MOOCULUS
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CALCULUS

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This text is based on David Guichard's open-source calculus text which in turn is a modification and expansion of notes written by Neal Koblitz at the University of Washington. New material has been added, and old material has been modified, so some portions now bear little resemblance to the original.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at http://www.math.wisc.edu/~keisler/calc.html under a Creative Commons license. In addition, the chapter on differential equations and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

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1 Limits

1.1 The Basic Ideas of Limits

Consider the function:

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

While $f(x)$ is undefined at $x = 2$, we can still plot $f(x)$ at other values, see Figure 1.1. Examining Table 1.1, we see that as x approaches 2, $f(x)$ approaches 1. We write this:

$$\text{As } x \to 2, f(x) \to 1 \qquad \text{or} \qquad \lim_{x \to 2} f(x) = 1.$$

Intuitively, $\lim_{x \to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. This leads us to the formal definition of a *limit*.

Definition The **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x \to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta, \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

If no such value of L can be found, then we say that $f(x)$ **does not exist** at $x = a$.

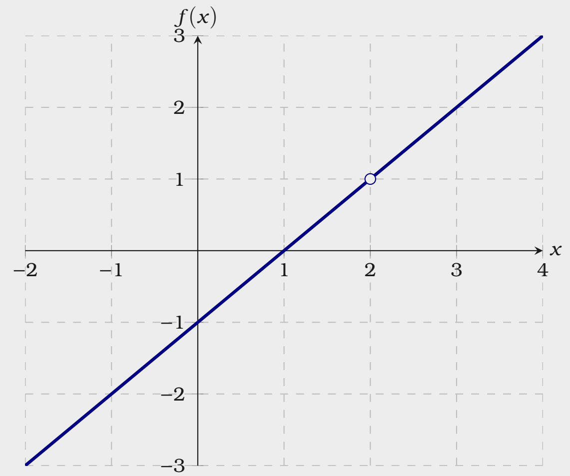

Figure 1.1: A plot of $f(x) = \frac{x^2-3x+2}{x-2}$.

x	$f(x)$	x	$f(x)$
1.7	0.7	2	undefined
1.9	0.9	2.001	1.001
1.99	0.99	2.01	1.01
1.999	0.999	2.1	1.1
2	undefined	2.3	1.3

Table 1.1: Values of $f(x) = \frac{x^2-3x+2}{x-2}$.

Equivalently, $\lim_{x \to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and $a - \delta < x < a + \delta$, we have $L - \varepsilon < f(x) < L + \varepsilon$.

In Figure 2.1, we see a geometric interpretation of this definition.

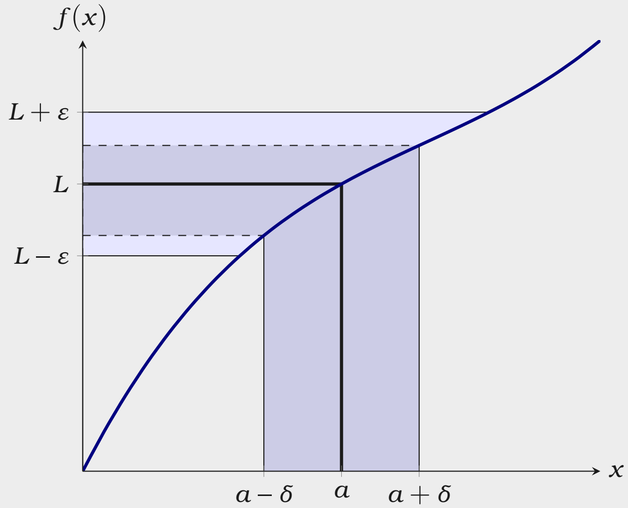

Figure 1.2: A geometric interpretation of the (ε, δ)-criterion for limits. If $0 < |x-a| < \delta$, then we have that $a-\delta < x < a+\delta$. In our diagram, we see that for all such x we are sure to have $L-\varepsilon < f(x) < L+\varepsilon$, and hence $|f(x)-L| < \varepsilon$.

Limits need not exist, let's examine two cases of this.

Example 1.1.1 Let $f(x) = \lfloor x \rfloor$. Explain why the limit

$$\lim_{x\to 2} f(x)$$

does not exist.

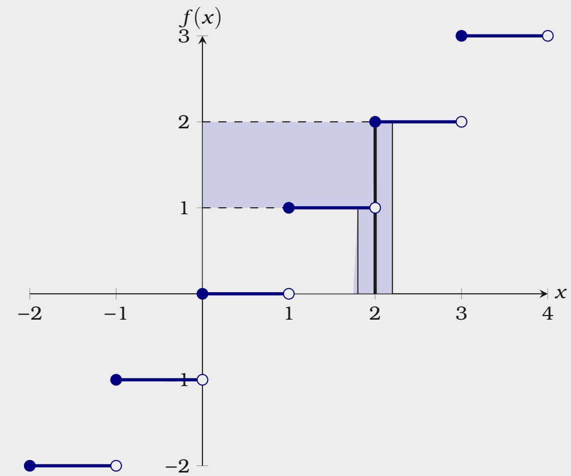

Figure 1.3: A plot of $f(x) = \lfloor x \rfloor$. Note, no matter which $\delta > 0$ is chosen, we can only at best bound $f(x)$ in the interval $[1, 2]$.

Solution *This is the function that returns the greatest integer less than or equal to x. Since $f(x)$ is defined for all real numbers, one might be tempted to think that the limit above is simply $f(2) = 2$. However, this is not the case. If $x < 2$, then $f(x) = 1$. Hence if $\varepsilon = .5$, we can* ***always*** *find a value for x (just to the left of 2) such that*

$$0 < |x-2| < \delta, \qquad \textit{where} \qquad \varepsilon < |f(x)-2|.$$

*On the other hand, $\lim_{x\to 2} f(x) \neq 1$, as in this case if $\varepsilon = .5$, we can **always** find a value for x (just to the right of 2) such that*

$$0 < |x-2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x) - 1|.$$

We've illustrated this in Figure 1.3. Moreover, no matter what value one chooses for $\lim_{x\to 2} f(x)$, we will always have a similar issue.

With the example of $f(x) = \lfloor x \rfloor$, we see that taking limits is truly different from evaluating functions.

Limits may not exist even if the function looks innocent.

Example 1.1.2 Let $f(x) = \sin\left(\frac{1}{x}\right)$. Explain why the limit

$$\lim_{x\to 0} f(x)$$

does not exist.

Solution *In this case $f(x)$ oscillates "wildly" as x approaches 0, see Figure 1.4. In fact, one can show that for any given δ, There is a value for x in the interval*

$$0 - \delta < x < 0 + \delta$$

*such that $f(x)$ is **any** value in the interval $[-1, 1]$. Hence the limit does not exist.*

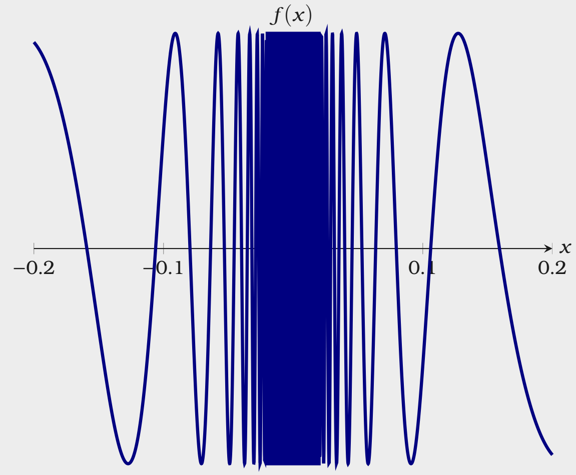

Figure 1.4: A plot of $f(x) = \sin\left(\frac{1}{x}\right)$.

Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of a *one-sided limit*:

Definition We say that the **limit** of $f(x)$ as x goes to a from the **left** is L,

$$\lim_{x\to a^-} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and

$$a - \delta < x \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

We say that the **limit** of $f(x)$ as x goes to a from the **right** is L,

$$\lim_{x \to a+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and

$$x < a + \delta \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

Limits from the left, or from the right, are collectively called **one-sided limits**.

Example 1.1.3 Let $f(x) = \lfloor x \rfloor$. Discuss

$$\lim_{x \to 2-} f(x), \qquad \lim_{x \to 2+} f(x), \qquad \text{and} \qquad \lim_{x \to 2+} f(x).$$

Solution *From the plot of $f(x)$, see Figure 1.3, we see that*

$$\lim_{x \to 2-} f(x) = 1, \qquad \textit{and} \qquad \lim_{x \to 2+} f(x) = 2.$$

Since these limits are different, $\lim_{x \to 2} f(x)$ does not exist.

Exercises for Section 1.1

(1) Evaluate the expressions by reference to the plot in Figure 1.5.

(a) $\lim_{x\to 4} f(x)$ (e) $\lim_{x\to 0+} f(x)$ (i) $\lim_{x\to 0} f(x+1)$

(b) $\lim_{x\to -3} f(x)$ (f) $f(-2)$ (j) $f(0)$

(c) $\lim_{x\to 0} f(x)$ (g) $\lim_{x\to 2-} f(x)$ (k) $\lim_{x\to 1-} f(x-4)$

(d) $\lim_{x\to 0-} f(x)$ (h) $\lim_{x\to -2-} f(x)$ (l) $\lim_{x\to 0+} f(x-2)$

(2) Use a table and a calculator to estimate $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$.

(3) Use a table and a calculator to estimate $\displaystyle\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$.

(4) Use a table and a calculator to estimate $\displaystyle\lim_{x\to 0} \frac{2^x - 1}{x}$.

(5) Sketch a plot of $f(x) = \dfrac{x}{|x|}$ and explain why $\displaystyle\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

(6) Let $f(x) = \sin\left(\dfrac{\pi}{x}\right)$. Construct three tables of the following form

x	$f(x)$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	

where $d = 1, 3, 7$. What do you notice? How do you reconcile the entries in your tables with the value of $\lim_{x\to 0} f(x)$?

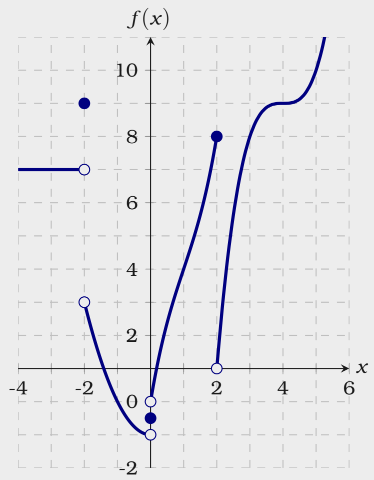

Figure 1.5: A piecewise defined function.

1.2 Limits by the Definition

Now we are going to get our hands dirty, and really use the definition of a limit.

Recall, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| + \delta$, we have $|f(x) - L| < \varepsilon$.

Example 1.2.1 Show that $\lim_{x\to 2} x^2 = 4$.

Solution *We want to show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that*

$$|x^2 - 4| < \varepsilon$$

whenever $0 < |x-2| < \delta$. Start by factoring the LHS of the inequality above

$$|x+2||x-2| < \varepsilon.$$

Since we are going to assume that $0 < |x-2| < \delta$, we will focus on the factor $|x+2|$. Since x is assumed to be close to 2, suppose that $x \in [1,3]$. In this case

$$|x+2| \leq 3+2 = 5,$$

and so we want

$$5 \cdot |x-2| < \varepsilon$$
$$|x-2| < \frac{\varepsilon}{5}$$

Recall, we assumed that $x \in [1,3]$, which is equivalent to $|x-2| < 1$. Hence we must set $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

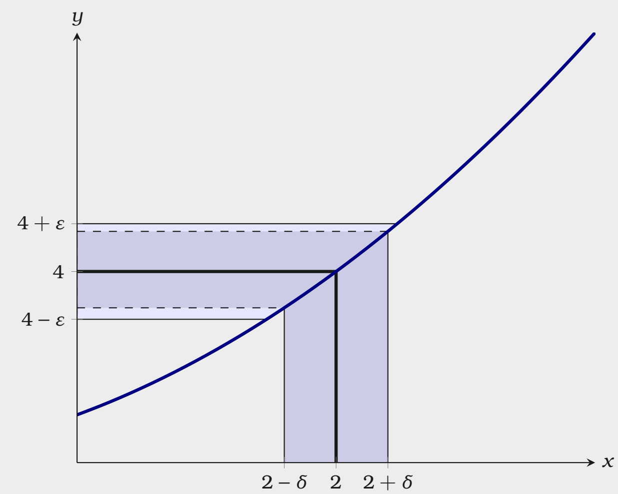

Figure 1.6: The (ε, δ)-criterion for $\lim_{x\to 2} x^2 = 4$. Here $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

When dealing with limits of polynomials, the general strategy is always the same. Let $p(x)$ be a polynomial. If showing

$$\lim_{x\to a} p(x) = L,$$

one must first factor out $|x-a|$ from $|p(x) - L|$. Next bound $x \in [a-1, a+1]$ and estimate the largest possible value of

$$\left|\frac{p(x) - L}{x-a}\right|$$

for $x \in [a-1, a+1]$. Call this estimation M. Finally, one must set $\delta = \min\left(\frac{\varepsilon}{M}, 1\right)$.

As you work with limits, you find that you need to do the same procedures again and again. The next theorems will expedite this process.

Theorem 1.2.2 *Suppose* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then*

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof *Given any* ε *we need to find a* δ *such that*

$$0 < |x < a| < \delta$$

implies

$$|f(x)g(x) - LM| < \varepsilon.$$

Here we use an algebraic trick, add $0 = -f(x)M + f(x)M$*:*

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)(g(x) - M)| + |(f(x) - L)M| \\ &= |f(x)||g(x) - M| + |f(x) - L||M|. \end{aligned}$$

This is all straightforward except perhaps for the "$\leq$". This follows from the *Triangle Inequality*. The **Triangle Inequality** states: If a and b are any real numbers then $|a + b| \leq |a| + |b|$.

