MOOCULUS
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CALCULUS

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This text is based on David Guichard's open-source calculus text which in turn is a modification and expansion of notes written by Neal Koblitz at the University of Washington. David Guichard's text is available at http://www.whitman.edu/mathematics/calculus/ under a Creative Commons license.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at http://www.math.wisc.edu/~keisler/calc.html under a Creative Commons license. In addition, the chapter on differential equations and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

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We will be glad to receive corrections and suggestions for improvement at fowler@math.osu.edu or snapp@math.osu.edu.

Contents

List of Main Theorems

How to Read Mathematics

Reading mathematics is **not** the same as reading a novel. To read mathematics you need:

(a) A pen.

(b) Plenty of blank paper.

(c) A willingness to write things down.

As you read mathematics, you must work along side of the text itself. You must **write** down each expression, **sketch** each graph, and **think** about what you are doing. You should work examples and fill-in the details. This is not an easy task, it is in fact **hard** work. However, mathematics is not a passive endeavor. You, the reader, must become a doer of mathematics.

1 Limits

1.1 The Basic Ideas of Limits

Consider the function:

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

While $f(x)$ is undefined at $x = 2$, we can still plot $f(x)$ at other values, see Figure 1.1. Examining Table 1.1, we see that as x approaches 2, $f(x)$ approaches 1. We write this:

$$\text{As } x \to 2,\ f(x) \to 1 \qquad \text{or} \qquad \lim_{x\to 2} f(x) = 1.$$

Intuitively, $\lim_{x\to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. This leads us to the formal definition of a *limit*.

Definition The **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x\to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta, \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

If no such value of L can be found, then we say that $\lim_{x\to a} f(x)$ **does not exist**.

In Figure 1.2, we see a geometric interpretation of this definition.

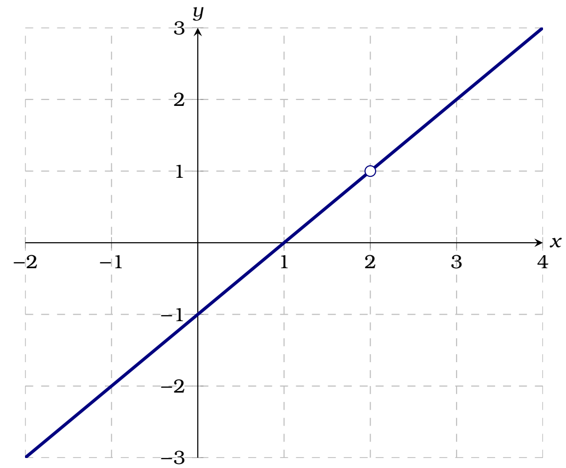


Figure 1.1: A plot of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

x	$f(x)$	x	$f(x)$
1.7	0.7	2	undefined
1.9	0.9	2.001	1.001
1.99	0.99	2.01	1.01
1.999	0.999	2.1	1.1
2	undefined	2.3	1.3

Table 1.1: Values of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

Equivalently, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and $a - \delta < x < a + \delta$, we have $L - \varepsilon < f(x) < L + \varepsilon$.

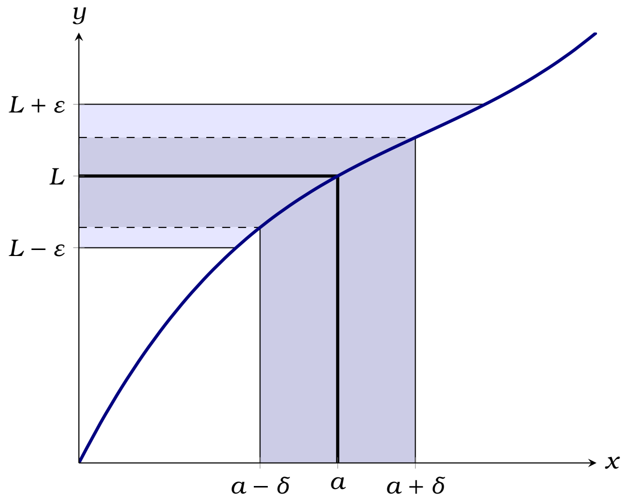


Figure 1.2: A geometric interpretation of the (ε, δ)-criterion for limits. If $0 < |x - a| < \delta$, then we have that $a - \delta < x < a + \delta$. In our diagram, we see that for all such x we are sure to have $L - \varepsilon < f(x) < L + \varepsilon$, and hence $|f(x) - L| < \varepsilon$.

Limits need not exist, let's examine two cases of this.

Example 1.1.1 Let $f(x) = \lfloor x \rfloor$. Explain why the limit

$$\lim_{x \to 2} f(x)$$

does not exist.

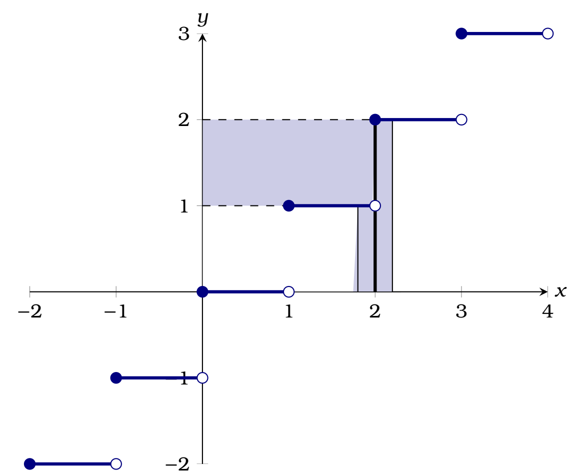


Figure 1.3: A plot of $f(x) = \lfloor x \rfloor$. Note, no matter which $\delta > 0$ is chosen, we can only at best bound $f(x)$ in the interval $[1, 2]$.

Solution *The function $\lfloor x \rfloor$ is the function that returns the greatest integer less than or equal to x. Since $f(x)$ is defined for all real numbers, one might be tempted to think that the limit above is simply $f(2) = 2$. However, this is not the case. If $x < 2$, then $f(x) = 1$. Hence if $\varepsilon = .5$, we can* ***always*** *find a value for x (just to the left of 2) such that*

$$0 < |x - 2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x) - 2|.$$

On the other hand, $\lim_{x \to 2} f(x) \neq 1$, as in this case if $\varepsilon = .5$, we can ***always*** *find a value for x (just to the right of 2) such that*

$$0 < |x - 2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x) - 1|.$$

With the example of $f(x) = \lfloor x \rfloor$, we see that taking limits is truly different from evaluating functions.

We've illustrated this in Figure 1.3. Moreover, no matter what value one chooses for $\lim_{x\to 2} f(x)$*, we will always have a similar issue.*

Limits may not exist even if the formula for the function looks innocent.

Example 1.1.2 Let $f(x) = \sin\left(\frac{1}{x}\right)$. Explain why the limit

$$\lim_{x\to 0} f(x)$$

does not exist.

Solution *In this case* $f(x)$ *oscillates "wildly" as* x *approaches* 0*, see Figure 1.4. In fact, one can show that for any given* δ*, There is a value for* x *in the interval*

$$0 - \delta < x < 0 + \delta$$

such that $f(x)$ *is* ***any*** *value in the interval* $[-1, 1]$*. Hence the limit does not exist.*

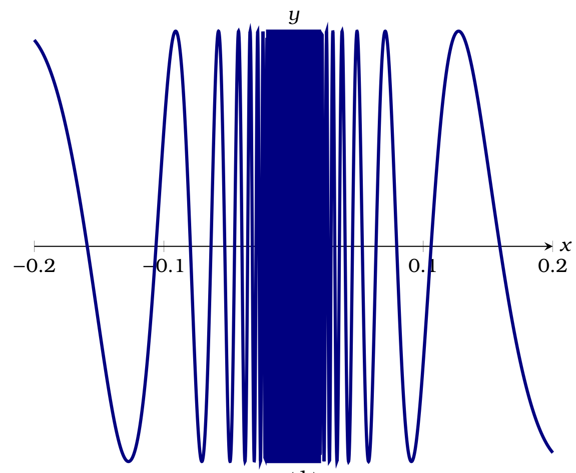


Figure 1.4: A plot of $f(x) = \sin\left(\frac{1}{x}\right)$.

Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of a *one-sided limit*:

Definition We say that the **limit** of $f(x)$ as x goes to a from the **left** is L,

$$\lim_{x\to a-} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x < a$ and

$$a - \delta < x \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

We say that the **limit** of $f(x)$ as x goes to a from the **right** is L,

$$\lim_{x\to a+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x > a$ and

$$x < a + \delta \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

Limits from the left, or from the right, are collectively called **one-sided limits**.

Example 1.1.3 Let $f(x) = \lfloor x \rfloor$. Discuss

$$\lim_{x \to 2-} f(x), \qquad \lim_{x \to 2+} f(x), \qquad \text{and} \qquad \lim_{x \to 2} f(x).$$

Solution *From the plot of $f(x)$, see Figure 1.3, we see that*

$$\lim_{x \to 2-} f(x) = 1, \qquad \textit{and} \quad \lim_{x \to 2+} f(x) = 2.$$

Since these limits are different, $\lim_{x \to 2} f(x)$ does not exist.

Exercises for Section 1.1

(1) Evaluate the expressions by reference to the plot in Figure 1.5.

(a) $\lim_{x \to 4} f(x)$ (e) $\lim_{x \to 0+} f(x)$ (i) $\lim_{x \to 0} f(x+1)$

(b) $\lim_{x \to -3} f(x)$ (f) $f(-2)$ (j) $f(0)$

(c) $\lim_{x \to 0} f(x)$ (g) $\lim_{x \to 2-} f(x)$ (k) $\lim_{x \to 1-} f(x-4)$

(d) $\lim_{x \to 0-} f(x)$ (h) $\lim_{x \to -2-} f(x)$ (l) $\lim_{x \to 0+} f(x-2)$

(2) Use a table and a calculator to estimate $\lim_{x \to 0} \frac{\sin(x)}{x}$.

(3) Use a table and a calculator to estimate $\lim_{x \to 0} \frac{\sin(2x)}{x}$.

(4) Use a table and a calculator to estimate $\lim_{x \to 0} \frac{x}{\sin\left(\frac{x}{3}\right)}$.

(5) Use a table and a calculator to estimate $\lim_{x \to 0} \frac{\tan(3x)}{\tan(5x)}$.

(6) Use a table and a calculator to estimate $\lim_{x \to 0} \frac{2^x - 1}{x}$.

(7) Use a table and a calculator to estimate $\lim_{x \to 0} (1+x)^{1/x}$.

(8) Sketch a plot of $f(x) = \frac{x}{|x|}$ and explain why $\lim_{x \to 0} \frac{x}{|x|}$ does not exist.

(9) Let $f(x) = \sin\left(\frac{\pi}{x}\right)$. Construct three tables of the following form

x	$f(x)$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	

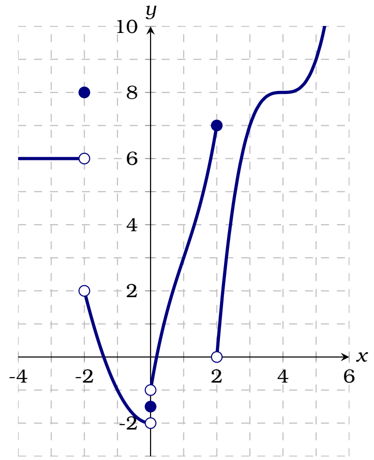


Figure 1.5: A plot of $f(x)$, a piecewise defined function.

where $d = 1, 3, 7$. What do you notice? How do you reconcile the entries in your tables with the value of $\lim_{x \to 0} f(x)$?

(10) In the theory of special relativity, a moving clock ticks slower than a stationary observer's clock. If the stationary observer records that t_s seconds have passed, then the clock moving at velocity v has recorded that

$$t_v = t_s \sqrt{1 - v^2/c^2}$$

seconds have passed, where c is the speed of light. What happens as $v \to c$ from below?

1.2 Limits by the Definition

Now we are going to get our hands dirty, and really use the definition of a limit.

Recall, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Example 1.2.1 Show that $\lim_{x\to 2} x^2 = 4$.

