Laplace Transform

Pierre-Simon Laplace {DoB $\overset{23/3}{1749}$, DoM 1788, DoD $\overset{5/3}{1827}$}

هو عالم رياضيات وفلكي فرنسي طور علم الرياضيات الفلكية وبفضله تغيرت دراسة الهندسة من العلوم القديمة الى التفاضل والتكامل، وله مؤلفات من خمس مجلدات في ميكانيكا الأجرام السماوية

Laplace Transform is a way for studying and analyzing LTI systems.

The idea behind any such transform is trying to analyze the inputs into a set of basic signals that have known and simple output, then the output will be the linear combination of such single outputs of these basic signals

<u>Benefits</u>:- ① provide a set of rich tools to study and analyze the continuous-time LTI systems

② Extend the Fourier Transform to study more systems even if they are unstable.

③ Convert the convolution, time-shifts, differentiations and integrations into algebraic operations {+, -, /, *}

if the input to cts LTI system is $x(t) = e^{st}$
then the output is $y(t) = \int_{-\infty}^{+\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{+\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$

$$\therefore y(t) = \int_{-\infty}^{+\infty} h(\tau) . e^{st} . e^{-s\tau} d\tau = e^{st} . \int_{-\infty}^{+\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

i.e. the output of e^{st} is the same e^{st} multiplied by a coefficient $H(s)$
So; e^{st} is Eigen function to LTI systems
and $H(s)$ is called Eigen-value

$$H(s) = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt$$

if s is pure imaginary $s = j\omega$ — if s is complex

$$\therefore \underbrace{H(j\omega) = \int_{-\infty}^{+\infty} h(t)\, e^{-j\omega t}\, dt}_{FT} \qquad \underbrace{H(s) = \int_{-\infty}^{+\infty} h(t)\, e^{-st}\, dt}_{\mathcal{L}T}$$

Ex1 :-

$$x(t) = e^{-at}\,u(t)$$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{-\infty}^{+\infty} e^{-at}\,.\,u(t)\,.\,e^{-st}\,dt$$

$$= \int_{0}^{+\infty} e^{-(s+a)t}\,dt = \frac{e^{-(s+a)t}}{-(s+a)}\Big|_{0}^{\infty}$$

$$= \frac{1}{-(s+a)}\left\{ e^{-(s+a)\infty} - 1 \right\}$$

$X(s)$ to converge $e^{-(s+a)\infty}$ must be converge

as $s = \sigma + j\omega \Rightarrow e^{-(s+a)\infty} = e^{-(\sigma+a+j\omega)\infty} = e^{-(\sigma+a)\infty}\,.\,e^{-j\omega\infty}$

this is a complex number in polar form $|z|\,e^{j\angle z}$

this converge only if its magnitude $|z| < \infty$

as $e^{-j\omega\infty}$ is not going to ∞ as it is just rotating object in the unit circle $\Rightarrow$ i.e. determing the angle of z only.

then $X(s)$ to converge we need $\sigma + a > 0$

OR $Re\{s\} > -a$

if converges then $X(s) = \frac{1}{-(s+a)}\{0-1\} = \frac{-1}{-(s+a)} = \frac{1}{s+a}$

$$\Rightarrow \boxed{X(s) = \frac{1}{s+a} \quad ; \quad Re\{s\} > -a} \longrightarrow \text{(I)}$$

Ex 2 :-

$$x(t) = -e^{-at}\,u(-t)$$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt = \int_{-\infty}^{+\infty} (-)\,e^{-at}\,.\,u(-t)\,.\,e^{-st}\,dt$$

$$= \int_{-\infty}^{0} -e^{-at}\,.\,e^{-st}\,dt = -\int_{-\infty}^{0} e^{-(s+a)t}\,dt$$

$$= -\frac{1}{-(s+a)}\,e^{-(s+a)t}\Big|_{-\infty}^{0} = \frac{1}{s+a}\left\{1 - e^{+(s+a)\infty}\right\}$$

to converge $Re\{s\} + a < 0 \Rightarrow Re\{s\} < -a$

$$\Rightarrow \boxed{X(s) = \frac{1}{s+a}\{1-0\} = \frac{1}{s+a} \quad ; \quad Re\{s\} < -a} \rightarrow \text{(II)}$$

From Ex1 and Ex2 same algebraic expressions as stated in equations (I) and (II) but different region of convergence (abbreviated as RoC)

$\Rightarrow$ ROC is needed to specify Laplace transform

Ex 3 :- $x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t)$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st}dt = \int_{-\infty}^{+\infty}[3e^{-2t}u(t) - 2e^{-t}u(t)]e^{-st}dt$$

$$= \int_{-\infty}^{+\infty} 3e^{-2t}u(t)e^{-st}dt - \int_{-\infty}^{+\infty} 2e^{-t}u(t)e^{-st}dt$$

$$= 3\int_{0}^{+\infty} e^{-(s+2)t}dt - 2\int_{0}^{\infty} e^{-(s+1)t}dt$$

$$= 3\,X_1(s) - 2\,X_2(s)$$

From Ex1 by similarity $\Rightarrow$ $X_1(s) = \frac{1}{s+2}$; $Re\{s\} > -2 \equiv ROC1$

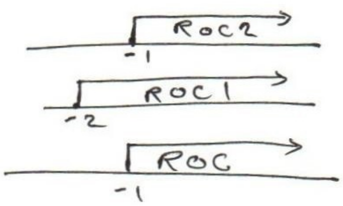
and $X_2(s) = \frac{1}{s+1}$; $Re\{s\} > -1 \equiv ROC2$

As $X(s) = 3X_1(s) - 2X_2(s)$ $Re\{s\}$ should satisfy both Roc1 & Roc2

i.e. ROC is the intersection of ROC1 and ROC2

$\therefore ROC \equiv ROC1 \cap ROC2$

$\therefore ROC \equiv Re\{s\} > -1$



$$\therefore X(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{3s+3-2s-4}{(s+2)(s+1)}$$

$$\therefore \boxed{X(s) = \frac{s-1}{s^2+3s+2} \quad ; \quad Re\{s\} > -1} \Longrightarrow \text{III}$$

Ex 4 :- $x(t) = e^{-2t}u(t) + e^{-t}[\cos(3t)]u(t)$

$\cos(3t) = \frac{1}{2}e^{j3t} + \frac{1}{2}e^{-j3t}$ as $e^{j\theta} = \cos(\theta) + j\sin(\theta)$

$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$

$\Rightarrow \cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta}$

$$\therefore x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-j3)t} + \frac{1}{2}e^{-(1+j3)t}\right]u(t)$$

$$\therefore X(s) = \int_0^{+\infty} e^{-(s+2)t}dt + \frac{1}{2}\int_0^{+\infty} e^{-(s+1-j3)t}dt + \frac{1}{2}\int_0^{+\infty} e^{-(s+1+j3)t}dt$$

using Ex1 $\Rightarrow$ $\frac{1}{s+2}$; $Re\{s\} > -2$ $\quad$ $\frac{1}{2}\cdot\frac{1}{s+1-j3}$; $Re\{s\} > -1$ $\quad$ $\frac{1}{2}\cdot\frac{1}{s+1+j3}$; $Re\{s\} -1$

as $-j3$ and $j3$ go with $j\omega$ to be $e^{-(3j+j\omega)t}$ and $e^{(-3j-j\omega)t}$

remember $s = \sigma + j\omega$

$$\therefore X(s) = \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s+(1-j3)}\right] + \frac{1}{2}\left[\frac{1}{s+(1+j3)}\right] ; \; Re\{s\} > -1$$

as $ROC = ROC1 \cap ROC2 \cap ROC3$

$$\therefore X(s) = \frac{1}{s+2} + \frac{1}{2}\left[\frac{s+(1+j3) + s+(1-j3)}{(s+1-j3)(s+1+j3)}\right]$$

$$= \frac{1}{s+2} + \frac{1}{2}\left[\frac{2s+2}{s^2+s+j3s+s-j3s+1+j3-j3+9}\right] = \frac{1}{s+2} + \frac{1}{2}\left[\frac{2(s+1)}{s^2+2s+10}\right]$$

$$\therefore X(s) = \frac{1}{s+2} + \frac{s+1}{s^2+2s+10} = \frac{s^2+2s+10+(s+1)(s+2)}{(s+2)(s^2+2s+10)} = \frac{s^2+2s+10+s^2+3s+2}{(s+2)(s^2+2s+10)}$$

$$\therefore \boxed{X(s) = \frac{2s^2+5s+12}{(s+2)(s^2+2s+10)} \quad ; \quad Re\{s\} > -1} \Longrightarrow \text{IV}$$

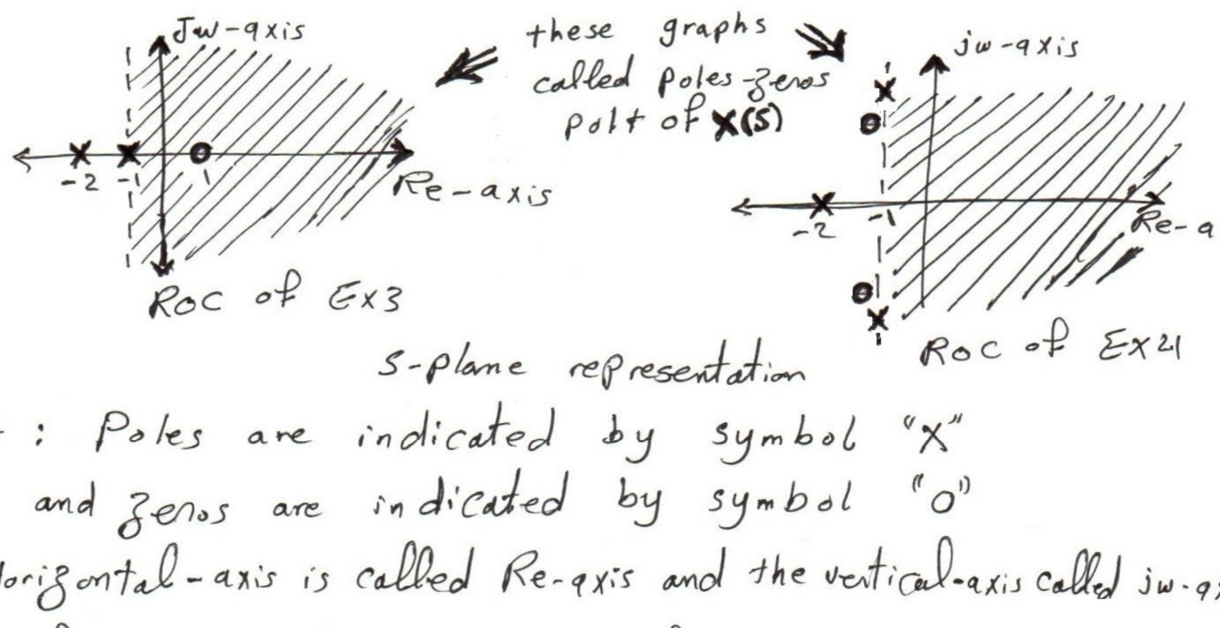
From EX1, EX2, EX3, and EX4 we conclude that
if the signal is a linear combination of Real OR Complex exponentials $\Rightarrow$ then its Laplace transform will be in the Form $\Rightarrow$ $X(s) = \frac{N(s)}{D(s)}$ = rational expression.

- Roots of $N(s)$ are called $\boxed{\text{Zeros}}$ of $X(s)$ $\Rightarrow$ as it makes $X(s) = 0$
- Roots of $D(s)$ are called $\boxed{\text{Poles}}$ of $X(s)$ $\Rightarrow$ as it makes $X(s) = \infty$

where $N(s)$ is the numerator polynomial in s
and $D(s)$ is the denominator polynomial in s

$\Rightarrow$ ROC Can NOT include poles; but may include zeros

- Poles and zeros are used to pictorialyze the ROC in s-plane $\Rightarrow$ i.e. gives us some information about the laplace transform in a graphical way
- For example the ROC's of EX3 and EX4 Can be explained in s-plane as Follow :-



s-plane representation

where : Poles are indicated by symbol "X"
and zeros are indicated by symbol "O"

and Horizontal-axis is called Re-axis and the vertical-axis called jw-axis

- if order of $D(s)$ is greater than the order of $N(s)$ by K $\Rightarrow$ there are K-zeros at ∞
- if " " $N(s)$ " " " " " " $D(s)$ by K $\Rightarrow$ ther are K-poles at ∞
- Poles and zeros at the same location Cancel each other

ROC properties

[1] $\boxed{\text{The ROC of } X(S) \text{ consists of strips parallel to the } j\omega\text{-axis in S-plane}}$

Explanation :- as $X(S) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-st} dt = \int_{-\infty}^{+\infty} x(t) . e^{-\sigma t} . e^{-j\omega} dt$

as $S = \sigma + j\omega$

- if $x(t)$ does not go to infinity {i.e. absolutely integrable} then the only possibly variable that control the convergence is σ which is the $Re\{s\}$ whatever ω is.
- So if the value σ_o is in the ROC then ALL the Line from $\{\sigma_o - j\infty\}$ to $\{\sigma_o + j\omega\}$ will be in the ROC too
- all these Lines/strips are parallel to the $j\omega$-axis.