Since $\lim_{x\to a} f(x) = L$, *there is a value* δ_1 *so that* $0 < |x - a| < \delta_1$ *implies* $|f(x) - L| < |\varepsilon/(2M)|$. *This means that* $0 < |x - a| < \delta_1$ *implies* $|f(x) - L||M| < \varepsilon/2$.

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + \underbrace{|f(x) - L||M|}_{<\frac{\varepsilon}{2}}.$$

If we can make $|f(x)||g(x) - M| < \varepsilon/2$, *then we'll be done. We can make* $|g(x) - M|$ *smaller than any fixed number by making* x *close enough to* a. *Unfortunately,* $\varepsilon/(2f(x))$ *is not a fixed number since* x *is a variable.*

Here we need another trick. We can find a δ_2 so that $|x-a|<\delta_2$ implies that $|f(x)-L|<1$, meaning that $L-1<f(x)<L+1$. This means that $|f(x)|<N$, where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0<|x-a|<\delta_3$ implies $|g(x)-M|<\varepsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a|<\delta$ implies that

$$|f(x)g(x)-LM|\leq \underbrace{|f(x)|}_{<N}\underbrace{|g(x)-M|}_{<\dfrac{\varepsilon}{2N}}+\underbrace{|f(x)-L||M|}_{<\dfrac{\varepsilon}{2}}.$$

so

$$\begin{aligned}|f(x)g(x)-LM| &\leq |f(x)||g(x)-M|+|f(x)-L||M|\\ &< N\frac{\varepsilon}{2N}+\left|\frac{\varepsilon}{2M}\right||M|\\ &= \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This is just what we needed, so by the definition of a limit, $\lim_{x\to a} f(x)g(x)=LM$.

Another useful way to put functions together is composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x)=\sqrt{x}$ and $g(x)=x^2+5$, then $f(g(x))=\sqrt{x^2+5}$ and $g(f(x))=(\sqrt{x})^2+5=x+5$. Here is a companion to theorem 3.3.4 for composition:

Theorem 1.2.3 *Suppose that $\lim_{x\to a} g(x)=L$ and $\lim_{x\to L} f(x)=f(L)$. Then*

$$\lim_{x\to a} f(g(x))=f(L).$$

Note the special form of the condition on f: it is not enough to know that $\lim_{x\to L} f(x) = M$, though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

Theorem 1.2.4 *Suppose that n is a positive integer. Then*

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Exercises for Section 1.2

(1) Use the definition of limits to explain why $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. Hint: Use the fact that $|\sin a| < 1$ for any real number a.

(2) Use the definition of limits to explain why $\lim_{x\to 4}(2x-5) = 3$.

(3) For each of the following limits, $\lim_{x\to a} f(x) = L$, use a graphing device to find δ such that $0 < |x-a| < \delta$ implies that $|f(x)-L| < \varepsilon$ where $\varepsilon = .1$.

(a) $\lim_{x\to 2}(3x+1) = 7$

(b) $\lim_{x\to 1}(x^2+2) = 3$

(c) $\lim_{x\to \pi}\sin(x) = 0$

(d) $\lim_{x\to 0}\tan(x) = 0$

(e) $\lim_{x\to 1}\sqrt{3x+1} = 2$

(f) $\lim_{x\to -2}\sqrt[3]{1-4x} = 3$

1.3 Limit Laws

In this section, we present a handful of tools to compute many limits without explicitly working with the definition of limit. Each of these could be proved directly as we did in the previous section.

Theorem 1.3.1 (Limit Laws) *Suppose that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, k *is some constant, and* n *is a positive integer. Then*

- $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = kL$
- $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$
- $\lim_{x\to a} (f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$
- $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ *provided if* n *is even, then* $f(x) \geq 0$ *near* a.

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 1.3.2 Compute $\lim_{x\to 1} \dfrac{x^2 - 3x + 5}{x - 2}$.

Solution *Using limit laws,*

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-3x+5}{x-2} &= \frac{\lim\limits_{x\to 1} x^2-3x+5}{\lim\limits_{x\to 1}(x-2)} \\
&= \frac{\lim\limits_{x\to 1} x^2-\lim\limits_{x\to 1} 3x+\lim\limits_{x\to 1} 5}{\lim\limits_{x\to 1} x-\lim\limits_{x\to 1} 2} \\
&= \frac{\left(\lim\limits_{x\to 1} x\right)^2-3\lim\limits_{x\to 1} x+5}{\lim\limits_{x\to 1} x-2} \\
&= \frac{1^2-3\cdot 1+5}{1-2} \\
&= \frac{1-3+5}{-1}=-3.
\end{aligned}$$

It is worth commenting on the trivial limit $\lim\limits_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x)=5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

We're primarily interested in limits that aren't so easy, namely limits in which a denominator approaches zero. The basic idea is to "divide out" by the offending factor. This is often easier said than done—here we give two examples of algebraic tricks that work on many of these limits.

Example 1.3.3 Compute $\lim\limits_{x\to 1}\dfrac{x^2+2x-3}{x-1}$.

Solution *We can't simply plug in $x = 1$ because that makes the denominator zero. However, when taking limits we assume $x \neq 1$:*

$$\lim_{x\to 1} \frac{x^2+2x-3}{x-1} = \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1}$$
$$= \lim_{x\to 1}(x+3) = 4$$

Limits allow us to examine functions where they are not defined.

Example 1.3.4 Compute $\displaystyle\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}$.

Solution *Using limit laws,*

$$\begin{aligned}\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\ &= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\ &= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\ &= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}.\end{aligned}$$

Here we are rationalizing the numerator by multiplying by the conjugate.

Exercises for Section 1.3

Compute the limits. If a limit does not exist, explain why.

(1) $\displaystyle\lim_{x\to 3} \frac{x^2+x-12}{x-3}$

(2) $\displaystyle\lim_{x\to 1} \frac{x^2+x-12}{x-3}$

(3) $\displaystyle\lim_{x\to -4} \frac{x^2+x-12}{x-3}$

(4) $\displaystyle\lim_{x\to 2} \frac{x^2+x-12}{x-2}$

(5) $\lim_{x\to 1} \frac{\sqrt{x+8}-3}{x-1}$

(6) $\lim_{x\to 0+} \sqrt{\frac{1}{x}+2} - \sqrt{\frac{1}{x}}$.

(7) $\lim_{x\to 2} 3$

(8) $\lim_{x\to 4} 3x^3 - 5x$

(9) $\lim_{x\to 0} \frac{4x-5x^2}{x-1}$

(10) $\lim_{x\to 1} \frac{x^2-1}{x-1}$

(11) $\lim_{x\to 0+} \frac{\sqrt{2-x^2}}{x}$

(12) $\lim_{x\to 0+} \frac{\sqrt{2-x^2}}{x+1}$

(13) $\lim_{x\to a} \frac{x^3-a^3}{x-a}$

(14) $\lim_{x\to 2} (x^2+4)^3$

(15) $\lim_{x\to 1} \begin{cases} x-5 & x \neq 1, \\ 7 & x = 1. \end{cases}$

1.4 Infinite Limits

Definition $\lim_{x->a} f(x) = \infty$

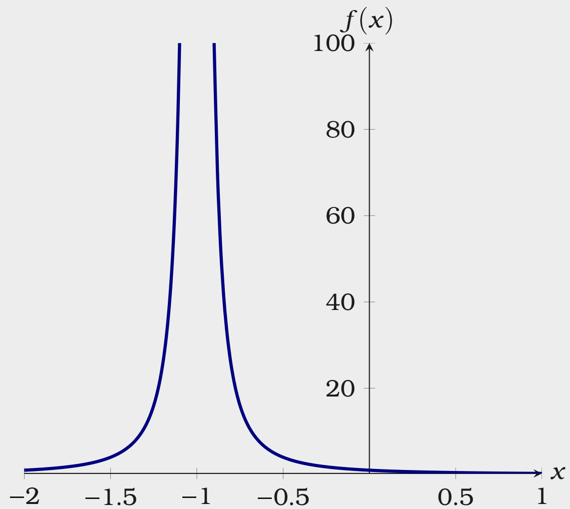

Figure 1.7: A plot of $f(x) = \frac{1}{(x+1)^2}$.

Another example of a function whose domain is not the entire x-axis is: $y = f(x) = 1/x$, the reciprocal function. We cannot substitute $x = 0$ in this formula. The function makes sense, however, for any nonzero x, so we take the domain to be: $\{x \in \mathbb{R} \mid x \neq 0\}$. The graph of this function does not have any point (x, y) with $x = 0$. As x gets close to 0 from either side, the graph goes off toward infinity. We call the vertical line $x = 0$ an **asymptote**.

1.5 Limits to Infinity

1.6 Continuity

At this point, we will discuss a few features of functions.

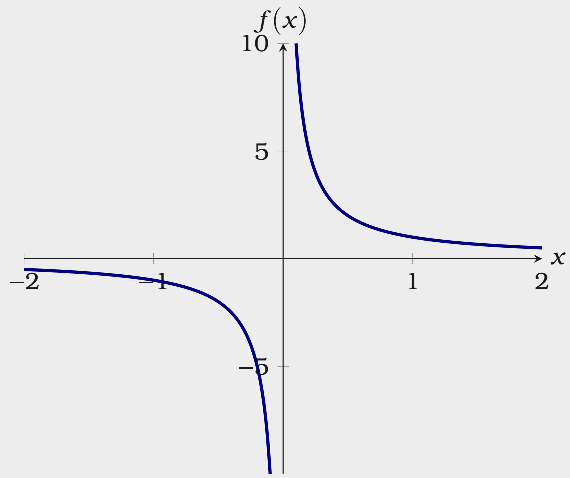

Figure 1.8: A plot of $f(x) = \frac{1}{x}$.

> **Definition** A function f is **bounded** if there is a number M such that $|f(x)| < M$ for every x in the domain of f.

Bounded. The graph in (c) appears to approach zero as x goes to both positive and negative infinity. It also never exceeds the value 1 or drops below the value 0. Because the graph never increases or decreases without bound, we say that the function represented by the graph in (c) is a **bounded** function.

For the function in (c), one such choice for M would be 10. However, the smallest (optimal) choice would be $M = 1$. In either case, simply finding an M is enough to establish boundedness. No such M exists for the hyperbola in (d) and hence we can say that it is **unbounded**.

PICTURES

> **Definition** A function f is **continuous at a point** a if $\lim_{x\to a} f(x) = f(a)$.

Definition A function f is **continuous** if it is continuous at every point in its domain.

Continuity. The graphs shown in (b) and (c) both represent **continuous** functions. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near $x = -1$ on the graph in (a) which is not continuous at that location.

Strangely, we can also say that (d) is continuous even though there is a vertical asymptote. A careful reading of the definition of continuous reveals the phrase "*at every point in its domain.*" Because the location of the asymptote, $x = 0$, is not in the domain of the function, and because the rest of the function is **well-behaved**, we can say that (d) is continuous.

We now have a collection of adjectives to describe the very rich and complex set of objects known as functions.

We close with a useful theorem about continuous functions:

Theorem 1.6.1 (Intermediate Value Theorem) *If $f(x)$ is a function that is continuous for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*

The Intermediate Value Theorem is most frequently used when $d = 0$.

For a nice proof of this theorem, see: Walk, Stephen M. *The intermediate value theorem is NOT obvious—and I am going to prove it to you.* College Math. J. 42 (2011), no. 4, 254–259.

In Figure 1.9, we see a geometric interpretation of this theorem.

Example 1.6.2 Explain why the function $f = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

By theorem 3.3.4, f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in [0, 1]$ such that $f(c) = 0$.

This example also points the way to a simple method for approximating roots.

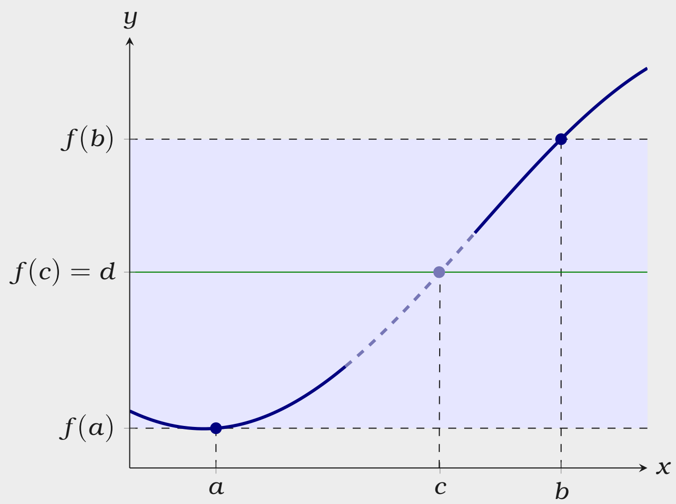

Figure 1.9: A geometric interpretation of the Intermediate Value Theorem. The function $f(x)$ is continuous on the interval $[a, b]$. Since d is in the interval $[f(a), f(b)]$, there exists a value c in $[a, b]$ such that $f(c) = d$.

Example 1.6.3 Approximate the root of the previous example to one decimal place. THIS IS A GOOD EXAMPLE - CF WITH THE PAPER LIST ABOVE.

If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6) < 0$ and $f(0.7) > 0$, so by the Intermediate Value Theorem, f has a root between 0.6 and 0.7. Repeating the process with $f(0.61)$, $f(0.62)$, and so on, we find that $f(0.61) < 0$ and $f(0.62) > 0$, so f has a root between 0.61 and 0.62, and the root is 0.6 rounded to one decimal place.