Solution *We want to show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that*

$$|x^2 - 4| < \varepsilon$$

whenever $0 < |x - 2| < \delta$. Start by factoring the left-hand side of the inequality above

$$|x + 2||x - 2| < \varepsilon.$$

Since we are going to assume that $0 < |x - 2| < \delta$, we will focus on the factor $|x + 2|$. Since x is assumed to be close to 2, suppose that $x \in [1, 3]$. In this case

$$|x + 2| \leq 3 + 2 = 5,$$

and so we want

$$5 \cdot |x - 2| < \varepsilon$$
$$|x - 2| < \frac{\varepsilon}{5}$$

Recall, we assumed that $x \in [1, 3]$, which is equivalent to $|x - 2| < 1$. Hence we must set $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

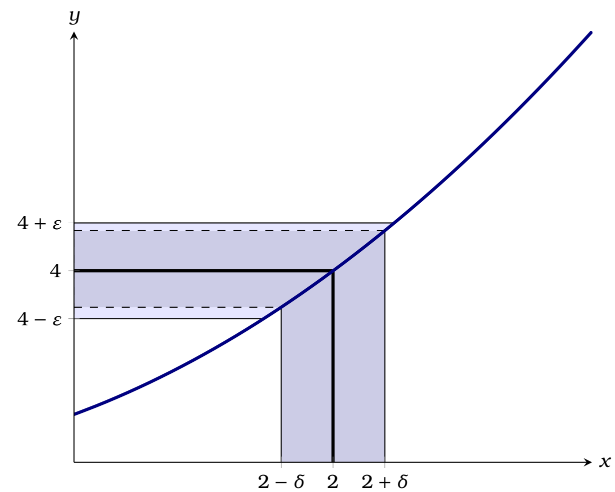


Figure 1.6: The (ε, δ)-criterion for $\lim_{x\to 2} x^2 = 4$. Here $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

When dealing with limits of polynomials, the general strategy is always the same. Let $p(x)$ be a polynomial. If showing

$$\lim_{x\to a} p(x) = L,$$

one must first factor out $|x - a|$ from $|p(x) - L|$. Next bound $x \in [a - 1, a + 1]$ and estimate the largest possible value of

$$\left|\frac{p(x) - L}{x - a}\right|$$

for $x \in [a-1, a+1]$. Call this estimation M. Finally, one must set $\delta = \min\left(\frac{\varepsilon}{M}, 1\right)$.

As you work with limits, you find that you need to do the same procedures again and again. The next theorems will expedite this process.

Theorem 1.2.2 (Limit Product Law) *Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$. Then*

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof *Given any ε we need to find a δ such that*

$$0 < |x-a| < \delta$$

implies

$$|f(x)g(x) - LM| < \varepsilon.$$

Here we use an algebraic trick, add $0 = -f(x)M + f(x)M$:

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)(g(x) - M)| + |(f(x) - L)M| \\ &= |f(x)||g(x) - M| + |f(x) - L||M|. \end{aligned}$$

We will use this same trick again of "adding 0" in the proof of Theorem 4.1.1.

This is all straightforward except perhaps for the "$\leq$". This follows from the *Triangle Inequality*. The **Triangle Inequality** states: If a and b are any real numbers then $|a+b| \leq |a| + |b|$.

Since $\lim_{x\to a} f(x) = L$, there is a value δ_1 so that $0 < |x-a| < \delta_1$ implies $|f(x) - L| < |\varepsilon/(2M)|$. This means that $0 < |x-a| < \delta_1$ implies $|f(x) - L||M| < \varepsilon/2$.

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + \underbrace{|f(x) - L||M|}_{<\frac{\varepsilon}{2}}.$$

If we can make $|f(x)||g(x) - M| < \varepsilon/2$, then we'll be done. We can make $|g(x) - M|$ smaller than any fixed number by making x close enough to a. Unfortunately, $\varepsilon/(2f(x))$ is not a fixed number since x is a variable.

Here we need another trick. We can find a δ_2 *so that* $|x-a|<\delta_2$ *implies that* $|f(x)-L|<1$, *meaning that* $L-1<f(x)<L+1$. *This means that* $|f(x)|<N$, *where* N *is either* $|L-1|$ *or* $|L+1|$, *depending on whether* L *is negative or positive. The important point is that* N *doesn't depend on* x. *Finally, we know that there is a* δ_3 *so that* $0<|x-a|<\delta_3$ *implies* $|g(x)-M|<\varepsilon/(2N)$. *Now we're ready to put everything together. Let* δ *be the smallest of* δ_1, δ_2, *and* δ_3. *Then* $|x-a|<\delta$ *implies that*

$$|f(x)g(x)-LM| \le \underbrace{|f(x)|}_{<N}\underbrace{|g(x)-M|}_{<\frac{\varepsilon}{2N}}+\underbrace{|f(x)-L||M|}_{<\frac{\varepsilon}{2}}.$$

so

$$\begin{aligned}|f(x)g(x)-LM| &\le |f(x)||g(x)-M|+|f(x)-L||M|\\ &< N\frac{\varepsilon}{2N}+\left|\frac{\varepsilon}{2M}\right||M|\\ &=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.\end{aligned}$$

This is just what we needed, so by the definition of a limit, $\lim_{x\to a} f(x)g(x)=LM$.

Another useful way to put functions together is composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x)=\sqrt{x}$ and $g(x)=x^2+5$, then $f(g(x))=\sqrt{x^2+5}$ and $g(f(x))=(\sqrt{x})^2+5=x+5$. This brings us to our next theorem.

Theorem 1.2.3 (Limit Composition Law) *Suppose that* $\lim_{x\to a} g(x)=M$ *and* $\lim_{x\to M} f(x)=f(M)$. *Then*

$$\lim_{x\to a} f(g(x))=f(M).$$

This is sometimes written as

$$\lim_{x\to a} f(g(x))=\lim_{g(x)\to M} f(g(x)).$$

Note the special form of the condition on $f(x)$: it is not enough to know that $\lim_{x\to L} f(x)$ exists, though it is a bit tricky to see why. Consider

$$f(x) = \begin{cases} 3 & \text{if } x = 2, \\ 4 & \text{if } x \neq 2. \end{cases}$$

and $g(x) = 2$. Now the conditions of Theorem 1.2.3 are not satisfied, and

$$\lim_{x\to 1} f(g(x)) = 3 \qquad \textit{but} \qquad \lim_{x\to 2} f(x) = 4.$$

Many of the most familiar functions do satisfy the conditions of Theorem 1.2.3. For example:

Theorem 1.2.4 (Limit Root Law) *Suppose that n is a positive integer. Then*

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Exercises for Section 1.2

(1) For each of the following limits, $\lim_{x\to a} f(x) = L$, use a graphing device to find δ such that $0 < |x-a| < \delta$ implies that $|f(x) - L| < \varepsilon$ where $\varepsilon = .1$.

(a) $\lim_{x\to 2}(3x+1) = 7$

(b) $\lim_{x\to 1}(x^2+2) = 3$

(c) $\lim_{x\to \pi}\sin(x) = 0$

(d) $\lim_{x\to 0}\tan(x) = 0$

(e) $\lim_{x\to 1}\sqrt{3x+1} = 2$

(f) $\lim_{x\to -2}\sqrt{1-4x} = 3$

The next set of exercises are for advanced students and can be skipped on first reading.

(2) Use the definition of limits to explain why $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. Hint: Use the fact that $|\sin(a)| < 1$ for any real number a.

(3) Use the definition of limits to explain why $\lim_{x\to 4}(2x-5) = 3$.

(4) Use the definition of limits to explain why $\lim_{x\to -3}(-4x-11) = 1$.

(5) Use the definition of limits to explain why $\lim_{x\to -2}\pi = \pi$.

(6) Use the definition of limits to explain why $\lim_{x\to -2}\frac{x^2-4}{x+2} = -4$.

(7) Use the definition of limits to explain why $\lim_{x\to 4} x^3 = 64$.

(8) Use the definition of limits to explain why $\lim_{x\to 1}(x^2+3x-1) = 3$.

(9) Use the definition of limits to explain why $\lim_{x\to 9}\frac{x-9}{\sqrt{x}-3} = 6$.

(10) Use the definition of limits to explain why $\lim_{x\to 2}\frac{1}{x} = \frac{1}{2}$.

1.3 Limit Laws

In this section, we present a handful of tools to compute many limits without explicitly working with the definition of limit. Each of these could be proved directly as we did in the previous section.

Theorem 1.3.1 (Limit Laws) *Suppose that* $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, k *is some constant, and* n *is a positive integer.*

Constant Law $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = kL$.

Sum Law $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$.

Product Law $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$.

Quotient Law $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$.

Power Law $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$.

Root Law $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ *provided if* n *is even, then* $f(x) \geq 0$ *near* a.

Composition Law *If* $\lim_{x\to a} g(x) = M$ *and* $\lim_{x\to M} f(x) = f(M)$, *then* $\lim_{x\to a} f(g(x)) = f(M)$.

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 1.3.2 Compute $\lim_{x\to 1} \dfrac{x^2 - 3x + 5}{x - 2}$.

Solution *Using limit laws,*

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1} x^2-3x+5}{\lim_{x\to 1}(x-2)}\\
&= \frac{\lim_{x\to 1} x^2-\lim_{x\to 1} 3x+\lim_{x\to 1} 5}{\lim_{x\to 1} x-\lim_{x\to 1} 2}\\
&= \frac{(\lim_{x\to 1} x)^2-3\lim_{x\to 1} x+5}{\lim_{x\to 1} x-2}\\
&= \frac{1^2-3\cdot 1+5}{1-2}\\
&= \frac{1-3+5}{-1}=-3.
\end{aligned}$$

It is worth commenting on the trivial limit $\lim\limits_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

We're primarily interested in limits that aren't so easy, namely limits in which a denominator approaches zero. The basic idea is to "divide out" by the offending factor. This is often easier said than done—here we give two examples of algebraic tricks that work on many of these limits.

Example 1.3.3 Compute $\lim\limits_{x\to 1}\dfrac{x^2+2x-3}{x-1}$.

Solution *We can't simply plug in $x = 1$ because that makes the denominator zero. However, when taking limits we assume $x \neq 1$:*

Limits allow us to examine functions where they are not defined.

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2+2x-3}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+3)}{x-1}\\
&= \lim_{x\to 1}(x+3)=4
\end{aligned}$$

Example 1.3.4 Compute $\lim_{x \to -1} \frac{\sqrt{x+5}-2}{x+1}$.

Solution *Using limit laws,*

$$\begin{aligned}\lim_{x\to-1}\frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to-1}\frac{\sqrt{x+5}-2}{x+1}\,\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2}\\ &= \lim_{x\to-1}\frac{x+5-4}{(x+1)(\sqrt{x+5}+2)}\\ &= \lim_{x\to-1}\frac{x+1}{(x+1)(\sqrt{x+5}+2)}\\ &= \lim_{x\to-1}\frac{1}{\sqrt{x+5}+2} = \frac{1}{4}.\end{aligned}$$

Here we are rationalizing the numerator by multiplying by the conjugate.

We'll conclude with one more theorem that will allow us to compute more difficult limits.

Theorem 1.3.5 (Squeeze Theorem) *Suppose that $g(x) \le f(x) \le h(x)$ for all x close to a but not necessarily equal to a. If*

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

then $\lim_{x\to a} f(x) = L$.

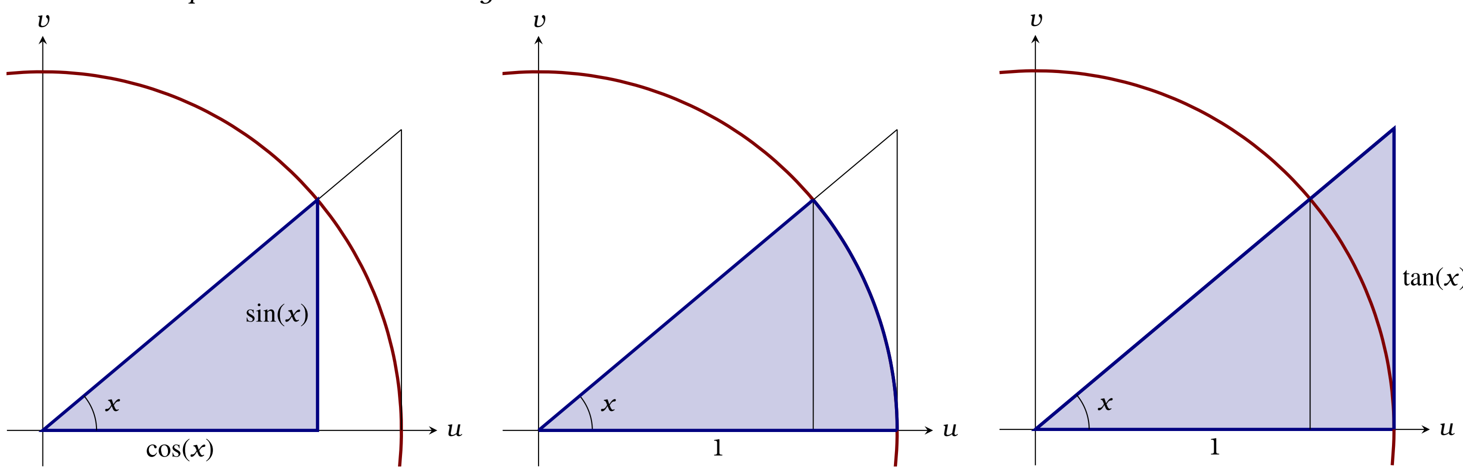
Example 1.3.6 theorem:deriv sin Compute

$$\lim_{x\to 0}\frac{\sin(x)}{x}.$$

For a nice discussion of this limit, see: Richman, Fred. *A circular argument*. College Math. J. 24 (1993), no. 2, 160–162.

