MOOCULUS

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CALCULUS

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This text is based on David Guichard's open-source calculus text which in turn is a modification and expansion of notes written by Neal Koblitz at the University of Washington. David Guichard's text is available at http://www.whitman.edu/mathematics/calculus/ under a Creative Commons license.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at http://www.math.wisc.edu/~keisler/calc.html under a Creative Commons license. In addition, the chapter on differential equations and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

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We will be glad to receive corrections and suggestions for improvement at fowler@math.osu.edu or snapp@math.osu.edu.

Contents

List of Main Theorems

How to Read Mathematics

Reading mathematics is **not** the same as reading a novel. To read mathematics you need:

(a) A pen.

(b) Plenty of blank paper.

(c) A willingness to write things down.

As you read mathematics, you must work along side of the text itself. You must **write** down each expression, **sketch** each graph, and **think** about what you are doing. You should work examples and fill-in the details. This is not an easy task, it is in fact **hard** work. However, mathematics is not a passive endeavor. You, the reader, must become a doer of mathematics.

1 Limits

1.1 The Basic Ideas of Limits

Consider the function:

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

While $f(x)$ is undefined at $x = 2$, we can still plot $f(x)$ at other values, see Figure 1.1. Examining Table 1.1, we see that as x approaches 2, $f(x)$ approaches 1. We write this:

$$\text{As } x \to 2, f(x) \to 1 \qquad \text{or} \qquad \lim_{x \to 2} f(x) = 1.$$

Intuitively, $\lim_{x \to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. This leads us to the formal definition of a *limit*.

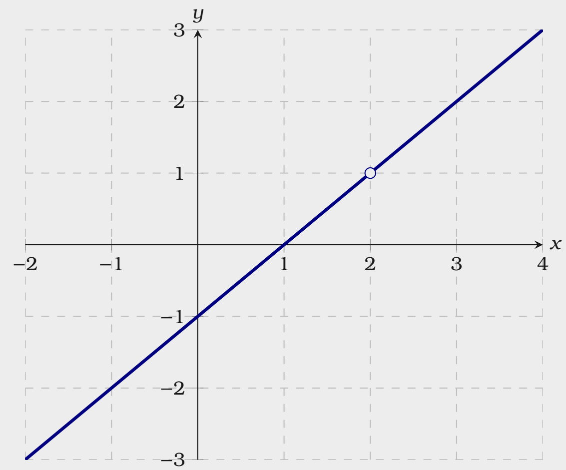


Figure 1.1: A plot of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

x	$f(x)$
1.7	0.7
1.9	0.9
1.99	0.99
1.999	0.999
2	undefined

x	$f(x)$
2	undefined
2.001	1.001
2.01	1.01
2.1	1.1
2.3	1.3

Table 1.1: Values of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

Definition The **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x \to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta, \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

If no such value of L can be found, then we say that $\lim_{x \to a} f(x)$ **does not exist**.

Equivalently, $\lim_{x \to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and $a - \delta < x < a + \delta$, we have $L - \varepsilon < f(x) < L + \varepsilon$.

In Figure 1.2, we see a geometric interpretation of this definition.

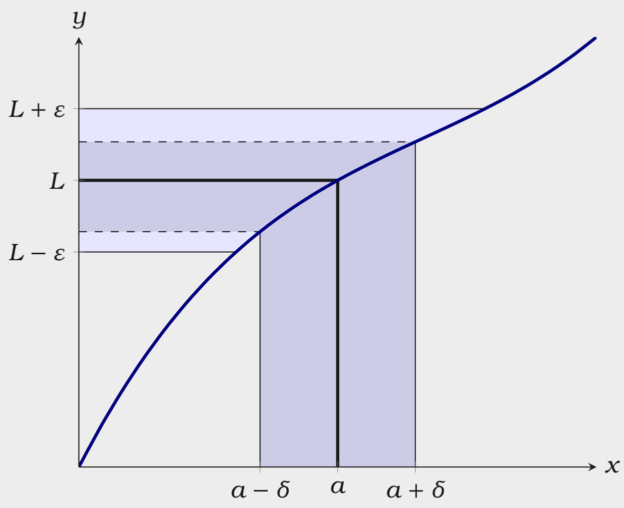


Figure 1.2: A geometric interpretation of the (ε, δ)-criterion for limits. If $0 < |x - a| < \delta$, then we have that $a - \delta < x < a + \delta$. In our diagram, we see that for all such x we are sure to have $L - \varepsilon < f(x) < L + \varepsilon$, and hence $|f(x) - L| < \varepsilon$.

Limits need not exist, let's examine two cases of this.

Example 1.1.1 Let $f(x) = \lfloor x \rfloor$. Explain why the limit

$$\lim_{x \to 2} f(x)$$

does not exist.

Solution *The function $\lfloor x \rfloor$ is the function that returns the greatest integer less than or equal to x. Since $f(x)$ is defined for all real numbers, one might be tempted to think that the limit above is simply $f(2) = 2$. However, this is not the case. If $x < 2$, then $f(x) = 1$. Hence if $\varepsilon = .5$, we can* ***always*** *find a value for x (just to the left of 2) such that*

$$0 < |x - 2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x) - 2|.$$

On the other hand, $\lim_{x \to 2} f(x) \neq 1$, as in this case if $\varepsilon = .5$, we can ***always*** *find a value for x (just to the right of 2) such that*

$$0 < |x - 2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x) - 1|.$$

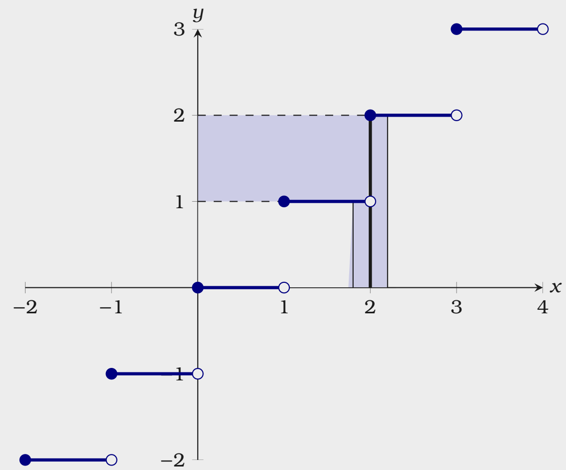


Figure 1.3: A plot of $f(x) = \lfloor x \rfloor$. Note, no matter which $\delta > 0$ is chosen, we can only at best bound $f(x)$ in the interval $[1, 2]$.

With the example of $f(x) = \lfloor x \rfloor$, we see that taking limits is truly different from evaluating functions.

We've illustrated this in Figure 1.3. Moreover, no matter what value one chooses for $\lim_{x\to 2} f(x)$, we will always have a similar issue.

Limits may not exist even if the formula for the function looks innocent.

Example 1.1.2 Let $f(x) = \sin\left(\frac{1}{x}\right)$. Explain why the limit

$$\lim_{x\to 0} f(x)$$

does not exist.

Solution *In this case $f(x)$ oscillates "wildly" as x approaches 0, see Figure 1.4. In fact, one can show that for any given δ, There is a value for x in the interval*

$$0 - \delta < x < 0 + \delta$$

such that $f(x)$ is ***any*** *value in the interval $[-1, 1]$. Hence the limit does not exist.*

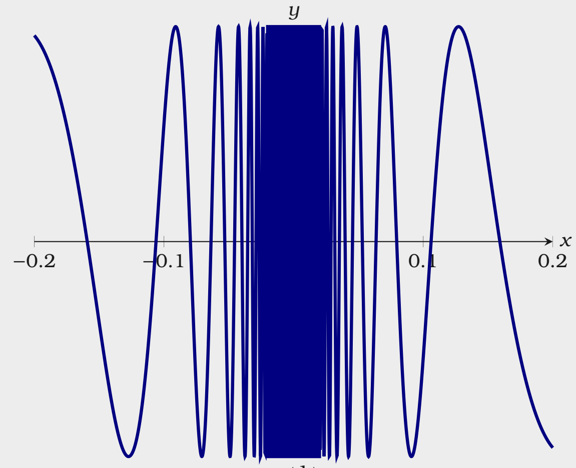


Figure 1.4: A plot of $f(x) = \sin\left(\frac{1}{x}\right)$.

Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of a *one-sided limit*:

Definition We say that the **limit** of $f(x)$ as x goes to a from the **left** is L,

$$\lim_{x\to a-} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x < a$ and

$$a - \delta < x \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

We say that the **limit** of $f(x)$ as x goes to a from the **right** is L,

$$\lim_{x\to a+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x > a$ and

$$x < a + \delta \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

Limits from the left, or from the right, are collectively called **one-sided limits**.

Example 1.1.3 Let $f(x) = \lfloor x \rfloor$. Discuss

$$\lim_{x\to 2-} f(x), \qquad \lim_{x\to 2+} f(x), \qquad \text{and} \qquad \lim_{x\to 2} f(x).$$

Solution *From the plot of $f(x)$, see Figure 1.3, we see that*

$$\lim_{x\to 2-} f(x) = 1, \qquad \textit{and} \qquad \lim_{x\to 2+} f(x) = 2.$$

Since these limits are different, $\lim_{x\to 2} f(x)$ does not exist.

Exercises for Section 1.1

(1) Evaluate the expressions by reference to the plot in Figure 1.5.

(a) $\lim_{x\to 4} f(x)$
(b) $\lim_{x\to -3} f(x)$
(c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to 0-} f(x)$
(e) $\lim_{x\to 0+} f(x)$
(f) $f(-2)$
(g) $\lim_{x\to 2-} f(x)$
(h) $\lim_{x\to -2-} f(x)$
(i) $\lim_{x\to 0} f(x+1)$
(j) $f(0)$
(k) $\lim_{x\to 1-} f(x-4)$
(l) $\lim_{x\to 0+} f(x-2)$

(2) Use a table and a calculator to estimate $\lim_{x\to 0} \frac{\sin(x)}{x}$.

(3) Use a table and a calculator to estimate $\lim_{x\to 0} \frac{\sin(2x)}{x}$.

(4) Use a table and a calculator to estimate $\lim_{x\to 0} \frac{x}{\sin\left(\frac{x}{3}\right)}$.

(5) Use a table and a calculator to estimate $\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$.

(6) Use a table and a calculator to estimate $\lim_{x\to 0} \frac{2^x - 1}{x}$.

(7) Use a table and a calculator to estimate $\lim_{x\to 0}(1+x)^{1/x}$.

