MOOCulus

massive open online calculus

CALCULUS

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This text is based on David Guichard's open-source calculus text which in turn is a modification and expansion of notes written by Neal Koblitz at the University of Washington. New material has been added, and old material has been modified, so some portions now bear little resemblance to the original.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at `http://www.math.wisc.edu/~keisler/calc.html` under a Creative Commons license. In addition, the chapter on differential equations and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

We will be glad to receive corrections and suggestions for improvement at `fowler@math.osu.edu` or `snapp@math.osu.edu`.

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Introduction

BADBAD

What is calculus?

1

Limits

1.1 Functions

This should be one of the last sections filled-in.

1.2 The Basic Ideas of Limits

Consider the function:

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

While $f(x)$ is undefined at $x = 2$, we can still plot $f(x)$ at other values, see Figure 1.1. Examining Table 1.1, we see that as x approaches 2, $f(x)$ approaches 1. We write this:

$$\text{As } x \to 2,\ f(x) \to 1 \qquad \text{or} \qquad \lim_{x \to 2} f(x) = 1.$$

Intuitively, $\lim_{x \to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. This leads us to the formal definition of a *limit*.

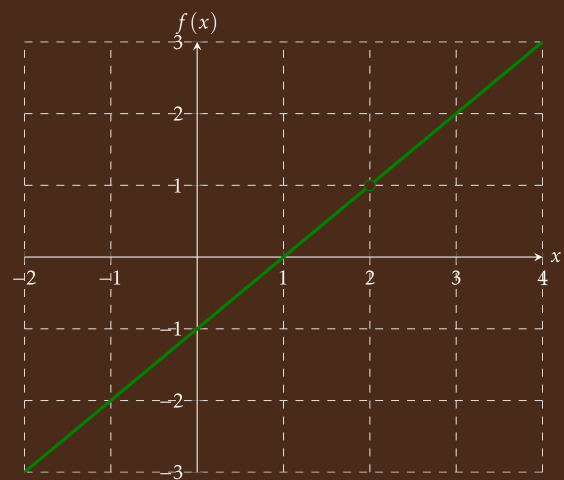

Figure 1.1: A plot of $f(x) = \frac{x^2-3x+2}{x-2}$.

x	$f(x)$
1.7	0.7
1.9	0.9
1.99	0.99
1.999	0.999
2	undefined

x	$f(x)$
2	undefined
2.001	1.001
2.01	1.01
2.1	1.1
2.3	1.3

Table 1.1: Values of $f(x) = \frac{x^2-3x+2}{x-2}$.

Definition The **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x \to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta, \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

If no such value of L can be found, then we say that $f(x)$ **does not exist** at $x = a$.

Equivalently, $\lim_{x \to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and $a - \delta < x < a + \delta$, we have $L - \varepsilon < f(x) < L + \varepsilon$.

In Figure 1.2, we see a geometric interpretation of this definition.

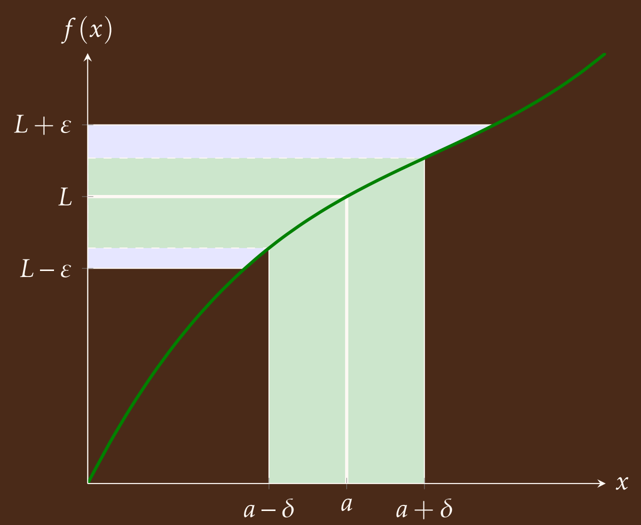

Figure 1.2: A geometric interpretation of the (ε, δ)-criterion for limits. If $0 < |x - a| < \delta$, then we have that $a - \delta < x < a + \delta$. In our diagram, we see that for all such x we are sure to have $L - \varepsilon < f(x) < L + \varepsilon$, and hence $|f(x) - L| < \varepsilon$.

Limits need not exist, let's examine two cases of this.

Example Let $f(x) = \lfloor x \rfloor$. Explain why the limit

$$\lim_{x \to 2} f(x)$$

does not exist.

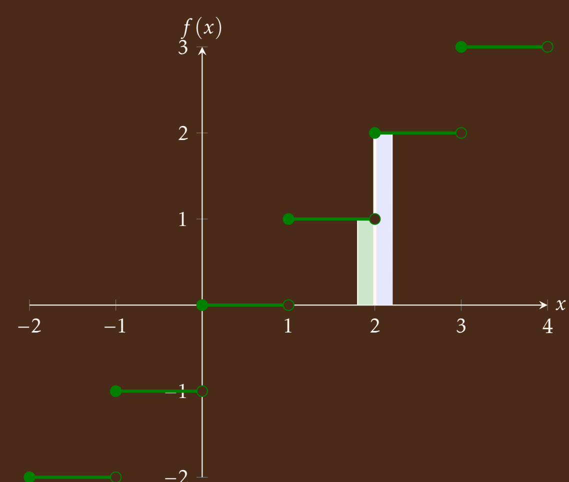

Figure 1.3: A plot of $f(x) = \lfloor x \rfloor$.

Solution This is the function that returns the greatest integer less than or equal to x. Since $f(x)$ is defined for all real numbers, one might be tempted to think that the limit above is simply $f(2) = 2$. However, this is not the case. If $x < 2$, then $f(x) = 1$. Hence if $\varepsilon = .5$, we can **always** find a value for x (just to the left of 2) such that

$$0 < |x-2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x)-2|.$$

On the other hand, $\lim_{x\to 2} f(x) \neq 1$, as in this case if $\varepsilon = .5$, we can **always** find a value for x (just to the right of 2) such that

$$0 < |x-2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x)-1|.$$

We've illustrated this in Figure 1.3. Moreover, no matter what value one chooses for $\lim\limits_{x\to 2} f(x)$, we will always have a similar issue.

With the example of $f(x) = \lfloor x \rfloor$, we see that taking limits is truly different from evaluating functions.

Limits may not exist even if the function looks innocent.

Example Let $f(x) = \sin\left(\frac{1}{x}\right)$. Explain why the limit

$$\lim_{x\to 0} f(x)$$

does not exist.

Solution In this case $f(x)$ oscillates "wildly" as x approaches 0, see Figure 1.4. In fact, one can show that for any given δ, There is a value for x in the interval

$$0-\delta < x < 0+\delta$$

such that $f(x)$ is **any** value in the interval $[-1, 1]$. Hence the limit does not exist.

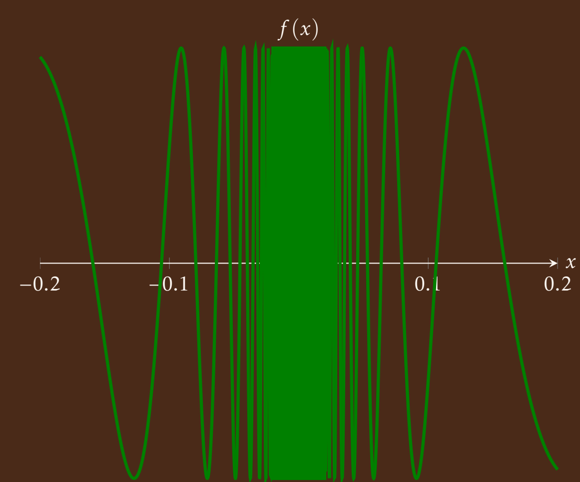

Figure 1.4: A plot of $f(x) = \sin\left(\frac{1}{x}\right)$.

Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of a *one-sided limit*:

Definition We say that the **limit** of $f(x)$ as x goes to a from the **left** is L,

$$\lim_{x\to a-} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and

$$a - \delta < x \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

We say that the **limit** of $f(x)$ as x goes to a from the **right** is L,

$$\lim_{x\to a+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and

$$x < a + \delta \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

Example Let $f(x) = \lfloor x \rfloor$. We now have one sided limits

$$\lim_{x\to 2-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 2+} f(x) = 2.$$

Exercises for Section 1.2

(1) Evaluate the expressions by reference to the plot in Figure 1.5.

(a) $\lim_{x\to 4} f(x)$
(b) $\lim_{x\to -3} f(x)$
(c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to 0-} f(x)$
(e) $\lim_{x\to 0+} f(x)$
(f) $f(-2)$
(g) $\lim_{x\to 2-} f(x)$
(h) $\lim_{x\to -2-} f(x)$
(i) $\lim_{x\to 0} f(x+1)$
(j) $f(0)$
(k) $\lim_{x\to 1-} f(x-4)$
(l) $\lim_{x\to 0+} f(x-2)$

(2) Use a calculator to estimate $\lim_{x\to 0} \frac{\sin x}{x}$.

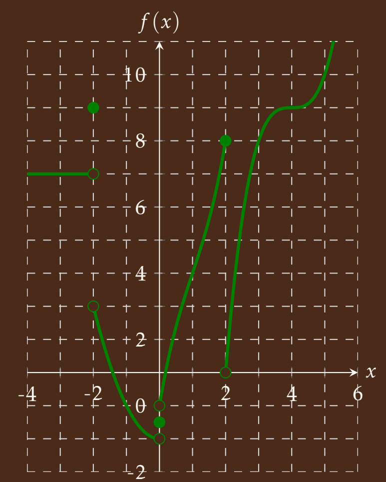

Figure 1.5: A piecewise defined function.

(3) Use a calculator to estimate $\lim_{x\to 0} \dfrac{\tan(3x)}{\tan(5x)}$.

(4) Use a calculator to estimate $\lim_{x\to 0} \dfrac{2^x - 1}{x}$.

(5) Sketch a plot of $f(x) = \dfrac{x}{|x|}$ and explain why $\lim_{x\to 0} \dfrac{x}{|x|}$ does not exist.

(6) Let $f(x) = \sin\left(\dfrac{\pi}{x}\right)$. Construct three tables of the following form

x	$f(x)$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	

where $d = 1, 3, 7$. What do you notice? How do you reconcile the entries in your tables with the value of $\lim_{x\to 0} f(x)$?

1.3 Limits by the Definition

Now we are going to get our hands dirty, and really use the definition of a limit.

Recall, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| + \delta$, we have $|f(x) - L| < \varepsilon$.

Example Show that $\lim_{x\to 2} x^2 = 4$.

Solution We want to show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that

$$|x^2 - 4| < \varepsilon$$

whenever $0 < |x-2| < \delta$. Start by factoring the LHS of the inequality above

$$|x+2||x-2| < \varepsilon.$$

Since we are going to assume that $0 < |x-2| < \delta$, we will focus on the factor $|x+2|$. Since x is assumed to be close to 2, suppose that $x \in [1,3]$. In this case

$$|x+2| \leq 3 + 2 = 5,$$

and so we want

$$5 \cdot |x-2| < \varepsilon$$
$$|x-2| < \frac{\varepsilon}{5}$$

Recall, we assumed that $x \in [1,3]$, which is equivalent to $|x-2| < 1$. Hence we must set $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

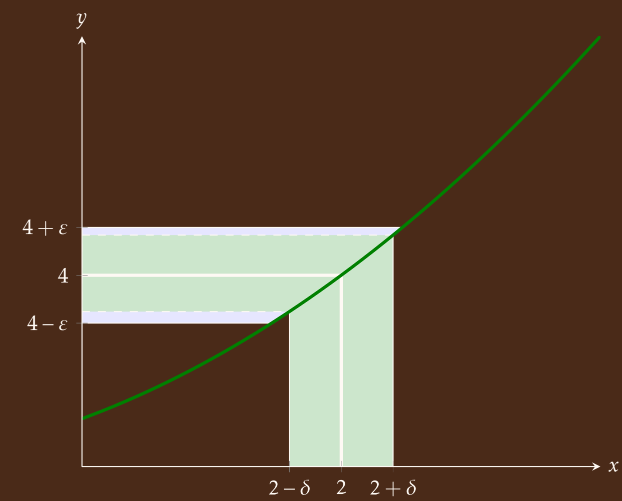

Figure 1.6: The (ε,δ)-criterion for $\lim_{x\to 2} x^2 = 4$. Here $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

PROSE!!!

Theorem Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof. Given any ε we need to find a δ such that

$$0 < |x < a| < \delta$$

implies

$$|f(x)g(x) - LM| < \varepsilon.$$

Here we use an algebraic trick, add $0 = -f(x)M + f(x)M$:

$$\begin{aligned}|f(x)g(x) - LM| &= |f(x)g(x) \boldsymbol{- f(x)M + f(x)M} - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)(g(x) - M)| + |(f(x) - L)M| \\ &= |f(x)||g(x) - M| + |f(x) - L||M|.\end{aligned}$$

This is all straightforward except perhaps for the "$\leq$". This follows from the *triangle inequality*. The **Triangle Inequality** states: If a and b are any real numbers then $|a + b| \leq |a| + |b|$.

Since $\lim_{x\to a} f(x) = L$, there is a value δ_1 so that $0 < |x - a| < \delta_1$ implies $|f(x) - L| < |\varepsilon/(2M)|$, This means that $0 < |x - a| < \delta_1$ implies $|f(x) - L||M| < \varepsilon/2$. You can see where this is going: if we can make $|f(x)||g(x) - M| < \varepsilon/2$ also, then we'll be done.

We can make $|g(x) - M|$ smaller than any fixed number by making x close enough to a; unfortunately, $\varepsilon/(2f(x))$ is not a fixed number, since x is a variable. Here we need another little trick, just like the one we used in analyzing x^2. We can find a δ_2 so that $|x - a| < \delta_2$ implies that $|f(x) - L| < 1$, meaning that $L - 1 < f(x) < L + 1$. This means that $|f(x)| < N$, where N is either $|L - 1|$ or $|L + 1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x - a| < \delta_3$ implies $|g(x) - M| < \varepsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x - a| < \delta$ implies that $|f(x) - L| < |\varepsilon/(2M)|$, $|f(x)| < N$, and $|g(x) - M| < \varepsilon/(2N)$. Then

$$\begin{aligned}|f(x)g(x) - LM| &\leq |f(x)||g(x) - M| + |f(x) - L||M| \\ &< N\frac{\varepsilon}{2N} + \left|\frac{\varepsilon}{2M}\right||M| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This is just what we needed, so by the official definition, $\lim_{x\to a} f(x)g(x) = LM$. $\square$

While theorem 3.3 is very helpful, we need a bit more to work easily with limits. Since the theorem applies when some limits are already known, we need to know the behavior of some functions that cannot themselves be constructed from the simple arithmetic operations of the theorem, such as $\sqrt{x}$. Also, there is one other extraordinarily useful way to put functions together: composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2 + 5$, then $f(g(x)) = \sqrt{x^2 + 5}$ and $g(f(x)) = (\sqrt{x})^2 + 5 = x + 5$. Here is a companion to theorem 3.3 for composition:

Theorem Suppose that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$. Then

$$\lim_{x\to a} f(g(x)) = f(L).$$

Note the special form of the condition on f: it is not enough to know that $\lim_{x\to L} f(x) = M$, though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

Theorem Suppose that n is a positive integer. Then

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Exercises for Section 1.3

(1) Use the definition of limits to explain why $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. Hint: Use the fact that $|\sin a| < 1$ for any real number a.

(2) Use the definition of limits to explain why $\lim_{x\to 4}(2x-5) = 3$.

(3) For each of the following limits, $\lim_{x\to a} f(x) = L$, use a graphing device to find δ such that $0 < |x-a|\delta$ implies that $|f(x)-L| < \varepsilon$ where $\varepsilon = .1$.

(a) $\lim_{x \to 2} (3x + 1) = 7$

(b) $\lim_{x \to 1} (x^2 + 2) = 3$

(c) $\lim_{x \to \pi} \sin(x) = 0$

(d) $\lim_{x \to 0} \tan(x) = 0$

(e) $\lim_{x \to 1} \sqrt{3x + 1} = 2$

(f) $\lim_{x \to -2} \sqrt[3]{1 - 4x} = 3$

1.4 Limit Laws

limit A handful of such theorems give us the tools to compute many limits without explicitly working with the definition of limit.

Limit Laws Suppose that $\lim\limits_{x \to a} f(x) = L$, $\lim\limits_{x \to a} g(x) = M$, k is some constant, and n is a positive integer. Then

- $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x) = kL$
- $\lim\limits_{x \to a} (f(x) + g(x)) = \lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x) = L + M$
- $\lim\limits_{x \to a} (f(x)g(x)) = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x) = LM$
- $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)} = \dfrac{L}{M}$, if $M \neq 0$
- $\lim\limits_{x \to a} f(x)^n = \left(\lim\limits_{x \to a} f(x) \right)^n = L^n$
- $\lim\limits_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to a} f(x)} = \sqrt[n]{L}$ provided if n is even, then $f(x) \geq 0$ near a.

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim\limits_{x \to a} x = a$.

Example Compute $\lim_{x\to 1} \frac{x^2-3x+5}{x-2}$. If we apply the theorem in all its gory detail, we get

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1}(x^2-3x+5)}{\lim_{x\to 1}(x-2)} \\
&= \frac{(\lim_{x\to 1} x^2) - (\lim_{x\to 1} 3x) + (\lim_{x\to 1} 5)}{(\lim_{x\to 1} x) - (\lim_{x\to 1} 2)} \\
&= \frac{(\lim_{x\to 1} x)^2 - 3(\lim_{x\to 1} x) + 5}{(\lim_{x\to 1} x) - 2} \\
&= \frac{1^2 - 3\cdot 1 + 5}{1-2} \\
&= \frac{1-3+5}{-1} = -3
\end{aligned}$$

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

Of course, as we've already seen, we're primarily interested in limits that aren't so easy, namely, limits in which a denominator approaches zero. There are a handful of algebraic tricks that work on many of these limits.

Example Compute $\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$. We can't simply plug in $x = 1$ because that makes the denominator zero. However:

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+2x-3}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} \\
&= \lim_{x\to 1} (x+3) = 4
\end{aligned}$$

Another of the most common algebraic tricks was used in section 3.1. Here's another example:

Example Compute $\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}$.

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}\,\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\
&= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}
\end{aligned}$$

At the very last step we have used theorems 3.3 and 3.3.

Exercises for Section 1.4

Compute the limits. If a limit does not exist, explain why.

(1) $\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

(2) $\lim_{x\to 1} \frac{x^2+x-12}{x-3}$

(3) $\lim_{x\to -4} \frac{x^2+x-12}{x-3}$

(4) $\lim_{x\to 2} \frac{x^2+x-12}{x-2}$

(5) $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$

(6) $\lim_{x\to 0+} \sqrt{\frac{1}{x}+2} - \sqrt{\frac{1}{x}}$.

(7) $\lim_{x\to 2} 3$

(8) $\lim_{x\to 4} 3x^3 - 5x$

(9) $\lim_{x\to 0} \frac{4x-5x^2}{x-1}$

(10) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(11) $\lim_{x\to 0+} \frac{\sqrt{2-x^2}}{x}$

(12) $\lim_{x\to 0+} \frac{\sqrt{2-x^2}}{x+1}$

(13) $\lim_{x\to a} \frac{x^3-a^3}{x-a}$

(14) $\lim_{x\to 2}(x^2+4)^3$

(15) $\lim_{x\to 1}\begin{cases} x-5 & x\neq 1, \\ 7 & x=1. \end{cases}$

1.5 Limits to and from Infinity

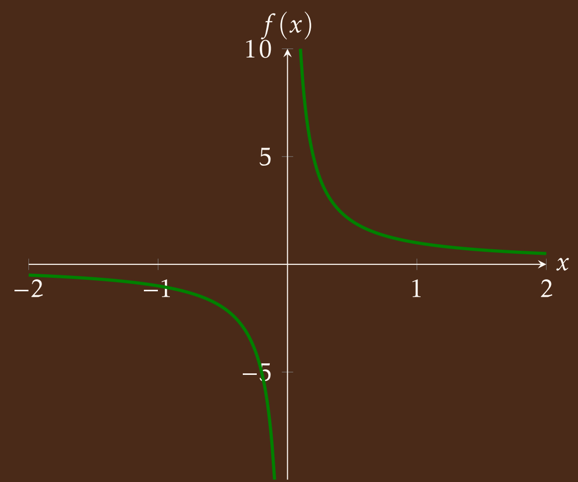

Figure 1.7: A plot of $f(x) = \frac{1}{x}$.

2

Analytic Geometry

Much of the mathematics in this chapter will be review for you. However, the examples will be oriented toward applications and so will take some thought.

In the (x,y) coordinate system we normally write the x-axis horizontally, with positive numbers to the right of the origin, and the y-axis vertically, with positive numbers above the origin. That is, unless stated otherwise, we take "rightward" to be the positive x-direction and "upward" to be the positive y-direction. In a purely mathematical situation, we normally choose the same scale for the x- and y-axes. For example, the line joining the origin to the point (a,a) makes an angle of 45° with the x-axis (and also with the y-axis).

In applications, often letters other than x and y are used, and often different scales are chosen in the horizontal and vertical directions. For example, suppose you drop something from a window, and you want to study how its height above the ground changes from second to second. It is natural to let the letter t denote the time (the number of seconds since the object was released) and to let the letter h denote the height. For each t (say, at one-second intervals) you have a corresponding height h. This information can be tabulated, and then plotted on the (t,h) coordinate plane, as shown in Figure 2.1.

We use the word "quadrant" for each of the four regions into which the plane is divided by the axes: the first quadrant is where points have both coordinates positive, or the "northeast" portion of the plot, and the second, third, and fourth quadrants are counted off counterclockwise, so the second quadrant is the north-

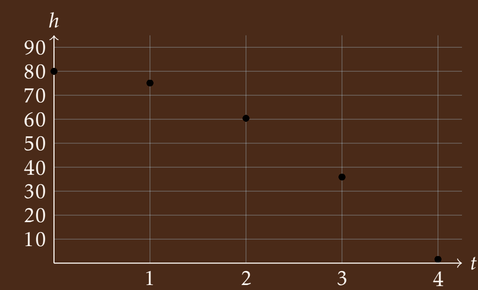

Figure 2.1: Height versus time.

t (sec)	s (meters)
0	80.0
1	75.1
2	60.4
3	35.9
4	1.6

Table 2.1: A data table.

west, the third is the southwest, and the fourth is the southeast.

Suppose we have two points A and B in the (x,y)-plane. We often want to know the change in x-coordinate (also called the "horizontal distance") in going from A to B. This is often written Δx, where the meaning of Δ (a capital delta in the Greek alphabet) is "change in". (Thus, Δx can be read as "change in x" although it usually is read as "delta x". The point is that Δx denotes a single number, and should not be interpreted as "delta times x".) For example, if $A = (2,1)$ and $B = (3,3)$, $\Delta x = 3-2 = 1$. Similarly, the "change in y" is written Δy. In our example, $\Delta y = 3-1 = 2$, the difference between the y-coordinates of the two points. It is the vertical distance you have to move in going from A to B. The general formulas for the change in x and the change in y between a point (x_1, y_1) and a point (x_2, y_2) are:

$$\Delta x = x_2 - x_1, \qquad \Delta y = y_2 - y_1.$$

Note that either or both of these might be negative.

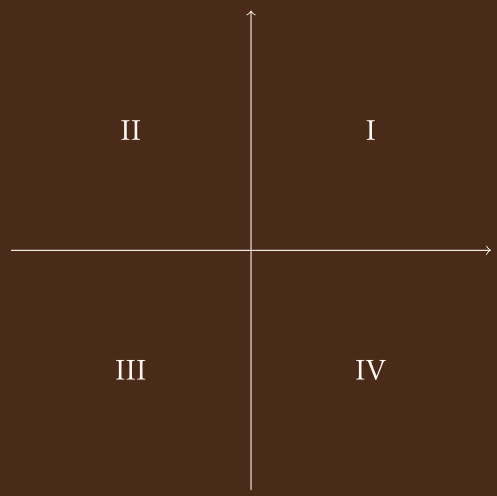

Figure 2.2: The four quadrants.

2.1 Lines

If we have two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then we can draw one and only one line through both points. By the *slope* of this line we mean the ratio of Δy to Δx. The slope is often denoted m: $m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$. For example, the line joining the points $(1,-2)$ and $(3,5)$ has slope $(5+2)/(3-1) = 7/2$.

Example According to the 1990 U.S. federal income tax schedules, a head of household paid 15% on taxable income up to \$26050. If taxable income was between \$26050 and \$134930, then, in addition, 28% was to be paid on the amount between \$26050 and \$67200, and 33% paid on the amount over \$67200 (if any). Interpret the tax bracket information (15%, 28%, or 33%) using mathematical terminology, and graph the tax on the y-axis against the taxable income on the x-axis.

The percentages, when converted to decimal values 0.15, 0.28, and 0.33, are the *slopes* of the straight lines which form the graph of the tax for the corresponding tax brackets. The tax graph is what's called a *polygonal line*, i.e., it's made up of several straight line segments of different slopes. The first line starts at the point (0,0) and heads upward with slope 0.15 (i.e., it goes upward

15 for every increase of 100 in the x-direction), until it reaches the point above $x = 26050$. Then the graph "bends upward," i.e., the slope changes to 0.28. As the horizontal coordinate goes from $x = 26050$ to $x = 67200$, the line goes upward 28 for each 100 in the x-direction. At $x = 67200$ the line turns upward again and continues with slope 0.33. See Figure **??**.

The most familiar form of the equation of a straight line is: $y = mx + b$. Here m is the slope of the line: if you increase x by 1, the equation tells you that you have to increase y by m. If you increase x by Δx, then y increases by $\Delta y = m\Delta x$. The number b is called the *y-intercept*, because it is where the line crosses the y-axis. If you know two points on a line, the formula $m = (y_2 - y_1)/(x_2 - x_1)$ gives you the slope. Once you know a point and the slope, then the y-intercept can be found by substituting the coordinates of either point in the equation: $y_1 = mx_1 + b$, i.e., $b = y_1 - mx_1$. Alternatively, one can use the "point-slope" form of the equation of a straight line: start with $(y - y_1)/(x - x_1) = m$ and then multiply to get $(y - y_1) = m(x - x_1)$, the point-slope form. Of course, this may be further manipulated to get $y = mx - mx_1 + y_1$, which is essentially the "$mx + b$" form.

It is possible to find the equation of a line between two points directly from the relation $(y - y_1)/(x - x_1) = (y_2 - y_1)/(x_2 - x_1)$, which says "the slope measured between the point (x_1, y_1) and the point (x_2, y_2) is the same as the slope measured between the point (x_1, y_1) and any other point (x, y) on the line." For example, if we want to find the equation of the line joining our earlier points $A = (2,1)$ and $B = (3,3)$, we can use this formula:

$$\frac{y-1}{x-2} = \frac{3-1}{3-2} = 2, \qquad \text{so that} \qquad y - 1 = 2(x-2), \qquad \text{i.e.,} \qquad y = 2x - 3.$$

Of course, this is really just the point-slope formula, except that we are not computing m in a separate step.