The limit in this example will be used in Theorem 6.1.1.

Solution *To compute this limit, use the Squeeze Theorem, Theorem 1.3.5. First note that we only need to examine* $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *and for the present time, we'll assume that* x *is positive—consider the diagrams below:*



Triangle A *Sector* *Triangle B*

From our diagrams above we see that

$$\textit{Area of Triangle A} \leq \textit{Area of Sector} \leq \textit{Area of Triangle B}$$

and computing these areas we find

$$\frac{\cos(x)\sin(x)}{2} \leq \left(\frac{x}{2\pi}\right)\cdot\pi \leq \frac{\tan(x)}{2}.$$

Multiplying through by 2*, and recalling that* $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ *we obtain*

$$\cos(x)\sin(x) \leq x \leq \frac{\sin(x)}{\cos(x)}.$$

Dividing through by $\sin(x)$ *and taking the reciprocals, we find*

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}.$$

Note, $\cos(-x) = \cos(x)$ *and* $\dfrac{\sin(-x)}{-x} = \dfrac{\sin(x)}{x}$*, so these inequalities hold for all* $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$*. Additionally, we know*

$$\lim_{x\to 0} \cos(x) = 1 = \lim_{x\to 0} \frac{1}{\cos(x)},$$

and so we conclude by the Squeeze Theorem, Theorem 1.3.5, $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$.

Exercises for Section 1.3

Compute the limits. If a limit does not exist, explain why.

(1) $\lim\limits_{x\to 3} \dfrac{x^2+x-12}{x-3}$

(2) $\lim\limits_{x\to 1} \dfrac{x^2+x-12}{x-3}$

(3) $\lim\limits_{x\to -4} \dfrac{x^2+x-12}{x-3}$

(4) $\lim\limits_{x\to 2} \dfrac{x^2+x-12}{x-2}$

(5) $\lim\limits_{x\to 1} \dfrac{\sqrt{x+8}-3}{x-1}$

(6) $\lim\limits_{x\to 0+} \sqrt{\dfrac{1}{x}+2} - \sqrt{\dfrac{1}{x}}$

(7) $\lim\limits_{x\to 2} 3$

(8) $\lim\limits_{x\to 4} 3x^3-5x$

(9) $\lim\limits_{x\to 0} \dfrac{4x-5x^2}{x-1}$

(10) $\lim\limits_{x\to 1} \dfrac{x^2-1}{x-1}$

(11) $\lim\limits_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x}$

(12) $\lim\limits_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x+1}$

(13) $\lim\limits_{x\to a} \dfrac{x^3-a^3}{x-a}$

(14) $\lim\limits_{x\to 2} (x^2+4)^3$

(15) $\lim\limits_{x\to 1} \begin{cases} x-5 & \text{if } x \neq 1, \\ 7 & \text{if } x = 1. \end{cases}$

1.4 Infinite Limits

Consider the function
$$f(x) = \frac{1}{(x+1)^2}$$
While the $\lim_{x \to -1} f(x)$ does not exist, see Figure 1.7, something can still be said.

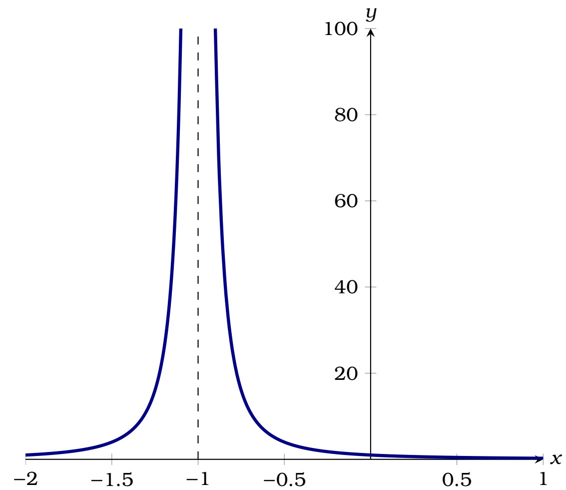


Figure 1.7: A plot of $f(x) = \dfrac{1}{(x+1)^2}$.

Definition If $f(x)$ grows arbitrarily large as x approaches a, we write
$$\lim_{x \to a} f(x) = \infty$$
and say that the limit of $f(x)$ **approaches infinity** as x goes to a.

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write
$$\lim_{x \to a} f(x) = -\infty$$
and say that the limit of $f(x)$ **approaches negative infinity** as x goes to a.

On the other hand, if we consider the function
$$f(x) = \frac{1}{(x-1)}$$
While we have $\lim_{x \to 1} f(x) \neq \pm\infty$, we do have one-sided limits, $\lim_{x \to 1+} f(x) = \infty$ and $\lim_{x \to 1-} f(x) = -\infty$, see Figure 1.8.

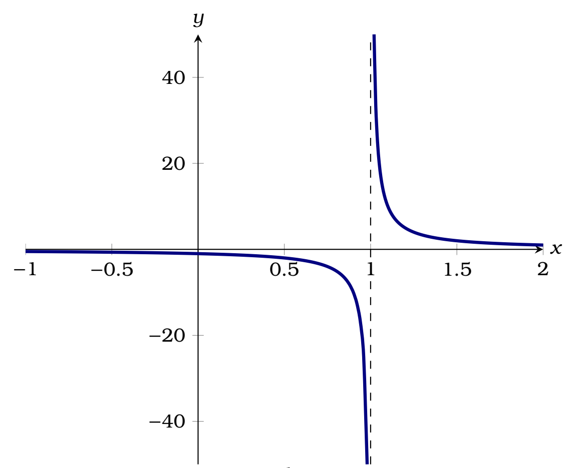


Figure 1.8: A plot of $f(x) = \dfrac{1}{x-1}$.

Definition If
$$\lim_{x \to a} f(x) = \pm\infty, \qquad \lim_{x \to a+} f(x) = \pm\infty, \qquad \text{or} \qquad \lim_{x \to a-} f(x) = \pm\infty,$$
then the line $x = a$ is a **vertical asymptote** of $f(x)$.

Example 1.4.1 Find the vertical asymptotes of

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

Solution *Start by factoring both the numerator and the denominator:*

$$\frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

Using limits, we must investigate when $x \to 2$ *and* $x \to 3$. *Write*

$$\begin{aligned}\lim_{x\to 2} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 2} \frac{(x-7)}{(x-3)} \\ &= \frac{-5}{-1} \\ &= 5.\end{aligned}$$

Now write

$$\begin{aligned}\lim_{x\to 3} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 3} \frac{(x-7)}{(x-3)} \\ &= \lim_{x\to 3} \frac{-4}{x-3}.\end{aligned}$$

Since $\lim_{x\to 3+} x - 3$ *approaches* 0 *from the right and the numerator is negative,* $\lim_{x\to 3+} f(x) = -\infty$. *Since* $\lim_{x\to 3-} x - 3$ *approaches* 0 *from the left and the numerator is negative,* $\lim_{x\to 3-} f(x) = \infty$. *Hence we have a vertical asymptote at* $x = 3$, *see Figure 1.9.*

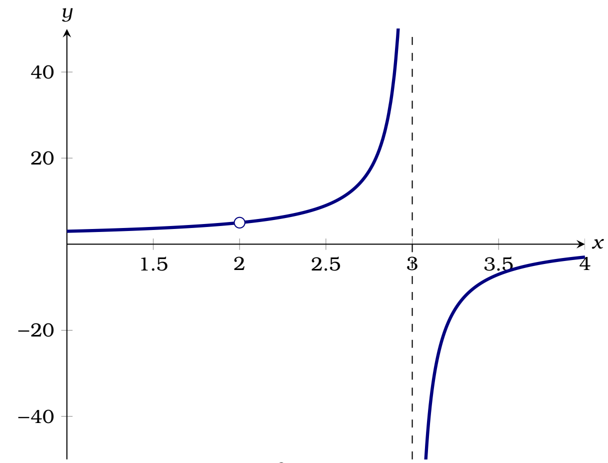


Figure 1.9: A plot of $f(x) = \dfrac{x^2 - 9x + 14}{x^2 - 5 + 6}$.

Exercises for Section 1.4

Compute the limits. If a limit does not exist, explain why.

(1) $\displaystyle\lim_{x\to 1-} \frac{1}{x^2-1}$

(2) $\displaystyle\lim_{x\to 4-} \frac{3}{x^2-2}$

(3) $\displaystyle\lim_{x\to -1+} \frac{1+2x}{x^3-1}$

(4) $\displaystyle\lim_{x\to 3+} \frac{x-9}{x^2-6x+9}$

(5) $\displaystyle\lim_{x\to 5} \frac{1}{(x-5)^4}$

(6) $\displaystyle\lim_{x\to -2} \frac{1}{(x^2+3x+2)^2}$

(7) $\displaystyle\lim_{x\to 0} \frac{1}{\frac{x}{x^5}-\cos(x)}$

(8) $\displaystyle\lim_{x\to 0+} \frac{x-11}{\sin(x)}$

(9) Find the vertical asymptotes of

$$f(x) = \frac{x-3}{x^2+2x-3}.$$

(10) Find the vertical asymptotes of

$$f(x) = \frac{x^2-x-6}{x+4}.$$

1.5 Limits at Infinity

Consider the function:

$$f(x) = \frac{6x-9}{x-1}$$

As x approaches infinity, it seems like $f(x)$ approaches a specific value. This is a *limit at infinity*.

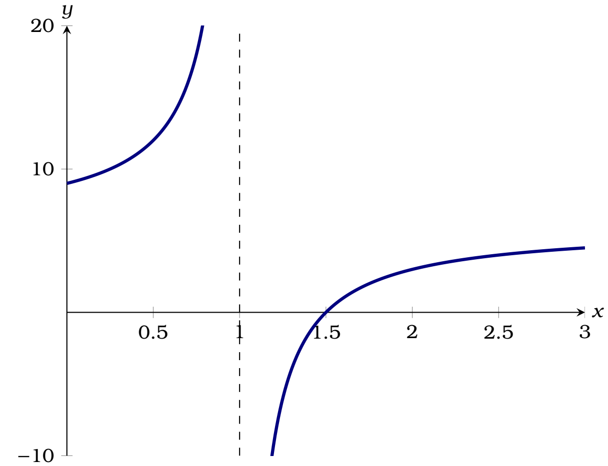


Figure 1.10: A plot of $f(x) = \dfrac{6x-9}{x-1}$.

Definition If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

$$\lim_{x\to\infty} f(x) = L$$

and we say, the **limit at infinity** of $f(x)$ is L.

If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x\to-\infty} f(x) = L$$

and we say, the **limit at negative infinity** of $f(x)$ is L.

Example 1.5.1 Compute

$$\lim_{x\to\infty} \frac{6x-9}{x-1}.$$

Solution *Write*

$$\begin{aligned}
\lim_{x\to\infty} \frac{6x-9}{x-1} &= \lim_{x\to\infty} \frac{6x-9}{x-1}\,\frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{\frac{6x}{x}-\frac{9}{x}}{\frac{x}{x}-\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{6}{1} \\
&= 6.
\end{aligned}$$

Here is a somewhat different example of a limit at infinity.

Example 1.5.2 Compute

$$\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4.$$

Solution *We can bound our function*

$$-1/x + 4 \le \frac{\sin(7x)}{x} + 4 \le 1/x + 4.$$

Since

$$\lim_{x\to\infty} -1/x + 4 = 4 = \lim_{x\to\infty} 1/x + 4$$

we conclude by the Squeeze Theorem, Theorem 1.3.5, $\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4 = 4.$

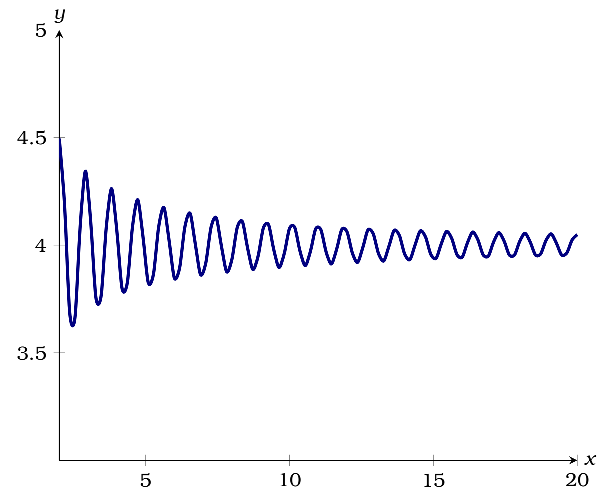


Figure 1.11: A plot of $f(x) = \frac{\sin(7x)}{x} + 4$.

Definition If

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$

then the line $y = L$ is a **horizontal asymptote** of $f(x)$.

Example 1.5.3 Give the horizontal asymptotes of

$$f(x) = \frac{6x-9}{x-1}$$

Solution *From our previous work, we see that* $\lim_{x\to\infty} f(x) = 6$, *and upon further inspection, we see that* $\lim_{x\to-\infty} f(x) = 6$. *Hence the horizontal asymptote of* $f(x)$ *is the line* $y = 6$.