[2] $\boxed{\text{For rational Laplace Transform; the ROC does not contain any pole}}$

Explanation :- the pole makes $X(s) = \infty \Rightarrow$ i.e. not converge

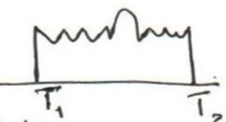
$\therefore$ ROC cannot contain poles at all

[3] $\boxed{\text{if } x(t) \text{ is of finite-duration and is absolutely integrable then the ROC is the entire S-plane}}$

Explanation :- $x(t)$ is finite-duration means finite-domain

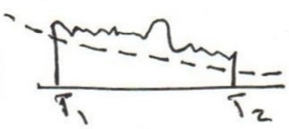
$x(t)$ is absolutely integrable means $|x(t)| < \infty \ \forall t$

$\Rightarrow$ $x(t)$ will be in general something like $\Rightarrow$



$\Rightarrow e^{-\sigma t}$ will be either decaying OR growing exponential

dot-line = decaying exponential



OR

dot-line = growing exponential

(A) $\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt$

σ is +ve $\equiv$ decaying exponential

$\therefore \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)| dt$

as $x(t)$ is absolutely integrable $\Rightarrow \int_{-\infty}^{+\infty} |x(t)| dt < \infty$

and $e^{-\sigma T_1} < \infty$ as σ +ve and $T_1 < \infty$

$\therefore \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} < \infty$

Except only if T_1 is -ve and $\sigma = +\infty$ needs check if there are poles at $\boxed{\infty = \sigma}$ not cancelled by zeros at ∞

(B) $\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt$

σ is -ve $\equiv$ growing exponential

$\therefore \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)| dt$

as $x(t)$ is absolutely integrable $\Rightarrow \int_{-\infty}^{+\infty} |x(t)| dt < \infty$

and $e^{-\sigma T_2} < \infty$ too as σ -ve and $T_2 < \infty$

$\therefore \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty$

Excep only if T_2 is +ve and $\sigma = -\infty$ needs check if there are poles at $\boxed{\sigma = -\infty}$ not cancelled by zeros at $\sigma = -\infty$

(C) $\boxed{\text{Note}}$ if σ is zero $\therefore \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt$ will be $\int_{T_1}^{T_2} |x(t)| dt$ which is $< \infty$ as $x(t)$ is absolutely integrable if $\sigma = 0 \Rightarrow \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt = \int_{T_1}^{T_2} |x(t)| dt < \infty$

as $x(t)$ is absolutely integrable

Note:- if $x(t)$ is in general complex $\Rightarrow x(t) = |x(t)| e^{j \angle x(t)}$

then the only parameter that controls the converges is $|x(t)|$

$\therefore \int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty$ Except if $(-\sigma t) = \infty$ needs check
or if $\sigma t = -\infty$

$\boxed{4}$ $\boxed{\text{if } x(t) \text{ is right-sided, and if the line } Re\{s\} = \sigma_0 \text{ is in the ROC then ALL values of } s \text{ for which } Re\{s\} > \sigma_0 \text{ will also be in the ROC}}$

Explanations- right-sided means the domain of non-zero of $x(t)$ will start from a value $t = T$, and extends to $t = +\infty$

$$\therefore \int_{-\infty}^{+\infty} |x(t)| e^{-st} dt = \int_{T}^{+\infty} |x(t)| e^{-st} dt$$

if $\sigma_0 \in ROC \Rightarrow \int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$

and if $\sigma_1 > \sigma_0$ then σ_1 is more positivity than σ_0 $\Rightarrow \boxed{\sigma_1 - \sigma_0 = +ve}$

$$\therefore \int_{T}^{+\infty} |x(t)| e^{-\sigma_1 t} dt = \int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} \cdot e^{-(\sigma_1 - \sigma_0) t} dt$$

then the expression $-(\sigma_1 - \sigma_0) t$ decreases as t increases towards $+\infty$

$$\therefore \int_{T}^{+\infty} |x(t)| e^{-\sigma_1 t} dt \leq e^{-(\sigma_1 - \sigma_0) T} \int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} dt$$

as $(\sigma_1 - \sigma_0) T < \infty$ and as $\int_{T}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$; as $\sigma_0 \in ROC$

then $\int_{T}^{+\infty} |x(t)| e^{-\sigma_1 t} dt < \infty$

$\Rightarrow \therefore \sigma_1 \in ROC$ which is called $\boxed{\text{right-half plane}}$ in this case

$\boxed{5}$ $\boxed{\text{if } x(t) \text{ is left-sided, and if the line } Re\{s\} = \sigma_0 \text{ is in the ROC Then ALL values of } s \text{ for which } Re\{s\} < \sigma_0 \text{ will also be in the ROC}}$

Explanation:- left-sided means $\int_{-\infty}^{+\infty} x(t) e^{-st} dt = \int_{-\infty}^{T} x(t) e^{-st} dt$

if $\sigma_0 \in ROC \Rightarrow \int_{-\infty}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$

if $\sigma_1 < \sigma_0$ then σ_1 is more negativity than $\sigma_0 \Rightarrow \sigma_0 - \sigma_1 = +ve$
OR $\Rightarrow \boxed{\sigma_1 - \sigma_0 = -ve}$

$$\therefore \int_{-\infty}^{T} |x(t)| e^{-\sigma_1 t} dt = \int_{-\infty}^{T} |x(t)| e^{-\sigma_0 t} \cdot e^{-(\sigma_1 - \sigma_0) t} dt$$

$$\leq e^{-(\sigma_1 - \sigma_0) T} \int_{-\infty}^{T} |x(t)| e^{-\sigma_0 t} dt < \infty$$

as $-(\sigma_1 - \sigma_0) t$ decreases as t decreases towards $-\infty$

as $-(\sigma_1 - \sigma_0) T < \infty$ and $\int_{-\infty}^{T} |x(t)| e^{-\sigma_0 t} dt < \infty$ as σ_0 is in ROC

$\Rightarrow \therefore \sigma_1$ also $\in ROC$ which is called $\boxed{\text{left-half plane}}$ in this case

6 if $x(t)$ is two-sided, and if the line $Re\{s\} = \sigma_0$ is in the ROC Then the ROC will consist of a strip in s-plane that includes line $Re\{s\} = \sigma_0$

Explanation:

The two-sided signal is a signal that is of infinite extent for both $t > 0$ and $t < 0$

then $x(t)$ can be divided into two sub-signals.

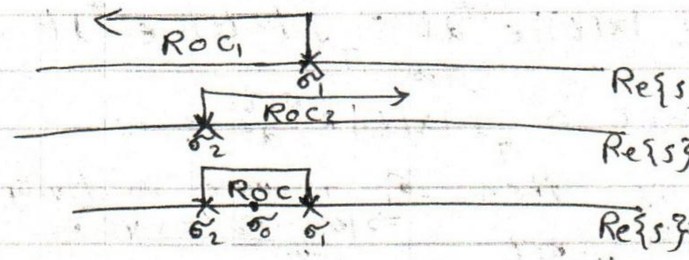
one $x_1(t)$ that is left-sided with ROC_1 left-half plane

and one $x_2(t)$ that is right-sided with ROC_2 is right-half plane

i.e. ROC_1 comes from $-\infty$ and bounded by a pole at σ_1

and ROC_2 comes from specific pole at σ_2 and goes to $+\infty$.

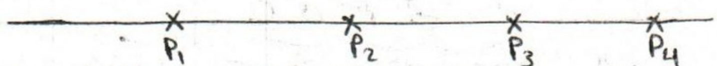
and as $X(s)$ exist then $ROC = ROC_1 \cap ROC_2$



Then ROC will be strip bounded by two poles that includes σ_0

7 if the Laplace transform $X(s)$ is rational the its ROC is bounded by poles or extends to infinity, no poles are contained in the ROC

Explanation: if $X(s)$ is rational $\frac{N(s)}{D(s)}$ and there are poles



and as ROC never includes poles

then ROC will stopped by a pole or goes to infinity

stopped here ≡ bounded

8 if $X(s) = \frac{N(s)}{D(s)}$; then if $x(t)$ is right-sided then ROC right to the rightmost pole

And if $x(t)$ is lef-sided then ROC left to the leftmost pole

it is a conclusion of previous properties of ROC

1 Linearity

if $x_1(t) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s)$, ROC $= R_1$

and $x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} X_2(s)$, ROC $= R_2$

Then $a x_1(t) \pm b x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} a X_1(s) \pm b X_2(s)$

with ROC containing $R_1 \cap R_2$

proof

let $x_3(t) = a x_1(t) \pm b x_2(t) \overset{\mathcal{L}}{\longleftrightarrow} \int_{-\infty}^{+\infty} x_3(t) e^{-st} dt = X_3(s)$

$$\therefore X_3(s) = \int_{-\infty}^{+\infty} \{a x_1(t) \pm b x_2(t)\} e^{-st} dt$$

$$\therefore X_3(s) = \int_{-\infty}^{+\infty} a x_1(t) e^{-st} dt \pm \int_{-\infty}^{+\infty} b x_2(t) e^{-st} dt$$

$$\therefore \boxed{X_3(s) = a \; X_1(s) \quad \pm b \; X_2(s)}$$

with ROC_3 should be satisfy both terms at the same time

i.e. ROC_3 should contain $R_1 \cap R_2$

But: may be when terms be added together

Some zeros cancel some poles

$\Rightarrow$ ROC_3 may be larger than $R_1 \cap R_2$

So, $\boxed{ROC_3 \text{ should contain at least } R_1 \cap R_2}$

2 Time-shift if $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, ROC $= R$

then $x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} e^{-st_0} X(s)$; ROC $= R$

proof

let $x_1(t) = x(t - t_0) \overset{\mathcal{L}}{\longleftrightarrow} X_1(s) = \int_{-\infty}^{+\infty} x_1(t) e^{-st} dt$

$$\therefore X_1(s) = \int_{-\infty}^{+\infty} x(t - t_0) e^{-st} dt$$

let $\tau = t - t_0 \Rightarrow t = \tau + t_0 \Rightarrow dt = d\tau \Rightarrow$ $t = -\infty \Rightarrow \tau = -\infty - t_0 \;\therefore \tau = -\infty$; $t = +\infty \Rightarrow \tau = +\infty - t_0 \;\therefore \tau = +\infty$

$$\therefore X_1(s) = \int_{-\infty}^{+\infty} x(\tau) e^{-s(\tau + t_0)} d\tau$$

$$= \int_{-\infty}^{+\infty} x(\tau) e^{-s\tau} e^{-st_0} d\tau = e^{-st_0} \cdot \int_{-\infty}^{+\infty} x(\tau) e^{-s\tau} d\tau$$

$\therefore \boxed{X_1(s) = e^{-st_0} \; X(s)}$ with values of s to converge is the same as $\in R$

as $X(s)$ exist $\Rightarrow \therefore e^{-s\tau}$ converge for some values of s for any τ including $\tau = t_0$ $\Rightarrow \boxed{\therefore ROC_1 = ROC = R}$

3 Shifting in S-Domain:- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$; ROC $= R$

Then $e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0)$; ROC $= R + Re\{s_0\}$

Proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$\Rightarrow X(s-s_0) = \int_{-\infty}^{+\infty} x(t)\, e^{-(s-s_0)t}\, dt = \int_{-\infty}^{+\infty} x(t)\, e^{-st} e^{s_0 t}\, dt$$

$$= \int_{-\infty}^{+\infty} \{x(t) e^{s_0 t}\}\, e^{-st}\, dt$$

$$\therefore \boxed{e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s-s_0)}$$

as $X(s)$ converges with $s \in R \Rightarrow$ we need converging for $\{s_0\}$ Too

$\therefore$ $\boxed{\text{ROC of } X(s-s_0) \text{ should be } R + Re\{s_0\}}$

i.e. if values of s in R then values of $s + Re\{s_0\}$ in ROC of $X(s-s_0)$,

4 Time-scaling:- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$; ROC $= R$

Then $x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right)$; ROC $= aR$

Proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

let $x_1(t) = x(at) \Rightarrow X_1(s) = \int_{-\infty}^{+\infty} x(at)\, e^{-st}\, dt$

let $\tau = at \Rightarrow t = \frac{\tau}{a} \Rightarrow d\tau = a\, dt$

at $t = -\infty \Rightarrow \tau = a(-\infty) = -\infty$ if $a > 0$

$= +\infty$ if $a < 0$

at $t = +\infty \Rightarrow \tau = a(+\infty) = +\infty$ if $a > 0$

$= -\infty$ if $a < 0$

$$\therefore X_1(s) = \int_{-\infty}^{+\infty} x(at)\, e^{-\frac{s}{a}\tau} \cdot a \cdot \frac{dt}{a}$$

$$= \begin{cases} \frac{1}{a} \int_{-\infty}^{+\infty} x(at)\, e^{-\frac{s}{a}\tau}\, d\tau & \text{if } a > 0 \\ \frac{1}{a} \int_{+\infty}^{-\infty} x(at)\, e^{-\frac{s}{a}\tau}\, d\tau & \text{if } a < 0 \end{cases}$$

$$= -\frac{1}{a} \int_{-\infty}^{+\infty} x(at)\, e^{-\frac{s}{a}\tau}\, d\tau \quad \text{if } a < 0 \text{ (Note limits of } \int_l^u)$$

as $at = \tau \Rightarrow \boxed{X_1(s) = \frac{1}{|a|} \int_{-\infty}^{+\infty} x(\tau)\, e^{-\left(\frac{s}{a}\right)\tau}\, d\tau = \frac{1}{|a|} X\left(\frac{s}{a}\right)}$