The slope m of a line in the form $y = mx + b$ tells us the direction in which the line is pointing. If m is positive, the line goes into the 1st quadrant as you go from left to right. If m is large and positive, it has a steep incline, while if m is small and positive, then the line has a small angle of inclination. If m is negative, the line goes into the 4th quadrant as you go from left to right. If m is a large negative number (large in absolute value), then the line points steeply downward; while if m is negative but near zero, then it points only a little downward. These four possibilities are illustrated in figure **??**.

If $m = 0$, then the line is horizontal: its equation is simply $y = b$.

There is one type of line that cannot be written in the form $y = mx + b$, namely, vertical lines. A vertical line has an equation of the form $x = a$. Sometimes one says that a vertical line has an "infinite" slope.

Sometimes it is useful to find the x-intercept of a line $y = mx + b$. This is the x-value when $y = 0$. Setting $mx + b$ equal to 0 and solving for x gives: $x = -b/m$. For example, the line $y = 2x - 3$ through the points $A = (2, 1)$ and $B = (3, 3)$ has x-intercept $3/2$.

Example Suppose that you are driving to Seattle at constant speed, and notice that after you have been traveling for 1 hour (i.e., $t = 1$), you pass a sign saying it is 110 miles to Seattle, and after driving another half-hour you pass a sign saying it is 85 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from Seattle, graph and find the equation $y = mt + b$ for your distance from Seattle. Find the slope, y-intercept, and t-intercept, and describe the practical meaning of each.

The graph of y versus t is a straight line because you are traveling at constant speed. The line passes through the two points $(1, 110)$ and $(1.5, 85)$, so its slope is $m = (85 - 110)/(1.5 - 1) = -50$. The meaning of the slope is that you are traveling at 50 mph; m is negative because you are traveling *toward* Seattle, i.e., your distance y is *decreasing*. The word "velocity" is often used for $m = -50$, when we want to indicate direction, while the word "speed" refers to the magnitude (absolute value) of velocity, which is 50 mph. To find the equation of the line, we use the point-slope formula:

$$\frac{y - 110}{t - 1} = -50, \qquad \text{so that}$$

$$y = -50(t - 1) + 110 = -50t + 160.$$

The meaning of the y-intercept 160 is that when $t = 0$ (when you started the trip) you were 160 miles from Seattle. To find the t-intercept, set $0 = -50t + 160$, so that $t = 160/50 = 3.2$. The meaning of the t-intercept is the duration of your trip, from the start until you arrive in Seattle. After traveling 3 hours and 12 minutes, your distance y from Seattle will be 0.

Exercises for Section 2.1

(1) Find the equation of the line through $(1,1)$ and $(-5,-3)$ in the form $y = mx+b$.

(2) Find the equation of the line through $(-1,2)$ with slope -2 in the form $y = mx+b$.

(3) Find the equation of the line through $(-1,1)$ and $(5,-3)$ in the form $y = mx+b$.

(4) Change the equation $y-2x = 2$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept.

(5) Change the equation $x+y = 6$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept.

(6) Change the equation $x = 2y-1$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept.

(7) Change the equation $3 = 2y$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept.

(8) Change the equation $2x+3y+6 = 0$ to the form $y = mx+b$, graph the line, and find the y-intercept and x-intercept.

(9) Determine whether the lines $3x+6y = 7$ and $2x+4y = 5$ are parallel.

(10) Suppose a triangle in the x,y–plane has vertices $(-1,0)$, $(1,0)$ and $(0,2)$. Find the equations of the three lines that lie along the sides of the triangle in $y = mx+b$ form.

(11) Suppose that you are driving to Seattle at constant speed. After you have been traveling for an hour you pass a sign saying it is 130 miles to Seattle, and after driving another 20 minutes you pass a sign saying it is 105 miles to Seattle. Using the horizontal axis for the time t and the vertical axis for the distance y from your starting point, graph and find the equation $y = mt+b$ for your distance from your starting point. How long does the trip to Seattle take?

(12) Let x stand for temperature in degrees Celsius (centigrade), and let y stand for temperature in degrees Fahrenheit. A temperature of 0°C corresponds to 32°F, and a temperature of 100°C corresponds to 212°F. Find the equation of the line that relates temperature Fahrenheit y to temperature Celsius x in the form $y = mx + b$. Graph the line, and find the point at which this line intersects $y = x$. What is the practical meaning of this point?

(13) A car rental firm has the following charges for a certain type of car: \$25 per day with 100 free miles included, \$0.15 per mile for more than 100 miles. Suppose you want to rent a car for one day, and you know you'll use it for more than 100 miles. What is the equation relating the cost y to the number of miles x that you drive the car?

(14) A photocopy store advertises the following prices: 5cents per copy for the first 20 copies, 4cents per copy for the 21st through 100th copy, and 3cents per copy after the 100th copy. Let x be the number of copies, and let y be the total cost of photocopying. (a) Graph the cost as x goes from 0 to 200 copies. (b) Find the equation in the form $y = mx + b$ that tells you the cost of making x copies when x is more than 100.

(15) In the Kingdom of Xyg the tax system works as follows. Someone who earns less than 100 gold coins per month pays no tax. Someone who earns between 100 and 1000 gold coins pays tax equal to 10% of the amount over 100 gold coins that he or she earns. Someone who earns over 1000 gold coins must hand over to the King all of the money earned over 1000 in addition to the tax on the first 1000. (a) Draw a graph of the tax paid y versus the money earned x, and give formulas for y in terms of x in each of the regions $0 \leq x \leq 100$, $100 \leq x \leq 1000$, and $x \geq 1000$. (b) Suppose that the King of Xyg decides to use the second of these line segments (for $100 \leq x \leq 1000$) for $x \leq 100$ as well. Explain in practical terms what the King is doing, and what the meaning is of the y-intercept.

(16) The tax for a single taxpayer is described in the figure **??**. Use this information to graph tax versus taxable income (i.e., x is the amount on Form 1040, line 37, and y is the amount on Form 1040, line 38). Find the slope and y-intercept of each line that makes up the polygonal graph, up to $x = 97620$.

(17) Market research tells you that if you set the price of an item at \$1.50, you will be able to sell 5000 items; and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Let x be the number of items you can sell, and let P be the price of an item. (a) Express P linearly in terms of x, in other words, express P in the form $P = mx + b$. (b) Express x linearly in terms of P.

(18) An instructor gives a 100-point final exam, and decides that a score 90 or above will be a grade of 4.0, a score of 40 or below will be a grade of 0.0, and between 40 and 90 the grading will be linear. Let x be the exam score, and let y be the corresponding grade. Find a formula of the form $y = mx + b$ which applies to scores x between 40 and 90.

2.2 Distance Between Two Points; Circles

Given two points (x_1, y_1) and (x_2, y_2), recall that their horizontal distance from one another is $\Delta x = x_2 - x_1$ and their vertical distance from one another is $\Delta y = y_2 - y_1$. (Actually, the word "distance" normally denotes "positive distance". Δx and Δy are *signed* distances, but this is clear from context.) The actual (positive) distance from one point to the other is the length of the hypotenuse of a right triangle with legs $|\Delta x|$ and $|\Delta y|$, as shown in figure **??**. The Pythagorean theorem then says that the distance between the two points is the square root of the sum of the squares of the horizontal and vertical sides:

$$\text{distance} = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

For example, the distance between points $A = (2, 1)$ and $B = (3, 3)$ is $\sqrt{(3-2)^2 + (3-1)^2} = \sqrt{5}$.

As a special case of the distance formula, suppose we want to know the distance of a point (x, y) to the origin. According to the distance formula, this is $\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$.

A point (x, y) is at a distance r from the origin if and only if $\sqrt{x^2 + y^2} = r$, or, if we square both sides: $x^2 + y^2 = r^2$. This is the equation of the circle of radius r centered at the origin. The special case $r = 1$ is called the unit circle; its equation is $x^2 + y^2 = 1$.

Similarly, if $C(h, k)$ is any fixed point, then a point (x, y) is at a distance r from the point C if and only if $\sqrt{(x-h)^2 + (y-k)^2} = r$, i.e., if and only if

$$(x-h)^2 + (y-k)^2 = r^2.$$

This is the equation of the circle of radius r centered at the point (h, k). For example, the circle of radius 5 centered at the point $(0, -6)$ has equation $(x-0)^2 + (y--6)^2 = 25$, or $x^2 + (y+6)^2 = 25$. If we expand this we get $x^2 + y^2 + 12y + 36 = 25$ or $x^2 + y^2 + 12y + 11 = 0$, but the original form is usually more useful.

Example Graph the circle $x^2 - 2x + y^2 + 4y - 11 = 0$. With a little thought we convert this to $(x-1)^2 + (y+2)^2 - 16 = 0$ or $(x-1)^2 + (y+2)^2 = 16$. Now we see that this is the circle with radius 4 and center $(1, -2)$, which is easy to graph.

Exercises for Section 2.2

(1) Find the equation of the circle of radius 3 centered at:

(a) $(0,0)$
(b) $(5,6)$
(c) $(-5,-6)$
(d) $(0,3)$
(e) $(0,-3)$
(f) $(3,0)$

(2) For each pair of points $A=(x_1,y_1)$ and $B=(x_2,y_2)$ find (i) Δx and Δy in going from A to B, (ii) the slope of the line joining A and B, (iii) the equation of the line joining A and B in the form $y=mx+b$, (iv) the distance from A to B, and (v) an equation of the circle with center at A that goes through B.

(a) $A=(2,0)$, $B=(4,3)$
(b) $A=(1,-1)$, $B=(0,2)$
(c) $A=(0,0)$, $B=(-2,-2)$
(d) $A=(-2,3)$, $B=(4,3)$
(e) $A=(-3,-2)$, $B=(0,0)$
(f) $A=(0.01,-0.01)$, $B=(-0.01,0.05)$

(3) Graph the circle $x^2+y^2+10y=0$.

(4) Graph the circle $x^2-10x+y^2=24$.

(5) Graph the circle $x^2-6x+y^2-8y=0$.

(6) Find the standard equation of the circle passing through $(-2,1)$ and tangent to the line $3x-2y=6$ at the point $(4,3)$. Sketch. (Hint: The line through the center of the circle and the point of tangency is perpendicular to the tangent line.)

2.3 Functions

A **function** $y = f(x)$ is a rule for determining y when we're given a value of x. For example, the rule $y = f(x) = 2x + 1$ is a function. Any line $y = mx + b$ is called a **linear** function. The graph of a function looks like a curve above (or below) the x-axis, where for any value of x the rule $y = f(x)$ tells us how far to go above (or below) the x-axis to reach the curve.

Functions can be defined in various ways: by an algebraic formula or several algebraic formulas, by a graph, or by an experimentally determined table of values. (In the latter case, the table gives a bunch of points in the plane, which we might then interpolate with a smooth curve, if that makes sense.)

Given a value of x, a function must give at most one value of y. Thus, vertical lines are not functions. For example, the line $x = 1$ has infinitely many values of y if $x = 1$. It is also true that if x is any number not 1 there is no y which corresponds to x, but that is not a problem—only multiple y values is a problem.

In addition to lines, another familiar example of a function is the parabola $y = f(x) = x^2$. We can draw the graph of this function by taking various values of x (say, at regular intervals) and plotting the points $(x, f(x)) = (x, x^2)$. Then connect the points with a smooth curve. (See figure **??**.)

The two examples $y = f(x) = 2x+1$ and $y = f(x) = x^2$ are both functions which can be evaluated at *any* value of x from negative infinity to positive infinity. For many functions, however, it only makes sense to take x in some interval or outside of some "forbidden" region. The interval of x-values at which we're allowed to evaluate the function is called the **domain** of the function.

For example, the square-root function $y = f(x) = \sqrt{x}$ is the rule which says, given an x-value, take the nonnegative number whose square is x. This rule only makes sense if x is positive or zero. We say that the domain of this function is $x \geq 0$, or more formally $\{x \in \mathbb{R} \mid x \geq 0\}$. Alternately, we can use interval notation, and write that the domain is $[0, \infty)$. (In interval notation, square brackets mean that the endpoint is included, and a parenthesis means that the endpoint is not included.) The fact that the domain of $y = \sqrt{x}$ is $[0, \infty)$ means that in the graph of this function ((see figure **??**) we have points (x, y) only above x-values on the right side of the x-axis.

Another example of a function whose domain is not the entire x-axis is: $y =$

$f(x) = 1/x$, the reciprocal function. We cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero x, so we take the domain to be: $\{x \in \mathbb{R} \mid x \neq 0\}$. The graph of this function does not have any point (x, y) with $x = 0$. As x gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an **asymptote**.

To summarize, two reasons why certain x-values are excluded from the domain of a function are that (i) we cannot divide by zero, and (ii) we cannot take the square root of a negative number. We will encounter some other ways in which functions might be undefined later.

Another reason why the domain of a function might be restricted is that in a given situation the x-values outside of some range might have no practical meaning. For example, if y is the area of a square of side x, then we can write $y = f(x) = x^2$. In a purely mathematical context the domain of the function $y = x^2$ is all of $\mathbb{R}$. But in the story-problem context of finding areas of squares, we restrict the domain to positive values of x, because a square with negative or zero side makes no sense.

In a problem in pure mathematics, we usually take the domain to be all values of x at which the formulas can be evaluated. But in a story problem there might be further restrictions on the domain because only certain values of x are of interest or make practical sense.

In a story problem, often letters different from x and y are used. For example, the volume V of a sphere is a function of the radius r, given by the formula $V = f(r) = \frac{4}{7}3\pi r^3$. Also, letters different from f may be used. For example, if y is the velocity of something at time t, we may write $y = v(t)$ with the letter v (instead of f) standing for the velocity function (and t playing the role of x).

The letter playing the role of x is called the **independent variable**, and the letter playing the role of y is called the **dependent variable** (because its value "depends on" the value of the independent variable). In story problems, when one has to translate from English into mathematics, a crucial step is to determine what letters stand for variables. If only words and no letters are given, then we have to decide which letters to use. Some letters are traditional. For example, almost always, t stands for time.

Example An open-top box is made from an $a \times b$ rectangular piece of cardboard by cutting out a square of side x from each of the four corners, and then folding the sides up and sealing them with duct tape. Find a formula for the volume V of the box as a function of x, and find the domain of this function.

The box we get will have height x and rectangular base of dimensions $a-2x$ by $b-2x$. Thus,

$$V = f(x) = x(a-2x)(b-2x).$$

Here a and b are constants, and V is the variable that depends on x, i.e., V is playing the role of y.

This formula makes mathematical sense for any x, but in the story problem the domain is much less. In the first place, x must be positive. In the second place, it must be less than half the length of either of the sides of the cardboard. Thus, the domain is

$$\{x \in \mathbb{R} \mid 0 < x < \frac{1}{2}(\text{minimum of } a \text{ and } b)\}.$$

In interval notation we write: the domain is the interval $(0, \min(a,b)/2)$. (You might think about whether we could allow 0 or (minimum of a and b) to be in the domain. They make a certain physical sense, though we normally would not call the result a box. If we were to allow these values, what would the corresponding volumes be? Does that make sense?)

Example (Circle of radius r centered at the origin) The equation for this circle is usually given in the form $x^2 + y^2 = r^2$. To write the equation in the form $y = f(x)$ we solve for y, obtaining $y = \pm\sqrt{r^2 - x^2}$. But *this is not a function*, because when we substitute a value in $(-r, r)$ for x there are two corresponding values of y. To get a function, we must choose one of the two signs in front of the square root. If we choose the positive sign, for example, we get the upper semicircle $y = f(x) = \sqrt{r^2 - x^2}$ (see figure **??**). The domain of this function is the interval $[-r, r]$, i.e., x must be between $-r$ and r (including the endpoints). If x is outside of that interval, then $r^2 - x^2$ is negative, and we cannot take the square root. In terms of the graph, this just means that there are no points on the curve whose x-coordinate is greater than r or less than $-r$.

Example Find the domain of

$$y = f(x) = \frac{1}{\sqrt{4x - x^2}}.$$

To answer this question, we must rule out the x-values that make $4x - x^2$ negative (because we cannot take the square root of a negative number) and also the x-values that make $4x - x^2$ zero (because if $4x - x^2 = 0$, then when we take the square root we get 0, and we cannot divide by 0). In other words, the domain consists of all x for which $4x - x^2$ is strictly positive. We give two different methods to find out when $4x - x^2 > 0$.

First method. Factor $4x - x^2$ as $x(4 - x)$. The product of two numbers is positive when either both are positive or both are negative, i.e., if either $x > 0$ and $4 - x > 0$, or else $x < 0$ and $4 - x < 0$. The latter alternative is impossible, since if x is negative, then $4 - x$ is greater than 4, and so cannot be negative. As for the first alternative, the condition $4 - x > 0$ can be rewritten (adding x to both sides) as $4 > x$, so we need: $x > 0$ and $4 > x$ (this is sometimes combined in the form $4 > x > 0$, or, equivalently, $0 < x < 4$). In interval notation, this says that the domain is the interval $(0, 4)$.

Second method. Write $4x - x^2$ as $-(x^2 - 4x)$, and then complete the square, obtaining $-\left((x-2)^2 - 4\right) = 4 - (x-2)^2$. For this to be positive we need $(x-2)^2 < 4$, which means that $x - 2$ must be less than 2 and greater than -2: $-2 < x - 2 < 2$. Adding 2 to everything gives $0 < x < 4$. Both of these methods are equally correct; you may use either in a problem of this type.

A function does not always have to be given by a single formula, as we have already seen (in the income tax problem, for example). Suppose that $y = v(t)$ is the velocity function for a car which starts out from rest (zero velocity) at time $t = 0$; then increases its speed steadily to 20 m/sec, taking 10 seconds to do this; then travels at constant speed 20 m/sec for 15 seconds; and finally applies the brakes to decrease speed steadily to 0, taking 5 seconds to do this. The formula for $y = v(t)$ is different in each of the three time intervals: first $y = 2x$, then $y = 20$, then $y = -4x + 120$. The graph of this function is shown in figure **??**.

Not all functions are given by formulas at all. A function can be given by an experimentally determined table of values, or by a description other than a formula. For example, the population y of the U.S. is a function of the time t: we can write $y = f(t)$. This is a perfectly good function—we could graph it (up to the present) if we had data for various t—but we can't find an algebraic formula for it.

Exercises for Section 2.3

Find the domain of each of the following functions:

(1) $y = f(x) = \sqrt{2x-3}$

(2) $y = f(x) = 1/(x+1)$

(3) $y = f(x) = 1/(x^2-1)$

(4) $y = f(x) = \sqrt{-1/x}$

(5) $y = f(x) = \sqrt[3]{x}$

(6) $y = f(x) = \sqrt[4]{x}$

(7) $y = f(x) = \sqrt{r^2-(x-h)^2}$, where r and h are positive constants.

(8) $y = f(x) = \sqrt{1-(1/x)}$

(9) $y = f(x) = 1/\sqrt{1-(3x)^2}$

(10) $y = f(x) = \sqrt{x} + 1/(x-1)$

(11) $y = f(x) = 1/(\sqrt{x}-1)$

(12) Find the domain of $h(x) = \begin{cases} (x^2-9)/(x-3) & x \neq 3 \\ 6 \text{ if } x = 3. \end{cases}$

(13) Suppose $f(x) = 3x-9$ and $g(x) = \sqrt{x}$. What is the domain of the composition $(g \circ f)(x)$? (Recall that **composition** is defined as $(g \circ f)(x) = g(f(x))$.) What is the domain of $(f \circ g)(x)$?

(14) A farmer wants to build a fence along a river. He has 500 feet of fencing and wants to enclose a rectangular pen on three sides (with the river providing the fourth side). If x is the length of the side perpendicular to the river, determine the area of the pen as a function of x. What is the domain of this function?

(15) A can in the shape of a cylinder is to be made with a total of 100 square centimeters of material in the side, top, and bottom; the manufacturer wants the can to hold the maximum possible volume. Write the volume as a function of the radius r of the can; find the domain of the function.

(16) A can in the shape of a cylinder is to be made to hold a volume of one liter (1000 cubic centimeters). The manufacturer wants to use the least possible material for the can. Write the surface area of the can (total of the top, bottom, and side) as a function of the radius r of the can; find the domain of the function.

2.4 Shifts and Dilations

Many functions in applications are built up from simple functions by inserting constants in various places. It is important to understand the effect such constants have on the appearance of the graph.

Horizontal shifts. *If we replace x by $x-C$ everywhere it occurs in the formula for $f(x)$, then the graph shifts over C to the right.* (If C is negative, then this means that the graph shifts over $|C|$ to the left.) For example, the graph of $y=(x-2)^2$ is the x^2-parabola shifted over to have its vertex at the point 2 on the x-axis. The graph of $y=(x+1)^2$ is the same parabola shifted over to the left so as to have its vertex at -1 on the x-axis. Note well: when replacing x by $x-C$ we must pay attention to meaning, not merely appearance. Starting with $y=x^2$ and literally replacing x by $x-2$ gives $y=x-2^2$. This is $y=x-4$, a line with slope 1, not a shifted parabola.

Vertical shifts. *If we replace y by $y-D$, then the graph moves up D units.* (If D is negative, then this means that the graph moves down $|D|$ units.) If the formula is written in the form $y=f(x)$ and if y is replaced by $y-D$ to get $y-D=f(x)$, we can equivalently move D to the other side of the equation and write $y=f(x)+D$. Thus, this principle can be stated: *to get the graph of $y=f(x)+D$, take the graph of $y=f(x)$ and move it D units up.* For example, the function $y=x^2-4x=(x-2)^2-4$ can be obtained from $y=(x-2)^2$ (see the last paragraph) by moving the graph 4 units down. The result is the x^2-parabola shifted 2 units to the right and 4 units down so as to have its vertex at the point $(2,-4)$.

WARNING Do not confuse $f(x)+D$ and $f(x+D)$. For example, if $f(x)$ is the function x^2, then $f(x)+2$ is the function x^2+2, while $f(x+2)$ is the function $(x+2)^2=x^2+4x+4$.

Example (Circles) An important example of the above two principles starts with the circle $x^2+y^2=r^2$. This is the circle of radius r centered at the origin. (As we saw, this is not a single function $y=f(x)$, but rather two functions $y=\pm\sqrt{r^2-x^2}$ put together; in any case, the two shifting principles apply to equations like this one that are not in the form $y=f(x)$.) If we replace x by $x-C$ and replace y by $y-D$—getting the equation $(x-C)^2+(y-D)^2=r^2$—the effect on the circle is to move it C to the right and D up, thereby obtaining the circle of radius r centered at the point (C, D). This tells us how to write the equation of any circle, not necessarily centered at the origin.

We will later want to use two more principles concerning the effects of constants on the appearance of the graph of a function.

Horizontal dilation. *If x is replaced by x/A in a formula and $A > 1$, then the effect on the graph is to expand it by a factor of A in the x-direction (away from the y-axis).* If A is between 0 and 1 then the effect on the graph is to contract by a factor of $1/A$ (towards the y-axis). We use the word "dilate" to mean expand or contract.

For example, replacing x by $x/0.5 = x/(1/2) = 2x$ has the effect of contracting toward the y-axis by a factor of 2. If A is negative, we dilate by a factor of $|A|$ and then flip about the y-axis. Thus, replacing x by $-x$ has the effect of taking the mirror image of the graph with respect to the y-axis. For example, the function $y = \sqrt{-x}$, which has domain $\{x \in \mathbb{R} \mid x \leq 0\}$, is obtained by taking the graph of $\sqrt{x}$ and flipping it around the y-axis into the second quadrant.

Vertical dilation. *If y is replaced by y/B in a formula and $B > 0$, then the effect on the graph is to dilate it by a factor of B in the vertical direction.* As before, this is an expansion or contraction depending on whether B is larger or smaller than one. Note that if we have a function $y = f(x)$, replacing y by y/B is equivalent to multiplying the function on the right by B: $y = Bf(x)$. The effect on the graph is to expand the picture away from the x-axis by a factor of B if $B > 1$, to contract it toward the x-axis by a factor of $1/B$ if $0 < B < 1$, and to dilate by $|B|$ and then flip about the x-axis if B is negative.

Example (Ellipses) A basic example of the two expansion principles is given by an **ellipse of semimajor axis** a **and semiminor axis** b. We get such an ellipse by starting with the unit circle—the circle of radius 1 centered at the origin, the equation of which is $x^2 + y^2 = 1$—and dilating by a factor of a horizontally and by a factor of b vertically. To get the equation of the resulting ellipse, which crosses the x-axis at $\pm a$ and crosses the y-axis at $\pm b$, we replace x by x/a and y by y/b in the equation for the unit circle. This gives

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Finally, if we want to analyze a function that involves both shifts and dilations, it is usually simplest to work with the dilations first, and then the shifts. For instance, if we want to dilate a function by a factor of A in the x-direction and then shift C to the right, we do this by replacing x first by x/A and then by $(x-C)$ in the formula. As an example, suppose that, after dilating our unit circle by a in the x-direction and by b in the y-direction to get the ellipse in the last paragraph, we then wanted to shift it a distance h to the right and a distance k upward, so as to be centered at the point (h,k). The new ellipse would have equation

$$\left(\frac{x-h}{a}\right)^2+\left(\frac{y-k}{b}\right)^2=1.$$

Note well that this is different than first doing shifts by h and k and then dilations by a and b:

$$\left(\frac{x}{a}-h\right)^2+\left(\frac{y}{b}-k\right)^2=1.$$

See figure **??**.

Exercises for Section 2.4

Starting with the graph of $y=\sqrt{x}$, the graph of $y=1/x$, and the graph of $y=\sqrt{1-x^2}$ (the upper unit semicircle), sketch the graph of each of the following functions:

(1) $f(x)=\sqrt{x-2}$

(2) $f(x)=-1-1/(x+2)$

(3) $f(x)=4+\sqrt{x+2}$

(4) $y=f(x)=x/(1-x)$

(5) $y=f(x)=-\sqrt{-x}$

(6) $f(x)=2+\sqrt{1-(x-1)^2}$

(7) $f(x)=-4+\sqrt{-(x-2)}$

(8) $f(x)=2\sqrt{1-(x/3)^2}$

(9) $f(x)=1/(x+1)$

(10) $f(x)=4+2\sqrt{1-(x-5)^2/9}$

(11) $f(x)=1+1/(x-1)$

(12) $f(x)=\sqrt{100-25(x-1)^2}+2$

The graph of $f(x)$ is shown below. Sketch the graphs of the following functions.

(13) $y = 1 + f(x+2)$

(14) $y = 1 + 2f(x)$

(15) $y = 2f(3x)$

(16) $y = 2f(3(x-2)) + 1$

(17) $y = (1/2)f(3x-3)$

(18) $y = f(1 + x/3) + 2$

3

Instantaneous Rate of Change: The Derivative

3.1 The slope of a function

Suppose that y is a function of x, say $y = f(x)$. It is often necessary to know how sensitive the value of y is to small changes in x.