It is a common misconception that a function cannot cross an asymptote. As the next example shows, a function can cross an asymptote, and in this case this occurs an infinite number of times!

Example 1.5.4 Give a horizontal asymptote of

$$f(x) = \frac{\sin(7x)}{x} + 4.$$

Solution *Again from previous work, we see that* $\lim_{x\to\infty} f(x) = 4$. *Hence* $y = 4$ *is a horizontal asymptote of* $f(x)$.

We conclude with an infinite limit at infinity.

Example 1.5.5 Compute

$$\lim_{x\to\infty} \ln(x)$$

Solution *The function* $\ln(x)$ *grows very slowly, and seems like it may have a horizontal asymptote, see Figure 1.12. However, if we consider the definition of the natural log*

$$\ln(x) = y \qquad \Leftrightarrow \qquad e^y = x$$

Since we need to raise e to higher and higher values to obtain larger numbers, we see that $\ln(x)$ *is unbounded, and hence* $\lim_{x\to\infty} \ln(x) = \infty$.

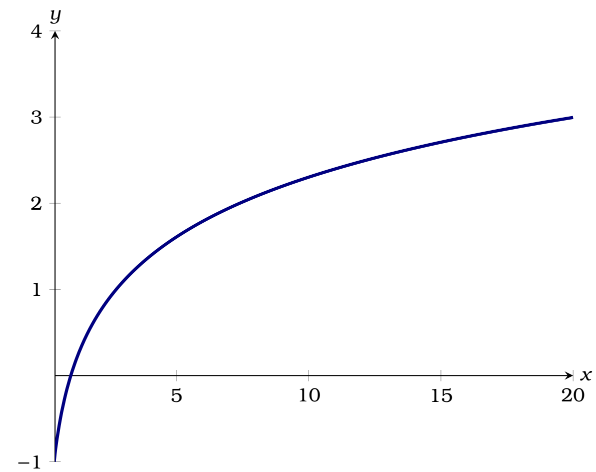


Figure 1.12: A plot of $f(x) = \ln(x)$.

Exercises for Section 1.5

Compute the limits.

(1) $\displaystyle\lim_{x\to\infty} \frac{1}{x}$

(2) $\displaystyle\lim_{x\to\infty} \frac{-x}{\sqrt{4+x^2}}$

(3) $\displaystyle\lim_{x\to\infty} \frac{2x^2-x+1}{4x^2-3x-1}$

(4) $\displaystyle\lim_{x\to-\infty} \frac{x^3-4}{3x^2+4x-1}$

(5) $\displaystyle\lim_{x\to\infty} \left(\frac{4}{x}+\pi\right)$

(6) $\displaystyle\lim_{x\to\infty} \frac{\cos(x)}{\ln(x)}$

(7) $\displaystyle\lim_{x\to\infty} \frac{\sin\left(x^7\right)}{\sqrt{x}}$

(8) $\displaystyle\lim_{x\to\infty} \left(17+\frac{32}{x}-\frac{(\sin(x/2))^2}{x^3}\right)$

(9) Suppose a population of feral cats on a certain college campus t years from now is approximated by

$$p(t) = \frac{1000}{5+2e^{-0.1t}}.$$

Approximately how many feral cats are on campus 10 years from now? 50 years from now? 100 years from now? 1000 years from now? What do you notice about the prediction—is this realistic?

(10) The amplitude of an oscillating spring is given by

$$a(t) = \frac{\sin(t)}{t}.$$

What happens to the amplitude of the oscillation over a long period of time?

1.6 Continuity

Informally, a function is continuous if you can "draw it" without "lifting your pencil." We need a formal definition.

Definition A function f is **continuous at a point** a if $\lim_{x\to a} f(x) = f(a)$.

Example 1.6.1 Find the discontinuities (the values for x where a function is not continuous) for the function given in Figure 1.13.

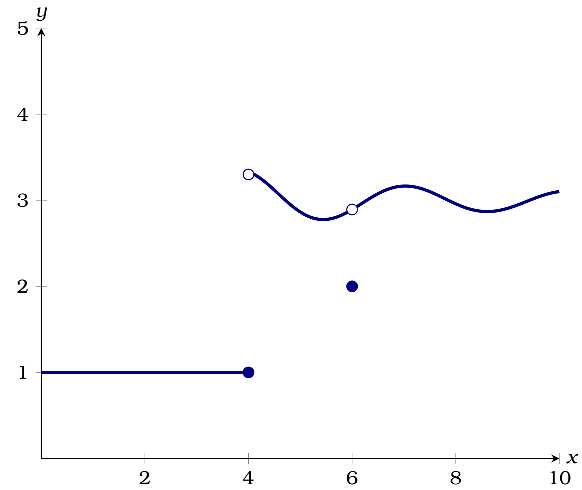


Figure 1.13: A plot of a function with discontinuities at $x = 4$ and $x = 6$.

Solution *From Figure 1.13 we see that* $\lim_{x\to 4} f(x)$ *does not exist as*

$$\lim_{x\to 4-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 4+} f(x) \approx 3.5$$

Hence $\lim_{x\to 4} f(x) \neq f(4)$*, and so* $f(x)$ *is not continuous at* $x = 4$*.*

We also see that $\lim_{x\to 6} f(x) \approx 3$ *while* $f(6) = 2$*. Hence* $\lim_{x\to 6} f(x) \neq f(6)$*, and so* $f(x)$ *is not continuous at* $x = 6$*.*

Building from the definition of *continuous at a point*, we can now define what it means for a function to be *continuous* on an interval.

Definition A function f is **continuous on an interval** if it is continuous at every point in the interval.

In particular, we should note that if a function is not defined on an interval, then it **cannot** be continuous on that interval.

Example 1.6.2 Consider the function

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

see Figure 1.14. Is this function continuous?

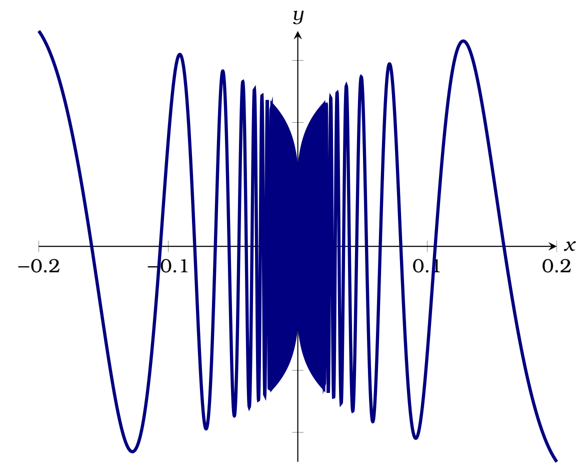


Figure 1.14: A plot of

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Solution *Considering $f(x)$, the only issue is when $x = 0$. We must show that $\lim_{x\to 0} f(x) = 0$. Note*

$$-|\sqrt[5]{x}| \le f(x) \le |\sqrt[5]{x}|.$$

Since

$$\lim_{x\to 0} -|\sqrt[5]{x}| = 0 = \lim_{x\to 0} |\sqrt[5]{x}|,$$

we see by the Squeeze Theorem, Theorem 1.3.5, that $\lim_{x\to 0} f(x) = 0$. Hence $f(x)$ is continuous.

Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.

We close with a useful theorem about continuous functions:

Theorem 1.6.3 (Intermediate Value Theorem) *If $f(x)$ is a function that is continuous for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*

The Intermediate Value Theorem is most frequently used when $d = 0$.

For a nice proof of this theorem, see: Walk, Stephen M. *The intermediate value theorem is NOT obvious—and I am going to prove it to you.* College Math. J. 42 (2011), no. 4, 254–259.

In Figure 1.15, we see a geometric interpretation of this theorem.

Example 1.6.4 Explain why the function $f(x) = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

Solution *By Theorem 1.3.1, $\lim_{x\to a} f(x) = f(a)$, for all real values of a, and hence f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, there is a $c \in [0, 1]$ such that $f(c) = 0$.*

This example also points the way to a simple method for approximating roots.

Example 1.6.5 Approximate a root of $f(x) = x^3 + 3x^2 + x - 2$ to one decimal place.

Solution *If we compute $f(0.1)$, $f(0.2)$, and so on, we find that $f(0.6) < 0$ and $f(0.7) > 0$, so by the Intermediate Value Theorem, f has a root between 0.6*

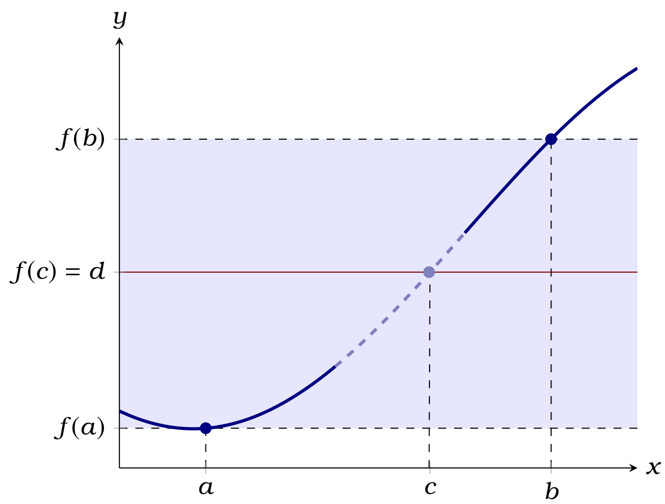


Figure 1.15: A geometric interpretation of the Intermediate Value Theorem. The function $f(x)$ is continuous on the interval $[a, b]$. Since d is in the interval $[f(a), f(b)]$, there exists a value c in $[a, b]$ such that $f(c) = d$.

and 0.7. *Repeating the process with* $f(0.61)$, $f(0.62)$, *and so on, we find that* $f(0.61) < 0$ *and* $f(0.62) > 0$, *so* f *has a root between* 0.61 *and* 0.62, *and the root is* 0.6 *rounded to one decimal place.*

Exercises for Section 1.6

(1) Consider the function

$$f(x) = \sqrt{x - 4}$$

Is $f(x)$ continuous at the point $x = 4$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(2) Consider the function

$$f(x) = \frac{1}{x + 3}$$

Is $f(x)$ continuous at the point $x = 3$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(3) Consider the function

$$f(x) = \begin{cases} 2x - 3 & \text{if } x < 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

Is $f(x)$ continuous at the point $x = 1$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(4) Consider the function

$$f(x) = \begin{cases} \dfrac{x^2 + 10x + 25}{x - 5} & \text{if } x \neq 5, \\ 10 & \text{if } x = 5. \end{cases}$$

Is $f(x)$ continuous at the point $x = 5$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(5) Consider the function

$$f(x) = \begin{cases} \dfrac{x^2 + 10x + 25}{x + 5} & \text{if } x \neq -5, \\ 0 & \text{if } x = -5. \end{cases}$$

Is $f(x)$ continuous at the point $x = -5$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(6) Determine the interval(s) on which the function $f(x) = x^7 + 3x^5 - 2x + 4$ is continuous.

(7) Determine the interval(s) on which the function $f(x) = \dfrac{x^2 - 2x + 1}{x + 4}$ is continuous.

(8) Determine the interval(s) on which the function $f(x) = \dfrac{1}{x^2 - 9}$ is continuous.

(9) Approximate a root of $f(x) = x^3 - 4x^2 + 2x + 2$ to two decimal places.

(10) Approximate a root of $f(x) = x^4 + x^3 - 5x + 1$ to two decimal places.

2 Basics of Derivatives

2.1 Slopes of Tangent Lines via Limits

Suppose that $f(x)$ is a function. It is often useful to know how sensitive the value of $f(x)$ is to small changes in x. To give you a feeling why this is true, consider the following:

- If $p(t)$ represents the position of an object with respect to time, the rate of change gives the velocity of the object.
- If $v(t)$ represents the velocity of an object with respect to time, the rate of change gives the acceleration of the object.
- The rate of change of a function can help us approximate a complicated function with a simple function.
- The rate of change of a function can be used to help us solve equations that we would not be able to solve via other methods.

The rate of change of a function is the slope of the tangent line. Part of our goal will be to give a formal definition of a tangent line. For now, consider the following informal definition:

> Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

While this is merely an informal definition of a tangent line, it contains the essence of how the formal definition will be constructed. We illustrate this informal definition with Figure 2.1.