(8) Sketch a plot of $f(x) = \frac{x}{|x|}$ and explain why $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

(9) Let $f(x) = \sin\left(\frac{\pi}{x}\right)$. Construct three tables of the following form

x	$f(x)$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	

where $d = 1, 3, 7$. What do you notice? How do you reconcile the entries in your tables with the value of $\lim_{x\to 0} f(x)$?

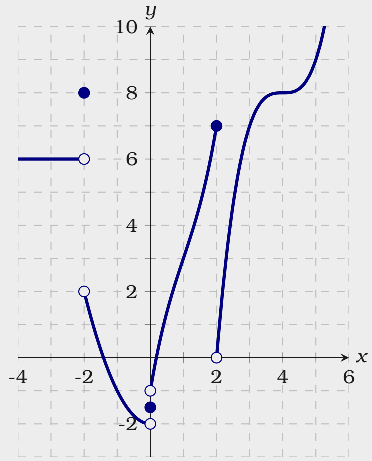


Figure 1.5: A plot of $f(x)$, a piecewise defined function.

(10) In the theory of special relativity, a moving clock ticks slower than a stationary observer's clock. If the stationary observer records that t_s seconds have passed, then the clock moving at velocity v has recorded that

$$t_v = t_s \sqrt{1 - v^2/c^2}$$

seconds have passed, where c is the speed of light. What happens as $v \to c$ from below?

1.2 Limits by the Definition

Now we are going to get our hands dirty, and really use the definition of a limit.

Recall, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Example 1.2.1 Show that $\lim_{x\to 2} x^2 = 4$.

Solution *We want to show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that*

$$|x^2 - 4| < \varepsilon$$

whenever $0 < |x-2| < \delta$. Start by factoring the left-hand side of the inequality above

$$|x+2||x-2| < \varepsilon.$$

Since we are going to assume that $0 < |x-2| < \delta$, we will focus on the factor $|x+2|$. Since x is assumed to be close to 2, suppose that $x \in [1,3]$. In this case

$$|x+2| \leq 3 + 2 = 5,$$

and so we want

$$5 \cdot |x-2| < \varepsilon$$
$$|x-2| < \frac{\varepsilon}{5}$$

Recall, we assumed that $x \in [1,3]$, which is equivalent to $|x-2| < 1$. Hence we must set $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

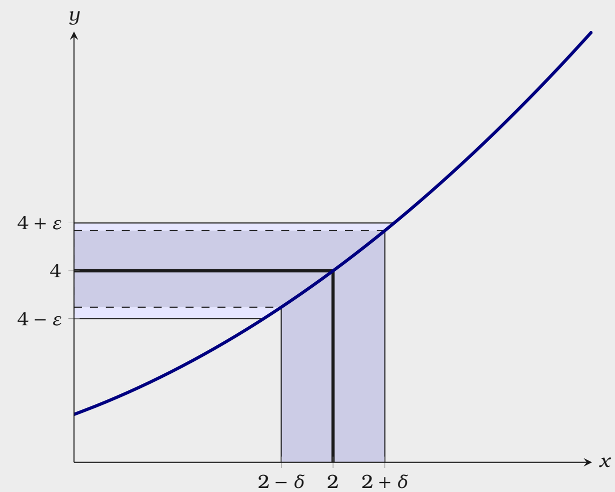


Figure 1.6: The (ε, δ)-criterion for $\lim_{x\to 2} x^2 = 4$. Here $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

When dealing with limits of polynomials, the general strategy is always the same. Let $p(x)$ be a polynomial. If showing

$$\lim_{x\to a} p(x) = L,$$

one must first factor out $|x-a|$ from $|p(x) - L|$. Next bound $x \in [a-1, a+1]$ and estimate the largest possible value of

$$\left|\frac{p(x) - L}{x - a}\right|$$

for $x \in [a-1, a+1]$. Call this estimation M. Finally, one must set $\delta = \min\left(\frac{\varepsilon}{M}, 1\right)$.

As you work with limits, you find that you need to do the same procedures again and again. The next theorems will expedite this process.

Theorem 1.2.2 (Limit Product Law) *Suppose* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then*

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof *Given any* ε *we need to find a* δ *such that*

$$0 < |x-a| < \delta$$

implies

$$|f(x)g(x) - LM| < \varepsilon.$$

Here we use an algebraic trick, add $0 = -f(x)M + f(x)M$*:*

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\
&= |f(x)(g(x)-M) + (f(x)-L)M| \\
&\le |f(x)(g(x)-M)| + |(f(x)-L)M| \\
&= |f(x)||g(x)-M| + |f(x)-L||M|.
\end{aligned}$$

Since $\lim_{x\to a} f(x) = L$, *there is a value* δ_1 *so that* $0 < |x-a| < \delta_1$ *implies* $|f(x)-L| < |\varepsilon/(2M)|$. *This means that* $0 < |x-a| < \delta_1$ *implies* $|f(x)-L||M| < \varepsilon/2$.

$$|f(x)g(x) - LM| \le |f(x)||g(x)-M| + \underbrace{|f(x)-L||M|}_{<\frac{\varepsilon}{2}}.$$

If we can make $|f(x)||g(x)-M| < \varepsilon/2$, *then we'll be done. We can make* $|g(x)-M|$ *smaller than any fixed number by making* x *close enough to* a. *Unfortunately,* $\varepsilon/(2f(x))$ *is not a fixed number since* x *is a variable.*

We will use this same trick again of "adding 0" in the proof of Theorem **??**.

This is all straightforward except perhaps for the "$\le$". This follows from the *Triangle Inequality*. The **Triangle Inequality** states: If a and b are any real numbers then $|a+b| \le |a| + |b|$.

Here we need another trick. We can find a δ_2 so that $|x-a| < \delta_2$ implies that $|f(x)-L| < 1$, meaning that $L-1 < f(x) < L+1$. This means that $|f(x)| < N$,

where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x)-M| < \varepsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that

$$|f(x)g(x)-LM| \le \underbrace{|f(x)|}_{<N}\underbrace{|g(x)-M|}_{<\frac{\varepsilon}{2N}} + \underbrace{|f(x)-L||M|}_{<\frac{\varepsilon}{2}}.$$

so

$$\begin{aligned}|f(x)g(x)-LM| &\le |f(x)||g(x)-M| + |f(x)-L||M| \\ &< N\frac{\varepsilon}{2N} + \left|\frac{\varepsilon}{2M}\right||M| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This is just what we needed, so by the definition of a limit, $\lim_{x\to a} f(x)g(x) = LM$.

Another useful way to put functions together is composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2+5$, then $f(g(x)) = \sqrt{x^2+5}$ and $g(f(x)) = (\sqrt{x})^2 + 5 = x+5$. This brings us to our next theorem.

Theorem 1.2.3 (Limit Composition Law) *Suppose that $\lim_{x\to a} g(x) = M$ and $\lim_{x\to M} f(x) = f(M)$. Then*

$$\lim_{x\to a} f(g(x)) = f(M).$$

This is sometimes written as

$$\lim_{x\to a} f(g(x)) = \lim_{g(x)\to M} f(g(x)).$$

Note the special form of the condition on $f(x)$: it is not enough to know that $\lim_{x\to L} f(x)$ exists, though it is a bit tricky to see why. Consider

$$f(x) = \begin{cases} 3 & \text{if } x = 2, \\ 4 & \text{if } x \neq 2. \end{cases}$$

and $g(x) = 2$. Now the conditions of Theorem 1.2.3 are not satisfied, and

$$\lim_{x\to 1} f(g(x)) = 3 \qquad \textit{but} \qquad \lim_{x\to 2} f(x) = 4.$$

Many of the most familiar functions do satisfy the conditions of Theorem 1.2.3. For example:

Theorem 1.2.4 (Limit Root Law) *Suppose that n is a positive integer. Then*

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Exercises for Section 1.2

(1) For each of the following limits, $\lim_{x\to a} f(x) = L$, use a graphing device to find δ such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$ where $\varepsilon = .1$.

(a) $\lim_{x\to 2}(3x + 1) = 7$ (c) $\lim_{x\to \pi} \sin(x) = 0$ (e) $\lim_{x\to 1} \sqrt{3x+1} = 2$

(b) $\lim_{x\to 1}(x^2 + 2) = 3$ (d) $\lim_{x\to 0} \tan(x) = 0$ (f) $\lim_{x\to -2} \sqrt{1-4x} = 3$

The next set of exercises are for advanced students and can be skipped on first reading.

(2) Use the definition of limits to explain why $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$. Hint: Use the fact that $|\sin(a)| < 1$ for any real number a.

(3) Use the definition of limits to explain why $\lim_{x\to 4}(2x - 5) = 3$.

(4) Use the definition of limits to explain why $\lim_{x\to -3}(-4x - 11) = 1$.

(5) Use the definition of limits to explain why $\lim_{x\to -2} \pi = \pi$.

(6) Use the definition of limits to explain why $\lim_{x\to -2} \dfrac{x^2 - 4}{x + 2} = -4$.

(7) Use the definition of limits to explain why $\lim_{x\to 4} x^3 = 64$.

(8) Use the definition of limits to explain why $\lim_{x\to 1}(x^2 + 3x - 1) = 3$.

(9) Use the definition of limits to explain why $\lim_{x\to 9} \dfrac{x - 9}{\sqrt{x} - 3} = 6$.

(10) Use the definition of limits to explain why $\lim_{x\to 2} \dfrac{1}{x} = \dfrac{1}{2}$.

1.3 Limit Laws

In this section, we present a handful of tools to compute many limits without explicitly working with the definition of limit. Each of these could be proved directly as we did in the previous section.

Theorem 1.3.1 (Limit Laws) *Suppose that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, k *is some constant, and* n *is a positive integer.*

Constant Law $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = kL.$

Sum Law $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M.$

Product Law $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM.$

Quotient Law $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}$, *if* $M \neq 0$.

Power Law $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n = L^n.$

Root Law $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ *provided if* n *is even, then* $f(x) \geq 0$ *near* a.

Composition Law *If* $\lim_{x\to a} g(x) = M$ *and* $\lim_{x\to M} f(x) = f(M)$, *then* $\lim_{x\to a} f(g(x)) = f(M)$.

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 1.3.2 Compute $\lim_{x\to 1} \frac{x^2 - 3x + 5}{x - 2}$.

Solution *Using limit laws,*

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1} x^2-3x+5}{\lim_{x\to 1}(x-2)} \\
&= \frac{\lim_{x\to 1} x^2 - \lim_{x\to 1} 3x + \lim_{x\to 1} 5}{\lim_{x\to 1} x - \lim_{x\to 1} 2} \\
&= \frac{(\lim_{x\to 1} x)^2 - 3\lim_{x\to 1} x + 5}{\lim_{x\to 1} x - 2} \\
&= \frac{1^2 - 3\cdot 1 + 5}{1-2} \\
&= \frac{1-3+5}{-1} = -3.
\end{aligned}$$

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

We're primarily interested in limits that aren't so easy, namely limits in which a denominator approaches zero. The basic idea is to "divide out" by the offending factor. This is often easier said than done—here we give two examples of algebraic tricks that work on many of these limits.