Exercises for Section 1.6

(1) For each part below sketch the graph of a function that is:

(a) Bounded, but not continuous.

(b) Differentiable and unbounded.

(c) Continuous at $x = 0$, not continuous at $x = 1$, and bounded.

(d) Differentiable everywhere except at $x = -1$, continuous, and unbounded.

(2) Is $f(x) = \sin(x)$ a bounded function? If so, find the bound—the smallest M as described in the definition of *bounded*.

(3) Is $s(t) = 1/(1+t^2)$ a bounded function? If so, find the bound—the smallest M as described in the definition of *bounded*.

(4) Is $v(u) = 2\ln|u|$ a bounded function? If so, find the bound—the smallest M as described in the definition of *bounded*.

(5) Consider the function

$$h(x) = \begin{cases} 2x-3, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x = 0$. Is h a continuous function?

(6) Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place.

(7) Approximate a root of $f = x^4 + x^3 - 5x + 1$ to one decimal place.

2 Derivatives

Suppose that $f(x)$ is a function. It is often useful to know how sensitive the value of $f(x)$ is to small changes in x. To give you a feeling why this is true, consider the following:

- If the change is zero, then x gives a local maximal or minimal values for $f(x)$.
- If $p(t)$ determines the position of an object with resepct to time, the change gives the velocity of the object.
- If $v(t)$ determines the velocity of an object with resepct to time, the change gives the accelleration of the object.
- The change can help us approximate a complicated function with a simple function.
- The change can be used to help us solve equations that we would not be able to solve via other methods.

The rate of change of a function is the slope of the tangent line. Informally, a tangent line to a curve is a line moving in the direction of the curve that "just touches" the curve at that point.

Definition Given a function $f(x)$ a line $\ell(x)$ is **tangent** FILL IN

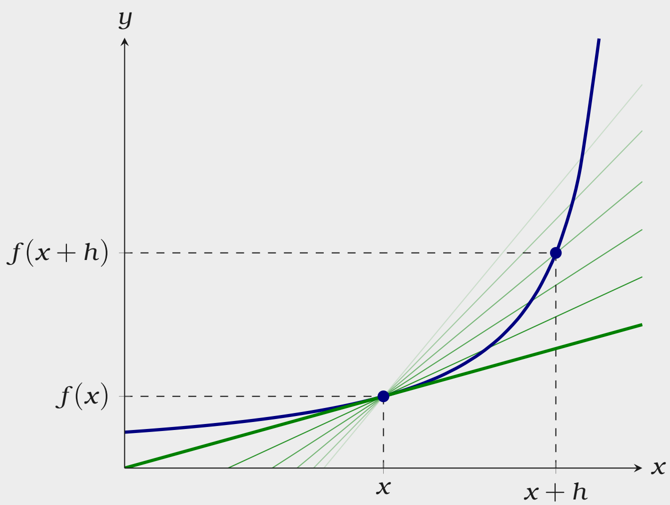

Figure 2.1: Tangent lines can be found as the limit of secant lines. The slope of the tangent line is given by $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

3 Instantaneous Rate of Change: The Derivative

3.1 The slope of a function

Suppose that y is a function of x, say $y = f(x)$. It is often necessary to know how sensitive the value of y is to small changes in x.

Example 3.1.1 Take, for example, $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin). When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

In the case of a straight line $y = mx + b$, the slope $m = \Delta y/\Delta x$ measures the change in y per unit change in x. This can be interpreted as a measure of "sensitivity"; for example, if $y = 100x + 5$, a small change in x corresponds to a change one hundred times as large in y, so y is quite sensitive to changes in x.

Let us look at the same ratio $\Delta y/\Delta x$ for our function $y = f(x) = \sqrt{625 - x^2}$ when x changes from 7 to 7.1. Here $\Delta x = 7.1 - 7 = 0.1$ is the change in x, and

$$\begin{aligned}\Delta y = f(x + \Delta x) - f(x) &= f(7.1) - f(7)\\ &= \sqrt{625 - 7.1^2} - \sqrt{625 - 7^2} \approx 23.9706 - 24 = -0.0294.\end{aligned}$$

Thus, $\Delta y/\Delta x \approx -0.0294/0.1 = -0.294$. This means that y changes by less than one third the change in x, so apparently y is not very sensitive to changes in x at $x = 7$. We say "apparently" here because we don't really know what happens between 7 and 7.1. Perhaps y changes dramatically as x runs through the values from 7 to 7.1, but at 7.1 y just happens to be close to its value at 7. This is not in fact the case for this particular function, but we don't yet know why.

One way to interpret the above calculation is by reference to a line. We have computed the slope of the line through $(7,24)$ and $(7.1,23.9706)$, called a **chord** of the circle. In general, if we draw the chord from the point $(7,24)$ to a nearby point on the semicircle $(7+\Delta x, f(7+\Delta x))$, the slope of this chord is the so-called **difference quotient**

$$\text{slope of chord} = \frac{f(7+\Delta x) - f(7)}{\Delta x} = \frac{\sqrt{625-(7+\Delta x)^2} - 24}{\Delta x}.$$

For example, if x changes only from 7 to 7.01, then the difference quotient (slope of the chord) is approximately equal to $(23.997081-24)/0.01 = -0.2919$. This is slightly less steep than the chord from $(7,24)$ to $(7.1,23.9706)$.

As the second value $7+\Delta x$ moves in towards 7, the chord joining $(7,f(7))$ to $(7+\Delta x, f(7+\Delta x))$ shifts slightly. As indicated in figure **??**, as Δx gets smaller and smaller, the chord joining $(7,24)$ to $(7+\Delta x, f(7+\Delta x))$ gets closer and closer to the **tangent line** to the circle at the point $(7,24)$. (Recall that the tangent line is the line that just grazes the circle at that point, i.e., it doesn't meet the circle at any second point.) Thus, as Δx gets smaller and smaller, the slope $\Delta y/\Delta x$ of the chord gets closer and closer to the slope of the tangent line. This is actually quite difficult to see when Δx is small, because of the scale of the graph. The values of Δx used for the figure are 1, 5, 10 and 15, not really very small values. The tangent line is the one that is uppermost at the right hand endpoint.

So far we have found the slopes of two chords that should be close to the slope of the tangent line, but what is the slope of the tangent line exactly? Since the tangent line touches the circle at just one point, we will never be able to calculate its slope directly, using two "known" points on the line. What we need is a way to capture what happens to the slopes of the chords as they get "closer and closer" to the tangent line.

Instead of looking at more particular values of Δx, let's see what happens if we do some algebra with the difference quotient using just Δx. The slope of a chord from $(7, 24)$ to a nearby point is given by

$$\begin{aligned}
\frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x} &= \frac{\sqrt{625-(7+\Delta x)^2}-24}{\Delta x}\,\frac{\sqrt{625-(7+\Delta x)^2}+24}{\sqrt{625-(7+\Delta x)^2}+24} \\
&= \frac{625-(7+\Delta x)^2-24^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{49-49-14\Delta x-\Delta x^2}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{\Delta x(-14-\Delta x)}{\Delta x(\sqrt{625-(7+\Delta x)^2}+24)} \\
&= \frac{-14-\Delta x}{\sqrt{625-(7+\Delta x)^2}+24}
\end{aligned}$$

Now, can we tell by looking at this last formula what happens when Δx gets very close to zero? The numerator clearly gets very close to -14 while the denominator gets very close to $\sqrt{625-7^2}+24 = 48$. Is the fraction therefore very close to $-14/48 = -7/24 \cong -0.29167$? It certainly seems reasonable, and in fact it is true: as Δx gets closer and closer to zero, the difference quotient does in fact get closer and closer to $-7/24$, and so the slope of the tangent line is exactly $-7/24$.

What about the slope of the tangent line at $x = 12$? Well, 12 can't be all that different from 7; we just have to redo the calculation with 12 instead of 7. This won't be hard, but it will be a bit tedious. What if we try to do all the algebra without using a specific value for x? Let's copy from above, replacing 7 by x. We'll have to do a bit more than that—for example, the "24" in the calculation came from $\sqrt{625-7^2}$, so we'll need to fix that too.

$$\begin{aligned}
&\frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x} = \\
&= \frac{\sqrt{625-(x+\Delta x)^2}-\sqrt{625-x^2}}{\Delta x}\,\frac{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}} \\
&= \frac{625-(x+\Delta x)^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{625-x^2-2x\Delta x-\Delta x^2-625+x^2}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{\Delta x(-2x-\Delta x)}{\Delta x(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{-2x-\Delta x}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}
\end{aligned}$$

Now what happens when Δx is very close to zero? Again it seems apparent that the quotient will be very close to

$$\frac{-2x}{\sqrt{625-x^2}+\sqrt{625-x^2}} = \frac{-2x}{2\sqrt{625-x^2}} = \frac{-x}{\sqrt{625-x^2}}.$$

Replacing x by 7 gives $-7/24$, as before, and now we can easily do the computation for 12 or any other value of x between -25 and 25.

So now we have a single, simple formula, $-x/\sqrt{625-x^2}$, that tells us the slope of the tangent line for any value of x. This slope, in turn, tells us how sensitive the value of y is to changes in the value of x.

What do we call such a formula? That is, a formula with one variable, so that substituting an "input" value for the variable produces a new "output" value? This is a function. Starting with one function, $\sqrt{625-x^2}$, we have derived, by means of some slightly nasty algebra, a new function, $-x/\sqrt{625-x^2}$, that gives us important information about the original function. This new function in fact is called the **derivative** of the original function. If the original is referred to as f or y then the derivative is often written f' or y' and pronounced "f prime" or "y prime", so in this case we might write $f'(x) = -x/\sqrt{625-x^2}$. At a particular point, say $x = 7$, we say that $f'(7) = -7/24$ or "f prime of 7 is $-7/24$" or "the derivative of f at 7 is $-7/24$."

To summarize, we compute the derivative of $f(x)$ by forming the difference quotient

$$\frac{f(x+\Delta x)-f(x)}{\Delta x},$$

which is the slope of a line, then we figure out what happens when Δx gets very close to 0.

We should note that in the particular case of a circle, there's a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining $(0,0)$ to $(7,24)$ has slope $24/7$. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{625-x^2})$ has slope $\sqrt{625-x^2}/x$, so the slope of the tangent line is $-x/\sqrt{625-x^2}$, as before. It is **NOT** always true that a tangent line is perpendicular to a line from the origin—don't use this shortcut in any other circumstance.

As above, and as you might expect, for different values of x we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of f, or the slope of its tangent line, is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line to a line, but if it makes sense at all the tangent line must be the line itself. It is not hard to see that the derivative of $f(x) = mx + b$ is $f'(x) = m$; see exercise 6 .

Exercises for Section 3.1

(1) Draw the graph of the function $y = f(x) = \sqrt{169-x^2}$ between $x = 0$ and $x = 13$. Find the slope $\Delta y/\Delta x$ of the chord between the points of the circle lying over (a) $x = 12$ and $x = 13$, (b) $x = 12$ and $x = 12.1$, (c) $x = 12$ and $x = 12.01$, (d) $x = 12$ and $x = 12.001$. Now use the geometry of tangent lines on a circle to find (e) the exact value of the derivative $f'(12)$. Your answers to (a)–(d) should be getting closer and closer to your answer to (e).

(2) Use geometry to find the derivative $f'(x)$ of the function $f(x) = \sqrt{625 - x^2}$ in the text for each of the following x: (a) 20, (b) 24, (c) −7, (d) −15. Draw a graph of the upper semicircle, and draw the tangent line at each of these four points.

(3) Draw the graph of the function $y = f(x) = 1/x$ between $x = 1/2$ and $x = 4$. Find the slope of the chord between (a) $x = 3$ and $x = 3.1$, (b) $x = 3$ and $x = 3.01$, (c) $x = 3$ and $x = 3.001$. Now use algebra to find a simple formula for the slope of the chord between $(3, f(3))$ and $(3 + \Delta x, f(3 + \Delta x))$. Determine what happens when Δx approaches 0. In your graph of $y = 1/x$, draw the straight line through the point $(3, 1/3)$ whose slope is this limiting value of the difference quotient as Δx approaches 0.

(4) Find an algebraic expression for the difference quotient $\big(f(1 + \Delta x) - f(1)\big)/\Delta x$ when $f(x) = x^2 - (1/x)$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(1)$.

(5) Draw the graph of $y = f(x) = x^3$ between $x = 0$ and $x = 1.5$. Find the slope of the chord between (a) $x = 1$ and $x = 1.1$, (b) $x = 1$ and $x = 1.001$, (c) $x = 1$ and $x = 1.00001$. Then use algebra to find a simple formula for the slope of the chord between 1 and $1 + \Delta x$. (Use the expansion $(A + B)^3 = A^3 + 3A^2B + 3AB^2 + B^3$.) Determine what happens as Δx approaches 0, and in your graph of $y = x^3$ draw the straight line through the point $(1, 1)$ whose slope is equal to the value you just found.

(6) Find an algebraic expression for the difference quotient $(f(x + \Delta x) - f(x))/\Delta x$ when $f(x) = mx + b$. Simplify the expression as much as possible. Then determine what happens as Δx approaches 0. That value is $f'(x)$.

(7) Sketch the unit circle. Discuss the behavior of the slope of the tangent line at various angles around the circle. Which trigonometric function gives the slope of the tangent line at an angle ϑ? Why? Hint: think in terms of ratios of sides of triangles.

(8) Sketch the parabola $y = x^2$. For what values of x on the parabola is the slope of the tangent line positive? Negative? What do you notice about the graph at the point(s) where the sign of the slope changes from positive to negative and vice versa?

3.2 An example

We started the last section by saying, "It is often necessary to know how sensitive the value of y is to small changes in x." We have seen one purely mathematical example of this: finding the "steepness" of a curve at a point is precisely this problem. Here is a more applied example.