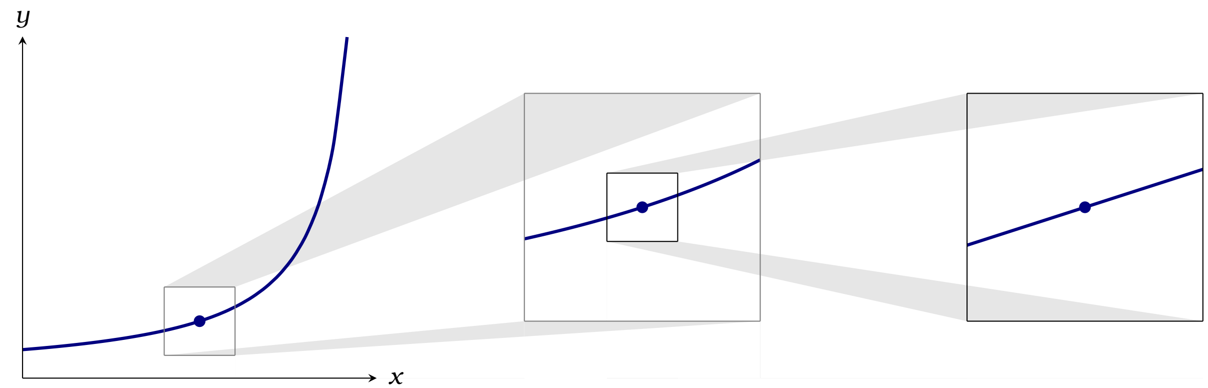


Figure 2.1: Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

The *derivative* of a function $f(x)$ at x, is the slope of the tangent line at x. To find the slope of this line, we consider *secant* lines, lines that locally intersect the curve at two points. The slope of any secant line that passes through the points $(x, f(x))$ and $(x+h, f(x+h))$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h},$$

see Figure 2.2. This leads to the *limit definition of the derivative*:

Definition of the Derivative The **derivative** of $f(x)$ is the function

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If this limit does not exist for a given value of x, then $f(x)$ is not **differentiable** at x.

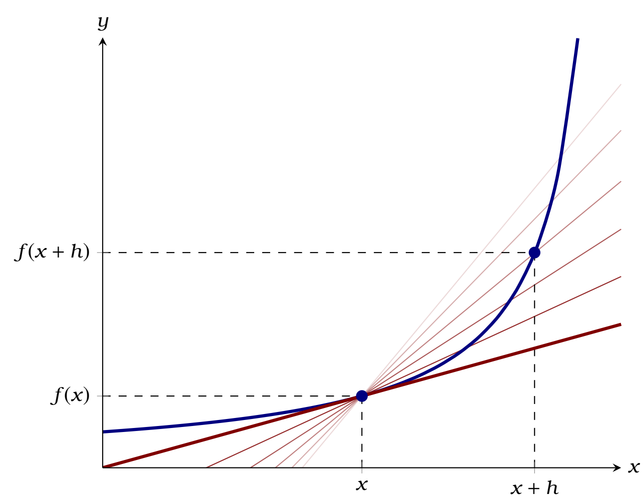


Figure 2.2: Tangent lines can be found as the limit of secant lines. The slope of the tangent line is given by $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Definition There are several different notations for the derivative, we'll mainly use
$$\frac{d}{dx}f(x) = f'(x).$$
If one is working with a function of a variable other than x, say t we write
$$\frac{d}{dt}f(t) = f'(t).$$
However, if $y = f(x)$, $\frac{dy}{dx}$, $\dot{y}$, and $D_x f(x)$ are also used.

Now we will give a number of examples, starting with a basic example.

Example 2.1.1 Compute
$$\frac{d}{dx}(x^3 + 1).$$

Solution *Using the definition of the derivative,*
$$\begin{aligned}\frac{d}{dx}f(x) &= \lim_{h\to 0}\frac{(x+h)^3 + 1 - (x^3+1)}{h}\\ &= \lim_{h\to 0}\frac{x^3 + 3x^2h + 3xh^2 + h^3 + 1 - x^3 - 1}{h}\\ &= \lim_{h\to 0}\frac{3x^2h + 3xh^2 + h^3}{h}\\ &= \lim_{h\to 0}(3x^2 + 3xh + h^2)\\ &= 3x^2.\end{aligned}$$

See Figure 2.3.

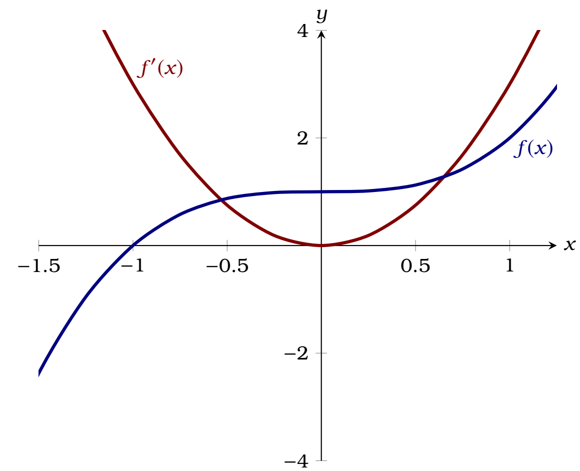


Figure 2.3: A plot of $f(x) = x^3 + 1$ and $f'(x) = 3x^2$.

Next we will consider the derivative a function that is not continuous on $\mathbb{R}$.

Example 2.1.2 Compute
$$\frac{d}{dt}\frac{1}{t}.$$

Solution *Using the definition of the derivative,*

$$\begin{aligned}\frac{d}{dt}\frac{1}{t} &= \lim_{h\to 0}\frac{\frac{1}{t+h}-\frac{1}{t}}{h}\\ &= \lim_{h\to 0}\frac{\frac{t}{t(t+h)}-\frac{t+h}{t(t+h)}}{h}\\ &= \lim_{h\to 0}\frac{\frac{t-(t+h)}{t(t+h)}}{h}\\ &= \lim_{h\to 0}\frac{t-t-h}{t(t+h)h}\\ &= \lim_{h\to 0}\frac{-h}{t(t+h)h}\\ &= \lim_{h\to 0}\frac{-1}{t(t+h)}\\ &= \frac{-1}{t^2}.\end{aligned}$$

This function is differentiable at all real numbers except for $t = 0$, see Figure 2.4.

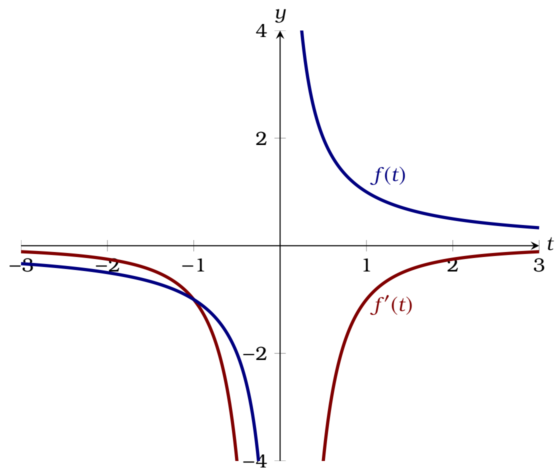


Figure 2.4: A plot of $f(t) = \frac{1}{t}$ and $f'(t) = \frac{-1}{t^2}$.

As you may have guessed, there is some connection to continuity and differentiability.

Theorem 2.1.3 (Differentiability implies Continuity) *If $f(x)$ is a differentiable function at $x = a$, then $f(x)$ is continuous at $x = a$.*

Proof *We want to show that $f(x)$ is continuous at $x = a$, hence we must show that*

$$\lim_{x\to a} f(x) = f(a).$$

Consider

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\left((x-a)\frac{f(x)-f(a)}{x-a}\right) && \textit{Multiply and divide by } (x-a).\\
&= \lim_{h\to 0} h\cdot\frac{f(a+h)-f(a)}{h} && \textit{Set } x=a+h.\\
&= \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right) && \textit{Limit Law.}\\
&= 0\cdot f'(a) = 0.
\end{aligned}$$

Since

$$\lim_{x\to a}(f(x)-f(a)) = 0$$

we see that $\lim_{x\to a} f(x) = f(a)$, *and so* $f(x)$ *is continuous.*

This theorem is often written as its contrapositive:

If $f(x)$ is not continuous at $x=a$, then $f(x)$ is not differentiable at $x=a$.

Let's see a function that is continuous whose derivative does not exist everywhere.

Example 2.1.4 Compute

$$\frac{d}{dx}|x|.$$

Solution *Using the definition of the derivative,*

$$\frac{d}{dx}|x| = \lim_{h\to 0}\frac{|x+h|-|x|}{h}.$$

If x *is positive we may assume that* x *is larger than* h*, as we are taking the limit as* h *goes to* 0,

$$\begin{aligned}
\lim_{h\to 0}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0}\frac{x+h-x}{h}\\
&= \lim_{h\to 0}\frac{h}{h}\\
&= 1.
\end{aligned}$$

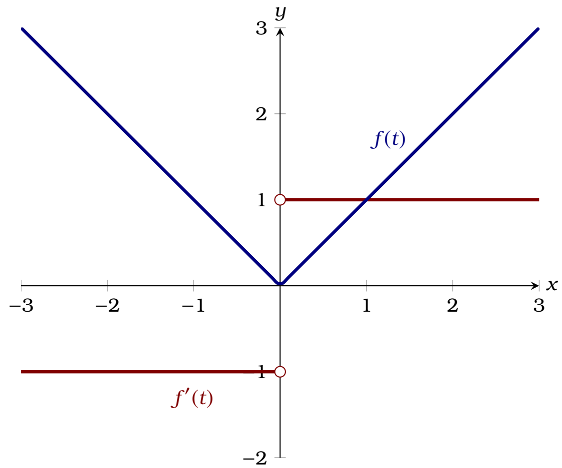


Figure 2.5: A plot of $f(x)=|x|$ and

$$f'(x) = \begin{cases} 1 & \text{if } x>0,\\ -1 & \text{if } x<0.\end{cases}$$

If x is negative we may assume that $|x|$ is larger than h, as we are taking the limit as h goes to 0,

$$\begin{aligned}\lim_{h\to 0}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0}\frac{x-h-x}{h}\\ &= \lim_{h\to 0+}\frac{-h}{h}\\ &= -1.\end{aligned}$$

However we still have one case left, when $x = 0$. In this situation, we must consider the one-sided limits:

$$\lim_{h\to 0+}\frac{|x+h|-|x|}{h} \qquad \textit{and} \qquad \lim_{h\to 0-}\frac{|x+h|-|x|}{h}.$$

In the first case,

$$\begin{aligned}\lim_{h\to 0+}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0+}\frac{0+h-0}{h}\\ &= \lim_{h\to 0+}\frac{h}{h}\\ &= 1.\end{aligned}$$

On the other hand

$$\begin{aligned}\lim_{h\to 0-}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0-}\frac{|0+h|-0}{h}\\ &= \lim_{h\to 0-}\frac{|h|}{h}\\ &= -1.\end{aligned}$$

Hence we see that the derivative is

$$f'(x) = \begin{cases} 1 & \textit{if } x > 0,\\ -1 & \textit{if } x < 0.\end{cases}$$

Note this function is undefined at 0, *see Figure 2.5.*

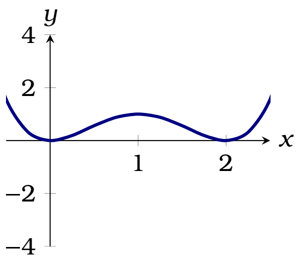
Thus from Theorem 2.1.3, we see that all differentiable functions on $\mathbb{R}$ are continuous on $\mathbb{R}$. Nevertheless as the previous example shows, there are continuous functions on $\mathbb{R}$ that are not differentiable on $\mathbb{R}$.

Exercises for Section 2.1

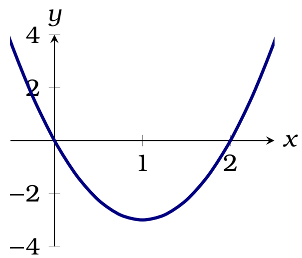
These exercises are conceptual in nature and require one to think about what the derivative means.

(1) If the line $y = 7x - 4$ is tangent to $f(x)$ at $x = 2$, find $f(2)$ and $f'(2)$.

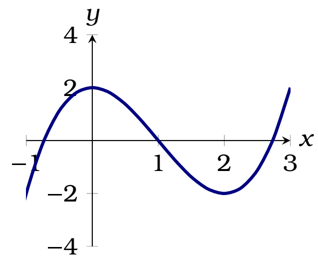
(2) Here are plots of four functions.



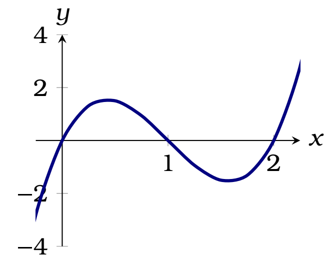
$p(x)$



$q(x)$



$r(x)$



$s(x)$

Two of these functions are the derivatives of the other two, identify which functions are the derivatives of the others.

(3) If $f(3) = 6$ and $f(3.1) = 6.4$, estimate $f'(3)$.

(4) If $f(-2) = 4$ and $f(-2 + h) = (h + 2)^2$, compute $f'(-2)$.

(5) If $f'(x) = x^3$ and $f(1) = 2$, approximate $f(1.2)$.

(6) Consider the plot of $f(x)$ in Figure 2.6.