Example 1.3.3 Compute $\lim_{x\to 1} \dfrac{x^2+2x-3}{x-1}$.

Solution *We can't simply plug in $x = 1$ because that makes the denominator zero. However, when taking limits we assume $x \neq 1$:*

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+2x-3}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} \\
&= \lim_{x\to 1}(x+3) = 4
\end{aligned}$$

Limits allow us to examine functions where they are not defined.

Example 1.3.4 Compute $\displaystyle\lim_{x\to-1} \frac{\sqrt{x+5}-2}{x+1}$.

Solution *Using limit laws,*

$$\begin{aligned}
\lim_{x\to-1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to-1} \frac{\sqrt{x+5}-2}{x+1}\,\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\
&= \lim_{x\to-1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to-1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to-1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}.
\end{aligned}$$

Here we are rationalizing the numerator by multiplying by the conjugate.

We'll conclude with one more theorem that will allow us to compute more difficult limits.

Theorem 1.3.5 (Squeeze Theorem) *Suppose that $g(x) \leq f(x) \leq h(x)$ for all x close to a but not necessarily equal to a. If*

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

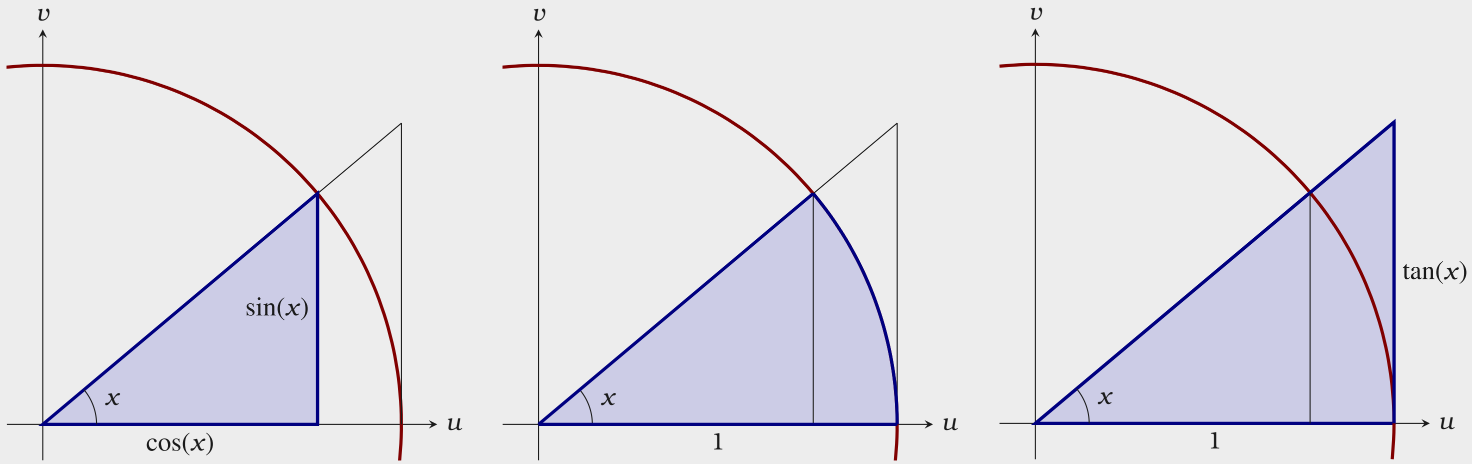
then $\displaystyle\lim_{x\to a} f(x) = L$.

Example 1.3.6 Compute

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

For a nice discussion of this limit, see: Richman, Fred. *A circular argument*. College Math. J. 24 (1993), no. 2, 160–162.

Solution *To compute this limit, use the Squeeze Theorem, Theorem 1.3.5. First note that we only need to examine* $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *and for the present time, we'll assume that* x *is positive—consider the diagrams below:*



Triangle A *Sector* *Triangle B*

From our diagrams above we see that

$$\text{Area of Triangle A} \leq \text{Area of Sector} \leq \text{Area of Triangle B}$$

and computing these areas we find

$$\frac{\cos(x)\sin(x)}{2} \leq \left(\frac{x}{2\pi}\right) \cdot \pi \leq \frac{\tan(x)}{2}.$$

Multiplying through by 2*, and recalling that* $\tan(x) = \frac{\sin(x)}{\cos(x)}$ *we obtain*

$$\cos(x)\sin(x) \leq x \leq \frac{\sin(x)}{\cos(x)}.$$

Dividing through by $\sin(x)$ *and taking the reciprocals, we find*

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}.$$

Note, $\cos(-x) = \cos(x)$ *and* $\dfrac{\sin(-x)}{-x} = \dfrac{\sin(x)}{x}$*, so these inequalities hold for all* $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$*. Additionally, we know*

$$\lim_{x\to 0} \cos(x) = 1 = \lim_{x\to 0} \frac{1}{\cos(x)},$$

and so we conclude by the Squeeze Theorem, Theorem 1.3.5, $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$.

Exercises for Section 1.3

Compute the limits. If a limit does not exist, explain why.

(1) $\lim_{x\to 3} \dfrac{x^2+x-12}{x-3}$

(2) $\lim_{x\to 1} \dfrac{x^2+x-12}{x-3}$

(3) $\lim_{x\to -4} \dfrac{x^2+x-12}{x-3}$

(4) $\lim_{x\to 2} \dfrac{x^2+x-12}{x-2}$

(5) $\lim_{x\to 1} \dfrac{\sqrt{x+8}-3}{x-1}$

(6) $\lim_{x\to 0+} \sqrt{\dfrac{1}{x}+2} - \sqrt{\dfrac{1}{x}}$

(7) $\lim_{x\to 2} 3$

(8) $\lim_{x\to 4} 3x^3-5x$

(9) $\lim_{x\to 0} \dfrac{4x-5x^2}{x-1}$

(10) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

(11) $\lim_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x}$

(12) $\lim_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x+1}$

(13) $\lim_{x\to a} \dfrac{x^3-a^3}{x-a}$

(14) $\lim_{x\to 2} (x^2+4)^3$

(15) $\lim_{x\to 1} \begin{cases} x-5 & \text{if } x \neq 1, \\ 7 & \text{if } x = 1. \end{cases}$

1.4 Infinite Limits

Consider the function

$$f(x) = \frac{1}{(x+1)^2}$$

While the $\lim_{x\to -1} f(x)$ does not exist, see Figure 1.7, something can still be said.

Definition If $f(x)$ grows arbitrarily large as x approaches a, we write

$$\lim_{x\to a} f(x) = \infty$$

and say that the limit of $f(x)$ **approaches infinity** as x goes to a.

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x\to a} f(x) = -\infty$$

and say that the limit of $f(x)$ **approaches negative infinity** as x goes to a.

On the other hand, if we consider the function

$$f(x) = \frac{1}{(x-1)}$$

While we have $\lim_{x\to 1} f(x) \neq \pm\infty$, we do have one-sided limits, $\lim_{x\to 1+} f(x) = \infty$ and $\lim_{x\to 1-} f(x) = -\infty$, see Figure 1.8.

Definition If

$$\lim_{x\to a} f(x) = \pm\infty, \qquad \lim_{x\to a+} f(x) = \pm\infty, \qquad \text{or} \qquad \lim_{x\to a-} f(x) = \pm\infty,$$

then the line $x = a$ is a **vertical asymptote** of $f(x)$.

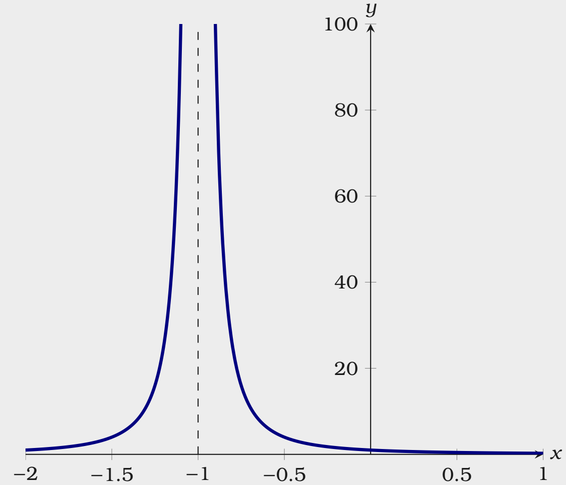


Figure 1.7: A plot of $f(x) = \dfrac{1}{(x+1)^2}$.

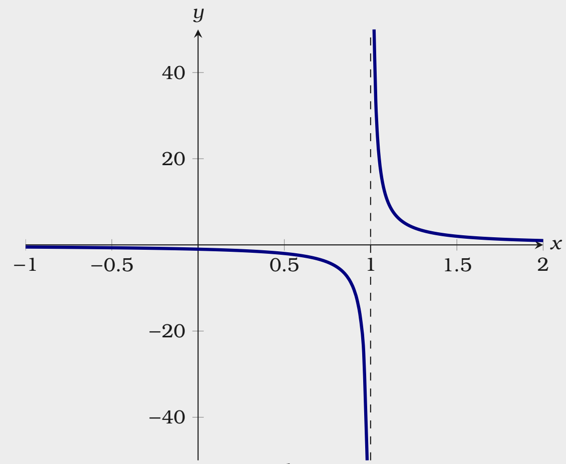


Figure 1.8: A plot of $f(x) = \dfrac{1}{x-1}$.

Example 1.4.1 Find the vertical asymptotes of

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

Solution *Start by factoring both the numerator and the denominator:*

$$\frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

Using limits, we must investigate when $x \to 2$ and $x \to 3$. Write

$$\begin{aligned}\lim_{x\to 2} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 2} \frac{(x-7)}{(x-3)} \\ &= \frac{-5}{-1} \\ &= 5.\end{aligned}$$

Now write

$$\begin{aligned}\lim_{x\to 3} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 3} \frac{(x-7)}{(x-3)} \\ &= \lim_{x\to 3} \frac{-4}{x-3}.\end{aligned}$$

Since $\lim_{x\to 3+} x - 3$ approaches 0 from the right and the numerator is negative, $\lim_{x\to 3+} f(x) = -\infty$. Since $\lim_{x\to 3-} x - 3$ approaches 0 from the left and the numerator is negative, $\lim_{x\to 3-} f(x) = \infty$. Hence we have a vertical asymptote at $x = 3$, see Figure 1.9.