With careful measurement it might be possible to discover that a dropped ball has height $h(t) = h_0 - kt^2$, t seconds after it is released. (Here h_0 is the initial height of the ball, when $t = 0$, and k is some number determined by the experiment.) A natural question is then, "How fast is the ball going at time t?" We can certainly get a pretty good idea with a little simple arithmetic. To make the calculation more concrete, let's say $h_0 = 100$ meters and $k = 4.9$ and suppose we're interested in the speed at $t = 2$. We know that when $t = 2$ the height is $100 - 4 \cdot 4.9 = 80.4$. A second later, at $t = 3$, the height is $100 - 9 \cdot 4.9 = 55.9$, so in that second the ball has traveled $80.4 - 55.9 = 24.5$ meters. This means that the *average* speed during that time was 24.5 meters per second. So we might guess that 24.5 meters per second is not a terrible estimate of the speed at $t = 2$. But certainly we can do better. At $t = 2.5$ the height is $100 - 4.9(2.5)^2 = 69.375$. During the half second from $t = 2$ to $t = 2.5$ the ball dropped $80.4 - 69.375 = 11.025$ meters, at an average speed of $11.025/(1/2) = 22.05$ meters per second; this should be a better estimate of the speed at $t = 2$. So it's clear now how to get better and better approximations: compute average speeds over shorter and shorter time intervals. Between $t = 2$ and $t = 2.01$, for example, the ball drops 0.19649 meters in one hundredth of a second, at an average speed of 19.649 meters per second.

We can't do this forever, and we still might reasonably ask what the actual speed precisely at $t = 2$ is. If Δt is some tiny amount of time, what we want to know is what happens to the average speed $(h(2) - h(2 + \Delta t))/\Delta t$ as Δt gets smaller and smaller. Doing a bit of algebra:

$$\begin{aligned}\frac{h(2)-h(2+\Delta t)}{\Delta t} &= \frac{80.4-(100-4.9(2+\Delta t)^2)}{\Delta t}\\ &= \frac{80.4-100+19.6+19.6\Delta t+4.9\Delta t^2}{\Delta t}\\ &= \frac{19.6\Delta t+4.9\Delta t^2}{\Delta t}\\ &= 19.6+4.9\Delta t\end{aligned}$$

When Δt is very small, this is very close to 19.6, and indeed it seems clear that as Δt goes to zero, the average speed goes to 19.6, so the exact speed at $t = 2$ is 19.6 meters per second. This calculation should look very familiar. In the language of the previous section, we might have started with $f(x) = 100-4.9x^2$ and asked for the slope of the tangent line at $x = 2$. We would have answered that question by computing

$$\frac{f(2+\Delta x)-f(2)}{\Delta x} = \frac{-19.6\Delta x-4.9\Delta x^2}{\Delta x} = -19.6-4.9\Delta x$$

The algebra is the same, except that following the pattern of the previous section the subtraction would be reversed, and we would say that the slope of the tangent line is -19.6. Indeed, in hindsight, perhaps we should have subtracted the other way even for the dropping ball. At $t = 2$ the height is 80.4; one second later the height is 55.9. The usual way to compute a “distance traveled” is to subtract the earlier position from the later one, or $55.9-80.4 = -24.5$. This tells us that the distance traveled is 24.5 meters, and the negative sign tells us that the height went down during the second. If we continue the original calculation we then get -19.6 meters per second as the exact speed at $t = 2$. If we interpret the negative sign as meaning that the motion is downward, which seems reasonable, then in fact this is the same answer as before, but with even more information, since the numerical answer contains the direction of motion as well as the speed. Thus, the speed of the ball is the value of the derivative of a certain function, namely, of the function that gives the position of the ball. (More properly, this is the *velocity* of the ball; velocity is signed speed, that is, speed with a direction indicated by the sign.)

The upshot is that this problem, finding the speed of the ball, is *exactly* the same problem mathematically as finding the slope of a curve. This may already be enough evidence to convince you that whenever some quantity is changing (the height of a curve or the height of a ball or the size of the economy or the distance of a space probe from earth or the population of the world) the rate at which the quantity is changing can, in principle, be computed in exactly the same way, by finding a derivative.

Exercises for Section 3.2

(1) An object is traveling in a straight line so that its position (that is, distance from some fixed point) is given by this table:

time (seconds)	0	1	2	3
distance (meters)	0	10	25	60

Find the average speed of the object during the following time intervals: $[0,1]$, $[0,2]$, $[0,3]$, $[1,2]$, $[1,3]$, $[2,3]$. If you had to guess the speed at $t=2$ just on the basis of these, what would you guess?

(2) Let $y=f(t)=t^2$, where t is the time in seconds and y is the distance in meters that an object falls on a certain airless planet. Draw a graph of this function between $t=0$ and $t=3$. Make a table of the average speed of the falling object between (a) 2 sec and 3 sec, (b) 2 sec and 2.1 sec, (c) 2 sec and 2.01 sec, (d) 2 sec and 2.001 sec. Then use algebra to find a simple formula for the average speed between time 2 and time $2+\Delta t$. (If you substitute $\Delta t=1$, 0.1, 0.01, 0.001 in this formula you should again get the answers to parts (a)–(d).) Next, in your formula for average speed (which should be in simplified form) determine what happens as Δt approaches zero. This is the instantaneous speed. Finally, in your graph of $y=t^2$ draw the straight line through the point $(2,4)$ whose slope is the instantaneous velocity you just computed; it should of course be the tangent line.

(3) If an object is dropped from an 80-meter high window, its height y above the ground at time t seconds is given by the formula $y = f(t) = 80 - 4.9t^2$. (Here we are neglecting air resistance; the graph of this function was shown in figure **??**.) Find the average velocity of the falling object between (a) 1 sec and 1.1 sec, (b) 1 sec and 1.01 sec, (c) 1 sec and 1.001 sec. Now use algebra to find a simple formula for the average velocity of the falling object between 1 sec and $1 + \Delta t$ sec. Determine what happens to this average velocity as Δt approaches 0. That is the instantaneous velocity at time $t = 1$ second (it will be negative, because the object is falling).

3.3 Limits

In the previous two sections we computed some quantities of interest (slope, velocity) by seeing that some expression "goes to" or "approaches" or "gets really close to" a particular value. In the examples we saw, this idea may have been clear enough, but it is too fuzzy to rely on in more difficult circumstances. In this section we will see how to make the idea more precise.

There is an important feature of the examples we have seen. Consider again the formula

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

We wanted to know what happens to this fraction as "Δx goes to zero." Because we were able to simplify the fraction, it was easy to see the answer, but it was not quite as simple as "substituting zero for Δx," as that would give

$$\frac{-19.6 \cdot 0 - 4.9 \cdot 0}{0},$$

which is meaningless. The quantity we are really interested in does not make sense "at zero," and this is why the answer to the original problem (finding a velocity or a slope) was not immediately obvious. In other words, we are generally going to want to figure out what a quantity "approaches" in situations where we can't merely plug in a value. If you would like to think about a hard example (which we will analyze later) consider what happens to $(\sin x)/x$ as x approaches zero.

Example 3.3.1 Does $\sqrt{x}$ approach 1.41 as x approaches 2? In this case it is possible to compute the actual value $\sqrt{2}$ to a high precision to answer the question. But since in general we won't be able to do that, let's not. We might start by computing $\sqrt{x}$ for values of x close to 2, as we did in the previous sections. Here are some values: $\sqrt{2.05} = 1.431782106$, $\sqrt{2.04} = 1.428285686$, $\sqrt{2.03} = 1.424780685$, $\sqrt{2.02} = 1.421267040$, $\sqrt{2.01} = 1.417744688$, $\sqrt{2.005} = 1.415980226$, $\sqrt{2.004} = 1.415627070$, $\sqrt{2.003} = 1.415273825$, $\sqrt{2.002} = 1.414920492$, $\sqrt{2.001} = 1.414567072$. So it looks at least possible that indeed these values "approach" 1.41—already $\sqrt{2.001}$ is quite close. If we continue this process, however, at some point we will appear to "stall." In fact, $\sqrt{2} = 1.414213562\ldots$, so we will never even get as far as 1.4142, no matter how long we continue the sequence.

So in a fuzzy, everyday sort of sense, it is true that $\sqrt{x}$ "gets close to" 1.41, but it does not "approach" 1.41 in the sense we want. To compute an exact slope or an exact velocity, what we want to know is that a given quantity becomes "arbitrarily close" to a fixed value, meaning that the first quantity can be made "as close as we like" to the fixed value. Consider again the quantities

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6 - 4.9\Delta x.$$

These two quantities are equal as long as Δx is not zero; if Δx is zero, the left hand quantity is meaningless, while the right hand one is -19.6. Can we say more than we did before about why the right hand side "approaches" -19.6, in the desired sense? Can we really make it "as close as we want" to -19.6? Let's try a test case. Can we make $-19.6 - 4.9\Delta x$ within one millionth (0.000001) of -19.6? The values within a millionth of -19.6 are those in the interval $(-19.600001, -19.599999)$. As Δx approaches zero, does $-19.6 - 4.9\Delta x$ eventually reside inside this interval? If Δx is positive, this would require that $-19.6 - 4.9\Delta x > -19.600001$. This is something we can manipulate with a little algebra:

$$\begin{aligned} -19.6 - 4.9\Delta x &> -19.600001 \\ -4.9\Delta x &> -0.000001 \\ \Delta x &< -0.000001/-4.9 \\ \Delta x &< 0.0000002040816327\ldots \end{aligned}$$

Thus, we can say with certainty that if Δx is positive and less than 0.0000002, then $\Delta x < 0.0000002040816327\ldots$ and so $-19.6 - 4.9\Delta x > -19.600001$. We could do a similar calculation if Δx is negative.

So now we know that we can make $-19.6 - 4.9\Delta x$ within one millionth of -19.6. But can we make it "as close as we want"? In this case, it is quite simple to see that the answer is yes, by modifying the calculation we've just done. It may be helpful to think of this as a game. I claim that I can make $-19.6 - 4.9\Delta x$ as close as you desire to -19.6 by making Δx "close enough" to zero. So the game is: you give me a number, like 10^{-6}, and I have to come up with a number representing how close Δx must be to zero to guarantee that $-19.6 - 4.9\Delta x$ is at least as close to -19.6 as you have requested.

Now if we actually play this game, I could redo the calculation above for each new number you provide. What I'd like to do is somehow see that I will always succeed, and even more, I'd like to have a simple strategy so that I don't have to do all that algebra every time. A strategy in this case would be a formula that gives me a correct answer no matter what you specify. So suppose the number you give me is ε. How close does Δx have to be to zero to guarantee that $-19.6 - 4.9\Delta x$ is in $(-19.6 - \varepsilon, -19.6 + \varepsilon)$? If Δx is positive, we need:

$$\begin{aligned} -19.6 - 4.9\Delta x &> -19.6 - \varepsilon \\ -4.9\Delta x &> -\varepsilon \\ \Delta x &< -\varepsilon / -4.9 \\ \Delta x &< \varepsilon / 4.9 \end{aligned}$$

So if I pick any number δ that is less than $\varepsilon/4.9$, the algebra tells me that whenever $\Delta x < \delta$ then $\Delta x < \varepsilon/4.9$ and so $-19.6 - 4.9\Delta x$ is within ε of -19.6. (This is exactly what I did in the example: I picked $\delta = 0.0000002 < 0.0000002040816327\ldots$.) A similar calculation again works for negative Δx. The important fact is that this is now a completely general result—it shows that I can always win, no matter what "move" you make.

Now we can codify this by giving a precise definition to replace the fuzzy, "gets closer and closer" language we have used so far. Henceforward, we will say something like "the limit of $(-19.6\Delta x - 4.9\Delta x^2)/\Delta x$ as Δx goes to zero is -19.6," and abbreviate this mouthful as

$$\lim_{\Delta x \to 0} \frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x} = -19.6.$$

Here is the actual, official definition of "limit".

Definition Suppose f is a function. We say that the **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x\to a} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| < \delta$, $|f(x)-L| < \varepsilon$.

The ε and δ here play exactly the role they did in the preceding discussion. The definition says, in a very precise way, that $f(x)$ can be made as close as desired to L (that's the $|f(x)-L| < \varepsilon$ part) by making x close enough to a (the $0 < |x-a| < \delta$ part). Note that we specifically make no mention of what must happen if $x = a$, that is, if $|x-a| = 0$. This is because in the cases we are most interested in, substituting a for x doesn't even make sense.

Make sure you are not confused by the names of important quantities. The generic definition talks about $f(x)$, but the function and the variable might have other names. In the discussion above, the function we analyzed was

$$\frac{-19.6\Delta x - 4.9\Delta x^2}{\Delta x}.$$

and the variable of the limit was not x but Δx. The x was the variable of the original function; when we were trying to compute a slope or a velocity, x was essentially a fixed quantity, telling us at what point we wanted the slope. (In the velocity problem, it was literally a fixed quantity, as we focused on the time 2.) The quantity a of the definition in all the examples was zero: we were always interested in what happened as Δx became very close to zero.

Armed with a precise definition, we can now prove that certain quantities behave in a particular way. The bad news is that even proofs for simple quantities can be quite tedious and complicated; the good news is that we rarely need to do such proofs, because most expressions act the way you would expect, and this can be proved once and for all.

Let's show carefully that $\lim_{x\to 2} x + 4 = 6$. This is not something we "need" to prove, since it is "obviously" true. But if we couldn't prove it using our official definition there would be something very wrong with the definition.

As is often the case in mathematical proofs, it helps to work backwards. We want to end up showing that under certain circumstances $x+4$ is close to 6; precisely, we want to show that $|x+4-6| < \varepsilon$, or $|x-2| < \varepsilon$. Under what circumstances? We want this to be true whenever $0 < |x-2| < \delta$. So the question becomes: can we choose a value for δ that guarantees that $0 < |x-2| < \delta$ implies $|x-2| < \varepsilon$? Of course: no matter what ε is, $\delta = \varepsilon$ works.

