) is $\sqrt{-1}$ (the square root of negative one). In mathematics we use (*i*) (for imaginary) but in electronics use (*j*) (because (*i*) already means current, and the next letter after (*i*) is (*j*)). We will use (*j*) in this course!

It is "*imaginary* " and useful ?!

- **imaginary numbers give us the ability to find solutions (roots) for *quadratic equations* like : $x^2 + 1 = 0$**

- **using imaginary numbers and real numbers together makes it a lot easier to do the calculations in many applications as they *encode the magnitude and phase together in a compact simpler form*.**

- **the imaginary numbers are *not imaginary, they are exist* and *fill a gab in math*. For example: *imaginary* x *imaginary* = *real* ➔ i.e. exist !**

➢Review of Complex Numbers.

- Dealing with imaginaries gives you the ability to work with the square root of negatives. BUT, in the same time you lose something !

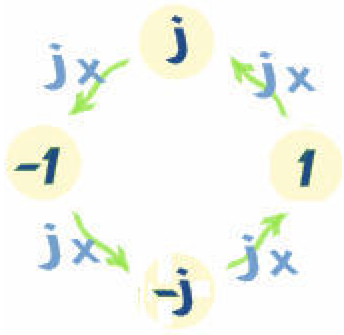
$$\because \quad j = \sqrt{-1}$$

$$and\ as\ \ (A^a)^b = \left(A^b\right)^a$$

$$then: j^2 = \left(\sqrt{-1}\right)^2 = \sqrt{(-1)^2} = \sqrt{1} = \pm 1$$

$$HERE\ this\ is\ not\ true\ , as\ you\ should\ always\ make\ j^2 = -1 \quad not\ +1$$

- Interesting Property:



$$\underline{\underline{Note}}: \quad j = j^5 = j^9 = \sqrt{-1} \quad ; \quad j^2 = j^6 = j^{10} = -1$$

$$j^3 = j^7 = j^{11} = -j \quad ; \quad j^4 = j^8 = j^{12} = 1$$

$$in\ general,\ j^m = j^{4\times n}.j^k = j^k, where \quad m, n, and\ k\ are\ \ integers$$

$$for\ example: \quad j^{99} = j^{96+3} = j^{4\times 24}.j^3 = j^3 = -j$$

➢ Review of Complex Numbers.

The complex number:

is made up of both real and imaginary components and its usually represented as: $Z = a + j\,b$

Where a **is called the real part of the complex number** Z**, or** $a = Re\{Z\}$**,**

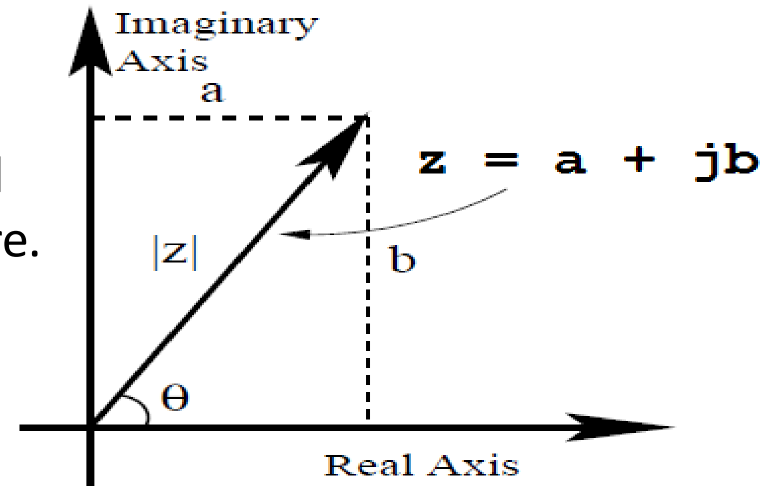
b **is called the imaginary part of the complex number** Z**, or** $b = Im\{Z\}$**,**

and (j **) is the square root of (**-1**), i.e.** $j = \sqrt{-1}$

The Complex Plane:

• A complex number can be visualized in a two-dimensional number plane, known as an **Argand diagram**, or the complex plane as shown in Figure.

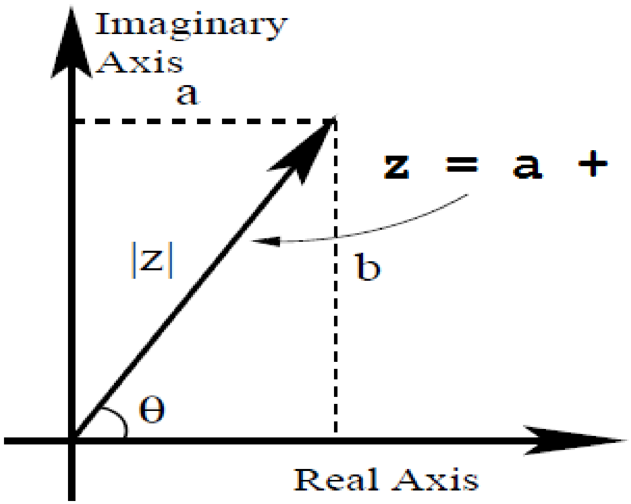
• It is conventional to represent a complex number as a **vector** in the complex plane, usually called a **phasor**.



If $Z_1 = a + j\,b$ **and** $Z_2 = c + j\,d$ **, then** $Z_1 = Z_2$ **iff** $a = c$ **and** $b = d$

➢Review of Complex Numbers.

The Complex Magnitude :



From the Figure, it can be easily seen (using the Pythagorean theorem) that the magnitude, or length, of the vector representing the complex number is:

$$(|z|)^2 = a^2 + b^2$$

$$|z| = \sqrt{a^2 + b^2}$$

It is *also* called **absolute value** or **modulus**.

The Phase Angle of a complex number:

It is the angle of the phasor of the complex number with the positive Real-Axis:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

It is usually called the Argument of the complex number. $\theta = Arg\{Z\} = \angle Z$

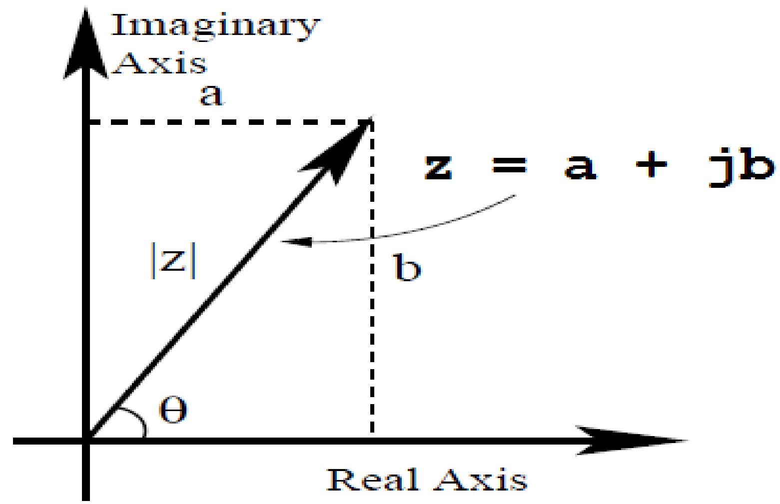
$$0 \leq \theta < 2\pi$$

➢Review of Complex Numbers.

Polar Form:

If you have Z = a + j b (Rectangular Form)

$$from\ the\ graph:$$



$$\sin(\theta)=\frac{b}{|Z|} \quad \Rightarrow \quad b=|Z|\sin(\theta)$$

$$and \quad \cos(\theta)=\frac{a}{|Z|} \quad \Rightarrow \quad a=|Z|\cos(\theta)$$

Then Z can be rewritten as :

$$Z = \ |Z|\cos(\theta)+j\ |Z|\sin(\theta)$$

$$= \ |Z|\left(\cos(\theta)+j\ \sin(\theta)\right)$$

$$From\ Euler's\ Formula: e^{j\theta}=\cos(\theta)+j\ \sin(\theta)$$

$$Then \quad Z=|Z|e^{j\theta}=|Z|e^{j\angle Z}, \quad if \quad r=|Z| \quad \Rightarrow \quad \underline{\underline{Z=re^{j\theta}}}$$

$$which\ is\ \underline{\underline{called\ the\ polar\ form}}\ of\ the\ complex\ number\ (Z)$$

➢Review of Complex Numbers.

Addition and Subtraction:

If you have $Z_1 = a + j\,b$ and $Z_2 = c + j\,d$

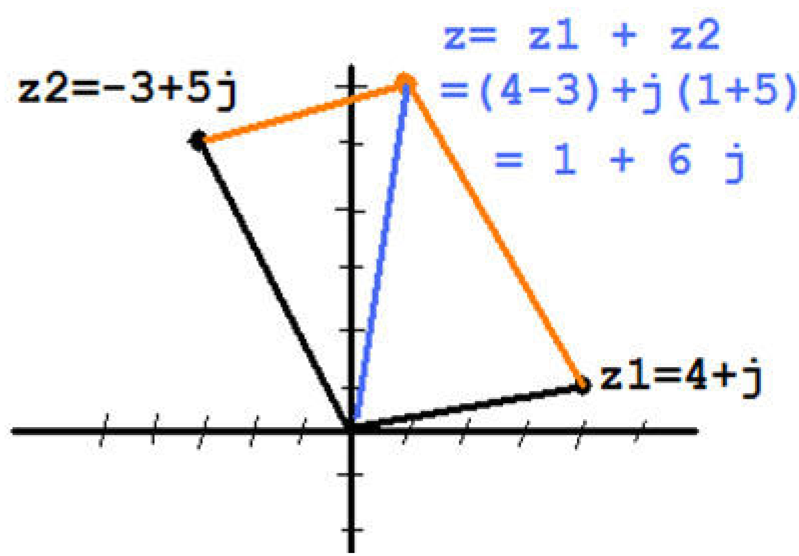
Then: $Z_1 + Z_2 = a + j\,b + c + j\,d = (a + c) + j\,(b + d)$

i.e. add the real part to the real part and the imaginary part to the imaginary part.

And : $Z_1 - Z_2 = (a - c) + j\,(b - d)$

i.e. subtract the real part from the real part and the imaginary part from the imaginary part

Complex numbers addition is similar to vectors sum.



➢ Review of Complex Numbers.

Multiplication:

If you have $Z_1 = a + j\,b$ and $Z_2 = c + j\,d$

Then: $Z_1 \,.\, Z_2 = (a + j\,b)\,.\,(c + j\,d)$

$$= a\,(c + j\,d) + j\,b\,(c + j\,d) = a\,c + j\,a\,d + j\,b\,c - b\,d$$

$$= a\,c - b\,d + j\,(a\,d + b\,c)$$

The commutative and distributive properties hold for the product of complex numbers.

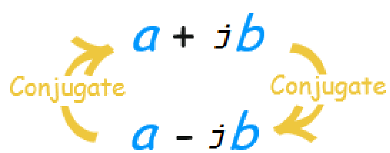
Complex Conjugation:

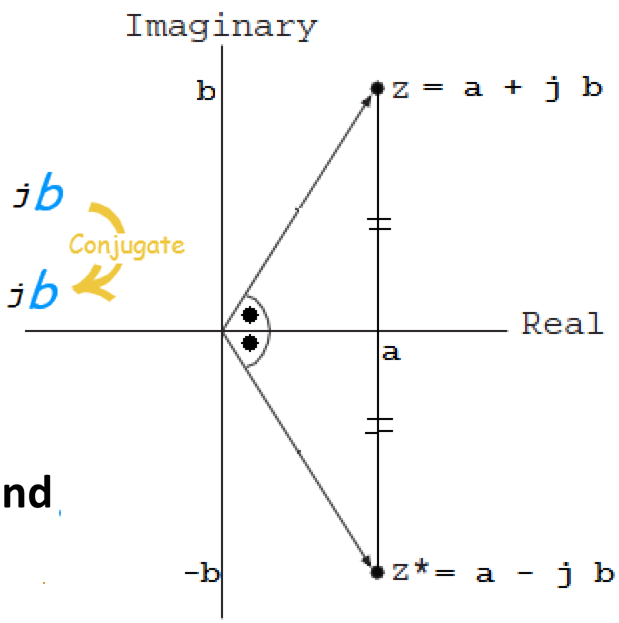
The complex conjugate of a complex number

$$Z = a + j\,b,$$

is denoted Z^*, and is defined by:

$$Z^* = a - j\,b$$





The complex conjugate Z* has:

- the same real part but opposite imaginary part, and
- the same magnitude but opposite phase.

➢Review of Complex Numbers.

Complex Conjugation Properties:

- $(Z^*)^* = Z$
- $(Z_1 + Z_2)^* = Z_1^* + Z_2^*$
- $(Z_1 . Z_2)^* = Z_1^* . Z_2^*$
- if $Z_2 \neq 0$, $\left(\dfrac{z_1}{z_2}\right)^* = \dfrac{z_1^*}{z_2^*}$
- $\left(z^n\right)^* = \left(z^*\right)^n$
- if Z is real, then $Z = Z^*$

➢Review of Complex Numbers.

Complex Conjugation Properties:

$$prove\, that: \quad \left(Z^n\right)^* = \left(Z^*\right)^n$$

$$let \quad Z = r\left(\cos(\theta) + j\sin(\theta)\right)$$

$$then\ by\ DeMoivre's\ Theroem:$$

$$Z^n = \left[r\left(\cos(\theta) + j\sin(\theta)\right)\right]^n = r^n\left(\cos(n\theta) + j\sin(n\theta)\right)$$

$$\therefore \left(Z^n\right)^* = \left(r^n\left(\cos(n\theta) + j\sin(n\theta)\right)\right)^* = r^n\left(\cos(n\theta) - j\sin(n\theta)\right)$$

$$= \left[r\left(\cos(\theta) - j\sin(\theta)\right)\right]^n$$

$$= \left(Z^*\right)^n$$

➢ Review of Complex Numbers.

<u>The Complex Magnitude Properties:</u>

- $|Z| = 0 \quad \text{iff} \quad Z = 0$
- $|Z| = |Z^*|$
- $|Z_1 \, . \, Z_2| = |Z_1| \, . \, |Z_2|$
- if $Z \neq 0$, then $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$
- $|Z_1 + Z_2| \leq |Z_1| + |Z_2|$
- $Z \, . \, Z^* = |Z|^2$

➢Review of Complex Numbers.

Division:

If you have $Z_1 = a + j\,b$ and $Z_2 = c + j\,d$

Then:

$$\frac{Z_1}{Z_2} = \frac{a + jb}{c + jd}$$

Is usually rewritten by rationalizing the denominator to make it simpler.
i.e. can be represented as a ratio.

$$\frac{Z_1}{Z_2} = \frac{a + jb}{c + jd} \times \frac{c - jd}{c - jd}$$

$$= \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

$\underline{Example:}$

$$Express \quad \frac{5 + 5j}{1 + 3j} \quad in\ the\ form\ of\ (a + jb)$$

We multiply the numerator and denominator by the conjugate of the denominator (1 - 3j).

$$\left(\frac{5 + 5j}{1 + 3j}\right) \times \left(\frac{1 - 3j}{1 - 3j}\right) = \frac{5 - 15j + 5j + 15}{1 - 3j + 3j + 9}$$

$$= \frac{20 - 10j}{10}$$

$$= \frac{20}{10} - \frac{10}{10} j = 2 - j$$

➢Review of Complex Numbers.

Examples to be solved on the board:

$$1-\text{show that the}\quad Arg\{Z^*\}=-Arg\{Z\}?$$

$$2-\text{Get the value of}\quad e^{-1+j\frac{\pi}{6}}\quad ?$$

$$3-\text{Simplify}\quad \frac{-20+\sqrt{-75}}{5}\quad ?$$

$4-$ *Show that when multiplying two complex numbers, actually we multiply their magnitudes and add their phase angles?*

$$5-\text{Proof that}:\qquad \cos^2(\theta)+\sin^2(\theta)=1$$

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

$1 - show\ that\ the \quad Arg\{Z^*\} = -Arg\{Z\}?$ $\qquad 2 - Get\ the\ value\ of \quad e^{-1 + j\frac{\pi}{6}} \quad ?$

$3 - Simplify\ \dfrac{-20+\sqrt{-75}}{5}$?

$4 - Show\ that\ when\ multiplying\ two\ complex\ numbers,\ actually$

$we\ multiply\ their\ magnitudes\ and\ add\ their\ phase\ angles$?

$5-\mathit{Proof\ that:}$ $\cos^2(\theta)+\sin^2(\theta)=1$

$$\cos(\alpha+\beta)=\cos(\alpha)\cos(\beta)-\sin(\alpha)\sin(\beta)$$

➢Review of Complex Numbers.

Assignment:

1- Proof the complex conjugate properties?

2- Proof the complex magnitude properties?

$$3-Proof\ that: \quad \sin(\alpha+\beta) = \sin(\alpha)\cos(\beta)+\cos(\alpha)\sin(\beta)$$

➢Review of Complex Numbers.

4- Consider a series AC electrical circuit with two resistors and a capacitor. The output complex voltage

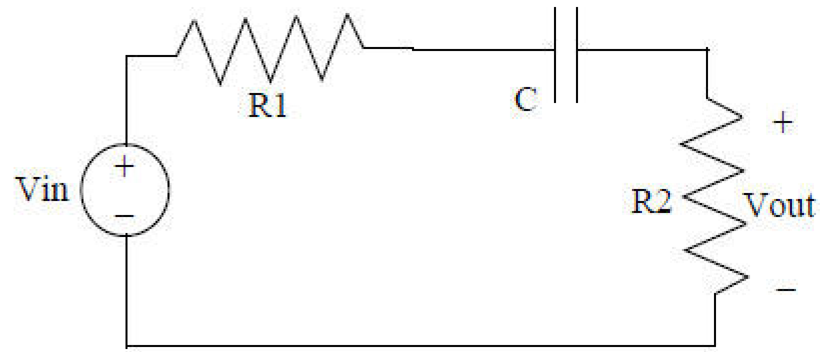


Figure : A simple AC circuit.

is related to the input complex voltage by the voltage divider law

$$\hat{V}_{out} = \frac{R_2}{R_1 + R_2 - i/(\omega C)} \hat{V}_{in}$$

If $R_1 = 100\Omega$, $R_2 = 200\Omega$, $C = 50\mu F$, and $\omega = 2\pi(60)$ cycles/s, and $\hat{V}_{in} = 100V$, then what is the

a) magnitude of the output voltage,
b) phase of the output voltage.
c) plot $|\hat{V}_{out}/\hat{V}_{in}|$ as a function of differents ω's.

Signals and Systems

Lecture # 2

Signals Classification and Transformation

Prepared by:

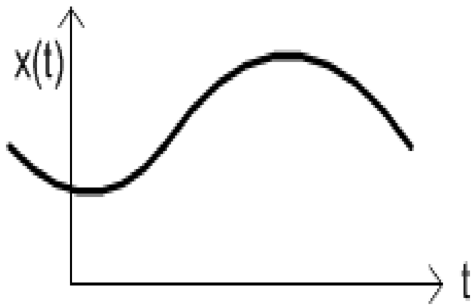
Dr. Mohammed Refaey

Topics of the lecture:

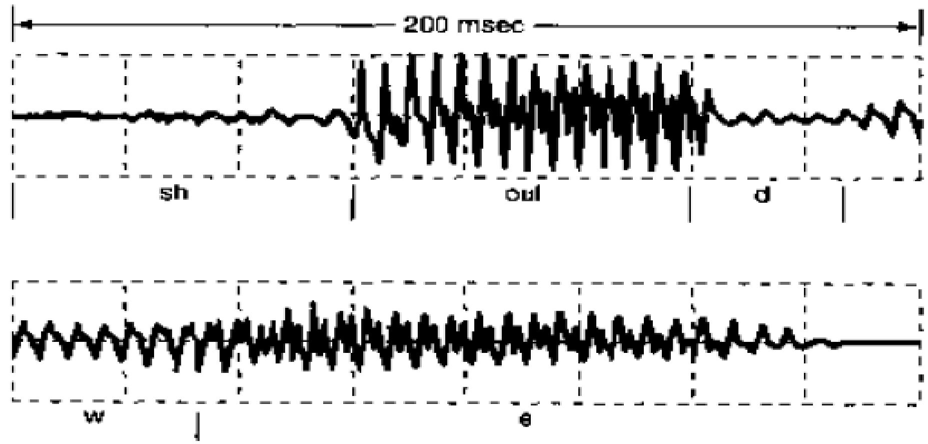
- **Signals Classification according to the Independent Variable.**
- **Signal’s Energy and Power.**
- **Signals Transformations of Independent Variable.**

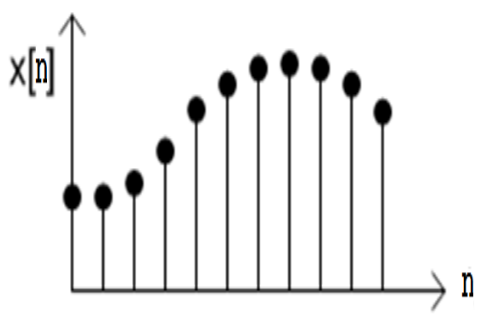
➢ Signal Classification according the independent variable

Signals are either *continuous in time* , that are defined at any time instant in its time domain (e.g. voltages and currents in electrical circuits or sound signals), OR, *discrete in time* , that are defined at integer time instants only in its time domain(e.g. closed stock market average , crime rate , or total population).

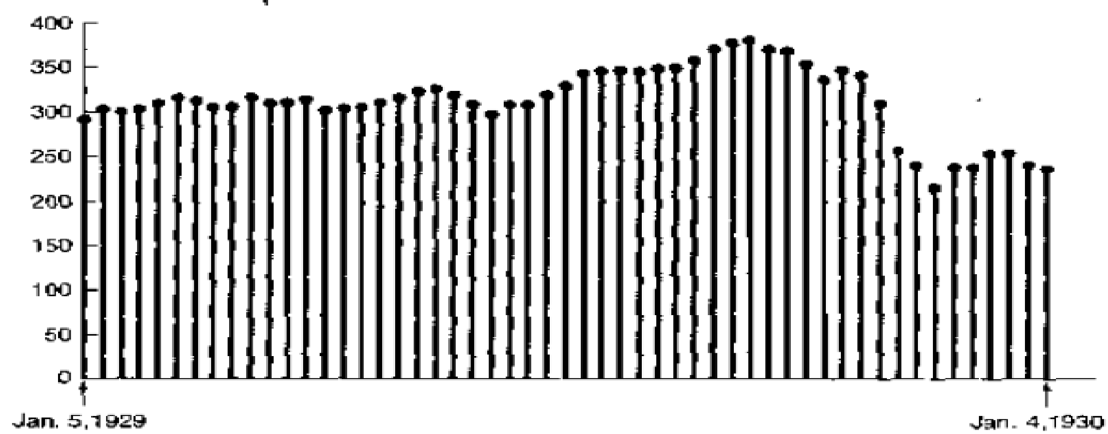


Continuous-Time





Discrete-Time



An example of a discrete-time signal: The weekly Dow-Jones stock market index from January 5, 1929, to January 4, 1930.

➢ Signal Classification according the independent variable

	Continuous-Time Signals	Discrete-Time Signals
- Domain:	Its Domain is a continuous interval of Time f(t)=x , t1 < t < t2	Its Domain is a sequence of Time samples f[n] = y , n ϵ { n1 , n1+1 ,n1+2 , …, n2}
- Range:	Its range is a real valued e.g. f(t)=1.43	Its range is real valued e.g. f[n]=9.32
- Symbol of IV:	t	n
- Function form:	( )	[ ]
- Examples:	• Speech Signal • Voltage and Current	• Digital Images • Stock Market Index

- In some books, if the domain and range are discrete the signal is called digital signal

➢ Signal's Energy and Power

- As too many signals are related to *physical quantities capturing energy and power*, it is useful to define and measure the signal's energy and power.

$$\text{in electrical circuit, the power consumed:}$$

$$P(t) = V(t).I(t) = \frac{1}{R}V^2(t), \qquad V \text{ is the voltage}$$

$$\text{in autombile, the power consumed through friction:}$$

$$P(t) = b.V^2(t), \qquad V \text{ is the automobile speed}$$

- In the above two examples, though they are different they have something in common, that is the *power is* a *constant* (that could be ignored for analysis purposes) *times a square of the system variable*.

➢ Signal's Energy and Power

Energy: is the capacity for doing work. You must have energy to accomplish work - it is like the "currency" for performing work.

Power: is the rate of doing work or the rate of using energy.

- For most of this class we will use a broad definition of power and energy that applies to any signal $x(t)$ or $x[n]$
- **Instantaneous signal power**

$$P(t) = |x(t)|^2 \qquad\qquad P[n] = |x[n]|^2$$

- **Signal energy**

$$E(t_0, t_1) = \int_{t_0}^{t_1} |x(t)|^2 \, \mathrm{d}t \qquad\qquad E(n_0, n_1) = \sum_{n=n_0}^{n_1} |x[n]|^2$$

- **Average signal power**

$$P(t_0, t_1) = \frac{1}{t_1 - t_0} \int_{t_0}^{t_1} |x(t)|^2 \, \mathrm{d}t$$

$$P(n_0, n_1) = \frac{1}{n_1 - n_0 + 1} \sum_{n=n_0}^{n_1} |x[n]|^2$$

|.| means magnitude of possibly complex valued x(t) or x[n]

➢ Signal's Energy and Power

Usually, the limits are taken over an infinite time interval to get the total Energy:

$$E_\infty \triangleq \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |x(t)|^2\, dt,$$

in discrete time→ $$E_\infty \triangleq \lim_{N\to\infty} \sum_{n=-N}^{+N} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |x[n]|^2$$

and total Average Power: $$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$

The division may lead to undefined limits

in discrete time→ $$P_\infty \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2$$

➢ Signal's Energy and Power

We will encounter many types of signals :

- Some have infinite average power, energy, or both
- A signal is called an **energy signal** if $E_\infty < \infty$
- A signal is called a **power signal** if $0 < P_\infty < \infty$
- A signal can be an energy signal, a power signal, or neither type
- A signal can not be both an energy signal and a power signal

Signal Energy & Power Tips

- There are a few rules that can help you determine whether a signal has finite energy and average power
- Signals with finite energy have zero average power:
  $E_\infty < \infty \Rightarrow P_\infty = 0$
- Signals of finite duration and amplitude have finite energy:
  $x(t) \leq K$ for $|t| > c$, $K < \infty \ \Rightarrow E_\infty < \infty$
- Signals with finite average power have infinite energy:
  $P_\infty > 0 \Rightarrow E_\infty = \infty$

➢ Signal's Energy and Power

Determine whether the energy and average power of each of the following signals is finite.

a- $$x(t) = \begin{cases} 8 & |t| < 5 \\ 0 & \text{otherwise} \end{cases}$$

b- $$x[n] = j$$

c- $$x[n] = A\cos(\omega n + \phi)$$

d- $$x(t) = \begin{cases} e^{at} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

e- $$x[n] = e^{j\omega n}$$

➢ Signal's Energy and Power

a-

$$E_\infty = \int_{-\infty}^{+\infty} |x(\sigma)|^2 d\sigma = \int_{-5}^{+5} |8|^2 d\sigma = \int_{-5}^{+5} 64 d\sigma = 640$$

$$P_\infty = 0 \quad as\ E_\infty\ is\ finite$$

b-

$$E_\infty = \sum_{n=-\infty}^{+\infty} |x[n]|^2 = \sum_{n=-\infty}^{+\infty} |j|^2 = \sum_{n=-\infty}^{+\infty} 1^2 = \infty$$

$$P_\infty = \lim_{N->\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |x[n]|^2 = \lim_{N->\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} |j|^2$$

$$= \lim_{N->\infty} \frac{1}{2N+1} \sum_{n=-N}^{+N} 1^2 = \lim_{N->\infty} \frac{1}{2N+1} (2N+1)$$

$$= \lim_{N->\infty} 1 = 1$$

➢ Signal's Energy and Power

a- $$x(t) = \begin{cases} 8 & |t| < 5 \\ 0 & \text{otherwise} \end{cases}$$

➢ Signal's Energy and Power

b- $x[n] = j$

➢ Signal's Energy and Power

c− $$x[n] = A\cos(\omega n + \phi)$$

➢ Signal's Energy and Power

d-

$$x(t) = \begin{cases} e^{at} & t > 0 \\ 0 & \text{otherwise} \end{cases}$$

➢ Signal’s Energy and Power

e-

$$x[n] = e^{j\omega n}$$

➢ Signals Transformation

Signal Transformation plays a central concept in signals and systems analysis. Where you need to know the shape of signal after deformation(s)/processing or how to construct a signal from another group of signals

Examples:

Pilot Actions → **Through electrical and mechanical system** → **Changes in aircraft thrust قوة الدفع, OR Positions in the control surfaces**

Through Dynamics and Kinematics → **Changes in velocity or heading توجيه**

Elements of a communication system. The transmitter changes the message signal into a form suitable for transmission over the channel. The receiver processes the channel output (i.e., the received signal) to produce an estimate of the message signal.