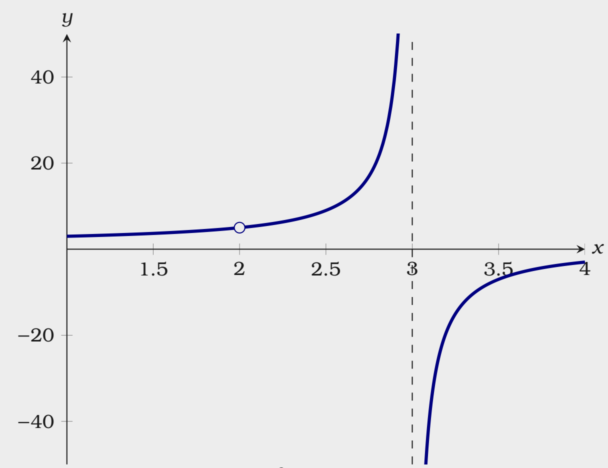


Figure 1.9: A plot of $f(x) = \dfrac{x^2 - 9x + 14}{x^2 - 5 + 6}$.

Exercises for Section 1.4

Compute the limits. If a limit does not exist, explain why.

(1) $\lim\limits_{x\to 1-} \dfrac{1}{x^2-1}$

(2) $\lim\limits_{x\to 4-} \dfrac{3}{x^2-2}$

(3) $\lim\limits_{x\to -1+} \dfrac{1+2x}{x^3-1}$

(4) $\lim\limits_{x\to 3+} \dfrac{x-9}{x^2-6x+9}$

(5) $\lim\limits_{x\to 5} \dfrac{1}{(x-5)^4}$

(6) $\lim\limits_{x\to -2} \dfrac{1}{(x^2+3x+2)^2}$

(7) $\lim\limits_{x\to 0} \dfrac{1}{\frac{x}{x^5}-\cos(x)}$

(8) $\lim\limits_{x\to 0+} \dfrac{x-11}{\sin(x)}$

(9) Find the vertical asymptotes of

$$f(x) = \frac{x-3}{x^2+2x-3}.$$

(10) Find the vertical asymptotes of

$$f(x) = \frac{x^2-x-6}{x+4}.$$

1.5 Limits at Infinity

Consider the function:

$$f(x) = \frac{6x-9}{x-1}$$

As x approaches infinity, it seems like $f(x)$ approaches a specific value. This is a *limit at infinity*.

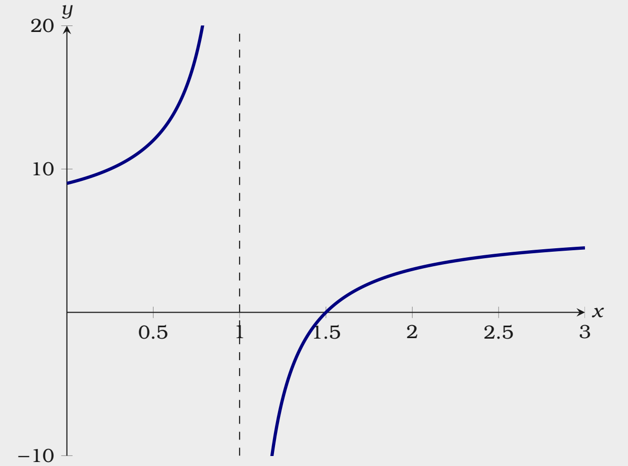


Figure 1.10: A plot of $f(x) = \frac{6x-9}{x-1}$.

Definition If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

$$\lim_{x\to\infty} f(x) = L$$

and we say, the **limit at infinity** of $f(x)$ is L.

If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x\to-\infty} f(x) = L$$

and we say, the **limit at negative infinity** of $f(x)$ is L.

Example 1.5.1 Compute

$$\lim_{x\to\infty} \frac{6x-9}{x-1}.$$

Solution *Write*

$$\begin{aligned}
\lim_{x\to\infty} \frac{6x-9}{x-1} &= \lim_{x\to\infty} \frac{6x-9}{x-1}\,\frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{\frac{6x}{x} - \frac{9}{x}}{\frac{x}{x} - \frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{6}{1} \\
&= 6.
\end{aligned}$$

Here is a somewhat different example of a limit at infinity.

Example 1.5.2 Compute
$$\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4.$$

Solution *We can bound our function*
$$-1/x + 4 \le \frac{\sin(7x)}{x} + 4 \le 1/x + 4.$$
Since
$$\lim_{x\to\infty} -1/x + 4 = 4 = \lim_{x\to\infty} 1/x + 4$$
we conclude by the Squeeze Theorem, Theorem 1.3.5, $\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4 = 4.$

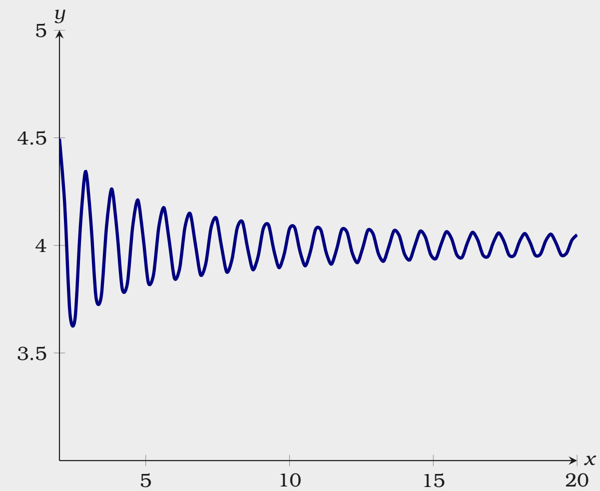


Figure 1.11: A plot of $f(x) = \frac{\sin(7x)}{x} + 4$.

Definition If
$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$
then the line $y = L$ is a **horizontal asymptote** of $f(x)$.

Example 1.5.3 Give the horizontal asymptotes of
$$f(x) = \frac{6x-9}{x-1}$$

Solution *From our previous work, we see that* $\lim_{x\to\infty} f(x) = 6$, *and upon further inspection, we see that* $\lim_{x\to-\infty} f(x) = 6$. *Hence the horizontal asymptote of* $f(x)$ *is the line* $y = 6$.

It is a common misconception that a function cannot cross an asymptote. As the next example shows, a function can cross an asymptote, and in this case this occurs an infinite number of times!

Example 1.5.4 Give a horizontal asymptote of
$$f(x) = \frac{\sin(7x)}{x} + 4.$$

Solution *Again from previous work, we see that* $\lim_{x\to\infty} f(x) = 4$. *Hence* $y = 4$ *is a horizontal asymptote of* $f(x)$.

We conclude with an infinite limit at infinity.

Example 1.5.5 Compute

$$\lim_{x\to\infty} \ln(x)$$

Solution *The function* $\ln(x)$ *grows very slowly, and seems like it may have a horizontal asymptote, see Figure 1.12. However, if we consider the definition of the natural log*

$$\ln(x) = y \qquad \Leftrightarrow \qquad e^y = x$$

Since we need to raise e to higher and higher values to obtain larger numbers, we see that $\ln(x)$ *is unbounded, and hence* $\lim_{x\to\infty} \ln(x) = \infty$.

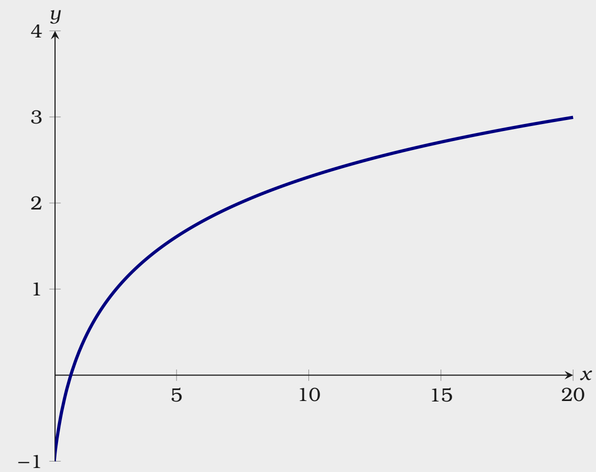


Figure 1.12: A plot of $f(x) = \ln(x)$.

Exercises for Section 1.5

Compute the limits.

(1) $\lim_{x\to\infty} \frac{1}{x}$

(2) $\lim_{x\to\infty} \frac{-x}{\sqrt{4+x^2}}$

(3) $\lim_{x\to\infty} \frac{2x^2-x+1}{4x^2-3x-1}$

(4) $\lim_{x\to-\infty} \frac{x^3-4}{3x^2+4x-1}$

(5) $\lim_{x\to\infty} \left(\frac{4}{x}+\pi\right)$

(6) $\lim_{x\to\infty} \frac{\cos(x)}{\ln(x)}$

(7) $\lim_{x\to\infty} \frac{\sin\left(x^7\right)}{\sqrt{x}}$

(8) $\lim_{x\to\infty} \left(17+\frac{32}{x}-\frac{(\sin(x/2))^2}{x^3}\right)$

(9) Suppose a population of feral cats on a certain college campus t years from now is approximated by

$$p(t) = \frac{1000}{5+2e^{-0.1t}}.$$

Approximately how many feral cats are on campus 10 years from now? 50 years from now? 100 years from now? 1000 years from now? What do you notice about the prediction—is this realistic?

(10) The amplitude of an oscillating spring is given by

$$a(t) = \frac{\sin(t)}{t}.$$

What happens to the amplitude of the oscillation over a long period of time?

1.6 Continuity

Informally, a function is continuous if you can "draw it" without "lifting your pencil." We need a formal definition.

Definition A function f is **continuous at a point** a if $\lim_{x\to a} f(x) = f(a)$.

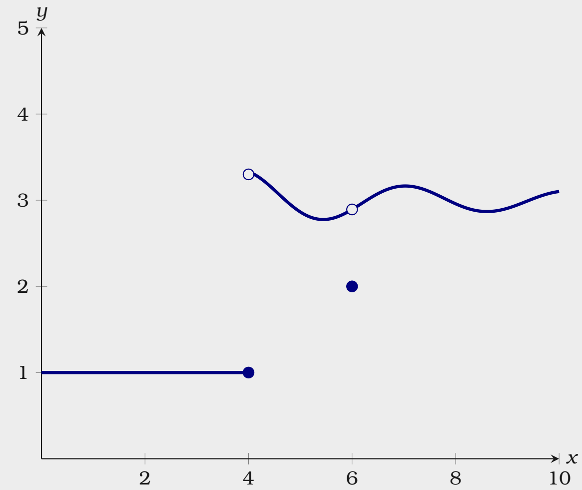


Figure 1.13: A plot of a function with discontinuities at $x = 4$ and $x = 6$.