Example Take, for example, $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin). When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

In the case of a straight line $y = mx + b$, the slope $m = \Delta y/\Delta x$ measures the change in y per unit change in x. This can be interpreted as a measure of "sensitivity"; for example, if $y = 100x + 5$, a small change in x corresponds to a change one hundred times as large in y, so y is quite sensitive to changes in x.

Let us look at the same ratio $\Delta y/\Delta x$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\begin{aligned} \Delta y = f(x + \Delta x) - f(x) = f(7.1) - f(7) \\ = \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2} \approx 23.9706 - 24 = -0.0294. \end{aligned}$$

Thus, $\Delta y/\Delta x \approx -0.0294/0.1 = -0.294$. This means that y changes by less than one third the change in x, so apparently y is not very sensitive to changes in x

at $x = 7$. We say "apparently" here because we don't really know what happens between 7 and 7.1. Perhaps y changes dramatically as x runs through the values from 7 to 7.1, but at 7.1 y just happens to be close to its value at 7. This is not in fact the case for this particular function, but we don't yet know why.

One way to interpret the above calculation is by reference to a line. We have computed the slope of the line through $(7,24)$ and $(7.1,23.9706)$, called a **chord** of the circle. In general, if we draw the chord from the point $(7,24)$ to a nearby point on the semicircle $(7+\Delta x, f(7+\Delta x))$, the slope of this chord is the so-called **difference quotient**

$$\text{slope of chord} = \frac{f(7+\Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625-(7+\Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the chord) is approximately equal to $(23.997081 - 24)/0.01 = -0.2919$. This is slightly less steep than the chord from $(7,24)$ to $(7.1,23.9706)$.

As the second value $7+\Delta x$ moves in towards 7, the chord joining $(7, f(7))$ to $(7+\Delta x, f(7+\Delta x))$ shifts slightly. As indicated in figure **??**, as Δx gets smaller and smaller, the chord joining $(7,24)$ to $(7+\Delta x, f(7+\Delta x))$ gets closer and closer to the **tangent line** to the circle at the point $(7,24)$. (Recall that the tangent line is the line that just grazes the circle at that point, i.e., it doesn't meet the circle at any second point.) Thus, as Δx gets smaller and smaller, the slope $\Delta y/\Delta x$ of the chord gets closer and closer to the slope of the tangent line. This is actually quite difficult to see when Δx is small, because of the scale of the graph. The values of Δx used for the figure are 1, 5, 10 and 15, not really very small values. The tangent line is the one that is uppermost at the right hand endpoint.

So far we have found the slopes of two chords that should be close to the slope of the tangent line, but what is the slope of the tangent line exactly? Since the tangent line touches the circle at just one point, we will never be able to calculate its slope directly, using two "known" points on the line. What we need is a way to capture what happens to the slopes of the chords as they get "closer and closer" to the tangent line.

Instead of looking at more particular values of Δx, let's see what happens if we do some algebra with the difference quotient using just Δx. The slope of a chord from $(7,24)$ to a nearby point is given by

$$\begin{aligned}
\frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x} &= \frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x}\frac{\sqrt{625-(7+\Delta x)^2}+24}{\sqrt{625-(7+\Delta x)^2}+24} \\
&= \frac{625-(7+\Delta x)^2-24^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{49-49-14\Delta x-\Delta x^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{\Delta x(-14-\Delta x)}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{-14-\Delta x}{\sqrt{625-(7+\Delta x)^2}+24}
\end{aligned}$$

Now, can we tell by looking at this last formula what happens when Δx gets very close to zero? The numerator clearly gets very close to -14 while the denominator gets very close to $\sqrt{625-7^2}+24=48$. Is the fraction therefore very close to $-14/48=-7/24\cong-0.29167$? It certainly seems reasonable, and in fact it is true: as Δx gets closer and closer to zero, the difference quotient does in fact get closer and closer to $-7/24$, and so the slope of the tangent line is exactly $-7/24$.

What about the slope of the tangent line at $x=12$? Well, 12 can't be all that different from 7; we just have to redo the calculation with 12 instead of 7. This won't be hard, but it will be a bit tedious. What if we try to do all the algebra without using a specific value for x? Let's copy from above, replacing 7 by x. We'll have to do a bit more than that—for example, the "24" in the calculation came from

$\sqrt{625-7^2}$, so we'll need to fix that too.

$$
\begin{aligned}
&\frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x} = \\
&= \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x}\,\frac{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}} \\
&= \frac{625-(x+\Delta x)^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{625-x^2-2x\Delta x-\Delta x^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{\Delta x(-2x-\Delta x)}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{-2x-\Delta x}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}
\end{aligned}
$$

Now what happens when Δx is very close to zero? Again it seems apparent that the quotient will be very close to

$$\frac{-2x}{\sqrt{625-x^2}+\sqrt{625-x^2}} = \frac{-2x}{2\sqrt{625-x^2}} = \frac{-x}{\sqrt{625-x^2}}.$$

Replacing x by 7 gives $-7/24$, as before, and now we can easily do the computation for 12 or any other value of x between -25 and 25.

So now we have a single, simple formula, $-x/\sqrt{625-x^2}$, that tells us the slope of the tangent line for any value of x. This slope, in turn, tells us how sensitive the value of y is to changes in the value of x.

What do we call such a formula? That is, a formula with one variable, so that substituting an "input" value for the variable produces a new "output" value? This is a function. Starting with one function, $\sqrt{625-x^2}$, we have derived, by means of some slightly nasty algebra, a new function, $-x/\sqrt{625-x^2}$, that gives us important information about the original function. This new function in fact is called the **derivative** of the original function. If the original is referred to as f or y then the derivative is often written f' or y' and pronounced "f prime" or "y prime", so in this case we might write $f'(x) = -x/\sqrt{625-x^2}$. At a particular point, say $x = 7$, we say that $f'(7) = -7/24$ or "f prime of 7 is $-7/24$" or "the derivative of f at 7 is $-7/24$."

To summarize, we compute the derivative of $f(x)$ by forming the difference quotient

$$\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

which is the slope of a line, then we figure out what happens when Δx gets very close to 0.

We should note that in the particular case of a circle, there's a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining $(0,0)$ to $(7,24)$ has slope 24/7. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{625-x^2})$ has slope $\sqrt{625-x^2}/x$, so the slope of the tangent line is $-x/\sqrt{625-x^2}$, as before. It is **NOT** always true that a tangent line is perpendicular to a line from the origin—don't use this shortcut in any other circumstance.

As above, and as you might expect, for different values of x we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of f, or the slope of its tangent line, is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line to a line, but if it makes sense at all the tangent line must be the line itself. It is not hard to see that the derivative of $f(x) = mx + b$ is $f'(x) = m$; see exercise 6 .

Exercises for Section 3.1

(1) Draw the graph of the function $y = f(x) = \sqrt{169-x^2}$ between $x = 0$ and $x = 13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x = 12$ and $x = 13$, (b) $x = 12$ and $x = 12.1$, (c) $x = 12$ and $x = 12.01$, (d) $x = 12$ and $x = 12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)–(d) should be getting closer and closer to your answer to (e).

(2) Use geometry to find the derivative $f'(x)$ of the function $f(x) = \sqrt{625 - x^2}$ in the text for each of the following x: (a) 20, (b) 24, (c) −7, (d) −15. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points.

(3) Draw the graph of the function $y = f(x) = 1/x$ between $x = 1/2$ and $x = 4$. Find the slope of the chord between (a) $x = 3$ and $x = 3.1$, (b) $x = 3$ and $x = 3.01$, (c) $x = 3$ and $x = 3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$. Determine what happens when Δx approaches 0. In your graph of $y = 1/x$, draw the straight line through the point $(3, 1/3)$ whose slope is this limiting value of the difference quotient as Δx approaches 0.

(4) Find an algebraic expression for the difference quotient $\big(f(1 + \Delta x) - f(1)\big)/\Delta x$ when $f(x) = x^2 - (1/x)$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(1)$.

(5) Draw the graph of $y = f(x) = x^3$ between $x = 0$ and $x = 1.5$. Find the slope of the chord between (a) $x = 1$ and $x = 1.1$, (b) $x = 1$ and $x = 1.001$, (c) $x = 1$ and $x = 1.00001$. Then use algebra to find a simple formula for the slope of the chord between 1 and $1 + \Delta x$. (Use the expansion $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.) Determine what happens as Δx approaches 0, and in your graph of $y = x^3$ draw the straight line through the point $(1, 1)$ whose slope is equal to the value you just found.

(6) Find an algebraic expression for the difference quotient $(f(x + \Delta x) - f(x))/\Delta x$ when $f(x) = mx + b$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(x)$.

(7) Sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle θ? Why? Hint: think in terms of ratios of sides of triangles.

(8) Sketch the parabola $y = x^2$. For what values of x on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?

3.2 An example

We started the last section by saying, "It is often necessary to know how sensitive the value of y is to small changes in x." We have seen one purely mathematical example of this: finding the "steepness" of a curve at a point is precisely this problem. Here is a more applied example.

With careful measurement it might be possible to discover that a dropped ball has height $h(t) = h_0 - kt^2$, t seconds after it is released. (Here h_0 is the initial height of the ball, when $t = 0$, and k is some number determined by the experiment.) A natural question is then, "How fast is the ball going at time t?" We can certainly get a pretty good idea with a little simple arithmetic. To make the calculation more concrete, let's say $h_0 = 100$ meters and $k = 4.9$ and suppose we're interested in the speed at $t = 2$. We know that when $t = 2$ the height is $100 - 4 \cdot 4.9 = 80.4$. A second later, at $t = 3$, the height is $100 - 9 \cdot 4.9 = 55.9$, so in that second the ball has traveled $80.4 - 55.9 = 24.5$ meters. This means that the *average* speed during that time was 24.5 meters per second. So we might guess that 24.5 meters per second is not a terrible estimate of the speed at $t = 2$. But certainly we can do better. At $t = 2.5$ the height is $100 - 4.9(2.5)^2 = 69.375$. During the half second from $t = 2$ to $t = 2.5$ the ball dropped $80.4 - 69.375 = 11.025$ meters, at an average speed of $11.025/(1/2) = 22.05$ meters per second; this should be a better estimate of the speed at $t = 2$. So it's clear now how to get better and better approximations: compute average speeds over shorter and shorter time intervals. Between $t = 2$ and $t = 2.01$, for example, the ball drops 0.19649 meters in one hundredth of a second, at an average speed of 19.649 meters per second.

We can't do this forever, and we still might reasonably ask what the actual speed precisely at $t = 2$ is. If Δt is some tiny amount of time, what we want to know is what happens to the average speed $(h(2) - h(2 + \Delta t))/\Delta t$ as Δt gets smaller and smaller. Doing a bit of algebra:

$$\begin{aligned}
\frac{h(2)-h(2+\Delta t)}{\Delta t} &= \frac{80.4-(100-4.9(2+\Delta t)^2)}{\Delta t} \\
&= \frac{80.4-100+19.6+19.6\Delta t+4.9\Delta t^2}{\Delta t} \\
&= \frac{19.6\Delta t+4.9\Delta t^2}{\Delta t} \\
&= 19.6+4.9\Delta t
\end{aligned}$$

When Δt is very small, this is very close to 19.6, and indeed it seems clear that as Δt goes to zero, the average speed goes to 19.6, so the exact speed at $t = 2$ is 19.6 meters per second. This calculation should look very familiar. In the language of the previous section, we might have started with $f(x) = 100 - 4.9x^2$ and asked for the slope of the tangent line at $x = 2$. We would have answered that question by computing

$$\frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{-19.6\Delta x-4.9\Delta x^2}{\Delta x} = -19.6-4.9\Delta x$$

The algebra is the same, except that following the pattern of the previous section the subtraction would be reversed, and we would say that the slope of the tangent line is -19.6. Indeed, in hindsight, perhaps we should have subtracted the other way even for the dropping ball. At $t = 2$ the height is 80.4; one second later the height is 55.9. The usual way to compute a "distance traveled" is to subtract the earlier position from the later one, or $55.9 - 80.4 = -24.5$. This tells us that the distance traveled is 24.5 meters, and the negative sign tells us that the height went down during the second. If we continue the original calculation we then get -19.6 meters per second as the exact speed at $t = 2$. If we interpret the negative sign as meaning that the motion is downward, which seems reasonable, then in fact this is the same answer as before, but with even more information, since the numerical answer contains the direction of motion as well as the speed. Thus, the speed of the ball is the value of the derivative of a certain function, namely, of the function that gives the position of the ball. (More properly, this is the *velocity* of the ball; velocity is signed speed, that is, speed with a direction indicated by the sign.)

The upshot is that this problem, finding the speed of the ball, is *exactly* the same problem mathematically as finding the slope of a curve. This may already be enough evidence to convince you that whenever some quantity is changing (the height of a curve or the height of a ball or the size of the economy or the distance of a space probe from earth or the population of the world) the rate at which the quantity is changing can, in principle, be computed in exactly the same way, by finding a derivative.

Exercises for Section 3.2

(1) An object is traveling in a straight line so that its position (that is, distance from some fixed point) is given by this table:

time (seconds)	0	1	2	3
distance (meters)	0	10	25	60

Find the average speed of the object during the following time intervals: $[0,1]$, $[0,2]$, $[0,3]$, $[1,2]$, $[1,3]$, $[2,3]$. If you had to guess the speed at $t = 2$ just on the basis of these, what would you guess?

(2) Let $y = f(t) = t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t = 0$ and $t = 3$. Make a table of the average speed of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average speed between time 2 and time $2+\Delta t$. (If you substitute $\Delta t = 1, 0.1, 0.01, 0.001$ in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average speed (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous speed. Finally, in your graph of $y = t^2$ draw the straight line through the point $(2,4)$ whose slope is the instantaneous velocity you just computed; it should of course be the tangent line.

(3) If an object is dropped from an 80-meter high window, its height y above the ground at time t seconds is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown in figure 2.1.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and $1 + \Delta t$ sec. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ second (it will be negative, because the object is falling).

3.3 Limits

In the previous two sections we computed some quantities of interest (slope, velocity) by seeing that some expression "goes to" or "approaches" or "gets really close to" a particular value. In the examples we saw, this idea may have been clear enough, but it is too fuzzy to rely on in more difficult circumstances. In this section we will see how to make the idea more precise.

There is an important feature of the examples we have seen. Consider again the formula

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

We wanted to know what happens to this fraction as "Δx goes to zero." Because we were able to simplify the fraction, it was easy to see the answer, but it was not quite as simple as "substituting zero for Δx," as that would give

$$\frac{-19.6 \cdot 0 - 4.9 \cdot 0}{0},$$

which is meaningless. The quantity we are really interested in does not make sense "at zero," and this is why the answer to the original problem (finding a velocity or a slope) was not immediately obvious. In other words, we are generally going to want to figure out what a quantity "approaches" in situations where we can't merely plug in a value. If you would like to think about a hard example (which we will analyze later) consider what happens to $(\sin x)/x$ as x approaches zero.

Example Does $\sqrt{x}$ approach 1.41 as x approaches 2? In this case it is possible to compute the actual value $\sqrt{2}$ to a high precision to answer the question. But since in general we won't be able to do that, let's not. We might start by computing $\sqrt{x}$ for values of x close to 2, as we did in the previous sections. Here are some values: $\sqrt{2.05} = 1.431782106$, $\sqrt{2.04} = 1.428285686$, $\sqrt{2.03} = 1.424780685$, $\sqrt{2.02} = 1.421267040$, $\sqrt{2.01} = 1.417744688$, $\sqrt{2.005} = 1.415980226$, $\sqrt{2.004} = 1.415627070$, $\sqrt{2.003} = 1.415273825$, $\sqrt{2.002} = 1.414920492$, $\sqrt{2.001} = 1.414567072$. So it looks at least possible that indeed these values "approach" 1.41—already $\sqrt{2.001}$ is quite close. If we continue this process, however, at some point we will appear to "stall." In fact, $\sqrt{2} = 1.414213562\ldots$, so we will never even get as far as 1.4142, no matter how long we continue the sequence.

So in a fuzzy, everyday sort of sense, it is true that $\sqrt{x}$ "gets close to" 1.41, but it does not "approach" 1.41 in the sense we want. To compute an exact slope or an exact velocity, what we want to know is that a given quantity becomes "arbitrarily close" to a fixed value, meaning that the first quantity can be made "as close as we like" to the fixed value. Consider again the quantities

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6 - 4.9\Delta x.$$

These two quantities are equal as long as Δx is not zero; if Δx is zero, the left hand quantity is meaningless, while the right hand one is -19.6. Can we say more than we did before about why the right hand side "approaches" -19.6, in the desired sense? Can we really make it "as close as we want" to -19.6? Let's try a test case. Can we make $-19.6 - 4.9\Delta x$ within one millionth (0.000001) of -19.6? The values within a millionth of -19.6 are those in the interval $(-19.600001, -19.599999)$. As Δx approaches zero, does $-19.6 - 4.9\Delta x$ eventually reside inside this interval? If Δx is positive, this would require that $-19.6 - 4.9\Delta x > -19.600001$. This is something we can manipulate with a little algebra:

$$\begin{aligned}
-19.6 - 4.9\Delta x &> -19.600001 \\
-4.9\Delta x &> -0.000001 \\
\Delta x &< -0.000001 / -4.9 \\
\Delta x &< 0.0000002040816327\ldots
\end{aligned}$$

Thus, we can say with certainty that if Δx is positive and less than 0.0000002, then $\Delta x < 0.0000002040816327\ldots$ and so $-19.6 - 4.9\Delta x > -19.600001$. We could do a similar calculation if Δx is negative.

So now we know that we can make $-19.6 - 4.9\Delta x$ within one millionth of -19.6. But can we make it "as close as we want"? In this case, it is quite simple to see that the answer is yes, by modifying the calculation we've just done. It may be helpful to think of this as a game. I claim that I can make $-19.6 - 4.9\Delta x$ as close as you desire to -19.6 by making Δx "close enough" to zero. So the game is: you give me a number, like 10^{-6}, and I have to come up with a number representing how close Δx must be to zero to guarantee that $-19.6 - 4.9\Delta x$ is at least as close to -19.6 as you have requested.

Now if we actually play this game, I could redo the calculation above for each new number you provide. What I'd like to do is somehow see that I will always succeed, and even more, I'd like to have a simple strategy so that I don't have to do all that algebra every time. A strategy in this case would be a formula that gives me a correct answer no matter what you specify. So suppose the number you give me is ε. How close does Δx have to be to zero to guarantee that $-19.6 - 4.9\Delta x$ is in $(-19.6 - \varepsilon, -19.6 + \varepsilon)$? If Δx is positive, we need:

$$\begin{aligned} -19.6 - 4.9\Delta x &> -19.6 - \varepsilon \\ -4.9\Delta x &> -\varepsilon \\ \Delta x &< -\varepsilon / -4.9 \\ \Delta x &< \varepsilon / 4.9 \end{aligned}$$

So if I pick any number δ that is less than $\varepsilon/4.9$, the algebra tells me that whenever $\Delta x < \delta$ then $\Delta x < \varepsilon/4.9$ and so $-19.6 - 4.9\Delta x$ is within ε of -19.6. (This is exactly what I did in the example: I picked $\delta = 0.0000002 < 0.0000002040816327\ldots$.) A similar calculation again works for negative Δx. The important fact is that this is now a completely general result—it shows that I can always win, no matter what "move" you make.

Now we can codify this by giving a precise definition to replace the fuzzy, "gets closer and closer" language we have used so far. Henceforward, we will say something like "the limit of $(-19.6\Delta x - 4.9\Delta x^2)/\Delta x$ as Δx goes to zero is -19.6," and abbreviate this mouthful as

$$\lim_{\Delta x \to 0} \frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6.$$

Here is the actual, official definition of "limit".

Definition Suppose f is a function. We say that the **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x \to a} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$.

The ε and δ here play exactly the role they did in the preceding discussion. The definition says, in a very precise way, that $f(x)$ can be made as close as desired to L (that's the $|f(x) - L| < \varepsilon$ part) by making x close enough to a (the $0 < |x - a| < \delta$ part). Note that we specifically make no mention of what must happen if $x = a$, that is, if $|x - a| = 0$. This is because in the cases we are most interested in, substituting a for x doesn't even make sense.

Make sure you are not confused by the names of important quantities. The generic definition talks about $f(x)$, but the function and the variable might have other names. In the discussion above, the function we analyzed was

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

and the variable of the limit was not x but Δx. The x was the variable of the original function; when we were trying to compute a slope or a velocity, x was essentially a fixed quantity, telling us at what point we wanted the slope. (In the velocity problem, it was literally a fixed quantity, as we focused on the time 2.) The quantity a of the definition in all the examples was zero: we were always interested in what happened as Δx became very close to zero.

Armed with a precise definition, we can now prove that certain quantities behave in a particular way. The bad news is that even proofs for simple quantities can be quite tedious and complicated; the good news is that we rarely need to do such proofs, because most expressions act the way you would expect, and this can be proved once and for all.

Let's show carefully that $\lim_{x \to 2} x + 4 = 6$. This is not something we "need" to prove, since it is "obviously" true. But if we couldn't prove it using our official definition there would be something very wrong with the definition.

As is often the case in mathematical proofs, it helps to work backwards. We want to end up showing that under certain circumstances $x + 4$ is close to 6; precisely, we want to show that $|x + 4 - 6| < \varepsilon$, or $|x - 2| < \varepsilon$. Under what circumstances? We want this to be true whenever $0 < |x - 2| < \delta$. So the question becomes: can we choose a value for δ that guarantees that $0 < |x - 2| < \delta$ implies $|x - 2| < \varepsilon$? Of course: no matter what ε is, $\delta = \varepsilon$ works.

So it turns out to be very easy to prove something "obvious," which is nice. It doesn't take long before things get trickier, however.

Example It seems clear that $\lim_{x\to 2} x^2 = 4$. Let's try to prove it. We will want to be able to show that $|x^2-4| < \varepsilon$ whenever $0 < |x-2| < \delta$, by choosing δ carefully. Is there any connection between $|x-2|$ and $|x^2-4|$? Yes, and it's not hard to spot, but it is not so simple as the previous example. We can write $|x^2-4| = |(x+2)(x-2)|$. Now when $|x-2|$ is small, part of $|(x+2)(x-2)|$ is small, namely $(x-2)$. What about $(x+2)$? If x is close to 2, $(x+2)$ certainly can't be too big, but we need to somehow be precise about it. Let's recall the "game" version of what is going on here. You get to pick an ε and I have to pick a δ that makes things work out. Presumably it is the really tiny values of ε I need to worry about, but I have to be prepared for anything, even an apparently "bad" move like $\varepsilon = 1000$. I expect that ε is going to be small, and that the corresponding δ will be small, certainly less than 1. If $\delta \le 1$ then $|x+2| < 5$ when $|x-2| < \delta$ (because if x is within 1 of 2, then x is between 1 and 3 and $x+2$ is between 3 and 5). So then I'd be trying to show that $|(x+2)(x-2)| < 5|x-2| < \varepsilon$. So now how can I pick δ so that $|x-2| < \delta$ implies $5|x-2| < \varepsilon$? This is easy: use $\delta = \varepsilon/5$, so $5|x-2| < 5(\varepsilon/5) = \varepsilon$. But what if the ε you choose is not small? If you choose $\varepsilon = 1000$, should I pick $\delta = 200$? No, to keep things "sane" I will never pick a δ bigger than 1. Here's the final "game strategy:" When you pick a value for ε I will pick $\delta = \varepsilon/5$ or $\delta = 1$, whichever is smaller. Now when $|x-2| < \delta$, I know both that $|x+2| < 5$ and that $|x-2| < \varepsilon/5$. Thus $|(x+2)(x-2)| < 5(\varepsilon/5) = \varepsilon$.

This has been a long discussion, but most of it was explanation and scratch work. If this were written down as a proof, it would be quite short, like this:

Proof that $\lim_{x\to 2} x^2 = 4$. Given any ε, pick $\delta = \varepsilon/5$ or $\delta = 1$, whichever is smaller. Then when $|x-2| < \delta$, $|x+2| < 5$ and $|x-2| < \varepsilon/5$. Hence $|x^2-4| = |(x+2)(x-2)| < 5(\varepsilon/5) = \varepsilon$.

It probably seems obvious that $\lim_{x\to 2} x^2 = 4$, and it is worth examining more closely why it seems obvious. If we write $x^2 = x \cdot x$, and ask what happens when x approaches 2, we might say something like, "Well, the first x approaches 2, and the second x approaches 2, so the product must approach $2 \cdot 2$." In fact this is pretty much right on the money, except for that word "must." Is it really true that if x

approaches a and y approaches b then xy approaches ab? It is, but it is not really obvious, since x and y might be quite complicated. The good news is that we can see that this is true once and for all, and then we don't have to worry about it ever again. When we say that x might be "complicated" we really mean that in practice it might be a function. Here is then what we want to know:

Theorem Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then $\lim_{x\to a} f(x)g(x) = LM$.

Proof. We have to use the official definition of limit to make sense of this. So given any ε we need to find a δ so that $0 < |x-a| < \delta$ implies $|f(x)g(x) - LM| < \varepsilon$. What do we have to work with? We know that we can make $f(x)$ close to L and $g(x)$ close to M, and we have to somehow connect these facts to make $f(x)g(x)$ close to LM.

We use, as is so often the case, a little algebraic trick:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\
&= |f(x)(g(x) - M) + (f(x) - L)M| \\
&\le |f(x)(g(x) - M)| + |(f(x) - L)M| \\
&= |f(x)||g(x) - M| + |f(x) - L||M|.
\end{aligned}$$

This is all straightforward except perhaps for the "$\le$". That is an example of the **triangle inequalitypage:triangle inequality**, which says that if a and b are any real numbers then $|a+b| \le |a| + |b|$. If you look at a few examples, using positive and negative numbers in various combinations for a and b, you should quickly understand why this is true; we will not prove it formally.

Since $\lim_{x\to a} f(x) = L$, there is a value δ_1 so that $0 < |x-a| < \delta_1$ implies $|f(x) - L| < |\varepsilon/(2M)|$, This means that $0 < |x-a| < \delta_1$ implies $|f(x) - L||M| < \varepsilon/2$. You can see where this is going: if we can make $|f(x)||g(x) - M| < \varepsilon/2$ also, then we'll be done.

We can make $|g(x) - M|$ smaller than any fixed number by making x close enough to a; unfortunately, $\varepsilon/(2f(x))$ is not a fixed number, since x is a variable. Here we need another little trick, just like the one we used in analyzing x^2. We can find a δ_2 so that $|x-a| < \delta_2$ implies that $|f(x) - L| < 1$, meaning that $L - 1 < f(x) < L + 1$. This means that $|f(x)| < N$, where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x) - M| < \varepsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that $|f(x) - L| < |\varepsilon/(2M)|$, $|f(x)| < N$, and $|g(x) - M| < \varepsilon/(2N)$. Then

$$\begin{aligned}|f(x)g(x) - LM| &\leq |f(x)||g(x) - M| + |f(x) - L||M| \\ &< N\frac{\varepsilon}{2N} + \left|\frac{\varepsilon}{2M}\right||M| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This is just what we needed, so by the official definition, $\lim\limits_{x\to a} f(x)g(x) = LM$. □

A handful of such theorems give us the tools to compute many limits without explicitly working with the definition of limit.