➢ Signals Transformation (1- Of Independent Variable (time))

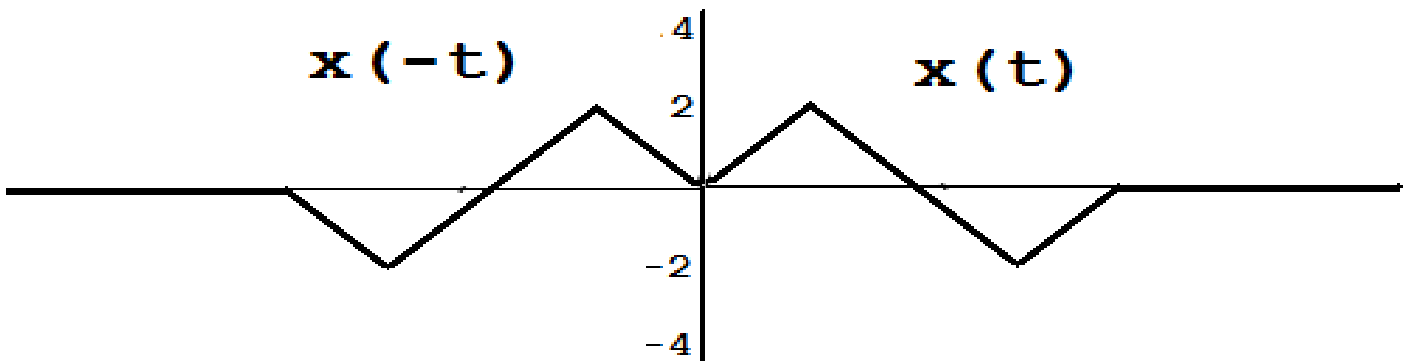
A- Basic Operations on the independent Variable:

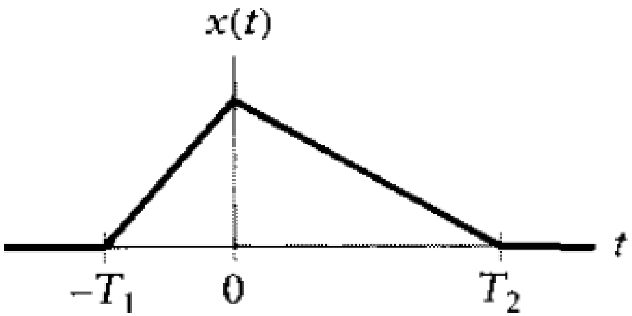
1- Time Reversal:

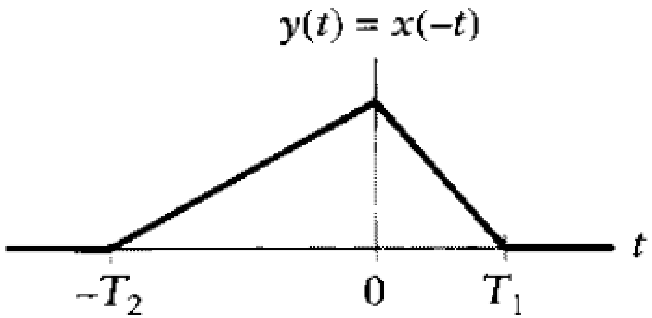
It is a reflection of the signal around the vertical axis (i.e. reversing it) at t=0 for the continuous-time signals, OR at n=0 for discrete-time signals.

e.g. if x(t) is an audio file, x(-t) is the same file but played backward.

Mirror





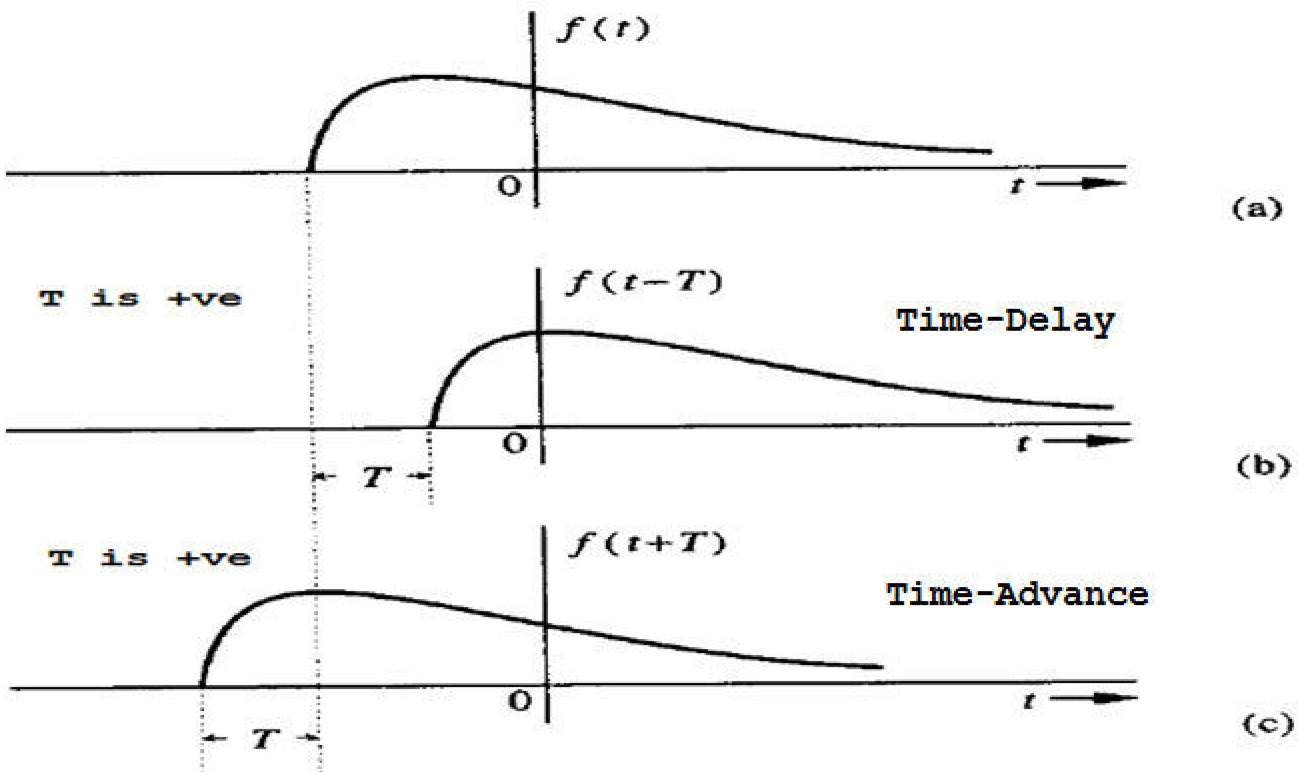


➢ Signals Transformation (1- Of Independent Variable (time))

2- Time Shift: $y(t) = x(t - t_o)$

It is the same signal in shape but moved either to right ($t_o > 0$, and called Time-Delay) or to the left ($t_o < 0$, and called Advance) of the original signal.
e.g. Any application has a transmitter and multiple receivers.

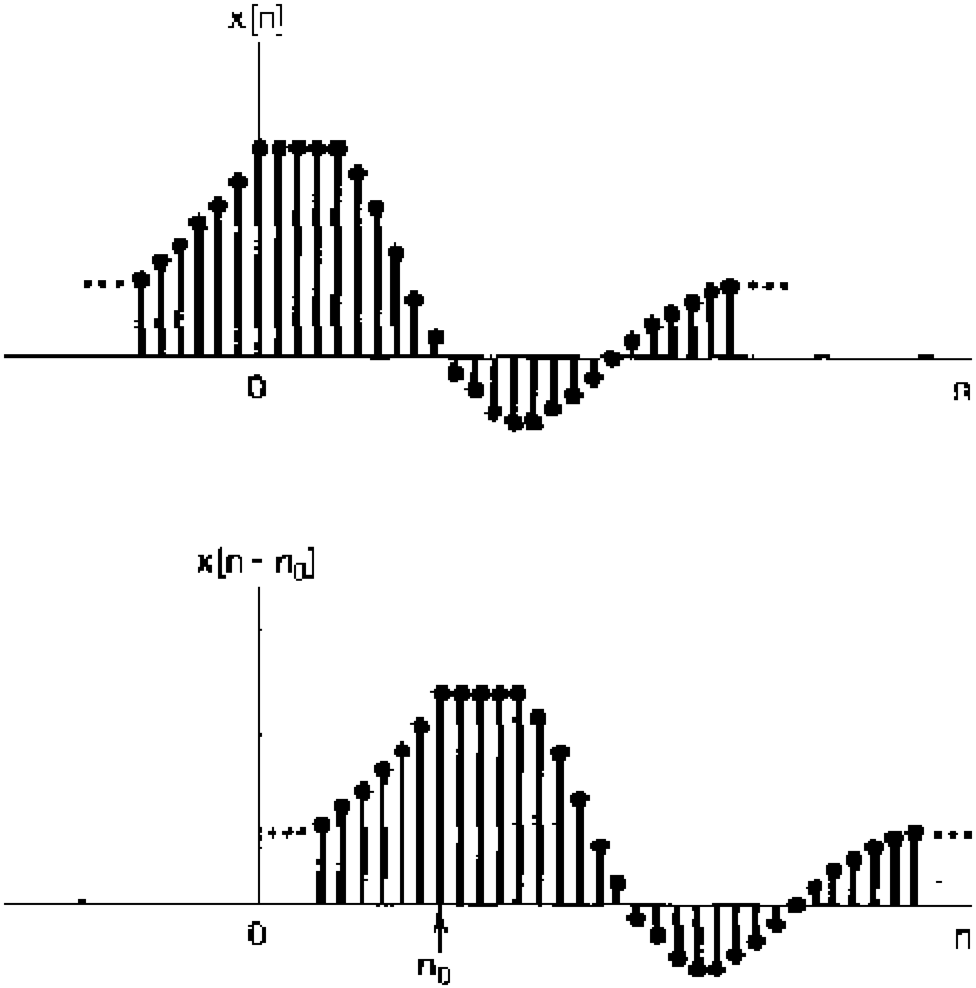
Example:
The original signal f(t) at (t = -5) will occur at a new location of the signal f(t-2) at:
t-2 = -5 → t= -3
i.e. the new location at (t=-3)
Similarly, its new location of the signal f(t+2) at:
t+2=-5 → t= -7
i.e. the new location will be at (t= -7)



Time shifting a signal.

➢ Signals Transformation (1- Of Independent Variable (time))

2- Time Shift:



Discrete-time signals related by a time shift. In this figure $n_0 > 0$, so that $x[n - n_0]$ is a delayed verson of $x[n]$ (i.e., each point in $x[n]$ occurs later in $x[n - n_0]$).

➢ Signals Transformation (1- Of Independent Variable (time))

3- Time Scaling:

$$y(t) = x(at)$$

It is a signal similar to the original signal in shape but it is either compressed **(a > 1, and called Time-Shrinking or Time-Compression)** or stretched **(a < 1, and called Time-Expansion or Time-Stretch)** version of the original signal.

e.g. An audio file either played at double speed or half speed.

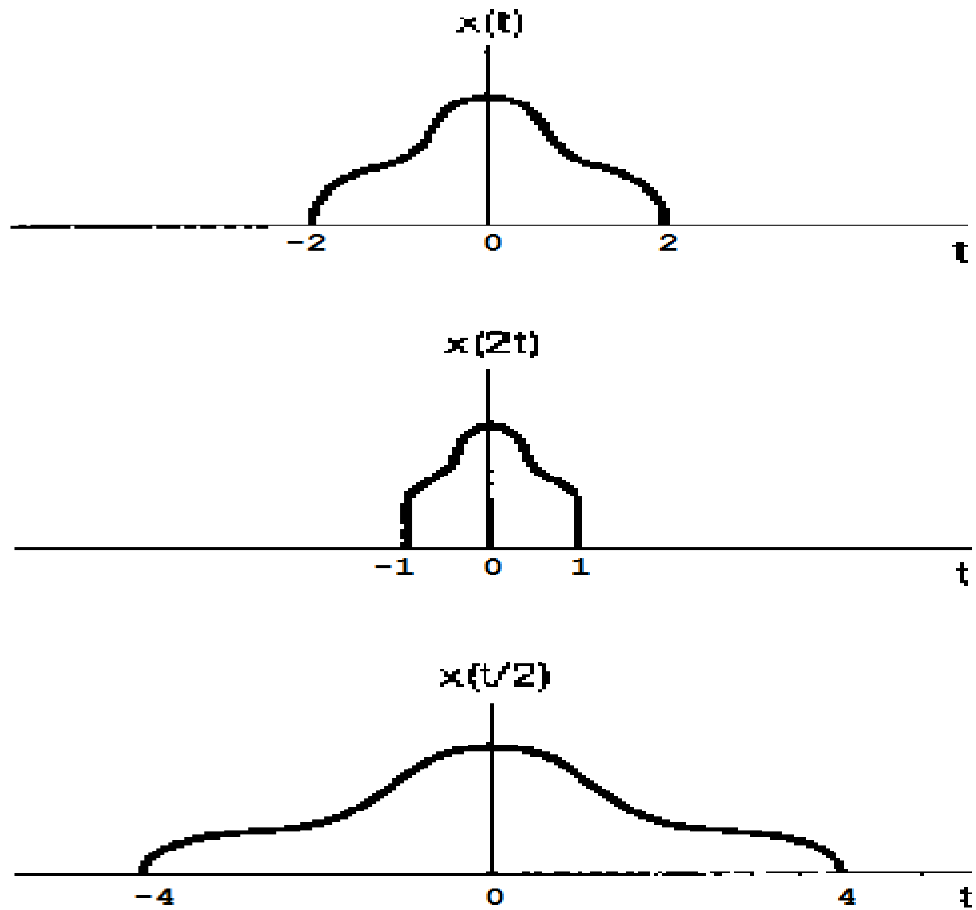


Figure 1.12 Continuous-time signals related by time scaling.

➢ Signals Transformation (1- Of Independent Variable (time))

In general:

$y(t) = x(a.t - b)$

If $b \neq 0$ →

There is a Time-Shift:
- Time-delay (move to right) if (t)and (b) have different signs
- Time advance(move to left) if (t)and (b) have same signs

If a = -ve →

There is a Time-Reversal:
The result is similar in shape to the original signal but mirrored/reversed version of it

If $|a| \neq 1$ →

There is a Time-Scaling:
- Time-Shrinking/Time-Compression if $|a| > 1$
- Time-Expansion or Time-Stretch if $|a| < 1$

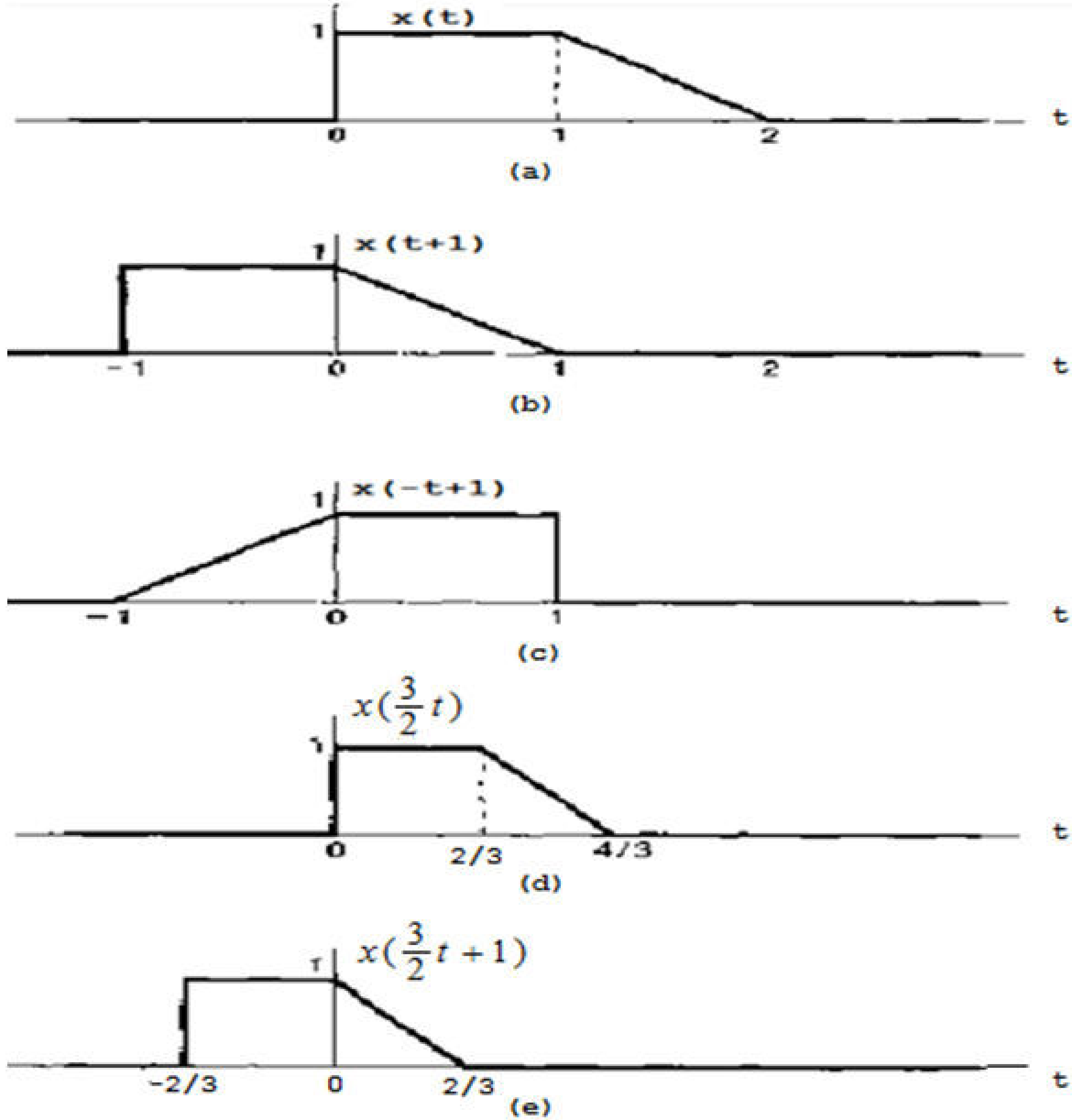
A systematic approach to get $y(t) = x(a.t - b)$ from $x(t)$:

1- Do the time-shift i.e. get → $y1(t) = x(t - b)$

2- Do the time-reverse/scaling on the resulting signal, i.e get → $y(t) = y1(a.t)$

➢ Signals Transformation (1- Of Independent Variable (time))

Example:



➢ Signals Transformation (1- Of Independent Variable (time))

Key-points approach to compute the independent transformations result:

1- Determine the Key-points in time.

Where the key-point is the point at which there is a change in the signal behavior.

In the last example in the previous slide for example three key-points:

At $t=0$, $t=1$, and $t=2$

2- let the required transformation as a Left-Hand-Side of an equation.

In the last example the second required transformation was $(-t+1)$.

And sequentially find the new key-points by let each old key-point in the right-Hand-Side of the equation and solve for t to get the new key-point's location.

For the last example:

For $t=0$ → $-t+1=0$ → $t=1$, then what was happened at $t=0$ will happen at $t=1$

For $t=1$ → $-t+1=1$ → $t=0$, then what was happened at $t=1$ will happen at $t=0$

For $t=2$ → $-t+1=2$ → $t=-1$, then what was happened at $t=2$ will happen at $t=-1$

3- draw the result through connecting between key-points.

➢ Signals Transformation (1- Of Independent Variable (time))

If you have to perform composite transformation x(a.t + b) on the independent variable for a given signal x(t), what is the correct order:
1- perform time-shift then time-scaling/reverse ?
2- perform time-scaling/reverse then time-shift ?
And How to perform either choice? Justify your answer ...

➢ Signals Transformation (1- Of Independent Variable (time))

If you have to perform composite transformation x(a.t + b) on the independent variable for a given signal x(t), what is the correct order:
1- perform time-shift then time-scaling/reverse ?
2- perform time-scaling/reverse then time-shift ?
And How to perform either choice? Justify your answer ...

Signals and Systems

Lecture # 3

Signal Transformations (followed), Odd and Even Signals, and Signal Periodicity

Prepared by:

Dr. Mohammed Refaey

Topics of the lecture:

- **Signals Transformations of Independent Variable.**
- **Even and Odd Signals.**
- **Signal Periodicity.**

➢ Signals Transformation (2- Of Dependent Variable)

An issue of fundamental importance in the study of signals and systems is the use of systems to process or manipulate signals. This issue usually involves a combination of some basic operations. In particular, we may identify two classes of operations, as described here.

B. *Operations performed on dependent variables.*

Amplitude scaling Let $x(t)$ denote a continuous-time signal. The signal $y(t)$ resulting from amplitude scaling applied to $x(t)$ is defined by

$$y(t) = cx(t) \qquad (1)$$

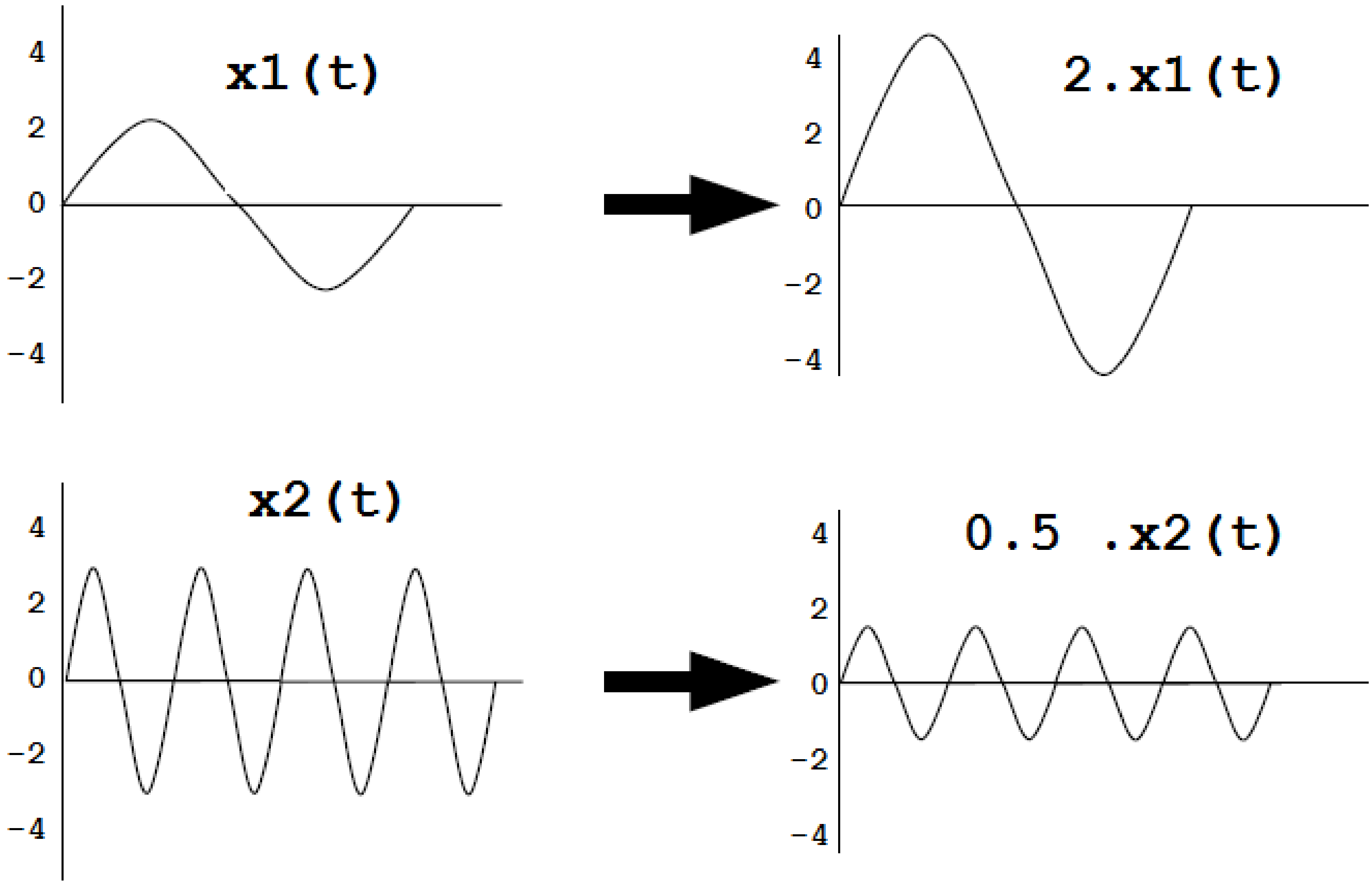
where c is the scaling factor. According to Eq. (1), the value of $y(t)$ is obtained by multiplying the corresponding value of $x(t)$ by the scalar c. A physical example of a device that performs amplitude scaling is an electronic *amplifier*. A resistor also performs amplitude scaling when $x(t)$ is a current, c is the resistance, and $y(t)$ is the output voltage.

In a manner similar to Eq. (1), for discrete-time signals we write

$$y[n] = cx[n]$$

➢ Signals Transformation (2- Of Dependent Variable)

Signal Scaling Example:



➢ Signals Transformation (2- Of Dependent Variable)

Addition Let $x_1(t)$ and $x_2(t)$ denote a pair of continuous-time signals. The signal $y(t)$ obtained by the addition of $x_1(t)$ and $x_2(t)$ is defined by

$$y(t) = x_1(t) + x_2(t) \qquad (2)$$

A physical example of a device that adds signals is an audio *mixer*, which combines music and voice signals.

In a manner similar to Eq. (2), for discrete-time signals we write

$$y[n] = x_1[n] + x_2[n]$$

Multiplication Let $x_1(t)$ and $x_2(t)$ denote a pair of continuous-time signals. The signal $y(t)$ resulting from the multiplication of $x_1(t)$ by $x_2(t)$ is defined by

$$y(t) = x_1(t)x_2(t) \qquad (3)$$

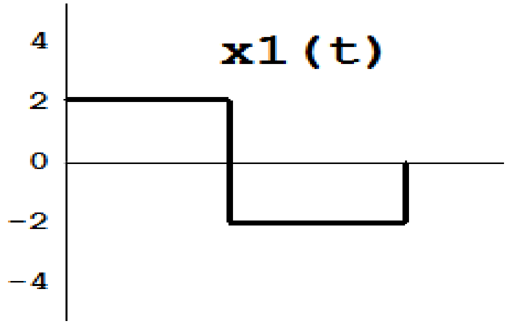
That is, for each prescribed time t the value of $y(t)$ is given by the product of the corresponding values of $x_1(t)$ and $x_2(t)$. A physical example of $y(t)$ is an *AM radio signal*, in which $x_1(t)$ consists of an audio signal plus a dc component, and $x_2(t)$ consists of a sinusoidal signal called a carrier wave.

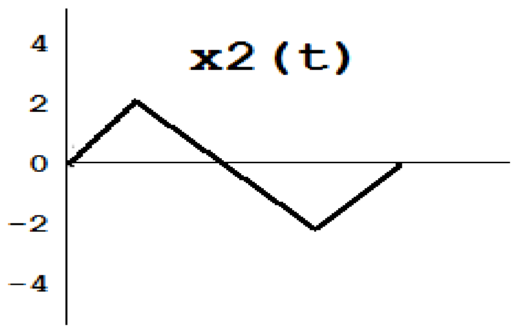
In a manner similar to Eq. (3), for discrete-time signals we write

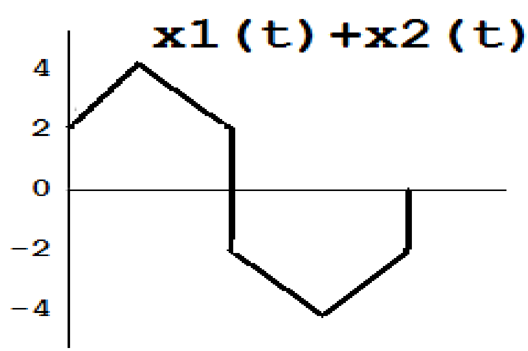
$$y[n] = x_1[n]x_2[n]$$

➢ Signals Transformation (2- Of Dependent Variable)

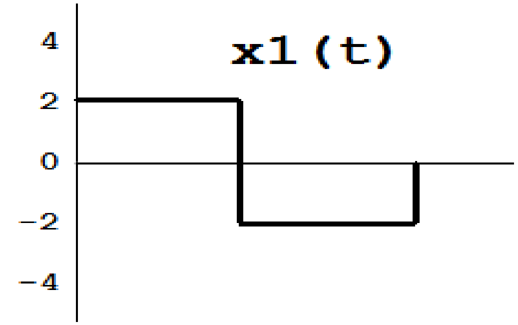
Signal Addition Example:

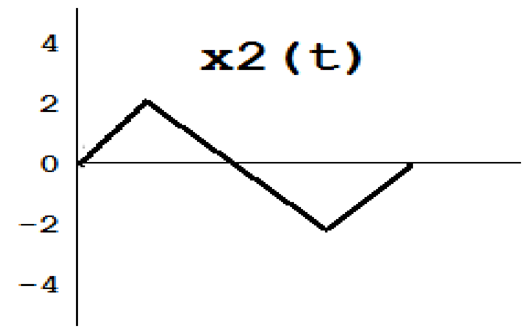


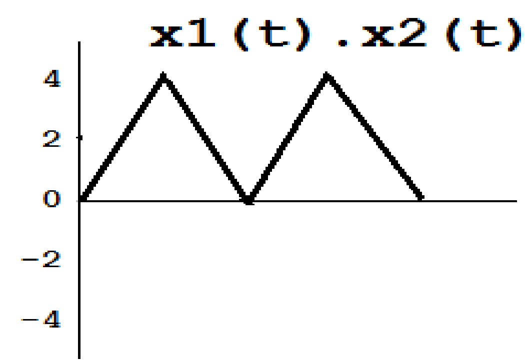




Signal Multiplication Example:







➢ Signals Transformation (2- Of Dependent Variable)

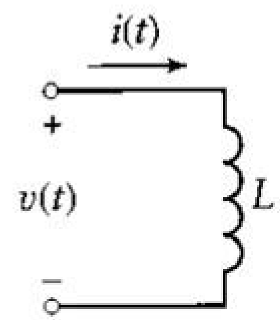
Differentiation Let $x(t)$ denote a continuous-time signal. The derivative of $x(t)$ with respect to time is defined by

$$y(t) = \frac{d}{dt} x(t) \qquad (4)$$

For example, an *inductor* performs differentiation. Let $i(t)$ denote the current flowing through an inductor of inductance L, as shown in Fig.   . The voltage $v(t)$ developed across the inductor is defined by

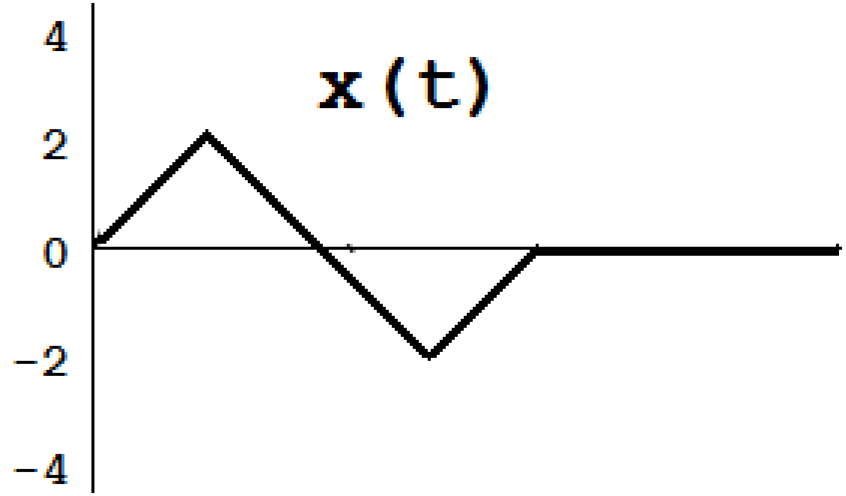
$$v(t) = L \frac{d}{dt} i(t) \qquad (5)$$

An **inductor**, also called a **coil** or **reactor**, is a passive two-terminal electrical component which resists changes in electric current passing through it. The Voltage across the coil is built according to **the rate of change in the input current**.