Example 1.6.1 Find the discontinuities (the values for x where a function is not continuous) for the function given in Figure 1.13.

Solution *From Figure 1.13 we see that* $\lim_{x\to 4} f(x)$ *does not exist as*

$$\lim_{x\to 4-} f(x) = 1 \qquad \textit{and} \qquad \lim_{x\to 4+} f(x) \approx 3.5$$

Hence $\lim_{x\to 4} f(x) \neq f(4)$*, and so* $f(x)$ *is not continuous at* $x = 4$.

We also see that $\lim_{x\to 6} f(x) \approx 3$ *while* $f(6) = 2$. *Hence* $\lim_{x\to 6} f(x) \neq f(6)$*, and so* $f(x)$ *is not continuous at* $x = 6$.

Building from the definition of *continuous at a point*, we can now define what it means for a function to be *continuous* on an interval.

Definition A function f is **continuous on an interval** if it is continuous at every point in the interval.

In particular, we should note that if a function is not defined on an interval, then it **cannot** be continuous on that interval.

Example 1.6.2 Consider the function

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

see Figure 1.14. Is this function continuous?

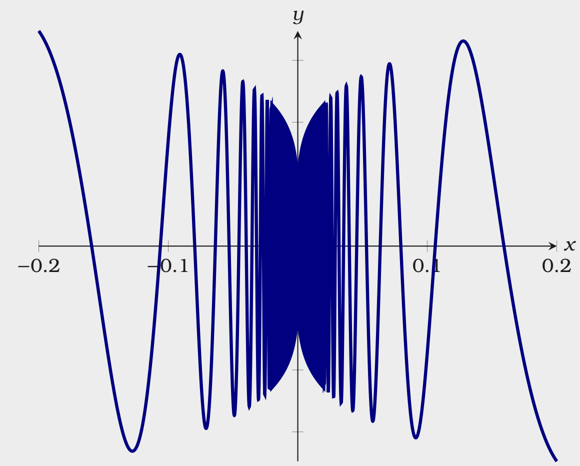


Figure 1.14: A plot of

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Solution *Considering $f(x)$, the only issue is when $x = 0$. We must show that $\lim_{x\to 0} f(x) = 0$. Note*

$$-|\sqrt[5]{x}| \leq f(x) \leq |\sqrt[5]{x}|.$$

Since

$$\lim_{x\to 0} -|\sqrt[5]{x}| = 0 = \lim_{x\to 0} |\sqrt[5]{x}|,$$

we see by the Squeeze Theorem, Theorem 1.3.5, that $\lim_{x\to 0} f(x) = 0$. Hence $f(x)$ is continuous.

Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.

We close with a useful theorem about continuous functions:

Theorem 1.6.3 (Intermediate Value Theorem) *If $f(x)$ is a function that is continuous for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*

The Intermediate Value Theorem is most frequently used when $d = 0$.

For a nice proof of this theorem, see: Walk, Stephen M. *The intermediate value theorem is NOT obvious—and I am going to prove it to you.* College Math. J. 42 (2011), no. 4, 254–259.

In Figure 1.15, we see a geometric interpretation of this theorem.

Example 1.6.4 Explain why the function $f(x) = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

Solution *By Theorem 1.3.1, $\lim_{x\to a} f(x) = f(a)$, for all real values of a, and hence f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in [0, 1]$ such that $f(c) = 0$.*

This example also points the way to a simple method for approximating roots.

Example 1.6.5 Approximate a root of $f(x) = x^3 + 3x^2 + x - 2$ to one decimal place.

Solution *If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6) < 0$ and $f(0.7) > 0$, so by the Intermediate Value Theorem, f has a root between 0.6*

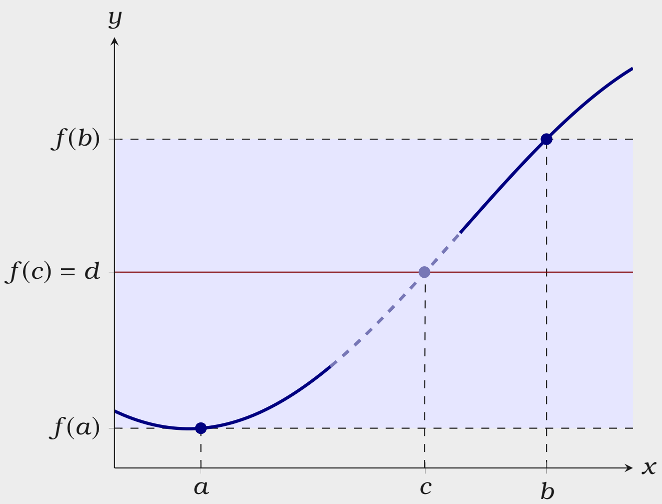


Figure 1.15: A geometric interpretation of the Intermediate Value Theorem. The function $f(x)$ is continuous on the interval $[a, b]$. Since d is in the interval $[f(a), f(b)]$, there exists a value c in $[a, b]$ such that $f(c) = d$.

and 0.7. *Repeating the process with* $f(0.61)$, $f(0.62)$, *and so on, we find that* $f(0.61) < 0$ *and* $f(0.62) > 0$, *so* f *has a root between* 0.61 *and* 0.62, *and the root is* 0.6 *rounded to one decimal place.*

Exercises for Section 1.6

(1) Consider the function

$$f(x) = \sqrt{x-4}$$

Is $f(x)$ continuous at the point $x = 4$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(2) Consider the function

$$f(x) = \frac{1}{x+3}$$

Is $f(x)$ continuous at the point $x = 3$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(3) Consider the function

$$f(x) = \begin{cases} 2x-3 & \text{if } x < 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

Is $f(x)$ continuous at the point $x = 1$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(4) Consider the function

$$f(x) = \begin{cases} \dfrac{x^2+10x+25}{x-5} & \text{if } x \neq 5, \\ 10 & \text{if } x = 5. \end{cases}$$

Is $f(x)$ continuous at the point $x = 5$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(5) Consider the function

$$f(x) = \begin{cases} \dfrac{x^2+10x+25}{x+5} & \text{if } x \neq -5, \\ 0 & \text{if } x = -5. \end{cases}$$

Is $f(x)$ continuous at the point $x = -5$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(6) Determine the interval(s) on which the function $f(x) = x^7 + 3x^5 - 2x + 4$ is continuous.

(7) Determine the interval(s) on which the function $f(x) = \dfrac{x^2-2x+1}{x+4}$ is continuous.

(8) Determine the interval(s) on which the function $f(x) = \dfrac{1}{x^2 - 9}$ is continuous.

(9) Approximate a root of $f(x) = x^3 - 4x^2 + 2x + 2$ to two decimal places.

(10) Approximate a root of $f(x) = x^4 + x^3 - 5x + 1$ to two decimal places.

2 Basics of Derivatives

2.1 Slopes of Tangent Lines via Limits

Suppose that $f(x)$ is a function. It is often useful to know how sensitive the value of $f(x)$ is to small changes in x. To give you a feeling why this is true, consider the following:

- If $p(t)$ represents the position of an object with respect to time, the rate of change gives the velocity of the object.
- If $v(t)$ represents the velocity of an object with respect to time, the rate of change gives the acceleration of the object.
- The rate of change of a function can help us approximate a complicated function with a simple function.
- The rate of change of a function can be used to help us solve equations that we would not be able to solve via other methods.

The rate of change of a function is the slope of the tangent line. Part of our goal will be to give a formal definition of a tangent line. For now, consider the following informal definition:

> Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

While this is merely an informal definition of a tangent line, it contains the essence of how the formal definition will be constructed. We illustrate this informal definition with Figure 2.1.

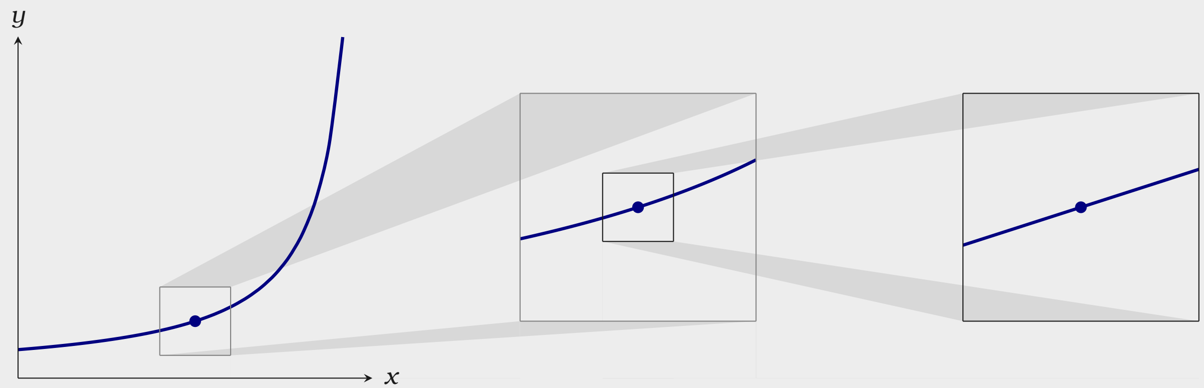


Figure 2.1: Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

The *derivative* of a function $f(x)$ at x, is the slope of the tangent line at x. To find the slope of this line, we consider *secant* lines, lines that locally intersect the curve at two points. The slope of any secant line that passes through the points $(x, f(x))$ and $(x+h, f(x+h))$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h},$$

see Figure 2.2. This leads to the *limit definition of the derivative*:

Definition The **derivative** of $f(x)$ is the function

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If this limit does not exist for a given value of x, then $f(x)$ is not **differentiable** at x.

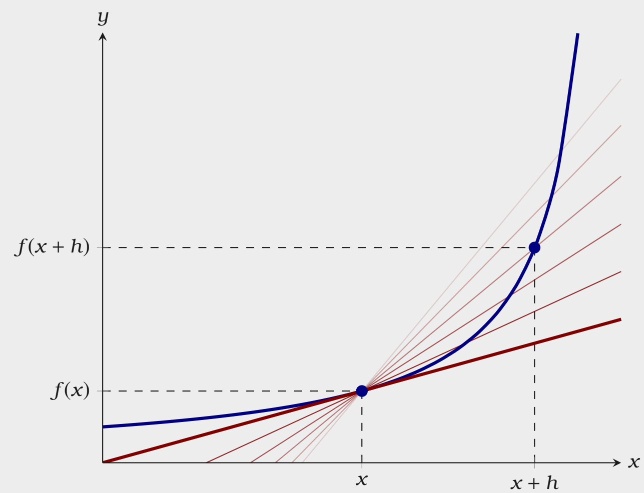


Figure 2.2: Tangent lines can be found as the limit of secant lines. The slope of the tangent line is given by $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Definition There are several different notations for the derivative, we'll mainly use

$$\frac{d}{dx}f(x) = f'(x).$$

If one is working with a function of a variable other than x, say t we write

$$\frac{d}{dt}f(t) = f'(t).$$

However, if $y = f(x)$, $\frac{dy}{dx}$, $\dot{y}$, and $D_x f(x)$ are also used.