Theorem Suppose that $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} g(x) = M$ and k is some constant. Then

$$\begin{aligned}&\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL \\ &\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M \\ &\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L - M \\ &\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM \\ &\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M}, \text{ if } M \text{ is not } 0\end{aligned}$$

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim\limits_{x\to a} x = a$.

Example Compute $\lim_{x\to 1} \dfrac{x^2-3x+5}{x-2}$. If we apply the theorem in all its gory detail, we get

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1}(x^2-3x+5)}{\lim_{x\to 1}(x-2)} \\
&= \frac{(\lim_{x\to 1} x^2) - (\lim_{x\to 1} 3x) + (\lim_{x\to 1} 5)}{(\lim_{x\to 1} x) - (\lim_{x\to 1} 2)} \\
&= \frac{(\lim_{x\to 1} x)^2 - 3(\lim_{x\to 1} x) + 5}{(\lim_{x\to 1} x) - 2} \\
&= \frac{1^2 - 3\cdot 1 + 5}{1-2} \\
&= \frac{1-3+5}{-1} = -3
\end{aligned}$$

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

Of course, as we've already seen, we're primarily interested in limits that aren't so easy, namely, limits in which a denominator approaches zero. There are a handful of algebraic tricks that work on many of these limits.

Example Compute $\lim_{x\to 1} \dfrac{x^2+2x-3}{x-1}$. We can't simply plug in $x = 1$ because that makes the denominator zero. However:

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2+2x-3}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} \\
&= \lim_{x\to 1} (x+3) = 4
\end{aligned}$$

While theorem 3.3 is very helpful, we need a bit more to work easily with limits. Since the theorem applies when some limits are already known, we need to know the behavior of some functions that cannot themselves be constructed from the simple arithmetic operations of the theorem, such as $\sqrt{x}$. Also, there is one other extraordinarily useful way to put functions together: composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2 + 5$, then $f(g(x)) = \sqrt{x^2 + 5}$ and $g(f(x)) = (\sqrt{x})^2 + 5 = x + 5$. Here is a companion to theorem 3.3 for composition:

Theorem Suppose that $\lim_{x \to a} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$. Then

$$\lim_{x \to a} f(g(x)) = f(L).$$

Note the special form of the condition on f: it is not enough to know that $\lim_{x \to L} f(x) = M$, though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

Theorem Suppose that n is a positive integer. Then

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Another of the most common algebraic tricks was used in section 3.1. Here's another example:

Example Compute $\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}$.

$$\begin{aligned}\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} \frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\ &= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\ &= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\ &= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}\end{aligned}$$

At the very last step we have used theorems 3.3 and 3.3.

Occasionally we will need a slightly modified version of the limit definition. Consider the function $f(x) = \sqrt{1-x^2}$, the upper half of the unit circle. What can we say about $\lim_{x\to 1} f(x)$? It is apparent from the graph of this familiar function that as x gets close to 1 from the left, the value of $f(x)$ gets close to zero. It does not even make sense to ask what happens as x approaches 1 from the right, since $f(x)$ is not defined there. The definition of the limit, however, demands that $f(1+\Delta x)$ be close to $f(1)$ whether Δx is positive or negative. Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of **one sided limit**:

Definition (One-sided limit) Suppose that $f(x)$ is a function. We say that $\lim_{x\to a^-} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < a - x < \delta$, $|f(x) - L| < \varepsilon$. We say that $\lim_{x\to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < x - a < \delta$, $|f(x) - L| < \varepsilon$.

Usually $\lim_{x\to a^-} f(x)$ is read "the limit of $f(x)$ from the left" and $\lim_{x\to a^+} f(x)$ is read "the limit of $f(x)$ from the right".

Example Discuss $\lim_{x\to 0}\frac{x}{|x|}$, $\lim_{x\to 0^-}\frac{x}{|x|}$, and $\lim_{x\to 0^+}\frac{x}{|x|}$.

The function $f(x) = x/|x|$ is undefined at 0; when $x > 0$, $|x| = x$ and so $f(x) = 1$; when $x < 0$, $|x| = -x$ and $f(x) = -1$. Thus $\lim_{x\to 0^-}\frac{x}{|x|} = \lim_{x\to 0^-} -1 = -1$ while $\lim_{x\to 0^+}\frac{x}{|x|} = \lim_{x\to 0^+} 1 = 1$. The limit of $f(x)$ must be equal to both the left and right limits; since they are different, the limit $\lim_{x\to 0}\frac{x}{|x|}$ does not exist.

Exercises for Section 3.3

Compute the limits. If a limit does not exist, explain why.

(1) $\lim_{x\to 3}\frac{x^2+x-12}{x-3}$

(2) $\lim_{x\to 1}\frac{x^2+x-12}{x-3}$

(3) $\lim_{x\to -4}\frac{x^2+x-12}{x-3}$

(4) $\lim_{x\to 2}\frac{x^2+x-12}{x-2}$

(5) $\lim_{x\to 1}\frac{\sqrt{x+8}-3}{x-1}$

(6) $\lim_{x\to 0^+}\sqrt{\frac{1}{x}+2}-\sqrt{\frac{1}{x}}$.

(7) $\lim_{x\to 2} 3$

(8) $\lim_{x\to 4} 3x^3-5x$

(9) $\lim_{x\to 0}\frac{4x-5x^2}{x-1}$

(10) $\lim_{x\to 1}\frac{x^2-1}{x-1}$

(11) $\lim_{x\to 0^+}\frac{\sqrt{2-x^2}}{x}$

(12) $\lim_{x\to 0^+}\frac{\sqrt{2-x^2}}{x+1}$

(13) $\lim_{x\to a}\frac{x^3-a^3}{x-a}$

(14) $\lim_{x\to 2}(x^2+4)^3$

(15) $\lim_{x\to 1}\begin{cases} x-5 & x\neq 1,\\ 7 & x=1.\end{cases}$

(16) $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$ (Hint: Use the fact that $|\sin a| < 1$ for any real number a. You should probably use the definition of a limit here.)

(17) Give an ε–δ proof, similar to example 3.3, of the fact that $\lim_{x\to 4}(2x-5)=3$.

(18) Evaluate the expressions by reference to this graph:

(19) Use a calculator to estimate $\lim_{x\to 0}\frac{\sin x}{x}$.

(20) Use a calculator to estimate $\lim_{x\to 0}\frac{\tan(3x)}{\tan(5x)}$.

3.4 The Derivative Function

We have seen how to create, or derive, a new function $f'(x)$ from a function $f(x)$, and that this new function carries important information. In one example we saw that $f'(x)$ tells us how steep the graph of $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ tells us the position of the object at time x. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "rate" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

We will begin to use different notations for the derivative of a function. While initially confusing, each is often useful so it is worth maintaining multiple versions of the same thing.

Consider again the function $f(x) = \sqrt{625 - x^2}$. We have computed the derivative $f'(x) = -x/\sqrt{625 - x^2}$, and have already noted that if we use the alternate notation $y = \sqrt{625 - x^2}$ then we might write $y' = -x/\sqrt{625 - x^2}$. Another notation is quite different, and in time it will become clear why it is often a useful one. Recall that to compute the the derivative of f we computed

$$\lim_{\Delta x \to 0} \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

The denominator here measures a distance in the x direction, sometimes called the "run", and the numerator measures a distance in the y direction, sometimes called the "rise," and "rise over run" is the slope of a line. Recall that sometimes such a numerator is abbreviated Δy, exchanging brevity for a more detailed expression. So in general, a derivative is given by

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

To recall the form of the limit, we sometimes say instead that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In other words, dy/dx is another notation for the derivative, and it reminds us that it is related to an actual slope between two points. This notation is called *Leibniz notation*, after Gottfried Leibniz, who developed the fundamentals of calculus independently, at about the same time that Isaac Newton did. Again, since we often use f and $f(x)$ to mean the original function, we sometimes use df/dx and $df(x)/dx$ to refer to the derivative. If the function $f(x)$ is written out in full we often write the last of these something like this

$$f'(x) = \frac{d}{dx}\sqrt{625 - x^2}$$

with the function written to the side, instead of trying to fit it into the numerator.

Example Find the derivative of $y = f(t) = t^2$.
We compute

$$\begin{aligned} y' = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} &= \lim_{\Delta t \to 0} \frac{(t + \Delta t)^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} 2t + \Delta t = 2t. \end{aligned}$$

Remember that Δt is a single quantity, not a "Δ" times a "t", and so Δt^2 is $(\Delta t)^2$ not $\Delta(t^2)$.

Example Find the derivative of $y = f(x) = 1/x$.

The computation:

$$
\begin{aligned}
y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x - x - \Delta x}{x(x+\Delta x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{x(x+\Delta x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x^2}
\end{aligned}
$$

Note. If you happen to know some "derivative formulas" from an earlier course, for the time being you should pretend that you do not know them. In examples like the ones above and the exercises below, you are required to know how to find the derivative formula starting from basic principles. We will later develop some formulas so that we do not always need to do such computations, but we will continue to need to know how to do the more involved computations.

Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is **no derivative**, because there is no tangent line. In order for the notion of the tangent line at a point to make sense, the curve must be "smooth" at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are two types of situations you should be aware of—corners and cusps—where there's a sudden change of direction and hence no derivative.

Example Discuss the derivative of the absolute value function $y = f(x) = |x|$.

If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve—but there are *two* directions of the curve that come together at the origin. We can summarize this as

$$y' = \begin{cases} 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0; \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

Example

Discuss the derivative of the function $y = x^{2/3}$, shown in figure **??**. We will later see how to compute this derivative; for now we use the fact that $y' = (2/3)x^{-1/3}$. Visually this looks much like the absolute value function, but it technically has a cusp, not a corner. The absolute value function has no tangent line at 0 because there are (at least) two obvious contenders—the tangent line of the left side of the curve and the tangent line of the right side. The function $y = x^{2/3}$ does not have a tangent line at 0, but unlike the absolute value function it can be said to have a single direction: as we approach 0 from either side the tangent line becomes closer and closer to a vertical line; the curve is vertical at 0. But as before, if you imagine traveling along the curve, an abrupt change in direction is required at 0: a full 180 degree turn.

In practice we won't worry much about the distinction between these examples; in both cases the function has a "sharp point" where there is no tangent line and no derivative.

Exercises for Section 3.4

(1) Find the derivative of $y = f(x) = \sqrt{169 - x^2}$.

(2) Find the derivative of $y = f(t) = 80 - 4.9t^2$.

(3) Find the derivative of $y = f(x) = x^2 - (1/x)$.

(4) Find the derivative of $y = f(x) = ax^2 + bx + c$ (where a, b, and c are constants).

(5) Find the derivative of $y = f(x) = x^3$.

(6) Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.

(7) Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.

(8) Find the derivative of $y = f(x) = 2/\sqrt{2x+1}$

(9) Find the derivative of $y = g(t) = (2t-1)/(t+2)$

(10) Find an equation for the tangent line to the graph of $f(x) = 5 - x - 3x^2$ at the point $x = 2$

(11) Find a value for a so that the graph of $f(x) = x^2 + ax - 3$ has a horizontal tangent line at $x = 4$.

3.5 Adjectives For Functions

As we have defined it in Section 2.3, a function is a very general object. At this point, it is useful to introduce a collection of adjectives to describe certain kinds of functions; these adjectives name useful properties that functions may have. Consider the graphs of the functions in Figure **??**. It would clearly be useful to have words to help us describe the distinct features of each of them. We will point out and define a few adjectives (there are many more) for the functions pictured here. For the sake of the discussion, we will assume that the graphs do not exhibit any unusual behavior off-stage (i.e., outside the view of the graphs).

Functions. Each graph in Figure **??** certainly represents a function—since each passes the *vertical line test*. In other words, as you sweep a vertical line across the graph of each function, the line never intersects the graph more than once. If it did, then the graph would not represent a function.

Bounded. The graph in (c) appears to approach zero as x goes to both positive and negative infinity. It also never exceeds the value 1 or drops below the value 0. Because the graph never increases or decreases without bound, we say that the function represented by the graph in (c) is a **bounded** function.

Definition (Bounded) A function f is bounded if there is a number M such that $|f(x)| < M$ for every x in the domain of f.

For the function in (c), one such choice for M would be 10. However, the smallest (optimal) choice would be $M = 1$. In either case, simply finding an M is enough to establish boundedness. No such M exists for the hyperbola in (d) and hence we can say that it is **unbounded**.

Continuity. The graphs shown in (b) and (c) both represent **continuous** functions. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near $x = -1$ on the graph in (a) which is not continuous at that location.

Definition (Continuous at a Point) A function f is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$.

Definition (Continuous) A function f is continuous if it is continuous at every point in its domain.

Strangely, we can also say that (d) is continuous even though there is a vertical asymptote. A careful reading of the definition of continuous reveals the phrase "*at every point in its domain.*" Because the location of the asymptote, $x = 0$, is not in the domain of the function, and because the rest of the function is *well-behaved*, we can say that (d) is continuous.

Differentiability. Now that we have introduced the derivative of a function at a point, we can begin to use the adjective **differentiable**. We can see that the tangent line is well-defined at every point on the graph in (c). Therefore, we can say that (c) is a differentiable function.

Definition (Differentiable at a Point) A function f is differentiable at point a if $f'(a)$ exists.

Definition (Differentiable) A function f is differentiable if is differentiable at every point (excluding endpoints and isolated points in the domain of f) in the domain of f.

Take note that, for technical reasons not discussed here, both of these definitions exclude endpoints and isolated points in the domain from consideration.

We now have a collection of adjectives to describe the very rich and complex set of objects known as functions.

We close with a useful theorem about continuous functions:

Theorem (Intermediate Value Theorem) If f is continuous on the interval $[a,b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ such that $f(c) = d$.

This is most frequently used when $d = 0$.

Example Explain why the function $f = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

By theorem 3.3, f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in [0,1]$ such that $f(c) = 0$.

This example also points the way to a simple method for approximating roots.

Example Approximate the root of the previous example to one decimal place.

If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6) < 0$ and $f(0.7) > 0$, so by the Intermediate Value Theorem, f has a root between 0.6 and 0.7. Repeating the process with $f(0.61)$, $f(0.62)$, and so on, we find that $f(0.61) < 0$ and $f(0.62) > 0$, so f has a root between 0.61 and 0.62, and the root is 0.6 rounded to one decimal place.

Exercises for Section 3.5

(1) Along the lines of Figure **??**, for each part below sketch the graph of a function that is:

- a. bounded, but not continuous.
- b. differentiable and unbounded.
- c. continuous at $x = 0$, not continuous at $x = 1$, and bounded.
- d. differentiable everywhere except at $x = -1$, continuous, and unbounded.

(2) Is $f(x) = \sin(x)$ a bounded function? If so, find the smallest M.

(3) Is $s(t) = 1/(1 + t^2)$ a bounded function? If so, find the smallest M.

(4) Is $v(u) = 2\ln|u|$ a bounded function? If so, find the smallest M.

(5) Consider the function

$$h(x) = \begin{cases} 2x - 3, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x = 0$. Is h a continuous function?

(6) Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place.

(7) Approximate a root of $f = x^4 + x^3 - 5x + 1$ to one decimal place.

4

Rules for Finding Derivatives

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. Many functions involve quantities raised to a constant power, such as polynomials and more complicated combinations like $y = (\sin x)^4$. So we start by examining powers of a single variable; this gives us a building block for more complicated examples.

4.1 The Power Rule

We start with the derivative of a power function, $f(x) = x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, as in x^π. We have already computed some simple examples, so the formula should not be a complete surprise:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

It is not easy to show this is true for any n. We will do some of the easier cases now, and discuss the rest later.

The easiest, and most common, is the case that n is a positive integer. To compute the derivative we need to compute the following limit:

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}.$$

For a specific, fairly small value of n, we could do this by straightforward algebra.

Example Find the derivative of $f(x) = x^3$.

$$\begin{aligned}
\frac{d}{dx}x^3 &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}. \\
&= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 = 3x^2.
\end{aligned}$$

The general case is really not much harder as long as we don't try to do too much. The key is understanding what happens when $(x+\Delta x)^n$ is multiplied out:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + + a_{n-1} x\Delta x^{n-1} + \Delta x^n.$$

We know that multiplying out will give a large number of terms all of the form $x^i \Delta x^j$, and in fact that $i + j = n$ in every term. One way to see this is to understand that one method for multiplying out $(x+\Delta x)^n$ is the following: In every $(x+\Delta x)$ factor, pick either the x or the Δx, then multiply the n choices together; do this in all possible ways. For example, for $(x+\Delta x)^3$, there are eight possible ways to do this:

$$\begin{aligned}
(x+\Delta x)(x+\Delta x)(x+\Delta x) &= xxx + xx\Delta x + x\Delta x x + x\Delta x\Delta x \\
&\qquad + \Delta x xx + \Delta x x\Delta x + \Delta x\Delta x x + \Delta x\Delta x\Delta x \\
&= x^3 + x^2\Delta x + x^2\Delta x + x\Delta x^2 \\
&\quad + x^2\Delta x + x\Delta x^2 + x\Delta x^2 + \Delta x^3 \\
&= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3
\end{aligned}$$

No matter what n is, there are n ways to pick Δx in one factor and x in the remaining $n-1$ factors; this means one term is $nx^{n-1}\Delta x$. The other coefficients are somewhat harder to understand, but we don't really need them, so in the formula above they have simply been called a_2, a_3, and so on. We know that every one of these terms contains Δx to at least the power 2. Now let's look at the limit:

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{x^n + nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n - x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0} nx^{n-1} + a_2x^{n-2}\Delta x + \cdots + a_{n-1}x\Delta x^{n-2} + \Delta x^{n-1} = nx^{n-1}.\end{aligned}$$

Now without much trouble we can verify the formula for negative integers. First let's look at an example:

Example Find the derivative of $y = x^{-3}$. Using the formula, $y' = -3x^{-3-1} = -3x^{-4}$.

Here is the general computation. Suppose n is a negative integer; the algebra is easier to follow if we use $n = -m$ in the computation, where m is a positive integer.

$$
\begin{aligned}
\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^{-m} - x^{-m}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^m - (x+\Delta x)^m}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^m - (x^m + m x^{m-1}\Delta x + a_2 x^{m-2}\Delta x^2 + \cdots + a_{m-1} x \Delta x^{m-1} + \Delta x^m)}{(x+\Delta x)^m x^m \Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-m x^{m-1} - a_2 x^{m-2}\Delta x - \cdots - a_{m-1} x \Delta x^{m-2} - \Delta x^{m-1})}{(x+\Delta x)^m x^m} \\
&= \frac{-m x^{m-1}}{x^m x^m} = \frac{-m x^{m-1}}{x^{2m}} = -m x^{m-1-2m} = n x^{-m-1} = n x^{n-1}.
\end{aligned}
$$

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever n is any real number. Let's note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that $f(x) = 1$; remember that this "1" is a function, not "merely" a number, and that $f(x) = 1$ has a graph that is a horizontal line, with slope zero everywhere. So we know that $f'(x) = 0$. We might also write $f(x) = x^0$, though there is some question about just what this means at $x = 0$. If we apply the power rule, we get $f'(x) = 0x^{-1} = 0/x = 0$, again noting that there is a problem at $x = 0$. So the power rule "works" in this case, but it's really best to just remember that the derivative of any constant function is zero.

Exercises for Section 4.1

Find the derivatives of the given functions.

(1) x^{100}

(2) x^{-100}

(3) $\frac{1}{x^5}$

(4) x^{π}

(5) $x^{3/4}$

(6) $x^{-9/7}$

4.2 Linearity of the Derivative

An operation is linear if it behaves "nicely" with respect to multiplication by a constant and addition. The name comes from the equation of a line through the origin, $f(x) = mx$, and the following two properties of this equation. First, $f(cx) = m(cx) = c(mx) = cf(x)$, so the constant c can be "moved outside" or "moved through" the function f. Second, $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$, so the addition symbol likewise can be moved through the function.

The corresponding properties for the derivative are:

$$(cf(x))' = \frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = cf'(x),$$

and

$$(f(x)+g(x))' = \frac{d}{dx}(f(x)+g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x).$$

It is easy to see, or at least to believe, that these are true by thinking of the distance/speed interpretation of derivatives. If one object is at position $f(t)$ at time t, we know its speed is given by $f'(t)$. Suppose another object is at position $5f(t)$ at time t, namely, that it is always 5 times as far along the route as the first object. Then it "must" be going 5 times as fast at all times.

The second rule is somewhat more complicated, but here is one way to picture it. Suppose a flat bed railroad car is at position $f(t)$ at time t, so the car is traveling at a speed of $f'(t)$ (to be specific, let's say that $f(t)$ gives the position on the track of the rear end of the car). Suppose that an ant is crawling from the back of the car to the front so that its position *on the car* is $g(t)$ and its speed *relative to the car* is $g'(t)$. Then in reality, at time t, the ant is at position $f(t) + g(t)$ along the track, and its speed is "obviously" $f'(t) + g'(t)$.

We don't want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let's see how to verify these rules by computation. We'll do one and leave the other for the exercises.

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-(f(x)+g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&= f'(x)+g'(x)
\end{aligned}$$

This is sometimes called the **sum rule** for derivatives.

Example Find the derivative of $f(x)=x^5+5x^2$. We have to invoke linearity twice here:

$$f'(x)=\frac{d}{dx}(x^5+5x^2)=\frac{d}{dx}x^5+\frac{d}{dx}(5x^2)=5x^4+5\frac{d}{dx}(x^2)=5x^4+5\cdot 2x^1=5x^4+10x.$$

Because it is so easy with a little practice, we can usually combine all uses of linearity into a single step. The following example shows an acceptably detailed computation.

Example Find the derivative of $f(x)=3/x^4-2x^2+6x-7$.

$$f'(x)=\frac{d}{dx}\left(\frac{3}{x^4}-2x^2+6x-7\right)=\frac{d}{dx}(3x^{-4}-2x^2+6x-7)=-12x^{-5}-4x+6.$$

Exercises for Section 4.2

Find the derivatives of the functions in 1–6.

(1) $5x^3+12x^2-15$

(2) $-4x^5 + 3x^2 - 5/x^2$

(3) $5(-3x^2 + 5x + 1)$

(4) $f(x) + g(x)$, where $f(x) = x^2 - 3x + 2$ and $g(x) = 2x^3 - 5x$

(5) $(x+1)(x^2 + 2x - 3)$

(6) $\sqrt{625 - x^2} + 3x^3 + 12$ (See section 3.1.)

(7) Find an equation for the tangent line to $f(x) = x^3/4 - 1/x$ at $x = -2$.

(8) Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$.

(9) Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t.

(10) Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.

(11) The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P?

(12) Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$.

(13) Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

(14) Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.

4.3 The Product Rule

Consider the product of two simple functions, say $f(x) = (x^2+1)(x^3-3x)$. An obvious guess for the derivative of f is the product of the derivatives of the constituent functions: $(2x)(3x^2-3) = 6x^3-6x$. Is this correct? We can easily check, by rewriting f and doing the calculation in a way that is known to work. First, $f(x) = x^5-3x^3+x^3-3x = x^5-2x^3-3x$, and then $f'(x) = 5x^4-6x^2-3$. Not even close! What went "wrong"? Well, nothing really, except the guess was wrong.

So the derivative of $f(x)g(x)$ is NOT as simple as $f'(x)g'(x)$. Surely there is some rule for such a situation? There is, and it is instructive to "discover" it by trying to do the general calculation even without knowing the answer in advance.

$$\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)+f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}g(x)\\
&= f(x)g'(x)+f'(x)g(x)
\end{aligned}$$

A couple of items here need discussion. First, we used a standard trick, "add and subtract the same thing", to transform what we had into a more useful form. After some rewriting, we realize that we have two limits that produce $f'(x)$ and $g'(x)$. Of course, $f'(x)$ and $g'(x)$ must actually exist for this to make sense. We also replaced $\lim_{\Delta x\to 0} f(x+\Delta x)$ with $f(x)$—why is this justified?

What we really need to know here is that $\lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$, or in the language of section 3.5, that f is continuous at x. We already know that $f'(x)$ exists (or the whole approach, writing the derivative of fg in terms of f' and g', doesn't make sense). This turns out to imply that f is continuous as well. Here's why:

$$\begin{aligned}\lim_{\Delta x\to 0} f(x+\Delta x) &= \lim_{\Delta x\to 0}(f(x+\Delta x)-f(x)+f(x))\\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\Delta x+\lim_{\Delta x\to 0} f(x)\\ &= f'(x)\cdot 0+f(x)=f(x)\end{aligned}$$

To summarize: the product rule says that

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x)+f'(x)g(x).$$

Returning to the example we started with, let $f(x)=(x^2+1)(x^3-3x)$. Then $f'(x)=(x^2+1)(3x^2-3)+(2x)(x^3-3x)=3x^4-3x^2+3x^2-3+2x^4-6x^2=5x^4-6x^2-3$, as before. In this case it is probably simpler to multiply $f(x)$ out first, then compute the derivative; here's an example for which we really need the product rule.

Example Compute the derivative of $f(x)=x^2\sqrt{625-x^2}$. We have already computed $\frac{d}{dx}\sqrt{625-x^2}=\frac{-x}{\sqrt{625-x^2}}$. Now

$$f'(x)=x^2\frac{-x}{\sqrt{625-x^2}}+2x\sqrt{625-x^2}=\frac{-x^3+2x(625-x^2)}{\sqrt{625-x^2}}=\frac{-3x^3+1250x}{\sqrt{625-x^2}}.$$

Exercises for Section 4.3

In 1–4, find the derivatives of the functions using the product rule.

(1) $x^3(x^3-5x+10)$

(2) $(x^2+5x-3)(x^5-6x^3+3x^2-7x+1)$

(3) $\sqrt{x}\sqrt{625-x^2}$

(4) $\frac{\sqrt{625-x^2}}{x^{20}}$

(5) Use the product rule to compute the derivative of $f(x) = (2x-3)^2$. Sketch the function. Find an equation of the tangent line to the curve at $x = 2$. Sketch the tangent line at $x = 2$.

(6) Suppose that f, g, and h are differentiable functions. Show that $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

(7) State and prove a rule to compute $(fghi)'(x)$, similar to the rule in the previous problem.

Remark: Product notation

Suppose $f_1, f_2, \ldots f_n$ are functions. The product of all these functions can be written

$$\prod_{k=1}^{n} f_k.$$

This is similar to the use of $\sum$ to denote a sum. For example,

$$\prod_{k=1}^{5} f_k = f_1 f_2 f_3 f_4 f_5$$

and

$$\prod_{k=1}^{n} k = 1 \cdot 2 \cdot \ldots \cdot n = n!.$$

We sometimes use somewhat more complicated conditions; for example

$$\prod_{k=1, k\neq j}^{n} f_k$$

denotes the product of f_1 through f_n except for f_j. For example,

$$\prod_{k=1, k\neq 4}^{5} x^k = x \cdot x^2 \cdot x^3 \cdot x^5 = x^{11}.$$

(8) The **generalized product rule** says that if $f_1, f_2, \ldots, f_n$ are differentiable functions at x then

$$\frac{d}{dx}\prod_{k=1}^{n} f_k(x) = \sum_{j=1}^{n}\left(f_j'(x)\prod_{k=1,k\neq j}^{n} f_k(x)\right).$$

Verify that this is the same as your answer to the previous problem when $n = 4$, and write out what this says when $n = 5$.