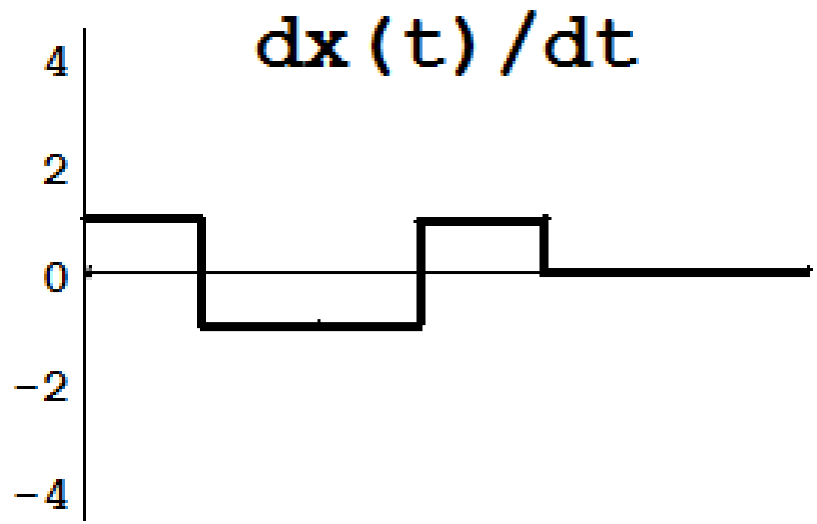


Inductor with current $i(t)$, inducing voltage $v(t)$ across its terminals.

➢ Signals Transformation (2- Of Dependent Variable)



Signal Differentiation Example:



➢ Signals Transformation (2- Of Dependent Variable)

Integration Let $x(t)$ denote a continuous-time signal. The integral of $x(t)$ with respect to time t is defined by

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \qquad (6)$$

where τ is the integration variable. For example, a capacitor performs integration. Let $i(t)$ denote the current flowing through a capacitor of capacitance C, as shown in Fig. 2 The voltage $v(t)$ developed across the capacitor is defined by

$$v(t) = \frac{1}{C} \int_{-\infty}^{t} i(\tau)\, d\tau \qquad (7)$$

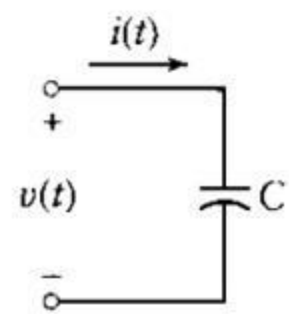
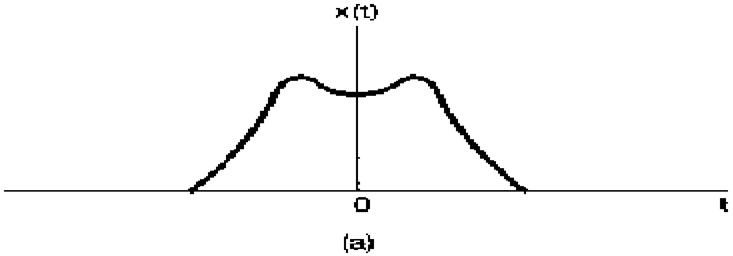


FIGURE 2 Capacitor with voltage $v(t)$ across its terminals, inducing current $i(t)$.

➢ Even and Odd Signals

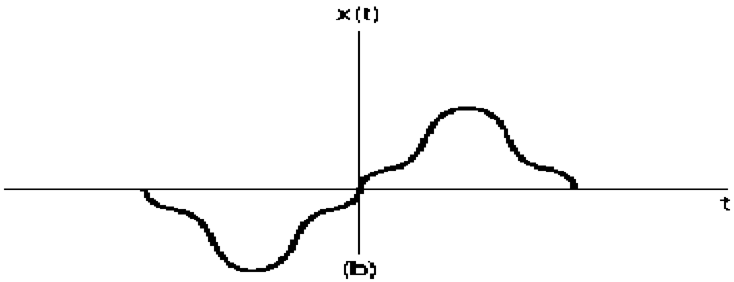
The signal is called “Even” if
it is identical to its time-reverse.
i.e. The even signal has the property that:

$$x(t) = x(-t)$$



The signal is called “Odd” if
it is mirrored around the point (0,0)
odd signal has the property that:

$$x(t) = -x(-t)$$



Odd signals must be zero at t=0

Any signal can be written as the sum of an odd signal and an even signal

$$x(t) = x_e(t) + x_o(t)$$

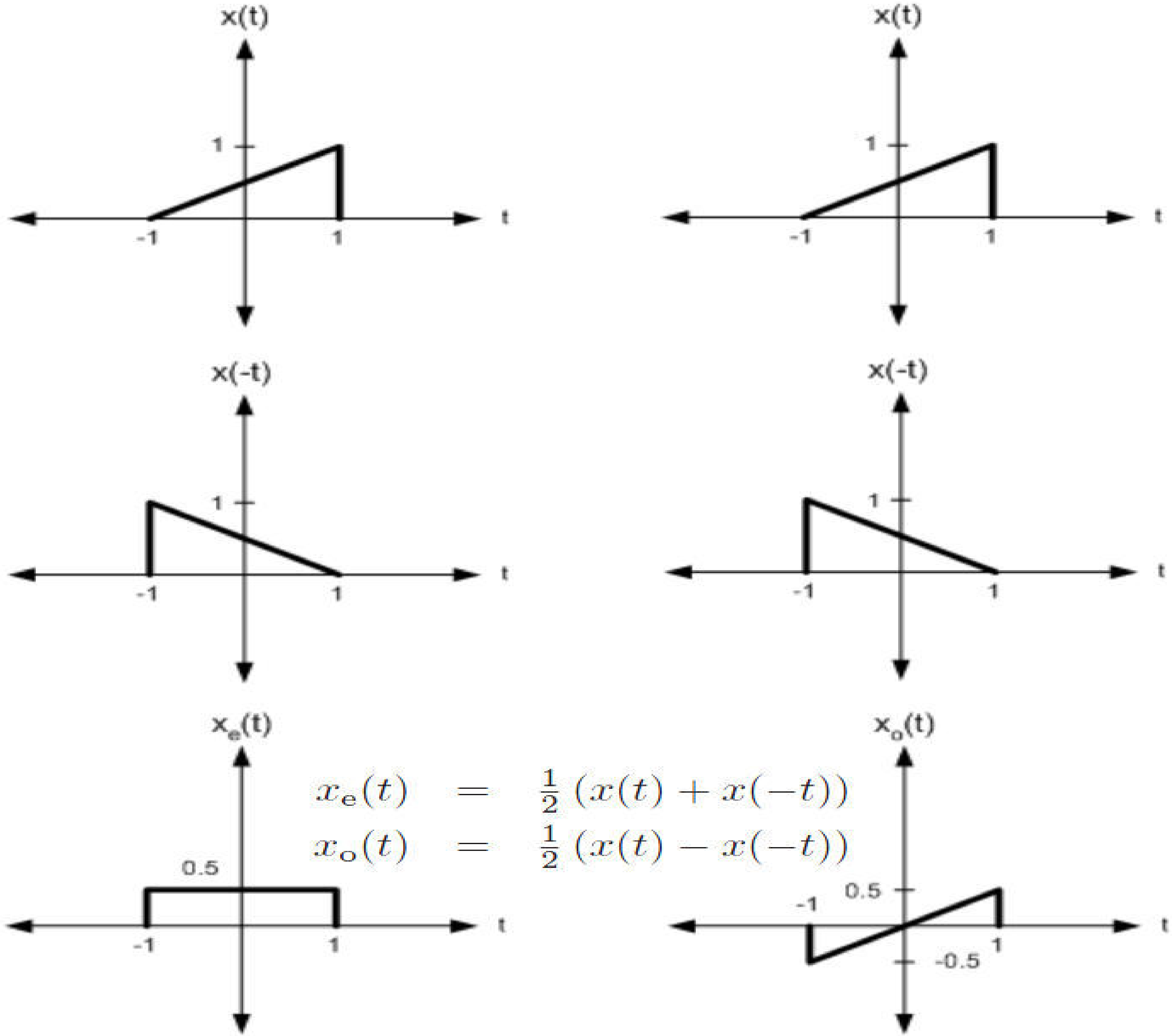
Where,

$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right) \longrightarrow \mathcal{E}v\{x(t)\}$$

$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right) \longrightarrow \mathcal{O}d\{x(t)\}$$

➢ Even and Odd Signals

Example:



$$x_e(t) = \frac{1}{2}\left(x(t) + x(-t)\right)$$
$$x_o(t) = \frac{1}{2}\left(x(t) - x(-t)\right)$$

➢ Even and Odd Signals

Note that

the product of two even signals or of two odd signals is an even signal

and that

the product of an even signal and an odd signal is an odd signal

➢ Even and Odd Signals

EXAMPLE

Determine whether or not the following functions are even or odd:

(a) $x(t) = t \cos t$
(b) $x(t) = \cos t \sin^2 t$
(c) $x(t) = t \sin t$

SOLUTION

(a) We have an odd function t multiplied by an even function $\cos t$. The product of an odd function with an even function is odd, so in this case $x(t)$ is odd.

(b) Let's look at this one in two steps. First, we note that $\cos t$ is even. Now $\sin t$ is odd, and we have $\sin^2 t = \sin t \sin t$, which is an odd function times an odd function. An odd function times an odd function is even, and so $\sin^2 t$ is even. In the end we have $x(t) = \cos t \sin^2 t$, which is an even function times an even function, so this signal is even.

(c) In this case we have an odd function times an odd function. We have $x(-t) = -t \sin(-t) = t \sin t$, so this is an even function.

➢ Periodic Signals

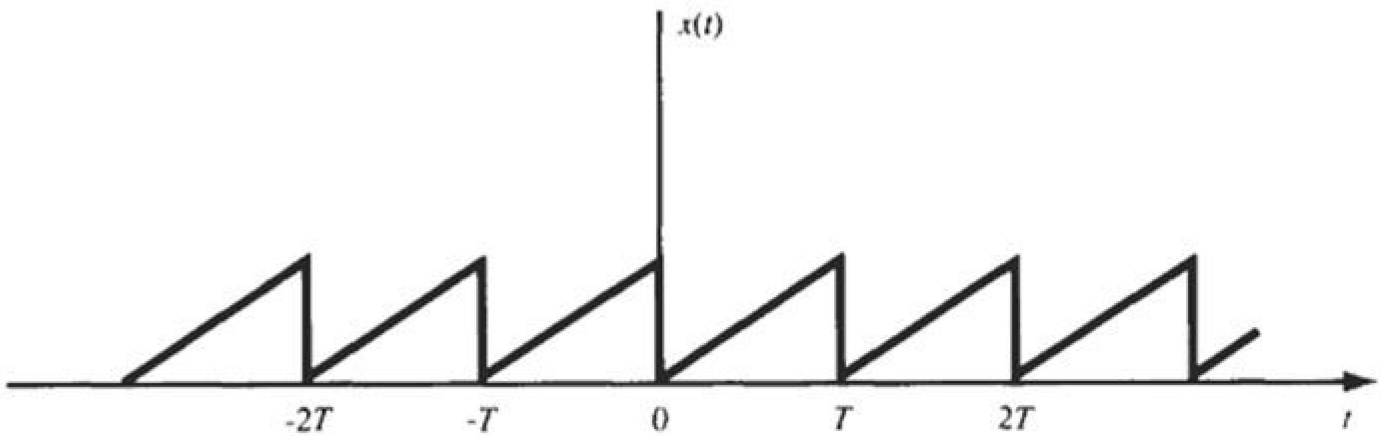
A continuous-time signal $x(t)$ is said to be *periodic with period* T if there is a positive nonzero value of T for which

$$x(t+T) = x(t) \qquad \text{all } t \tag{1}$$

An example of such a signal is given in Fig. (*a*). From Eq. (1) or Fig. (*a*) it follows that

$$x(t+mT) = x(t) \tag{2}$$

for all t and any integer m. The *fundamental period* T_0 of $x(t)$ is the smallest positive value of T for which Eq. (1) holds.

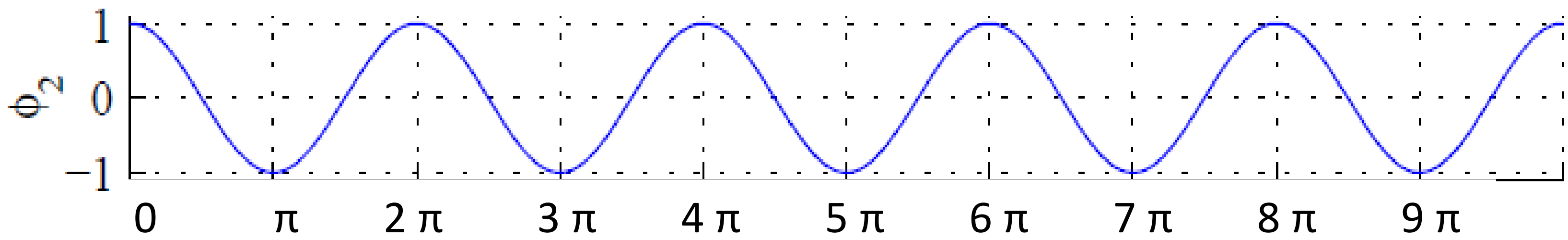


(*a*)

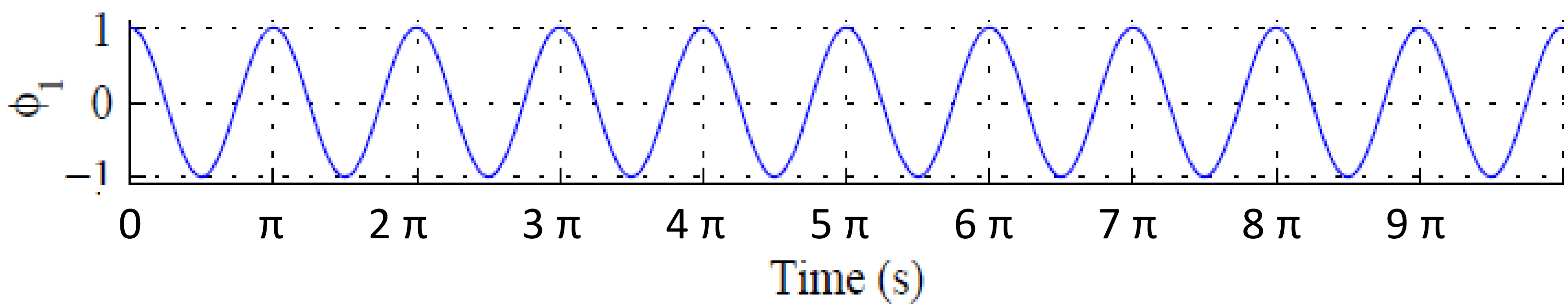
➢ Periodic Signals

Examples:

Cos(t) $T_o = 2\pi$



Cos(2t) $T_o = \pi$



Periodic Signals

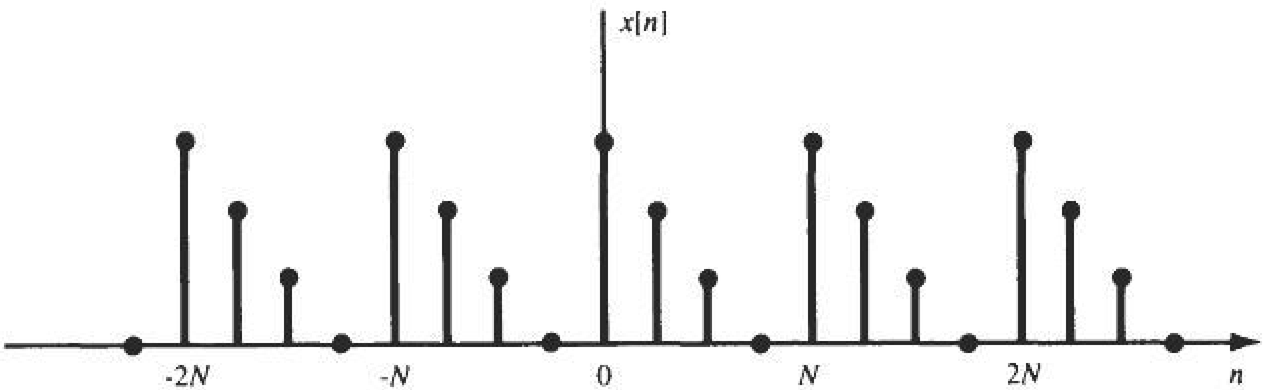
Periodic discrete-time signals are defined analogously. A sequence (discrete-time signal) $x[n]$ is *periodic with period N* if there is a positive integer N for which

$$x[n+N]=x[n] \qquad \text{all } n \qquad (3)$$

An example of such a sequence is given in Fig. (*b*). From Eq. (3) and Fig. (*b*) it follows that

$$x[n+mN]=x[n] \qquad (4)$$

for all n and any integer m. The fundamental period N_0 of $x[n]$ is the smallest positive integer N for which Eq. (3) holds.



(*b*)

Periodic Signals

Note 1: the fundamental period of a constant continuous-time signal is undefined! As there is no smallest positive value of T. if you take any choice you find another smaller one e.g. 0.1, 0.01, 0.001, 0.0001…etc

Note 2: the fundamental period of a constant discrete-time signal is 1.

Note 3: the uniform sampling of a periodic continuous-time signal may not be periodic!

Note 4: the sum of two continuous-time periodic signals may not be periodic BUT, the sum of two discrete-time periodic signals is always periodic

Note 5: the sum of two periodic signals is periodic only if :

K T1 = L T2 ➔ $$\frac{T1}{T2} = \frac{L}{K} = \frac{Integer}{Integer} = Rational\ \ Number$$

➢ Periodic Signals

LCM: the Least Common Multiplier

To compute the LCM of a list of numbers:

1- begins by listing all of the numbers vertically in a table

2- The process begins by dividing all of the numbers by 2. If a number does not divide evenly, just rewrite the number again.

3- Now, check if 2 divides again.

4- Once 2 no longer divides, divide by 3.

5- If 3 no longer divides, try 5 then 7 ... etc (i.e. Prime numbers). Keep going until all of the numbers have been reduced to 1.

	2	2	3	7
4	2	1	1	1
7	7	7	7	1
12	6	3	1	1
21	21	21	7	1
42	21	21	7	1

Note: if any prime can not divide any row it is neglected. Like 5 in the above mentioned example.

6- Now, LCM is the multiplications of the numbers in the top row: **2 x 2 x 3 x 7 = 84**

Signals and Systems

Lecture # 4

Exponentials and Sinusoidal Signals Relationship

Prepared by:

Dr. Mohammed Refaey

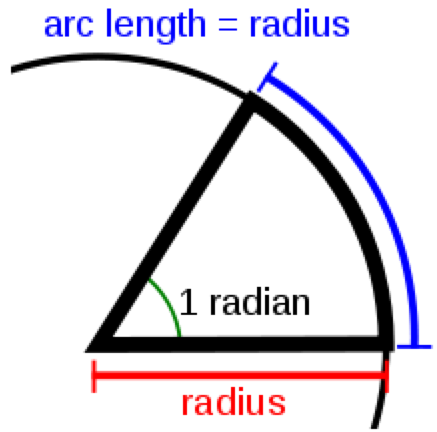
Topics of the lecture:

- **Fundamental Concepts.**
- **Exponential and Sinusoidal Signals Relationship.**

➢ Fundamental Concepts.

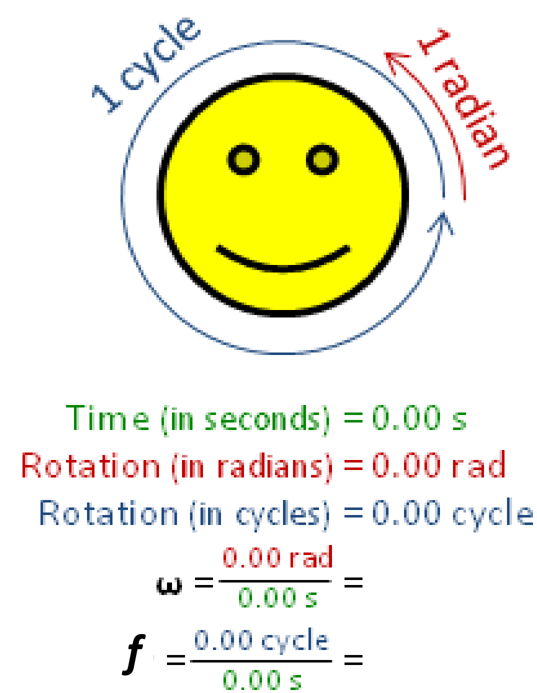
Let us first to know what is meant by the radians? And what is the difference between the angular frequency (w) and the frequency (f)?

- **Radian is the ratio between the length of an arc and its radius. The radian is the standard unit of angular measure in many areas of math.**



- **Angular Frequency (*w*) is the number of radians per second. rad/sec**

- **Frequency (*f*): is the number of cycles per second. cyc/sec**



$$f = \frac{cyc}{sec} = \frac{cyc}{rad} \times \frac{rad}{sec} = \frac{1}{2\pi} \times w = \frac{w}{2\pi} \Rightarrow w = 2\pi f$$

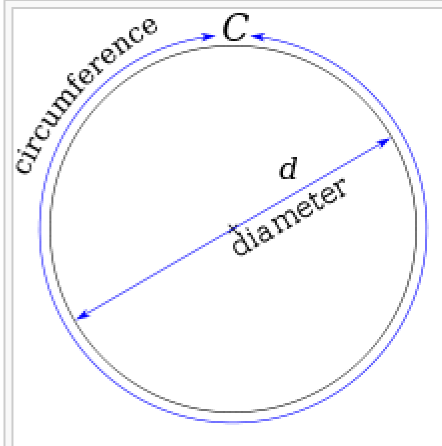
$$as\ \frac{cyc(in\ one\ second)}{rad(in\ one\ second)} = \frac{one\ cycle}{no.of\ radians\ per\ cycle} = \frac{1}{2\pi}$$

$$\frac{\theta(=1\,rad)}{arc(=r)} = \frac{2\pi}{circumference} \Rightarrow circumference = 2\pi r$$

$$as\ \frac{\theta 1}{arc\ of\ \theta 1} = \frac{\theta 2}{arc\ of\ \theta 2}$$

➢ Fundamental Concepts.

The constant π **(pi):** is the ratio of a circle's circumference to its diameter = 3.14159265...



The circumference of a circle is slightly more than three times as long as its diameter. The exact ratio is called π.

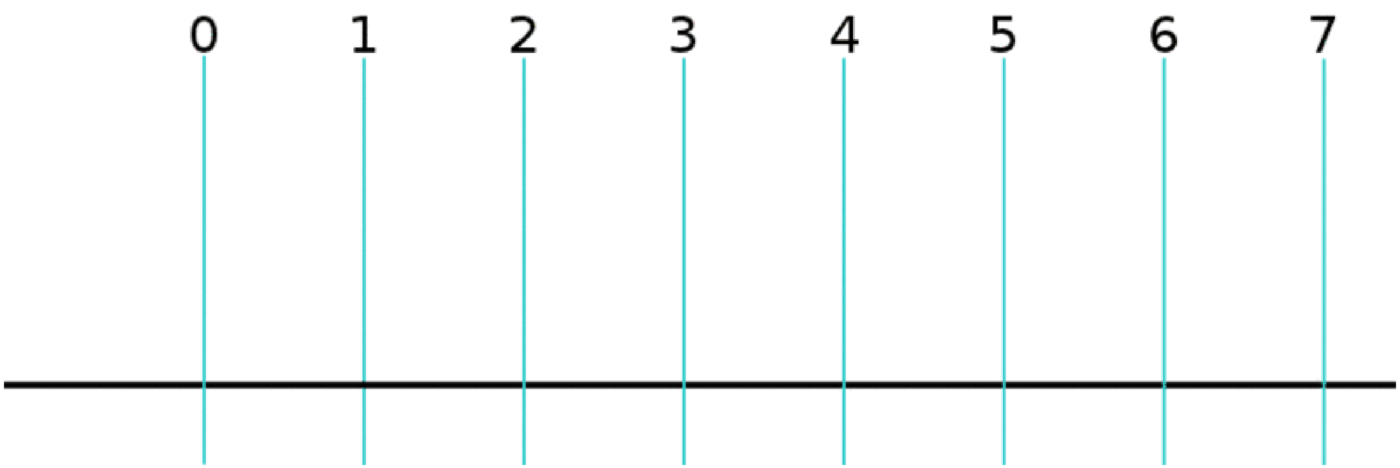
It is not 22/7 !

$$\frac{22}{7} = 3.\overline{142\,857},$$

$$\pi \approx 3.141\,592\,65\ldots$$

<u>It is NOT rational number!</u>

Its decimal representation never ends and never settles into a permanent repeating pattern.



When a circle's radius is 1 unit, its circumference is 2π.