- For the values of s that $\in R \Rightarrow$ those values make the integration converges of $X(s)$

$\therefore (a.s)$ make X_1 converge $\Rightarrow \boxed{ROC_1 = aR}$ $\Rightarrow$ scaling in S-Domain $\Rightarrow$ if pole at s_0 $\Rightarrow$ its new location as $= s_0$ is $s = \frac{s_0}{a}$

5 Conjugation :- if $x(t) \xleftrightarrow{\mathcal{L}} X(s)$; ROC = R

then $x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)$; ROC = R

proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt$$

$$X^*(s) = \left[\int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt\right]^*$$

$$= \int_{-\infty}^{+\infty} x^*(t)\, e^{-s^*t}\, dt$$

Replace s by $s^* \Rightarrow X^*(s^*) = \int_{-\infty}^{+\infty} x^*(t)\, e^{-st}\, dt = \mathcal{L}\{x^*(t)\}$

$\therefore \boxed{x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*)}$

if $x(t)$ is real $\Rightarrow x(t) = x^*(s)$

then $X^*(s^*) = \int_{-\infty}^{+\infty} x(t)\, e^{-st}\, dt = X(s)$

$\Rightarrow \boxed{X(s) = X^*(s^*) \text{ when } x(t) \text{ is real}}$

Then as $\{x(t)\, e^{-st}\}$ converge for some values of s which are R

Then $\{x^*(t)\, e^{-st}\}$ will converge too for the same set of values of s = R

as $|x(t)| = |x^*(t)|$ $\Rightarrow \boxed{\text{ROC of } X^*(s^*) = R}$

Magnitude

6 Convolution :- if $x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s)$, $ROC_1 = R_1$

and $x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s)$, $ROC_2 = R_2$

then $x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)\, X_2(s)$; RoC containing $R_1 \cap R_2$

proof

let $x_3(t) = x_1(t) * x_2(t) = \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, dt$ as $*$ means convolution

$\therefore \quad X_3(s) = \int_{-\infty}^{+\infty} x_3(t)\, e^{-st}\, dt = \int_{-\infty}^{+\infty} \left\{ \int_{-\infty}^{+\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right\} e^{-st}\, dt$

$$= \int_{-\infty}^{+\infty} x_1(\tau) \left\{ \int_{-\infty}^{+\infty} x_2(t-\tau)\, e^{-st}\, dt \right\} d\tau$$

using Time-shift property $\Rightarrow \int_{-\infty}^{+\infty} x_2(t-\tau)\, e^{-st}\, dt = e^{-s\tau} X_2(s)$

$$\therefore \quad X_3(s) = \int_{-\infty}^{+\infty} x_1(\tau) \cdot e^{-s\tau} \cdot X_2(s)\, d\tau$$

$\therefore \boxed{X_3(s) = X_2(s) \cdot \int_{-\infty}^{+\infty} x_1(\tau)\, e^{-s\tau}\, d\tau = X_2(s)\, X_1(s)} = X_1(s)\, X_2(s)$

Here the RoC should contain set of values of s for which Both $X_1(s)$ and $X_2(s)$ Converge $\Rightarrow \boxed{ROC_3 \text{ Containing } R_1 \cap R_2}$

as the multiplication of $X_1(s) \cdot X_2(s)$ could leads to zeros cancel poles which may lead to that ROC_3 may be bigger than $R_1 \cap R_2$

[7] Differentiation in Time-Domain: if $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$, ROC = R

then $\frac{d}{dt}x(t) \overset{\mathcal{L}}{\longleftrightarrow} s\,X(s)$; ROC containing R

Proof

$$x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds \Rightarrow \text{From inverse Laplace}$$

differentiate both sides with respect to t $\Rightarrow \frac{d}{dt}x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} s\,X(s)e^{st}ds$

Note X(s) is constant

$$\therefore \frac{dx(t)}{dt} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}[s\,X(s)]\,e^{st}ds \equiv \text{definition of inverse Laplace for } s\,X(s)$$

$$\therefore \boxed{\frac{d\,x(t)}{dt} \overset{\mathcal{L}}{\longleftrightarrow} s\,X(s)}$$

So if X(s) has a pole at s=0 will be cancelled, then ROC of sX(s) may be larger than R but at least ROC contain R

[8] Differentiation in S-Domain: if $x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s)$; ROC = R

then $-t\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d}{ds}X(s)$; ROC = R

Proof

$$X(s) = \int_{-\infty}^{+\infty} x(t)\,e^{-st}\,dt$$

differentiate both sides with respect to s $\Rightarrow \frac{d}{ds}X(s) = \int_{-\infty}^{+\infty}(-t)\,x(t)\,e^{-st}dt$

Note x(t) is constant

$$= \mathcal{L}\{(-t)\,x(t)\}$$

$$\Rightarrow \boxed{(-t)\,x(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{d\,X(s)}{ds}}$$

Note that the same set of s required for X(s) to be converge is the same set required here for sX(s) to converge

$$\therefore \boxed{\text{ROC of } \frac{d\,X(s)}{ds} = R}$$

[9] Integration in Time: as $e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}$; $Re\{s\} > -a$

$\Rightarrow$ as $a=0 \Rightarrow u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}$; $Re\{s\} > 0$

And as $\boxed{\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s}X(s)}$ with $\boxed{\text{ROC containing } R \cap \{Re\{s\} > 0\}}$

with ROC may be larger if X(s) has zero at s=0

<u>Ex</u>

$$x(t) = u(t)$$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-st} dt = \int_{-\infty}^{+\infty} u(t) e^{-st} dt = \int_0^{\infty} e^{-st} dt = \frac{1}{-s}\left[e^{-st} \Big|_0^{\infty} \right]$$

$$\therefore X(s) = \frac{-1}{s}[0-1] = \frac{1}{s} \quad ; \quad Re\{s\} > 0$$

$$\therefore \quad \boxed{u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \quad ; \quad Re\{s\} > 0}$$

<u>Ex</u>

$$x(t) = \delta(t)$$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-st} dt = \int_{-\infty}^{+\infty} \delta(t) e^{-st} dt = 1 \; ; \; ROC \equiv \text{entire s-plane}$$

$$\therefore \quad \boxed{\delta(t) \overset{\mathcal{L}}{\longleftrightarrow} 1 \quad ; \quad ROC = \text{entire s-plane}}$$

The variable s is complex in general i.e. $s = \sigma + j\omega$

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t) e^{-st} dt = \int_{-\infty}^{+\infty} [x(t) e^{-\sigma t}] e^{-j\omega t} dt$$

which means Fourier Transform $X(j\omega) = X(s)\big|_{s=j\omega}$

and Laplace Transform is $X(\sigma + j\omega) = FT\{x(t) e^{-\sigma t}\}$

$\therefore$ The Inverse Laplace Transform can be defined as

$$FT^{-1}\{X(\sigma + j\omega)\} = x(t) e^{-\sigma t}$$

$$\therefore \quad x(t) e^{-\sigma t} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{(j\omega)t} d\omega$$

multiplying both sides by $e^{\sigma t}$

$$\therefore \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{\sigma t} \cdot e^{j\omega t} d\omega$$

$$\because \quad s = \sigma + j\omega \Rightarrow ds = j\,d\omega \Rightarrow d\omega = \frac{ds}{j}$$

$$\therefore \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega) e^{(\sigma + j\omega)t} \cdot \frac{ds}{j}$$

$$\therefore \quad \boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma - \infty}^{\sigma + \infty} X(s) e^{st} ds}$$

is the inverse Laplace Transform.

as when $\omega = -\infty \Rightarrow s = \sigma + j(-\infty) = \sigma - j\infty$

at $\omega = +\infty \Rightarrow s = \sigma + j(+\infty) = \sigma + j\infty$

which can be calculated for any $\sigma \in ROC$

i.e. The inverse laplace transform can be calculated using previous Formula using any vertical line lies in the ROC

Partial Fractions

$$X(s) = \frac{N(s)}{D(s)}$$

① if the degree of $D(s)$ is NOT GREATER than the degree of $N(s)$, then Do Long Division.

② Factor out $D(s)$:-
A- Find Any common factors.
B- Try to reduce using Known identies.
C- Do NOT reduce to complex factors.

③ write out a partial fraction for each factor and every exponent of each factor.
- Some times you may get a factor with an exponent like $(x-2)^3$ ...etc. You need a partial fraction for each exponent from 1 up ... $\frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3}$

④ Multiply the whole equation by $D(s)$

⑤ Solve for the unknown coefficients by
A- substituting zeros of $D(s)$ {i.e. roots}
B- making a system of linear equations (of each power) and solving.

⑥ write out your answer.

⚠ A proposed approach to get the Inverse Laplace Transform is to try to decompose $X(s)$ using partial fractions

$$X(s) = \sum_{i=1}^{m} \frac{A_i}{s+a_i}$$

and using results of EX1 and EX2, and the ROC of each individual term can be infered from the ROC of $X(s)$ and the properties of ROC in general.

A Big Example Bringing It All Together

Here is a nice big example for you!

$$\frac{x^2+15}{(x+3)^2\,(x^2+3)}$$

- Because $(x+3)^2$ has an exponent of 2, it needs two terms (A_1 and A_2).
- And (x^2+3) is a quadratic, so it will need $Bx + C$:

$$\frac{x^2+15}{(x+3)^2(x^2+3)} = \frac{A_1}{x+3} + \frac{A_2}{(x+3)^2} + \frac{Bx + C}{x^2+3}$$

Now multiply through by $\mathbf{(x+3)^2(x^2+3)}$:

$$x^2+15 = (x+3)(x^2+3)A_1 + (x^2+3)A_2 + (x+3)^2(Bx + C)$$

There is a zero at $\mathbf{x} = \mathbf{-3}$ (because x+3=0), so let us try that:

$$\mathbf{(-3)}^2+15 = 0 + (\mathbf{(-3)}^2+3)A_2 + 0$$

And simplify it to:

$$24 = 12A_2$$

$$\text{so } A_2=2$$

Let us replace A_2 with 2:

$$x^2+15 = (x+3)(x^2+3)A_1 + 2x^2+6 + (x+3)^2(Bx + C)$$

Now expand the whole thing:

$$x^2+15 = (x^3+3x+3x^2+9)A_1 + 2x^2+6 + (x^3+6x^2+9x)B + (x^2+6x+9)C$$

Gather powers of x together:

$$x^2+15 = x^3(A_1+B)+x^2(3A_1+6B+C+2)+x(3A_1+9B+6C)+(9A_1+6+9C)$$

Separate the powers and write as a Systems of Linear Equations:

$$\begin{aligned} x^3\text{:} &\quad 0 = A_1+B \\ x^2\text{:} &\quad 1 = 3A_1+6B+C+2 \\ x\text{:} &\quad 0 = 3A_1+9B+6C \\ \text{Constants:} &\quad 15 = 9A_1+6+9C \end{aligned}$$

Simplify, and arrange neatly:

$$\begin{array}{rclllll} 0 &=& A_1 &+& B && \\ -1 &=& 3A_1 &+& 6B &+& C \\ 0 &=& 3A_1 &+& 9B &+& 6C \\ 1 &=& A_1 &&&+& C \end{array}$$

Now solve.

You can choose your own way to solve this ... I decided to subtract the 4th equation from the 2nd to begin with:

$$\begin{array}{rclllll} 0 &=& A_1 &+& B && \\ \mathbf{-2} &\mathbf{=}& \mathbf{2A_1} &\mathbf{+}& \mathbf{6B} && \\ 0 &=& 3A_1 &+& 9B &+& 6C \\ 1 &=& A_1 &&&+& C \end{array}$$

Then subtract 2 times the 1st equation from the 2nd:

$$\begin{array}{rclllll} 0 &=& A_1 &+& B && \\ \mathbf{-2} &\mathbf{=}& && \mathbf{4B} && \\ 0 &=& 3A_1 &+& 9B &+& 6C \\ 1 &=& A_1 &&&+& C \end{array}$$

Now I know that $\mathbf{B} = \mathbf{-(1/2)}$.

We are getting somewhere!