4.4 The Quotient Rule

What is the derivative of $(x^2+1)/(x^3-3x)$? More generally, we'd like to have a formula to compute the derivative of $f(x)/g(x)$ if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x)\cdot(1/g(x))$, that is, this is "really" a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. So really the only new bit of information we need is $(1/g(x))'$ in terms of $g'(x)$. As with the product rule, let's set this up and see how far we can get:

$$\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{\Delta x\to 0}\frac{\frac{1}{g(x+\Delta x)}-\frac{1}{g(x)}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}\\
&= \lim_{\Delta x\to 0}-\frac{g(x+\Delta x)-g(x)}{\Delta x}\frac{1}{g(x+\Delta x)g(x)}\\
&= -\frac{g'(x)}{g(x)^2}
\end{aligned}$$

Now we can put this together with the product rule:

$$\frac{d}{dx}\frac{f(x)}{g(x)} = f(x)\frac{-g'(x)}{g(x)^2}+f'(x)\frac{1}{g(x)} = \frac{-f(x)g'(x)+f'(x)g(x)}{g(x)^2} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}.$$

Example Compute the derivative of $(x^2+1)/(x^3-3x)$.

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x} = \frac{2x(x^3-3x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2} = \frac{-x^4-6x^2+3}{(x^3-3x)^2}.$$

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

Example Find the derivative of $\sqrt{625-x^2}/\sqrt{x}$ in two ways: using the quotient rule, and using the product rule.

Quotient rule:

$$\frac{d}{dx}\frac{\sqrt{625-x^2}}{\sqrt{x}} = \frac{\sqrt{x}(-x/\sqrt{625-x^2}) - \sqrt{625-x^2}\cdot 1/(2\sqrt{x})}{x}.$$

Note that we have used $\sqrt{x} = x^{1/2}$ to compute the derivative of $\sqrt{x}$ by the power rule.

Product rule:

$$\frac{d}{dx}\sqrt{625-x^2}x^{-1/2} = \sqrt{625-x^2}\frac{-1}{2}x^{-3/2} + \frac{-x}{\sqrt{625-x^2}}x^{-1/2}.$$

With a bit of algebra, both of these simplify to

$$-\frac{x^2+625}{2\sqrt{625-x^2}x^{3/2}}.$$

Occasionally you will need to compute the derivative of a quotient with a constant numerator, like $10/x^2$. Of course you can use the quotient rule, but it is usually not the easiest method. If we do use it here, we get

$$\frac{d}{dx}\frac{10}{x^2} = \frac{x^2\cdot 0 - 10\cdot 2x}{x^4} = \frac{-20}{x^3},$$

since the derivative of 10 is 0. But it is simpler to do this:

$$\frac{d}{dx}\frac{10}{x^2} = \frac{d}{dx}10x^{-2} = -20x^{-3}.$$

Admittedly, x^2 is a particularly simple denominator, but we will see that a similar calculation is usually possible. Another approach is to remember that

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2},$$

but this requires extra memorization. Using this formula,

$$\frac{d}{dx}\frac{10}{x^2} = 10\frac{-2x}{x^4}.$$

Note that we first use linearity of the derivative to pull the 10 out in front.

Exercises for Section 4.4

Find the derivatives of the functions in 1–4 using the quotient rule.

(1) $\dfrac{x^3}{x^3 - 5x + 10}$

(2) $\dfrac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1}$

(3) $\dfrac{\sqrt{x}}{\sqrt{625 - x^2}}$

(4) $\dfrac{\sqrt{625 - x^2}}{x^{20}}$

(5) Find an equation for the tangent line to $f(x) = (x^2 - 4)/(5 - x)$ at $x = 3$.

(6) Find an equation for the tangent line to $f(x) = (x - 2)/(x^3 + 4x - 1)$ at $x = 1$.

(7) Let P be a polynomial of degree n and let Q be a polynomial of degree m (with Q not the zero polynomial). Using sigma notation we can write

$$P = \sum_{k=0}^{n} a_k x^k, \qquad Q = \sum_{k=0}^{m} b_k x^k.$$

Use sigma notation to write the derivative of the **rational function** P/Q.

(8) The curve $y = 1/(1 + x^2)$ is an example of a class of curves each of which is called a **witch of Agnesi**. Sketch the curve and find the tangent line to the curve at $x = 5$. (The word *witch* here is a mistranslation of the original Italian, as described at

(9) If $f'(4) = 5$, $g'(4) = 12$, $(fg)(4) = f(4)g(4) = 2$, and $g(4) = 6$, compute $f(4)$ and $\dfrac{d}{dx}\dfrac{f}{g}$ at 4.

4.5 The Chain Rule

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine simple functions to make more complicated functions: function composition, as discussed in section 3.3. For example, consider $\sqrt{625-x^2}$. This function has many simpler components, like 625 and x^2, and then there is that square root symbol, so the square root function $\sqrt{x} = x^{1/2}$ is involved. The obvious question is: can we compute the derivative using the derivatives of the constituents $625-x^2$ and $\sqrt{x}$? We can indeed. In general, if $f(x)$ and $g(x)$ are functions, we can compute the derivatives of $f(g(x))$ and $g(f(x))$ in terms of $f'(x)$ and $g'(x)$.

> **Example** Form the two possible compositions of $f(x) = \sqrt{x}$ and $g(x) = 625 - x^2$ and compute the derivatives. First, $f(g(x)) = \sqrt{625-x^2}$, and the derivative is $-x/\sqrt{625-x^2}$ as we have seen. Second, $g(f(x)) = 625-(\sqrt{x})^2 = 625-x$ with derivative -1. Of course, these calculations do not use anything new, and in particular the derivative of $f(g(x))$ was somewhat tedious to compute from the definition.

Suppose we want the derivative of $f(g(x))$. Again, let's set up the derivative and play some algebraic tricks:

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{\Delta x\to 0}\frac{f(g(x+\Delta x))-f(g(x))}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{f(g(x+\Delta x))-f(g(x))}{g(x+\Delta x))-g(x)}\frac{g(x+\Delta x))-g(x)}{\Delta x}\end{aligned}$$

Now we see immediately that the second fraction turns into $g'(x)$ when we take the limit. The first fraction is more complicated, but it too looks something like a derivative. The denominator, $g(x+\Delta x))-g(x)$, is a change in the value of g, so let's abbreviate it as $\Delta g = g(x+\Delta x))-g(x)$, which also means $g(x+\Delta x) = g(x)+\Delta g$. This gives us

$$\lim_{\Delta x\to 0}\frac{f(g(x)+\Delta g)-f(g(x))}{\Delta g}.$$

As Δx goes to 0, it is also true that Δg goes to 0, because $g(x+\Delta x)$ goes to $g(x)$. So we can rewrite this limit as

$$\lim_{\Delta g \to 0} \frac{f(g(x)+\Delta g) - f(g(x))}{\Delta g}.$$

Now this looks exactly like a derivative, namely $f'(g(x))$, that is, the function $f'(x)$ with x replaced by $g(x)$. If this all withstands scrutiny, we then get

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Unfortunately, there is a small flaw in the argument. Recall that what we mean by $\lim_{\Delta x \to 0}$ involves what happens when Δx is close to 0 *but not equal to 0.* The qualification is very important, since we must be able to divide by Δx. But when Δx is close to 0 but not equal to 0, $\Delta g = g(x+\Delta x)) - g(x)$ is close to 0 *and possibly equal to 0.* This means it doesn't really make sense to divide by Δg. Fortunately, it is possible to recast the argument to avoid this difficulty, but it is a bit tricky; we will not include the details, which can be found in many calculus books. Note that many functions g do have the property that $g(x+\Delta x) - g(x) \neq 0$ when Δx is small, and for these functions the argument above is fine.

The chain rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity $f'(g(x))$ is the derivative of f with x replaced by g; this can be written df/dg. As usual, $g'(x) = dg/dx$. Then the chain rule becomes

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

This looks like trivial arithmetic, but it is not: dg/dx is not a fraction, that is, not literal division, but a single symbol that means $g'(x)$. Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

Example Compute the derivative of $\sqrt{625-x^2}$. We already know that the answer is $-x/\sqrt{625-x^2}$, computed directly from the limit. In the context of the chain rule, we have $f(x) = \sqrt{x}$, $g(x) = 625 - x^2$. We know that $f'(x) =$

$(1/2)x^{-1/2}$, so $f'(g(x)) = (1/2)(625-x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x) = -2x$ we have

$$f'(g(x))g'(x) = \frac{1}{2\sqrt{625-x^2}}(-2x) = \frac{-x}{\sqrt{625-x^2}}.$$

Example Compute the derivative of $1/\sqrt{625-x^2}$. This is a quotient with a constant numerator, so we could use the quotient rule, but it is simpler to use the chain rule. The function is $(625-x^2)^{-1/2}$, the composition of $f(x) = x^{-1/2}$ and $g(x) = 625-x^2$. We compute $f'(x) = (-1/2)x^{-3/2}$ using the power rule, and then

$$f'(g(x))g'(x) = \frac{-1}{2(625-x^2)^{3/2}}(-2x) = \frac{x}{(625-x^2)^{3/2}}.$$

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

Example Compute the derivative of

$$f(x) = \frac{x^2-1}{x\sqrt{x^2+1}}.$$

The "last" operation here is division, so to get started we need to use the quotient rule first. This gives

$$\begin{aligned} f'(x) &= \frac{(x^2-1)'x\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)} \\ &= \frac{2x^2\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \end{aligned}$$

Now we need to compute the derivative of $x\sqrt{x^2+1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}x\sqrt{x^2+1} = x\frac{d}{dx}\sqrt{x^2+1} + \sqrt{x^2+1}.$$

Finally, we use the chain rule:

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(x^2+1)^{1/2} = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

And putting it all together:

$$\begin{aligned} f'(x) &= \frac{2x^2\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}. \\ &= \frac{2x^2\sqrt{x^2+1} - (x^2-1)\left(x\frac{x}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right)}{x^2(x^2+1)}. \end{aligned}$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left.

Example Compute the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$. Here we have a more complicated chain of compositions, so we use the chain rule twice. At the outermost "layer" we have the function $g(x) = 1+\sqrt{1+\sqrt{x}}$ plugged into $f(x) = \sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{d}{dx}\left(1+\sqrt{1+\sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1+\sqrt{x}}$. Using the chain rule again:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\begin{aligned} \frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} &= \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2}\frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2}\frac{1}{2}x^{-1/2}. \\ &= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}} \end{aligned}$$

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

Example Compute the derivative of $f(x) = \dfrac{x^3}{x^2+1}$. Write $f(x) = x^3(x^2+1)^{-1}$, then

$$\begin{aligned}
f'(x) &= x^3 \frac{d}{dx}(x^2+1)^{-1} + 3x^2(x^2+1)^{-1} \\
&= x^3(-1)(x^2+1)^{-2}(2x) + 3x^2(x^2+1)^{-1} \\
&= -2x^4(x^2+1)^{-2} + 3x^2(x^2+1)^{-1} \\
&= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2}{x^2+1} \\
&= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2(x^2+1)}{(x^2+1)^2} \\
&= \frac{-2x^4+3x^4+3x^2}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}
\end{aligned}$$

Note that we already had the derivative on the second line; all the rest is simplification. It is easier to get to this answer by using the quotient rule, so there's a trade off: more work for fewer memorized formulas.

Exercises for Section 4.5

Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

(1) $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$

(2) $x^3 - 2x^2 + 4\sqrt{x}$

(3) $(x^2+1)^3$

(4) $x\sqrt{169-x^2}$

(5) $(x^2 - 4x + 5)\sqrt{25-x^2}$

(6) $\sqrt{r^2-x^2}$, r is a constant

(7) $\sqrt{1+x^4}$

(8) $\dfrac{1}{\sqrt{5-\sqrt{x}}}$.

(9) $(1+3x)^2$

(10) $\dfrac{(x^2+x+1)}{(1-x)}$

(11) $\dfrac{\sqrt{25-x^2}}{x}$

(12) $\sqrt{\dfrac{169}{x}-x}$

(13) $\sqrt{x^3-x^2-(1/x)}$

(14) $100/(100-x^2)^{3/2}$

(15) $\sqrt[3]{x+x^3}$

(16) $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$

(17) $(x+8)^5$

(18) $(4-x)^3$

(19) $(x^2+5)^3$

(20) $(6-2x^2)^3$

(21) $(1-4x^3)^{-2}$

(22) $5(x+1-1/x)$

(23) $4(2x^2-x+3)^{-2}$

(24) $\dfrac{1}{1+1/x}$

(25) $\dfrac{-3}{4x^2-2x+1}$

(26) $(x^2+1)(5-2x)/2$

(27) $(3x^2+1)(2x-4)^3$

(28) $\dfrac{x+1}{x-1}$

(29) $\dfrac{x^2-1}{x^2+1}$

(30) $\dfrac{(x-1)(x-2)}{x-3}$

(31) $\dfrac{2x^{-1}-x^{-2}}{3x^{-1}-4x^{-2}}$

(32) $3(x^2+1)(2x^2-1)(2x+3)$

(33) $\dfrac{1}{(2x+1)(x-3)}$

(34) $((2x+1)^{-1}+3)^{-1}$

(35) $(2x+1)^3(x^2+1)^2$

(36) Find an equation for the tangent line to $f(x)=(x-2)^{1/3}/(x^3+4x-1)^2$ at $x=1$.

(37) Find an equation for the tangent line to $y=9x^{-2}$ at $(3,1)$.

(38) Find an equation for the tangent line to $(x^2-4x+5)\sqrt{25-x^2}$ at $(3,8)$.

(39) Find an equation for the tangent line to $\dfrac{(x^2+x+1)}{(1-x)}$ at $(2,-7)$.

(40) Find an equation for the tangent line to $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ at $(1,\sqrt{4+\sqrt{5}})$.

5

Transcendental Functions

So far we have used only algebraic functions as examples when finding derivatives, that is, functions that can be built up by the usual algebraic operations of addition, subtraction, multiplication, division, and raising to constant powers. Both in theory and practice there are other functions, called transcendental, that are very useful. Most important among these are the trigonometric functions, the inverse trigonometric functions, exponential functions, and logarithms.

5.1 Trigonometric Functions

When you first encountered the trigonometric functions it was probably in the context of "triangle trigonometry," defining, for example, the sine of an angle as the "side opposite over the hypotenuse." While this will still be useful in an informal way, we need to use a more expansive definition of the trigonometric functions. First an important note: while degree measure of angles is sometimes convenient because it is so familiar, it turns out to be ill-suited to mathematical calculation, so (almost) everything we do will be in terms of **radian measure** of angles.

To define the radian measurement system, we consider the unit circle in the xy-plane:

An angle, x, at the center of the circle is associated with an arc of the circle which is said to **subtend** the angle. In the figure, this arc is the portion of the circle from point $(1,0)$ to point A. The length of this arc is the radian measure of the angle x; the fact that the radian measure is an actual geometric length is largely responsible for the usefulness of radian measure. The circumference of the unit circle is $2\pi r = 2\pi(1) = 2\pi$, so the radian measure of the full circular angle (that is, of the 360 degree angle) is 2π.

While an angle with a particular measure can appear anywhere around the circle, we need a fixed, conventional location so that we can use the coordinate system to define properties of the angle. The standard convention is to place the starting radius for the angle on the positive x-axis, and to measure positive angles counterclockwise around the circle. In the figure, x is the standard location of the angle $\pi/6$, that is, the length of the arc from $(1,0)$ to A is $\pi/6$. The angle y in the picture is $-\pi/6$, because the distance from $(1,0)$ to B along the circle is also $\pi/6$, but in a clockwise direction.

Now the fundamental trigonometric definitions are: the cosine of x and the sine of x are the first and second coordinates of the point A, as indicated in the figure. The angle x shown can be viewed as an angle of a right triangle, meaning the usual triangle definitions of the sine and cosine also make sense. Since the hypotenuse of the triangle is 1, the "side opposite over hypotenuse" definition of the sine is the second coordinate of point A over 1, which is just the second coordinate; in other words, both methods give the same value for the sine.

The simple triangle definitions work only for angles that can "fit" in a right triangle, namely, angles between 0 and $\pi/2$. The coordinate definitions, on the other hand, apply to any angles, as indicated in this figure:

The angle x is subtended by the heavy arc in the figure, that is, $x = 7\pi/6$. Both coordinates of point A in this figure are negative, so the sine and cosine of $7\pi/6$ are both negative.

The remaining trigonometric functions can be most easily defined in terms of the sine and cosine, as usual:

$$\tan x = \frac{\sin x}{\cos x}$$
$$\cot x = \frac{\cos x}{\sin x}$$
$$\sec x = \frac{1}{\cos x}$$
$$\csc x = \frac{1}{\sin x}$$

and they can also be defined as the corresponding ratios of coordinates.

Although the trigonometric functions are defined in terms of the unit circle, the unit circle diagram is not what we normally consider the graph of a trigonometric function. (The unit circle is the graph of, well, the circle.) We can easily get a qualitatively correct idea of the graphs of the trigonometric functions from the unit circle diagram. Consider the sine function, $y = \sin x$. As x increases from 0 in the unit circle diagram, the second coordinate of the point A goes from 0 to a maximum of 1, then back to 0, then to a minimum of -1, then back to 0, and then it obviously repeats itself. So the graph of $y = \sin x$ must look something like this:

Similarly, as angle x increases from 0 in the unit circle diagram, the first coordinate of the point A goes from 1 to 0 then to -1, back to 0 and back to 1, so the graph of $y = \cos x$ must look something like this:

Exercises for Section 5.1

Some useful trigonometric identities are in appendix **??**.

(1) Find all values of θ such that $\sin(\theta) = -1$; give your answer in radians.

(2) Find all values of θ such that $\cos(2\theta) = 1/2$; give your answer in radians.

(3) Use an angle sum identity to compute $\cos(\pi/12)$.

(4) Use an angle sum identity to compute $\tan(5\pi/12)$.

(5) Verify the identity $\cos^2(t)/(1 - \sin(t)) = 1 + \sin(t)$.

(6) Verify the identity $2\csc(2\theta) = \sec(\theta)\csc(\theta)$.

(7) Verify the identity $\sin(3\theta) - \sin(\theta) = 2\cos(2\theta)\sin(\theta)$.

(8) Sketch $y = 2\sin(x)$.

(9) Sketch $y = \sin(3x)$.

(10) Sketch $y = \sin(-x)$.

(11) Find all of the solutions of $2\sin(t) - 1 - \sin^2(t) = 0$ in the interval $[0, 2\pi]$.

5.2 The Derivative of $\sin x$

The Derivative of the sine What about the derivative of the sine function? The rules for derivatives that we have are no help, since $\sin x$ is not an algebraic function. We need to return to the definition of the derivative, set up a limit, and try to compute it. Here's the definition:

$$\frac{d}{dx}\sin x = \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}.$$

Using some trigonometric identities, we can make a little progress on the quotient:

$$\begin{aligned}\frac{\sin(x+\Delta x)-\sin x}{\Delta x} &= \frac{\sin x\cos\Delta x+\sin\Delta x\cos x-\sin x}{\Delta x}\\ &= \sin x\frac{\cos\Delta x-1}{\Delta x}+\cos x\frac{\sin\Delta x}{\Delta x}.\end{aligned}$$

This isolates the difficult bits in the two limits

$$\lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x} \quad\text{and}\quad \lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}.$$

Here we get a little lucky: it turns out that once we know the second limit the first is quite easy. The second is quite tricky, however. Indeed, it is the hardest limit we will actually compute, and we devote a section to it.

5.3 A hard limit

We want to compute this limit:

$$\lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x}.$$

Equivalently, to make the notation a bit simpler, we can compute

$$\lim_{x\to 0}\frac{\sin x}{x}.$$

In the original context we need to keep x and Δx separate, but here it doesn't hurt to rename Δx to something more convenient.

To do this we need to be quite clever, and to employ some indirect reasoning. The indirect reasoning is embodied in a theorem, frequently called the **squeeze theorem.**

Theorem (Squeeze Theorem) Suppose that $g(x) \le f(x) \le h(x)$ for all x close to a but not equal to a. If $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$, then $\lim_{x\to a} f(x) = L$.

This theorem can be proved using the official definition of limit. We won't prove it here, but point out that it is easy to understand and believe graphically. The condition says that $f(x)$ is trapped between $g(x)$ below and $h(x)$ above, and that at $x = a$, both g and h approach the same value. This means the situation looks something like figure **??**. The wiggly curve is $x^2 \sin(\pi/x)$, the upper and lower curves are x^2 and $-x^2$. Since the sine function is always between -1 and 1, $-x^2 \le x^2 \sin(\pi/x) \le x^2$, and it is easy to see that $\lim_{x\to 0} -x^2 = 0 = \lim_{x\to 0} x^2$. It is not so easy to see directly, that is algebraically, that $\lim_{x\to 0} x^2 \sin(\pi/x) = 0$, because the π/x prevents us from simply plugging in $x = 0$. The squeeze theorem makes this "hard limit" as easy as the trivial limits involving x^2.

To do the hard limit that we want, $\lim_{x\to 0}(\sin x)/x$, we will find two simpler functions g and h so that $g(x) \le (\sin x)/x \le h(x)$, and so that $\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x)$. Not too surprisingly, this will require some trigonometry and geometry. Referring to figure **??**, x is the measure of the angle in radians. Since the circle has radius 1, the coordinates of point A are $(\cos x, \sin x)$, and the area of the small triangle is $(\cos x \sin x)/2$. This triangle is completely contained within the circular wedge-shaped region bordered by two lines and the circle from $(1,0)$ to point A. Comparing the areas of the triangle and the wedge we see $(\cos x \sin x)/2 \le x/2$, since the area of a circular region with angle θ and radius r is $\theta r^2/2$. With a little algebra this turns into $(\sin x)/x \le 1/\cos x$, giving us the h we seek.

To find g, we note that the circular wedge is completely contained inside the larger triangle. The height of the triangle, from $(1,0)$ to point B, is $\tan x$, so comparing areas we get $x/2 \le (\tan x)/2 = \sin x/(2\cos x)$. With a little algebra this becomes $\cos x \le (\sin x)/x$. So now we have

$$\cos x \le \frac{\sin x}{x} \le \frac{1}{\cos x}.$$

Finally, the two limits $\lim_{x\to 0}\cos x$ and $\lim_{x\to 0} 1/\cos x$ are easy, because $\cos(0) = 1$. By the squeeze theorem, $\lim_{x\to 0}(\sin x)/x = 1$ as well.

Before we can complete the calculation of the derivative of the sine, we need one other limit:

$$\lim_{x\to 0}\frac{\cos x - 1}{x}.$$

This limit is just as hard as $\sin x/x$, but closely related to it, so that we don't have to a similar calculation; instead we can do a bit of tricky algebra.

$$\frac{\cos x - 1}{x} = \frac{\cos x - 1}{x}\frac{\cos x + 1}{\cos x + 1} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = \frac{-\sin^2 x}{x(\cos x + 1)} = -\frac{\sin x}{x}\frac{\sin x}{\cos x + 1}.$$

To compute the desired limit it is sufficient to compute the limits of the two final fractions, as x goes to 0. The first of these is the hard limit we've just done, namely 1. The second turns out to be simple, because the denominator presents no problem:

$$\lim_{x\to 0}\frac{\sin x}{\cos x + 1} = \frac{\sin 0}{\cos 0 + 1} = \frac{0}{2} = 0.$$

Thus,

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Exercises for Section 5.3

(1) Compute $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{x}$

(2) Compute $\displaystyle\lim_{x\to 0}\frac{\sin(7x)}{\sin(2x)}$

(3) Compute $\displaystyle\lim_{x\to 0}\frac{\cot(4x)}{\csc(3x)}$

(4) Compute $\displaystyle\lim_{x\to 0}\frac{\tan x}{x}$

(5) Compute $\displaystyle\lim_{x\to \pi/4}\frac{\sin x - \cos x}{\cos(2x)}$

(6) For all $x \geq 0$, $4x - 9 \leq f(x) \leq x^2 - 4x + 7$. Find $\displaystyle\lim_{x\to 4} f(x)$.

(7) For all x, $2x \leq g(x) \leq x^4 - x^2 + 2$. Find $\displaystyle\lim_{x\to 1} g(x)$.

(8) Use the Squeeze Theorem to show that $\displaystyle\lim_{x\to 0} x^4\cos(2/x) = 0$.

5.4 The Derivative of $\sin x$, continued

The Derivative of the sine, continued Now we can complete the calculation of the derivative of the sine:

$$\begin{aligned}\frac{d}{dx}\sin x &= \lim_{\Delta x\to 0}\frac{\sin(x+\Delta x)-\sin x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\sin x\frac{\cos\Delta x-1}{\Delta x}+\cos x\frac{\sin\Delta x}{\Delta x}\\ &= \sin x\cdot 0+\cos x\cdot 1=\cos x.\end{aligned}$$

The derivative of a function measures the slope or steepness of the function; if we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true: Notice that where the cosine is zero the sine does appear to have a horizontal tangent line, and that the sine appears to be steepest where the cosine takes on its extreme values of 1 and -1.

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

Example Compute the derivative of $\sin(x^2)$.

$$\frac{d}{dx}\sin(x^2)=\cos(x^2)\cdot 2x=2x\cos(x^2).$$

Example Compute the derivative of $\sin^2(x^3-5x)$.

$$\begin{aligned}\frac{d}{dx}\sin^2(x^3-5x) &= \frac{d}{dx}(\sin(x^3-5x))^2\\ &= 2(\sin(x^3-5x))^1\cos(x^3-5x)(3x^2-5)\\ &= 2(3x^2-5)\cos(x^3-5x)\sin(x^3-5x).\end{aligned}$$

Exercises for Section 5.4

Find the derivatives of the following functions.

(1) $\sin^2(\sqrt{x})$

(2) $\sqrt{x}\sin x$

(3) $\dfrac{1}{\sin x}$

(4) $\dfrac{x^2+x}{\sin x}$

(5) $\sqrt{1-\sin^2 x}$

5.5 Derivatives of the Trigonometric Functions

All of the other trigonometric functions can be expressed in terms of the sine, and so their derivatives can easily be calculated using the rules we already have. For the cosine we need to use two identities,

$$\cos x = \sin(x + \frac{\pi}{2}),$$
$$\sin x = -\cos(x + \frac{\pi}{2}).$$

Now:

$$\frac{d}{dx}\cos x = \frac{d}{dx}\sin(x + \frac{\pi}{2}) = \cos(x + \frac{\pi}{2}) \cdot 1 = -\sin x$$
$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$
$$\frac{d}{dx}\sec x = \frac{d}{dx}(\cos x)^{-1} = -1(\cos x)^{-2}(-\sin x) = \frac{\sin x}{\cos^2 x} = \sec x \tan x$$

The derivatives of the cotangent and cosecant are similar and left as exercises.