Now we will give a number of examples, starting with a basic example.

Example 2.1.1 Compute

$$\frac{d}{dx}(x^3 + 1).$$

Solution *Using the definition of the derivative,*

$$\begin{aligned} \frac{d}{dx}f(x) &= \lim_{h\to 0} \frac{(x+h)^3 + 1 - (x^3+1)}{h} \\ &= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 1 - x^3 - 1}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\ &= 3x^2. \end{aligned}$$

See Figure 2.3.

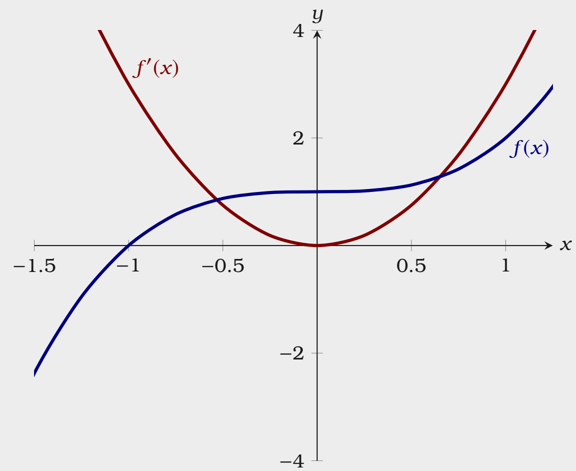


Figure 2.3: A plot of $f(x) = x^3 + 1$ and $f'(x) = 3x^2$.

Next we will consider the derivative a function that is not continuous on $\mathbb{R}$.

Example 2.1.2 Compute

$$\frac{d}{dt}\frac{1}{t}.$$

Solution *Using the definition of the derivative,*

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{t} &= \lim_{h\to 0}\frac{\frac{1}{t+h}-\frac{1}{t}}{h}\\
&= \lim_{h\to 0}\frac{\frac{t}{t(t+h)}-\frac{t+h}{t(t+h)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{t-(t+h)}{t(t+h)}}{h}\\
&= \lim_{h\to 0}\frac{t-t-h}{t(t+h)h}\\
&= \lim_{h\to 0}\frac{-h}{t(t+h)h}\\
&= \lim_{h\to 0}\frac{-1}{t(t+h)}\\
&= \frac{-1}{t^2}.
\end{aligned}
$$

This function is differentiable at all real numbers except for $t = 0$, see Figure 2.4.

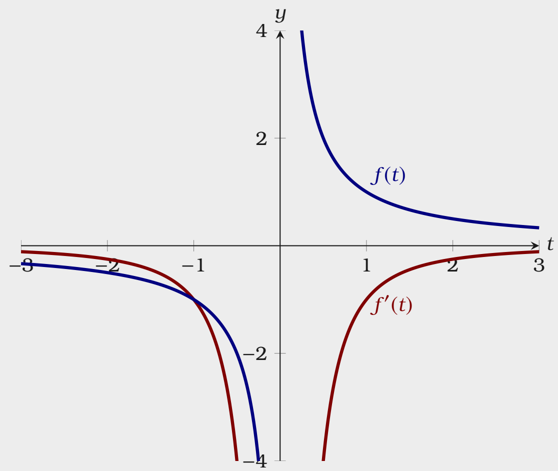


Figure 2.4: A plot of $f(t) = \frac{1}{t}$ and $f'(t) = \frac{-1}{t^2}$.

As you may have guessed, there is some connection to continuity and differentiability.

> **Theorem 2.1.3 (Differentiability implies Continuity)** *If $f(x)$ is a differentiable function at $x = a$, then $f(x)$ is continuous at $x = a$.*

Proof *We want to show that $f(x)$ is continuous at $x = a$, hence we must show that*

$$\lim_{x\to a} f(x) = f(a).$$

Consider

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\left((x-a)\frac{f(x)-f(a)}{x-a}\right) && \textit{Multiply and divide by } (x-a).\\
&= \lim_{x\to a}(x-a)\lim_{x\to a}\frac{f(x)-f(a)}{x-a} && \textit{Limit Law.}\\
&= \lim_{h\to 0}h\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} && \textit{Set } x=a+h.\\
&= 0\cdot f'(a)=0.
\end{aligned}$$

Since

$$\lim_{x\to a}(f(x)-f(a))=0$$

we see that $\lim_{x\to a} f(x) = f(a)$*, and so* $f(x)$ *is continuous.*

This theorem is often written as its contrapositive:

If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

Let's see a function that is continuous whose derivative does not exist everywhere.

Example 2.1.4 Compute

$$\frac{d}{dx}|x|.$$

Solution *Using the definition of the derivative,*

$$\frac{d}{dx}|x| = \lim_{h\to 0}\frac{|x+h|-|x|}{h}.$$

If x *is positive we may assume that* x *is larger than* h*, as we are taking the limit as* h *goes to* 0*,*

$$\begin{aligned}
\lim_{h\to 0}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0}\frac{x+h-x}{h}\\
&= \lim_{h\to 0}\frac{h}{h}\\
&= 1.
\end{aligned}$$

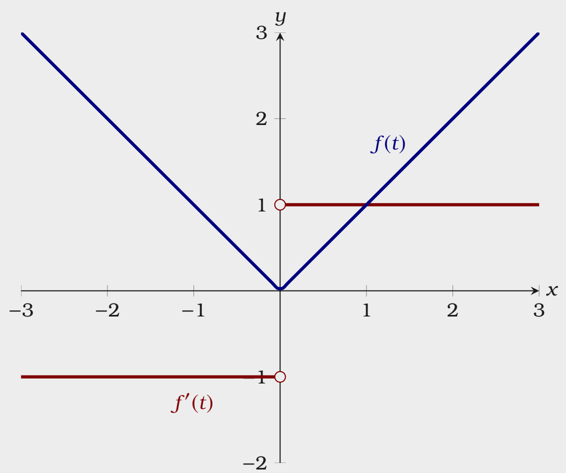


Figure 2.5: A plot of $f(x) = |x|$ and

$$f'(x)=\begin{cases}1 & \text{if } x>0,\\ -1 & \text{if } x<0.\end{cases}$$

If x is negative we may assume that $|x|$ is larger than h, as we are taking the limit as h goes to 0,

$$\begin{aligned}\lim_{h\to 0}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0}\frac{x-h-x}{h}\\ &= \lim_{h\to 0+}\frac{-h}{h}\\ &= -1.\end{aligned}$$

However we still have one case left, when $x = 0$. In this situation, we must consider the one-sided limits:

$$\lim_{h\to 0+}\frac{|x+h|-|x|}{h} \qquad \textit{and} \qquad \lim_{h\to 0-}\frac{|x+h|-|x|}{h}.$$

In the first case,

$$\begin{aligned}\lim_{h\to 0+}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0+}\frac{0+h-0}{h}\\ &= \lim_{h\to 0+}\frac{h}{h}\\ &= 1.\end{aligned}$$

On the other hand

$$\begin{aligned}\lim_{h\to 0-}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0-}\frac{|0+h|-0}{h}\\ &= \lim_{h\to 0-}\frac{|h|}{h}\\ &= -1.\end{aligned}$$

Hence we see that the derivative is

$$f'(x) = \begin{cases} 1 & \textit{if } x > 0,\\ -1 & \textit{if } x < 0.\end{cases}$$

Note this function is undefined at 0, *see Figure 2.5.*

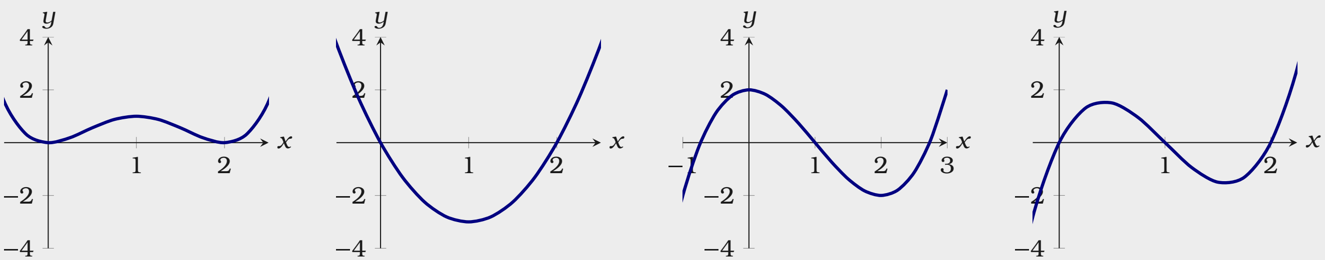
Thus from Theorem 2.1.3, we see that all differentiable functions on $\mathbb{R}$ are continuous on $\mathbb{R}$. Nevertheless as the previous example shows, there are continuous functions on $\mathbb{R}$ that are not differentiable on $\mathbb{R}$.

Exercises for Section 2.1

These exercises are conceptual in nature and require one to think about what the derivative means.

(1) If the line $y = 7x - 4$ is tangent to $f(x)$ at $x = 2$, find $f(2)$ and $f'(2)$.

(2) Here are plots of four functions.



$p(x)$ $q(x)$ $r(x)$ $s(x)$

Two of these functions are the derivatives of the other two, identify which functions are the derivatives of the others.

(3) If $f(3) = 6$ and $f(3.1) = 6.4$, estimate $f'(3)$.

(4) If $f(-2) = 4$ and $f(-2 + h) = (h + 2)^2$, compute $f'(-2)$.

(5) If $f'(x) = x^3$ and $f(1) = 2$, approximate $f(1.2)$.

(6) Consider the plot of $f(x)$ in Figure 2.6.

(a) On which subinterval(s) of $[0, 6]$ is $f(x)$ continuous?

(b) On which subinterval(s) of $[0, 6]$ is $f(x)$ differentiable?

(c) Sketch a plot of $f'(x)$.

Figure 2.6: A plot of $f(x)$.

These exercises are computational in nature.

(7) Let $f(x) = x^2 - 4$. Use the definition of the derivative to compute $f'(-3)$ and find the equation of the tangent line to the curve at $x = -3$.