And from the 1st equation I can figure that $\mathbf{A_1} = \mathbf{+(1/2)}$.

And from the 4th equation I can figure that $\mathbf{C} = \mathbf{+(1/2)}$.

Final Result:

$$A_1=1/2 \qquad A_2=2 \qquad B=-(1/2) \qquad C=1/2$$

And we can now write our partial fractions:

$$\frac{x^2+15}{(x+3)^2(x^2+3)} = \frac{1}{2(x+3)} + \frac{2}{(x+3)^2} + \frac{-x + 1}{2(x^2+3)}$$

Phew! Lots of work. But it can be done.

- For LTI systems the output $y(t)$ can be calculated using the input $x(t)$ and impulse response $h(t)$ using the equation $\Rightarrow$ $y(t) = x(t) * h(t)$; $* \equiv$ Convolution

- Using the convolution property of Laplace Transform, then

$$Y(s) = X(s) . H(s)$$

$$\therefore \; H(s) = \frac{Y(s)}{X(s)}$$

$H(s)$ is called "System Function" OR "Transfer Function"

- Causality:- LTI causal if impulse response is zero for $t<0$
  i.e. causal if impulse response is right sided

then $\boxed{\text{The ROC associated with the system function for a causal system is a right-half plane}}$

But the opposite is NOT always TRUE.

$\boxed{\text{if the } \cancel{X}(s) \text{ is rational, causality of the system is equivalent to the ROC being the right-half plane to the right of the rightmost pole}}$

Again the converse is NOT in general TRUE

- Stability:- LTI is stable if its impulse response is absolutely integrable which means Fourier Tranform of the impulse response converges

as $\mathcal{FT} \equiv \mathcal{LT}\Big|_{\sigma=0}$

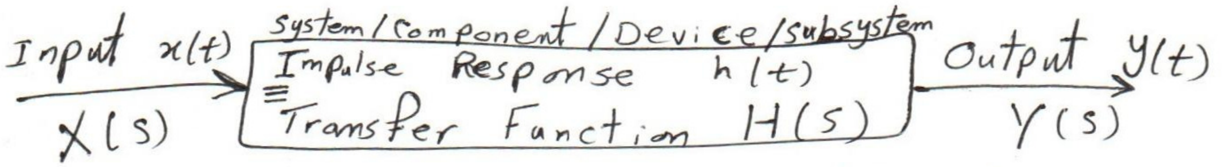
then $\boxed{\text{An LTI system is stable if and only if the ROC of its system function } H(s) \text{ includes the entire } j\omega\text{-axis [i.e. } Re\{s\}=0]}$

$\boxed{\text{A causal system is stable if and only if ALL poles lies left to } j\omega\text{-axis ; i.e. all poles have negative real parts}}$

Block Diagram and System Algebra

- <u>Block Diagram</u>:- it is a graphical representation of a system, subsystem, component, or device.
- it consists of a single block or a combination of blocks interconnected together in some form.
- <u>Basic Elements of Block Diagram</u>:-

① <u>Block</u> :- it is a box has a single input and a single output, and it is used to represent a single component with a specific Transfer Function.



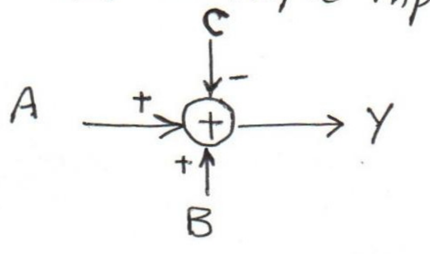
$$y(t) = x(t) * h(t) \quad \{* \equiv \text{Convolution}\}$$

$$Y(s) = X(s) \cdot H(s)$$

$$\therefore \; H(s) = \frac{Y(s)}{X(s)}$$

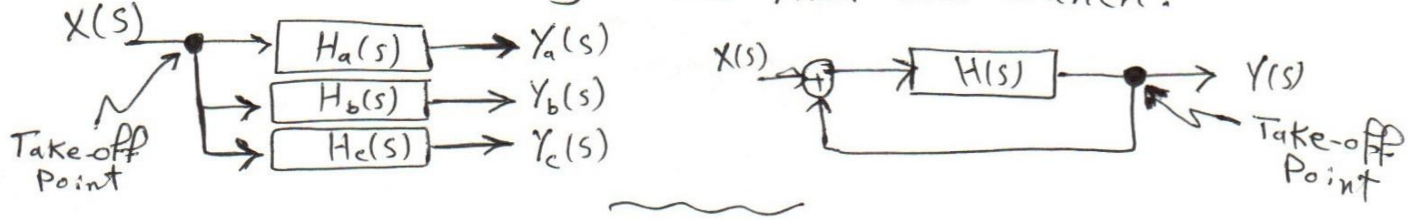
where the transfer function $H(s)$ is the Laplace Transform of the impulse response $h(t)$.

② <u>Summing Point</u>:- it is a circle with a cross inside it like ⊕ OR ⊗ ; and it is used to perform summation or subtraction or combination of summation and subtraction of the inputs based on the polarity of inputs
- it has multiple inputs and a single output.



$$\Longrightarrow \quad Y(s) = A(s) + B(s) - C(s)$$

(3) Take-of :- it is a point from which the input signal can be passed through more than one branch.



(4) Branch :- it is a line having a coefficient that is applied to its input to get its output.

$$A \xrightarrow{k} B = kA$$

it is equivalent to block with transfer function $= k$

(5) Arrow :- it is used to indicate the flow of signals through the system.

$$\longrightarrow$$

(6) Time-delay :- it is used to get a delayed version of a signal

$$x(t),\ X(s) \longrightarrow \boxed{D} \longrightarrow y(t) = x(t-1)$$
$$Y(s) = e^{-s}\, X(s)$$

(7) Integrator :-

$$x(t),\ X(s) \longrightarrow \boxed{\int} \longrightarrow y(t) = \int x(\tau)\, d\tau$$
$$Y(s) = \frac{1}{s}\, X(s)$$

(8) Differentiator :-

$$x(t),\ X(s) \longrightarrow \boxed{S} \longrightarrow y(t) = \frac{d}{dt}\, x(t)$$
$$Y(s) = s\, X(s)$$

Basic Interconnections

(1) Series connection :- $X(S) \longrightarrow \boxed{H_1(s)} \xrightarrow{Z(S)} \boxed{H_2(s)} \longrightarrow Y(s)$

$$Y(s) = H_2(s) \cdot Z(s)$$

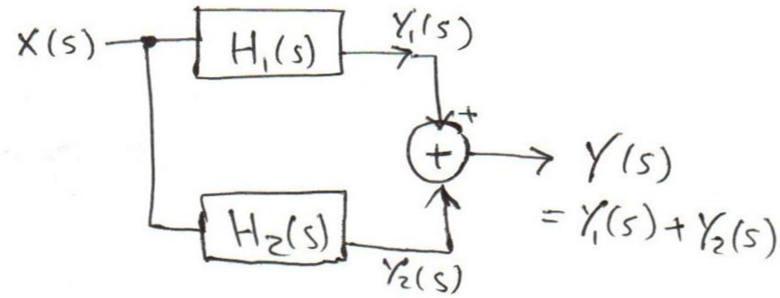
where $Z(s) = H_1(s) \cdot X(s) \Rightarrow Y(s) = \{H_2(s)\, H_1(s)\}\, X(s)$

an equivalent diagram $X(S) \longrightarrow \boxed{H_1(s) \cdot H_2(s)} \longrightarrow Y(S)$

OR $X(S) \longrightarrow \boxed{H(s)} \longrightarrow Y(s)$; with $H(s) = H_1(s) \cdot H_2(s)$.

This is TRUE with "n" blocks connected in series.

② Parallel connection:



$$Y_1(s) = X(s) \cdot H_1(s)$$

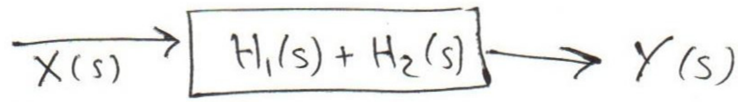
$$Y_2(s) = X(s) \cdot H_2(s)$$

$$\therefore Y(s) = X(s) \cdot H_1(s) + X(s) \cdot H_2(s)$$

$$= \{ H_1(s) + H_2(s) \} \cdot X(s)$$

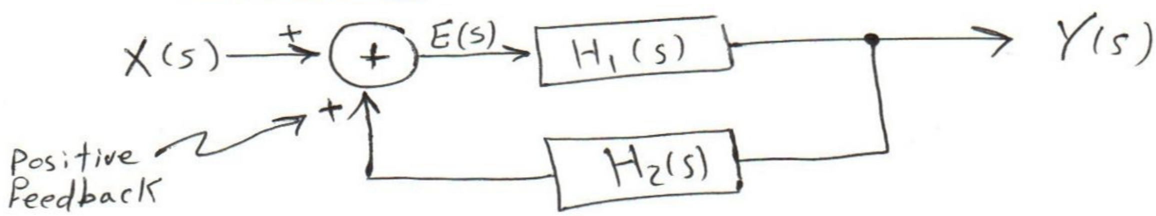
$$= H(s) \cdot X(s) \quad ; \quad H(s) = H_1(s) + H_2(s)$$

The equivalent diagram



IT also TRUE with "n" parallel blocks

③ Feedback connection:



$$Y(s) = E(s) \cdot H_1(s)$$

$$E(s) = X(s) + Y(s) \cdot H_2(s)$$

$$\therefore Y(s) = \{ X(s) + Y(s) \cdot H_2(s) \} \cdot H_1(s)$$

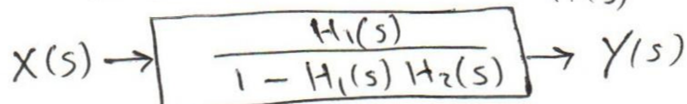
$$= H_1(s) \cdot X(s) + H_1(s) \cdot H_2(s) \cdot Y(s)$$

$$\therefore Y(s) - H_1(s) \cdot H_2(s) \cdot Y(s) = H_1(s) \cdot X(s)$$

$$Y(s) \cdot \{ 1 - H_1(s) H_2(s) \} = H_1(s) \cdot X(s)$$

$$\therefore \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 - H_1(s) H_2(s)} \equiv \underset{H(s)}{\text{Transfer Function}}$$

The equivalent diagram



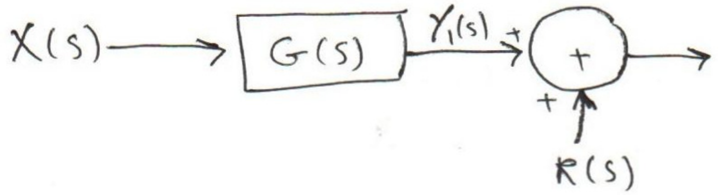
in case of Negative Feedback $\Rightarrow$ $H(s) = \dfrac{H_1(s)}{1 + H_1(s) H_2(s)}$