➢ Exponential Signals and sinusoidal Signals

The Relationship between the Complex Exponential Signals and Sinusoid Signals:

The *Complex Exponential Signals has the form*: $e^{jwt} \quad OR \quad e^{jwn}$

$$e^{jwt} = 1e^{j\angle wt} = \cos(wt) + j\sin(wt)$$

$$at \quad wt = 0, \quad e^{jwt} = 1 + 0j = 1$$

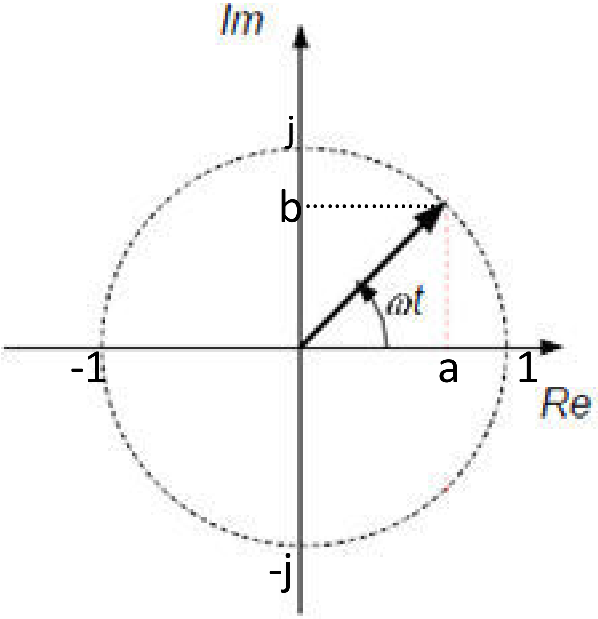
$$at \quad wt = \frac{\pi}{2}, \quad e^{jwt} = 0 + 1j = j$$

$$at \quad wt = \pi, \quad e^{jwt} = -1 + 0j = -1$$

$$at \quad wt = \frac{3\pi}{2}, \quad e^{jwt} = 0 - 1j = -j$$

$$at \quad wt = 2\pi, \quad e^{jwt} = 1 + 0j = 1$$

$$\ldots and \ so \ on$$



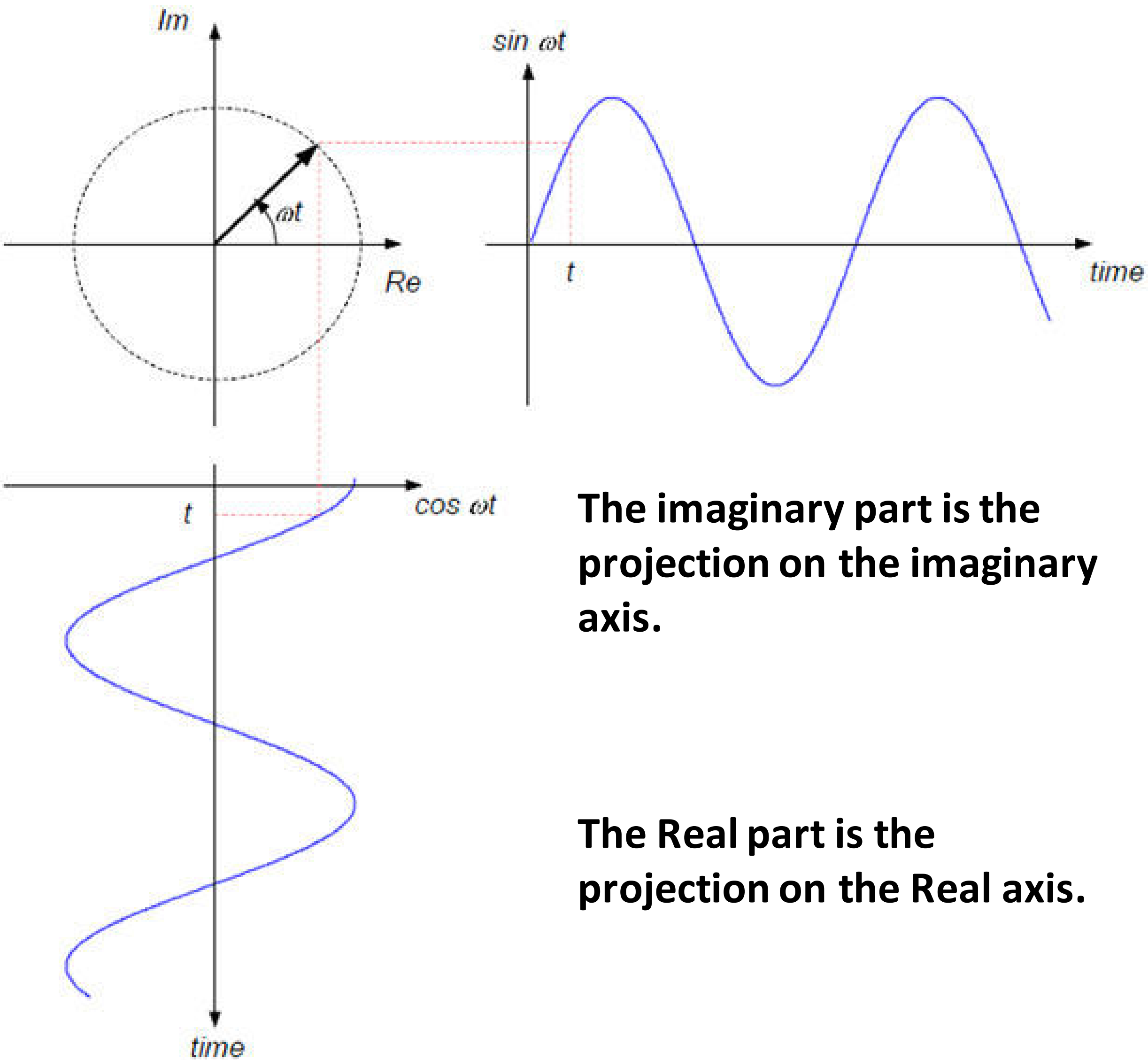
As the angle (wt) is increased, either by increasing the rotating frequency (w) or as the time (t) goes, the point representing e^{jwt} is rotating around the unit circle.

➢ Exponential Signals and sinusoidal Signals

See the first Flash Video. (http://www.fourier-series.com/fourierseries2/flash_programs/cong/index.html)

Play the slide to see the interactive flash

➢ Exponential Signals and sinusoidal Signals

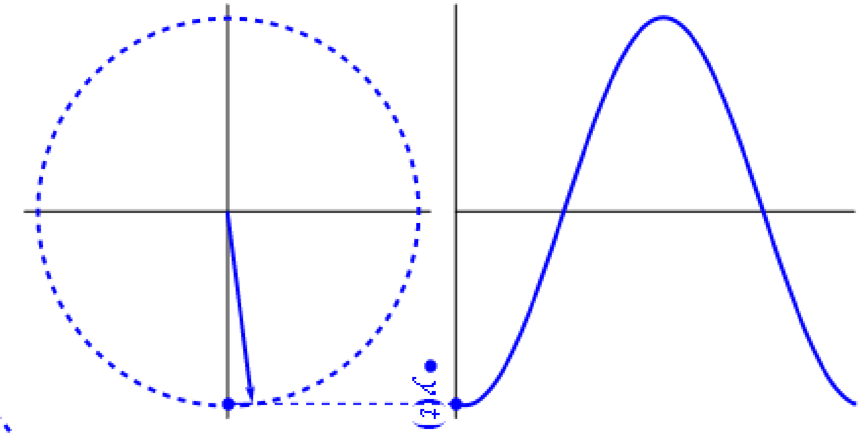


The relationship between the complex exponential and the sinusoidal signals.

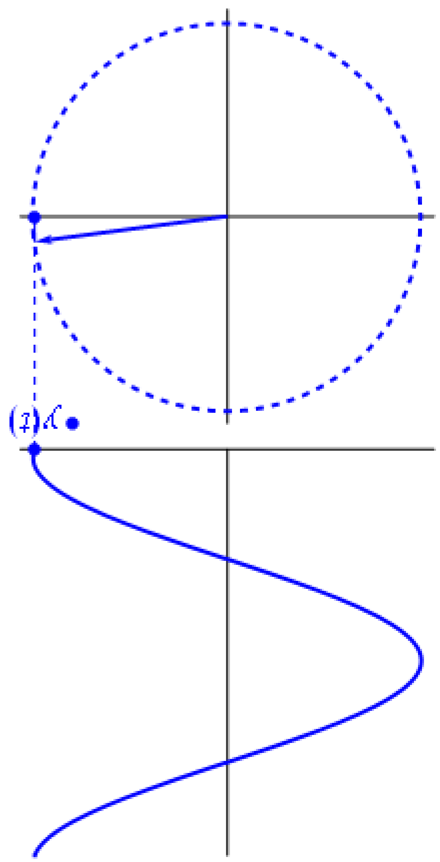
The imaginary part is the projection on the imaginary axis.

The Real part is the projection on the Real axis.

➢ Exponential Signals and sinusoidal Signals



The relationship between the complex exponential and the sinusoidal signals.



The imaginary part is the projection on the imaginary axis.

The Real part is the projection on the Real axis.

➢ Exponential Signals and sinusoidal Signals

See the second Flash Video. (http://www.fourier-series.com/fourierseries2/flash_programs/cong_with_time/index.html)

Play the slide to see the interactive flash

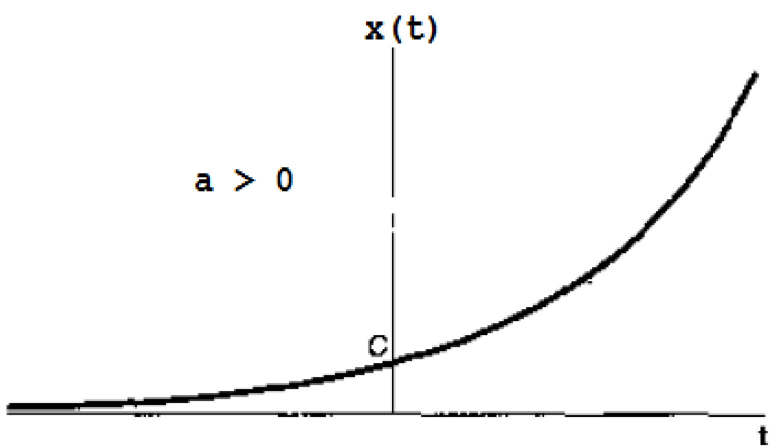
➢ Exponential Signals and sinusoidal Signals (continuous-time case)

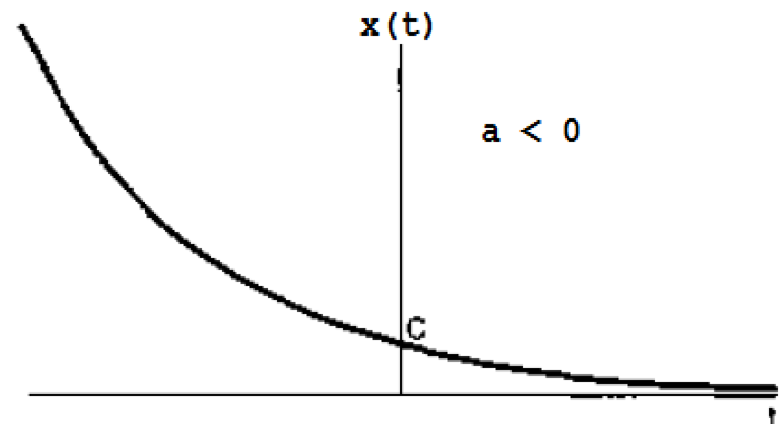
$$The\ general\ form \Rightarrow \quad x(t) = Ce^{at}$$

There are three important cases for ***C*** and ***a***.

<u>Case 1: Real Exponential Continuous-Time Signals</u>

If (*C*) and (*a*) are both real numbers then x(t) is called a real exponential signal.





e.g. Chain Reactions in atomic explosion **e.g. damped mechanical systems**

If (*C*) is < 0 then x(t) will be mirrored around the horizontal t-axis

If (*a*) is = 0 then x(t) will be constant signal

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

$$The\ general\ form\ \Rightarrow\quad x(t) = Ce^{at}$$

Case 2: periodic complex exponentials (and sinusoidal signals)

If (*a*) is purely imaginary, i.e. $a =$ jw. then x(t) = C e^{jwt} , and by ignoring the scaling factor, which not affect the periodicity property (it may only change phase and/or the magnitude), then: x(t) = e^{jwt}, which is always periodic as shown in the previous flash videos.

$$as\ the\ signal\quad x(t) = e^{jwt}\quad is\ periodic$$

$$\therefore\qquad x(t+T) = x(t)$$

$$\therefore\quad e^{jw(t+T)} = e^{jwt}.e^{jwT} = e^{jwt}$$

$$\Rightarrow\quad e^{jwT} = 1 \quad\Rightarrow\quad as\ e^{jwT} = \cos(wT) + j\sin(wT) \quad\Rightarrow\quad wT = k2\pi$$

$$\Rightarrow\quad T = k\frac{2\pi}{w},\qquad k\ is\ integer$$

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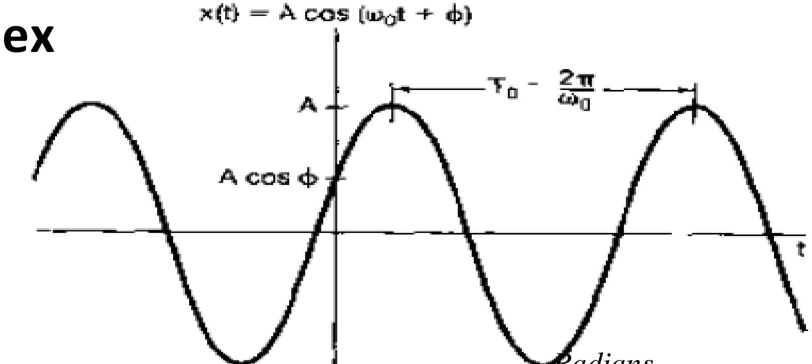
As the fundamental period is the *smallest positive* value of *T* for which $e^{jwT} = 1\quad OR\quad w_o T_o = 2\pi$ **:**

$$\overset{k=1}{\Rightarrow}\quad T_o = \frac{2\pi}{|w_o|} \quad\Rightarrow\quad (T_o\ for\ e^{jwt}) = (T_o\ for\ e^{-jwt}) \quad as\ |w_o| = |-w_o|$$

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

A closely related signals to the periodic complex exponential signal are the sinusoidal signals, a cosine sinusoidal general form is:



x(t)=A cos(wt + Ø)

Where (*A*) is the maximum amplitude, (*w*) is the angular frequency (rad/sec) (*w=2πf*), (*f*) the regular frequency (cyc/sec), and (*Ø*) is the phase angle (radians).

$$Angular\ Frequency = w = \frac{Radians}{Seconds}$$

if we have the smallest time of one cycle $(= T_o\ Seconds)$

then we have the smallest angle of one cycle $(= 2\pi\ Radians)$

then we have the fundamental frequency $w_o = \dfrac{2\pi}{T_o} \Rightarrow T_o = \dfrac{2\pi}{w_o}$

according to Euler's Formula: $\quad e^{jwt} = \cos(wt) + j\sin(wt)$

$$\therefore e^{-jwt} = \cos(-wt) + j\sin(-wt) = \cos(wt) - j\sin(wt)$$

$$\therefore A\cos(wt+\phi) = \frac{A}{2}e^{j(wt+\phi)} + \frac{A}{2}e^{-j(wt+\phi)},\ and\quad A\sin(wt+\phi) = \frac{A}{2j}e^{j(wt+\phi)} - \frac{A}{2j}e^{-j(wt+\phi)}$$

$$OR:\quad A\cos(wt+\phi) = A\,\mathrm{Re}\{e^{j(wt+\phi)}\}\ ,\ and\quad A\sin(wt+\phi) = A\,\mathrm{Im}\{e^{j(wt+\phi)}\}$$

ALL these signals can be written in terms of each other and have the same fundamental

$$period\quad T_o = \frac{2\pi}{|w_o|},\qquad where\ w_o\ is\ the\ fundamental\ frequency$$

they are also power signals?! Proof that...{applications: LC circuit, and acoustic signals}

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

Harmonically-Related Complex Exponentials:

is a set of periodic exponentials all of which are periodic with a common period (T_o)

$$e^{jwt} \quad to\ be\ periodic \quad \Rightarrow \quad e^{jwT} = 1$$

$$\Rightarrow \therefore \ wT = k2\pi \ ; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$\therefore w = k\frac{2\pi}{T}$$

$$the\ \ fundamental\ \ frequency\ (w_o)\ is\ the\ smallest\ value\ of\ \ (w)$$

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تردد أى أكثر من اشارة

$$\overset{k=1}{\Rightarrow} w_o = \frac{2\pi}{T_o} \quad , \quad as\, T_o\ is\ the\ Fundamental\ Period\ of\ \ signal\ having\ w_o\ and\ w_o T_o = 2\pi$$

$$\Rightarrow w = k\, w_o \qquad ; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$this\ represents\ \ a\ \underline{\underline{set}}\ of\ \ signals\ each\ one\ of\ \ them\ has\ a$$

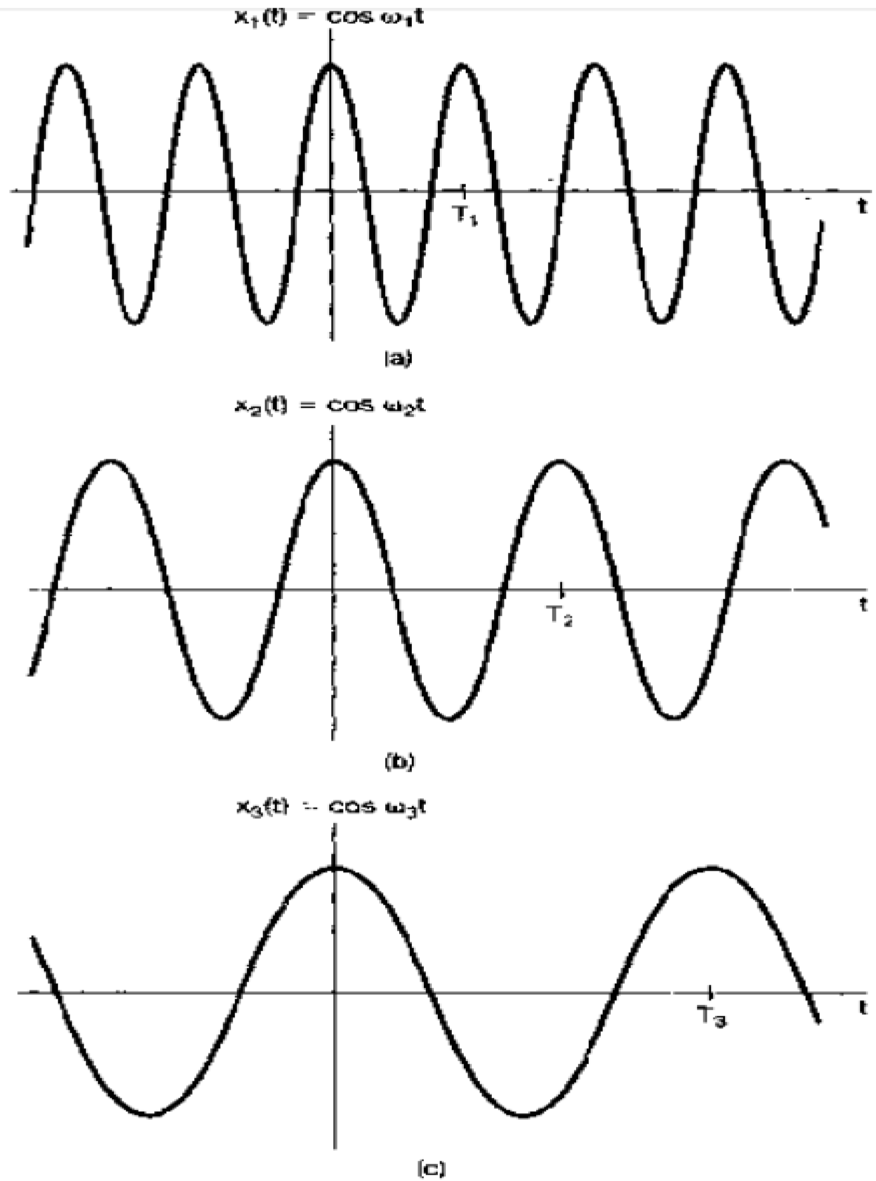
$$frequency\ (w)\ that\ is\ multiple\ of\ \ one\ common\ frequency\ (w_o)$$

$$\phi_k(t) = e^{jkw_o t} \quad ; \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

$$the\ \ fundamental\ \ period\ of\ \ k^{th}\ \ harmonic\ is \quad T_{o_k} = \frac{2\pi}{|kw_o|} = \frac{T_o}{|k|}$$

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

You can use the previous flashes for interactive clarification of the relationship between the frequency and the fundamental period of continuous-time sinusoids



Relationship between the fundamental frequency and period for continuous-time sinusoidal signals; here, $\omega_1 > \omega_2 > \omega_3$, which implies that $T_1 < T_2 < T_3$.

➢ Exponential Signals and sinusoidal Signals (continuous-time case)

$$The\ general\ \ form \Rightarrow \quad x(t) = Ce^{at}$$

Case 3: General Continuous-time Complex Exponential signals:

$$let\ (C)\ and\ (a)\ both\ as\ complex\ numbers$$

$$let \quad C = |C|e^{j\theta}, \quad and \quad a = r + jw_o$$

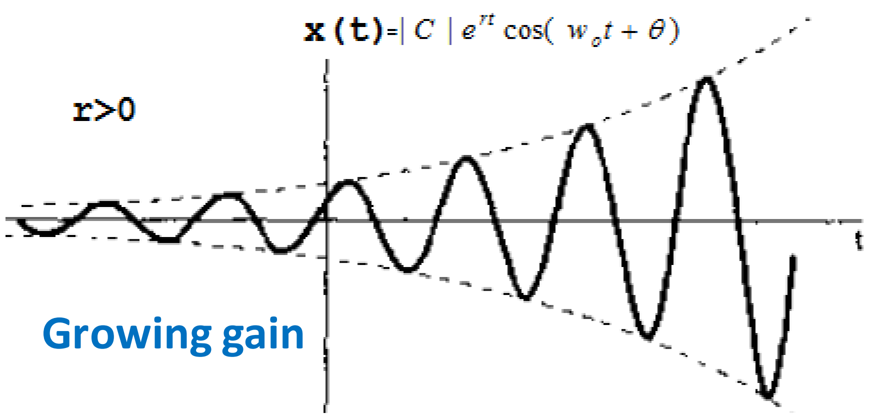
$$\Rightarrow \quad x(t) = Ce^{at} = |C|e^{j\theta}e^{(r+jw_o)t}$$

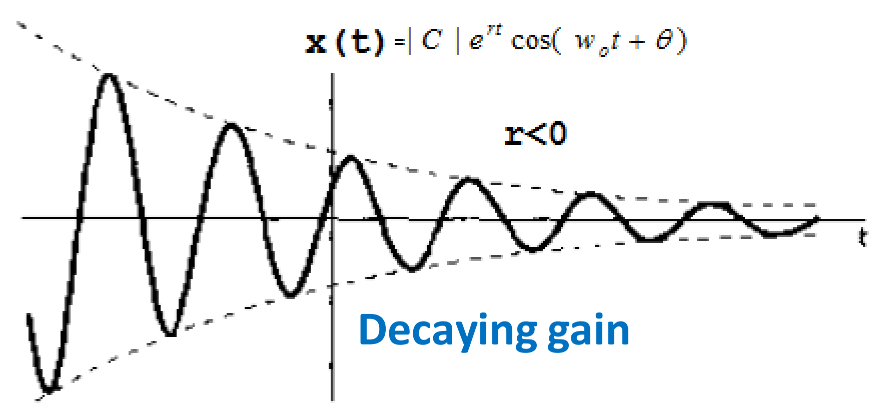
$$= |C|e^{j\theta}e^{rt}e^{jw_o t} = \quad |C|e^{rt}e^{j(w_o t+\theta)}$$

$$= |C|e^{rt}\{\cos(w_o t+\theta) + j\sin(w_o t+\theta)\}$$

$$= variable\ positive\ gain \times periodic\ signal$$

- As *a* is in the power so rectangular form is more suitable, while the polar form of C is more suitable to be able to add the phases together!





➢ Exponential Signals and sinusoidal Signals (discrete-time case)

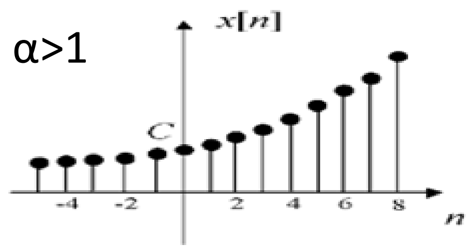
$$The\ general\ form \Rightarrow \quad x[n] = C\alpha^n = Ce^{\beta n}$$

Case 1: Real Exponential Discrete-Time Signals

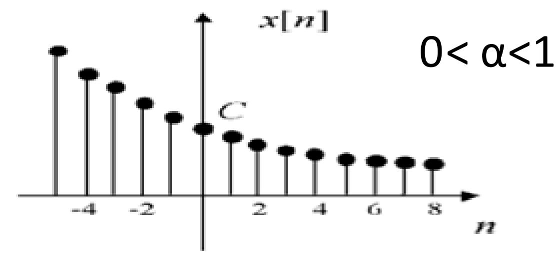
If (C) and (α) are both real numbers then x[n] is called a real exponential signal.

α is real is almost as if β is real

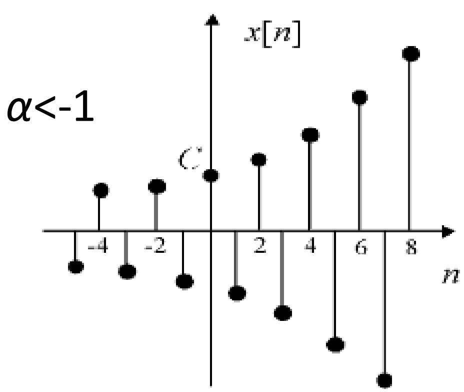
As e is real and real to the power real is real



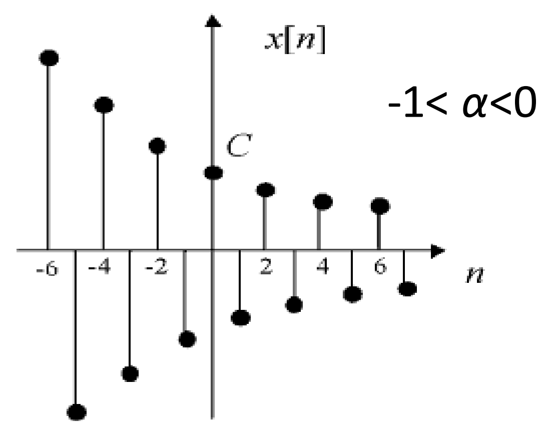
Discrete-time exponential signal growing unbounded with time.



Discrete-time exponential signal tapering off to zero with time.



Discrete-time exponential signal alternating and growing unbounded with time.



Discrete-time exponential signal alternating and tapering off to zero with time.

➢ Exponential Signals and sinusoidal Signals (discrete-time case)

$$\text{The general form} \Rightarrow \quad x[n] = C\alpha^{n} = Ce^{\beta n}$$

Case 2: Discrete-Time Complex Exponentials:

If (β) is purely imaginary (jw) and ignoring the scaling factor (C) → $x[n]=e^{jwn}$

$$\text{As before, according to Euler's Formula}$$

$$e^{jwn} = \cos(wn) + j\sin(wn)$$

$$\text{and} \quad A\cos(wn+\phi) = \frac{1}{2}\left\{e^{j(wn+\phi)} + e^{-j(wn+\phi)}\right\}$$

$$= \mathrm{Re}\left\{e^{j(wn+\phi)}\right\}$$

Again the signals $x[n]=e^{jwn}$ and $A\cos(wn + \emptyset)$ have same periodicity properties and parameters BUT THEY ARE NOT NECSSARILY PERIODIC (we will see it soon).

➢ Exponential Signals and sinusoidal Signals (discrete-time case)

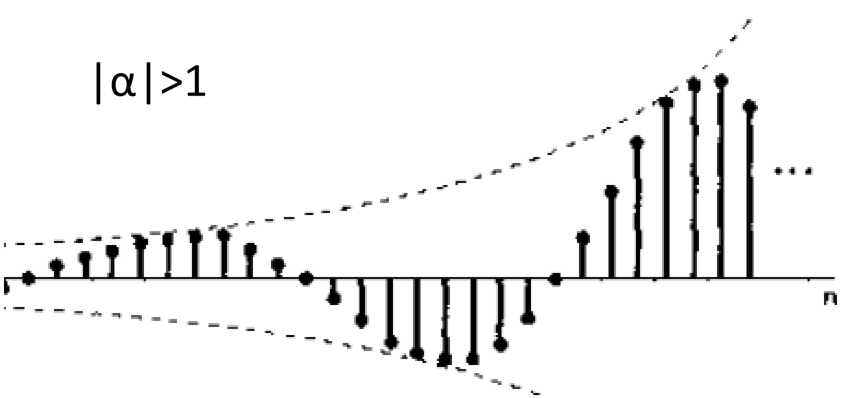
$$The\ general\ form \Rightarrow \quad x[n] = C\alpha^n = Ce^{\beta n}$$

<u>Case 3: General discrete-time complex exponentials:</u>

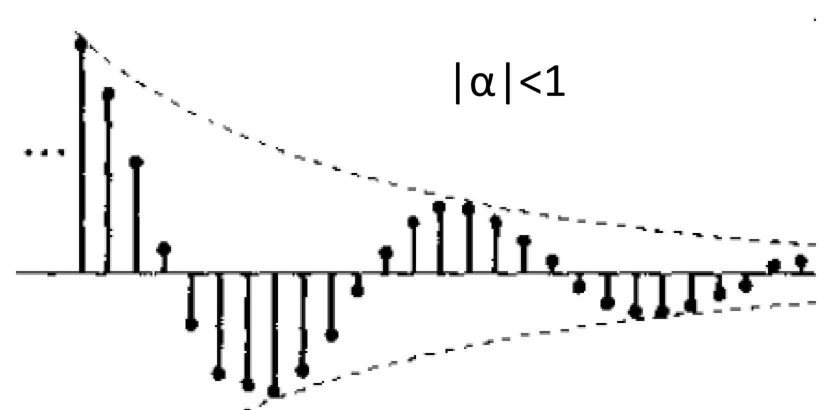
If (C) and (α) are both complex (in polar form), i.e. let $C = |c|e^{j\theta}$ and $\alpha = |\alpha|e^{jw}$

then $x[n] = |c|\,|\alpha|^n e^{j(wn+\theta)} = |c|\,|\alpha|^n \cos(wn+\theta) + j|c|\,|\alpha|^n \sin(wn+\theta)$

= variable positive gain x Sinusoidal signals



(a)



(b)

➢ Exponential Signals and sinusoidal Signals (discrete-time case)

Periodicity properties of discrete-time complex exponentials:

$consider\ the\ signal \qquad e^{jwn}$

$let \quad w = w_o + 2\pi \quad \Rightarrow$

$$e^{jwn} = \quad e^{j(w_o+2\pi)n} = \quad e^{jw_o n} e^{j2\pi n}$$

$2\pi n = integer \quad multiple \quad of \quad 2\pi$ $\quad for\ all\ values\ of\ n \Rightarrow \quad e^{j2\pi n} = 1$

$\Rightarrow \quad e^{j(w_o+2\pi)n} = \quad e^{jw_o n}$

$i.e. \quad the\ discrete - time\ complex\ exponentials\ seperated\ by\ (2\pi)$

$in\ frequency\ are\ identical$

$\Rightarrow then\ this\ means\ that\ there\ are\ only\ a\ range$

$of \quad 2\pi \quad for \quad w \quad of \quad e^{jwn} \quad to\ have\ distinct\ /\ different\ signals$

$commonly \Rightarrow \quad -\pi < w < \pi \quad OR \quad 0 < w < 2\pi$

$for\ continuous\ time\ exponential\ signals\ this\ is\ not\ the\ case\ as$

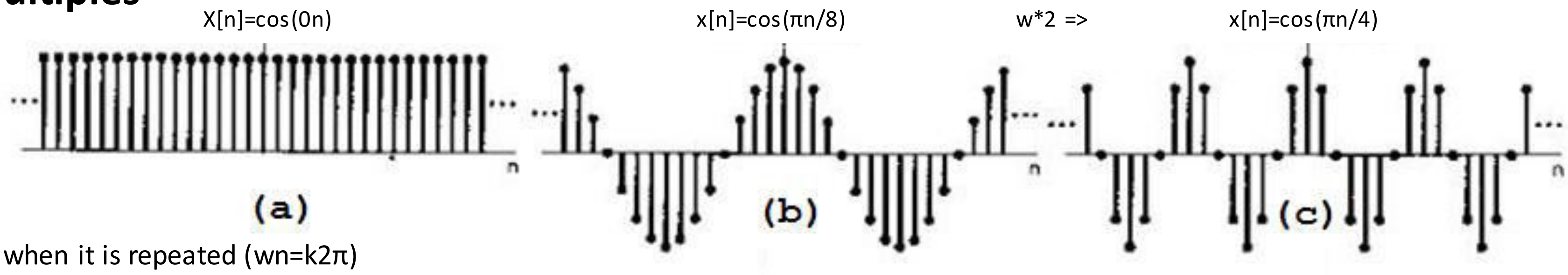
$e^{j2\pi t} \neq 1 \quad for\ all\ values\ of\ t, \quad as\ t\ is\ not\ an\ integer$

Even when t=integer it does not mean same signals but it means different signals meet at some values of t and they are totally different (one faster than another).