Exercises for Section 5.5

Find the derivatives of the following functions.

(1) $\sin x \cos x$

(2) $\sin(\cos x)$

(3) $\sqrt{x \tan x}$

(4) $\tan x/(1 + \sin x)$

(5) $\cot x$

(6) $\csc x$

(7) $x^3 \sin(23x^2)$

(8) $\sin^2 x + \cos^2 x$

(9) $\sin(\cos(6x))$

(10) Compute $\frac{d}{d\theta}\frac{\sec\theta}{1 + \sec\theta}$.

(11) Compute $\dfrac{d}{dt}t^5\cos(6t)$.

(12) Compute $\dfrac{d}{dt}\dfrac{t^3\sin(3t)}{\cos(2t)}$.

(13) Find all points on the graph of $f(x) = \sin^2(x)$ at which the tangent line is horizontal.

(14) Find all points on the graph of $f(x) = 2\sin(x) - \sin^2(x)$ at which the tangent line is horizontal.

(15) Find an equation for the tangent line to $\sin^2(x)$ at $x = \pi/3$.

(16) Find an equation for the tangent line to $\sec^2 x$ at $x = \pi/3$.

(17) Find an equation for the tangent line to $\cos^2 x - \sin^2(4x)$ at $x = \pi/6$.

(18) Find the points on the curve $y = x + 2\cos x$ that have a horizontal tangent line.

(19) Let C be a circle of radius r. Let A be an arc on C subtending a central angle θ. Let B be the chord of C whose endpoints are the endpoints of A. (Hence, B also subtends θ.) Let s be the length of A and let d be the length of B. Sketch a diagram of the situation and compute $\lim\limits_{\theta\to 0^+} s/d$.

5.6 Exponential and Logarithmic functions

An exponential function has the form a^x, where a is a constant; examples are 2^x, 10^x, e^x. The logarithmic functions are the **inverses** of the exponential functions, that is, functions that "undo" the exponential functions, just as, for example, the cube root function "undoes" the cube function: $\sqrt[3]{2^3} = 2$. Note that the original function also undoes the inverse function: $(\sqrt[3]{8})^3 = 8$.

Let $f(x) = 2^x$. The inverse of this function is called the logarithm base 2, denoted $\log_2(x)$ or (especially in computer science circles) $\lg(x)$. What does this really mean? The logarithm must undo the action of the exponential function, so for example it must be that $\lg(2^3) = 3$—starting with 3, the exponential function produces $2^3 = 8$, and the logarithm of 8 must get us back to 3. A little thought shows that it is not a coincidence that $\lg(2^3)$ simply gives the exponent—the exponent *is* the original value that we must get back to. In other words, *the logarithm is the exponent.* Remember this catchphrase, and what it means, and you won't go wrong. (You *do* have to remember what it means. Like any good mnemonic, "the logarithm is the exponent" leaves out a lot of detail, like "Which exponent?" and "Exponent of what?")

Example What is the value of $\log_{10}(1000)$? The "10" tells us the appropriate number to use for the base of the exponential function. The logarithm is the exponent, so the question is, what exponent E makes $10^E = 1000$? If we can find such an E, then $\log_{10}(1000) = \log_{10}(10^E) = E$; finding the appropriate exponent is the same as finding the logarithm. In this case, of course, it is easy: $E = 3$ so $\log_{10}(1000) = 3$.

Let's review some laws of exponents and logarithms; let a be a positive number. Since $a^5 = a{\cdot}a{\cdot}a{\cdot}a{\cdot}a$ and $a^3 = a{\cdot}a{\cdot}a$, it's clear that $a^5{\cdot}a^3 = a{\cdot}a{\cdot}a{\cdot}a{\cdot}a{\cdot}a{\cdot}a{\cdot}a = a^8 = a^{5+3}$, and in general that $a^m a^n = a^{m+n}$. Since "the logarithm is the exponent," it's no surprise that this translates directly into a fact about the logarithm function. Here are three facts from the example: $\log_a(a^5) = 5$, $\log_a(a^3) = 3$, $\log_a(a^8) = 8$. So $\log_a(a^5 a^3) = \log_a(a^8) = 8 = 5 + 3 = \log_a(a^5) + \log_a(a^3)$. Now let's make this a bit more general. Suppose A and B are two numbers, $A = a^x$, and $B = a^y$. Then $\log_a(AB) = \log_a(a^x a^y) = \log_a(a^{x+y}) = x + y = \log_a(A) + \log_a(B)$.

Now consider $(a^5)^3 = a^5 \cdot a^5 \cdot a^5 = a^{5+5+5} = a^{5\cdot 3} = a^{15}$. Again it's clear that more generally $(a^m)^n = a^{mn}$, and again this gives us a fact about logarithms. If $A = a^x$ then $A^y = (a^x)^y = a^{xy}$, so $\log_a(A^y) = xy = y\log_a(A)$—the exponent can be "pulled out in front."

We have cheated a bit in the previous two paragraphs. It is obvious that $a^5 = a \cdot a \cdot a \cdot a \cdot a$ and $a^3 = a \cdot a \cdot a$ and that the rest of the example follows; likewise for the second example. But when we consider an exponential function a^x we can't be limited to substituting integers for x. What does $a^{2.5}$ or $a^{-1.3}$ or a^π mean? And is it really true that $a^{2.5}a^{-1.3} = a^{2.5-1.3}$? The answer to the first question is actually quite difficult, so we will evade it; the answer to the second question is "yes."

We'll evade the full answer to the hard question, but we have to know something about exponential functions. You need first to understand that since it's not "obvious" what 2^x should mean, we are really free to make it mean whatever we want, so long as we keep the behavior that *is* obvious, namely, when x is a positive integer. What else do we want to be true about 2^x? We want the properties of the previous two paragraphs to be true for all exponents: $2^x 2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$.

After the positive integers, the next easiest number to understand is 0: $2^0 = 1$. You have presumably learned this fact in the past; why is it true? It is true precisely because we want $2^a 2^b = 2^{a+b}$ to be true about the function 2^x. We need it to be true that $2^0 2^x = 2^{0+x} = 2^x$, and this only works if $2^0 = 1$. The same argument implies that $a^0 = 1$ for any a.

The next easiest set of numbers to understand is the negative integers: for example, $2^{-3} = 1/2^3$. We know that whatever 2^{-3} means it must be that $2^{-3}2^3 = 2^{-3+3} = 2^0 = 1$, which means that 2^{-3} must be $1/2^3$. In fact, by the same argument, once we know what 2^x means for some value of x, 2^{-x} must be $1/2^x$ and more generally $a^{-x} = 1/a^x$.

Next, consider an exponent $1/q$, where q is a positive integer. We want it to be true that $(2^x)^y = 2^{xy}$, so $(2^{1/q})^q = 2$. This means that $2^{1/q}$ is a q-th root of 2, $2^{1/q} = \sqrt[q]{2}$. This is all we need to understand that $2^{p/q} = (2^{1/q})^p = (\sqrt[q]{2})^p$ and $a^{p/q} = (a^{1/q})^p = (\sqrt[q]{a})^p$.

What's left is the hard part: what does 2^x mean when x cannot be written as a fraction, like $x = \sqrt{2}$ or $x = \pi$? What we know so far is how to assign meaning to 2^x whenever $x = p/q$; if we were to graph this we'd see something like this: But this is a poor picture, because you can't see that the "curve" is really a whole lot of individual points, above the rational numbers on the x-axis. There are really a lot of "holes" in the curve, above $x = \pi$, for example. But (this is the hard part) it is possible to prove that the holes can be "filled in", and that the resulting function, called 2^x, really does have the properties we want, namely that $2^x 2^y = 2^{x+y}$ and $(2^x)^y = 2^{xy}$.

Exercises for Section 5.6

(1) Expand $\log_{10}((x+45)^7(x-2))$.

(2) Expand $\log_2 \dfrac{x^3}{3x-5+(7/x)}$.

(3) Write $\log_2 3x + 17\log_2(x-2) - 2\log_2(x^2+4x+1)$ as a single logarithm.

(4) Solve $\log_2(1+\sqrt{x}) = 6$ for x.

(5) Solve $2^{x^2} = 8$ for x.

(6) Solve $\log_2(\log_3(x)) = 1$ for x.

5.7 Derivatives of the exponential and logarithmic functions

As with the sine, we don't know anything about derivatives that allows us to compute the derivatives of the exponential and logarithmic functions without going back to basics. Let's do a little work with the definition again:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{\Delta x\to 0}\frac{a^{x+\Delta x}-a^x}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{a^x a^{\Delta x}-a^x}{\Delta x}\\ &= \lim_{\Delta x\to 0} a^x\frac{a^{\Delta x}-1}{\Delta x}\\ &= a^x\lim_{\Delta x\to 0}\frac{a^{\Delta x}-1}{\Delta x}\end{aligned}$$

There are two interesting things to note here: As in the case of the sine function we are left with a limit that involves Δx but not x, which means that whatever $\lim\limits_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ is, we know that it is a number, that is, a constant. This means that a^x has a remarkable property: its derivative is a constant times itself.

We earlier remarked that the hardest limit we would compute is $\lim\limits_{x\to 0}\sin x/x = 1$; we now have a limit that is just a bit too hard to include here. In fact the hard part is to see that $\lim\limits_{\Delta x\to 0}(a^{\Delta x}-1)/\Delta x$ even exists—does this fraction really get closer and closer to some fixed value? Yes it does, but we will not prove this fact.

We can look at some examples. Consider $(2^x-1)/x$ for some small values of x: 1, 0.828427124, 0.756828460, 0.724061864, 0.70838051, 0.70070877 when x is 1, 1/2, 1/4, 1/8, 1/16, 1/32, respectively. It looks like this is settling in around 0.7, which turns out to be true (but the limit is not exactly 0.7). Consider next $(3^x-1)/x$: 2, 1.464101616, 1.264296052, 1.177621520, 1.13720773, 1.11768854, at the same values of x. It turns out to be true that in the limit this is about 1.1. Two examples don't establish a pattern, but if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1;

the value at which this happens is called e, so that

$$\lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} = 1.$$

As you might guess from our two examples, e is closer to 3 than to 2, and in fact $e \approx 2.718$.

Now we see that the function e^x has a truly remarkable property:

$$\begin{aligned}
\frac{d}{dx} e^x &= \lim_{\Delta x \to 0} \frac{e^{x+\Delta x} - e^x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{e^x e^{\Delta x} - e^x}{\Delta x} \\
&= \lim_{\Delta x \to 0} e^x \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x \lim_{\Delta x \to 0} \frac{e^{\Delta x} - 1}{\Delta x} \\
&= e^x
\end{aligned}$$

That is, e^x is its own derivative, or in other words the slope of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (z, e^z) and has slope e^z there, no matter what z is. It is sometimes convenient to express the function e^x without an exponent, since complicated exponents can be hard to read. In such cases we use $\exp(x)$, e.g., $\exp(1 + x^2)$ instead of e^{1+x^2}.

What about the logarithm function? This too is hard, but as the cosine function was easier to do once the sine was done, so the logarithm is easier to do now that we know the derivative of the exponential function. Let's start with $\log_e x$, which as you probably know is often abbreviated $\ln x$ and called the "natural logarithm" function.

Consider the relationship between the two functions, namely, that they are inverses, that one "undoes" the other. Graphically this means that they have the same graph except that one is "flipped" or "reflected" through the line $y = x$:
This means that the slopes of these two functions are closely related as well: For example, the slope of e^x is e at $x = 1$; at the corresponding point on the $\ln(x)$ curve,

the slope must be $1/e$, because the "rise" and the "run" have been interchanged. Since the slope of e^x is e at the point $(1,e)$, the slope of $\ln(x)$ is $1/e$ at the point $(e,1)$.

More generally, we know that the slope of e^x is e^z at the point (z,e^z), so the slope of $\ln(x)$ is $1/e^z$ at (e^z,z). In other words, the slope of $\ln x$ is the reciprocal of the first coordinate at any point; this means that the slope of $\ln x$ at $(x,\ln x)$ is $1/x$. The upshot is:

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

We have discussed this from the point of view of the graphs, which is easy to understand but is not normally considered a rigorous proof—it is too easy to be led astray by pictures that seem reasonable but that miss some hard point. It is possible to do this derivation without resorting to pictures, and indeed we will see an alternate approach soon.

Note that $\ln x$ is defined only for $x > 0$. It is sometimes useful to consider the function $\ln|x|$, a function defined for $x \neq 0$. When $x < 0$, $\ln|x| = \ln(-x)$ and

$$\frac{d}{dx}\ln|x| = \frac{d}{dx}\ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}.$$

Thus whether x is positive or negative, the derivative is the same.

What about the functions a^x and $\log_a x$? We know that the derivative of a^x is some constant times a^x itself, but what constant? Remember that "the logarithm is the exponent" and you will see that $a = e^{\ln a}$. Then

$$a^x = (e^{\ln a})^x = e^{x\ln a},$$

and we can compute the derivative using the chain rule:

$$\frac{d}{dx}a^x = \frac{d}{dx}(e^{\ln a})^x = \frac{d}{dx}e^{x\ln a} = (\ln a)e^{x\ln a} = (\ln a)a^x.$$

The constant is simply $\ln a$. Likewise we can compute the derivative of the logarithm function $\log_a x$. Since

$$x = e^{\ln x}$$

we can take the logarithm base a of both sides to get

$$\log_a(x) = \log_a(e^{\ln x}) = \ln x \log_a e.$$

Then

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

This is a perfectly good answer, but we can improve it slightly. Since

$$\begin{aligned}
a &= e^{\ln a} \\
\log_a(a) &= \log_a(e^{\ln a}) = \ln a \log_a e \\
1 &= \ln a \log_a e \\
\frac{1}{\ln a} &= \log_a e,
\end{aligned}$$

we can replace $\log_a e$ to get

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

You may if you wish memorize the formulas

$$\frac{d}{dx}a^x = (\ln a)a^x \quad \text{and} \quad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}.$$

Because the "trick" $a = e^{\ln a}$ is often useful, and sometimes essential, it may be better to remember the trick, not the formula.

Example Compute the derivative of $f(x) = 2^x$.

$$\begin{aligned}
\frac{d}{dx}2^x &= \frac{d}{dx}(e^{\ln 2})^x \\
&= \frac{d}{dx}e^{x\ln 2} \\
&= \left(\frac{d}{dx}x\ln 2\right)e^{x\ln 2} \\
&= (\ln 2)e^{x\ln 2} = 2^x \ln 2
\end{aligned}$$

Example Compute the derivative of $f(x) = 2^{x^2} = 2^{(x^2)}$.

$$\begin{aligned}\frac{d}{dx}2^{x^2} &= \frac{d}{dx}e^{x^2\ln 2}\\ &= \left(\frac{d}{dx}x^2\ln 2\right)e^{x^2\ln 2}\\ &= (2\ln 2)xe^{x^2\ln 2}\\ &= (2\ln 2)x2^{x^2}\end{aligned}$$

Example Compute the derivative of $f(x) = x^x$. At first this appears to be a new kind of function: it is not a constant power of x, and it does not seem to be an exponential function, since the base is not constant. But in fact it is no harder than the previous example.

$$\begin{aligned}\frac{d}{dx}x^x &= \frac{d}{dx}e^{x\ln x}\\ &= \left(\frac{d}{dx}x\ln x\right)e^{x\ln x}\\ &= (x\frac{1}{x} + \ln x)x^x\\ &= (1+\ln x)x^x\end{aligned}$$

Example Recall that we have not justified the power rule except when the exponent is a positive or negative integer. We can use the exponential function to take care of other exponents.

$$\begin{aligned}\frac{d}{dx}x^r &= \frac{d}{dx}e^{r\ln x}\\ &= \left(\frac{d}{dx}r\ln x\right)e^{r\ln x}\\ &= (r\frac{1}{x})x^r\\ &= rx^{r-1}\end{aligned}$$

Exercises for Section 5.7

In 1–19, find the derivatives of the functions.

(1) 3^{x^2}

(2) $\dfrac{\sin x}{e^x}$

(3) $(e^x)^2$

(4) $\sin(e^x)$

(5) $e^{\sin x}$

(6) $x^{\sin x}$

(7) x^3e^x

(8) $x+2^x$

(9) $(1/3)^{x^2}$

(10) e^{4x}/x

(11) $\ln(x^3+3x)$

(12) $\ln(\cos(x))$

(13) $\sqrt{\ln(x^2)}/x$

(14) $\ln(\sec(x)+\tan(x))$

(15) $x^{\cos(x)}$

(16) $x\ln x$

(17) $\ln(\ln(3x))$

(18) $\dfrac{1+\ln(3x^2)}{1+\ln(4x)}$

(19) $\dfrac{x^8(x-23)^{1/2}}{27x^6(4x-6)^8}$

(20) Find the value of a so that the tangent line to $y = \ln(x)$ at $x = a$ is a line through the origin. Sketch the resulting situation.

(21) If $f(x) = \ln(x^3 + 2)$ compute $f'(e^{1/3})$.

(22) If $y = \log_a x$ then $a^y = x$. Use implicit differentiation to find y'.

5.8 Implicit Differentiation

As we have seen, there is a close relationship between the derivatives of e^x and $\ln x$ because these functions are inverses. Rather than relying on pictures for our understanding, we would like to be able to exploit this relationship computationally. In fact this technique can help us find derivatives in many situations, not just when we seek the derivative of an inverse function.

We will begin by illustrating the technique to find what we already know, the derivative of $\ln x$. Let's write $y = \ln x$ and then $x = e^{\ln x} = e^y$, that is, $x = e^y$. We say that this equation defines the function $y = \ln x$ implicitly because while it is not an explicit expression $y = \ldots$, it is true that if $x = e^y$ then y is in fact the natural logarithm function. Now, for the time being, pretend that all we know of y is that $x = e^y$; what can we say about derivatives? We can take the derivative of both sides of the equation:

$$\frac{d}{dx}x = \frac{d}{dx}e^y.$$

Then using the chain rule on the right hand side:

$$1 = \left(\frac{d}{dx}y\right)e^y = y'e^y.$$

Then we can solve for y':

$$y' = \frac{1}{e^y} = \frac{1}{x}.$$

There is one little difficulty here. To use the chain rule to compute $d/dx(e^y) = y'e^y$ we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $1/x$. When using this method we will always have to assume that the desired derivative exists, but fortunately this is a safe assumption for most such problems.

The example $y = \ln x$ involved an inverse function defined implicitly, but other functions can be defined implicitly, and sometimes a single equation can be used to implicitly define more than one function.

Here's a familiar example. The equation $r^2 = x^2 + y^2$ describes a circle of radius r. The circle is not a function $y = f(x)$ because for some values of x there are two corresponding values of y. If we want to work with a function, we can break the circle into two pieces, the upper and lower semicircles, each of which is a function. Let's call these $y = U(x)$ and $y = L(x)$; in fact this is a fairly simple example, and it's possible to give explicit expressions for these: $U(x) = \sqrt{r^2 - x^2}$ and $L(x) = -\sqrt{r^2 - x^2}$. But it's somewhat easier, and quite useful, to view both functions as given implicitly by $r^2 = x^2 + y^2$: both $r^2 = x^2 + U(x)^2$ and $r^2 = x^2 + L(x)^2$ are true, and we can think of $r^2 = x^2 + y^2$ as defining both $U(x)$ and $L(x)$.

Now we can take the derivative of both sides as before, remembering that y is not simply a variable but a function—in this case, y is either $U(x)$ or $L(x)$ but we're not yet specifying which one. When we take the derivative we just have to remember to apply the chain rule where y appears.

$$\begin{aligned}\frac{d}{dx}r^2 &= \frac{d}{dx}(x^2 + y^2)\\ 0 &= 2x + 2yy'\\ y' &= \frac{-2x}{2y} = -\frac{x}{y}\end{aligned}$$

Now we have an expression for y', but it contains y as well as x. This means that if we want to compute y' for some particular value of x we'll have to know or compute y at that value of x as well. It is at this point that we will need to know whether y is $U(x)$ or $L(x)$. Occasionally it will turn out that we can avoid explicit use of $U(x)$ or $L(x)$ by the nature of the problem

Example Find the slope of the circle $4 = x^2 + y^2$ at the point $(1, -\sqrt{3})$. Since we know both the x and y coordinates of the point of interest, we do not need to explicitly recognize that this point is on $L(x)$, and we do not need to use $L(x)$ to compute y—but we could. Using the calculation of y' from above,

$$y' = -\frac{x}{y} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

It is instructive to compare this approach to others.

We might have recognized at the start that $(1, -\sqrt{3})$ is on the function $y = L(x) = -\sqrt{4 - x^2}$. We could then take the derivative of $L(x)$, using the power rule and the chain rule, to get

$$L'(x) = -\frac{1}{2}(4-x^2)^{-1/2}(-2x) = \frac{x}{\sqrt{4-x^2}}.$$

Then we could compute $L'(1) = 1/\sqrt{3}$ by substituting $x = 1$.

Alternately, we could realize that the point is on $L(x)$, but use the fact that $y' = -x/y$. Since the point is on $L(x)$ we can replace y by $L(x)$ to get

$$y' = -\frac{x}{L(x)} = -\frac{x}{\sqrt{4-x^2}},$$

without computing the derivative of $L(x)$ explicitly. Then we substitute $x = 1$ and get the same answer as before.

In the case of the circle it is possible to find the functions $U(x)$ and $L(x)$ explicitly, but there are potential advantages to using implicit differentiation anyway. In some cases it is more difficult or impossible to find an explicit formula for y and implicit differentiation is the only way to find the derivative.

Example Find the derivative of any function defined implicitly by $yx^2 + e^y = x$. We treat y as an unspecified function and use the chain rule:

$$\begin{aligned}
\frac{d}{dx}(yx^2 + e^y) &= \frac{d}{dx}x \\
(y \cdot 2x + y' \cdot x^2) + y'e^y &= 1 \\
y'x^2 + y'e^y &= 1 - 2xy \\
y'(x^2 + e^y) &= 1 - 2xy \\
y' &= \frac{1 - 2xy}{x^2 + e^y}
\end{aligned}$$

You might think that the step in which we solve for y' could sometimes be difficult—after all, we're using implicit differentiation here because we can't solve the equation $yx^2 + e^y = x$ for y, so maybe after taking the derivative we get something that is hard to solve for y'. In fact, *this never happens.* All occurrences y' come from applying the chain rule, and whenever the chain rule is used it deposits a single y' multiplied by some other expression. So it will always be possible to group the terms containing y' together and factor out the y', just as in the previous example. If you ever get anything more difficult you have made a mistake and should fix it before trying to continue.

It is sometimes the case that a situation leads naturally to an equation that defines a function implicitly.

Example Consider all the points (x,y) that have the property that the distance from (x,y) to (x_1,y_1) plus the distance from (x,y) to (x_2,y_2) is $2a$ (a is some constant). These points form an ellipse, which like a circle is not a function but can viewed as two functions pasted together. Because we know how to write down the distance between two points, we can write down an implicit equation for the ellipse:

$$\sqrt{(x-x_1)^2+(y-y_1)^2}+\sqrt{(x-x_2)^2+(y-y_2)^2}=2a.$$

Then we can use implicit differentiation to find the slope of the ellipse at any point, though the computation is rather messy.

Example We have already justified the power rule by using the exponential function, but we could also do it for rational exponents by using implicit differentiation. Suppose that $y = x^{m/n}$, where m and n are positive integers. We can write this implicitly as $y^n = x^m$, then because we justified the power rule for integers, we can take the derivative of each side:

$$\begin{aligned} ny^{n-1}y' &= mx^{m-1} \\ y' &= \frac{m}{n}\frac{x^{m-1}}{y^{n-1}} \\ y' &= \frac{m}{n}\frac{x^{m-1}}{(x^{m/n})^{n-1}} \\ y' &= \frac{m}{n}x^{m-1-(m/n)(n-1)} \\ y' &= \frac{m}{n}x^{m-1-m+(m/n)} \\ y' &= \frac{m}{n}x^{(m/n)-1} \end{aligned}$$

Exercises for Section 5.8

In exercises 1–8, find a formula for the derivative y' at the point (x, y):

(1) $y^2 = 1 + x^2$

(2) $x^2 + xy + y^2 = 7$

(3) $x^3 + xy^2 = y^3 + yx^2$

(4) $4\cos x \sin y = 1$

(5) $\sqrt{x} + \sqrt{y} = 9$

(6) $\tan(x/y) = x + y$

(7) $\sin(x + y) = xy$

(8) $\dfrac{1}{x} + \dfrac{1}{y} = 7$

(9) A hyperbola passing through $(8,6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8,6)$.

(10) Compute y' for the ellipse of example 5.8.

(11) The graph of the equation $x^2 - xy + y^2 = 9$ is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel.

(12) Repeat the previous problem for the points at which the ellipse intersects the y-axis.

(13) Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.

(14) Find an equation for the tangent line to $x^4 = y^2 + x^2$ at $(2, \sqrt{12})$. (This curve is the **kampyle of Eudoxus**.)

(15) Find an equation for the tangent line to $x^{2/3} + y^{2/3} = a^{2/3}$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0$ and $y_1 \neq 0$. (This curve is an **astroid**.)

(16) Find an equation for the tangent line to $(x^2 + y^2)^2 = x^2 - y^2$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0, -1, 1$. (This curve is a **lemniscate**.)

Remark: Definition

Two curves are **orthogonal** if at each point of intersection, the angle between their tangent lines is $\pi/2$. Two families of curves, $\mathcal{A}$ and $\mathcal{B}$, are **orthogonal trajectories** of each other if given any curve C in $\mathcal{A}$ and any curve D in $\mathcal{B}$ the curves C and D are orthogonal. For example, the family of horizontal lines in the plane is orthogonal to the family of vertical lines in the plane.

(17) Show that $x^2 - y^2 = 5$ is orthogonal to $4x^2 + 9y^2 = 72$. (Hint: You need to find the intersection points of the two curves and then show that the product of the derivatives at each intersection point is -1.)

(18) Show that $x^2 + y^2 = r^2$ is orthogonal to $y = mx$. Conclude that the family of circles centered at the origin is an orthogonal trajectory of the family of lines that pass through the origin.

Note that there is a technical issue when $m = 0$. The circles fail to be differentiable when they cross the x-axis. However, the circles are orthogonal to the x-axis. Explain why. Likewise, the vertical line through the origin requires a separate argument.

(19) For $k \neq 0$ and $c \neq 0$ show that $y^2 - x^2 = k$ is orthogonal to $yx = c$. In the case where k and c are both zero, the curves intersect at the origin. Are the curves $y^2 - x^2 = 0$ and $yx = 0$ orthogonal to each other?

(20) Suppose that $m \neq 0$. Show that the family of curves $\{y = mx + b \mid b \in \mathbb{R}\}$ is orthogonal to the family of curves $\{y = -(x/m) + c \mid c \in \mathbb{R}\}$.