- Block Diagram Algebra for Summing Points :-

(A) shifting summing point after a block :-

$= G(s).\{X(s) + R(s)\}$

$= G(s)\,X(s) + G(s)\,R(s)$

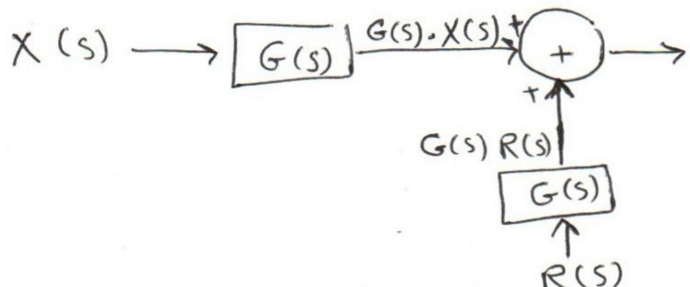


$Y(s) = Y_1(s) + R(s)$

$= G(s)X(s) + R(s)$

$\neq Y(s)$ in previous example

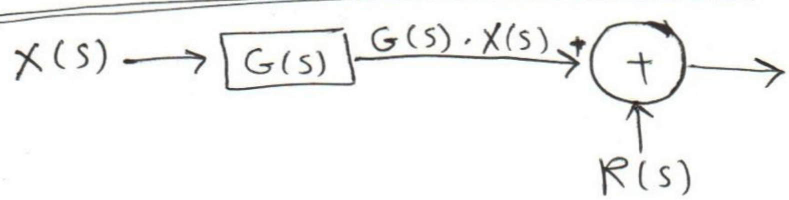
to be correct should be



$Y(s) = G(s)X(s) + G(s)R(s)$

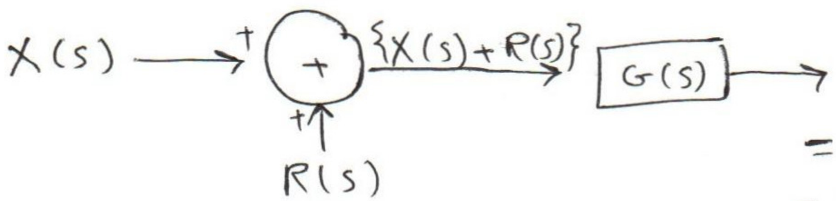
$=$ original $Y(s)$

(B) shifting summing point befor a block :-



$Y(s)$

$= G(s)\,X(s) + R(s)$



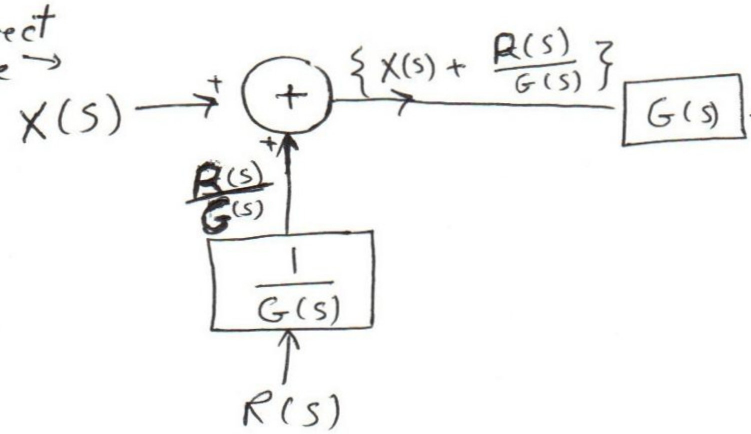
$Y(s)$

$= G(s).\{X(s) + R(s)\}$

$= G(s)\,X(s) + G(s)\,R(s)$

$\neq$ original $Y(s)$

To be correct it should be →



$Y(s)$

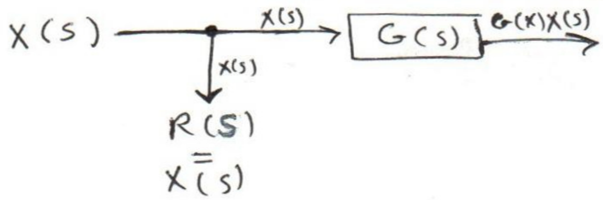
$= G(s).X(s) + G(s).\frac{R(s)}{G(s)}$

$= G(s).X(s) + R(s)$

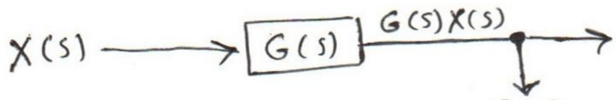
$=$ original $Y(s)$

Block Diagram Algebra for Take-off points :-

(A) shifting Take-off point after a block :-



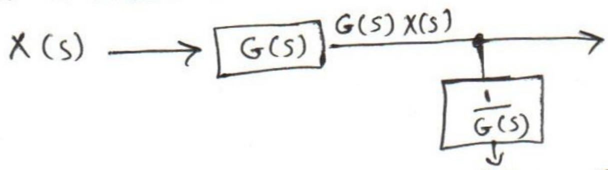
$Y(s) = G(s) \cdot X(s)$



$Y(s) = G(s) \cdot X(s)$

$R(s) = G(s) \cdot X(s) \neq$ original $R(s)$

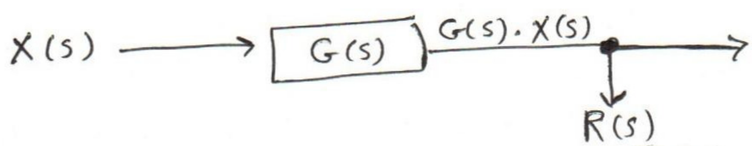
To be correct it should be



$Y(s) = G(s) \cdot X(s)$

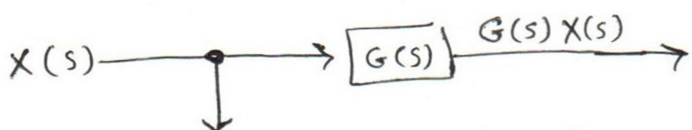
$R(s) = \dfrac{G(s)X(s)}{G(s)} = X(s) =$ original $R(s)$

(B) shifting Take-off point before a block :-



$Y(s) = G(s) \cdot X(s)$

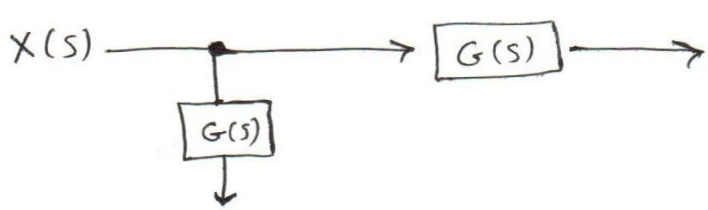
$R(s) = G(s) \cdot X(s)$



$Y(s) = G(s) \cdot X(s)$

$R(s) = X(s) \neq$ original $R(s)$

To be correct it should be

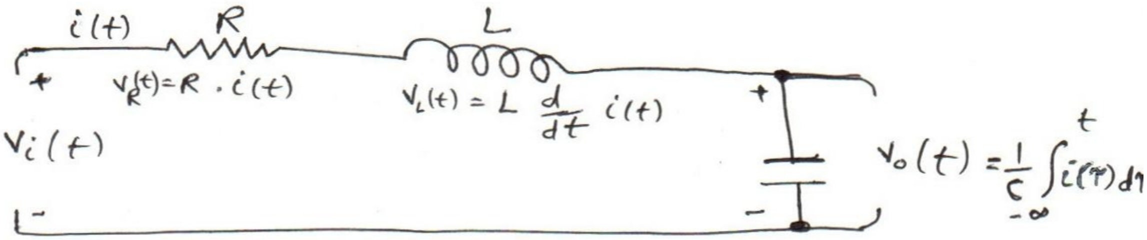


$Y(s) = G(s) \cdot X(s)$

$R(s) = G(s) \cdot X(s)$
$=$ original $R(s)$

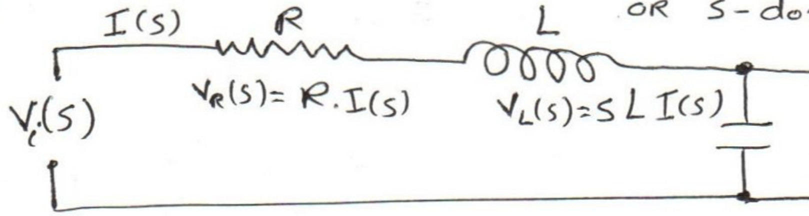
An Example

- Look at the following electrical circuit



to get the equivalent transfer function of such system in time-domain it is a headache !

But instead ① we convert it to {Laplace form} it will be OR s-domain



$$V_o(s) = \frac{1}{C} \cdot \frac{1}{s} \cdot I(s) = \frac{1}{sC} I(s)$$

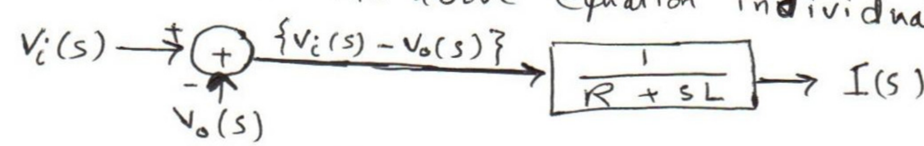
② write down the equations of the current passing through all series branch elements and voltage across all parallel branches.

$$\therefore \quad I(s) = \frac{V_i(s) - V_o(s)}{\{R + sL\}} = \left\{\frac{1}{R+sL}\right\}\{V_i(s) - V_o(s)\} \quad \text{(I)}$$

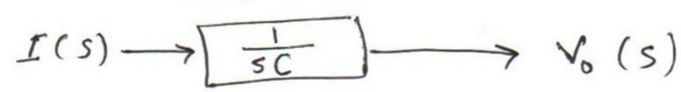
and $\quad V_o(s) = \frac{1}{sC} \cdot I(s) \longrightarrow$ (II)

③ Draw the block diagrams for all the above equation individually

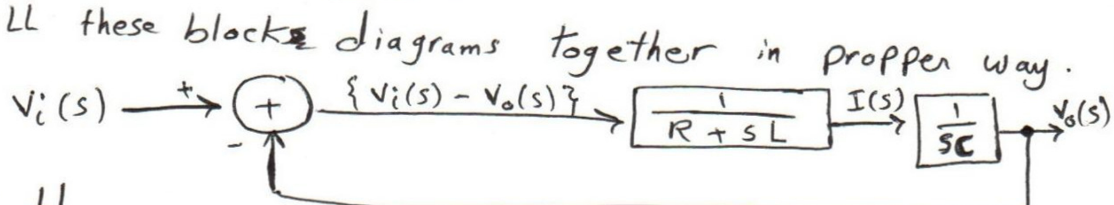
equation # (I) $\Rightarrow$



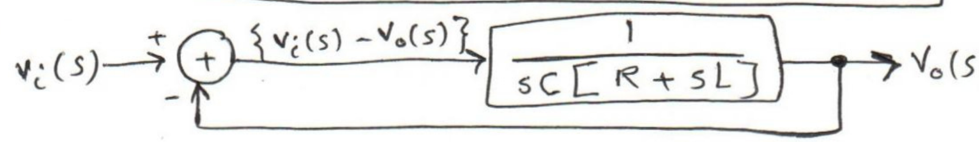
equation # (II) $\Rightarrow$



④ Combine ALL these blocks diagrams together in propper way.



OR equivalently



with equivalent transfer function $H(s) = \dfrac{1}{1 + sC \cdot \{R + sL\}}$

as it is a negative feedback connection with $H_2(s) = 1$

Follow the example

it is equivalent to negative feedback connection with $H_1(s) = \frac{1}{sC\{R+sL\}}$

and $H_2(s) = 1$

the equivalent transfer function $H(s) = \frac{H_1(s)}{1 + H_1(s).H_2(s)}$

$$\therefore H(s) = \frac{1}{\{sRC + s^2LC\}} \times \left\{ \frac{1}{1 + \left\{ \frac{1}{sRC + s^2LC} \right\}} \right\}$$

$$= \frac{1}{sRC + s^2LC} \times \left\{ \frac{1}{\left\{ \frac{sRC + s^2LC + 1}{sRC + s^2LC} \right\}} \right\}$$

$$= \frac{1}{\cancel{\{sRC + s^2LC\}}} \times \frac{\cancel{\{sRC + s^2LC\}}}{1 + sRC + s^2LC}$$

$$\therefore H(s) = \frac{1}{1 + sRC + s^2LC}$$

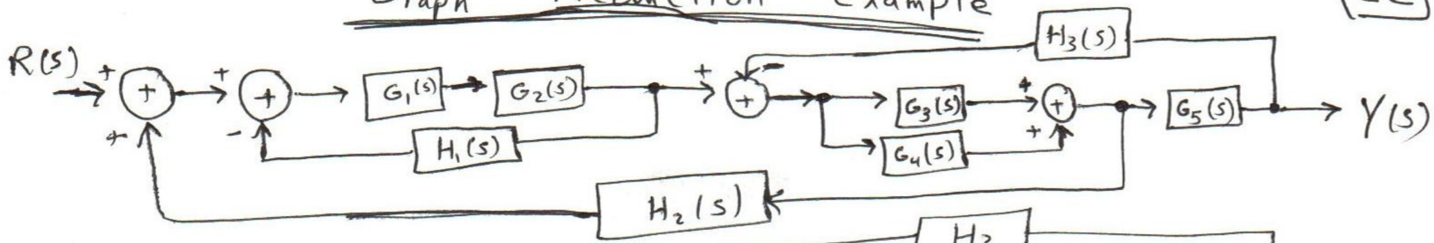
$$\therefore \quad V_i(s) \longrightarrow \boxed{\frac{1}{1 + sRC + s^2LC}} \longrightarrow V_o(s)$$