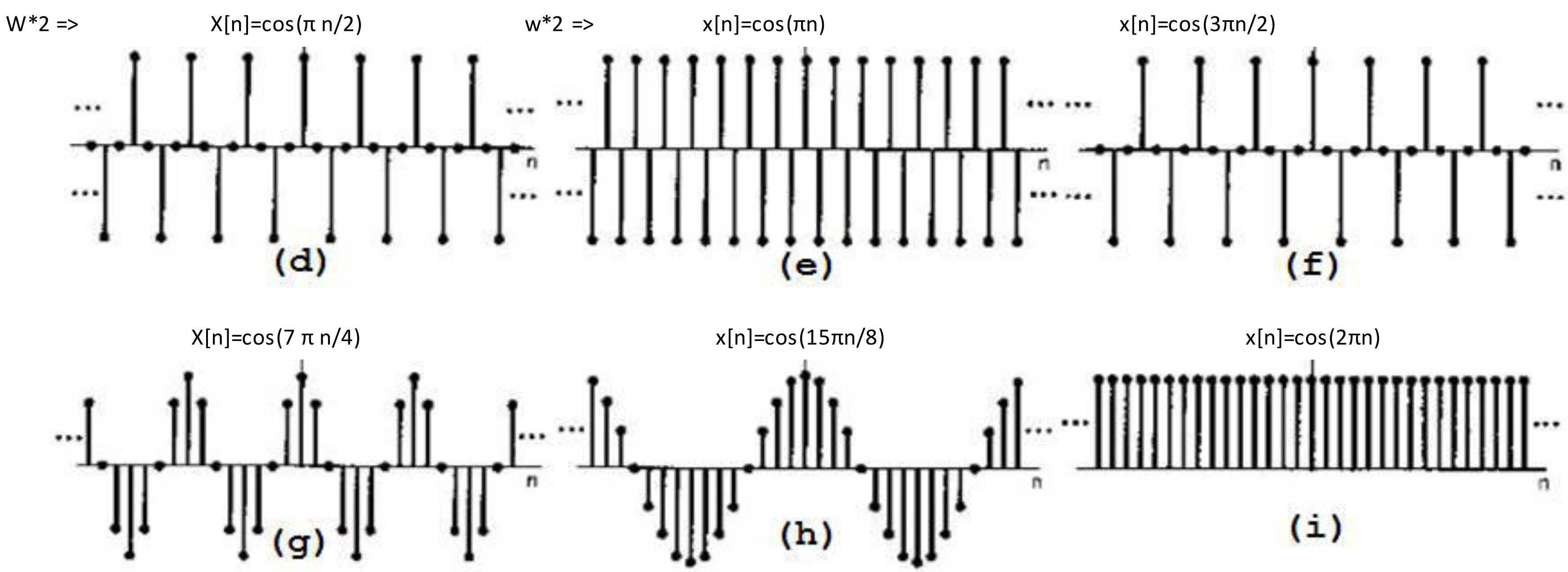
➢ Exponential Signals and sinusoidal Signals (discrete-time case)

Periodicity properties of discrete-time complex exponentials:

Note that the rate of fluctuation increases then decreases with the increase in w the high frequencies are exist around π and low frequencies exist around 2 π and their multiples



- Check when it is repeated (wn=k2π)
i.e. after how many samples



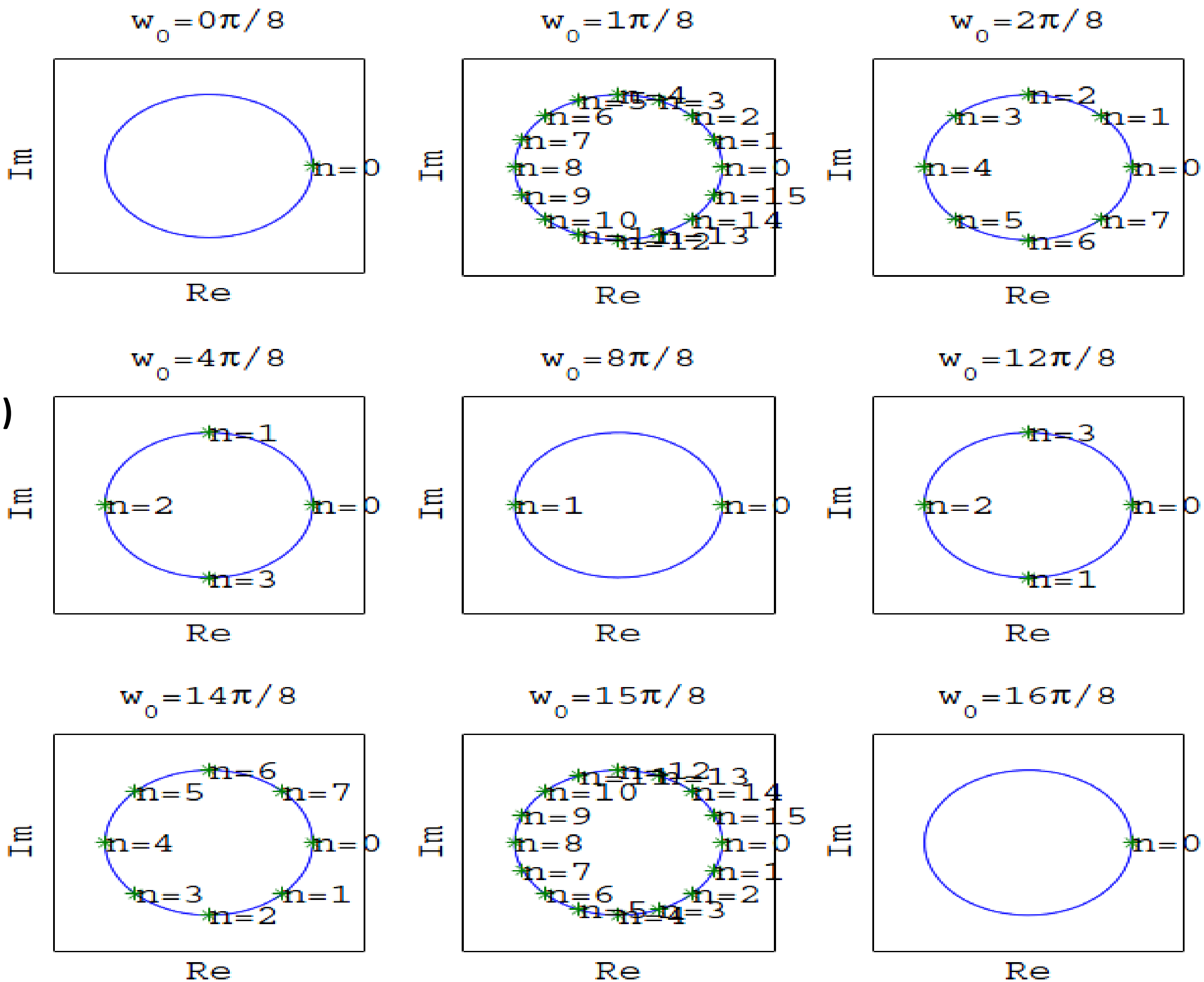
➢ Exponential Signals and sinusoidal Signals (discrete-time case)

Periodicity properties of discrete-time complex exponentials:

Number of Samples per cycle

=

the number (n) that makes $n W_o = k2\pi$



Exponential Signals and sinusoidal Signals (discrete-time case)

Periodicity properties of discrete-time complex exponentials:

$$The\ function\quad e^{jwn}\quad to\ be\ periodic$$

$$\therefore e^{jw_o(n+N)} = e^{jw_o n}e^{jw_o N} = e^{jw_o n}$$

$$e^{jw_o N} = 1 \;\Rightarrow\; w_o N = m2\pi\ ; \quad for\ \underline{\underline{m}}\ is\ an\ integer$$

$$\therefore the\ condition\ of\ e^{jw_o n}\ to\ be\ periodic\ is\ \left.\frac{w_o}{2\pi} = \frac{m}{N}\right\}\ to\ be\ simplified$$

$$rational\ number\{\ simplified \equiv no\ common\ factors\ between\ \underline{\underline{m}}\ and\ \underline{\underline{N}}\}$$

$$if\ this\ happened\ the\ signal\ e^{jw_o n}\ will\ be\ periodic\ and\ the\ fundamental$$

$$period\ will\ be\ (N),\ otherwise\ it\ will\ be\ not\ periodic$$

$$the\ same\ is\ true\ for\ discrete-time\ sinusoids$$

if you have a signal that is composed of a combination of discrete $-$ *time complex exponentials or sinusoidals then you should check every subsignal individually and if you find them ALL periodic with fundamentals {N1,N2,N3,...} then the container signal will be periodic with fundamental period* $N = LCM\{N1,N2,N3,\ldots\}$,

$LCM = least\ common\ multiplier.$

➢ Exponential Signals and sinusoidal Signals

The difference between the continuous-time complex exponential e^{jwt} and the discrete-time complex exponential e^{jwn}

e^{jwt}	e^{jwn}
Distinct signals for distinct (w)	Identical signals separated in frequency (w) by 2π
Periodic for any choice of (w)	Periodic only if $\frac{w}{2\pi} = \frac{m}{N}$ is a rational number
Fundamental frequency $w = \frac{2\pi}{T_o}$	Fundamental frequency $\frac{2\pi}{N} = \frac{w}{m}$
Fundamental Period: w=0 → undefined w≠0 → $T_o = \frac{2\pi}{w}$	Fundamental Period: w=0 → N=1 w≠0 → $N = \frac{2\pi}{w} m$

Signals and Systems

Lecture # 5

Basic Signals

Prepared by:

Dr. Mohammed Refaey

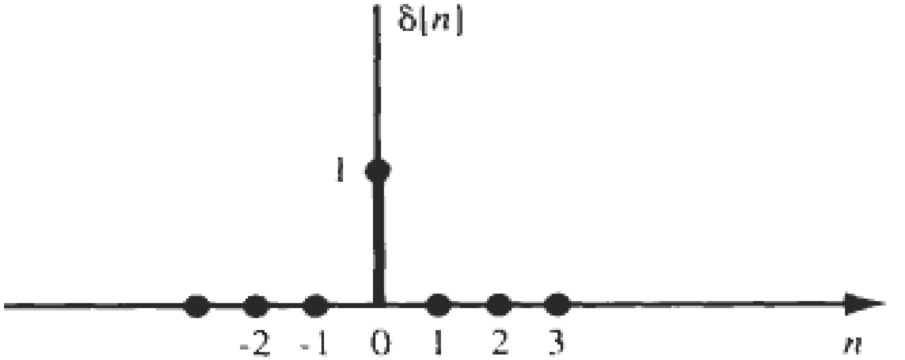
Topics of the lecture:

- **Discrete-Time Unit Impulse and Unit step Signals.**
- **Continuous-Time Unit Impulse and Unit step Signals.**

➢ Basic Signals

<u>The discrete-time unit Impulse Signal:</u>

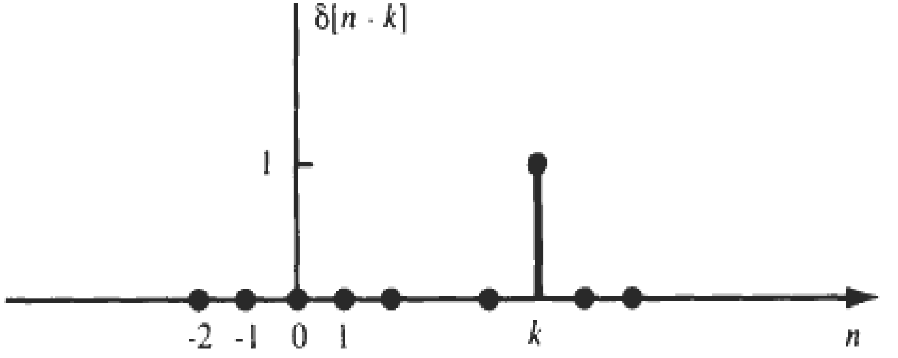
$$\delta[n]=\begin{cases}1 & ; \quad n=0\\ 0 & ; \quad n\neq 0\end{cases}$$



It is also called unit sample signal

<u>The discrete-time shifted unit impulse Signal:</u>

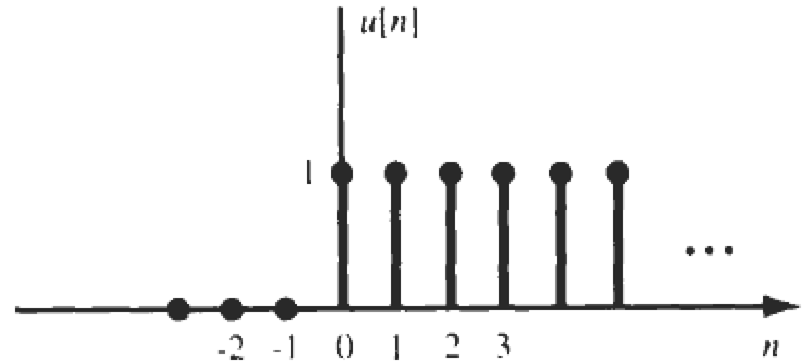
$$\delta[n-k]=\begin{cases}1 & ; \quad n=k\\ 0 & ; \quad n\neq k\end{cases}$$



Basic Signals

The discrete-time unit step Signal:

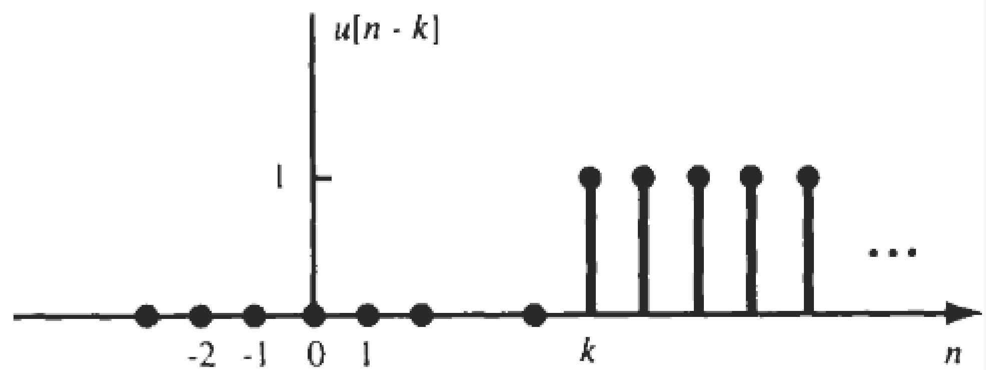
$$u[n] = \begin{cases} 1 & ; \quad n \geq 0 \\ 0 & ; \quad n < 0 \end{cases}$$



It is also called unit sequence **signal**

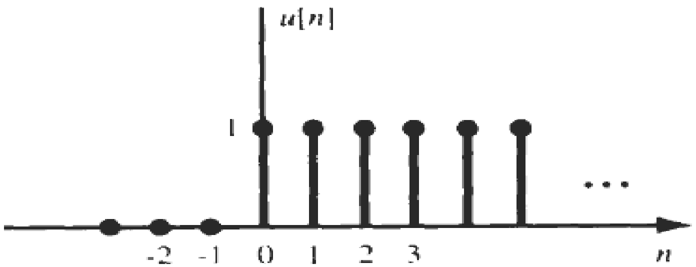
The discrete-time shifted unit step Signal:

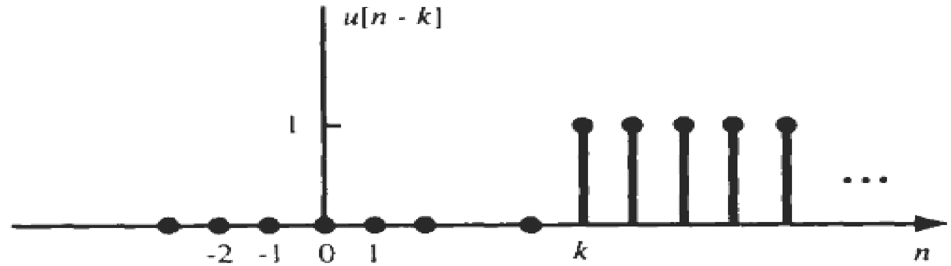
$$u[n-k] = \begin{cases} 1 & ; \quad n \geq k \\ 0 & ; \quad n < k \end{cases}$$



➢ Basic Signals

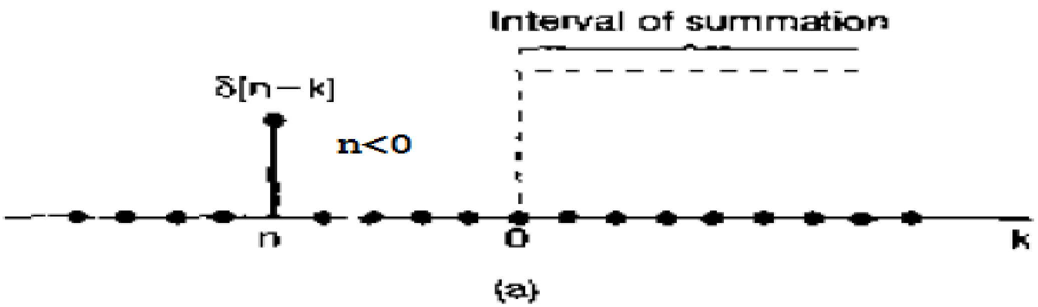
The relationship between discrete-time unit impulse and unit step signals:





$$\delta[n] = u[n] - u[n-1]$$

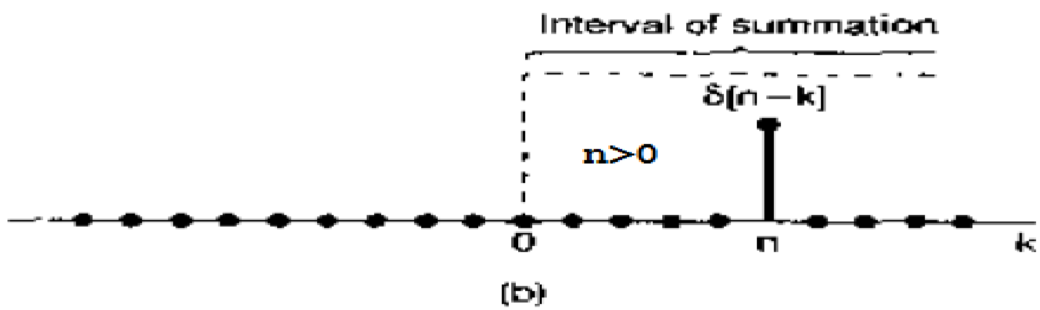
$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$



Let m = n – k OR k = n – m

$$\therefore \quad u[n] = \sum_{k=+\infty}^{0} \delta[n-k]$$

$$\therefore \quad u[n] = \sum_{k=0}^{\infty} \delta[n-k]$$



➢ Basic Signals

The unit impulse sequence can be used to sample the value of a signal at $n = 0$. In particular, since $\delta[n]$ is nonzero (and equal to 1) only for $n = 0$, it follows that

$$x[n]\delta[n] = x[0]\delta[n] = x[0]$$

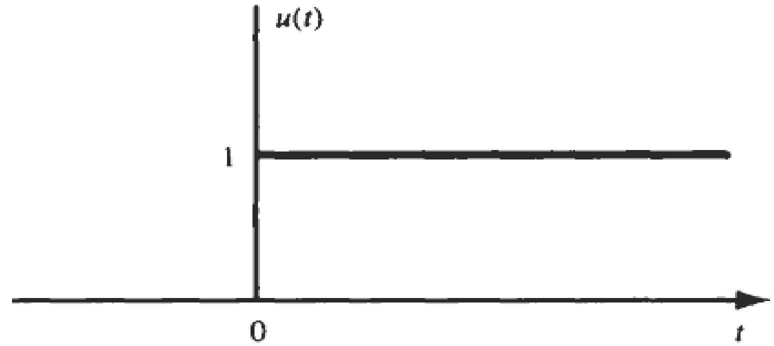
More generally, if we consider a unit impulse $\delta[n - n_0]$ at $n = n_0$, then

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0] = x[n_0]$$

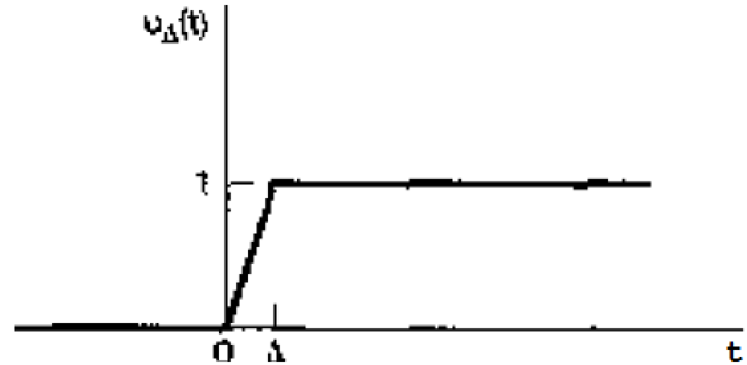
➢ Basic Signals

The continuous-time unit step Signal:

$$u(t) = \begin{cases} 1 & ; \quad t > 0 \\ 0 & ; \quad t < 0 \end{cases}$$



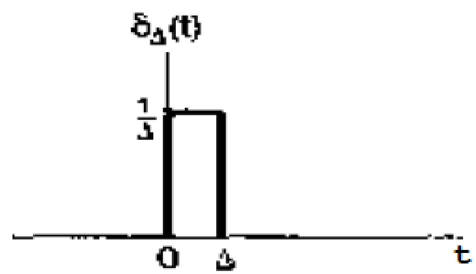
As the there is no such sudden change in real practical application. So an approximation of the ideal case is usually is what is happen.



Continuous approximation to the unit step, $u_\Delta(t)$.

Similarly to the discrete-time case the unit continuous-time impulse signal is the differentiation of the unit step signal

$$\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$$

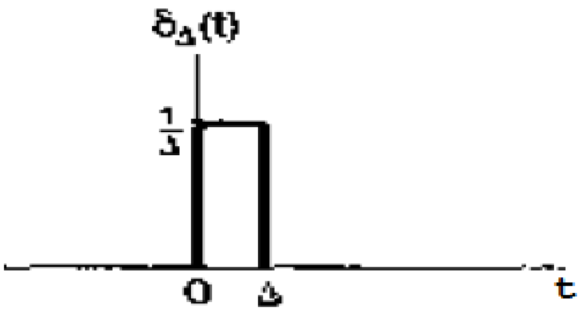


Derivative of $u_\Delta(t)$.

➢ Basic Signals

<u>The continuous-time unit impulse Signal:</u>

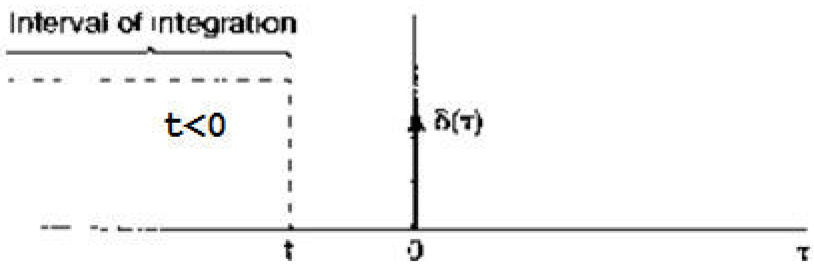
$$\delta_\Delta(t) = \frac{d}{dt} u_\Delta(t)$$



Derivative of $u_\Delta(t)$.

Note that $\delta_\Delta(t)$ is a short pulse of duration Δ and unit area. As the Δ becomes smaller the $\delta_\Delta(t)$ becomes narrower and higher <u>maintaining</u> the unit area. As the Δ goes to zero the $\delta_\Delta(t)$ goes to ∞

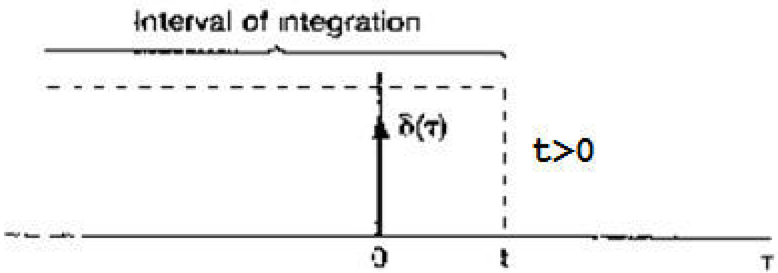
$$\delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$



(a)

Then u(t) can be thought as a running integral of δ(t)

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau$$



➢ Basic Signals

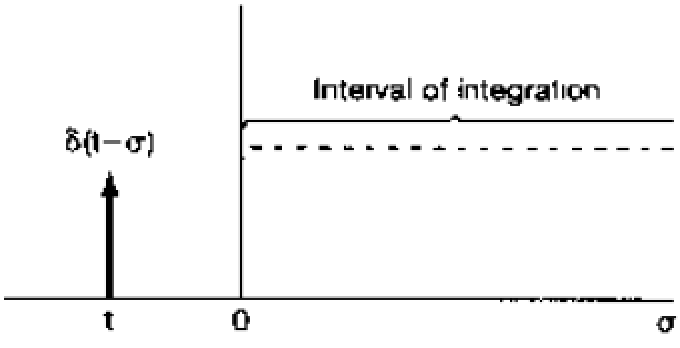
The relationship between continuous-time unit impulse and unit step signals:

$$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

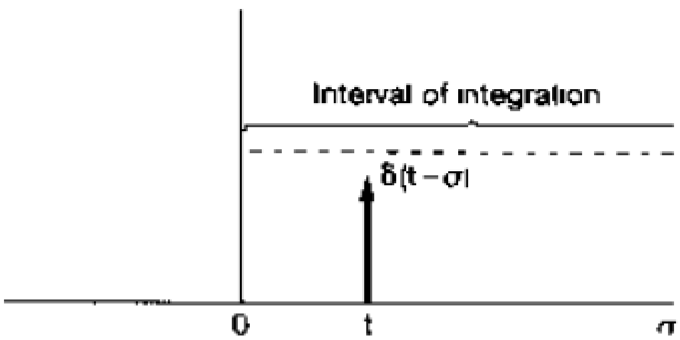
Let $\sigma = t - \tau$ **OR** $\tau = t - \sigma$

$$u(t) = \int_{\infty}^{0} \delta(t - \sigma)(-d\sigma)$$

$$u(t) = \int_{0}^{\infty} \delta(t - \sigma) d\sigma$$



(a)



➢ Basic Signals

$$x_1(t) = x(t)\delta_\Delta(t).$$

In Figure (a) we have depicted the two time functions $x(t)$ and $\delta_\Delta(t)$, and in Figure (b) we see an enlarged view of the nonzero portion of their product. By construction, $x_1(t)$ is zero outside the interval $0 \leq t \leq \Delta$. For Δ sufficiently small so that $x(t)$ is approximately constant over this interval,

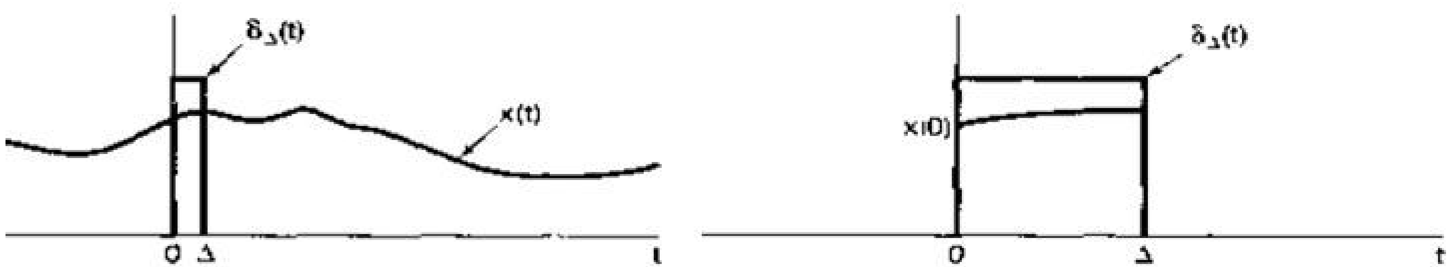
$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t).$$

Since $\delta(t)$ is the limit as $\Delta \rightarrow 0$ of $\delta_\Delta(t)$, it follows that

$$x(t)\delta(t) = x(0)\delta(t).$$

By the same argument, we have an analogous expression for an impulse concentrated at an arbitrary point, say, t_0. That is,

$$x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0).$$



(a) (b)

Signals and Systems

Lecture # 6

Systems Properties

Prepared by:

Dr. Mohammed Refaey

Topics of the lecture:

- **Systems Interconnections.**
- **Systems Properties.**
  1. **Memoryless**
  2. **Invertibility**
  3. **Causality**
  4. **Stability**

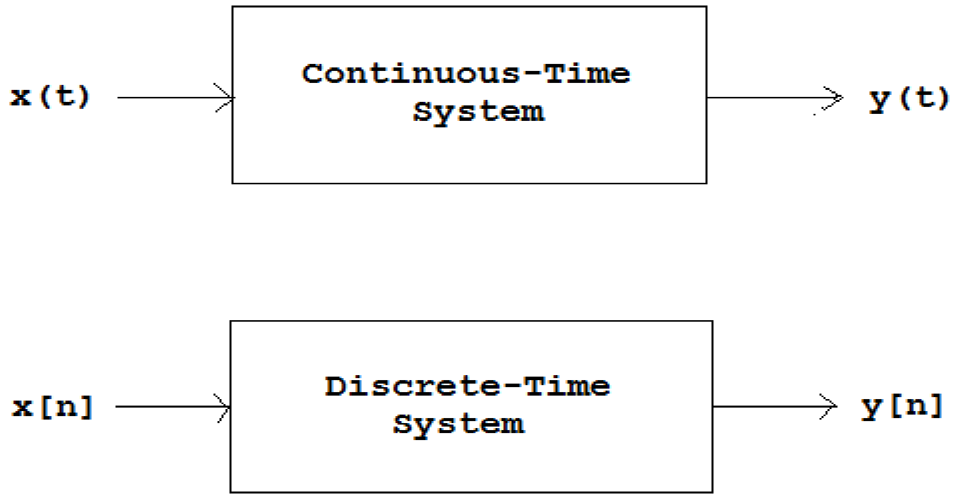
➢ Systems

Recall:

The system is the tool or the process through which the input signal is used to get another output signal or make the system to act in a certain behavior.