So it turns out to be very easy to prove something "obvious," which is nice. It doesn't take long before things get trickier, however.

Example 3.3.2 It seems clear that $\lim_{x\to 2} x^2 = 4$. Let's try to prove it. We will want to be able to show that $|x^2-4| < \varepsilon$ whenever $0 < |x-2| < \delta$, by choosing δ carefully. Is there any connection between $|x-2|$ and $|x^2-4|$? Yes, and it's not hard to spot, but it is not so simple as the previous example. We can write $|x^2-4| = |(x+2)(x-2)|$. Now when $|x-2|$ is small, part of $|(x+2)(x-2)|$ is small, namely $(x-2)$. What about $(x+2)$? If x is close to 2, $(x+2)$ certainly can't be too big, but we need to somehow be precise about it. Let's recall the "game" version of what is going on here. You get to pick an ε and I have to pick a δ that makes things work out. Presumably it is the really tiny values of ε I need to worry about, but I have to be prepared for anything, even an apparently "bad" move like $\varepsilon = 1000$. I expect that ε is going to be small, and that the corresponding δ will be small, certainly less than 1. If $\delta \le 1$ then $|x+2| < 5$ when $|x-2| < \delta$ (because if x is within 1 of 2, then x is between 1 and 3 and $x+2$ is between 3 and 5). So then I'd be trying to show that $|(x+2)(x-2)| < 5|x-2| < \varepsilon$. So now how can I pick δ so that $|x-2| < \delta$ implies $5|x-2| < \varepsilon$? This is easy: use $\delta = \varepsilon/5$, so $5|x-2| < 5(\varepsilon/5) = \varepsilon$. But what if the ε you choose is not small? If you choose $\varepsilon = 1000$, should I pick $\delta = 200$? No, to keep things "sane" I will never pick a δ bigger than 1. Here's the final "game strategy:" When you pick a value for ε I will pick $\delta = \varepsilon/5$ or $\delta = 1$, whichever is smaller. Now when $|x-2| < \delta$, I know both that $|x+2| < 5$ and that $|x-2| < \varepsilon/5$. Thus $|(x+2)(x-2)| < 5(\varepsilon/5) = \varepsilon$.

This has been a long discussion, but most of it was explanation and scratch work. If this were written down as a proof, it would be quite short, like this:

Proof that $\lim_{x\to 2} x^2 = 4$. Given any ε, pick $\delta = \varepsilon/5$ or $\delta = 1$, whichever is smaller. Then when $|x-2| < \delta$, $|x+2| < 5$ and $|x-2| < \varepsilon/5$. Hence

$|x^2 - 4| = |(x+2)(x-2)| < 5(\varepsilon/5) = \varepsilon.$

It probably seems obvious that $\lim_{x\to 2} x^2 = 4$, and it is worth examining more closely why it seems obvious. If we write $x^2 = x \cdot x$, and ask what happens when x approaches 2, we might say something like, "Well, the first x approaches 2, and the second x approaches 2, so the product must approach $2 \cdot 2$." In fact this is pretty much right on the money, except for that word "must." Is it really true that if x approaches a and y approaches b then xy approaches ab? It is, but it is not really obvious, since x and y might be quite complicated. The good news is that we can see that this is true once and for all, and then we don't have to worry about it ever again. When we say that x might be "complicated" we really mean that in practice it might be a function. Here is then what we want to know:

Theorem 3.3.3 *Suppose* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then* $\lim_{x\to a} f(x)g(x) = LM$.

Proof *We have to use the official definition of limit to make sense of this. So given any* ε *we need to find a* δ *so that* $0 < |x-a| < \delta$ *implies* $|f(x)g(x) - LM| < \varepsilon$. *What do we have to work with? We know that we can make* $f(x)$ *close to* L *and* $g(x)$ *close to* M, *and we have to somehow connect these facts to make* $f(x)g(x)$ *close to* LM.

We use, as is so often the case, a little algebraic trick:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\
&= |f(x)(g(x) - M) + (f(x) - L)M| \\
&\le |f(x)(g(x) - M)| + |(f(x) - L)M| \\
&= |f(x)||g(x) - M| + |f(x) - L||M|.
\end{aligned}$$

This is all straightforward except perhaps for the "$\le$*". That is an example of the* **triangle inequalitypage:triangle inequality**, *which says that if* a *and* b *are any real numbers then* $|a+b| \le |a| + |b|$. *If you look at a few examples, using positive and negative numbers in various combinations for* a *and* b, *you* *should quickly understand why this is true; we will not prove it formally.*

Since $\lim_{x\to a} f(x) = L$, *there is a value* δ_1 *so that* $0 < |x-a| < \delta_1$ *implies* $|f(x)-L| < |\varepsilon/(2M)|$, *This means that* $0 < |x-a| < \delta_1$ *implies* $|f(x)-L||M| < \varepsilon/2$. *You can see where this is going: if we can make* $|f(x)||g(x)-M| < \varepsilon/2$ *also, then we'll be done.*

We can make $|g(x)-M|$ *smaller than any fixed number by making* x *close enough to* a*; unfortunately,* $\varepsilon/(2f(x))$ *is not a fixed number, since* x *is a variable. Here we need another little trick, just like the one we used in analyzing* x^2. *We can find a* δ_2 *so that* $|x-a| < \delta_2$ *implies that* $|f(x)-L| < 1$, *meaning that* $L-1 < f(x) < L+1$. *This means that* $|f(x)| < N$, *where* N *is either* $|L-1|$ *or* $|L+1|$, *depending on whether* L *is negative or positive. The important point is that* N *doesn't depend on* x. *Finally, we know that there is a* δ_3 *so that* $0 < |x-a| < \delta_3$ *implies* $|g(x)-M| < \varepsilon/(2N)$. *Now we're ready to put everything together. Let* δ *be the smallest of* δ_1, δ_2, *and* δ_3. *Then* $|x-a| < \delta$ *implies that* $|f(x)-L| < |\varepsilon/(2M)|$, $|f(x)| < N$, *and* $|g(x)-M| < \varepsilon/(2N)$. *Then*

$$\begin{aligned}|f(x)g(x)-LM| &\le |f(x)||g(x)-M| + |f(x)-L||M| \\ &< N\frac{\varepsilon}{2N} + \left|\frac{\varepsilon}{2M}\right||M| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

This is just what we needed, so by the official definition, $\lim_{x\to a} f(x)g(x) = LM$.

A handful of such theorems give us the tools to compute many limits without explicitly working with the definition of limit.

Theorem 3.3.4 *Suppose that* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$ *and* k *is some constant. Then*

$$\lim_{x\to a} kf(x) = k\lim_{x\to a} f(x) = kL$$

$$\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L+M$$

$$\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L-M$$

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x) = LM$$

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L}{M},\ \textit{if } M \textit{ is not } 0$$

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 3.3.5 Compute $\lim_{x\to 1} \dfrac{x^2-3x+5}{x-2}$. If we apply the theorem in all its gory detail, we get

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1}(x^2-3x+5)}{\lim_{x\to 1}(x-2)} \\
&= \frac{(\lim_{x\to 1} x^2) - (\lim_{x\to 1} 3x) + (\lim_{x\to 1} 5)}{(\lim_{x\to 1} x) - (\lim_{x\to 1} 2)} \\
&= \frac{(\lim_{x\to 1} x)^2 - 3(\lim_{x\to 1} x) + 5}{(\lim_{x\to 1} x) - 2} \\
&= \frac{1^2 - 3\cdot 1 + 5}{1-2} \\
&= \frac{1-3+5}{-1} = -3
\end{aligned}$$

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

Of course, as we've already seen, we're primarily interested in limits that aren't so easy, namely, limits in which a denominator approaches zero. There are a handful of algebraic tricks that work on many of these limits.

Example 3.3.6 Compute $\lim_{x\to 1} \frac{x^2+2x-3}{x-1}$. We can't simply plug in $x = 1$ because that makes the denominator zero. However:

$$\begin{aligned} \lim_{x\to 1} \frac{x^2+2x-3}{x-1} &= \lim_{x\to 1} \frac{(x-1)(x+3)}{x-1} \\ &= \lim_{x\to 1}(x+3) = 4 \end{aligned}$$

While theorem 3.3.4 is very helpful, we need a bit more to work easily with limits. Since the theorem applies when some limits are already known, we need to know the behavior of some functions that cannot themselves be constructed from the simple arithmetic operations of the theorem, such as $\sqrt{x}$. Also, there is one other extraordinarily useful way to put functions together: composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2+5$, then $f(g(x)) = \sqrt{x^2+5}$ and $g(f(x)) = (\sqrt{x})^2+5 = x+5$. Here is a companion to theorem 3.3.4 for composition:

Theorem 3.3.7 *Suppose that* $\lim_{x\to a} g(x) = L$ *and* $\lim_{x\to L} f(x) = f(L)$. *Then*

$$\lim_{x\to a} f(g(x)) = f(L).$$

Note the special form of the condition on f: it is not enough to know that $\lim_{x\to L} f(x) = M$, though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

Theorem 3.3.8 *Suppose that* n *is a positive integer. Then*

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Another of the most common algebraic tricks was used in section 3.1. Here's another example:

Example 3.3.9 Compute $\lim_{x\to -1} \dfrac{\sqrt{x+5}-2}{x+1}$.

$$\begin{aligned}
\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}\,\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\
&= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\
&= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}
\end{aligned}$$

At the very last step we have used theorems 3.3.7 and 3.3.8.

Occasionally we will need a slightly modified version of the limit definition. Consider the function $f(x) = \sqrt{1-x^2}$, the upper half of the unit circle. What can we say about $\lim_{x\to 1} f(x)$? It is apparent from the graph of this familiar function that as x gets close to 1 from the left, the value of $f(x)$ gets close to zero. It does not even make sense to ask what happens as x approaches 1 from the right, since $f(x)$ is not defined there. The definition of the limit, however, demands that $f(1+\Delta x)$ be close to $f(1)$ whether Δx is positive or negative. Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of **one sided limit**:

Definition (One-sided limit) Suppose that $f(x)$ is a function. We say that $\lim_{x\to a^-} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < a - x < \delta$, $|f(x) - L| < \varepsilon$. We say that $\lim_{x\to a^+} f(x) = L$ if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < x - a < \delta$, $|f(x) - L| < \varepsilon$.

Usually $\lim_{x\to a^-} f(x)$ is read "the limit of $f(x)$ from the left" and $\lim_{x\to a^+} f(x)$ is read "the limit of $f(x)$ from the right".

Example 3.3.10 Discuss $\lim_{x\to 0} \frac{x}{|x|}$, $\lim_{x\to 0^-} \frac{x}{|x|}$, and $\lim_{x\to 0^+} \frac{x}{|x|}$.

The function $f(x) = x/|x|$ is undefined at 0; when $x > 0$, $|x| = x$ and so $f(x) = 1$; when $x < 0$, $|x| = -x$ and $f(x) = -1$. Thus $\lim_{x\to 0^-} \frac{x}{|x|} = \lim_{x\to 0^-} -1 = -1$ while $\lim_{x\to 0^+} \frac{x}{|x|} = \lim_{x\to 0^+} 1 = 1$. The limit of $f(x)$ must be equal to both the left and right limits; since they are different, the limit $\lim_{x\to 0} \frac{x}{|x|}$ does not exist.

Exercises for Section 3.3

Compute the limits. If a limit does not exist, explain why.

(1) $\lim_{x\to 3} \frac{x^2 + x - 12}{x - 3}$

(2) $\lim_{x\to 1} \frac{x^2 + x - 12}{x - 3}$

(3) $\lim_{x\to -4} \frac{x^2 + x - 12}{x - 3}$

(4) $\lim_{x\to 2} \frac{x^2 + x - 12}{x - 2}$

(5) $\lim_{x\to 1} \frac{\sqrt{x+8} - 3}{x - 1}$

(6) $\lim_{x\to 0^+} \sqrt{\frac{1}{x} + 2} - \sqrt{\frac{1}{x}}$.

(7) $\lim_{x\to 2} 3$

(8) $\lim_{x\to 4} 3x^3 - 5x$

(9) $\lim_{x\to 0} \frac{4x - 5x^2}{x - 1}$

(10) $\lim_{x\to 1} \frac{x^2 - 1}{x - 1}$

(11) $\lim_{x\to 0^+} \frac{\sqrt{2 - x^2}}{x}$

(12) $\lim_{x\to 0^+} \frac{\sqrt{2-x^2}}{x+1}$

(13) $\lim_{x\to a} \frac{x^3-a^3}{x-a}$

(14) $\lim_{x\to 2} (x^2+4)^3$

(15) $\lim_{x\to 1} \begin{cases} x-5 & x \neq 1, \\ 7 & x = 1. \end{cases}$

(16) $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$ (Hint: Use the fact that $|\sin a| < 1$ for any real number a. You should probably use the definition of a limit here.)

(17) Give an ε–δ proof, similar to example 3.3, of the fact that $\lim_{x\to 4}(2x-5) = 3$.

(18) Evaluate the expressions by reference to this graph:

(19) Use a calculator to estimate $\lim_{x\to 0} \frac{\sin x}{x}$.

(20) Use a calculator to estimate $\lim_{x\to 0} \frac{\tan(3x)}{\tan(5x)}$.