(a) On which subinterval(s) of $[0, 6]$ is $f(x)$ continuous?

(b) On which subinterval(s) of $[0, 6]$ is $f(x)$ differentiable?

(c) Sketch a plot of $f'(x)$.

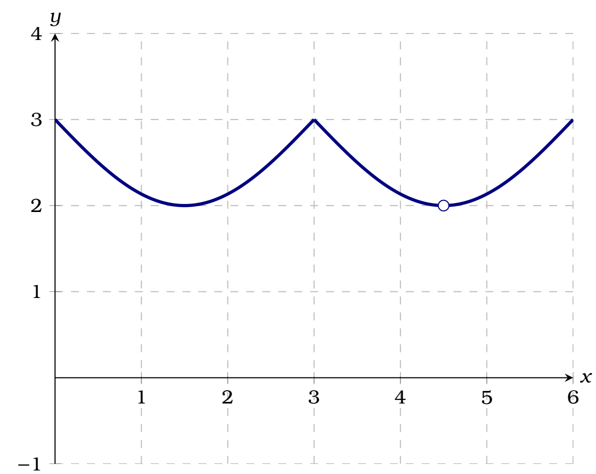


Figure 2.6: A plot of $f(x)$.

These exercises are computational in nature.

(7) Let $f(x) = x^2 - 4$. Use the definition of the derivative to compute $f'(-3)$ and find the equation of the tangent line to the curve at $x = -3$.

(8) Let $f(x) = \dfrac{1}{x+2}$. Use the definition of the derivative to compute $f'(1)$ and find the equation of the tangent line to the curve at $x = 1$.

(9) Let $f(x) = \sqrt{x-3}$. Use the definition of the derivative to compute $f'(5)$ and find the equation of the tangent line to the curve at $x = 5$.

(10) Let $f(x) = \dfrac{1}{\sqrt{x}}$. Use the definition of the derivative to compute $f'(4)$ and find the equation of the tangent line to the curve at $x = 4$.

2.2 Basic Derivative Rules

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. We will start simply and build-up to more complicated examples.

The Constant Rule

The simplest function is a constant function. Recall that derivatives measure the rate of change of a function at a given point. Hence, the derivative of a constant function is zero. For example:

- The constant function plots a horizontal line—so the slope of the tangent line is 0.
- If $p(t)$ represents the position of an object with respect to time and $p(t)$ is constant, then the object is not moving, so its velocity is zero. Hence $\frac{d}{dt}p(t) = 0$.
- If $v(t)$ represents the velocity of an object with respect to time and $v(t)$ is constant, then the object's acceleration is zero. Hence $\frac{d}{dt}v(t) = 0$.

The examples above lead us to our next theorem.

To gain intuition, you should compute the derivative of $f(x) = 6$ using the limit definition of the derivative.

Theorem 2.2.1 (The Constant Rule) *Given a constant c,*

$$\frac{d}{dx}c = 0.$$

Proof *From the limit definition of the derivative, write*

$$\begin{aligned}\frac{d}{dx}c &= \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= \lim_{h\to 0} 0 = 0.\end{aligned}$$

The Power Rule

Now let's examine derivatives of powers of a single variable. Here we have a nice rule.

To gain intuition, you should compute the derivative of $f(x) = x^3$ using the limit definition of the derivative.

Theorem 2.2.2 (The Power Rule) *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof *At this point we will only prove this theorem for n being a positive integer. Later in Section 5.3, we will give the complete proof. From the limit definition of the derivative, write*

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Start by expanding the term $(x+h)^n$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n - x^n}{h}$$

Note, by the Binomial Theorem, we write $\binom{n}{k}$ for the coefficients. Canceling the terms x^n and $-x^n$, and noting $\binom{n}{1} = \binom{n}{n-1} = n$, write

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n}{h}\\ &= \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}.\end{aligned}$$

Since every term but the first has a factor of h, we see

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Recall, the **Binomial Theorem** states that if n is a nonnegative integer, then

$$(a+b)^n = a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + a^0 b^n$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now we will show you several examples. We begin with something basic.

Example 2.2.3 Compute
$$\frac{d}{dx}x^{13}.$$

Solution *Applying the power rule, we write*
$$\frac{d}{dx}x^{13} = 13x^{12}.$$

Sometimes, it is not as obvious that one should apply the power rule.

Example 2.2.4 Compute
$$\frac{d}{dx}\frac{1}{x^4}.$$

Solution *Applying the power rule, we write*
$$\frac{d}{dx}\frac{1}{x^4} = \frac{d}{dx}x^{-4} = -4x^{-5}.$$

The power rule also applies to radicals once we rewrite them as exponents.

Example 2.2.5 Compute
$$\frac{d}{dx}\sqrt[5]{x}.$$

Solution *Applying the power rule, we write*
$$\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \frac{x^{-4/5}}{5}.$$

The Sum Rule

We want to be able to take derivatives of functions "one piece at a time." The *sum rule* allows us to do this. The sum rule says that we can add the rates of change of two functions to obtain the rate of change of the sum of both functions. For example, viewing the derivative as the velocity of an object, the sum rule states that the velocity of the person walking on a moving bus is the sum of the velocity of the bus and the walking person.

Theorem 2.2.6 (The Sum Rule) *If $f(x)$ and $g(x)$ are differentiable and c is a constant, then*

(a) $\dfrac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$,

(b) $\dfrac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$,

(c) $\dfrac{d}{dx}(c\cdot f(x)) = c\cdot f'(x)$.

Proof *We will only prove part (a) above, the rest are similar. Write*

$$\begin{aligned}\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\ &= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\ &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\ &= f'(x)+g'(x).\end{aligned}$$

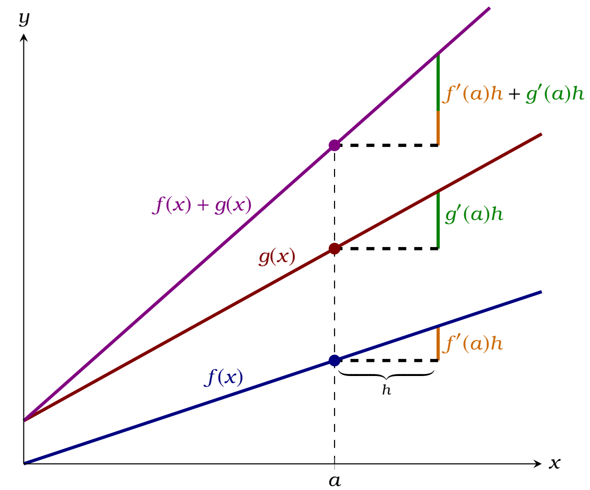


Figure 2.7: A geometric interpretation of the sum rule. Since every point on $f(x)+g(x)$ is the sum of the corresponding points on $f(x)$ and $g(x)$, increasing a by a "small amount" h, increases $f(a)+g(a)$ by the sum of $f'(a)h$ and $g'(a)h$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f'(a)h+g'(a)h}{h} = f'(a)+g'(a).$$

Example 2.2.7 Compute

$$\frac{d}{dx}\left(x^5+\frac{1}{x}\right).$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx}\left(x^5+\frac{1}{x}\right) &= \frac{d}{dx}x^5+\frac{d}{dx}x^{-1}\\ &= 5x^4-x^{-2}.\end{aligned}$$

Example 2.2.8 Compute

$$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right).$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right) &= 3\frac{d}{dx}x^{-1/3} - 2\frac{d}{dx}x^{1/2} + \frac{d}{dx}x^{-7}\\ &= -x^{-4/3} - x^{-1/2} - 7x^{-8}.\end{aligned}$$

The Derivative of e^x

We don't know anything about derivatives that allows us to compute the derivatives of exponential functions without getting our hands dirty. Let's do a little work with the definition of the derivative:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h} - a^x}{h}\\ &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\ &= \lim_{h\to 0} a^x\frac{a^h - 1}{h}\\ &= a^x \lim_{h\to 0}\frac{a^h - 1}{h}\\ &= a^x \cdot \underbrace{(\text{constant})}_{\lim_{h\to 0}\frac{a^h-1}{h}}\end{aligned}$$

There are two interesting things to note here: We are left with a limit that involves h but not x, which means that whatever $\lim_{h\to 0}(a^h - 1)/h$ is, we know that it is a number, that is, a constant. This means that a^x has a remarkable property: Its derivative is a constant times itself. Unfortunately it is beyond the scope of this text to compute the limit

$$\lim_{h\to 0}\frac{a^h - 1}{h}.$$

However, we can look at some examples. Consider $(2^h - 1)/h$ and $(3^h - 1)/h$:

h	$(2^h-1)/h$
-1	$.5$
-0.1	≈ 0.6700
-0.01	≈ 0.6910
-0.001	≈ 0.6929
-0.0001	≈ 0.6931
-0.00001	≈ 0.6932

h	$(2^h-1)/h$
1	1
0.1	≈ 0.7177
0.01	≈ 0.6956
0.001	≈ 0.6834
0.0001	≈ 0.6932
0.00001	≈ 0.6932

h	$(3^h-1)/h$
-1	≈ 0.6667
-0.1	≈ 1.0404
-0.01	≈ 1.0926
-0.001	≈ 1.0980
-0.0001	≈ 1.0986
-0.00001	≈ 1.0986

h	$(3^h-1)/h$
1	2
0.1	≈ 1.1612
0.01	≈ 1.1047
0.001	≈ 1.0992
0.0001	≈ 1.0987
0.00001	≈ 1.0986

While these tables don't prove a pattern, it turns out that

$$\lim_{h\to 0}\frac{2^h-1}{h}\approx .7 \qquad \text{and} \qquad \lim_{h\to 0}\frac{3^h-1}{h}\approx 1.1.$$

Moreover, if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a=2$ and $a=3$ the limit will be exactly 1. This happens when

$$a = e = 2.718281828459045\dots.$$

This brings us to our next definition.

Definition Euler's number is defined to be the number e such that

$$\lim_{h\to 0}\frac{e^h-1}{h} = 1.$$

Now we see that the function e^x has a truly remarkable property:

Theorem 2.2.9 (The Derivative of e^x)

$$\frac{d}{dx}e^x = e^x.$$

Proof *From the limit definition of the derivative, write*

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^x e^h - e^x}{h}\\ &= \lim_{h\to 0} e^x \frac{e^h-1}{h}\\ &= e^x \lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^x.\end{aligned}$$

Hence e^x is its own derivative. In other words, the slope of the plot of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (a, e^a) and has slope e^a there, no matter what a is.

Example 2.2.10 Compute:

$$\frac{d}{dx}\left(8\sqrt{x} + 7e^x\right)$$

Solution *Write:*

$$\begin{aligned}\frac{d}{dx}\left(8\sqrt{x} + 7e^x\right) &= 8\frac{d}{dx}x^{1/2} + 7\frac{d}{dx}e^x\\ &= 4x^{-1/2} + 7e^x.\end{aligned}$$

Exercises for Section 2.2

Compute:

(1) $\frac{d}{dx}5$

(2) $\frac{d}{dx}-7$

(3) $\frac{d}{dx}e^7$

(4) $\frac{d}{dx}\frac{1}{\sqrt{2}}$

(5) $\frac{d}{dx}x^{100}$

(6) $\frac{d}{dx}x^{-100}$

(7) $\frac{d}{dx}\frac{1}{x^5}$

(8) $\frac{d}{dx}x^{\pi}$

(9) $\frac{d}{dx}x^{3/4}$

(10) $\frac{d}{dx}\frac{1}{(\sqrt[7]{x})^9}$

(11) $\frac{d}{dx}\left(5x^3+12x^2-15\right)$

(12) $\frac{d}{dx}\left(-4x^5+3x^2-\frac{5}{x^2}\right)$

(13) $\frac{d}{dx}5(-3x^2+5x+1)$

(14) $\frac{d}{dx}\left(3\sqrt{x}+\frac{1}{x}-x^e\right)$

(15) $\frac{d}{dx}\left(\frac{x^2}{x^7}+\frac{\sqrt{x}}{x}\right)$

Expand or simplify to compute the following:

(16) $\frac{d}{dx}\left((x+1)(x^2+2x-3)\right)$

(17) $\frac{d}{dx}\frac{x^3-2x^2-5x+6}{(x-1)}$

(18) $\frac{d}{dx}\frac{x-5}{\sqrt{x}-\sqrt{5}}$

(19) $\frac{d}{dx}((x+1)(x+1)(x-1)(x-1))$

(20) Suppose the position of an object at time t is given by $f(t)=-49t^2/10+5t+10$. Find a function giving the velocity of the object at time t. The acceleration of an object is the rate at which its velocity is changing, which means it is given by the derivative of the velocity function. Find the acceleration of the object at time t.

(21) Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of $f(x)$, $cf(x)$, $f'(x)$, and $(cf(x))'$ on the same diagram.

(22) Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$.

(23) Find an equation for the tangent line to $f(x) = x^3/4 - 1/x$ at $x = -2$.

(24) Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$.