A system may consists of physical components (Hardware Realization) or may consists of an algorithm that compute the output signal from the input signal (Software Realization).

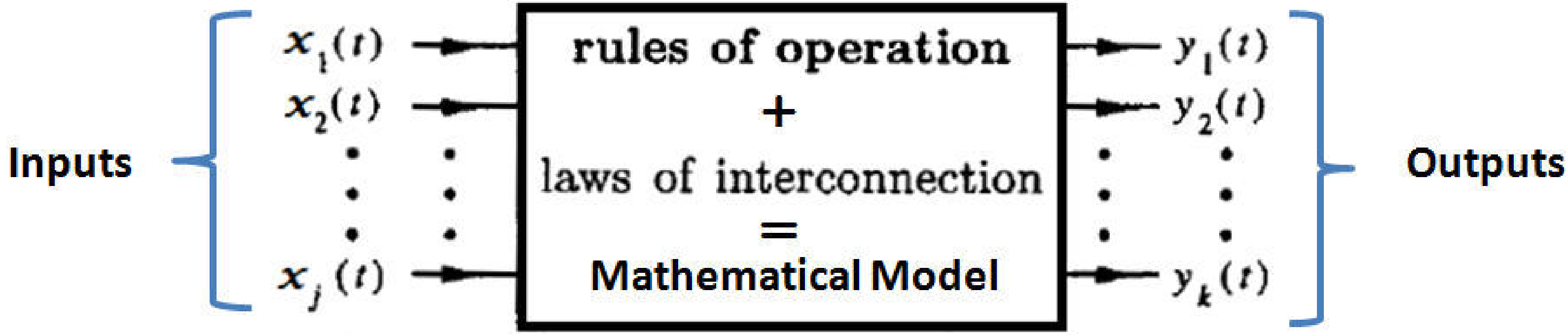
Physical systems in the broadest sense are an interconnection of components, devices, or subsystems.



➢ Systems

A system is characterized by its **inputs**, its **outputs** (or **responses**), and the **rules of operation** (or **laws**) adequate to describe its behavior. For example, in electrical systems, the laws of operation are the familiar voltage-current relationships for the resistors, capacitors, inductors, transformers, transistors, and so on, as well as the laws of interconnection (i.e., Kirchhoff's laws). Using these laws, we derive mathematical equations relating the outputs to the inputs. These equations then represent a **mathematical model** of the system. Thus a system is characterized by its inputs, its outputs, and its mathematical model.

A system can be conveniently illustrated by a "black box" with one set of accessible terminals where the input variables $x_1(t), x_2(t), \ldots, x_j(t)$ are applied and another set of accessible terminals where the output variables $y_1(t), y_2(t), \ldots, y_k(t)$ are observed. Note that the direction of the arrows for the variables in Fig. is always from cause to effect.



➢ Systems Interconnections

Many real systems are built as interconnection of *<u>subsystems</u>*.

Example:

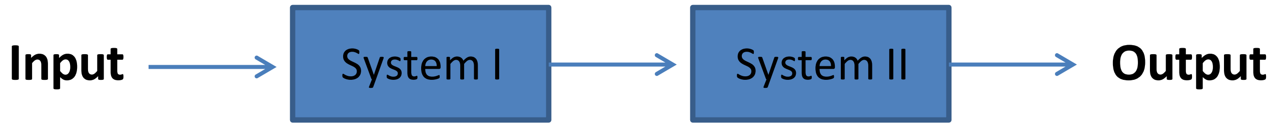
The Audio System Ξ **Receiver** **+** **Player** **+** **Amplifier** **+** **Speaker**

Viewing (through *block diagram*) a system as an interconnection of *subsystems* has the advantage of:

1- Facilitate the understanding of existing systems by understanding the function of each subsystem and how it is connected to other components.

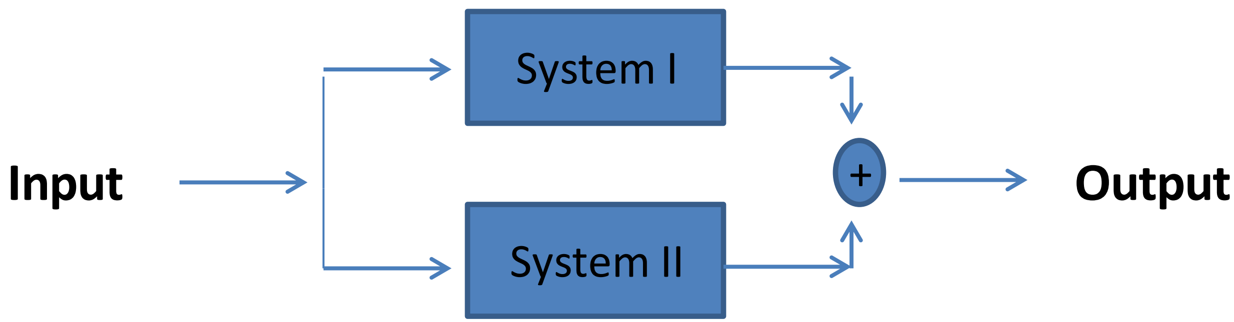
2- Help us to build complex systems

➢ Systems Interconnections



Series (cascade) Interconnection

Example: Radio Receiver → Amplifier

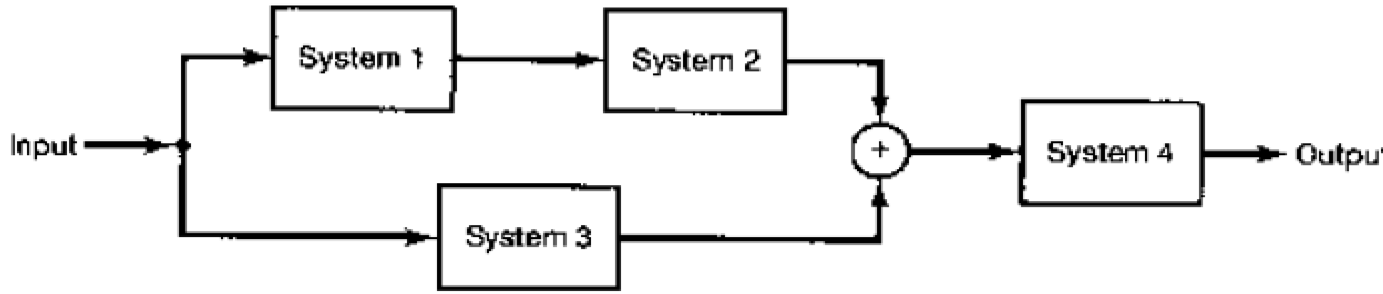


Parallel Interconnection

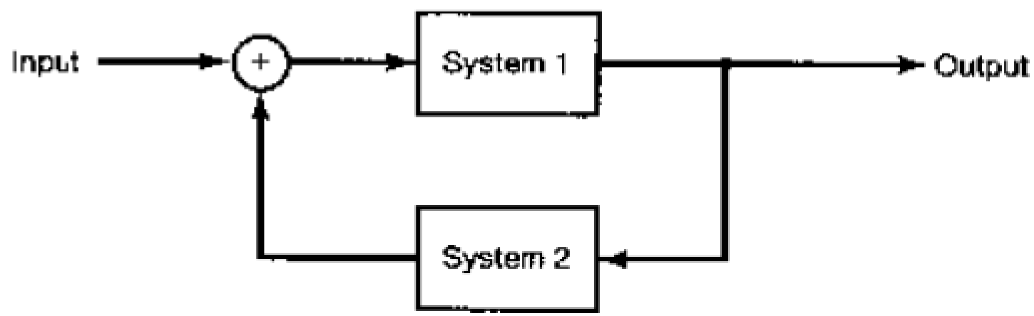
Example:

Mic#1
Mic#2 } **one Audio Stream**

➢ Systems Interconnections



Series-Parallel Interconnection



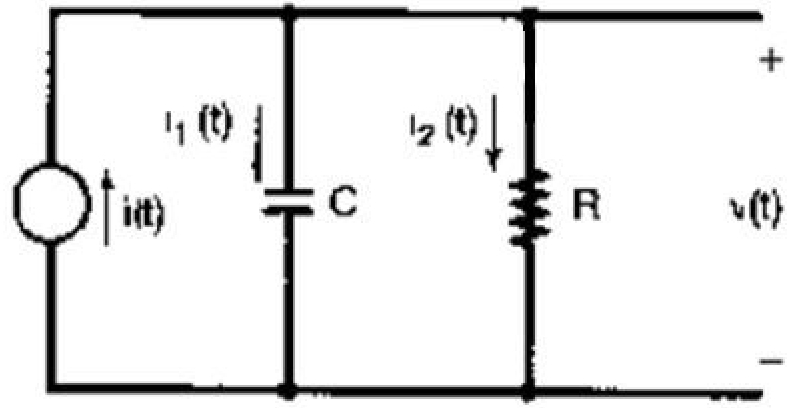
Feedback Interconnection

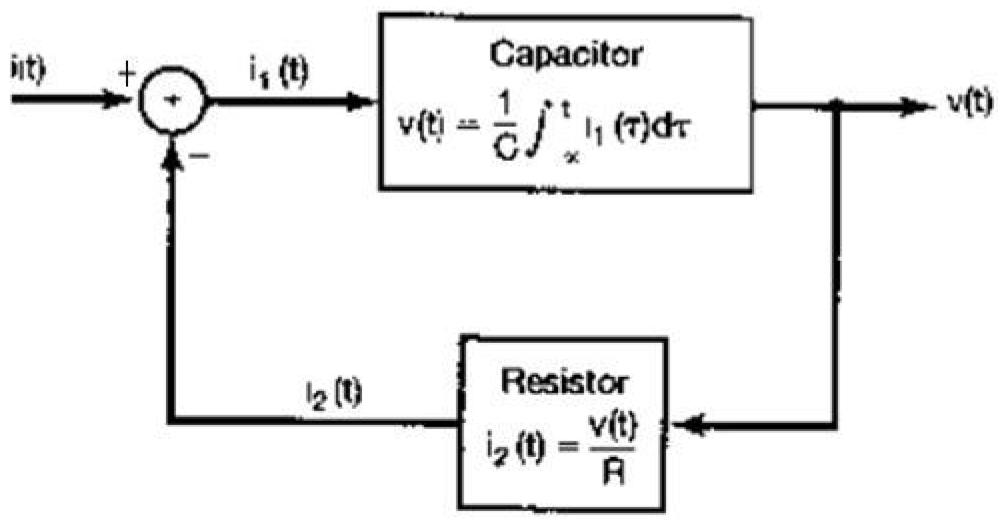
Examples:

-Fluid flow control in a car **- Autopilot and sensors in airplane**

➢ Systems Interconnections

Real Example:





1- Memoryless:

The system is said to be *Memoryless* if its output at any time instant depends only on the input at the same time instant.

i.e.

$$y(t)\,\big|_{t=t_o} \xleftarrow{\text{depends only on}} x(t)\,\big|_{t=t_o}$$

$CT:$

i.e. $x(t) \xrightarrow{S} y(t)$

$$y(t) = S\big(x(t)\big) = \textit{function} \;\; \text{in } x(t)$$

$DT:$

$x[n] \xrightarrow{S} y[n]$

$$y[n] = S\big(x[n]\big) = \textit{function} \;\; \text{in } x[n]$$

Examples:

1-

$$y[n] = \left(2x[n] + x^2[n]\right)^2$$

This system is *Memoryless* as the output at any time n_0 is a function of the input at that time instant only and there is no need for memory.

2- The voltage (*v*) across the resistor (*R*) depends on the electrical current passing through it at the same time instant (*i*):

$$v(t) = R.i(t)$$

This system is also *Memoryless*

➢ Systems Properties

1- Memoryless:

An example of a discrete-time system with memory is an *accumulator* or *summer*

$$y[n] = \sum_{k=-\infty}^{n} x[k],$$

and a second example is a *delay*

$$y[n] = x[n-1].$$

A capacitor is an example of a continuous-time system with memory, since if the input is taken to be the current and the output is the voltage, then

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau,$$

where C is the capacitance.

Roughly speaking, the concept of memory in a system corresponds to the presence of a mechanism in the system that retains or stores information about input values at times other than the current time.

In many physical systems, memory is directly associated with the storage of energy. For example, the capacitor stores energy by accumulating electrical charge.

➢ Systems Properties

1- Memoryless:

A useful strategy to check the memory property of a system is to use the counter example technique. To find this counter example, if any, check for different values representing or covering the whole real number scale.
You can use the 7 values:

($t<-1$), ($t=-1$), ($-1<t<0$), ($t=0$), ($0<t<1$), ($t=1$), ($t>1$)

And do not miss a complex values if applicable like (j), ($-j$)
Also take care of (π), ($\pi/4$), ($\pi/2$), ($3\pi/4$), ($3\pi/2$) if the system consider angles.

Check- $y(t) = x(t)\cos(t+3)$ is it *Memoryless?*

***Note I**: Memoryless system does not contain time-shift, time-reverse, nor time scaling for the input signal {may exist in the multiplying gain}. So, check if any one of them exist first, if exist then the system is not memoryless.*

***Note II**:* **do not be fooled by mathematical symbols like summation and integration as they need memory.**

2- Invertibility:

The system is said to be invertible if distinct values of the input lead to distinct values of the output.

It has a reverse system, when cascaded together with its origin system gives us the identity system. i.e. the output of the two systems connected in series is the input itself $x(t) \rightarrow \boxed{H} \rightarrow y(t) \rightarrow \boxed{H^{-1}} \rightarrow x(t)$

Examples of invertible systems:

1- y (t) = 2 x (t) where the inverse is w (t) = ½ y (t)

2- $y[n] = \sum_{k=-\infty}^{n} x[k]$ where the inverse is w[n] = y[n] - y[n-1]

Examples of noninvertible systems:

1- y (t) = 0

2- $y[n] = x^2[n]$

- The strategy of trying to find { *two inputs have the same output* } could be used to verify that a system is { *not* } invertible.

Invertibility is important in applications like data encoding for secure systems

➢ Systems Properties

3- Causality:

The system is said to be casual if the output at any time depends only on the present or past values of the input (not on future values).
Also called "*nonanticipative*" لا يتنبأ – لا يستشرف system.

Examples:
- The RC circuit, where the capacitor voltage responds only to present and past values of source voltage.
- Automobile motion, where it does not anticipate future actions of the driver.

Note: *All Memoryless systems are also causal systems*.

The noncausal systems are applicable when we are processing a stored data (like a recorded audio file) or when the independent variable is not the time (like in image processing), or when there is a time-delay in the system.

The counter example strategy stated previously for Memoryless is applicable to check causality of a given system.

➢ Systems Properties

3- Causality:

Self-Check Examples:

$$y[n] = x[n] + 3x[n+1]$$

$$y(t) = x(t)\cos(t+1)$$

$$y[n] = x[-n]$$

➢ Systems Properties

4- Stability:

The system is said to be stable if bounded inputs leads to bounded outputs BIBO.

$$x(t) \text{ Bounded} \Rightarrow |x(t)|<B \quad , \; B < \infty \quad , \text{ for all } t$$

***Stability of physical systems generally results from the presence of mechanics that dissipates energy.**

Examples:

- **The RC circuit is stable, as R dissipates energy.**
- **The Automobile represent stable system as the friction dissipates energy and the speed does not go to ∞ .**

-A bank account with initial deposit without further withdrawals represents unstable system as the balance increased without a bound.

➢ Systems Properties

4- Stability:

Self-Check Examples:

- $y(t) = t\,x(t)$

- $y(t) = e^{x(t)}$

Signals and Systems

Lecture # 7

System Properties (continued)

Prepared by:

Dr. Mohammed Refaey

Topics of the lecture:

- **Systems Properties.** **(continued)**
  5. **Time-Invariance**
  6. **Linearity**

➢ Systems Properties

5- Time-Invariance:

The system is said to be time-invariant if the behavior and characteristics of the system are (not change) fixed over time.

Example: in RC circuit if the values of resistance (R) and capacitance (C) are not changed over time then the RC circuit will be time-invariant. Then you expect to get the same results if you repeat the same experiment at two different times. But, if the values of (R) and (c) changed/fluctuate over time, then the results of repeated experiment will not be the same as the system becomes time-variant.

In signals and systems language, the system is said to be time-invariant if a time-shift in the input signal results in identical time-shift in the output signal.

So, any time-varying gain system is not time-invariant nor stable.

➢ Systems Properties

5- Time-Invariance:

1- let : $$x_1(t) \xrightarrow{S} y_1(t)$$

And let $$x_2(t) = x_1(t - t_o) \xrightarrow{S} y_2(t)$$

2- Get $y_2(t)$ in terms of $x_1(t)$ using the system equation. → I

3- Get the expression $y_1(t\text{-}t_o)$ by replacing each (t) by $(t\text{-}t_o)$ → II

4- check if the result of I and II are equal?

- **If Yes → time-invariant system.**
- **If No → not time-invariant system**

➢ Systems Properties

5- Time-Invariance:

Examples to be solved on the board :

$$y(t) = \sin(x(t))$$

$$y[n] = n\,x[n]$$

$$y(t) = x(2t)$$

➢ Systems Properties

5- Time-Invariance: $y(t) = \sin(x(t))$

➢ Systems Properties

<u>5- Time-Invariance:</u> $y[n] = n\,x[n]$

➢ Systems Properties

5- Time-Invariance: $y(t) = x(2t)$

➢ Systems Properties

6- Linearity:

The system is said to be linear if :

$$x_1(t) \xrightarrow{S} y_1(t)$$

$$x_2(t) \xrightarrow{S} y_2(t)$$

And system has the following two properties:

1- Additive property: $x_1(t) + x_2(t) \xrightarrow{S} y_1(t) + y_2(t)$

2- Scaling/Homogeneity property:

$$ax_1(t) \xrightarrow{S} ay_1(t)$$

The two conditions can be meet together through satisfying the superposition property that: $ax_1(t) + bx_2(t) \xrightarrow{S} ay_1(t) + by_2(t)$

i.e. a linear combination of inputs result in the same linear combination of outputs.

➢ Systems Properties

Linearity check algorithm:

1- let :

$$x_1(t) \xrightarrow{S} y_1(t)$$

$$x_2(t) \xrightarrow{S} y_2(t)$$

And let

$$x_3(t) = ax_1(t) + bx_2(t) \xrightarrow{S} y_3(t)$$

2- Get y_3(t) in terms of x_1(t) and x_2(t) using the system equation. → I

3- Get the expression ay_1(t) + b y_2(t) in terms of x_1(t) and x_2(t). → II

4- check if the result of I and II are equal?

- **If yes → linear system.**
- **If No → not linear system.**

➢ Systems Properties

6- Linearity:

Examples to be solved on the board:

1- $y(t) = t\,x(t)$

2- $y(t) = x^2(t)$

3- $y[n] = Re\{x[n]\}$

4- $y[n] = 2x[n] + 3$

➢ Systems Properties

6- Linearity:

1- $$y(t) = t\,x(t)$$

➢ Systems Properties

6- Linearity:

2- $y(t) = x^2(t)$

Systems Properties

6- Linearity:

$$y[n] = Re\{x[n]\}$$

➢ Systems Properties

6- Linearity:

$$y[n] = 2x[n] + 3$$

➢ Review on topics till now

Topics covered:

1- Complex Numbers.

2- Signals and Systems definitions and classifications.

3- Energy and Power.

4- Signals Transformations both for dependent and independent variables.

5- Even and Odd signals.

6- Periodic Signals.

7- Continuous-time Exponential Signals.

8- Discrete-time Exponential Signals.

9- The differences between Continuous-time Exponential and Discrete-time Exponential Signals.

10- Unit impulse and unit step signals.

11- System Properties.

Signals and Systems

Lectures # 8 & #9

Discrete-time LTI Systems

(Convolution Sum)

Prepared by:

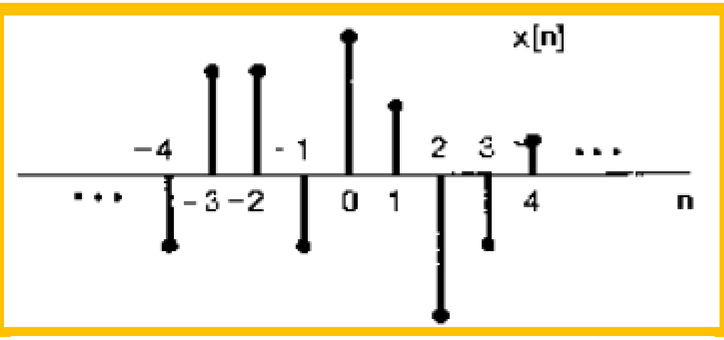
Dr. Mohammed Refaey

Topics of the lecture:

- **Convolution Sum Formula Derivation**
- **Convolution Sum Computation Algorithm**
- **Examples.**

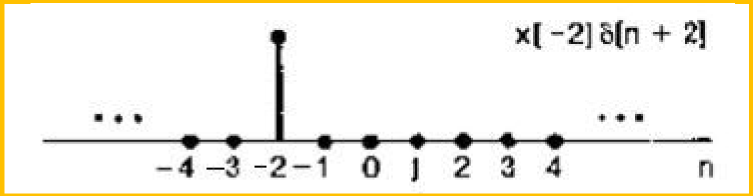
➢ Convolution Sum Formula Derivation

If the original discrete-time signal x[n] is as in figure →

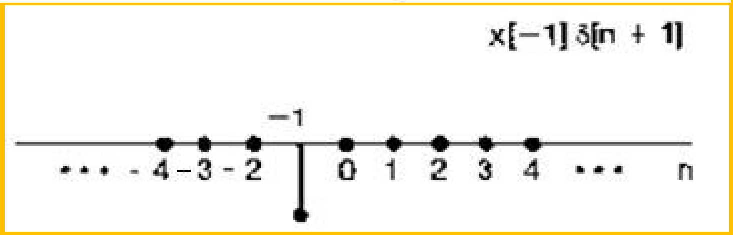


Recall: the unit impulse/sample signal

$$\delta[n] = \begin{cases} 1 & ; \quad n=0 \\ 0 & ; \quad n\neq 0 \end{cases}$$

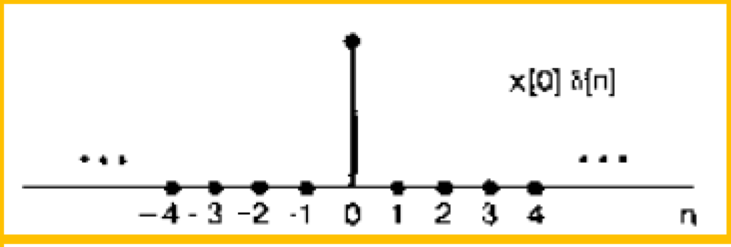


→ **x[n].δ[n+2] = x[-2].δ[n+2] = x[-2]**



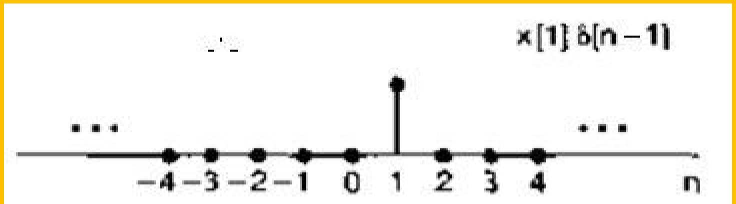
→ **x[n].δ[n+1] = x[-1].δ[n+1] = x[-1]**

→ **x[n].δ[n] = x[0].δ[n] = x[0].δ[n-0] = x[0]**



→ **x[n].δ[n-1] = x[1].δ[n-1] = x[1]**

→ **And so on, then: x[k] = x[k]. δ[n-k] = x[n] . δ[n-k]**



$$\therefore x[n] = sum \quad of \quad all \quad samples$$

$$= \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

➢ Convolution Sum Formula Derivation

$$\therefore x[n] = sum \quad of \quad all \quad samples$$

$$\therefore x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\because \qquad x[n] \xrightarrow{S} y[n]$$

$$let \qquad \delta[n] \xrightarrow{S} h[n]$$

$(as\ \ \delta\ \ is\ an\ impulse,\ h\ \ is\ called\ the\ impulse\ response)$

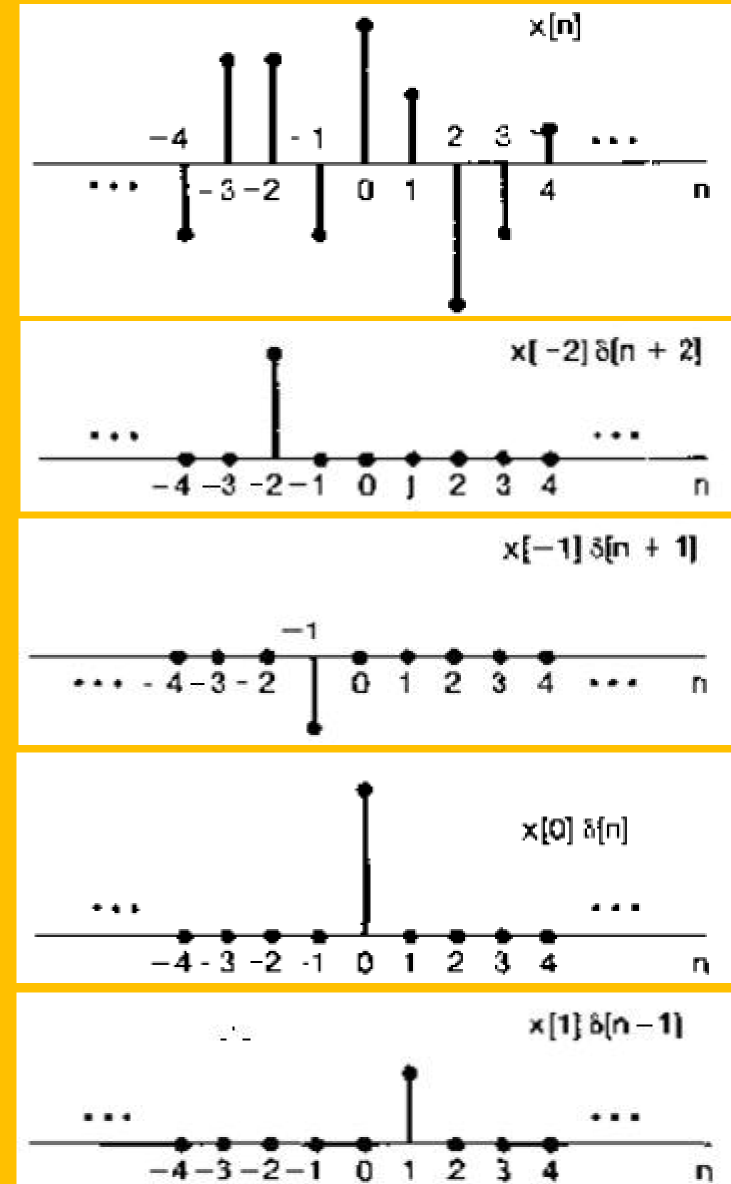
$as \quad S \quad LTI \quad system:$

$$\therefore \delta[n-k] \xrightarrow{S} h[n-k] \ (LTI \ \equiv \ same \ \ shift)$$

$$\because x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

$$\xrightarrow{S} \boxed{y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]} = x[n] * h[n]$$

Convolution Sum Formula



➢ Convolution Sum Formula Derivation

Another View:

n=-2	n=-1	n=0	n=1	…	n=k
$x[-2] = x[n].\delta[n+2] = x[-2].\delta[n+2]$	$x[-1] = x[n].\delta[n+1] = x[-1].\delta[n+1]$	$x[0] = x[n].\delta[n] = x[0].\delta[n]$	$x[1] = x[n].\delta[n-1] = x[1].\delta[n-1]$	…	*In general :* $x[k] = x[n].\delta[n-k] = x[k].\delta[n-k]$
$\downarrow S$	$\downarrow S$	$\downarrow S$	$\downarrow S$		$\downarrow S$
$x[-2].h_{-2}[n] = y[-2]$	$x[-1].h_{-1}[n] = y[-1]$	$x[0].h[n] = y[0]$	$x[1].h_1[n] = y[1]$	…	$x[k].h_k[n] = y[k]$

$$\therefore \quad x[n] = sum\ of\ all\ samples$$

$$\therefore \quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \xrightarrow{S}$$

$$\therefore \quad y[n] = sum \quad of \quad all \quad responses$$

$$\therefore \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] = \boxed{\sum_{k=-\infty}^{\infty} x[k]h[n-k]}$$

$$as\ S\ is\ LTI\ so\ h_k[n] = h[n-k]$$

As the system is LTI ➔ *$h_k[n] = h[n-k]$* ➔ to characterize/analyze the system *S* we need only: to know h[n]

➢ Convolution Sum Formula Derivation

$$\therefore\ x[n] = \sum_{k=-\infty}^{\infty} x[k].\delta[n-k]$$

$$\xrightarrow{\ s\ } y[n] = \sum_{k=-\infty}^{\infty} x[k].h[n-k]$$

The independent variable is renamed to ***k***

There is a time-reverse

The shift value is the value of ***n***

For each shift value ***n*** there are : multiplications + sum of multiplications

➢ Convolution Sum Computation Algorithm for short finite-domain signals

To compute the convolution sum of $x[n]$ and $h[n]$:

1- Let the two signals as functions in the independent variable *k* instead of *n*.
So, $x[k]$ instead of $x[n]$
and $h[k]$ instead of $h[n]$ (just renaming the independent variable will not make any difference)

2- Choose one of the two signals and *time-reverse* it to be either $x[-k]$ or $h[-k]$.