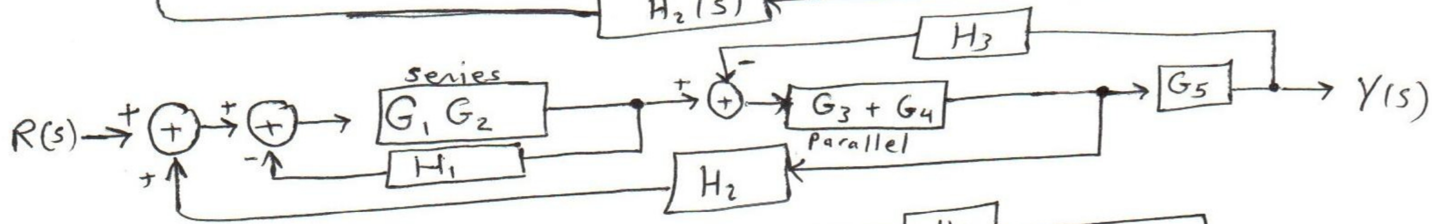
Block Diagram Reduction

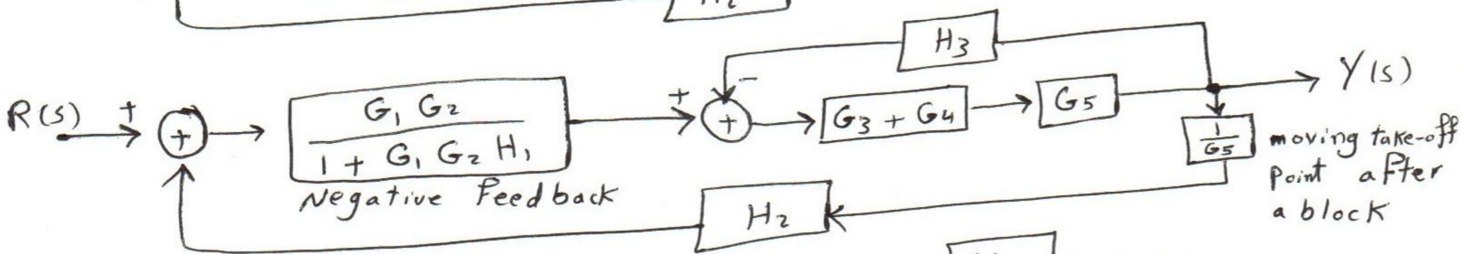
Rules :-

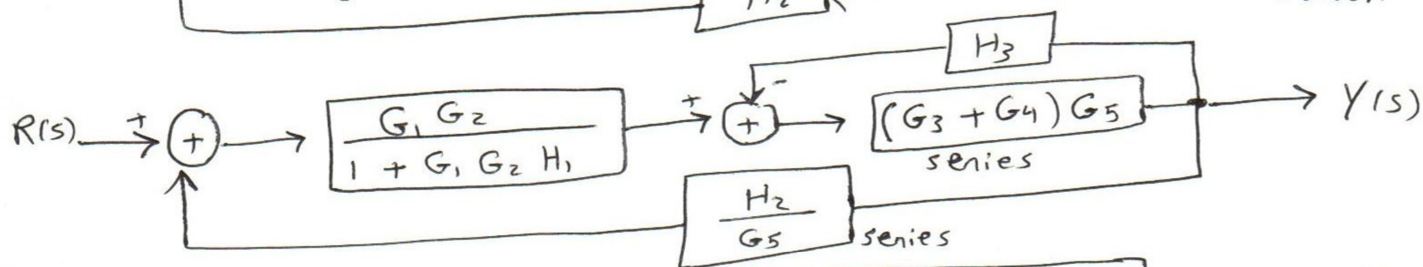
1. check for the blocks connected in series and simplify.
2. " " " " " " parallel " " .
3. " " " " " " feedback loops and simplify.
4. If there is a difficulty with take-off point shift it.
5. " " " " " " summing " " "
6. Repeat steps from 1 to 5 till there are No modification.
7. If there are multiple inputs → do the previous steps for each input individually and then get the final answer by adding them all together

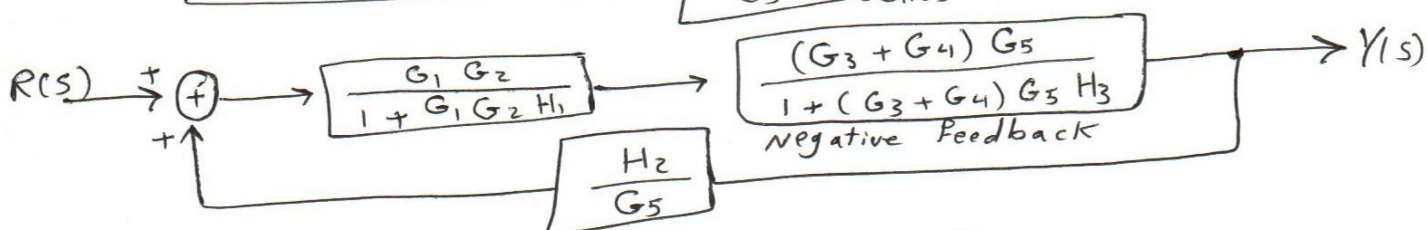
Graph Reduction Example

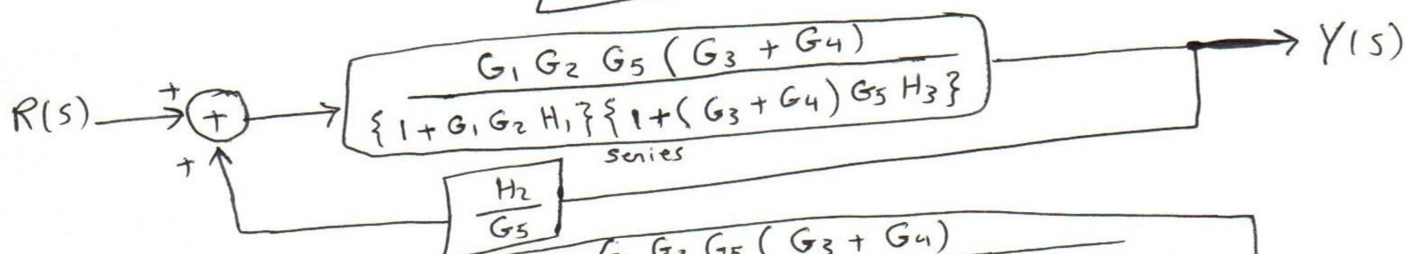


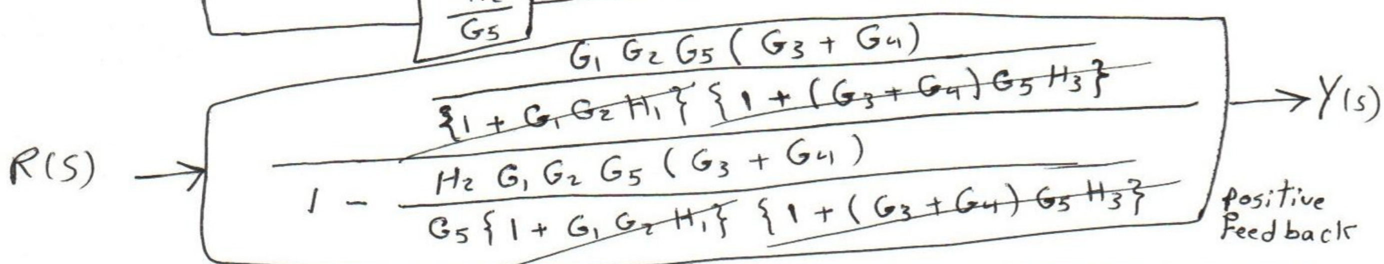


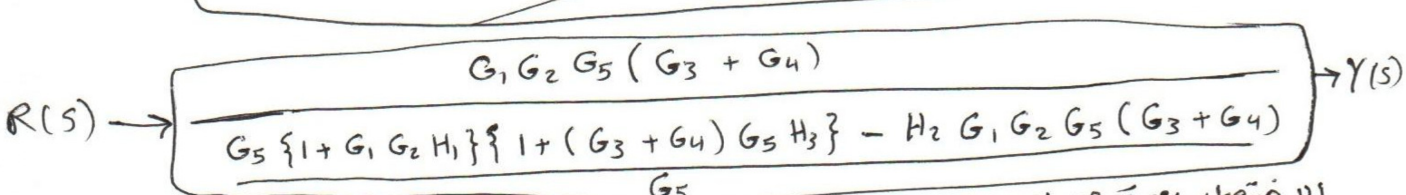




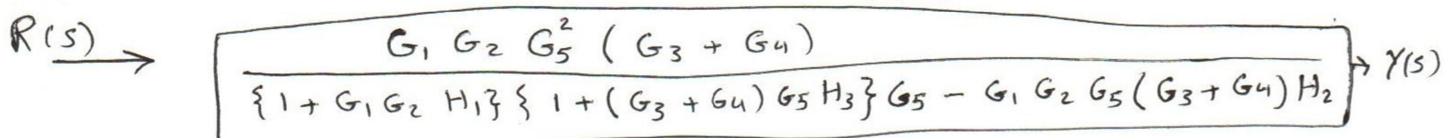








الاختصار بعد توحيد المقامات في المقام



$\therefore$ System Function

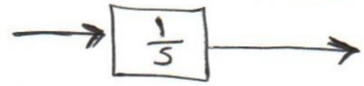
$$H(s) = \frac{Y(s)}{R(s)} = \frac{G_1 G_2 G_5^2 (G_3 + G_4)}{\{1 + G_1 G_2 H_1\}\{1 + (G_3 + G_4) G_5 H_3\} G_5 - G_1 G_2 G_5 (G_3 + G_4) H_2}$$

System Algebra and Block Diagram Examples

Recall:- differentiation $\frac{d\,x(t)}{ds} \equiv s\;X(s)$

Integration $\int_{-\infty}^{t} x(\tau)\,d\tau \equiv \frac{1}{s}\;X(s)$

$\therefore$ The differentiator is 

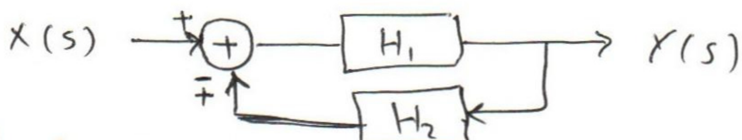
The integrator is 

Note:- The differentiator has two main disadvantages :-

1- Difficult to implement. ١- صعب التنفيذ

2- Prune to errors. and Sensitive to Noise. ٢- يتأثر بال Noise وعرضة للأخطاء

Recall Too:- in feedback interconnection



$H(s) = \frac{H_1(s)}{1 \pm H_1(s)H_2(s)}$

1- $H_1(s)$ is in the forward branch and $H_2(s)$ is in the backward/feedback branch

2- the sign in denominator is opposite to the polarity of the feedback branch

EX1:- $\frac{d}{dt}\,y(t) + 3\,y(t) = x(t)$

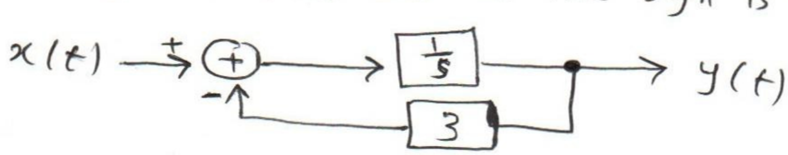
$\therefore\; S\;Y(s) + 3\;Y(s) = X(s)$

$\therefore\; Y(s)\,\{3 + S\} = X(s)$

$\therefore\; H(s) = \frac{Y(s)}{X(s)} = \frac{1}{S+3}$ {using differentiators}

OR $H(s) = \frac{\frac{1}{s}}{1 + 3\cdot\frac{1}{s}}$ {using integrators}

$\equiv$ Feedback Interconnection with $H_1(s) = \frac{1}{s}$ and $H_2(s) = 3$ (Negative feedback as the sign is +ve in denominator)

$\Rightarrow$ 

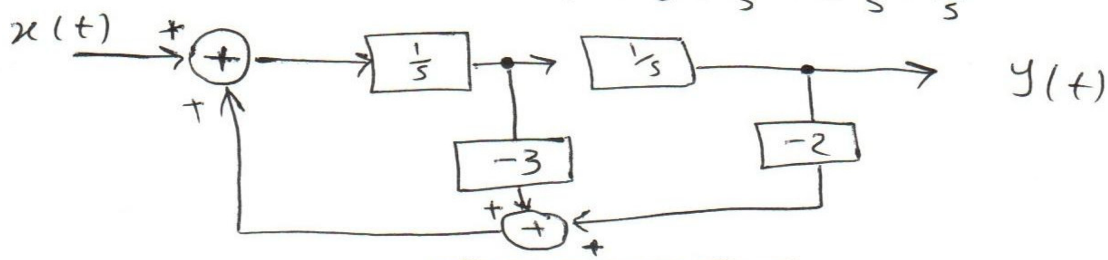
Ex2:- $H(s) = \dfrac{1}{s^2+3s+2} \equiv$ direct Form

$= \dfrac{1}{(s+1)} \cdot \dfrac{1}{(s+2)} \equiv$ Series Form

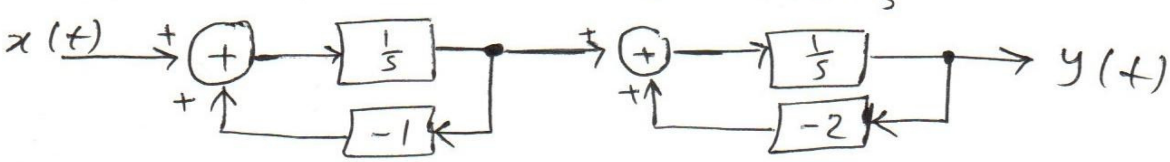
$= \dfrac{1}{s+1} - \dfrac{1}{s+2} \equiv$ parallel Form