(25) Prove that $\dfrac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

3 Curve Sketching

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

3.1 Extrema

Local *extrema* on a function are points on the graph where the y coordinate is larger (or smaller) than all other y coordinates on the graph at points "close to" (x, y).

Definition

(a) A point $(x, f(x))$ is a **local maximum** if there is an interval $a < x < b$ with $f(x) \geq f(z)$ for every z in (a, b).

(b) A point $(x, f(x))$ is a **local minimum** if there is an interval $a < x < b$ with $f(x) \leq f(z)$ for every z in (a, b).

A **local extremum** is either a local maximum or a local minimum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well.

If $(x, f(x))$ is a point where $f(x)$ reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

Theorem 3.1.1 (Fermat's Theorem) *If $f(x)$ has a local extremum at $x = a$ and $f(x)$ is differentiable at a, then $f'(a) = 0$.*

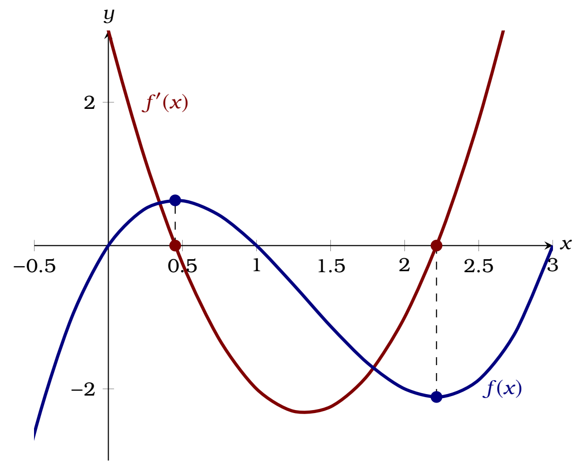


Figure 3.1: A plot of $f(x) = x^3 - 4x^2 + 3x$ and $f'(x) = 3x^2 - 8x + 3$.

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, see Figure 3.1, or the derivative is undefined, as in Figure 3.2. This brings us to our next definition.

Definition Any value of x for which $f'(x)$ is zero or undefined is called a **critical point** for $f(x)$.

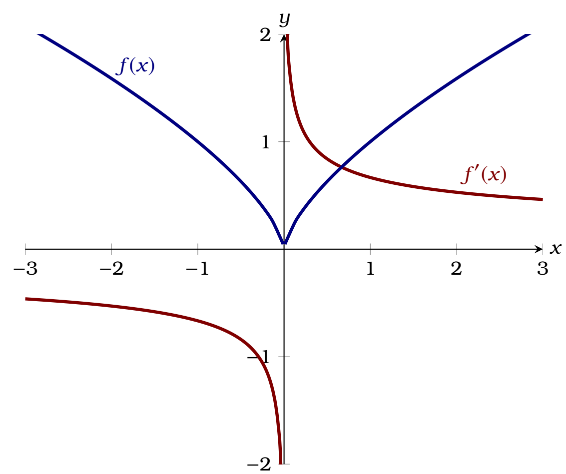


Figure 3.2: A plot of $f(x) = x^{2/3}$ and $f'(x) = \dfrac{2}{3x^{1/3}}$.

Warning When looking for local maximum and minimum points, you are likely to make two sorts of mistakes:

- You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere.
- You might assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true, see Figure 3.3.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach is to test directly whether the y coordinates near the potential maximum or minimum are above or below the y coordinate at the point of interest.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

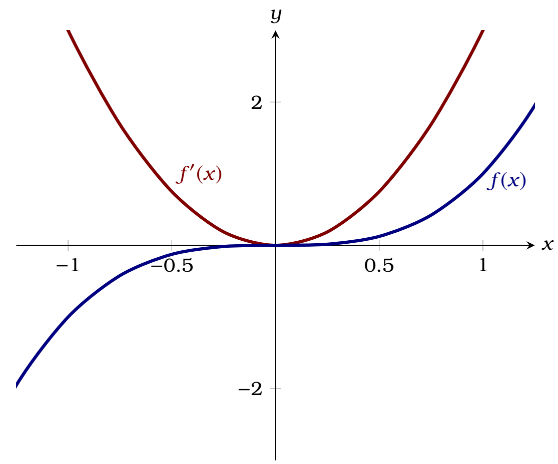


Figure 3.3: A plot of $f(x) = x^3$ and $f'(x) = 3x^2$. While $f'(0) = 0$, there is neither a maximum nor minimum at $(0, f(0))$.

Example 3.1.2 Find all local maximum and minimum points for the function $f(x) = x^3 - x$.

Solution *Write*

$$\frac{d}{dx}f(x) = 3x^2 - 1.$$

This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. *Looking first at* $x = \sqrt{3}/3$, *we see that*

$$f(\sqrt{3}/3) = -2\sqrt{3}/9.$$

Now we test two points on either side of $x = \sqrt{3}/3$, *making sure that neither is farther away than the nearest critical point; since* $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ *and we can use* $x = 0$ *and* $x = 1$. *Since*

$$f(0) = 0 > -2\sqrt{3}/9 \qquad \text{and} \qquad f(1) = 0 > -2\sqrt{3}/9,$$

there must be a local minimum at $x = \sqrt{3}/3$.

For $x = -\sqrt{3}/3$, *we see that* $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. *This time we can use* $x = 0$ *and* $x = -1$, *and we find that* $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, *so there must be a local maximum at* $x = -\sqrt{3}/3$, *see Figure 3.4.*

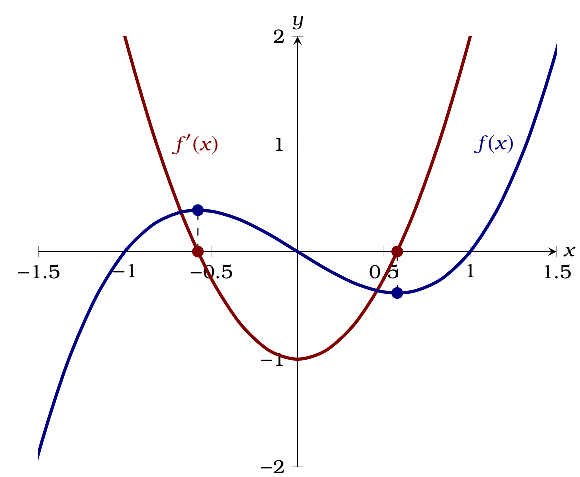


Figure 3.4: A plot of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$.

Exercises for Section 3.1

In the following problems, find the x values for local maximum and minimum points by the method of this section.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = -\dfrac{x^4}{4} + x^3 + x^2$

(8) $f(x) = \begin{cases} x - 1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$

(9) $f(x) = \begin{cases} x - 3 & x < 3 \\ x^3 & 3 \leq x \leq 5 \\ 1/x & x > 5 \end{cases}$

(10) $f(x) = x^2 - 98x + 4$

(11) $f(x) = \begin{cases} -2 & x = 0 \\ 1/x^2 & x \neq 0 \end{cases}$

(12) How many critical points can a quadratic polynomial function have?

(13) Explore the family of functions $f(x) = x^3 + cx + 1$ where c is a constant. How many and what types of local extrema are there? Your answer should depend on the value of c, that is, different values of c will give different answers.

3.2 The First Derivative Test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical point by testing "near-by" points is not always convenient. Instead, since we have already had to compute the derivative to find the critical points, we can use information about the derivative to decide. Recall that

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

So how exactly does the derivative tell us whether there is a maximum, minimum, or neither at a point? Use the *first derivative test*.

Theorem 3.2.1 (First Derivative Test) *Suppose that $f(x)$ is continuous on an interval, and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.*
- *If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.*
- *If $f'(x)$ has the same sign to the left and right of $f'(a)$, then $f'(a)$ is not a local extremum.*

Example 3.2.2 Consider the function

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Find the intervals on which $f(x)$ is increasing and decreasing and identify the local extrema of $f(x)$.

Solution *Start by computing*

$$\frac{d}{dx}f(x) = x^3 + x^2 - 2x.$$

Now we need to find when this function is positive and when it is negative. To do this, solve

$$f'(x) = x^3 + x^2 - 2x = 0.$$

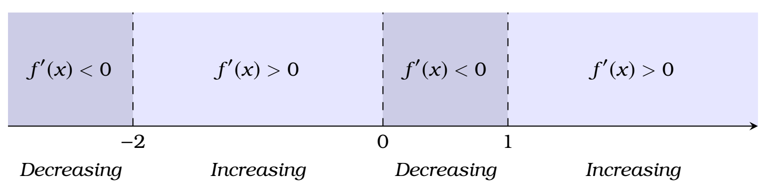
Factor $f'(x)$

$$\begin{aligned} f'(x) &= x^3 + x^2 - 2x \\ &= x(x^2 + x - 2) \\ &= x(x+2)(x-1). \end{aligned}$$

So the critical points (when $f'(x) = 0$*) are when* $x = -2$*,* $x = 0$*, and* $x = 1$*. Now we can check points* ***between*** *the critical points to find when* $f'(x)$ *is increasing and decreasing:*

$$f'(-3) = -12 \qquad f'(.5) = -0.625 \qquad f'(-1) = 2 \qquad f'(2) = 8$$

From this we can make a sign table:



Hence $f(x)$ *is increasing on* $(-2, 0) \cup (1, \infty)$ *and* $f(x)$ *is decreasing on* $(-\infty, -2) \cup (0, 1)$*. Moreover, from the first derivative test, Theorem 3.2.1, the local maximum is at* $x = 0$ *while the local minima are at* $x = -2$ *and* $x = 1$*, see Figure 3.5.*

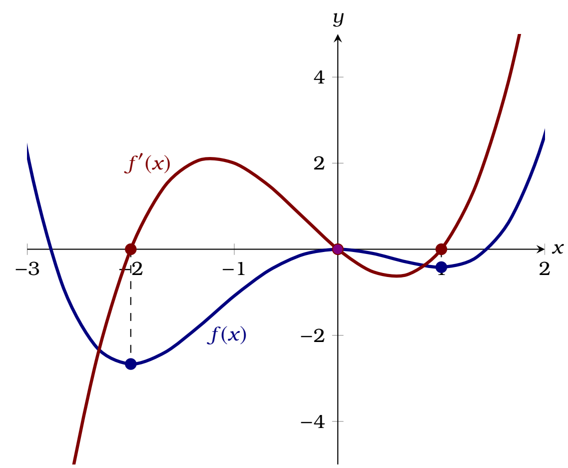


Figure 3.5: A plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f'(x) = x^3 + x^2 - 2x$.

Hence we have seen that if $f'(x)$ is zero and increasing at a point, then $f(x)$ has a local minimum at the point. If $f'(x)$ is zero and decreasing at a point then $f(x)$ has a local maximum at the point. Thus, we see that we can gain information about $f(x)$ by studying how $f'(x)$ changes. This leads us to our next section.

Exercises for Section 3.2

In the following exercises, find all critical points and identify them as local maximum points, local minimum points, or neither.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

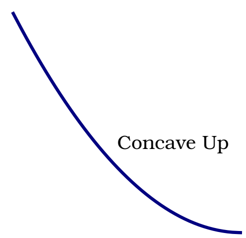
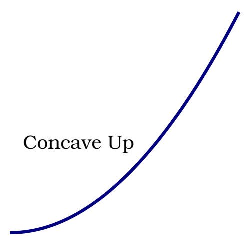
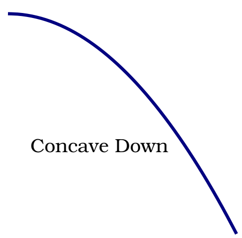
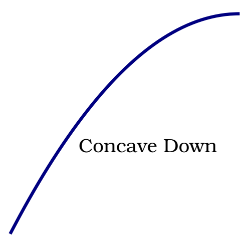
(6) $y = (x^2 - 1)/x$

(7) $f(x) = |x^2 - 121|$

(8) Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. Show that $f(x)$ has exactly one critical point using the first derivative test. Give conditions on a and b which guarantee that the critical point will be a maximum.

3.3 Concavity and Inflection Points

We know that the sign of the derivative tells us whether a function is increasing or decreasing. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing. We summarize this in the table below:

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) > 0$	 Here $f'(x) < 0$ and $f''(x) > 0$. This means that $f(x)$ slopes down and is getting *less steep*. In this case the curve is **concave up**.	 Here $f'(x) > 0$ and $f''(x) > 0$. This means that $f(x)$ slopes up and is getting *steeper*. In this case the curve is **concave up**.
$f''(x) < 0$	 Here $f'(x) < 0$ and $f''(x) < 0$. This means that $f(x)$ slopes down and is getting *steeper*. In this case the curve is **concave down**.	 Here $f'(x) > 0$ and $f''(x) < 0$. This means that $f(x)$ slopes up and is getting less *steep*. In this case the curve is **concave down**.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 3.3.1 (Test for Concavity) *Suppose that $f''(x)$ exists on an interval.*

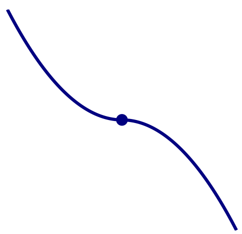
(a) *If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval.*

(b) *If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.*

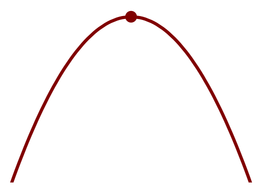
Of particular interest are points at which the concavity changes from up to down or down to up.