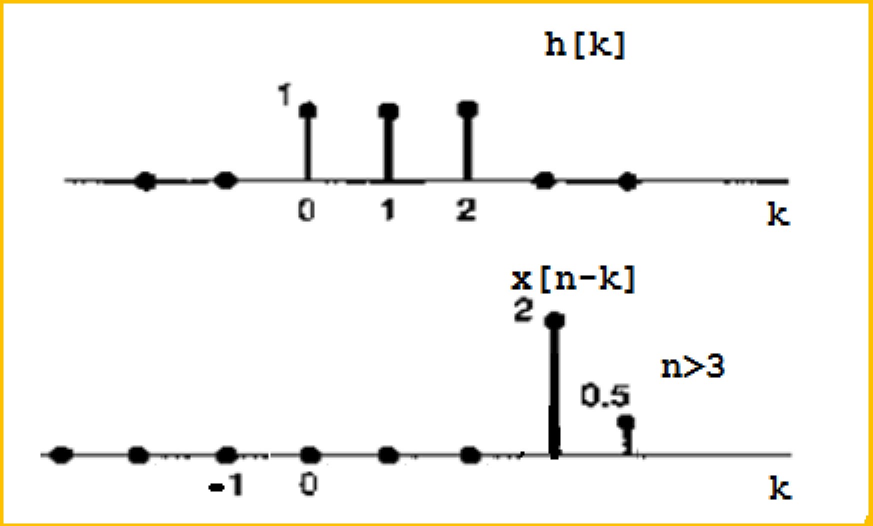
3- Determine the start of the *area of overlapping* ,in terms of shift value (n), between the resulted two signals. (area of overlapping means that this domain that both of the signals have non-zero values at the same time)

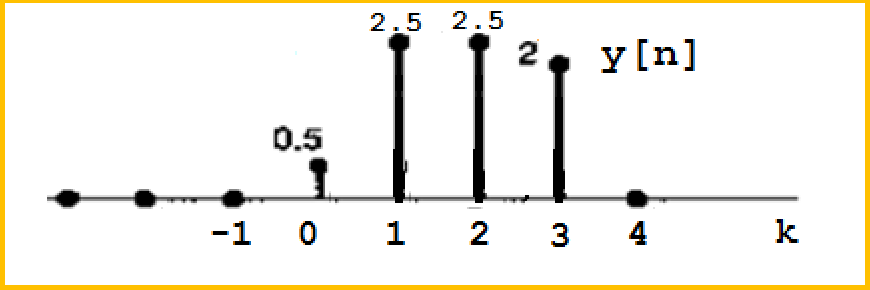
4- Compute the *boundaries* of the overlapping area in terms of shift value (n).
(the first and last point of overlapping)

5- for each shift value of the overlapping area (computed in 4) **compute the output at that time shift by multiplying each point with its corresponding point in the other signal and sum up ALL these multiplications and the result will be the value of the output at that time-shift .**

➢ Convolution Sum : Examples

[1]- Compute the convolution sum of the following input x[n] and system impulse response h[n]:





The answer following the algorithm:

1- let the two signals as functions in k instead of n.

2- let we time-reverse either one of the two signals. Let us choose x[k].

3- if n < 0. there is no overlapping between h[k] and x[n-k] → y[n]=0, n<0

4- the overlapping will start from n=0. and the length of overlapping will be
$(N_x + N_h) - 1 = 2 + 3 - 1 = 4$ points
Starting from n=0 then 1,2,and n=3

5- at n=0 → y[0]= 0x2 + 1x0.5 + 1x0 + 1x0 = 0.5
   at n=1 → y[1]= 1x2 + 1x0.5 + 1x0 = 2.5
   at n=2 → y[2]= 1x0 + 1x2 + 1x0.5 = 2.5
   at n=3 → y[3]= 1x0 + 1x0 + 1x2 + 0x0.5 = 2
   for n>3 → there is no overlapping between h[k] and x[n-k] → y[n]= 0; n>3

➢ Convolution Sum Computation Algorithm for lengthy/infinite-domain signals

To compute the convolution sum of x[n] and h[n]:

1- Let the two signals as functions in the independent variable k instead of n.

So, x[k] instead of x[n]

and h[k] instead of h[n] (just renaming the independent variable will not make any difference)

2- Choose one of the two signals and *time-reverse* it to be either x[-k] or h[-k].

3- Determine the *areas of overlapping* that have similar conditions of mathematical treatment, as the shift (n) will slide the time-reversed signal starting from (- ∞) towards (+∞). (area of overlapping means that this domain that both of the signals have non-zero values at the same time) (similar conditions of mathematical treatment means that there is one mathematical formula to compute the sum of multiplication of the two overlapping signals)

4- Compute the *boundaries* (*U*) and (*L*) of each overlapping area. (upper and lower limit of summation)

5- Compute the *mathematical formula* of each overlapping area using the formula:

$$y[n]=\sum_{k=L}^{U}x[k]h[n-k]_{(if\ \ you\ reversed\ h)}$$

OR

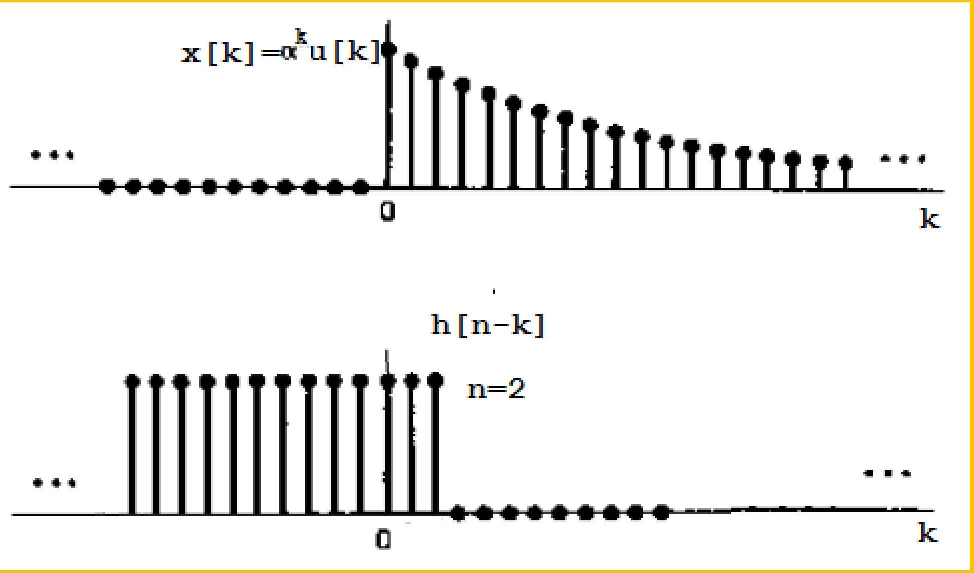
$$y[n]=\sum_{k=L}^{U}h[k]x[n-k]_{(if\ \ you\ reversed\ x)}$$

6- Repeat steps 4 and 5 for each overlapping area.

➢ Convolution Sum : Examples

[2]- Compute the convolution sum of the following input x[n] and system impulse response h[n], with 0<α<1:

$$x[n] = \alpha^n u[n]$$
$$h[n] = u[n]$$



The answer following the algorithm:

1- let the two signals as functions in k instead of n.

2- let we time-reverse either one of the two signals. Let us choose h[k].

3- if n < 0. there is no overlapping between x[k] and h[n-k] → y[n]=0, n<0

4- the overlapping will start from n=0. and slightly and gradually the signal h[n-k] will slides under x[k] as n becomes larger and larger.
at n=0 → there is overlapping from (0) to (0)
at n=1 → there is overlapping from (0) to (1)
at n=2 → there is overlapping from (0) to (2)
And so on … then the boundaries of overlapping are: L=0 (as the lower limit is fixed at 0)
U=n (as the upper limit is equal to n)

5- for n ≥ 0: $$y[n] = \sum_{k=L}^{U} x[k]h[n-k]$$

$$\therefore\ y[n] = \sum_{k=0}^{n} \alpha^k u[k]u[n-k]$$

$$\therefore\ y[n] = \sum_{k=0}^{n} \alpha^k \qquad \therefore\ y[n] = \frac{1-\alpha^{n+1}}{1-\alpha}; n \geq 0$$

$$Recall \Rightarrow \sum_{a}^{b} r^k = r^a \frac{(1-r^{b-a+1})}{1-r}, r \neq 1$$

➢ Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n]=\begin{cases}1 & ; 0\le n\le 4\\ 0 & ; otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n & ; 0\le n\le 6\\ 0 & ; otherwise\end{cases} ; \alpha>1$

The answer following the algorithm:

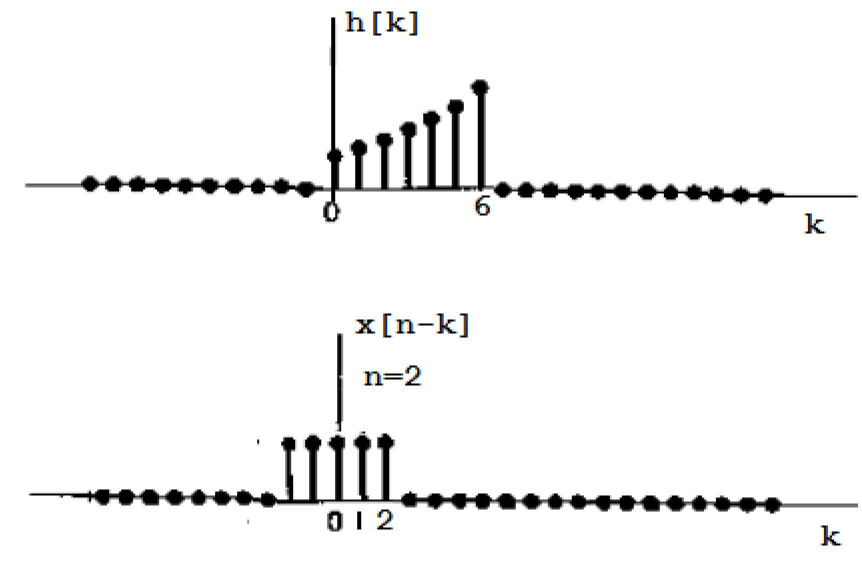
1- let the two signals as functions in k instead of n.

2- let we time-reverse either one of the two signals. Let us choose x[k].

3- if n < 0. there is no overlapping between h[k] and x[n-k] → y[n]=0, n<0

4- the overlapping will start from n=0. and slightly and gradually the signal x[n-k] will slides under h[k] as 0 ≤ n ≤ 3 . (partial overlapping)
at n=0 → there is overlapping from (0) to (0)
at n=1 → there is overlapping from (0) to (1)
at n=2 → there is overlapping from (0) to (2)
And so on … then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)
U=n (as the upper limit is equal to n)

5- for 0 ≤ n ≤3 : $$y[n]=\sum_{k=L}^{U}h[k]x[n-k]$$



$$y[n]=\sum_{k=0}^{n}\alpha^k.1=\sum_{k=0}^{n}\alpha^k$$

$$\therefore\ y[n]=\frac{1-\alpha^{n+1}}{1-\alpha} \quad ; \quad 0\le n\le 3$$

6- as n=4 there is total overlapping:
So, repeat steps 4 and 5 for this region too.

See next page

Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n]=\begin{cases}1 & ; \; 0\le n\le 4\\ 0 & ; otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n & ; \; 0\le n\le 6\\ 0 & ; otherwise\end{cases} ; \alpha>1$

(continued)

4'- the total overlapping will start from n=4 as slightly and gradually the signal x[n-k] will slides under h[k] as 4 ≤ n ≤ 6 . (total overlapping)

at n=4 → there is overlapping from (0) to (4)

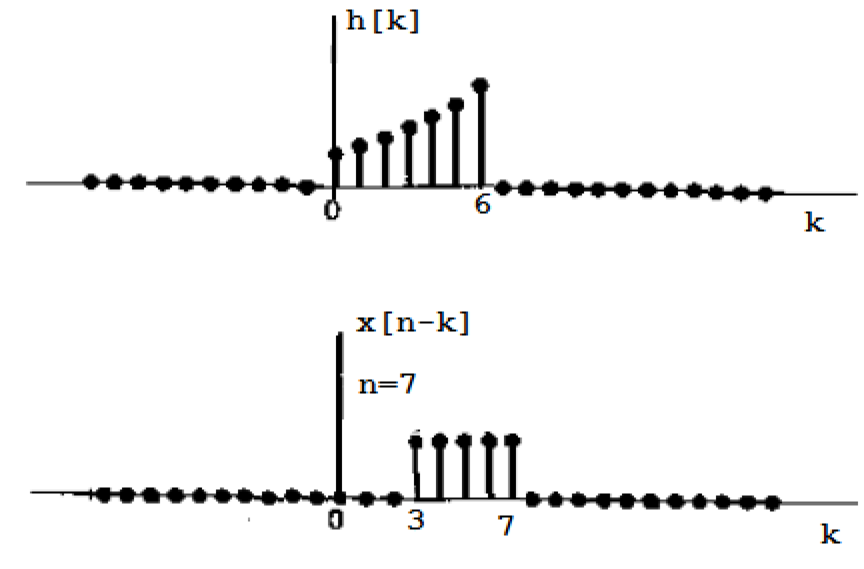
at n=5 → there is overlapping from (1) to (5)

at n=6 → there is overlapping from (2) to (6)

then the overlapping boundaries of this area are:

L=n-4 (as the lower limit is less n by 4)

U=n (as the upper limit is equal to n)



5'- for 4 ≤ n ≤ 6 : $$y[n]=\sum_{k=L}^{U} h[k]x[n-k]$$

$$y[n]=\sum_{k=n-4}^{n}\alpha^k .1=\sum_{k=n-4}^{n}\alpha^k$$

$$\therefore\; y[n]=\frac{\alpha^{n-4}(1-\alpha^5)}{1-\alpha}$$

$$\therefore\; y[n]=\frac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} \quad ; \quad 4\le n\le 6$$

6'- as n>6 the signal x[n-k] will slide out of h[k] (partial overlapping again):

So, repeat steps 4 and 5 for this region too.

See next page

Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n]=\begin{cases}1 & ; 0\le n\le 4\\ 0 & ; otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n & ; 0\le n\le 6\\ 0 & ; otherwise\end{cases};\alpha>1$

(continued)

4”- The second partial overlapping will start from n=7 as slightly and gradually the signal x[n-k] will slides out of h[k] as 7 ≤ n ≤ 10 . (partial overlapping)

at n=7 → there is overlapping from (3) to (6)

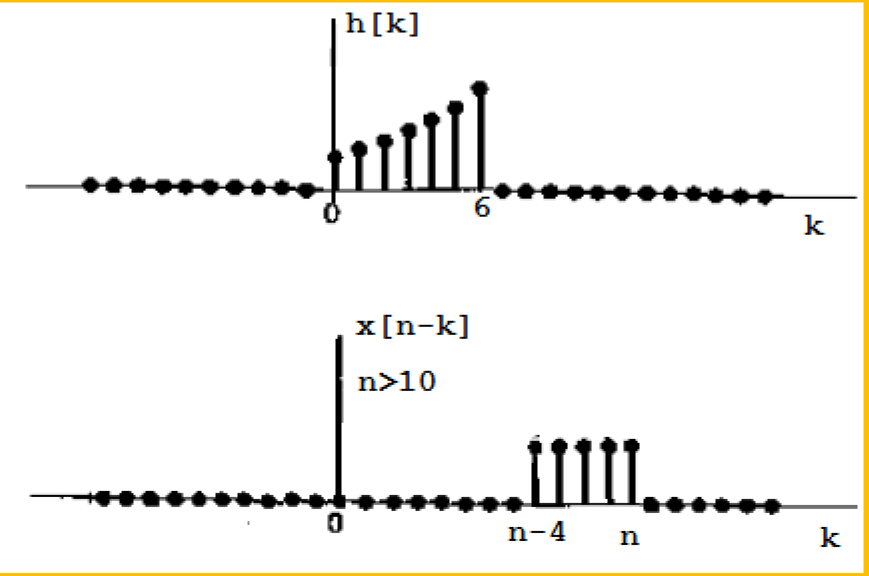
at n=8 → there is overlapping from (4) to (6)

at n=9 → there is overlapping from (5) to (6)

then the overlapping boundaries of this area are:

L=n-4 (as the lower limit is less n by 4)

U=6 (as the upper limit is fixed to 6)



5”- for 7 ≤ n ≤ 10 : $y[n]=\sum_{k=L}^{U}h[k]x[n-k]$

$$y[n]=\sum_{k=n-4}^{6}\alpha^k.1=\sum_{k=n-4}^{6}\alpha^k$$

$$\therefore\ y[n]=\frac{\alpha^{n-4}(1-\alpha^{6-n+4+1})}{1-\alpha}=\frac{\alpha^{n-4}(1-\alpha^{11-n})}{1-\alpha}$$

$$\therefore\ y[n]=\frac{\alpha^{n-4}-\alpha^{7}}{1-\alpha}\quad ;\quad 7\le n\le 10$$

6”- as n>10 there is no overlapping between the signal x[n-k] and h[k] → y[n]=0; n>10.

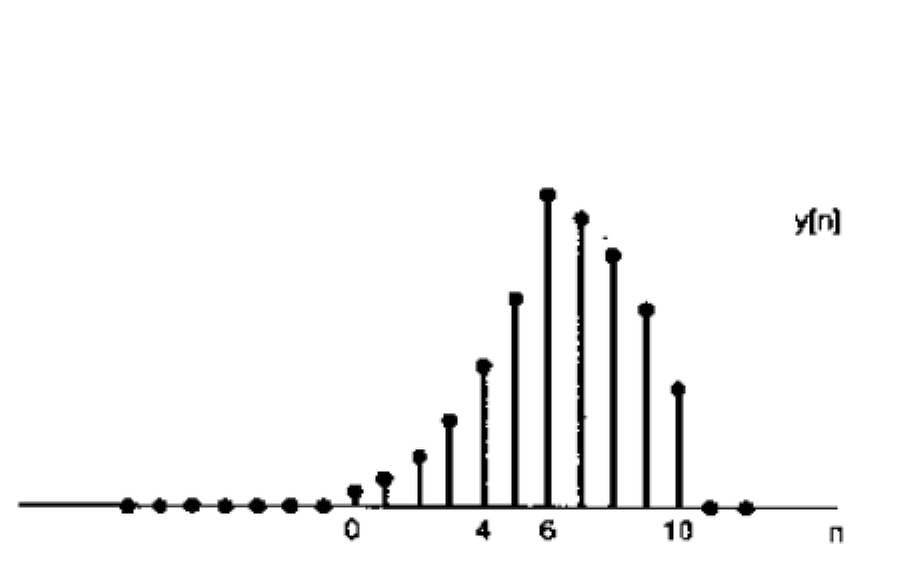
➢ Convolution Sum : Examples

[3]- Compute the convolution sum of the following input x[n] and system impulse response h[n]: $x[n]=\begin{cases}1 & ; 0\le n\le 4\\ 0 & ; otherwise\end{cases}$ $h[n]=\begin{cases}\alpha^n & ; 0\le n\le 6\\ 0 & ; otherwise\end{cases}; \alpha>1$

(continued)

Then collecting results of all areas gives us:

$$\therefore\ y[n] = \begin{cases} 0 & ; \quad n<0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha} & ; \quad 0\le n\le 3 \\ \dfrac{\alpha^{n-4}-\alpha^{n+1}}{1-\alpha} & ; \quad 4\le n\le 6 \\ \dfrac{\alpha^{n-4}-\alpha^{7}}{1-\alpha} & ; \quad 7\le n\le 10 \\ 0 & ; \quad n>10 \end{cases}$$



➢ Convolution Sum : Examples

[4]- Have fun with this applet in this web page { http://www.jhu.edu/~signals/discreteconv2/index.html }

Please wait the Web page to be downloaded

You should be connected to the INTERNET and configured the liveWeb Add in

Visit this site for help https://docs.google.com/a/fci-cu.edu.eg/document/d/1tOrwbl71BR42ZV561i1DKcxs0-tMETaXtj5JzikBZpU/edit

Also if the applet not working visit: htwww.java.com/en/download/help/java_blocked.xtps://ml

Please wait the Web page to be downloaded
http://www.jhu.edu/~signals/discreteconv2/index.html

You should be connected to the INTERNET
and
configured the liveWeb Add in

Signals and Systems

Lectures # 10 & # 11

Continuous-time LTI Systems

(Convolution Integral)

Prepared by:

Dr. Mohammed Refaey

Topics of the lecture:

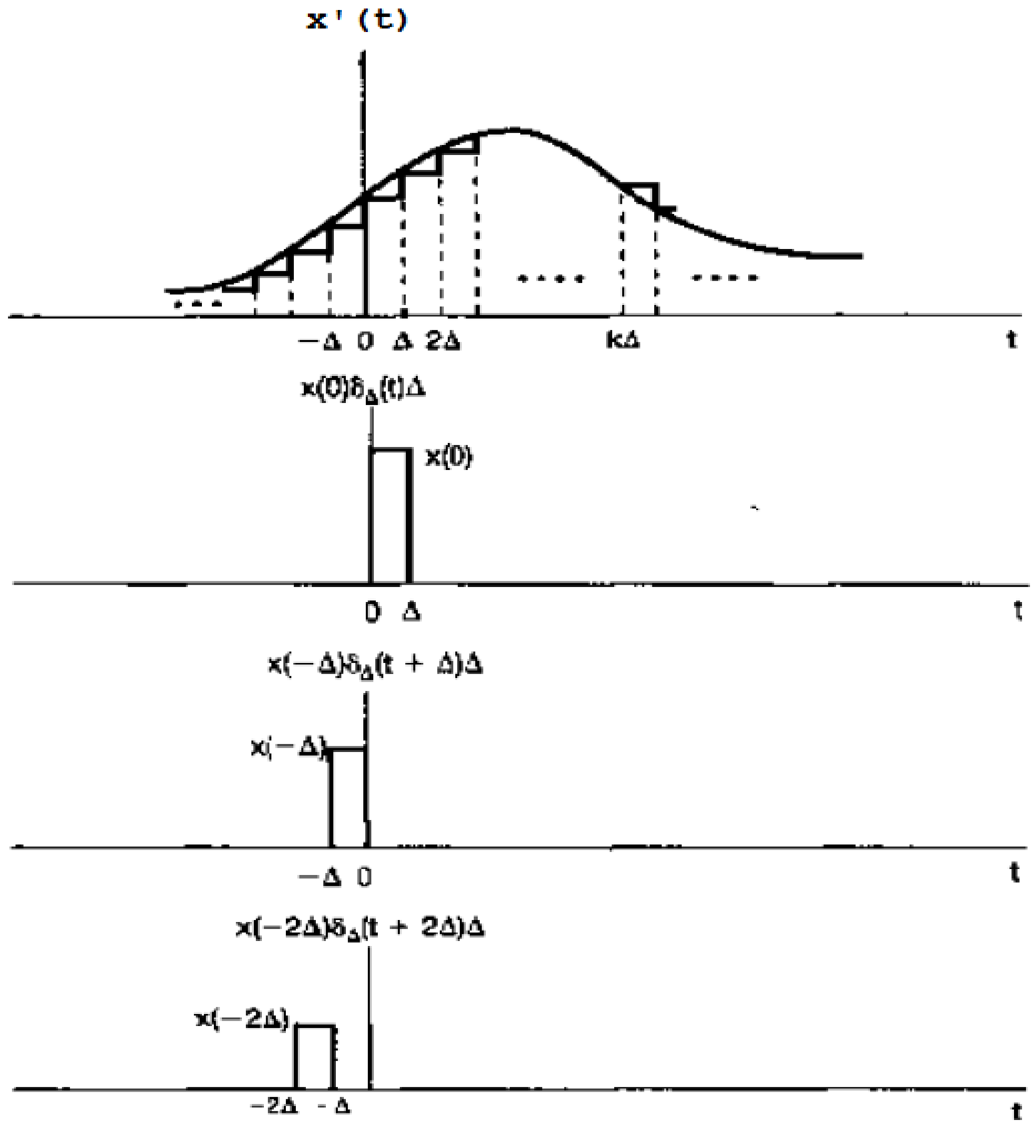
- **Convolution Integral Formula Derivation**

- **Convolution Integral Computation Algorithm**

- **Examples.**

➢ Convolution Integral Formula Derivation for LTI Systems

Then the pulse approximation x'(t) of x(t) will be as in figure →



$$Recall: \quad \delta_\Delta(t) = \begin{cases} \frac{1}{\Delta} & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$then \quad \delta_\Delta(t).\Delta = \begin{cases} 1 & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\Rightarrow \quad x'(t).\delta_\Delta(t).\Delta = \begin{cases} x'(0) & ; \quad 0 < t < \Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t).\Delta = x'(0).\delta_\Delta(t).\Delta$$

$$\Rightarrow \quad x'(t).\delta_\Delta(t+\Delta).\Delta = \begin{cases} x'(-\Delta) & ; \quad -\Delta < t < 0 \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t+\Delta).\Delta = x'(-\Delta).\delta_\Delta(t+\Delta).\Delta$$

$$\Rightarrow \quad x'(t).\delta_\Delta(t+2\Delta).\Delta = \begin{cases} x'(-2\Delta) & ; \quad -2\Delta < t < -\Delta \\ 0 & ; \quad Otherwise \end{cases}$$

$$\therefore \quad x'(t).\delta_\Delta(t+2\Delta).\Delta = x'(-2\Delta).\delta_\Delta(t+2\Delta).\Delta$$

→And so on, then:

$$\therefore x'(t) = sum\ of\ all\ pulses = \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_\Delta(t-k\Delta)\Delta$$

➢ Convolution Integral Formula Derivation for LTI Systems

$$x'(t) = sum \quad of \quad all \quad pulses$$

$$= \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

$$\because \quad x'(t) \xrightarrow{S} y'(t)$$

$$let \quad \delta_{\Delta}(t) \xrightarrow{S} h'(t)$$

$$as \quad S \quad LTI \quad system:$$

$$\therefore \delta_{\Delta}(t-k\Delta) \xrightarrow{S} h'(t-k\Delta) \; (LTI \equiv same \; shift)$$

$$\therefore x'(t) = \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

$$\xrightarrow{S} y'(t) = \sum_{k=-\infty}^{\infty} x'(k\Delta)h'(t-k\Delta)\Delta$$

$$\because \quad x(t) = \lim_{\Delta \to 0} \quad x'(t)$$

$$\therefore \quad y(t) = \lim_{\Delta \to 0} \quad y'(t)$$

$$and \quad h(t) = \lim_{\Delta \to 0} \quad h'(t)$$

$$then \quad as \quad \Delta \to 0$$

$$x(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x'(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta$$

$$\xrightarrow{S} y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x'(k\Delta)h'(t-k\Delta)\Delta$$

$as \; \Delta \to 0$ *the summation tends to be integration*

$$\sum_{-\infty}^{\infty} \quad \to \quad \int_{-\infty}^{\infty}$$

$$x'(k\Delta) \quad \to \quad x(k\Delta)$$

$$h'(t-k\Delta) \to h(t-k\Delta)$$

$$let \quad k.\Delta = \tau \qquad \therefore \Delta.dk = d\tau$$

as dk is the step of summation $\to$ *then* $dk = 1$

$$\therefore \Delta.1 = d\tau \qquad \to \qquad \Delta = d\tau$$

$$\boxed{\therefore \; y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

called Convolution Integral Formula

➢ Convolution Integral Formula Derivation for LTI Systems

$$\therefore\ y(t)=\int_{-\infty}^{\infty} x(\tau).h(t-\tau)d\tau$$

The independent variable is renamed to ***τ***

There is a time-reverse

The shift value is the value of ***t***

For each shift value ***t*** there are : multiplications + integral of multiplications

➢ Convolution Integral Computation Algorithm for LTI

<u>To compute the convolution integral of x(t) and h(t):</u>

1- Let the two signals as functions in the independent variable τ instead of t.

So, x(τ) instead of x(t)

and h(τ) instead of h(t) (just renaming the independent variable will not make any difference)

2- Choose one of the two signals and ***time-reverse*** it to be either x(-τ) or h(-τ).