Note it is better to convert to integrators instead of differentiators

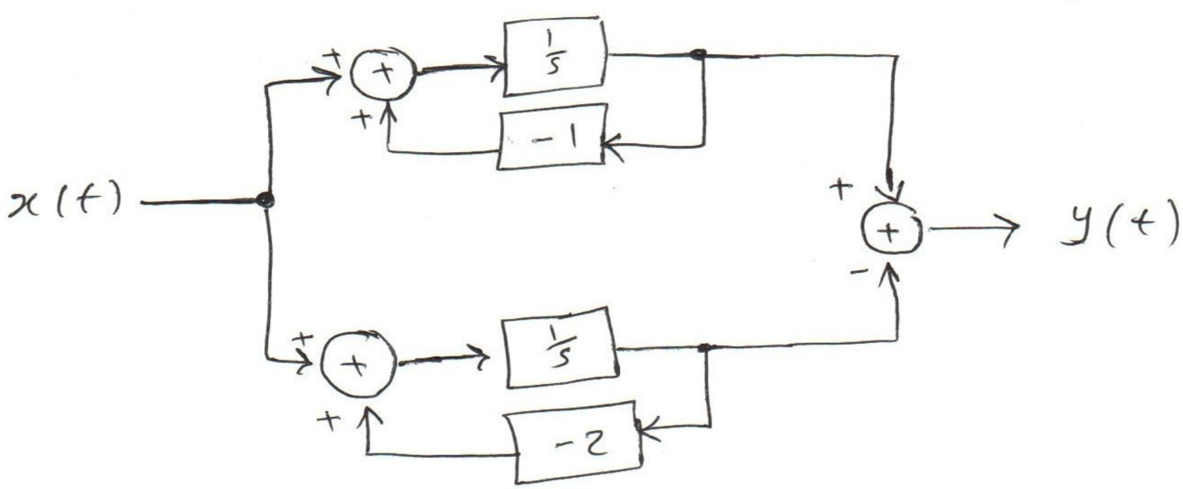
Ⓐ Direct Form:- $H(s) = \dfrac{\frac{1}{s} \cdot \frac{1}{s}}{1 + 3 \cdot \frac{1}{s} + 2 \frac{1}{s} \cdot \frac{1}{s}}$



Ⓑ Series Form:- $H(s) = \dfrac{\frac{1}{s}}{1+1 \cdot \frac{1}{s}} \cdot \dfrac{\frac{1}{s}}{1+2 \cdot \frac{1}{s}}$



Ⓒ Parallel Form:- $H(s) = \dfrac{\frac{1}{s}}{1+1 \cdot \frac{1}{s}} - \dfrac{\frac{1}{s}}{1+2 \cdot \frac{1}{s}}$



Ex 3:- Draw the block diagram of the transfer function

$$H(s) = \frac{1}{s^2+as+b}$$

Steps :-

① Remember that $H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+as+b}$

$\therefore \quad s^2 Y(s) + a s Y(s) + b Y(s) = X(s)$

which means :-

Ⓐ Numerator البسط is corresponding to forward path

Ⓑ Denominator المقام is corresponding to feedback path

Ⓒ You need double (OR TWO) differentiators.

② Divide by largest exponent of s of the denominator
Here divide by s^2

$$\Rightarrow \quad H(s) = \frac{\frac{1}{s^2}}{1+\frac{a}{s}+\frac{b}{s^2}} \equiv \frac{\frac{1}{s}\cdot\frac{1}{s}}{1+a\cdot\frac{1}{s}+b\cdot\frac{1}{s}\cdot\frac{1}{s}}$$

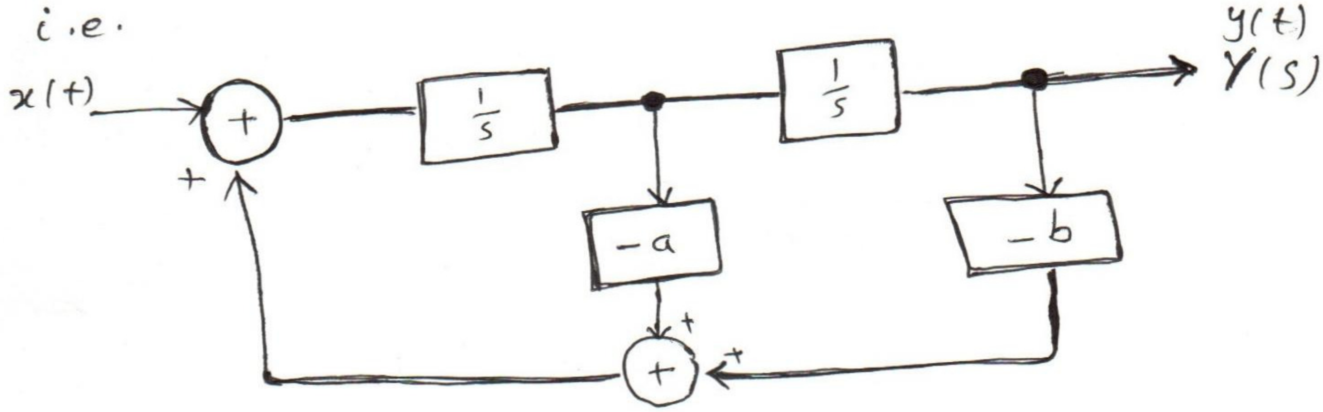
to replace All differentiators(s) by integrators ($\frac{1}{s}$)

③ Put the Denominator in the form $\{1 \pm \text{polynomial in} (\frac{1}{s})\}$
Here it is in the required form.

④ Draw the numerator with the same coefficients in the forward path

⑤ Draw the denominator with opposite-sign coefficients in the feedback loop path. {except the 1}

i.e.



Ex4: General Example

Draw the system with transfer Function

$$H(s) = \frac{cs^2 + ds + e}{2s^2 + as + b}$$

steps

① Divide the Numerator and Denominator by (s^2) {large exponent in denominator}

$$H(s) = \frac{c + d/s + e/s^2}{2 + a/s + b/s^2}$$

to replace differentiators by integrators.

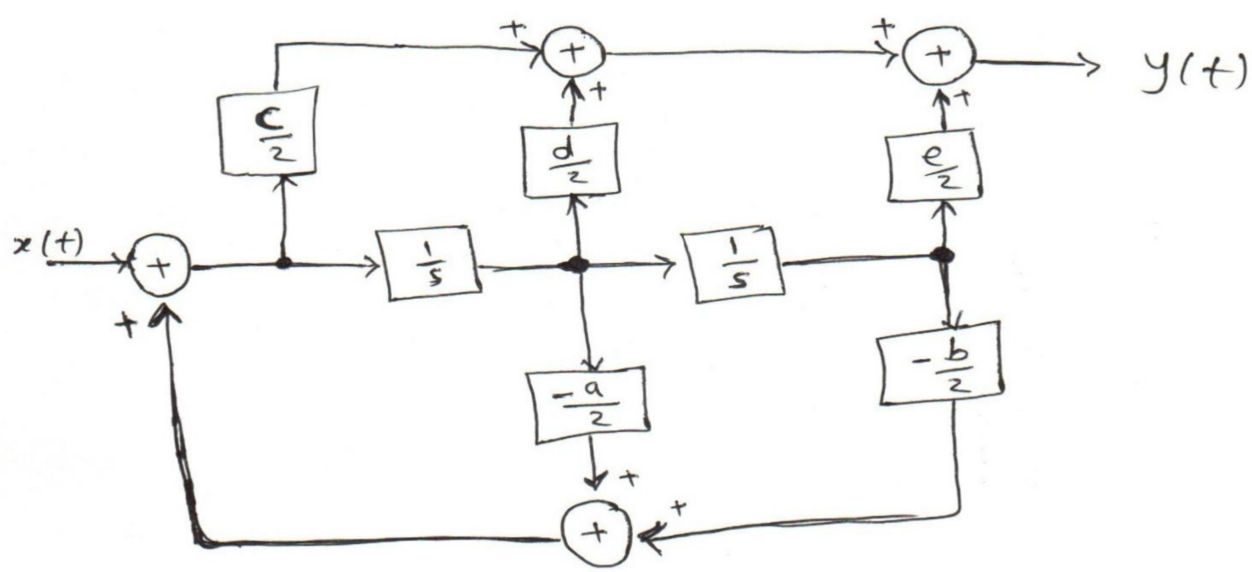
② Divide the Numerator and Denominator by (2)

$$H(s) = \frac{\frac{c}{2} + \frac{d}{2}\cdot\frac{1}{s} + \frac{e}{2}\cdot\frac{1}{s}\cdot\frac{1}{s}}{1 + \frac{a}{2}\cdot\frac{1}{s} + \frac{b}{2}\cdot\frac{1}{s}\cdot\frac{1}{s}}$$

to put $H(s)$ in propper form of Rational transfer Function that corresponding to a usual feedback " " .

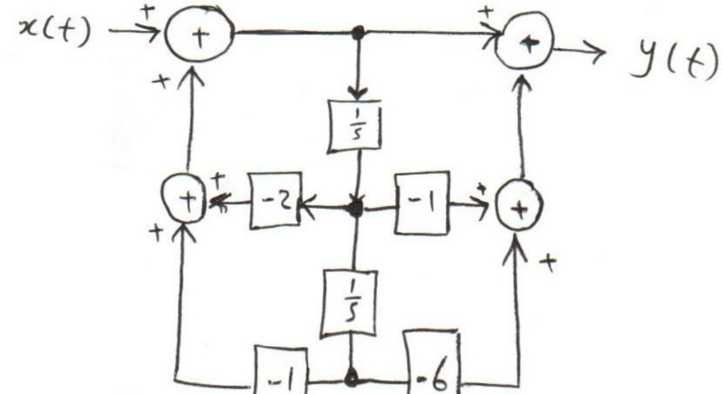
③ Draw All coefficients {as it is} of Numerator in the Forward path

④ Draw All coefficients except the 1 {with opposite sign} of Denominator in the feedback loop path



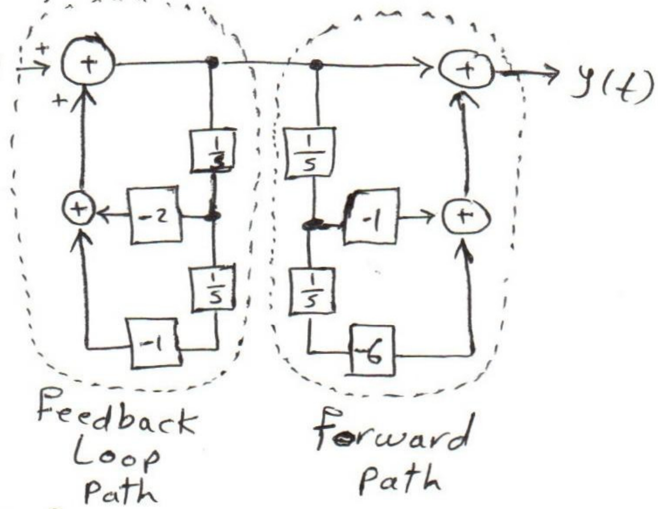
Direct Form

Ex 5:- Get the equation of the transfer function of this block diagram



Solution #1:- {directly from graph}

① write down the coefficients of the forward path in the numerator {with it's sign}



$$\therefore H_{FF}(s) = 1 - 1\cdot\frac{1}{s} - 6\cdot\frac{1}{s}\cdot\frac{1}{s}$$

② write down the coefficients of the feedback loop path in the denominator {with opposite sign} after you write 1 first

$$\therefore H_{FB}(s) = 1 + 2\frac{1}{s} + 1\cdot\frac{1}{s}\cdot\frac{1}{s} = \boxed{1 + \frac{2}{s} + \frac{1}{s^2}}$$