(8) Let $f(x) = \dfrac{1}{x+2}$. Use the definition of the derivative to compute $f'(1)$ and find the equation of the tangent line to the curve at $x = 1$.

(9) Let $f(x) = \sqrt{x-3}$. Use the definition of the derivative to compute $f'(5)$ and find the equation of the tangent line to the curve at $x = 5$.

(10) Let $f(x) = \dfrac{1}{\sqrt{x}}$. Use the definition of the derivative to compute $f'(4)$ and find the equation of the tangent line to the curve at $x = 4$.

2.2 Basic Derivative Rules

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. We will start simply and build-up to more complicated examples.

2.2.1 The Constant Rule

The simplest function is a constant function. Recall that derivatives measure the rate of change of a function at a given point. Hence, the derivative of a constant function is zero. For example:

- The constant function plots a horizontal line—so the slope of the tangent line is 0.
- If $p(t)$ represents the position of an object with respect to time and $p(t)$ is constant, then the object is not moving, so its velocity is zero. Hence $\frac{d}{dt}p(t) = 0$.
- If $v(t)$ represents the velocity of an object with respect to time and $v(t)$ is constant, then the object's acceleration is zero. Hence $\frac{d}{dt}v(t) = 0$.

The examples above lead us to our next theorem.

To gain intuition, you should compute the derivative of $f(x) = 6$ using the limit definition of the derivative.

Theorem 2.2.1 (The Constant Rule) *Given a constant c,*

$$\frac{d}{dx}c = 0.$$

Proof *From the limit definition of the derivative, write*

$$\begin{aligned}\frac{d}{dx}c &= \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= \lim_{h\to 0} 0 = 0.\end{aligned}$$

2.2.2 The Power Rule

Now let's examine derivatives of powers of a single variable. Here we have a nice rule.

To gain intuition, you should compute the derivative of $f(x) = x^3$ using the limit definition of the derivative.

Theorem 2.2.2 (The Power Rule) *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof *At this point we will only prove this theorem for n being a positive integer, later we will give the complete proof. From the limit definition of the derivative, write*

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Start by expanding the term $(x+h)^n$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n - x^n}{h}$$

Note, by the Binomial Theorem, we write $\binom{n}{k}$ for the coefficients. Canceling the terms x^n and $-x^n$, and noting $\binom{n}{1} = \binom{n}{n-1} = n$, write

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n}{h}\\ &= \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}.\end{aligned}$$

Since every term but the first has a factor of h, we see

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Recall, the **Binomial Theorem** states that if n is a nonnegative integer, then

$$(a+b)^n = a^nb^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1b^{n-1} + a^0b^n$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now we will show you several examples. We begin with something basic.

Example 2.2.3 Compute
$$\frac{d}{dx}x^{13}.$$

Solution *Applying the power rule, we write*
$$\frac{d}{dx}x^{13} = 13x^{12}.$$

Sometimes, it is not as obvious that one should apply the power rule.

Example 2.2.4 Compute
$$\frac{d}{dx}\frac{1}{x^4}.$$

Solution *Applying the power rule, we write*
$$\frac{d}{dx}\frac{1}{x^4} = \frac{d}{dx}x^{-4} = -4x^{-5}.$$

The power rule also applies to radicals once we rewrite them as exponents.

Example 2.2.5 Compute
$$\frac{d}{dx}\sqrt[5]{x}.$$

Solution *Applying the power rule, we write*
$$\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \frac{x^{-4/5}}{5}.$$

2.2.3 The Sum Rule

We want to be able to take derivatives of functions “one piece at a time.” The *sum rule* allows us to do this. The sum rule says that we can add the rates of change of two functions to obtain the rate of change of the sum of both functions. For example, viewing the derivative as the velocity of an object, the sum rule states that the velocity of the person walking on a moving bus is the sum of the velocity of the bus and the walking person.

Theorem 2.2.6 (The Sum Rule) *If $f(x)$ and $g(x)$ are differentiable and c is a constant, then*

(a) $\dfrac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$,

(b) $\dfrac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$,

(c) $\dfrac{d}{dx}(c\cdot f(x)) = c\cdot f'(x)$.

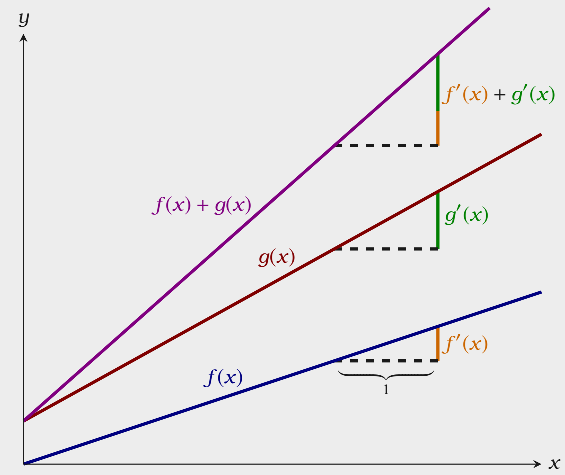


Figure 2.7: A geometric interpretation of the sum rule.

Proof *We will only prove part (a) above, the rest are similar. Write*

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x).
\end{aligned}$$

Example 2.2.7 Compute

$$\frac{d}{dx}\left(x^5+\frac{1}{x}\right).$$

Solution *Write*

$$\begin{aligned}
\frac{d}{dx}\left(x^5+\frac{1}{x}\right) &= \frac{d}{dx}x^5+\frac{d}{dx}x^{-1}\\
&= 5x^4 - x^{-2}.
\end{aligned}$$

Example 2.2.8 Compute

$$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right).$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right) &= 3\frac{d}{dx}x^{-1/3} - 2\frac{d}{dx}x^{1/2} + \frac{d}{dx}x^{-7}\\ &= -x^{-4/3} - x^{-1/2} - 7x^{-8}.\end{aligned}$$

2.2.4 The Derivative of e^x

We don't know anything about derivatives that allows us to compute the derivatives of exponential functions without getting our hands dirty. Let's do a little work with the definition of the derivative:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h} - a^x}{h}\\ &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\ &= \lim_{h\to 0} a^x\frac{a^h - 1}{h}\\ &= a^x \lim_{h\to 0}\frac{a^h - 1}{h}\\ &= a^x \cdot \underbrace{(\text{constant})}_{\lim_{h\to 0}\frac{a^h-1}{h}}\end{aligned}$$

There are two interesting things to note here: We are left with a limit that involves h but not x, which means that whatever $\lim\limits_{h\to 0}(a^h - 1)/h$ is, we know that it is a number, that is, a constant. This means that a^x has a remarkable property: Its derivative is a constant times itself. Unfortunately it is beyond the scope of this text to compute the limit

$$\lim_{h\to 0}\frac{a^h - 1}{h}.$$

However, we can look at some examples. Consider $(2^h - 1)/h$ and $(3^h - 1)/h$:

h	$(2^h-1)/h$
-1	$.5$
-0.1	≈ 0.6700
-0.01	≈ 0.6910
-0.001	≈ 0.6929
-0.0001	≈ 0.6931
-0.00001	≈ 0.6932

h	$(2^h-1)/h$
1	1
0.1	≈ 0.7177
0.01	≈ 0.6956
0.001	≈ 0.6834
0.0001	≈ 0.6932
0.00001	≈ 0.6932

h	$(3^h-1)/h$
-1	≈ 0.6667
-0.1	≈ 1.0404
-0.01	≈ 1.0926
-0.001	≈ 1.0980
-0.0001	≈ 1.0986
-0.00001	≈ 1.0986

h	$(3^h-1)/h$
1	2
0.1	≈ 1.1612
0.01	≈ 1.1047
0.001	≈ 1.0992
0.0001	≈ 1.0987
0.00001	≈ 1.0986

While these tables don't prove a pattern, it turns out that

$$\lim_{h\to 0}\frac{2^h-1}{h}\approx .7 \qquad \text{and} \qquad \lim_{h\to 0}\frac{3^h-1}{h}\approx 1.1.$$

Moreover, if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1. This happens when

$$a = e = 2.718281828459045\ldots.$$

This brings us to our next definition.

Definition Euler's number is defined to be the number e such that

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

Now we see that the function e^x has a truly remarkable property:

Theorem 2.2.9 (The Derivative of e^x)

$$\frac{d}{dx}e^x = e^x.$$

Proof *From the limit definition of the derivative, write*

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}\\ &= \lim_{h\to 0} e^x\frac{e^h-1}{h}\\ &= e^x \lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^x.\end{aligned}$$

Hence e^x is its own derivative. In other words, the slope of the plot of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (a, e^a) and has slope e^a there, no matter what a is.

Example 2.2.10 Compute:

$$\frac{d}{dx}\left(8\sqrt{x}+7e^x\right)$$

Solution *Write:*

$$\begin{aligned}\frac{d}{dx}\left(8\sqrt{x}+7e^x\right) &= 8\frac{d}{dx}x^{1/2}+7\frac{d}{dx}e^x\\ &= 4x^{-1/2}+7e^x.\end{aligned}$$

Exercises for Section 2.2

Compute:

(1) $\frac{d}{dx}5$

(2) $\frac{d}{dx}-7$

(3) $\frac{d}{dx}e^7$

(4) $\frac{d}{dx}\frac{1}{\sqrt{2}}$

(5) $\frac{d}{dx}x^{100}$

(6) $\frac{d}{dx}x^{-100}$

(7) $\frac{d}{dx}\frac{1}{x^5}$

(8) $\frac{d}{dx}x^{\pi}$

(9) $\frac{d}{dx}x^{3/4}$

(10) $\frac{d}{dx}\frac{1}{(\sqrt[7]{x})^9}$

(11) $\frac{d}{dx}\left(5x^3+12x^2-15\right)$

(12) $\frac{d}{dx}\left(-4x^5+3x^2-\frac{5}{x^2}\right)$

(13) $\frac{d}{dx}5(-3x^2+5x+1)$

(14) $\frac{d}{dx}\left(3\sqrt{x}+\frac{1}{x}-x^e\right)$

(15) $\frac{d}{dx}\left(\frac{x^2}{x^7}+\frac{\sqrt{x}}{x}\right)$

Expand or simplify to compute the following:

(16) $\frac{d}{dx}\left((x+1)(x^2+2x-3)\right)$

(17) $\frac{d}{dx}\frac{x^3-2x^2-5x+6}{(x-1)}$

(18) $\frac{d}{dx}\frac{x-5}{\sqrt{x}-\sqrt{5}}$

(19) $\frac{d}{dx}((x+1)(x+1)(x-1)(x-1))$

(20) Suppose the position of an object at time t is given by $f(t)=-49t^2/10+5t+10$. Find a function giving the velocity of the object at time t. The acceleration of an object is the rate at which its velocity is changing, which means it is given by the derivative of the velocity function. Find the acceleration of the object at time t.