5.9 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often it is necessary to invert the functions, for example, to find an angle with a specified sine. Of course, there are many angles with the same sine, so the sine function doesn't actually have an inverse that reliably "undoes" the sine function. If you know that $\sin x = 0.5$, you can't reverse this to discover x, that is, you can't solve for x, as there are infinitely many angles with sine 0.5. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function does have a proper inverse.

The sine takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$. If we truncate the sine, keeping only the interval $[-\pi/2, \pi/2]$, as shown in figure **??**, then this truncated sine has an inverse function. We call this the inverse sine or the arcsine, and write $y = \arcsin(x)$.

Recall that a function and its inverse undo each other in either order, for example, $(\sqrt[3]{x})^3 = x$ and $\sqrt[3]{x^3} = x$. This does not work with the sine and the "inverse sine" because the inverse sine is the inverse of the truncated sine function, not the real sine function. It is true that $\sin(\arcsin(x)) = x$, that is, the sine undoes the arcsine. It is not true that the arcsine undoes the sine, for example, $\sin(5\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$, so doing first the sine then the arcsine does not get us back where we started. This is because $5\pi/6$ is not in the domain of the truncated sine. If we start with an angle between $-\pi/2$ and $\pi/2$ then the arcsine does reverse the sine: $\sin(\pi/6) = 1/2$ and $\arcsin(1/2) = \pi/6$.

What is the derivative of the arcsine? Since this is an inverse function, we can discover the derivative by using implicit differentiation. Suppose $y = \arcsin(x)$. Then

$$\sin(y) = \sin(\arcsin(x)) = x.$$

Now taking the derivative of both sides, we get

$$y' \cos y = 1$$
$$y' = \frac{1}{\cos y}$$

As we expect when using implicit differentiation, y appears on the right hand side here. We would certainly prefer to have y' written in terms of x, and as in the case of $\ln x$ we can actually do that here. Since $\sin^2 y + \cos^2 y = 1$, $\cos^2 y = 1 - \sin^2 y = 1 - x^2$. So $\cos y = \pm\sqrt{1-x^2}$, but which is it—plus or minus? It could in general be either, but this isn't "in general": since $y = \arcsin(x)$ we know that $-\pi/2 \le y \le \pi/2$, and the cosine of an angle in this interval is always positive. Thus $\cos y = \sqrt{1-x^2}$ and

$$\frac{d}{dx}\arcsin(x) = \frac{1}{\sqrt{1-x^2}}.$$

Note that this agrees with figure **??**: the graph of the arcsine has positive slope everywhere.

We can do something similar for the cosine. As with the sine, we must first truncate the cosine so that it can be inverted, as shown in figure **??**. Then we use implicit differentiation to find that

$$\frac{d}{dx}\arccos(x) = \frac{-1}{\sqrt{1-x^2}}.$$

Note that the truncated cosine uses a different interval than the truncated sine, so that if $y = \arccos(x)$ we know that $0 \le y \le \pi$. The computation of the derivative of the arccosine is left as an exercise.

Finally we look at the tangent; the other trigonometric functions also have "partial inverses" but the sine, cosine and tangent are enough for most purposes. The tangent, truncated tangent and inverse tangent are shown in figure **??**; the derivative of the arctangent is left as an exercise.

Exercises for Section 5.9

(1) Show that the derivative of $\arccos x$ is $-\dfrac{1}{\sqrt{1-x^2}}$.

(2) Show that the derivative of $\arctan x$ is $\dfrac{1}{1+x^2}$.

(3) The inverse of cot is usually defined so that the range of arccot is $(0, \pi)$. Sketch the graph of $y = \operatorname{arccot} x$. In the process you will make it clear what the domain of arccot is. Find the derivative of the arccotangent.

(4) Show that $\operatorname{arccot} x + \arctan x = \pi/2$.

(5) Find the derivative of $\arcsin(x^2)$.

(6) Find the derivative of $\arctan(e^x)$.

(7) Find the derivative of $\arccos(\sin x^3)$

(8) Find the derivative of $\ln((\arcsin x)^2)$

(9) Find the derivative of $\arccos e^x$

(10) Find the derivative of $\arcsin x + \arccos x$

(11) Find the derivative of $\log_5(\arctan(x^x))$

5.10 Limits revisited

We have defined and used the concept of limit, primarily in our development of the derivative. Recall that $\lim_{x \to a} f(x) = L$ is true if, in a precise sense, $f(x)$ gets closer and closer to L as x gets closer and closer to a. While some limits are easy to see, others take some ingenuity; in particular, the limits that define derivatives are always difficult on their face, since in

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

both the numerator and denominator approach zero. Typically this difficulty can be resolved when f is a "nice" function and we are trying to compute a derivative. Occasionally such limits are interesting for other reasons, and the limit of a fraction in which both numerator and denominator approach zero can be difficult to analyze. Now that we have the derivative available, there is another technique that can sometimes be helpful in such circumstances.

Before we introduce the technique, we will also expand our concept of limit. We will occasionally want to know what happens to some quantity when a variable gets very large or "goes to infinity".

> **Example** What happens to the function $\cos(1/x)$ as x goes to infinity? It seems clear that as x gets larger and larger, $1/x$ gets closer and closer to zero, so $\cos(1/x)$ should be getting closer and closer to $\cos(0) = 1$.

As with ordinary limits, this concept of "limit at infinity" can be made precise. Roughly, we want $\lim_{x \to \infty} f(x) = L$ to mean that we can make $f(x)$ as close as we want to L by making x large enough. Compare this definition to the definition of limit in section 3.3.

Definition (Limit at infinity) If f is a function, we say that $\lim_{x \to \infty} f(x) = L$ if for every $\varepsilon > 0$ there is an $N > 0$ so that whenever $x > N$, $|f(x) - L| < \varepsilon$. We may similarly define $\lim_{x \to -\infty} f(x) = L$.

We include this definition for completeness, but we will not explore it in detail. Suffice it to say that such limits behave in much the same way that ordinary limits do; in particular there is a direct analog of theorem 3.3.

Now consider this limit:

$$\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x}.$$

As x approaches π, both the numerator and denominator approach zero, so it is not obvious what, if anything, the quotient approaches. We can often compute such limits by application of the following theorem.

Theorem (L'Hôpital's Rule) For "sufficiently nice" functions $f(x)$ and $g(x)$, if $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$ or $\lim_{x\to a} f(x) = \pm\infty = \lim_{x\to a} g(x)$, and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$. This remains true if "$x \to a$" is replaced by "$x \to \infty$" or "$x \to -\infty$".

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here. We also will not need to worry about the precise definition of "sufficiently nice", as the functions we encounter will be suitable.

Example Compute $\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x}$ in two ways. First we use L'Hôpital's Rule: Since the numerator and denominator both approach zero,

$$\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x} = \lim_{x\to\pi} \frac{2x}{\cos x},$$

provided the latter exists. But in fact this is an easy limit, since the denominator now approaches -1, so

$$\lim_{x\to\pi} \frac{x^2-\pi^2}{\sin x} = \frac{2\pi}{-1} = -2\pi.$$

We don't really need L'Hôpital's Rule to do this limit. Rewrite it as

$$\lim_{x\to\pi} (x+\pi)\frac{x-\pi}{\sin x}$$

and note that

$$\lim_{x\to\pi} \frac{x-\pi}{\sin x} = \lim_{x\to\pi} \frac{x-\pi}{-\sin(x-\pi)} = \lim_{x\to 0} -\frac{x}{\sin x}$$

since $x - \pi$ approaches zero as x approaches π. Now

$$\lim_{x \to \pi} (x + \pi) \frac{x - \pi}{\sin x} = \lim_{x \to \pi} (x + \pi) \lim_{x \to 0} -\frac{x}{\sin x} = 2\pi(-1) = -2\pi$$

as before.

Example Compute $\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1}$ in two ways. As x goes to infinity both the numerator and denominator go to infinity, so we may apply L'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x \to \infty} \frac{4x - 3}{2x + 47}.$$

In the second quotient, it is still the case that the numerator and denominator both go to infinity, so we are allowed to use L'Hôpital's Rule again:

$$\lim_{x \to \infty} \frac{4x - 3}{2x + 47} = \lim_{x \to \infty} \frac{4}{2} = 2.$$

So the original limit is 2 as well.

Again, we don't really need L'Hôpital's Rule, and in fact a more elementary approach is easier—we divide the numerator and denominator by x^2:

$$\lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} = \lim_{x \to \infty} \frac{2x^2 - 3x + 7}{x^2 + 47x + 1} \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \to \infty} \frac{2 - \frac{3}{x} + \frac{7}{x^2}}{1 + \frac{47}{x} + \frac{1}{x^2}}.$$

Now as x approaches infinity, all the quotients with some power of x in the denominator approach zero, leaving 2 in the numerator and 1 in the denominator, so the limit again is 2.

Example Compute $\lim_{x \to 0} \frac{\sec x - 1}{\sin x}$. Both the numerator and denominator approach zero, so applying L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{\sec x - 1}{\sin x} = \lim_{x \to 0} \frac{\sec x \tan x}{\cos x} = \frac{1 \cdot 0}{1} = 0.$$

Example Compute $\lim_{x\to 0^+} x \ln x$.

This doesn't appear to be suitable for L'Hôpital's Rule, but it also is not "obvious". As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like (something very small) · (something very large and negative). But this could be anything: it depends on *how small* and *how large*. For example, consider $(x^2)(1/x)$, $(x)(1/x)$, and $(x)(1/x^2)$. As x approaches zero, each of these is (something very small) · (something very large), yet the limits are respectively zero, 1, and ∞.

We can in fact turn this into a L'Hôpital's Rule problem:

$$x \ln x = \frac{\ln x}{1/x} = \frac{\ln x}{x^{-1}}.$$

Now as x approaches zero, both the numerator and denominator approach infinity (one $-\infty$ and one $+\infty$, but only the size is important). Using L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{\ln x}{x^{-1}} = \lim_{x\to 0^+} \frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} \frac{1}{x}(-x^2) = \lim_{x\to 0^+} -x = 0.$$

One way to interpret this is that since $\lim_{x\to 0^+} x \ln x = 0$, the x approaches zero much faster than the $\ln x$ approaches $-\infty$.

Exercises for Section 5.10

Compute the limits.

(1) $\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

(2) $\lim_{x\to\infty} \frac{e^x}{x^3}$

(3) $\lim_{x\to\infty} \sqrt{x^2+x} - \sqrt{x^2-x}$

(4) $\lim_{x\to\infty} \frac{\ln x}{x}$

(5) $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

(6) $\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(7) $\lim_{x\to 0} \frac{\sqrt{9+x} - 3}{x}$

(8) $\lim_{t\to 1^+} \frac{(1/t) - 1}{t^2 - 2t + 1}$

(9) $\lim_{x\to 2} \frac{2 - \sqrt{x+2}}{4 - x^2}$

(10) $\lim\limits_{t\to\infty} \dfrac{t+5-2/t-1/t^3}{3t+12-1/t^2}$

(11) $\lim\limits_{y\to\infty} \dfrac{\sqrt{y+1}+\sqrt{y-1}}{y}$

(12) $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{\sqrt[1/3]{x}-1}$

(13) $\lim\limits_{x\to 0} \dfrac{(1-x)^{1/4}-1}{x}$

(14) $\lim\limits_{t\to 0}\left(t+\dfrac{1}{t}\right)((4-t)^{3/2}-8)$

(15) $\lim\limits_{t\to 0^+}\left(\dfrac{1}{t}+\dfrac{1}{\sqrt{t}}\right)(\sqrt{t+1}-1)$

(16) $\lim\limits_{x\to 0} \dfrac{x^2}{\sqrt{2x+1}-1}$

(17) $\lim\limits_{u\to 1} \dfrac{(u-1)^3}{(1/u)-u^2+3/u-3}$

(18) $\lim\limits_{x\to 0} \dfrac{2+(1/x)}{3-(2/x)}$

(19) $\lim\limits_{x\to 0^+} \dfrac{1+5/\sqrt{x}}{2+1/\sqrt{x}}$

(20) $\lim\limits_{x\to 0^+} \dfrac{3+x^{-1/2}+x^{-1}}{2+4x^{-1/2}}$

(21) $\lim\limits_{x\to\infty} \dfrac{x+x^{1/2}+x^{1/3}}{x^{2/3}+x^{1/4}}$

(22) $\lim\limits_{t\to\infty} \dfrac{1-\sqrt{\frac{t}{t+1}}}{2-\sqrt{\frac{4t+1}{t+2}}}$

(23) $\lim\limits_{t\to\infty} \dfrac{1-\frac{t}{t-1}}{1-\sqrt{\frac{t}{t-1}}}$

(24) $\lim\limits_{x\to-\infty} \dfrac{x+x^{-1}}{1+\sqrt{1-x}}$

(25) $\lim\limits_{x\to 1} \dfrac{\int_1^x 1/t\,dt}{\int_1^x 1/(2t+1)\,dt}$

(26) $\lim\limits_{x\to\infty} \dfrac{\int_1^x \sqrt{t+(1/t)}\,dt}{x\sqrt{x}}$

(27) $\lim\limits_{x\to\pi/2} \dfrac{\cos x}{(\pi/2)-x}$

(28) $\lim\limits_{x\to 0} \dfrac{e^x-1}{x}$

(29) $\lim\limits_{x\to 0} \dfrac{x^2}{e^x-x-1}$

(30) $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$

(31) $\lim\limits_{x\to 0} \dfrac{\ln(x^2+1)}{x}$

(32) $\lim\limits_{x\to 1} \dfrac{x\ln x}{x^2-1}$

(33) $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{\ln(x+1)}$

(34) $\lim\limits_{x\to 1} \dfrac{x^{1/4}-1}{x}$

(35) $\lim\limits_{x\to 1^+} \dfrac{\sqrt{x}}{x-1}$

(36) $\lim\limits_{x\to 1} \dfrac{\sqrt{x}-1}{x-1}$

(37) $\lim_{x\to\infty} \frac{x^{-1}+x^{-1/2}}{x+x^{-1/2}}$

(38) $\lim_{x\to\infty} \frac{x+x^{-2}}{2x+x^{-2}}$

(39) $\lim_{x\to\infty} \frac{5+x^{-1}}{1+2x^{-1}}$

(40) $\lim_{x\to\infty} \frac{4x}{\sqrt{2x^2+1}}$

(41) $\lim_{x\to 0} \frac{3x^2+x+2}{x-4}$

(42) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{\sqrt{x+4}-2}$

(43) $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{\sqrt{x+2}-2}$

(44) $\lim_{x\to 0^+} \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}$

(45) $\lim_{x\to 0} \frac{\sqrt{x^2+1}-1}{\sqrt{x+1}-1}$

(46) $\lim_{x\to\infty} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(47) $\lim_{x\to 0^+} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(48) $\lim_{x\to 1} (x+5)\left(\frac{1}{2x}+\frac{1}{x+2}\right)$

(49) $\lim_{x\to 2} \frac{x^3-6x-2}{x^3+4}$

(50) $\lim_{x\to 2} \frac{x^3-6x-2}{x^3-4x}$

(51) $\lim_{x\to 1+} \frac{x^3+4x+8}{2x^3-2}$

(52) The function $f(x) = \frac{x}{\sqrt{x^2+1}}$ has two horizontal asymptotes. Find them and give a rough sketch of f with its horizontal asymptotes.

5.11 Hyperbolic Functions

The hyperbolic functions appear with some frequency in applications, and are quite similar in many respects to the trigonometric functions. This is a bit surprising given our initial definitions.

Definition The **hyperbolic cosine** is the function

$$\cosh x = \frac{e^x + e^{-x}}{2},$$

and the **hyperbolic sine** is the function

$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

Notice that cosh is even (that is, $\cosh(-x) = \cosh(x)$) while sinh is odd ($\sinh(-x) = -\sinh(x)$), and $\cosh x + \sinh x = e^x$. Also, for all x, $\cosh x > 0$, while $\sinh x = 0$ if and only if $e^x - e^{-x} = 0$, which is true precisely when $x = 0$.

Lemma

The range of $\cosh x$ is $[1, \infty)$.

Proof. Let $y = \cosh x$. We solve for x:

$$\begin{aligned} y &= \frac{e^x + e^{-x}}{2} \\ 2y &= e^x + e^{-x} \\ 2ye^x &= e^{2x} + 1 \\ 0 &= e^{2x} - 2ye^x + 1 \\ e^x &= \frac{2y \pm \sqrt{4y^2 - 4}}{2} \\ e^x &= y \pm \sqrt{y^2 - 1} \end{aligned}$$

From the last equation, we see $y^2 \geq 1$, and since $y \geq 0$, it follows that $y \geq 1$.

Now suppose $y \geq 1$, so $y \pm \sqrt{y^2 - 1} > 0$. Then $x = \ln(y \pm \sqrt{y^2 - 1})$ is a real number, and $y = \cosh x$, so y is in the range of $\cosh(x)$. $\square$

Definition The other hyperbolic functions are

$$\tanh x = \frac{\sinh x}{\cosh x}$$
$$\coth x = \frac{\cosh x}{\sinh x}$$
$$\operatorname{sech} x = \frac{1}{\cosh x}$$
$$\operatorname{csch} x = \frac{1}{\sinh x}$$

The domain of coth and csch is $x \neq 0$ while the domain of the other hyperbolic functions is all real numbers. Graphs are shown in figure **??**

Certainly the hyperbolic functions do not closely resemble the trigonometric functions graphically. But they do have analogous properties, beginning with the following identity.

Theorem For all x in $\mathbb{R}$, $\cosh^2 x - \sinh^2 x = 1$.

Proof. The proof is a straightforward computation:

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{e^{2x} + 2 + e^{-2x} - e^{2x} + 2 - e^{-2x}}{4} = \frac{4}{4} = 1.$$

$\square$

This immediately gives two additional identities:

$$1 - \tanh^2 x = \operatorname{sech}^2 x \qquad \text{and} \qquad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

The identity of the theorem also helps to provide a geometric motivation. Recall that the graph of $x^2 - y^2 = 1$ is a hyperbola with asymptotes $x = \pm y$ whose x-intercepts are ± 1. If (x, y) is a point on the right half of the hyperbola, and if we let $x = \cosh t$, then $y = \pm\sqrt{x^2-1} = \pm\sqrt{\cosh^2 x - 1} = \pm\sinh t$. So for some suitable t, $\cosh t$ and $\sinh t$ are the coordinates of a typical point on the hyperbola. In fact, it turns out that t is twice the area shown in the first graph of figure **??**. Even this is analogous to trigonometry; $\cos t$ and $\sin t$ are the coordinates of a typical point on the unit circle, and t is twice the area shown in the second graph of figure **??**.

Given the definitions of the hyperbolic functions, finding their derivatives is straightforward. Here again we see similarities to the trigonometric functions.

Theorem $\frac{d}{dx}\cosh x = \sinh x$ and $\frac{d}{dx}\sinh x = \cosh x$.

Proof. $\frac{d}{dx}\cosh x = \frac{d}{dx}\frac{e^x+e^{-x}}{2} = \frac{e^x-e^{-x}}{2} = \sinh x$, and $\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x-e^{-x}}{2} = \frac{e^x+e^{-x}}{2} = \cosh x$. □

Since $\cosh x > 0$, $\sinh x$ is increasing and hence injective, so $\sinh x$ has an inverse, $\operatorname{arcsinh} x$. Also, $\sinh x > 0$ when $x > 0$, so $\cosh x$ is injective on $[0,\infty)$ and has a (partial) inverse, $\operatorname{arccosh} x$. The other hyperbolic functions have inverses as well, though $\operatorname{arcsech} x$ is only a partial inverse. We may compute the derivatives of these functions as we have other inverse functions.

Theorem $\frac{d}{dx}\operatorname{arcsinh} x = \frac{1}{\sqrt{1+x^2}}$.

Proof. Let $y = \operatorname{arcsinh} x$, so $\sinh y = x$. Then $\frac{d}{dx}\sinh y = \cosh(y)\cdot y' = 1$, and so $y' = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}$. □

The other derivatives are left to the exercises.

Exercises for Section 5.11

(1) Show that the range of $\sinh x$ is all real numbers. (Hint: show that if $y = \sinh x$ then $x = \ln(y + \sqrt{y^2+1})$.)

(2) Compute the following limits:

- a. $\lim_{x\to\infty} \cosh x$
- b. $\lim_{x\to\infty} \sinh x$
- c. $\lim_{x\to\infty} \tanh x$
- d. $\lim_{x\to\infty} (\cosh x - \sinh x)$

(3) Show that the range of $\tanh x$ is $(-1, 1)$. What are the ranges of coth, sech, and csch? (Use the fact that they are reciprocal functions.)

(4) Prove that for every $x, y \in \mathbb{R}$, $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$. Obtain a similar identity for $\sinh(x-y)$.

(5) Prove that for every $x, y \in \mathbb{R}$, $\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y$. Obtain a similar identity for $\cosh(x-y)$.

(6) Use exercises 4 and 5 to show that $\sinh(2x) = 2 \sinh x \cosh x$ and $\cosh(2x) = \cosh^2 x + \sinh^2 x$ for every x. Conclude also that $(\cosh(2x) - 1)/2 = \sinh^2 x$.

(7) Show that $\dfrac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$. Compute the derivatives of the remaining hyperbolic functions as well.

(8) What are the domains of the six inverse hyperbolic functions?

(9) Sketch the graphs of all six inverse hyperbolic functions.

6

Curve Sketching

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

6.1 Maxima and Minima

A **local maximum point** on a function is a point (x,y) on the graph of the function whose y coordinate is larger than all other y coordinates on the graph at points "close to" (x,y). More precisely, $(x,f(x))$ is a local maximum if there is an interval (a,b) with $a<x<b$ and $f(x)\geq f(z)$ for every z in (a,b). Similarly, (x,y) is a **local minimum point** if it has locally the smallest y coordinate. Again being more precise: $(x,f(x))$ is a local minimum if there is an interval (a,b) with $a<x<b$ and $f(x)\leq f(z)$ for every z in (a,b). A **local extremum** is either a local minimum or a local maximum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well. Some examples of local maximum and minimum points are shown in figure **??**.

If $(x, f(x))$ is a point where $f(x)$ reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

Theorem (Fermat's Theorem) If $f(x)$ has a local extremum at $x = a$ and f is differentiable at a, then $f'(a) = 0$.

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, as in the left hand graph in figure **??**, or the derivative is undefined, as in the right hand graph. Any value of x for which $f'(x)$ is zero or undefined is called a **critical value** for f. When looking for local maximum and minimum points, you are likely to make two sorts of mistakes: You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere. You might also assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true. A portion of the graph of $f(x) = x^3$ is shown in figure **??**. The derivative of f is $f'(x) = 3x^2$, and $f'(0) = 0$, but there is neither a maximum nor minimum at $(0, 0)$.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach, but one that is often tedious or difficult, is to test directly whether the y coordinates "near" the potential maximum or minimum are above or below the y coordinate at the point of interest. Of course, there are too many points "near" the point to test, but a little thought shows we need only test two provided we know that f is continuous (recall that this means that the graph of f has no jumps or gaps).

Suppose, for example, that we have identified three points at which f' is zero or nonexistent: (x_1, y_1), (x_2, y_2), (x_3, y_3), and $x_1 < x_2 < x_3$ (see figure **??**). Suppose that we compute the value of $f(a)$ for $x_1 < a < x_2$, and that $f(a) < f(x_2)$. What can we say about the graph between a and x_2? Could there be a point $(b, f(b))$, $a < b < x_2$ with $f(b) > f(x_2)$? No: if there were, the graph would go up from $(a, f(a))$ to

$(b, f(b))$ then down to $(x_2, f(x_2))$ and somewhere in between would have a local maximum point. (This is not obvious; it is a result of the Extreme Value Theorem, theorem **??**.) But at that local maximum point the derivative of f would be zero or nonexistent, yet we already know that the derivative is zero or nonexistent only at x_1, x_2, and x_3. The upshot is that one computation tells us that $(x_2, f(x_2))$ has the largest y coordinate of any point on the graph near x_2 and to the left of x_2. We can perform the same test on the right. If we find that on both sides of x_2 the values are smaller, then there must be a local maximum at $(x_2, f(x_2))$; if we find that on both sides of x_2 the values are larger, then there must be a local minimum at $(x_2, f(x_2))$; if we find one of each, then there is neither a local maximum or minimum at x_2.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

Example Find all local maximum and minimum points for the function $f(x) = x^3 - x$. The derivative is $f'(x) = 3x^2 - 1$. This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. Looking first at $x = \sqrt{3}/3$, we see that $f(\sqrt{3}/3) = -2\sqrt{3}/9$. Now we test two points on either side of $x = \sqrt{3}/3$, making sure that neither is farther away than the nearest critical value; since $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ and we can use $x = 0$ and $x = 1$. Since $f(0) = 0 > -2\sqrt{3}/9$ and $f(1) = 0 > -2\sqrt{3}/9$, there must be a local minimum at $x = \sqrt{3}/3$. For $x = -\sqrt{3}/3$, we see that $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. This time we can use $x = 0$ and $x = -1$, and we find that $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, so there must be a local maximum at $x = -\sqrt{3}/3$.

Of course this example is made very simple by our choice of points to test, namely $x = -1, 0, 1$. We could have used other values, say $-5/4$, $1/3$, and $3/4$, but this would have made the calculations considerably more tedious.

Example Find all local maximum and minimum points for $f(x) = \sin x + \cos x$. The derivative is $f'(x) = \cos x - \sin x$. This is always defined and is zero whenever $\cos x = \sin x$. Recalling that the $\cos x$ and $\sin x$ are the x and y coordinates of points on a unit circle, we see that $\cos x = \sin x$ when x is $\pi/4$, $\pi/4 \pm \pi$, $\pi/4 \pm 2\pi$, $\pi/4 \pm 3\pi$, etc. Since both sine and cosine have a period of 2π, we need only determine the status of $x = \pi/4$ and $x = 5\pi/4$. We can use 0 and $\pi/2$ to test the critical value $x = \pi/4$. We find that $f(\pi/4) = \sqrt{2}$, $f(0) = 1 < \sqrt{2}$ and $f(\pi/2) = 1$, so there is a local maximum when $x = \pi/4$ and also when $x = \pi/4 \pm 2\pi$, $\pi/4 \pm 4\pi$, etc. We can summarize this more neatly by saying that there are local maxima at $\pi/4 \pm 2k\pi$ for every integer k.

We use π and 2π to test the critical value $x = 5\pi/4$. The relevant values are $f(5\pi/4) = -\sqrt{2}$, $f(\pi) = -1 > -\sqrt{2}$, $f(2\pi) = 1 > -\sqrt{2}$, so there is a local minimum at $x = 5\pi/4$, $5\pi/4 \pm 2\pi$, $5\pi/4 \pm 4\pi$, etc. More succinctly, there are local minima at $5\pi/4 \pm 2k\pi$ for every integer k.