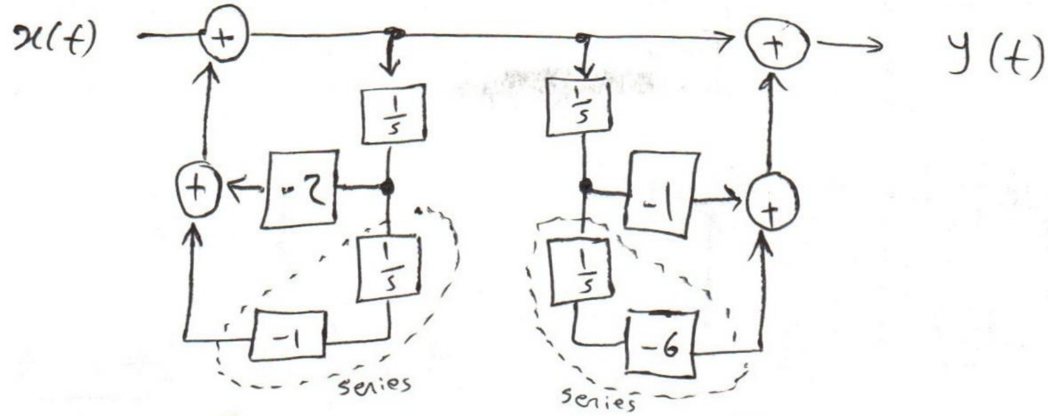
③ write down the overall equation of the transfer function

$$H(s) = \frac{H_{FF}(s)}{H_{FB}(s)}$$

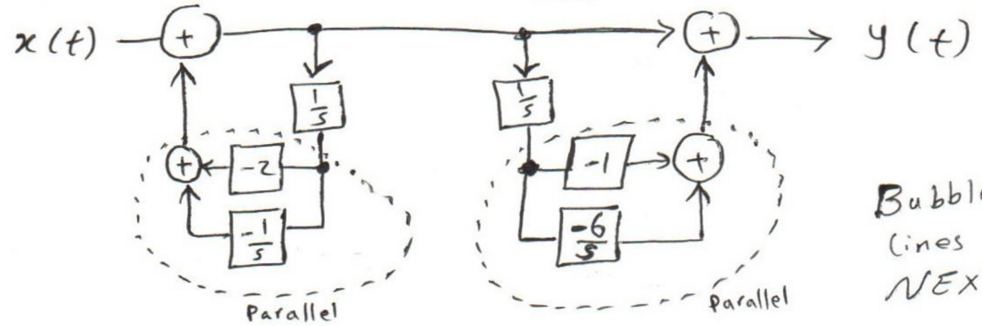
$$\therefore H(s) = \frac{1 - \frac{1}{s} - \frac{6}{s^2}}{1 + \frac{2}{s} + \frac{1}{s^2}}$$

multiply by s^2 $\Rightarrow$ $\boxed{H(s) = \frac{s^2 - s - 6}{s^2 + 2s + 1}}$

Solution #2 :- { using graph reduction }

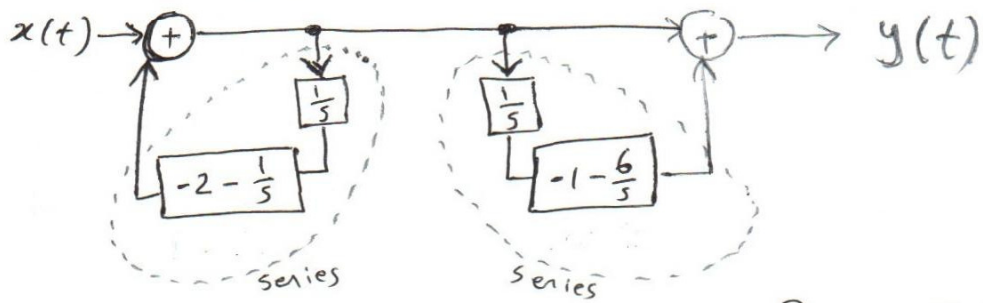


$\equiv$

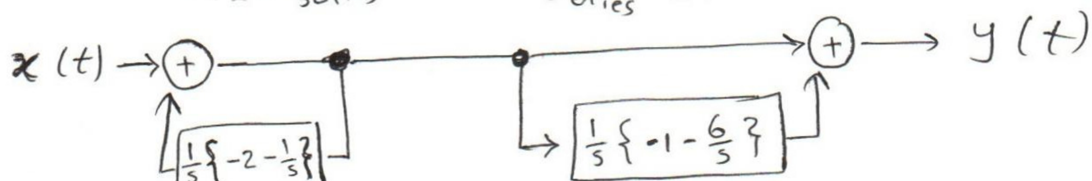


Bubbles of dotted lines prepare for NEXT step.

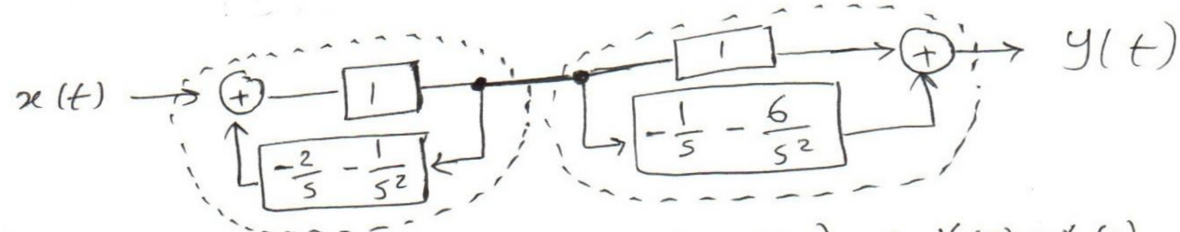
$\equiv$



$\equiv$



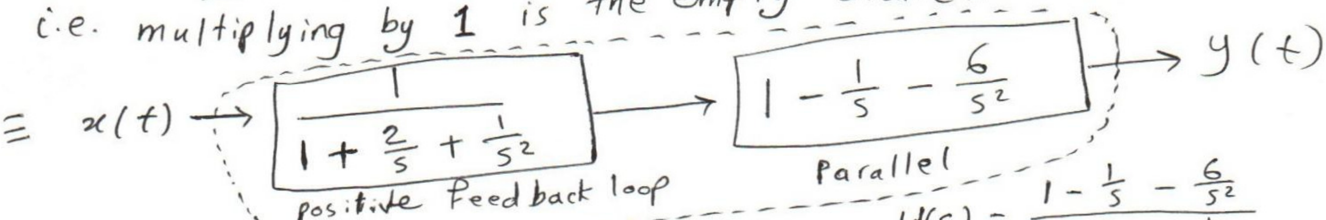
$\equiv$



as $X_1(s) \to \boxed{1} \to Y_1(s) \equiv Y_1(s) = 1 \cdot X_1(s) = X_1(s) \Rightarrow Y_1(s) = X_1(s)$

i.e. multiplying by **1** is the empty branch

$\equiv$



$$\therefore H(s) = \frac{1 - \frac{1}{s} - \frac{6}{s^2}}{1 + \frac{2}{s} + \frac{1}{s^2}}$$

$$\equiv x(t) \longrightarrow \boxed{\frac{1 - \frac{1}{s} - \frac{6}{s^2}}{1 + \frac{2}{s} + \frac{1}{s^2}}} \longrightarrow y(t)$$

Series

$$\boxed{\therefore H(s) = \frac{s^2 - s - 6}{s^2 + 2s + 1}}$$

EX 6

IF the transfer Function is :- $H(s) = \dfrac{4s^2+8s-12}{2s^2+6s+4}$

Draw it using :-

A- Direct Form

B- Series Form

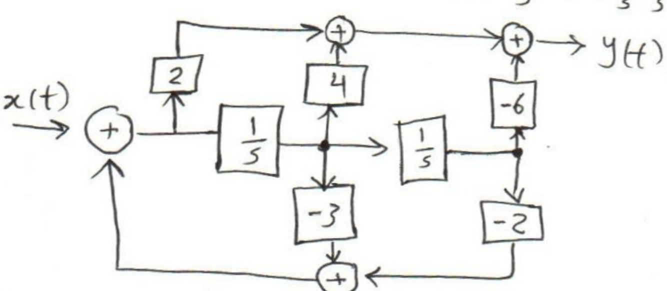
C- parallel Form

Solution

(A) $H(s) = \dfrac{4s^2+8s-12}{2s^2+6s+4}$

divide by s^2 $\Rightarrow H(s) = \dfrac{4+\frac{8}{s}-\frac{12}{s^2}}{2+\frac{6}{s}+\frac{4}{s^2}}$

divide by 2 $\Rightarrow H(s) = \dfrac{2+4\cdot\frac{1}{s}-6\cdot\frac{1}{s}\cdot\frac{1}{s}}{1+3\cdot\frac{1}{s}+2\cdot\frac{1}{s}\cdot\frac{1}{s}}$

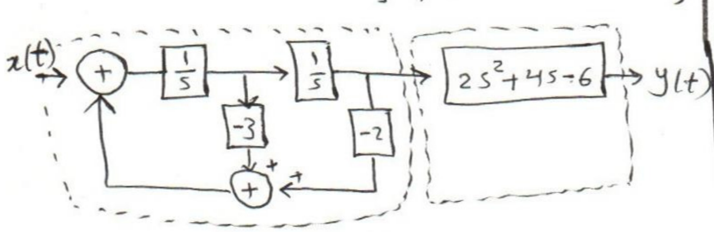


direct Form

(B) $H(s) = \dfrac{4s^2+8s-12}{2s^2+6s+4}$

divide by 2 $\Rightarrow H(s) = \dfrac{1}{\{s^2+3s+2\}} \cdot \{2s^2+4s-6\}$

$\therefore H(s) = \left\{\dfrac{\frac{1}{s}\cdot\frac{1}{s}}{1+3\cdot\frac{1}{s}+2\frac{1}{s}\cdot\frac{1}{s}}\right\}\left\{2s^2+4s-6\right\}$



Series Form

(C) $H(s) = \dfrac{2s^2+4s-6}{s^2+3s+2}$

after dividing by 2

- using long division

$$\begin{array}{r|l} & 2 \\ s^2+3s+2 & 2s^2+4s-6 \\ & 2s^2+6s+4 \\ \hline & -2s-10 \end{array}$$

$\therefore H(s) = 2 + \dfrac{-2s-10}{s^2+3s+2}$

long division used to make degree of denominator > degree of numerator to be valid for partial Fraction

- Do partial fraction for $\dfrac{-2s-10}{s^2+3s+2}$

$\therefore \dfrac{-2s-10}{s^2+3s+2} = \dfrac{-2s-10}{(s+2)(s+1)}$

$\therefore \dfrac{-2s-10}{(s+1)(s+2)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$

$\therefore -2s-10 = A(s+2) + B(s+1)$

let $s=-2 \Rightarrow -2(-2)-10 = B(-1)$

$\Rightarrow -B = -6 \Rightarrow \boxed{B=6}$

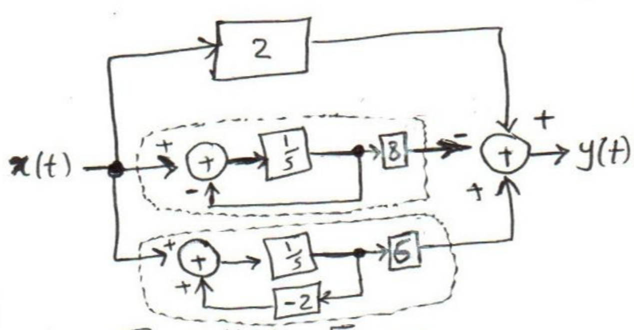
let $s=-1 \Rightarrow -2(-1)-10 = A$

$\Rightarrow \boxed{A=-8}$

$\therefore H(s) = 2 - \dfrac{8}{s+1} + \dfrac{6}{s+2}$

$= 2 - \dfrac{8\cdot\frac{1}{s}}{1+1\cdot\frac{1}{s}} + \dfrac{6\cdot\frac{1}{s}}{1+2\cdot\frac{1}{s}}$

$\therefore H(s) = 2 - 8\cdot\dfrac{\frac{1}{s}}{1+\frac{1}{s}} + 6\cdot\dfrac{\frac{1}{s}}{1+2\cdot\frac{1}{s}}$



Parallel Form

Follow EX 6

- The series Form is called "Cascade Form" Too which means a representation of $H(s)$ as a multiplication of smaller terms, Note :- it has more than one eqn.

$$H(s) = \frac{4s^2+8s-12}{2s^2+6s+4} = \frac{2s^2+4s-6}{s^2+3s+2} \quad \text{after dividing by 2}$$

$$H(s) = \left\{\frac{1}{s^2+3s+2}\right\}\left\{2s^2+4s-6\right\} \Longrightarrow \text{eqn \# 1}$$

$$= \left\{\frac{1}{s^2+3s+2}\right\}\cdot\left\{2\right\}\cdot\left\{s^2+2s-3\right\} \Rightarrow \text{eqn \# 2}$$

$$= \left\{\frac{1}{s^2+3s+2}\right\}\cdot\left\{2\right\}\cdot\left\{(s-1)(s+3)\right\} \Rightarrow \text{eqn \# 3}$$

$$= \left\{2\right\}\cdot\left\{\frac{1}{s^2+3s+2}\right\}\cdot\left\{(s-1)\right\}\cdot\left\{(s+3)\right\} \Rightarrow \text{eqn \# 4}$$

$$= \left\{2\right\}\cdot\left\{\frac{1}{(s+1)(s+2)}\right\}\cdot\left\{(s-1)\right\}\cdot\left\{(s+3)\right\}$$

$$= \left\{2\right\}\cdot\left\{\frac{(s-1)}{(s+1)(s+2)}\right\}\cdot\left\{(s+3)\right\} \Longrightarrow \text{eqn \# 5}$$

$$= \left\{2\right\}\cdot\left\{\frac{1}{(s+2)}\right\}\cdot\left\{(s+3)\right\}\cdot\left\{\frac{s-1}{s+1}\right\} \Rightarrow \text{eqn \# 6}$$

$$= \left\{2\right\}\left\{s-1\right\}\left\{\frac{1}{s+1}\right\}\left\{s+3\right\}\left\{\frac{1}{s+2}\right\} \Rightarrow \text{eqn \# 7}$$

⋮ It is a funny flexible game ☺

etc ...

and ALL these equations are equal for cascade / series Form representation,

Good luck