(21) Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of $f(x)$, $cf(x)$, $f'(x)$, and $(cf(x))'$ on the same diagram.

(22) Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$.

(23) Find an equation for the tangent line to $f(x) = x^3/4 - 1/x$ at $x = -2$.

(24) Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$.

(25) Prove that $\dfrac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

Answers to Exercises

Answers for 1.1

1. (a) 8, (b) 6, (c) DNE, (d) -2, (e) -1, (f) 8, (g) 7, (h) 6, (i) 3, (j) $-3/2$, (k) 6, (l) 2 2. 1 3. 2 4. 3 5. $3/5$ 6. $0.6931 \approx \ln(2)$ 7. $2.718 \approx e$ 8. Consider what happens when x is near zero and positive, as compared to when x is near zero and negative. 9. The limit does not exist, so it is not surprising that the resulting values are so different. 10. When v approaches c from below, then t_v approaches zero—meaning that one second to the stationary observations seems like very little time at all for our traveler.

Answers for 1.2

1. For these problems, there are many possible values of δ, so we provide an inequality that δ must satisfy when $\varepsilon = 0.1$. (a) $\delta < 1/30$, (b) $\delta < \frac{\sqrt{110}}{10} - 1 \approx 0.0488$, (c) $\delta < \arcsin(1/10) \approx 0.1002$, (d) $\delta < \arctan(1/10) \approx 0.0997$ (e) $\delta < 13/100$, (f) $\delta < 59/400$ 2. Let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $0 < |x - 0| < \delta$, then $|x \cdot 1| < \varepsilon$, since $\sin\left(\frac{1}{x}\right) \leq 1$, $\left|x \sin\left(\frac{1}{x}\right) - 0\right| < \varepsilon$. 3. Let $\varepsilon > 0$. Set $\delta = \varepsilon/2$. If $0 < |x - 4| < \delta$, then $|2x - 8| < 2\delta = \varepsilon$, and then because $|2x - 8| = |(2x - 5) - 3|$, we conclude $|(2x - 5) - 3| < \varepsilon$. 4. Let $\varepsilon > 0$. Set $\delta = \varepsilon/4$. If $0 < |x - (-3)| < \delta$, then $|-4x - 12| < 4\delta = \varepsilon$, and then because $|-4x - 12| = |(-4x - 11) - 1|$, we conclude $|(-4x - 11) - 1| < \varepsilon$. 5. Let $\varepsilon > 0$. No matter what I choose for δ, if x is within δ of -2, then π is within ε of π. 6. As long as $x \neq -2$, we have $\frac{x^2 - 4}{x + 2} = x - 2$, and the limit is not sensitive to the value of the function at the point -2; the limit

only depends on nearby values, so we really want to compute $\lim_{x\to -2}(x-2)$. Let $\varepsilon > 0$. Set $\delta = \varepsilon$. Then if $0 < |x-(-2)| < \delta$, we have $|(x-2)-(-4)| < \varepsilon$. 7. Let $\varepsilon > 0$. Pick δ so that $\delta < 1$ and $\delta < \dfrac{\varepsilon}{61}$. Suppose $0 < |x-4| < \delta$. Then $4-\delta < x < 4+\delta$. Cube to get $(4-\delta)^3 < x^3 < (4+\delta)^3$. Expanding the right-side inequality, we get $x^3 < \delta^3 + 12\cdot\delta^2 + 48\cdot\delta + 64 < \delta + 12\delta + 48\delta + 64 = 64 + \varepsilon$. The other inequality is similar. 8. Let $\varepsilon > 0$. Pick δ small enough so that $\delta < \varepsilon/6$ and $\delta < 1$. Assume $|x-1| < \delta$, so $6\cdot|x-1| < \varepsilon$. Since x is within $\delta < 1$ of 1, we know $0 < x < 2$. So $|x+4| < 6$. Putting it together, $|x+4|\cdot|x-1| < \varepsilon$, so $|x^2+3x-4| < \varepsilon$, and therefore $|(x^2+3x-1)-3| < \varepsilon$. 9. Let $\varepsilon > 0$. Set $\delta = 3\varepsilon$. Assume $0 < |x-9| < \delta$. Divide both sides by 3 to get $\dfrac{|x-9|}{3} < \varepsilon$. Note that $\sqrt{x}+3 > 3$, so $\dfrac{|x-9|}{\sqrt{x}+3} < \varepsilon$. This can be rearranged to conclude $\left|\dfrac{x-9}{\sqrt{x}-3} - 6\right| < \varepsilon$. 10. Let $\varepsilon > 0$. Set δ to be the minimum of 2ε and 1. Assume x is within δ of 2, so $|x-2| < 2\varepsilon$ and $1 < x < 3$. So $\left|\dfrac{x-2}{2}\right| < \varepsilon$. Since $1 < x < 3$, we also have $2x > 2$, so $\left|\dfrac{x-2}{2x}\right| < \varepsilon$. Simplifying, $\left|\dfrac{1}{2} - \dfrac{1}{x}\right| < \varepsilon$, which is what we wanted.

Answers for 1.3

1. 7 2. 5 3. 0 4. DNE 5. 1/6 6. 0 7. 3 8. 172 9. 0 10. 2 11. DNE 12. $\sqrt{2}$ 13. $3a^2$ 14. 512 15. -4

Answers for 1.4

1. $-\infty$ 2. 3/14 3. 1/2 4. $-\infty$ 5. ∞ 6. ∞ 7. 0 8. $-\infty$ 9. $x = 1$ and $x = -3$ 10. $x = -4$

Answers for 1.5

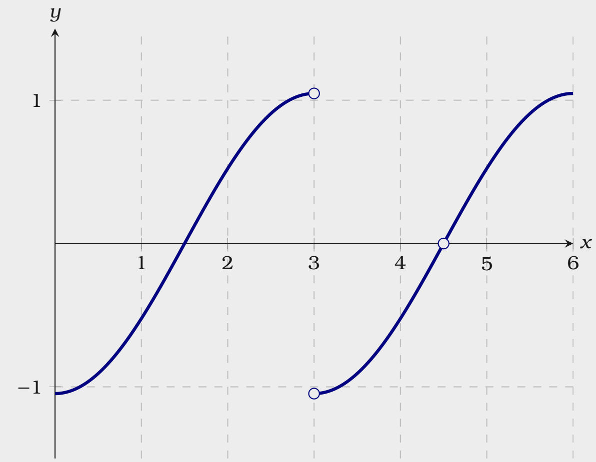
1. 0 2. -1 3. $\dfrac{1}{2}$ 4. $-\infty$ 5. π 6. 0 7. 0 8. 17 9. After 10 years, $\approx$ 174 cats; after 50 years, $\approx$ 199 cats; after 100 years, $\approx$ 200 cats; after 1000 years, $\approx$ 200 cats; in the sense that the population of cats cannot grow indefinitely this is somewhat realistic. 10. The amplitude goes to zero.

Answers for 1.6

1. $f(x)$ is continuous at $x = 4$ but it is not continuous on $\mathbb{R}$. 2. $f(x)$ is continuous at $x = 3$ but it is not continuous on $\mathbb{R}$. 3. $f(x)$ is not continuous at $x = 1$ and it is not continuous on $\mathbb{R}$. 4. $f(x)$ is not continuous at $x = 5$ and it is not continuous on $\mathbb{R}$. 5. $f(x)$ is continuous at $x = -5$ and it is also continuous on $\mathbb{R}$. 6. $\mathbb{R}$ 7. $(-\infty, -4) \cup (-4, \infty)$ 8. $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ 9. $x = -0.48$, $x = 1.31$, or $x = 3.17$ 10. $x = 0.20$, or $x = 1.35$

Answers for 2.1

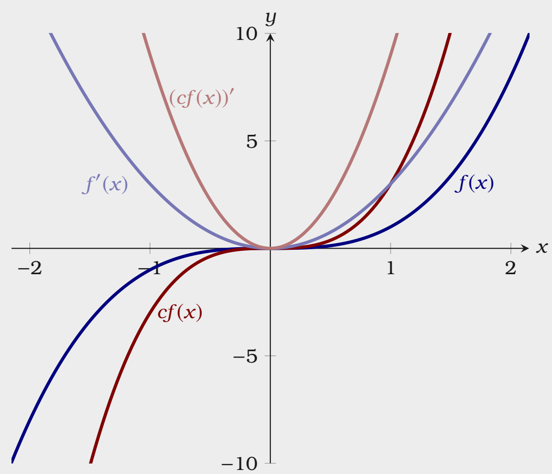
1. $f(2) = 10$ and $f'(2) = 7$ 2. $p'(x) = s(x)$ and $r'(x) = q(x)$ 3. $f'(3) \approx 4$ 4. $f'(-2) = 4$ 5. $f(1.2) \approx 2.2$ 6. (a) $[0, 4.5) \cup (4.5, 6]$, (b) $[0, 3) \cup (3, 6]$, (c) See Figure 7. $f'(-3) = -6$ with tangent line $y = -6x - 13$ 8. $f'(1) = -1/2$ with tangent line $y = \frac{-1}{2}x + \frac{5}{6}$ 9. $f'(5) = \frac{1}{2\sqrt{2}}$ with tangent line $y = \frac{1}{2\sqrt{2}}x - \frac{1}{2\sqrt{2}}$ 10. $f'(4) = \frac{-1}{16}$ with tangent line $y = \frac{-1}{16}x + \frac{3}{4}$



Answer 2.1.6: (c) a sketch of $f'(x)$.

Answers for 2.2

1. 0 2. 0 3. 0 4. 0 5. $100x^{99}$ 6. $-100x^{-101}$ 7. $-5x^{-6}$ 8. $\pi x^{\pi-1}$ 9. $(3/4)x^{-1/4}$ 10. $-(9/7)x^{-16/7}$ 11. $15x^2 + 24x$ 12. $-20x^4 + 6x + 10/x^3$ 13. $-30x + 25$ 14. $\frac{3}{2}x^{-1/2} - x^{-2} - ex^{e-1}$ 15. $-5x^{-6} - x^{-3/2}/2$ 16. $3x^2 + 6x - 1$ 17. $2x - 1$ 18. $x^{-1/2}/2$ 19. $4x^3 - 4x$ 20. $-49t/5 + 5$, $-49/5$ 21. See Figure 22. $x^3/16 - 3x/4 + 4$ 23. $y = 13x/4 + 5$ 24. $y = 24x - 48 - \pi^3$ 25. $\frac{d}{dx}cf(x) = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = cf'(x)$.



Answer 2.2.21.

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