3.4 The Derivative Function

We have seen how to create, or derive, a new function $f'(x)$ from a function $f(x)$, and that this new function carries important information. In one example we saw that $f'(x)$ tells us how steep the graph of $f(x)$ is; in another we saw that $f'(x)$ tells us the velocity of an object if $f(x)$ tells us the position of the object at time x. As we said earlier, this same mathematical idea is useful whenever $f(x)$ represents some changing quantity and we want to know something about how it changes, or roughly, the "rate" at which it changes. Most functions encountered in practice are built up from a small collection of "primitive" functions in a few simple ways, for example, by adding or multiplying functions together to get new, more complicated functions. To make good use of the information provided by $f'(x)$ we need to be able to compute it for a variety of such functions.

We will begin to use different notations for the derivative of a function. While initially confusing, each is often useful so it is worth maintaining multiple versions of the same thing.

Consider again the function $f(x) = \sqrt{625 - x^2}$. We have computed the derivative $f'(x) = -x/\sqrt{625 - x^2}$, and have already noted that if we use the alternate notation $y = \sqrt{625 - x^2}$ then we might write $y' = -x/\sqrt{625 - x^2}$. Another notation is quite different, and in time it will become clear why it is often a useful one. Recall that to compute the the derivative of f we computed

$$\lim_{\Delta x \to 0} \frac{\sqrt{625 - (7 + \Delta x)^2} - 24}{\Delta x}.$$

The denominator here measures a distance in the x direction, sometimes called the "run", and the numerator measures a distance in the y direction, sometimes called the "rise," and "rise over run" is the slope of a line. Recall that sometimes such a numerator is abbreviated Δy, exchanging brevity for a more detailed expression. So in general, a derivative is given by

$$y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

To recall the form of the limit, we sometimes say instead that

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}.$$

In other words, dy/dx is another notation for the derivative, and it reminds us that it is related to an actual slope between two points. This notation is called *Leibniz notation*, after Gottfried Leibniz, who developed the fundamentals of calculus independently, at about the same time that Isaac Newton did. Again, since we often use f and $f(x)$ to mean the original function, we sometimes use df/dx and $df(x)/dx$ to refer to the derivative. If the function $f(x)$ is written out in full we often write the last of these something like this

$$f'(x) = \frac{d}{dx}\sqrt{625 - x^2}$$

with the function written to the side, instead of trying to fit it into the numerator.

Example 3.4.1 Find the derivative of $y = f(t) = t^2$.

We compute

$$\begin{aligned} y' = \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t} &= \lim_{\Delta t \to 0} \frac{(t + \Delta t)^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{t^2 + 2t\Delta t + \Delta t^2 - t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} \frac{2t\Delta t + \Delta t^2}{\Delta t} \\ &= \lim_{\Delta t \to 0} 2t + \Delta t = 2t. \end{aligned}$$

Remember that Δt is a single quantity, not a "Δ" times a "t", and so Δt^2 is $(\Delta t)^2$ not $\Delta(t^2)$.

Example 3.4.2 Find the derivative of $y = f(x) = 1/x$.

The computation:

$$
\begin{aligned}
y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= \lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x}{x(x+\Delta x)} - \frac{x+\Delta x}{x(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{x-(x+\Delta x)}{x(x+\Delta x)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x - x - \Delta x}{x(x+\Delta x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{x(x+\Delta x)\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-1}{x(x+\Delta x)} = \frac{-1}{x^2}
\end{aligned}
$$

Note. If you happen to know some "derivative formulas" from an earlier course, for the time being you should pretend that you do not know them. In examples like the ones above and the exercises below, you are required to know how to find the derivative formula starting from basic principles. We will later develop some formulas so that we do not always need to do such computations, but we will continue to need to know how to do the more involved computations.

Sometimes one encounters a point in the domain of a function $y = f(x)$ where there is **no derivative**, because there is no tangent line. In order for the notion of the tangent line at a point to make sense, the curve must be "smooth" at that point. This means that if you imagine a particle traveling at some steady speed along the curve, then the particle does not experience an abrupt change of direction. There are two types of situations you should be aware of—corners and cusps—where there's a sudden change of direction and hence no derivative.

Example 3.4.3 Discuss the derivative of the absolute value function $y = f(x) = |x|$.

If x is positive, then this is the function $y = x$, whose derivative is the constant 1. (Recall that when $y = f(x) = mx + b$, the derivative is the slope m.) If x is negative, then we're dealing with the function $y = -x$, whose derivative is the constant -1. If $x = 0$, then the function has a corner, i.e., there is no tangent line. A tangent line would have to point in the direction of the curve—but there are *two* directions of the curve that come together at the origin. We can summarize this as

$$y' = \begin{cases} 1 & \text{if } x > 0; \\ -1 & \text{if } x < 0; \\ \text{undefined} & \text{if } x = 0. \end{cases}$$

Example 3.4.4

Discuss the derivative of the function $y = x^{2/3}$, shown in figure **??**. We will later see how to compute this derivative; for now we use the fact that $y' = (2/3)x^{-1/3}$. Visually this looks much like the absolute value function, but it technically has a cusp, not a corner. The absolute value function has no tangent line at 0 because there are (at least) two obvious contenders—the tangent line of the left side of the curve and the tangent line of the right side. The function $y = x^{2/3}$ does not have a tangent line at 0, but unlike the absolute value function it can be said to have a single direction: as we approach 0 from either side the tangent line becomes closer and closer to a vertical line; the curve is vertical at 0. But as before, if you imagine traveling along the curve, an abrupt change in direction is required at 0: a full 180 degree turn.

In practice we won't worry much about the distinction between these examples; in both cases the function has a "sharp point" where there is no tangent line and no derivative.

Exercises for Section 3.4

(1) Find the derivative of $y = f(x) = \sqrt{169 - x^2}$.

(2) Find the derivative of $y = f(t) = 80 - 4.9t^2$.

(3) Find the derivative of $y = f(x) = x^2 - (1/x)$.

(4) Find the derivative of $y = f(x) = ax^2 + bx + c$ (where a, b, and c are constants).

(5) Find the derivative of $y = f(x) = x^3$.

(6) Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.

(7) Shown is the graph of a function $f(x)$. Sketch the graph of $f'(x)$ by estimating the derivative at a number of points in the interval: estimate the derivative at regular intervals from one end of the interval to the other, and also at "special" points, as when the derivative is zero. Make sure you indicate any places where the derivative does not exist.

(8) Find the derivative of $y = f(x) = 2/\sqrt{2x+1}$

(9) Find the derivative of $y = g(t) = (2t-1)/(t+2)$

(10) Find an equation for the tangent line to the graph of $f(x) = 5 - x - 3x^2$ at the point $x = 2$

(11) Find a value for a so that the graph of $f(x) = x^2 + ax - 3$ has a horizontal tangent line at $x = 4$.

3.5 Adjectives For Functions

As we have defined it in Section **??**, a function is a very general object. At this point, it is useful to introduce a collection of adjectives to describe certain kinds of functions; these adjectives name useful properties that functions may have. Consider the graphs of the functions in Figure **??**. It would clearly be useful to have words to help us describe the distinct features of each of them. We will point out and define a few adjectives (there are many more) for the functions pictured here. For the sake of the discussion, we will assume that the graphs do not exhibit any unusual behavior off-stage (i.e., outside the view of the graphs).

Functions. Each graph in Figure **??** certainly represents a function—since each passes the *vertical line test*. In other words, as you sweep a vertical line across the graph of each function, the line never intersects the graph more than once. If it did, then the graph would not represent a function.

Bounded. The graph in (c) appears to approach zero as x goes to both positive and negative infinity. It also never exceeds the value 1 or drops below the value 0. Because the graph never increases or decreases without bound, we say that the function represented by the graph in (c) is a **bounded** function.

Definition (Bounded) A function f is bounded if there is a number M such that $|f(x)| < M$ for every x in the domain of f.

For the function in (c), one such choice for M would be 10. However, the smallest (optimal) choice would be $M = 1$. In either case, simply finding an M is enough to establish boundedness. No such M exists for the hyperbola in (d) and hence we can say that it is **unbounded**.

Continuity. The graphs shown in (b) and (c) both represent **continuous** functions. Geometrically, this is because there are no jumps in the graphs. That is, if you pick a point on the graph and approach it from the left and right, the values of the function approach the value of the function at that point. For example, we can see that this is not true for function values near $x = -1$ on the graph in (a) which is not continuous at that location.

Definition (Continuous at a Point) A function f is continuous at a point a if $\lim_{x \to a} f(x) = f(a)$.

Definition (Continuous) A function f is continuous if it is continuous at every point in its domain.

Strangely, we can also say that (d) is continuous even though there is a vertical asymptote. A careful reading of the definition of continuous reveals the phrase "*at every point in its domain.*" Because the location of the asymptote, $x = 0$, is not in the domain of the function, and because the rest of the function is *well-behaved*, we can say that (d) is continuous.

Differentiability. Now that we have introduced the derivative of a function at a point, we can begin to use the adjective **differentiable**. We can see that the tangent line is well-defined at every point on the graph in (c). Therefore, we can say that (c) is a differentiable function.

Definition (Differentiable at a Point) A function f is differentiable at point a if $f'(a)$ exists.

Definition (Differentiable) A function f is differentiable if is differentiable at every point (excluding endpoints and isolated points in the domain of f) in the domain of f.

Take note that, for technical reasons not discussed here, both of these definitions exclude endpoints and isolated points in the domain from consideration.

We now have a collection of adjectives to describe the very rich and complex set of objects known as functions.

We close with a useful theorem about continuous functions:

Theorem 3.5.1 *(Intermediate Value Theorem) If f is continuous on the interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*

This is most frequently used when $d = 0$.

Example 3.5.2 Explain why the function $f = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

By theorem 3.3.4, f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in [0, 1]$ such that $f(c) = 0$.

This example also points the way to a simple method for approximating roots.

Example 3.5.3 Approximate the root of the previous example to one decimal place.

If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6) < 0$ and $f(0.7) > 0$, so by the Intermediate Value Theorem, f has a root between 0.6 and 0.7. Repeating the process with $f(0.61)$, $f(0.62)$, and so on, we find that $f(0.61) < 0$ and $f(0.62) > 0$, so f has a root between 0.61 and 0.62, and the root is 0.6 rounded to one decimal place.

Exercises for Section 3.5

(1) Along the lines of Figure **??**, for each part below sketch the graph of a function that is:

- a. bounded, but not continuous.
- b. differentiable and unbounded.
- c. continuous at $x = 0$, not continuous at $x = 1$, and bounded.
- d. differentiable everywhere except at $x = -1$, continuous, and unbounded.

(2) Is $f(x) = \sin(x)$ a bounded function? If so, find the smallest M.

(3) Is $s(t) = 1/(1+t^2)$ a bounded function? If so, find the smallest M.

(4) Is $v(u) = 2\ln|u|$ a bounded function? If so, find the smallest M.

(5) Consider the function

$$h(x) = \begin{cases} 2x-3, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x = 0$. Is h a continuous function?

(6) Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place.

(7) Approximate a root of $f = x^4 + x^3 - 5x + 1$ to one decimal place.

4 Rules for Finding Derivatives

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. Many functions involve quantities raised to a constant power, such as polynomials and more complicated combinations like $y = (\sin x)^4$. So we start by examining powers of a single variable; this gives us a building block for more complicated examples.

4.1 The Power Rule

We start with the derivative of a power function, $f(x) = x^n$. Here n is a number of any kind: integer, rational, positive, negative, even irrational, as in x^π. We have already computed some simple examples, so the formula should not be a complete surprise:

$$\frac{d}{dx}x^n = nx^{n-1}.$$

It is not easy to show this is true for any n. We will do some of the easier cases now, and discuss the rest later.

The easiest, and most common, is the case that n is a positive integer. To compute the derivative we need to compute the following limit:

$$\frac{d}{dx}x^n = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^n - x^n}{\Delta x}.$$

For a specific, fairly small value of n, we could do this by straightforward algebra.

Example 4.1.1 Find the derivative of $f(x) = x^3$.

$$\begin{aligned} \frac{d}{dx}x^3 &= \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x}. \\ &= \lim_{\Delta x \to 0} \frac{x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - x^3}{\Delta x}. \\ &= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x\Delta x^2 + \Delta x^3}{\Delta x}. \\ &= \lim_{\Delta x \to 0} 3x^2 + 3x\Delta x + \Delta x^2 = 3x^2. \end{aligned}$$

The general case is really not much harder as long as we don't try to do too much. The key is understanding what happens when $(x+\Delta x)^n$ is multiplied out:

$$(x+\Delta x)^n = x^n + nx^{n-1}\Delta x + a_2 x^{n-2}\Delta x^2 + \cdots + + a_{n-1}x\Delta x^{n-1} + \Delta x^n.$$

We know that multiplying out will give a large number of terms all of the form $x^i \Delta x^j$, and in fact that $i+j=n$ in every term. One way to see this is to understand that one method for multiplying out $(x+\Delta x)^n$ is the following: In every $(x+\Delta x)$ factor, pick either the x or the Δx, then multiply the n choices together; do this in all possible ways. For example, for $(x+\Delta x)^3$, there are eight possible ways to do this:

$$\begin{aligned} (x+\Delta x)(x+\Delta x)(x+\Delta x) &= xxx + xx\Delta x + x\Delta x x + x\Delta x\Delta x \\ &\quad + \Delta x xx + \Delta x x\Delta x + \Delta x\Delta x x + \Delta x\Delta x\Delta x \\ &= x^3 + x^2\Delta x + x^2\Delta x + x\Delta x^2 \\ &\quad + x^2\Delta x + x\Delta x^2 + x\Delta x^2 + \Delta x^3 \\ &= x^3 + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 \end{aligned}$$

No matter what n is, there are n ways to pick Δx in one factor and x in the remaining $n-1$ factors; this means one term is $nx^{n-1}\Delta x$. The other coefficients are somewhat harder to understand, but we don't really need them, so in the formula above they have simply been called a_2, a_3, and so on. We know that every one of these terms contains Δx to at least the power 2. Now let's look at the limit:

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^n - x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{x^n + nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n - x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{nx^{n-1}\Delta x + a_2x^{n-2}\Delta x^2 + \cdots + a_{n-1}x\Delta x^{n-1} + \Delta x^n}{\Delta x}\\ &= \lim_{\Delta x\to 0} nx^{n-1} + a_2x^{n-2}\Delta x + \cdots + a_{n-1}x\Delta x^{n-2} + \Delta x^{n-1} = nx^{n-1}.\end{aligned}$$

Now without much trouble we can verify the formula for negative integers. First let's look at an example:

Example 4.1.2 Find the derivative of $y = x^{-3}$. Using the formula, $y' = -3x^{-3-1} = -3x^{-4}$.