3- Determine the ***areas of overlapping*** that have similar conditions of mathematical treatment, as the shift (t) will slide the time-reversed signal starting from (- ∞) towards (+∞). (area of overlapping means that this domain that both of the signals have non-zero values at the same time) (similar conditions of mathematical treatment means that there is one mathematical formula to compute the integration of multiplication of the two overlapping signals)

4- Compute the ***boundaries*** (*U*) and (*L*) of each overlapping area. (upper and lower limit of integration)

5- Compute the ***mathematical formula*** of each overlapping area using the formula:

$$\boxed{y(t) = \int_{L}^{U} x(\tau)h(t-\tau)\,d\tau \quad (\text{if you reversed } h)}$$

OR

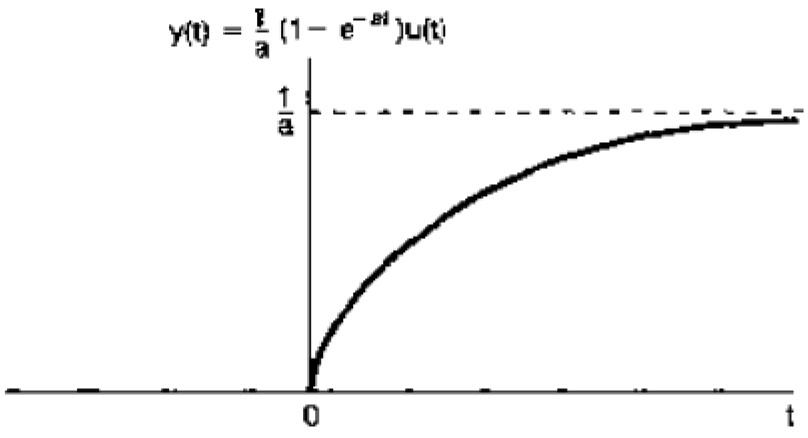
$$\boxed{y(t) = \int_{L}^{U} h(\tau)x(t-\tau)\,d\tau \quad (\text{if you reversed } x)}$$

6- Repeat steps 4 and 5 for each overlapping area.

➢ Convolution Integral : Examples

[1]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t) = e^{-at}u(t) \quad ; \quad a > 0$

$$h(t) = u(t)$$



The answer following the algorithm:

1- let the two signals as functions in (τ) instead of (t).

2- let we time-reverse either one of the two signals. Let us choose h(τ).

3- if t < 0. there is no overlapping between x(τ) and h(t-τ) → y(t)=0 , t<0

4- the overlapping will start from t=0. and slightly and gradually the signal h(t-τ) will slides under x(τ) as t is increasing:

at t=0 → there is overlapping from (0) to (0)

at t=1 → there is overlapping from (0) to (1)

at t=2 → there is overlapping from (0) to (2)

And so on … then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)

U=t (as the upper limit is equal to t)

5- for t ≥ 0: $y(t) = \int_{L}^{U} x(\tau)h(t-\tau)d\tau$

$$\therefore \quad y(t) = \int_{0}^{t} x(\tau)h(t-\tau)d\tau$$

$$\therefore \quad y(t) = \int_{0}^{t} e^{-a\tau}u(\tau)u(t-\tau)d\tau$$

$$\therefore \quad y(t) = \int_{0}^{t} e^{-a\tau}d\tau = \left.\frac{e^{-a\tau}}{-a}\right|_{0}^{t} = \frac{-1}{a}\{e^{-at} - 1\}$$

$$\therefore \quad y(t) = \frac{1}{a}\{1 - e^{-at}\} \quad ; \quad t \geq 0$$

➢ Convolution Integral : Examples

[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t) = \begin{cases} 1 & ;0<t<T \\ 0 & ;otherwise \end{cases}$; $h(t) = \begin{cases} t & ;0<t<2T \\ 0 & ;otherwise \end{cases}$

The answer following the algorithm:

1- let the two signals as functions in (τ) instead of (t).

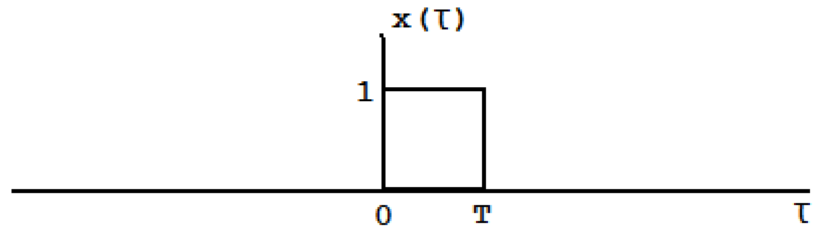
2- let we time-reverse either one of the two signals. Let us choose h(τ).

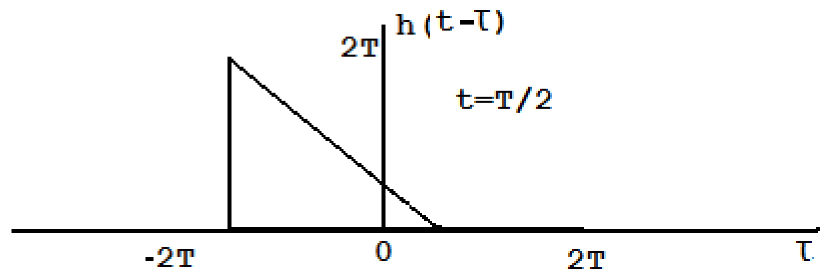
3- if t < 0. there is no overlapping between x(τ) and h(t-τ) → y(t)=0, t<0

4- the overlapping will start from t=0. and slightly and gradually the signal h(t-τ) will slides under x(τ) as 0<t<T . (partial overlapping)
at t=0 → there is overlapping from (0) to (0)
at t=T/4 → there is overlapping from (0) to (T/4)
at t=T/2 → there is overlapping from (0) to (T/2)
And so on … then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)
U=t (as the upper limit is equal to t)

5- for 0<t <T: $y(t) = \int_L^U x(\tau)h(t-\tau)d\tau$

6- Repeat steps 4 and 5 for T≤t<2T (total overlapping)





$$\therefore \quad y(t) = \int_0^t x(\tau)h(t-\tau)d\tau$$

$$\therefore \quad y(t) = \int_0^t 1.(t-\tau)d\tau$$

$$\therefore y(t) = \int_0^t t d\tau - \int_0^t \tau d\tau \quad = t\{\tau|_0^t\} - \{\frac{\tau^2}{2}\Big|_0^t\}$$

$$= t\{t-0\} - \{\frac{t^2}{2} - 0\} \quad = t^2 - \frac{t^2}{2} \quad = \frac{t^2}{2}$$

$$\therefore y(t) = \frac{t^2}{2} \quad ; 0 \le t < T$$

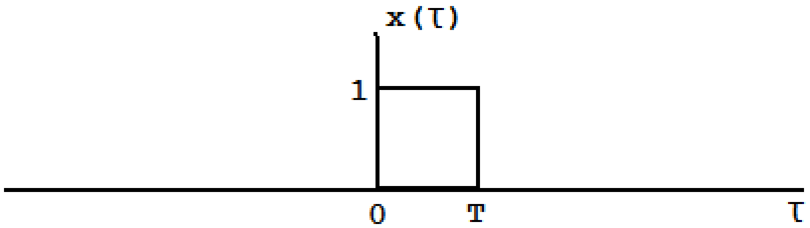
➢ Convolution Integral : Examples

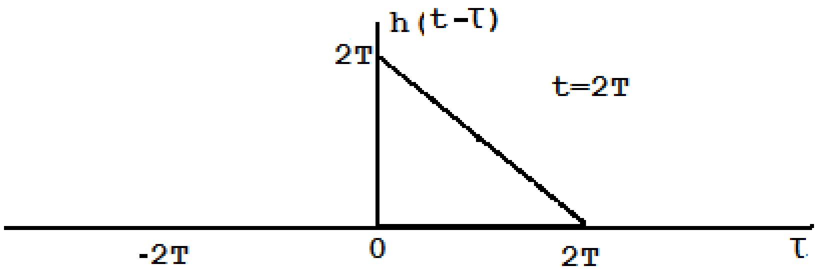
[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t) = \begin{cases} 1 & ;0<t<T \\ 0 & ;otherwise \end{cases}$; $h(t) = \begin{cases} t & ;0<t<2T \\ 0 & ;otherwise \end{cases}$

4'- after T till 2T there is a total overlapping :

at t=T → there is overlapping from (0) to (T)
at t=3T/2 → there is overlapping from (0) to (T)
at t=2T → there is overlapping from (0) to (T)

then the overlapping boundaries of this area are: L=0 (as the lower limit is fixed at 0)
U=T (as the upper limit is fixed to T)





5'- for T≤t <2T: $y(t) = \int_L^U x(\tau)h(t-\tau)d\tau$

$$\therefore \quad y(t) = \int_0^T x(\tau)h(t-\tau)d\tau$$

$$\therefore \quad y(t) = \int_0^T 1.(t-\tau)d\tau$$

$$\therefore y(t) = \int_0^T t d\tau - \int_0^T \tau d\tau$$

$$= t\{\tau|_0^T\} - \{\frac{\tau^2}{2}\Big|_0^T\}$$

$$= t\{T-0\} - \{\frac{T^2}{2} - 0\}$$

$$= tT - \frac{T^2}{2}$$

$$\therefore y(t) = Tt - \frac{T^2}{2} \quad ; T \le t < 2T$$

6'- Repeat steps 4 and 5 for 2T≤t<3T (partial overlapping)

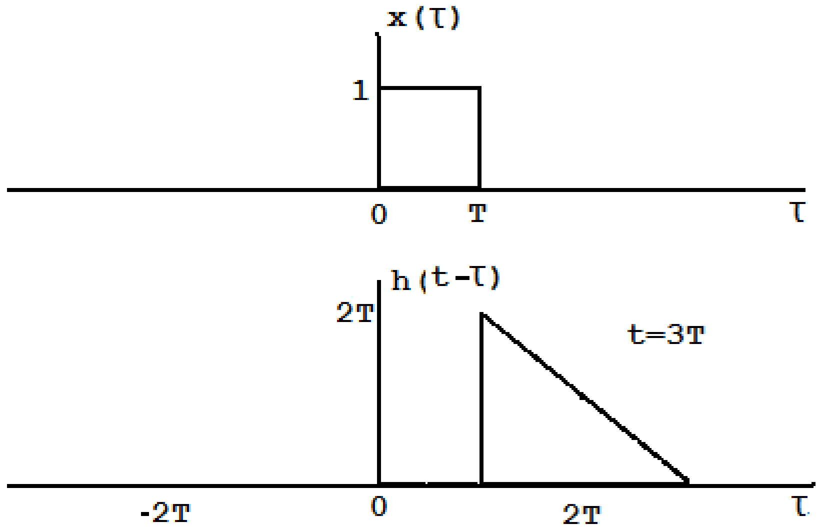
➢ Convolution Integral : Examples

[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t) = \begin{cases} 1 & ; 0<t<T \\ 0 & ; otherwise \end{cases}$; $h(t) = \begin{cases} t & ; 0<t<2T \\ 0 & ; otherwise \end{cases}$

4"- after 2T till 3T there is a partial overlapping :

at t=2T → there is overlapping from (0) to (T)
at t=5T/2 → there is overlapping from (T/2) to (T)
at t=3T → there is overlapping from (T) to (T)

then the overlapping boundaries of this area are: L=t-2T (as the lower limit is fixed at 0)
U=T (as the upper limit is fixed to T)



5"- for 2T≤t <3T: $y(t) = \int_{L}^{U} x(\tau)h(t-\tau)d\tau$

$$\therefore \quad y(t) = \int_{t-2T}^{T} x(\tau)h(t-\tau)d\tau$$

$$\therefore \quad y(t) = \int_{t-2T}^{T} 1.(t-\tau)d\tau$$

$$\therefore y(t) = \int_{t-2T}^{T} t d\tau - \int_{t-2T}^{T} \tau d\tau$$

$$= t\{\tau\Big|_{t-2T}^{T}\} - \{\frac{\tau^2}{2}\Big|_{t-2T}^{T}\}$$

$$= t\{T-(t-2T)\} - \{\frac{T^2}{2} - \frac{(t-2T)^2}{2}\}$$

$$= t\{-t+3T\} - \{\frac{T^2-(t^2-4tT+4T^2)}{2}\}$$

$$= -t^2 + 3tT + \frac{3T^2}{2} + \frac{t^2}{2} - 2tT$$

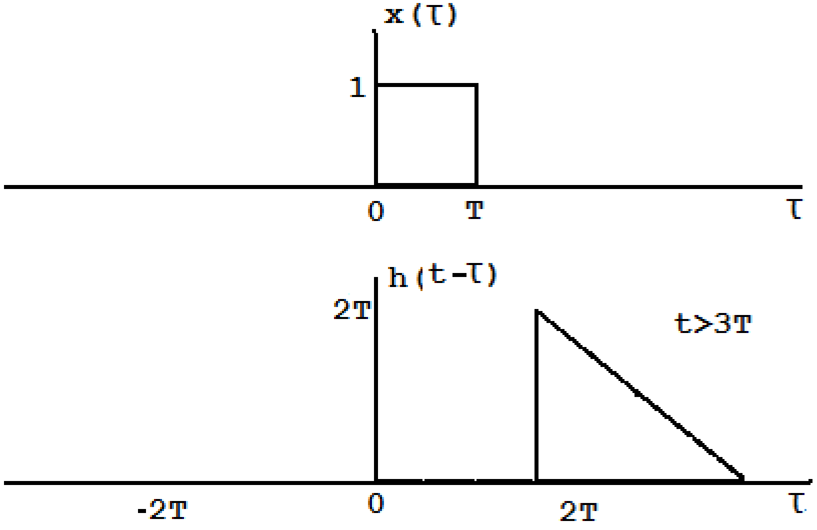
$$\therefore \quad y(t) = -\frac{t^2}{2} + Tt + \frac{3T^2}{2} \quad ; \quad 2T \le t < 3T$$

➢ Convolution Integral : Examples

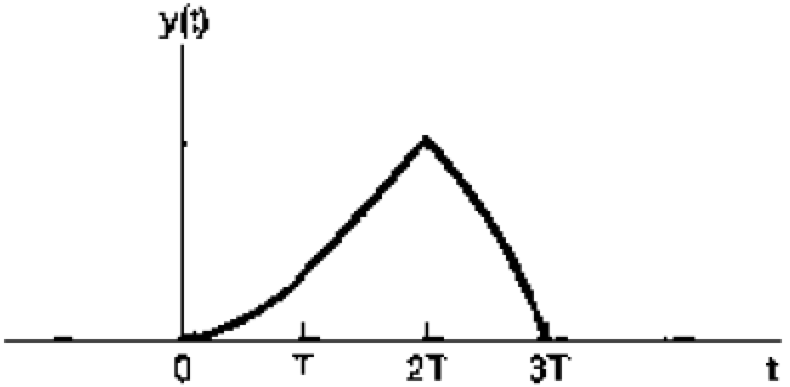
[2]- Compute the convolution sum of the following input x(t) and system impulse response h(t): $x(t)=\begin{cases}1 & ;0<t<T\\ 0 & ;otherwise\end{cases}$; $h(t)=\begin{cases}t & ;0<t<2T\\ 0 & ;otherwise\end{cases}$

4'''- after 3T there is NO overlapping

→ y(t)=0 , t>3T



$$\therefore y(t) = \begin{cases} 0 & ; \quad t<0 \\ \dfrac{t^2}{2} & ; \quad 0 \le t < T \\ Tt - \dfrac{T^2}{2} & ; \quad T \le t \le 2T \\ -\dfrac{t^2}{2} + Tt + \dfrac{3T^2}{2} & ; \quad 2T < t \le 3T \\ 0 & ; \quad t > 3T \end{cases}$$



Signals and Systems

Lecture # 12

LTI Systems Properties

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Topics of the lecture:

- **Properties of Linear Time-Invariant Systems .**
  - **Commutative**
  - **Distributive**
  - **Associative**
  - **Memoryless**
  - **Invertibility**
  - **Causality**
  - **Stability**

➢ Properties of Linear Time-Invariant Systems

We developed LTI System Representation as follow:

for discrete – time :

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n]*h[n]$$

for continuous – time :

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = x(t)*h(t)$$

1- The commutative property:

$$let \quad r = n-k \qquad \therefore \quad k = n-r$$

$$\therefore \quad when \quad k=-\infty \quad \rightarrow \quad r = n-(-\infty) = +\infty$$

$$and \quad when \quad k=+\infty \quad \rightarrow \quad r = n-(+\infty) = -\infty$$

The convolution sum in terms of (r) becomes:

$$\therefore \quad y[n] = \sum_{r=+\infty}^{-\infty} x[n-r]h[r]$$

As (r) is a running dummy variable, we can rename it as (k) again and exchange the sum limits without any effect:

$$\therefore \quad y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = h[n]*x[n]$$

Similarly:

$$let \quad w = t-\tau \quad \therefore \quad \tau = t-w$$

$$\therefore \quad dw = -d\tau$$

$$\therefore \quad when \quad \tau=-\infty \quad \rightarrow \quad w = t-(-\infty) = +\infty$$

$$and \quad when \quad \tau=+\infty \quad \rightarrow \quad w = t-(+\infty) = -\infty$$

The convolution sum in terms of (r) becomes:

$$\therefore \quad y(t) = \int_{+\infty}^{-\infty} x(t-w)h(w)(-dw)$$

As (w) is a running dummy variable, we can rename it as (τ) again and exchange the integration limits and multiply by (-1):

$$\therefore \quad y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau = h(t)*x(t)$$

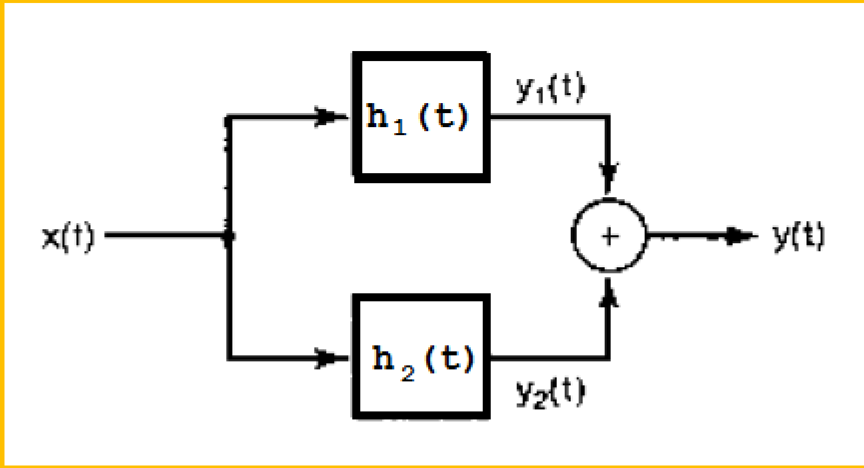
➢ Properties of Linear Time-Invariant Systems

2- The distributive property:

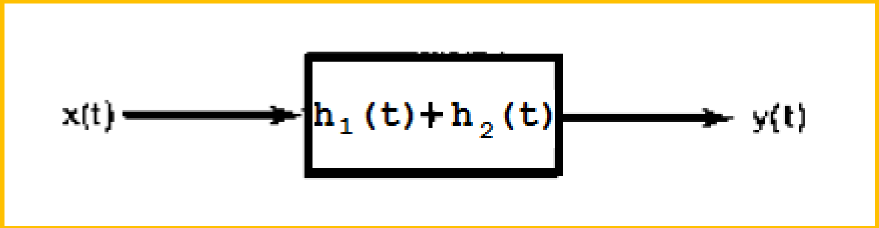
$$x(t) * \{ h_1(t) + h_2(t) \} = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x[n] * \{ h_1[n] + h_2[n] \} = x[n] * h_1[n] + x[n] * h_2[n]$$

Graphical Interpretation:



If we have two systems connected in parallel then we can build an equivalent system that have an impulse response equal to the addition of the impulse responses of the original two systems.



2- The distributive property: (follow)

Example:

$$x[n] = \left\{\frac{1}{2}\right\}^n u[n] + 2^n u[-n]$$

$$h[n] = u[n]$$

Then:

An idea to get the convolution between x[n] and h[n] is as follow:

$$x[n] = \left\{\frac{1}{2}\right\}^n u[n] + 2^n u[-n]$$
$$= x_1[n] + x_2[n]$$

$$\therefore\ y[n] = x[n] * h[n] = h[n] * x[n]$$
$$= h[n] * \{ x_1[n] + x_2[2] \}$$
$$= h[n] * x_1[n] + h[n] * x_2[2]$$

Assignment: **Complete the example?**

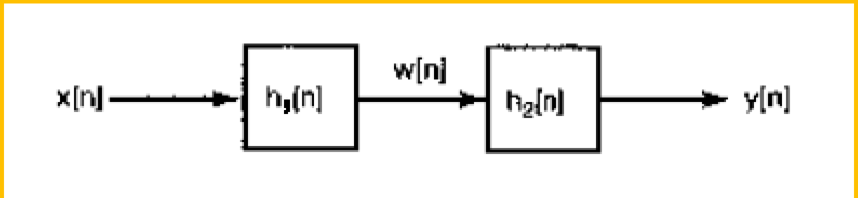
➢ Properties of Linear Time-Invariant Systems

3- The associative property:

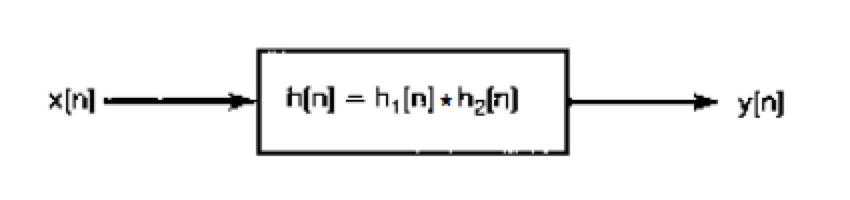
$$x(t) * \{ h_1(t) * h_2(t) \} = \{ x(t) * h_1(t) \} * h_2(t) = x(t) * h_1(t) * h_2(t)$$

$$x[n] * \{ h_1[n] * h_2[n] \} = \{ x[n] * h_1[n] \} * h_2[n] = x[n] * h_1[n] * h_2[n]$$

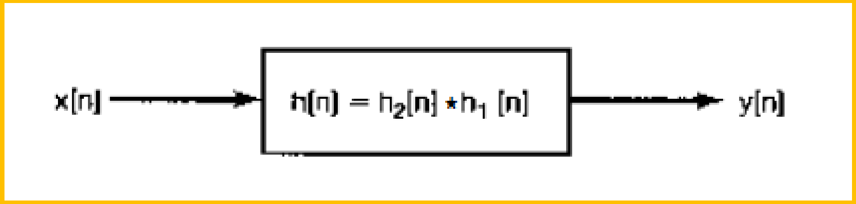
Graphical Interpretation:

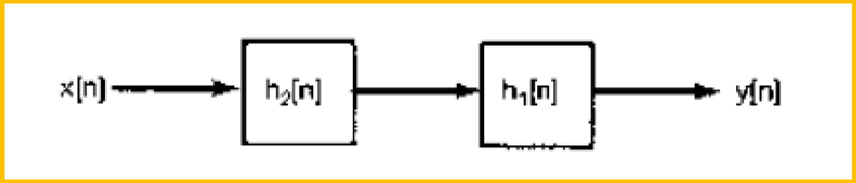


Associative Property →



Commutative Property →





Properties of Linear Time-Invariant Systems

4- LTI systems with and without memory:

Remember:
The "Memoryless" system its output at any time instance depends only on the input at that time instant.

Recall :
the convolution formula :

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

Then the only way to keep the output y[n] at any time instance (n_o) to depend only on the input $x[n_o]$ is to keep the running variable (k) at that time instance only i.e. $k = n_o$

Then the only condition that make that true is:

$$h[n_o-k]=0 \quad \text{for} \quad n_o \neq k$$

OR $\boxed{h[n]=0 \quad \text{for} \quad n \neq 0}$ $\longrightarrow$ **The condition of any system to be memoryless**

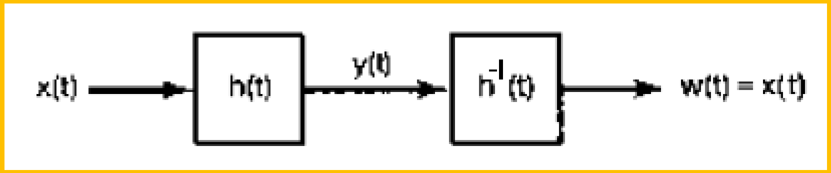
In this case $h[n] = W\,\delta[n]$,
where $W = h[0]$

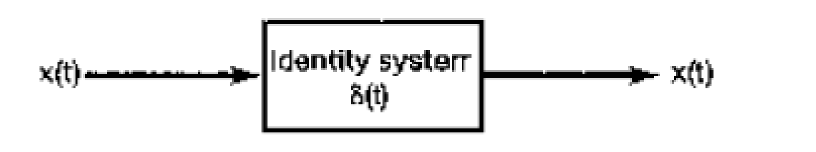
For continuous-time systems the condition to be memoryless is $\boxed{h(t)=0 \quad \text{for} \quad t \neq 0}$

➢ Properties of Linear Time-Invariant Systems

5- Invertibility of LTI systems :

The system has an inverse if →





Then:

The impulse response $h^{-1}(t)$ must satisfy the following condition: $h(t) * h^{-1}(t) = \delta(t)$

Example: $y(t) = x(t - t_o)$

If the input equal the impulse, then the output will be the impulse response → $h(t) = \delta(t - t_o)$

Then the output can be re-written as : $y(t) = x(t) * h(t)$

$$\therefore \quad x(t - t_o) = x(t) * \delta(t - t_o)$$

i.e. the convolution of a signal with a shifted impulse simply shifts the signal. Then:

As we need to shift back y(t) to return to x(t), we can do this by convolving it with a shifted impulse

$$\therefore \quad x(t) = y(t) * \delta(t + t_o)$$

$$\therefore \quad h^{-1}(t) = \delta(t + t_o)$$

$$\therefore \quad h(t) * h^{-1}(t) = \delta(t - t_o) * \delta(t + t_o) = \delta(t) \quad \#$$

➢ Properties of Linear Time-Invariant Systems

5- Invertibility of LTI systems : (followed)

Example II: If the impulse response → $h[n] = u[n]$

Then the output can be written as : $y[n] = x[n] * u[n]$

$$\therefore \quad y[n] = \sum_{k=-\infty}^{\infty} x[k] * u[n-k]$$

As u[n-k] =0 for n-k <0 i.e. for k >n and u[n-k]=1 otherwise → then the output can be re-written as :

$$\therefore \quad y[n] = \sum_{k=-\infty}^{n} x[k]$$

i.e. y[n] is the running sum of x[n]. If we need to get back x[n] , we can just subtract y[n] - y[n-1]

$$\therefore \quad x[n] = y[n] - y[n-1]$$

$$\therefore \quad x[n] = y[n] * \delta[n] - y[n] * \delta[n-1] = y[n] * \{ \delta[n] - \delta[n-1] \}$$

$$\therefore \quad h^{-1}[n] = \delta[n] - \delta[n-1]$$

$$\begin{aligned} \therefore \quad h[n] * h^{-1}[n] &= u[n] * \{ \delta[n] - \delta[n-1] \} \\ &= u[n] * \delta[n] - u[n] * \delta[n-1] \\ &= u[n] - u[n-1] \\ &= \delta[n] \qquad \# \end{aligned}$$

➢ Properties of Linear Time-Invariant Systems

6- Causality of LTI systems :

Recall: the output of a causal system does not depend on future inputs.

As the output of LTI system given by:

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] * h[n-k]$$

Then at any time instance n the RHS of the previous equation should not include x[k] for k > n

to make this possible then:

$$h[n-k] = 0 \quad ; \quad for\ k > n$$

$$\therefore \quad h[n-k] = 0 \quad ; \quad for\ n-k < 0$$

which can be re-written as:

$$\boxed{\therefore \quad h[n] = 0 \quad ; \quad for\ n < 0}$$

The same relation applies for causal continuous-time system for which :

$$\boxed{\therefore \quad h(t) = 0 \quad ; \quad for\ t < 0}$$

If we combine this with the definition casual *signal* (x[n]=0, for n<0), then:
The LTI system is causal if its impulse response is causal.

➢ Properties of Linear Time-Invariant Systems

7- Stability of LTI systems :

Recall: the system is causal if bounded inputs lead to bounded outputs (BIBO).

If the input signal is bounded: $|\, x[n] \,| \leq B \quad for\ all\ n$

If this input is applied to a LTI system with impulse response h[n] then :

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

$$as \quad |\, A + C \,| \leq |\, A \,| + |\, B \,| \;\; then \quad \therefore |y[n]| \;\leq \sum_{k=-\infty}^{\infty} \; |h[k]x[n-k]|$$

$$as \quad |\, A.C \,| = |\, A \,| . |\, B \,| \;\; then \quad \therefore |y[n]| \;\leq \sum_{k=-\infty}^{\infty} \; |h[k]||x[n-k]|$$

$$\therefore |y[n]| \;\leq \sum_{k=-\infty}^{\infty} \; B|h[k]| \quad \Longrightarrow \quad |y[n]| \leq B \sum_{k=-\infty}^{\infty} \; |h[k]|$$

Then for |y[n]| to be bounded this requires: $\sum_{k=-\infty}^{\infty} \; |h[k]| < \infty$ **OR** $\sum_{n=-\infty}^{\infty} \; |h[n]| < \infty$

The same relationship applies for continuous-time systems too: $\int_{-\infty}^{\infty} |h(t)| \; dt < \infty$