Exercises for Section 6.1

In problems 1–12, find all local maximum and minimum points (x, y) by the method of this section.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - (1/x^2)$

(8) $y = \cos(2x) - x$

(9) $f(x) = \begin{cases} x - 1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$

(10) $f(x) = \begin{cases} x - 3 & x < 3 \\ x^3 & 3 \leq x \leq 5 \\ 1/x & x > 5 \end{cases}$

(11) $f(x) = x^2 - 98x + 4$

(12) $f(x) = \begin{cases} -2 & x = 0 \\ 1/x^2 & x \neq 0 \end{cases}$

(13) For any real number x there is a unique integer n such that $n \leq x < n + 1$, and the greatest integer function is defined as $\lfloor x \rfloor = n$. Where are the critical values of the greatest integer function? Which are local maxima and which are local minima?

(14) Explain why the function $f(x) = 1/x$ has no local maxima or minima.

(15) How many critical points can a quadratic polynomial function have?

(16) Show that a cubic polynomial can have at most two critical points. Give examples to show that a cubic polynomial can have zero, one, or two critical points.

(17) Explore the family of functions $f(x) = x^3 + cx + 1$ where c is a constant. How many and what types of local extremes are there? Your answer should depend on the value of c, that is, different values of c will give different answers.

(18) We generalize the preceding two questions. Let n be a positive integer and let f be a polynomial of degree n. How many critical points can f have? (Hint: Recall the **Fundamental Theorem of Algebra**, which says that a polynomial of degree n has at most n roots.)

6.2 The first derivative test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical value is not always convenient. We can instead use information about the derivative $f'(x)$ to decide; since we have already had to compute the derivative to find the critical values, there is often relatively little extra work involved in this method.

How can the derivative tell us whether there is a maximum, minimum, or neither at a point? Suppose that $f'(a) = 0$. If there is a local maximum when $x = a$, the function must be lower near $x = a$ than it is right at $x = a$. If the derivative exists near $x = a$, this means $f'(x) > 0$ when x is near a and $x < a$, because the function must "slope up" just to the left of a. Similarly, $f'(x) < 0$ when x is near a and $x > a$, because f slopes down from the local maximum as we move to the right. Using the same reasoning, if there is a local minimum at $x = a$, the derivative of f must be negative just to the left of a and positive just to the right. If the derivative exists near a but does not change from positive to negative or negative to positive, that is, it is positive on both sides or negative on both sides, then there is neither a maximum nor minimum when $x = a$. See the first graph in figure **??** and the graph in figure **??** for examples.

Example Find all local maximum and minimum points for $f(x) = \sin x + \cos x$ using the first derivative test. The derivative is $f'(x) = \cos x - \sin x$ and from example 6.1 the critical values we need to consider are $\pi/4$ and $5\pi/4$.

The graphs of $\sin x$ and $\cos x$ are shown in figure **??**. Just to the left of $\pi/4$ the cosine is larger than the sine, so $f'(x)$ is positive; just to the right the cosine is smaller than the sine, so $f'(x)$ is negative. This means there is a local maximum at $\pi/4$. Just to the left of $5\pi/4$ the cosine is smaller than the sine, and to the right the cosine is larger than the sine. This means that the derivative $f'(x)$ is negative to the left and positive to the right, so f has a local minimum at $5\pi/4$.

Exercises for Section 6.2

In 1–13, find all critical points and identify them as local maximum points, local minimum points, or neither.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - (1/x^2)$

(8) $y = \cos(2x) - x$

(9) $f(x) = (5 - x)/(x + 2)$

(10) $f(x) = |x^2 - 121|$

(11) $f(x) = x^3/(x + 1)$

(12) $f(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

(13) $f(x) = \sin^2 x$

(14) Find the maxima and minima of $f(x) = \sec x$.

(15) Let $f(\theta) = \cos^2(\theta) - 2\sin(\theta)$. Find the intervals where f is increasing and the intervals where f is decreasing in $[0, 2\pi]$. Use this information to classify the critical points of f as either local maximums, local minimums, or neither.

(16) Let $r > 0$. Find the local maxima and minima of the function $f(x) = \sqrt{r^2 - x^2}$ on its domain $[-r, r]$.

(17) Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. Show that f has exactly one critical point using the first derivative test. Give conditions on a and b which guarantee that the critical point will be a maximum. It is possible to see this without using calculus at all; explain.

6.3 The second derivative test

The basis of the first derivative test is that if the derivative changes from positive to negative at a point at which the derivative is zero then there is a local maximum at the point, and similarly for a local minimum. If f' changes from positive to negative it is decreasing; this means that the derivative of f', f'', might be negative, and if in fact f'' is negative then f' is definitely decreasing, so there is a local maximum at the point in question. Note well that f' might change from positive to negative while f'' is zero, in which case f'' gives us no information about the critical value. Similarly, if f' changes from negative to positive there is a local minimum at the point, and f' is increasing. If $f'' > 0$ at the point, this tells us that f' is increasing, and so there is a local minimum.

Example Consider again $f(x) = \sin x + \cos x$, with $f'(x) = \cos x - \sin x$ and $f''(x) = -\sin x - \cos x$. Since $f''(\pi/4) = -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0$, we know there is a local maximum at $\pi/4$. Since $f''(5\pi/4) = --\sqrt{2}/2 - -\sqrt{2}/2 = \sqrt{2} > 0$, there is a local minimum at $5\pi/4$.

When it works, the second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails, and sometimes the second derivative is quite difficult to evaluate; in such cases we must fall back on one of the previous tests.

Example Let $f(x) = x^4$. The derivatives are $f'(x) = 4x^3$ and $f''(x) = 12x^2$. Zero is the only critical value, but $f''(0) = 0$, so the second derivative test tells us nothing. However, $f(x)$ is positive everywhere except at zero, so clearly $f(x)$ has a local minimum at zero. On the other hand, $f(x) = -x^4$ also has zero as its only critical value, and the second derivative is again zero, but $-x^4$ has a local maximum at zero.

Exercises for Section 6.3

Find all local maximum and minimum points by the second derivative test.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - (1/x^2)$

(8) $y = \cos(2x) - x$

(9) $y = 4x + \sqrt{1-x}$

(10) $y = (x+1)/\sqrt{5x^2+35}$

(11) $y = x^5 - x$

(12) $y = 6x + \sin 3x$

(13) $y = x + 1/x$

(14) $y = x^2 + 1/x$

(15) $y = (x+5)^{1/4}$

(16) $y = \tan^2 x$

(17) $y = \cos^2 x - \sin^2 x$

(18) $y = \sin^3 x$

6.4 Concavity and inflection points

We know that the sign of the derivative tells us whether a function is increasing or decreasing; for example, when $f'(x) > 0$, $f(x)$ is increasing. The sign of the second derivative $f''(x)$ tells us whether f' is increasing or decreasing; we have seen that if f' is zero and increasing at a point then there is a local minimum at the point, and if f' is zero and decreasing at a point then there is a local maximum at the point. Thus, we extracted information about f from information about f''.

We can get information from the sign of f'' even when f' is not zero. Suppose that $f''(a) > 0$. This means that near $x = a$, f' is increasing. If $f'(a) > 0$, this means that f slopes up and is getting steeper; if $f'(a) < 0$, this means that f slopes down and is getting *less* steep. The two situations are shown in figure **??**. A curve that is shaped like this is called **concave up.**

Now suppose that $f''(a) < 0$. This means that near $x = a$, f' is decreasing. If $f'(a) > 0$, this means that f slopes up and is getting less steep; if $f'(a) < 0$, this means that f slopes down and is getting steeper. The two situations are shown in figure **??**. A curve that is shaped like this is called **concave down.**

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. Of particular interest are points at which the concavity changes from up to down or down to up; such points are called **inflection points.** If the concavity changes from up to down at $x = a$, f'' changes from positive to the left of a to negative to the right of a, and usually $f''(a) = 0$. We can identify such points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points. Note that it is possible that $f''(a) = 0$ but the concavity is the same on both sides; $f(x) = x^4$ at $x = 0$ is an example.

> **Example** Describe the concavity of $f(x) = x^3 - x$. $f'(x) = 3x^2 - 1$, $f''(x) = 6x$. Since $f''(0) = 0$, there is potentially an inflection point at zero. Since $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$ the concavity does change from down to up at zero, and the curve is concave down for all $x < 0$ and concave up for all $x > 0$.

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

Exercises for Section 6.4

Describe the concavity of the functions in 1–18.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - (1/x^2)$

(8) $y = \sin x + \cos x$

(9) $y = 4x + \sqrt{1-x}$

(10) $y = (x+1)/\sqrt{5x^2+35}$

(11) $y = x^5 - x$

(12) $y = 6x + \sin 3x$

(13) $y = x + 1/x$

(14) $y = x^2 + 1/x$

(15) $y = (x+5)^{1/4}$

(16) $y = \tan^2 x$

(17) $y = \cos^2 x - \sin^2 x$

(18) $y = \sin^3 x$

(19) Identify the intervals on which the graph of the function $f(x) = x^4 - 4x^3 + 10$ is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

(20) Describe the concavity of $y = x^3 + bx^2 + cx + d$. You will need to consider different cases, depending on the values of the coefficients.

(21) Let n be an integer greater than or equal to two, and suppose f is a polynomial of degree n. How many inflection points can f have? Hint: Use the second derivative test and the fundamental theorem of algebra.

6.5 Asymptotes and Other Things to Look For

A vertical asymptote is a place where the function becomes infinite, typically because the formula for the function has a denominator that becomes zero. For example, the reciprocal function $f(x) = 1/x$ has a vertical asymptote at $x = 0$, and the function $\tan x$ has a vertical asymptote at $x = \pi/2$ (and also at $x = -\pi/2$, $x = 3\pi/2$, etc.). Whenever the formula for a function contains a denominator it is worth looking for a vertical asymptote by checking to see if the denominator can ever be zero, and then checking the limit at such points. Note that there is not always a vertical asymptote where the derivative is zero: $f(x) = (\sin x)/x$ has a zero denominator at $x = 0$, but since $\lim_{x\to 0}(\sin x)/x = 1$ there is no asymptote there.

A horizontal asymptote is a horizontal line to which $f(x)$ gets closer and closer as x approaches ∞ (or as x approaches $-\infty$). For example, the reciprocal function has the x-axis for a horizontal asymptote. Horizontal asymptotes can be identified by computing the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Since $\lim_{x\to\infty} 1/x = \lim_{x\to-\infty} 1/x = 0$, the line $y = 0$ (that is, the x-axis) is a horizontal asymptote in both directions.

Some functions have asymptotes that are neither horizontal nor vertical, but some other line. Such asymptotes are somewhat more difficult to identify and we will ignore them.

If the domain of the function does not extend out to infinity, we should also ask what happens as x approaches the boundary of the domain. For example, the function $y = f(x) = 1/\sqrt{r^2 - x^2}$ has domain $-r < x < r$, and y becomes infinite as x approaches either r or $-r$. In this case we might also identify this behavior because when $x = \pm r$ the denominator of the function is zero.

If there are any points where the derivative fails to exist (a cusp or corner), then we should take special note of what the function does at such a point.

Finally, it is worthwhile to notice any symmetry. A function $f(x)$ that has the same value for $-x$ as for x, i.e., $f(-x) = f(x)$, is called an "even function." Its graph is symmetric with respect to the y-axis. Some examples of even functions are: x^n when n is an even number, $\cos x$, and $\sin^2 x$. On the other hand, a function that satisfies the property $f(-x) = -f(x)$ is called an "odd function." Its graph is symmetric with respect to the origin. Some examples of odd functions are: x^n when n is an odd number, $\sin x$, and $\tan x$. Of course, most functions are neither even nor odd, and do not have any particular symmetry.

Exercises for Section 6.5

Sketch the curves. Identify clearly any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts.

(1) $y = x^5 - 5x^4 + 5x^3$

(2) $y = x^3 - 3x^2 - 9x + 5$

(3) $y = (x-1)^2(x+3)^{2/3}$

(4) $x^2 + x^2y^2 = a^2y^2$, $a > 0$.

(5) $y = 4x + \sqrt{1-x}$

(6) $y = (x+1)/\sqrt{5x^2+35}$

(7) $y = x^5 - x$

(8) $y = 6x + \sin 3x$

(9) $y = x + 1/x$

(10) $y = x^2 + 1/x$

(11) $y = (x+5)^{1/4}$

(12) $y = \tan^2 x$

(13) $y = \cos^2 x - \sin^2 x$

(14) $y = \sin^3 x$

(15) $y = x(x^2+1)$

(16) $y = x^3 + 6x^2 + 9x$

(17) $y = x/(x^2-9)$

(18) $y = x^2/(x^2+9)$

(19) $y = 2\sqrt{x} - x$

(20) $y = 3\sin(x) - \sin^3(x)$, for $x \in [0, 2\pi]$

(21) $y = (x-1)/(x^2)$

For each of the following five functions, identify any vertical and horizontal asymptotes, and identify intervals on which the function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

(22) $f(\theta) = \sec(\theta)$

(23) $f(x) = 1/(1+x^2)$

(24) $f(x) = (x-3)/(2x-2)$

(25) $f(x) = 1/(1-x^2)$

(26) $f(x) = 1+1/(x^2)$

(27) Let $f(x) = 1/(x^2-a^2)$, where $a \geq 0$. Find any vertical and horizontal asymptotes and the intervals upon which the given function is concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing. Discuss how the value of a affects these features.

Answers to selected exercises

1. (a) 8, (b) 6, (c) dne, (d) -2, (e) -1, (f) 8, (g) 7, (h) 6, (i) 3, (j) $-3/2$, (k) 6, (l) 2

Answers for 1.2

1. 0

Answers for 1.3

1. 7 2. 5 3. 0 4. undefined 5. 1/6 6. 0 7. 3 8. 172 9. 0 10. 2 11. does not exist 12. $\sqrt{2}$ 13. $3a^2$ 14. 512 15. -4

Answers for 1.4

1. $(2/3)x+(1/3)$ 2. $y=-2x$ 3. $(-2/3)x+(1/3)$ 4. $y=2x+2$, 2, -1 5. $y=-x+6$, 6, 6 6. $y=x/2+1/2$, $1/2$, -1 7. $y=3/2$, y-intercept: $3/2$, no x-intercept 8. $y=(-2/3)x-2$, -2, -3 9. yes 10. $y=0$, $y=-2x+2$, $y=2x+2$ 11. $y=75t$, 164 minutes 12. $y=(9/5)x+32$, $(-40,-40)$ 13. $y=0.15x+10$ 14. $0.03x+1.2$ 15. (a) $y=\begin{cases}0 & 0\le x<100\\ (x/10)-10 & 100\le x\le 1000\\ x-910 & 1000<x\end{cases}$ 16. $y=\begin{cases}0.15x & 0\le x\le 19450\\ 0.28x-2528.50 & 19450<x\le 47050\\ 0.33x-4881 & 47050<x\le 97620\end{cases}$ 17. (a) $P=-0.0001x+2$ (b) $x=-10000P+20000$ 18. $(2/25)x-(16/5)$

Answers for 2.1

1. (a) $x^2+y^2=9$ (b) $(x-5)^2+(y-6)^2=9$ (c) $(x+5)^2+(y+6)^2=9$ 2. (a) $\Delta x=2$, $\Delta y=3$, $m=3/2$, $y=(3/2)x-3$, $\sqrt{13}$
(b) $\Delta x=-1$, $\Delta y=3$, $m=-3$, $y=-3x+2$, $\sqrt{10}$
(c) $\Delta x=-2$, $\Delta y=-2$, $m=1$, $y=x$, $\sqrt{8}$ 6. $(x+2/7)^2+(y-41/7)^2=1300/49$

Answers for 2.2

1. $\{x \mid x \geq 3/2\}$ 2. $\{x \mid x \neq -1\}$ 3. $\{x \mid x \neq 1 \text{ and } x \neq -1\}$ 4. $\{x \mid x < 0\}$ 5. $\{x \mid x \in \mathbb{R}\}$, i.e., all x 6. $\{x \mid x \geq 0\}$ 7. $\{x \mid h-r \leq x \leq h+r\}$ 8. $\{x \mid x \geq 1\}$ 9. $\{x \mid -1/3 < x < 1/3\}$ 10. $\{x \mid x \geq 0 \text{ and } x \neq 1\}$ 11. $\{x \mid x \geq 0 \text{ and } x \neq 1\}$ 12. $\mathbb{R}$ 13. $\{x \mid x \geq 3\}$, $\{x \mid x \geq 0\}$ 14. $A=x(500-2x)$, $\{x \mid 0 \leq x \leq 250\}$ 15. $V=r(50-\pi r^2)$, $\{r \mid 0 < r \leq \sqrt{50/\pi}\}$ 16. $A=2\pi r^2+2000/r$, $\{r \mid 0 < r < \infty\}$

Answers for 2.3

Answers for 2.4

1. -5, -2.47106145, -2.4067927, -2.400676, -2.4 2. $-4/3$, $-24/7$, $7/24$, $3/4$ 3. -0.107526881, -0.11074197, -0.1110741, $\dfrac{-1}{3(3+\Delta x)} \to \dfrac{-1}{9}$ 4. $\dfrac{3+3\Delta x+\Delta x^2}{1+\Delta x} \to 3$
5. 3.31, 3.003001, 3.0000,
$3+3\Delta x+\Delta x^2 \to 3$ 6. m

Answers for 3.1

1. 10, 25/2, 20, 15, 25, 35. 2. 5, 4.1, 4.01, 4.001, $4+\Delta t \to 4$ 3. -10.29, -9.849, -9.8049,
$-9.8-4.9\Delta t \to -9.8$

Answers for 3.2

1. 7 2. 5 3. 0 4. undefined 5. 1/6 6. 0 7. 3 8. 172 9. 0 10. 2 11. does not exist 12. $\sqrt{2}$ 13. $3a^2$ 14. 512 15. -4 16. 0

Answers for 3.3

1. $-x/\sqrt{169-x^2}$ 2. $-9.8t$ 3. $2x+1/x^2$ 4. $2ax+b$ 5. $3x^2$ 8. $-2/(2x+1)^{3/2}$ 9. $5/(t+2)^2$ 10. $y=-13x+17$ 11. -8

Answers for 3.4

Answers for 3.5

1. $100x^{99}$ 2. $-100x^{-101}$ 3. $-5x^{-6}$ 4. $\pi x^{\pi-1}$ 5. $(3/4)x^{-1/4}$ 6. $-(9/7)x^{-16/7}$

Answers for 4.1

1. $15x^2+24x$ 2. $-20x^4+6x+10/x^3$ 3. $-30x+25$ 4. $6x^2+2x-8$ 5. $3x^2+6x-1$ 6. $9x^2-x/\sqrt{625-x^2}$ 7. $y=13x/4+5$ 8. $y=24x-48-\pi^3$ 9. $-49t/5+5, -49/5$ 11. $\displaystyle\sum_{k=1}^{n} ka_k x^{k-1}$ 12. $x^3/16-3x/4+4$

Answers for 4.2

1. $3x^2(x^3-5x+10)+x^3(3x^2-5)$ 2. $(x^2+5x-3)(5x^4-18x^2+6x-7)+(2x+5)(x^5-6x^3+3x^2-7x+1)$ 3. $\dfrac{\sqrt{625-x^2}}{2\sqrt{x}}-\dfrac{x\sqrt{x}}{\sqrt{625-x^2}}$ 4. $\dfrac{-1}{x^{19}\sqrt{625-x^2}}-\dfrac{20\sqrt{625-x^2}}{x^{21}}$ 5. $f'=4(2x-3), y=4x-7$

Answers for 4.3

1. $\dfrac{3x^2}{x^3-5x+10}-\dfrac{x^3(3x^2-5)}{(x^3-5x+10)^2}$ 2. $\dfrac{2x+5}{x^5-6x^3+3x^2-7x+1}-\dfrac{(x^2+5x-3)(5x^4-18x^2+6x-7)}{(x^5-6x^3+3x^2-7x+1)^2}$ 3. $\dfrac{1}{2\sqrt{x}\sqrt{625-x^2}}+\dfrac{x^{3/2}}{(625-x^2)^{3/2}}$ 4. $\dfrac{-1}{x^{19}\sqrt{625-x^2}}-\dfrac{20\sqrt{625-x^2}}{x^{21}}$ 5. $y=17x/4-41/4$ 6. $y=11x/16-15/16$ 8. $y=19/169-5x/338$ 9. $13/18$

Answers for 4.4

1. $4x^3-9x^2+x+7$ 2. $3x^2-4x+2/\sqrt{x}$ 3. $6(x^2+1)^2x$ 4. $\sqrt{169-x^2}-x^2/\sqrt{169-x^2}$
5. $(2x-4)\sqrt{25-x^2}-$
$(x^2-4x+5)x/\sqrt{25-x^2}$ 6. $-x/\sqrt{r^2-x^2}$ 7. $2x^3/\sqrt{1+x^4}$ 8. $\dfrac{1}{4\sqrt{x}(5-\sqrt{x})^{3/2}}$ 9. $6+18x$
10. $\dfrac{2x+1}{1-x}+\dfrac{x^2+x+1}{(1-x)^2}$ 11. $-1/\sqrt{25-x^2}-\sqrt{25-x^2}/x^2$ 12. $\dfrac{1}{2}\left(\dfrac{-169}{x^2}-1\right)\bigg/\sqrt{\dfrac{169}{x}-x}$
13. $\dfrac{3x^2-2x+1/x^2}{2\sqrt{x^3-x^2-(1/x)}}$ 14. $\dfrac{300x}{(100-x^2)^{5/2}}$ 15. $\dfrac{1+3x^2}{3(x+x^3)^{2/3}}$ 16. $\left(4x(x^2+1)+\dfrac{4x^3+4x}{2\sqrt{1+(x^2+1)^2}}\right)\bigg/$
$2\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ 17. $5(x+8)^4$ 18. $-3(4-x)^2$ 19. $6x(x^2+5)^2$ 20.
$-12x(6-2x^2)^2$ 21. $24x^2(1-4x^3)^{-3}$ 22. $5+5/x^2$ 23. $-8(4x-1)(2x^2-x+3)^{-3}$ 24.
$1/(x+1)^2$ 25. $3(8x-2)/(4x^2-2x+1)^2$ 26. $-3x^2+5x-1$ 27. $6x(2x-4)^3+6(3x^2+$
$1)(2x-4)^2$ 28. $-2/(x-1)^2$ 29. $4x/(x^2+1)^2$ 30. $(x^2-6x+7)/(x-3)^2$ 31. $-5/(3x-4)^2$
32. $60x^4+72x^3+18x^2+18x-6$ 33. $(5-4x)/((2x+1)^2(x-3)^2)$ 34. $1/(2(2+3x)^2)$
35. $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$ 36. $y=23x/96-29/96$ 37.
$y=3-2x/3$ 38. $y=13x/2-23/2$ 39. $y=2x-11$ 40. $y=\dfrac{20+2\sqrt{5}}{5\sqrt{4+\sqrt{5}}}x+\dfrac{3\sqrt{5}}{5\sqrt{4+\sqrt{5}}}$

Answers for 4.5

1. $2n\pi-\pi/2$, any integer n 2. $n\pi\pm\pi/6$, any integer n 3. $(\sqrt{2}+\sqrt{6})/4$ 4. $-(1+\sqrt{3})/(1-\sqrt{3})=2+\sqrt{3}$ 11. $t=\pi/2$

Answers for 5.1

1. 5 2. 7/2 3. 3/4 4. 1 5. $-\sqrt{2}/2$ 6. 7 7. 2

Answers for 5.3

1. $\sin(\sqrt{x})\cos(\sqrt{x})/\sqrt{x}$ 2. $\dfrac{\sin x}{2\sqrt{x}}+\sqrt{x}\cos x$ 3. $-\dfrac{\cos x}{\sin^2 x}$ 4. $\dfrac{(2x+1)\sin x-(x^2+x)\cos x}{\sin^2 x}$
5. $\dfrac{-\sin x\cos x}{\sqrt{1-\sin^2 x}}$

Answers for 5.4

1. $\cos^2 x – \sin^2 x$ 2. $-\sin x \cos(\cos x)$ 3. $\dfrac{\tan x + x \sec^2 x}{2\sqrt{x \tan x}}$ 4. $\dfrac{\sec^2 x(1+\sin x) - \tan x \cos x}{(1+\sin x)^2}$
5. $-\csc^2 x$ 6. $-\csc x \cot x$ 7. $3x^2 \sin(23x^2) + 46x^4 \cos(23x^2)$ 8. 0 9. $-6\cos(\cos(6x))\sin(6x)$
10. $\sin\theta/(\cos\theta+1)^2$ 11. $5t^4\cos(6t) - 6t^5\sin(6t)$ 12. $3t^2(\sin(3t)+t\cos(3t))/\cos(2t) + 2t^3\sin(3t)\sin(2t)/\cos^2(2t)$ 13. $n\pi/2$, any integer n 14. $\pi/2+n\pi$, any integer n 15. $\sqrt{3}x/2+3/4-\sqrt{3}\pi/6$ 16. $8\sqrt{3}x+4-8\sqrt{3}\pi/3$ 17. $3\sqrt{3}x/2-\sqrt{3}\pi/4$ 18. $\pi/6+2n\pi$, $5\pi/6+2n\pi$, any integer n

Answers for 5.5

Answers for 5.6

1. $2\ln(3)x3^{x^2}$ 2. $\dfrac{\cos x - \sin x}{e^x}$ 3. $2e^{2x}$ 4. $e^x\cos(e^x)$ 5. $\cos(x)e^{\sin x}$ 6. $x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$
7. $3x^2e^x + x^3e^x$ 8. $1+2^x\ln(2)$ 9. $-2x\ln(3)(1/3)^{x^2}$ 10. $e^{4x}(4x-1)/x^2$ 11. $(3x^2+3)/(x^3+3x)$ 12. $-\tan(x)$ 13. $(1-\ln(x^2))/(x^2\sqrt{\ln(x^2)})$ 14. $\sec(x)$ 15. $x^{\cos(x)}(\cos(x)/x - \cos(x)\ln(x))$ 20. e

Answers for 5.7

1. x/y 2. $-(2x+y)/(x+2y)$ 3. $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$ 4. $\sin(x)\sin(y)/(\cos(x)\cos(y))$
5. $-\sqrt{y}/\sqrt{x}$ 6. $(y\sec^2(x/y)-y^2)/(x\sec^2(x/y)+y^2)$ 7. $(y-\cos(x+y))/(\cos(x+y)-x)$ 8. $-y^2/x^2$ 9. 1 11. $y=2x\pm 6$ 12. $y=x/2\pm 3$ 13. $(\sqrt{3},2\sqrt{3})$, $(-\sqrt{3},-2\sqrt{3})$, $(2\sqrt{3},\sqrt{3})$, $(-2\sqrt{3},-\sqrt{3})$ 14. $y=7x/\sqrt{3}-8/\sqrt{3}$ 15. $y=(-y_1^{1/3}x+y_1^{1/3}x_1+x_1^{1/3}y_1)/x_1^{1/3}$
16. $(y-y_1)/(x-x_1)=(2x_1^3+2x_1y_1^2-x_1)/(2y_1^3+2y_1x_1^2+y_1)$

Answers for 5.8

3. $\dfrac{-1}{1+x^2}$ 5. $\dfrac{2x}{\sqrt{1-x^4}}$ 6. $\dfrac{e^x}{1+e^{2x}}$ 7.