Here is the general computation. Suppose n is a negative integer; the algebra is easier to follow if we use $n = -m$ in the computation, where m is a positive integer.

$$\begin{aligned}\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^{-m} - x^{-m}}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{\frac{1}{(x+\Delta x)^m} - \frac{1}{x^m}}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{x^m - (x+\Delta x)^m}{(x+\Delta x)^m x^m\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{x^m - (x^m + mx^{m-1}\Delta x + a_2x^{m-2}\Delta x^2 + \cdots + a_{m-1}x\Delta x^{m-1} + \Delta x^m)}{(x+\Delta x)^m x^m\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{-mx^{m-1} - a_2x^{m-2}\Delta x - \cdots - a_{m-1}x\Delta x^{m-2} - \Delta x^{m-1})}{(x+\Delta x)^m x^m}\\ &= \frac{-mx^{m-1}}{x^m x^m} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = nx^{-m-1} = nx^{n-1}.\end{aligned}$$

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever n is any real number. Let's note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that $f(x) = 1$; remember that this "1" is a function, not "merely" a number, and that $f(x) = 1$ has a graph that is a horizontal line, with slope zero everywhere. So we know that $f'(x) = 0$. We might also write $f(x) = x^0$, though there is some question about just what this means at $x = 0$. If we apply the power rule, we get $f'(x) = 0x^{-1} = 0/x = 0$, again noting that there is a problem at $x = 0$. So the power rule "works" in this case, but it's really best to just remember that the derivative of any constant function is zero.

Exercises for Section 4.1

Find the derivatives of the given functions.

(1) x^{100}

(2) x^{-100}

(3) $\dfrac{1}{x^5}$

(4) x^π

(5) $x^{3/4}$

(6) $x^{-9/7}$

4.2 Linearity of the Derivative

An operation is linear if it behaves "nicely" with respect to multiplication by a constant and addition. The name comes from the equation of a line through the origin, $f(x) = mx$, and the following two properties of this equation. First, $f(cx) = m(cx) = c(mx) = cf(x)$, so the constant c can be "moved outside" or "moved through" the function f. Second, $f(x+y) = m(x+y) = mx + my = f(x) + f(y)$, so the addition symbol likewise can be moved through the function.

The corresponding properties for the derivative are:

$$(cf(x))' = \frac{d}{dx}cf(x) = c\frac{d}{dx}f(x) = cf'(x),$$

and

$$(f(x) + g(x))' = \frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) = f'(x) + g'(x).$$

It is easy to see, or at least to believe, that these are true by thinking of the distance/speed interpretation of derivatives. If one object is at position $f(t)$ at time t, we know its speed is given by $f'(t)$. Suppose another object is at position $5f(t)$ at time t, namely, that it is always 5 times as far along the route as the first object. Then it "must" be going 5 times as fast at all times.

The second rule is somewhat more complicated, but here is one way to picture it. Suppose a flat bed railroad car is at position $f(t)$ at time t, so the car is traveling at a speed of $f'(t)$ (to be specific, let's say that $f(t)$ gives the position on the track of the rear end of the car). Suppose that an ant is crawling from the back of the car to the front so that its position *on the car* is $g(t)$ and its speed *relative to the car* is $g'(t)$. Then in reality, at time t, the ant is at position $f(t) + g(t)$ along the track, and its speed is "obviously" $f'(t) + g'(t)$.

We don't want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let's see how to verify these rules by computation. We'll do one and leave the other for the exercises.

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-(f(x)+g(x))}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)+g(x+\Delta x)-f(x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)+g(x+\Delta x)-g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\left(\frac{f(x+\Delta x)-f(x)}{\Delta x}+\frac{g(x+\Delta x)-g(x)}{\Delta x}\right)\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{g(x+\Delta x)-g(x)}{\Delta x}\\
&= f'(x)+g'(x)
\end{aligned}$$

This is sometimes called the **sum rule** for derivatives.

Example 4.2.1 Find the derivative of $f(x) = x^5 + 5x^2$. We have to invoke linearity twice here:

$$f'(x) = \frac{d}{dx}(x^5+5x^2) = \frac{d}{dx}x^5+\frac{d}{dx}(5x^2) = 5x^4+5\frac{d}{dx}(x^2) = 5x^4+5\cdot 2x^1 = 5$$

Because it is so easy with a little practice, we can usually combine all uses of linearity into a single step. The following example shows an acceptably detailed computation.

Example 4.2.2 Find the derivative of $f(x) = 3/x^4 - 2x^2 + 6x - 7$.

$$f'(x) = \frac{d}{dx}\left(\frac{3}{x^4}-2x^2+6x-7\right) = \frac{d}{dx}(3x^{-4}-2x^2+6x-7) = -12x^{-5}-4x+6.$$

Exercises for Section 4.2

Find the derivatives of the functions in 1–6.

(1) $5x^3 + 12x^2 - 15$

(2) $-4x^5 + 3x^2 - 5/x^2$

(3) $5(-3x^2 + 5x + 1)$

(4) $f(x) + g(x)$, where $f(x) = x^2 - 3x + 2$ and $g(x) = 2x^3 - 5x$

(5) $(x+1)(x^2 + 2x - 3)$

(6) $\sqrt{625 - x^2} + 3x^3 + 12$ (See section 3.1.)

(7) Find an equation for the tangent line to $f(x) = x^3/4 - 1/x$ at $x = -2$.

(8) Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$.

(9) Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t.

(10) Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.

(11) The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P?

(12) Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$.

(13) Prove that $\dfrac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

(14) Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.

4.3 The Product Rule

Consider the product of two simple functions, say $f(x) = (x^2+1)(x^3-3x)$. An obvious guess for the derivative of f is the product of the derivatives of the constituent functions: $(2x)(3x^2-3) = 6x^3-6x$. Is this correct? We can easily check, by rewriting f and doing the calculation in a way that is known to work. First, $f(x) = x^5-3x^3+x^3-3x = x^5-2x^3-3x$, and then $f'(x) = 5x^4-6x^2-3$. Not even close! What went "wrong"? Well, nothing really, except the guess was wrong.

So the derivative of $f(x)g(x)$ is NOT as simple as $f'(x)g'(x)$. Surely there is some rule for such a situation? There is, and it is instructive to "discover" it by trying to do the general calculation even without knowing the answer in advance.

$$
\begin{aligned}
\frac{d}{dx}(f(x)g(x)) &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)+f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x+\Delta x)g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x)-f(x)g(x)}{\Delta x}\\
&= \lim_{\Delta x\to 0}f(x+\Delta x)\frac{g(x+\Delta x)-g(x)}{\Delta x}+\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}g(x)\\
&= f(x)g'(x)+f'(x)g(x)
\end{aligned}
$$

A couple of items here need discussion. First, we used a standard trick, "add and subtract the same thing", to transform what we had into a more useful form. After some rewriting, we realize that we have two limits that produce $f'(x)$ and $g'(x)$. Of course, $f'(x)$ and $g'(x)$ must actually exist for this to make sense. We also replaced $\lim_{\Delta x\to 0} f(x+\Delta x)$ with $f(x)$—why is this justified?

What we really need to know here is that $\lim_{\Delta x\to 0} f(x+\Delta x) = f(x)$, or in the language of section 3.5, that f is continuous at x. We already know that $f'(x)$ exists (or the whole approach, writing the derivative of fg in terms of f' and g', doesn't make sense). This turns out to imply that f is continuous as well. Here's why:

$$\begin{aligned}\lim_{\Delta x\to 0} f(x+\Delta x) &= \lim_{\Delta x\to 0}(f(x+\Delta x)-f(x)+f(x))\\ &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}\Delta x+\lim_{\Delta x\to 0} f(x)\\ &= f'(x)\cdot 0+f(x)=f(x)\end{aligned}$$

To summarize: the product rule says that

$$\frac{d}{dx}(f(x)g(x)) = f(x)g'(x)+f'(x)g(x).$$

Returning to the example we started with, let $f(x) = (x^2+1)(x^3-3x)$. Then $f'(x) = (x^2+1)(3x^2-3)+(2x)(x^3-3x) = 3x^4-3x^2+3x^2-3+2x^4-6x^2 = 5x^4-6x^2-3$, as before. In this case it is probably simpler to multiply $f(x)$ out first, then compute the derivative; here's an example for which we really need the product rule.

Example 4.3.1 Compute the derivative of $f(x) = x^2\sqrt{625-x^2}$. We have already computed $\frac{d}{dx}\sqrt{625-x^2} = \frac{-x}{\sqrt{625-x^2}}$. Now

$$f'(x) = x^2\frac{-x}{\sqrt{625-x^2}}+2x\sqrt{625-x^2} = \frac{-x^3+2x(625-x^2)}{\sqrt{625-x^2}} = \frac{-3x^3+1250x}{\sqrt{625-x^2}}.$$

Exercises for Section 4.3

In 1–4, find the derivatives of the functions using the product rule.

(1) $x^3(x^3-5x+10)$

(2) $(x^2+5x-3)(x^5-6x^3+3x^2-7x+1)$

(3) $\sqrt{x}\sqrt{625-x^2}$

(4) $\dfrac{\sqrt{625-x^2}}{x^{20}}$

(5) Use the product rule to compute the derivative of $f(x) = (2x-3)^2$. Sketch the function. Find an equation of the tangent line to the curve at $x = 2$. Sketch the tangent line at $x = 2$.

(6) Suppose that f, g, and h are differentiable functions. Show that $(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

(7) State and prove a rule to compute $(fghi)'(x)$, similar to the rule in the previous problem.

Remark: Product notation

Suppose $f_1, f_2, \ldots f_n$ are functions. The product of all these functions can be written

$$\prod_{k=1}^{n} f_k.$$

This is similar to the use of $\sum$ to denote a sum. For example,

$$\prod_{k=1}^{5} f_k = f_1 f_2 f_3 f_4 f_5$$

and

$$\prod_{k=1}^{n} k = 1 \cdot 2 \cdot \ldots \cdot n = n!.$$

We sometimes use somewhat more complicated conditions; for example

$$\prod_{k=1, k\neq j}^{n} f_k$$

denotes the product of f_1 through f_n except for f_j. For example,

$$\prod_{k=1, k\neq 4}^{5} x^k = x \cdot x^2 \cdot x^3 \cdot x^5 = x^{11}.$$

(8) The **generalized product rule** says that if $f_1, f_2, \ldots, f_n$ are differentiable functions at x then

$$\frac{d}{dx}\prod_{k=1}^{n} f_k(x) = \sum_{j=1}^{n}\left(f_j'(x)\prod_{k=1,k\neq j}^{n} f_k(x)\right).$$

Verify that this is the same as your answer to the previous problem when $n = 4$, and write out what this says when $n = 5$.

4.4 The Quotient Rule

What is the derivative of $(x^2+1)/(x^3-3x)$? More generally, we'd like to have a formula to compute the derivative of $f(x)/g(x)$ if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x)\cdot(1/g(x))$, that is, this is "really" a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. So really the only new bit of information we need is $(1/g(x))'$ in terms of $g'(x)$. As with the product rule, let's set this up and see how far we can get:

$$
\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{\Delta x\to 0}\frac{\frac{1}{g(x+\Delta x)}-\frac{1}{g(x)}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)}}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)\Delta x}\\
&= \lim_{\Delta x\to 0}-\frac{g(x+\Delta x)-g(x)}{\Delta x}\frac{1}{g(x+\Delta x)g(x)}\\
&= -\frac{g'(x)}{g(x)^2}
\end{aligned}
$$

Now we can put this together with the product rule:

$$
\frac{d}{dx}\frac{f(x)}{g(x)} = f(x)\frac{-g'(x)}{g(x)^2} + f'(x)\frac{1}{g(x)} = \frac{-f(x)g'(x)+f'(x)g(x)}{g(x)^2} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}.
$$

Example 4.4.1 Compute the derivative of $(x^2+1)/(x^3-3x)$.

$$
\frac{d}{dx}\frac{x^2+1}{x^3-3x} = \frac{2x(x^3-3x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2} = \frac{-x^4-6x^2+3}{(x^3-3x)^2}.
$$

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

Example 4.4.2 Find the derivative of $\sqrt{625-x^2}/\sqrt{x}$ in two ways: using the quotient rule, and using the product rule.

Quotient rule:

$$\frac{d}{dx}\frac{\sqrt{625-x^2}}{\sqrt{x}} = \frac{\sqrt{x}(-x/\sqrt{625-x^2}) - \sqrt{625-x^2}\cdot 1/(2\sqrt{x})}{x}.$$

Note that we have used $\sqrt{x} = x^{1/2}$ to compute the derivative of $\sqrt{x}$ by the power rule.

Product rule:

$$\frac{d}{dx}\sqrt{625-x^2}x^{-1/2} = \sqrt{625-x^2}\frac{-1}{2}x^{-3/2} + \frac{-x}{\sqrt{625-x^2}}x^{-1/2}.$$

With a bit of algebra, both of these simplify to

$$-\frac{x^2+625}{2\sqrt{625-x^2}x^{3/2}}.$$

Occasionally you will need to compute the derivative of a quotient with a constant numerator, like $10/x^2$. Of course you can use the quotient rule, but it is usually not the easiest method. If we do use it here, we get

$$\frac{d}{dx}\frac{10}{x^2} = \frac{x^2\cdot 0 - 10\cdot 2x}{x^4} = \frac{-20}{x^3},$$

since the derivative of 10 is 0. But it is simpler to do this:

$$\frac{d}{dx}\frac{10}{x^2} = \frac{d}{dx}10x^{-2} = -20x^{-3}.$$

Admittedly, x^2 is a particularly simple denominator, but we will see that a similar calculation is usually possible. Another approach is to remember that

$$\frac{d}{dx}\frac{1}{g(x)} = \frac{-g'(x)}{g(x)^2},$$

but this requires extra memorization. Using this formula,

$$\frac{d}{dx}\frac{10}{x^2} = 10\frac{-2x}{x^4}.$$

Note that we first use linearity of the derivative to pull the 10 out in front.

Exercises for Section 4.4

Find the derivatives of the functions in 1–4 using the quotient rule.

(1) $\dfrac{x^3}{x^3-5x+10}$

(2) $\dfrac{x^2+5x-3}{x^5-6x^3+3x^2-7x+1}$

(3) $\dfrac{\sqrt{x}}{\sqrt{625-x^2}}$

(4) $\dfrac{\sqrt{625-x^2}}{x^{20}}$

(5) Find an equation for the tangent line to $f(x)=(x^2-4)/(5-x)$ at $x=3$.

(6) Find an equation for the tangent line to $f(x)=(x-2)/(x^3+4x-1)$ at $x=1$.

(7) Let P be a polynomial of degree n and let Q be a polynomial of degree m (with Q not the zero polynomial). Using sigma notation we can write

$$P=\sum_{k=0}^{n} a_k x^k, \qquad Q=\sum_{k=0}^{m} b_k x^k.$$

Use sigma notation to write the derivative of the **rational function** P/Q.

(8) The curve $y=1/(1+x^2)$ is an example of a class of curves each of which is called a **witch of Agnesi**. Sketch the curve and find the tangent line to the curve at $x=5$. (The word *witch* here is a mistranslation of the original Italian, as described at

(9) If $f'(4)=5$, $g'(4)=12$, $(fg)(4)=f(4)g(4)=2$, and $g(4)=6$, compute $f(4)$ and $\dfrac{d}{dx}\dfrac{f}{g}$ at 4.

4.5 The Chain Rule

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine simple functions to make more complicated functions: function composition, as discussed in section 3.3. For example, consider $\sqrt{625-x^2}$. This function has many simpler components, like 625 and x^2, and then there is that square root symbol, so the square root function $\sqrt{x}=x^{1/2}$ is involved. The obvious question is: can we compute the derivative using the derivatives of the constituents $625-x^2$ and $\sqrt{x}$? We can indeed. In general, if $f(x)$ and $g(x)$ are functions, we can compute the derivatives of $f(g(x))$ and $g(f(x))$ in terms of $f'(x)$ and $g'(x)$.

Example 4.5.1 Form the two possible compositions of $f(x)=\sqrt{x}$ and $g(x)=625-x^2$ and compute the derivatives. First, $f(g(x))=\sqrt{625-x^2}$, and the derivative is $-x/\sqrt{625-x^2}$ as we have seen. Second, $g(f(x))=625-(\sqrt{x})^2=625-x$ with derivative -1. Of course, these calculations do not use anything new, and in particular the derivative of $f(g(x))$ was somewhat tedious to compute from the definition.

Suppose we want the derivative of $f(g(x))$. Again, let's set up the derivative and play some algebraic tricks:

$$\begin{aligned}\frac{d}{dx}f(g(x)) &= \lim_{\Delta x\to 0}\frac{f(g(x+\Delta x))-f(g(x))}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{f(g(x+\Delta x))-f(g(x))}{g(x+\Delta x))-g(x)}\frac{g(x+\Delta x))-g(x)}{\Delta x}\end{aligned}$$

Now we see immediately that the second fraction turns into $g'(x)$ when we take the limit. The first fraction is more complicated, but it too looks something like a derivative. The denominator, $g(x+\Delta x))-g(x)$, is a change in the value of g, so let's abbreviate it as $\Delta g=g(x+\Delta x))-g(x)$, which also means $g(x+\Delta x)=g(x)+\Delta g$.

This gives us

$$\lim_{\Delta x \to 0} \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}.$$

As Δx goes to 0, it is also true that Δg goes to 0, because $g(x + \Delta x)$ goes to $g(x)$. So we can rewrite this limit as

$$\lim_{\Delta g \to 0} \frac{f(g(x) + \Delta g) - f(g(x))}{\Delta g}.$$

Now this looks exactly like a derivative, namely $f'(g(x))$, that is, the function $f'(x)$ with x replaced by $g(x)$. If this all withstands scrutiny, we then get

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Unfortunately, there is a small flaw in the argument. Recall that what we mean by $\lim_{\Delta x \to 0}$ involves what happens when Δx is close to 0 *but not equal to 0*. The qualification is very important, since we must be able to divide by Δx. But when Δx is close to 0 but not equal to 0, $\Delta g = g(x + \Delta x)) - g(x)$ is close to 0 *and possibly equal to 0*. This means it doesn't really make sense to divide by Δg. Fortunately, it is possible to recast the argument to avoid this difficulty, but it is a bit tricky; we will not include the details, which can be found in many calculus books. Note that many functions g do have the property that $g(x + \Delta x) - g(x) \neq 0$ when Δx is small, and for these functions the argument above is fine.

The chain rule has a particularly simple expression if we use the Leibniz notation for the derivative. The quantity $f'(g(x))$ is the derivative of f with x replaced by g; this can be written df/dg. As usual, $g'(x) = dg/dx$. Then the chain rule becomes

$$\frac{df}{dx} = \frac{df}{dg}\frac{dg}{dx}.$$

This looks like trivial arithmetic, but it is not: dg/dx is not a fraction, that is, not literal division, but a single symbol that means $g'(x)$. Nevertheless, it turns out that what looks like trivial arithmetic, and is therefore easy to remember, is really true.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen.

Example 4.5.2 Compute the derivative of $\sqrt{625-x^2}$. We already know that the answer is $-x/\sqrt{625-x^2}$, computed directly from the limit. In the context of the chain rule, we have $f(x) = \sqrt{x}$, $g(x) = 625-x^2$. We know that $f'(x) = (1/2)x^{-1/2}$, so $f'(g(x)) = (1/2)(625-x^2)^{-1/2}$. Note that this is a two step computation: first compute $f'(x)$, then replace x by $g(x)$. Since $g'(x) = -2x$ we have

$$f'(g(x))g'(x) = \frac{1}{2\sqrt{625-x^2}}(-2x) = \frac{-x}{\sqrt{625-x^2}}.$$

Example 4.5.3 Compute the derivative of $1/\sqrt{625-x^2}$. This is a quotient with a constant numerator, so we could use the quotient rule, but it is simpler to use the chain rule. The function is $(625-x^2)^{-1/2}$, the composition of $f(x) = x^{-1/2}$ and $g(x) = 625-x^2$. We compute $f'(x) = (-1/2)x^{-3/2}$ using the power rule, and then

$$f'(g(x))g'(x) = \frac{-1}{2(625-x^2)^{3/2}}(-2x) = \frac{x}{(625-x^2)^{3/2}}.$$

In practice, of course, you will need to use more than one of the rules we have developed to compute the derivative of a complicated function.

Example 4.5.4 Compute the derivative of

$$f(x) = \frac{x^2-1}{x\sqrt{x^2+1}}.$$

The "last" operation here is division, so to get started we need to use the quotient rule first. This gives

$$f'(x) = \frac{(x^2-1)'x\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}$$
$$= \frac{2x^2\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}.$$

Now we need to compute the derivative of $x\sqrt{x^2+1}$. This is a product, so we use the product rule:

$$\frac{d}{dx}x\sqrt{x^2+1} = x\frac{d}{dx}\sqrt{x^2+1} + \sqrt{x^2+1}.$$

Finally, we use the chain rule:

$$\frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}(x^2+1)^{1/2} = \frac{1}{2}(x^2+1)^{-1/2}(2x) = \frac{x}{\sqrt{x^2+1}}.$$

And putting it all together:

$$f'(x) = \frac{2x^2\sqrt{x^2+1} - (x^2-1)(x\sqrt{x^2+1})'}{x^2(x^2+1)}.$$
$$= \frac{2x^2\sqrt{x^2+1} - (x^2-1)\left(x\frac{x}{\sqrt{x^2+1}} + \sqrt{x^2+1}\right)}{x^2(x^2+1)}.$$

This can be simplified of course, but we have done all the calculus, so that only algebra is left.

Example 4.5.5 Compute the derivative of $\sqrt{1+\sqrt{1+\sqrt{x}}}$. Here we have a more complicated chain of compositions, so we use the chain rule twice. At the outermost "layer" we have the function $g(x) = 1 + \sqrt{1+\sqrt{x}}$ plugged into

$f(x) = \sqrt{x}$, so applying the chain rule once gives

$$\frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} = \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2} \frac{d}{dx}\left(1+\sqrt{1+\sqrt{x}}\right).$$

Now we need the derivative of $\sqrt{1+\sqrt{x}}$. Using the chain rule again:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2} \frac{1}{2}x^{-1/2}.$$

So the original derivative is

$$\begin{aligned} \frac{d}{dx}\sqrt{1+\sqrt{1+\sqrt{x}}} &= \frac{1}{2}\left(1+\sqrt{1+\sqrt{x}}\right)^{-1/2} \frac{1}{2}\left(1+\sqrt{x}\right)^{-1/2} \frac{1}{2}x^{-1/2}. \\ &= \frac{1}{8\sqrt{x}\sqrt{1+\sqrt{x}}\sqrt{1+\sqrt{1+\sqrt{x}}}} \end{aligned}$$

Using the chain rule, the power rule, and the product rule, it is possible to avoid using the quotient rule entirely.

Example 4.5.6 Compute the derivative of $f(x) = \dfrac{x^3}{x^2+1}$. Write $f(x) = x^3(x^2+1)^{-1}$, then

$$\begin{aligned}
f'(x) &= x^3\frac{d}{dx}(x^2+1)^{-1} + 3x^2(x^2+1)^{-1} \\
&= x^3(-1)(x^2+1)^{-2}(2x) + 3x^2(x^2+1)^{-1} \\
&= -2x^4(x^2+1)^{-2} + 3x^2(x^2+1)^{-1} \\
&= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2}{x^2+1} \\
&= \frac{-2x^4}{(x^2+1)^2} + \frac{3x^2(x^2+1)}{(x^2+1)^2} \\
&= \frac{-2x^4+3x^4+3x^2}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2}
\end{aligned}$$

Note that we already had the derivative on the second line; all the rest is simplification. It is easier to get to this answer by using the quotient rule, so there's a trade off: more work for fewer memorized formulas.

Exercises for Section 4.5

Find the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

(1) $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$

(2) $x^3 - 2x^2 + 4\sqrt{x}$

(3) $(x^2 + 1)^3$

(4) $x\sqrt{169 - x^2}$

(5) $(x^2 - 4x + 5)\sqrt{25 - x^2}$

(6) $\sqrt{r^2 - x^2}$, r is a constant

(7) $\sqrt{1 + x^4}$

(8) $\dfrac{1}{\sqrt{5 - \sqrt{x}}}$.

(9) $(1 + 3x)^2$

(10) $\dfrac{(x^2 + x + 1)}{(1 - x)}$

(11) $\dfrac{\sqrt{25 - x^2}}{x}$

(12) $\sqrt{\dfrac{169}{x} - x}$

(13) $\sqrt{x^3 - x^2 - (1/x)}$

(14) $100/(100 - x^2)^{3/2}$

(15) $\sqrt[3]{x + x^3}$

(16) $\sqrt{(x^2 + 1)^2 + \sqrt{1 + (x^2 + 1)^2}}$

(17) $(x + 8)^5$

(18) $(4 - x)^3$

(19) $(x^2 + 5)^3$

(20) $(6 - 2x^2)^3$

(21) $(1 - 4x^3)^{-2}$

(22) $5(x + 1 - 1/x)$

(23) $4(2x^2 - x + 3)^{-2}$

(24) $\dfrac{1}{1 + 1/x}$

(25) $\dfrac{-3}{4x^2 - 2x + 1}$

(26) $(x^2 + 1)(5 - 2x)/2$

(27) $(3x^2 + 1)(2x - 4)^3$

(28) $\dfrac{x + 1}{x - 1}$

(29) $\dfrac{x^2 - 1}{x^2 + 1}$

(30) $\dfrac{(x - 1)(x - 2)}{x - 3}$

(31) $\dfrac{2x^{-1} - x^{-2}}{3x^{-1} - 4x^{-2}}$

(32) $3(x^2 + 1)(2x^2 - 1)(2x + 3)$

(33) $\dfrac{1}{(2x + 1)(x - 3)}$

(34) $((2x + 1)^{-1} + 3)^{-1}$

(35) $(2x + 1)^3(x^2 + 1)^2$

(36) Find an equation for the tangent line to $f(x) = (x-2)^{1/3}/(x^3+4x-1)^2$ at $x = 1$.

(37) Find an equation for the tangent line to $y = 9x^{-2}$ at $(3,1)$.

(38) Find an equation for the tangent line to $(x^2-4x+5)\sqrt{25-x^2}$ at $(3,8)$.

(39) Find an equation for the tangent line to $\dfrac{(x^2+x+1)}{(1-x)}$ at $(2,-7)$.

(40) Find an equation for the tangent line to $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ at $(1, \sqrt{4+\sqrt{5}})$.

Answers to selected exercises