Definition If $f(x)$ is continuous and its concavity changes either from up to down or down to up at $x = a$, then $f(x)$ has an **inflection point** at $x = a$.

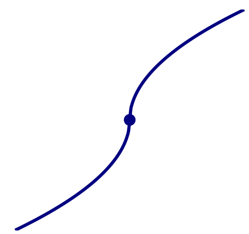
It is instructive to see some examples and nonexamples of inflection points.



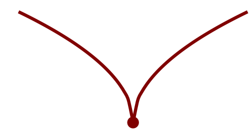
This is an inflection point. The concavity changes from concave up to concave down.



This is **not** an inflection point. The curve is concave down on either side of the point.



This is an inflection point. The concavity changes from concave up to concave down.



This is **not** an inflection point. The curve is concave down on either side of the point.

We identify inflection points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points.

Warning Even if $f''(a) = 0$, the point determined by $x = a$ might **not** be an inflection point.

Example 3.3.2 Describe the concavity of $f(x) = x^3 - x$.

Solution *To start, compute the first and second derivative of $f(x)$ with respect to x,*

$$f'(x) = 3x^2 - 1 \qquad \text{and} \qquad f''(x) = 6x.$$

Since $f''(0) = 0$, there is potentially an inflection point at zero. Since $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$ the concavity does change from down to up at zero—there is an inflection point at $x = 0$. The curve is concave down for all $x < 0$ and concave up for all $x > 0$, see Figure 3.6.

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

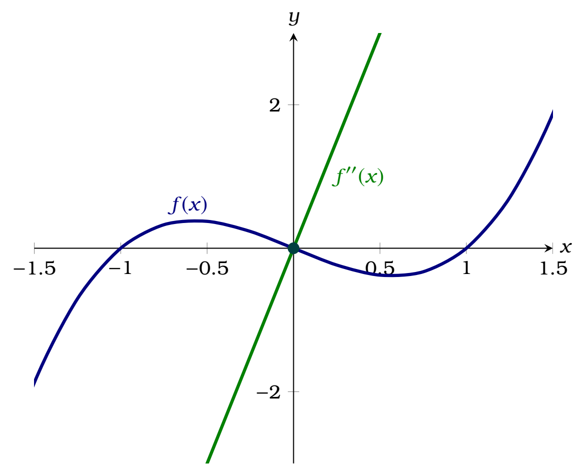


Figure 3.6: A plot of $f(x) = x^3 - x$ and $f''(x) = 6x$. We can see that the concavity change at $x = 0$.

Exercises for Section 3.3

In the following exercises, describe the concavity of the functions.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - \dfrac{1}{x^2}$

(8) $y = x^5 - x$

(9) $y = x + 1/x$

(10) $y = x^2 + 1/x$

(11) Identify the intervals on which the graph of the function $f(x) = x^4 - 4x^3 + 10$ is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

3.4 The Second Derivative Test

Recall the first derivative test, Theorem 3.2.1:

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.
- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.

If $f'(x)$ changes from positive to negative it is decreasing. In this case, $f''(x)$ might be negative, and if in fact $f''(x)$ is negative then $f'(x)$ is definitely decreasing, so there is a local maximum at the point in question. On the other hand, if $f'(x)$ changes from negative to positive it is increasing. Again, this means that $f''(x)$ might be positive, and if in fact $f''(x)$ is positive then $f'(x)$ is definitely increasing, so there is a local minimum at the point in question. We summarize this as the *second derivative test.*

Theorem 3.4.1 (Second Derivative Test) *Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f''(a) < 0$, then $f(x)$ has a local maximum at a.*
- *If $f''(a) > 0$, then $f(x)$ has a local minimum at a.*
- *If $f''(a) = 0$, then the test is inconclusive. In this case, $f(x)$ may or may not have a local extremum at $x = a$.*

The second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests.

Example 3.4.2 Once again, consider the function

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Use the second derivative test, Theorem 3.4.1, to locate the local extrema of $f(x)$.

Solution *Start by computing*

$$f'(x) = x^3 + x^2 - 2x \qquad \text{and} \qquad f''(x) = 3x^2 + 2x - 2.$$

Using the same technique as used in the solution of Example 3.2.2, we find that

$$f'(-2) = 0, \qquad f'(0) = 0, \qquad f'(1) = 0.$$

Now we'll attempt to use the second derivative test, Theorem 3.4.1,

$$f''(-2) = 6, \qquad f''(0) = -2, \qquad f''(1) = 3.$$

Hence we see that $f(x)$ has a local minimum at $x = -2$, a local maximum at $x = 0$, and a local minimum at $x = 1$, see Figure 3.7.

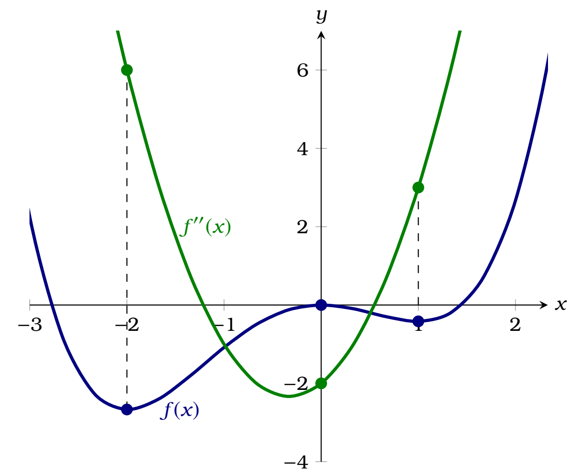


Figure 3.7: A plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f''(x) = 3x^2 + 2x - 2$.

Warning If $f''(a) = 0$, then the second derivative test gives no information on whether $x = a$ is a local extremum.

Exercises for Section 3.4

Find all local maximum and minimum points by the second derivative test.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - \dfrac{1}{x^2}$

(8) $y = x^5 - x$

(9) $y = x + 1/x$

(10) $y = x^2 + 1/x$

3.5 Sketching the Plot of a Function

In this section, we will give some general guidelines for sketching the plot of a function.

Procedure for Sketching the Plots of Functions

- Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.
- Find candidates for vertical asymptotes, these are points where $f(x)$ is undefined.
- Compute $f'(x)$ and $f''(x)$.
- Find the critical points, the points where $f'(x) = 0$.
- Use the second derivative test to identify local extrema and/or find the intervals where your function is increasing/decreasing.
- Find the candidates for inflection points, the points where $f''(x) = 0$.
- Identify inflection points and concavity.
- If possible find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.
- Find horizontal asymptotes.
- Determine an interval that shows all relevant behavior.

At this point you should be able to sketch the plot of your function.

Let's see this procedure in action. We'll sketch the plot of $2x^3 - 3x^2 - 12x$. Following our guidelines above, we start by computing $f(0) = 0$. Hence we see that the y-intercept is $(0, 0)$. Place this point on your plot, see Figure 3.8.

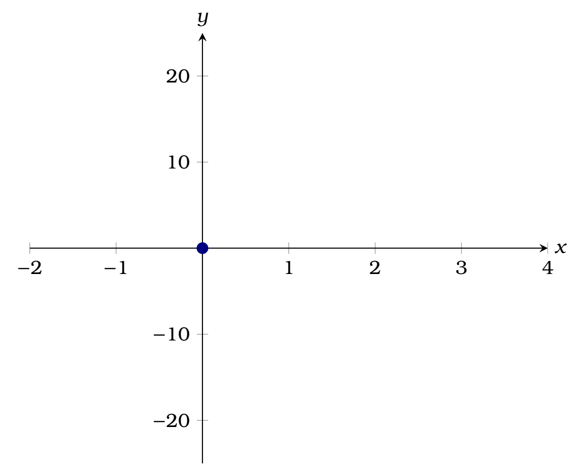


Figure 3.8: We start by placing the point $(0, 0)$.

Note that there are no vertical asymptotes as our function is defined for all real numbers. Now compute $f'(x)$ and $f''(x)$,

$$f'(x) = 6x^2 - 6x - 12 \qquad \text{and} \qquad f''(x) = 12x - 6.$$

The critical points are where $f'(x) = 0$, thus we need to solve $6x^2 - 6x - 12 = 0$ for x. Write

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0. \end{aligned}$$

Thus

$$f'(2) = 0 \qquad \text{and} \qquad f'(-1) = 0.$$

Mark the critical points $x = 2$ and $x = -1$ on your plot, see Figure 3.9.

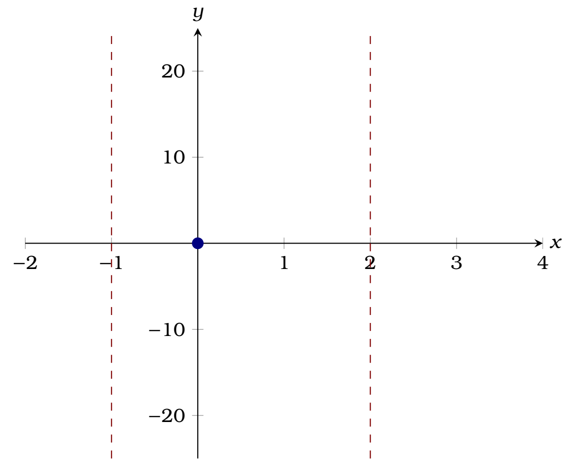


Figure 3.9: Now we add the critical points $x = -1$ and $x = 2$.

Check the second derivative evaluated at the critical points. In this case,

$$f''(-1) = -18 \qquad \text{and} \qquad f''(2) = 18,$$

hence $x = -1$, corresponding to the point $(-1, 7)$ is a local maximum and $x = 2$, corresponding to the point $(2, -20)$ is local minimum of $f(x)$. Moreover, this tells us that our function is increasing on $[-2, -1)$, decreasing on $(-1, 2)$, and increasing on $(2, 4]$. Identify this on your plot, see Figure 3.10.

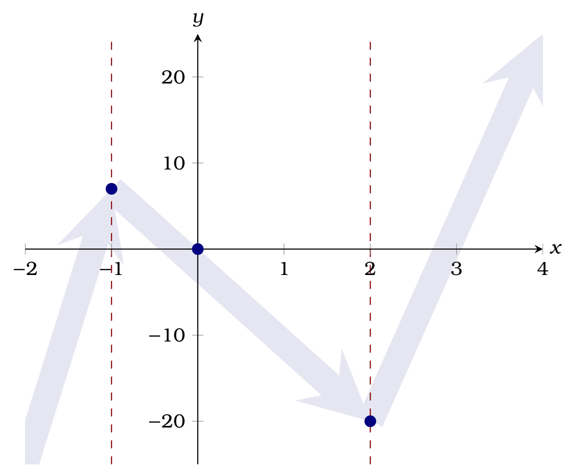


Figure 3.10: We have identified the local extrema of $f(x)$ and where this function is increasing and decreasing.

The candidates for the inflection points are where $f''(x) = 0$, thus we need to solve $12x - 6 = 0$ for x. Write

$$\begin{aligned} 12x - 6 &= 0 \\ x - 1/2 &= 0 \\ x &= 1/2. \end{aligned}$$

Thus $f''(1/2) = 0$. Checking points, $f''(0) = -6$ and $f''(1) = 6$. Hence $x = 1/2$ is an inflection point, with $f(x)$ concave down to the left of $x = 1/2$ and $f(x)$ concave up to the right of $x = 1/2$. We can add this information to our plot, see Figure 3.11.

Finally, in this case, $f(x) = 2x^3 - 3x^2 - 12x$, we can find the x-intercepts. Write

$$\begin{aligned} 2x^3 - 3x^2 - 12x &= 0 \\ x(2x^2 - 3x - 12) &= 0. \end{aligned}$$

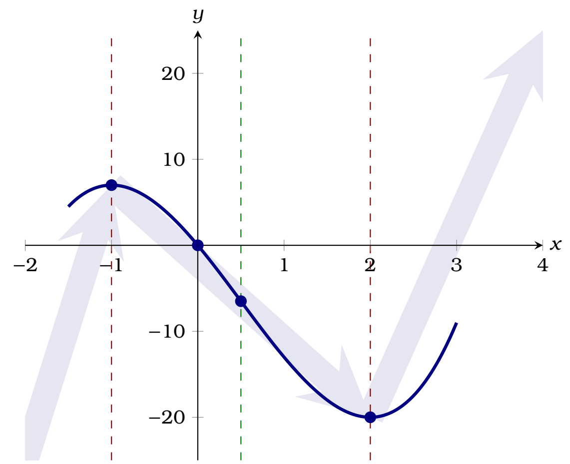
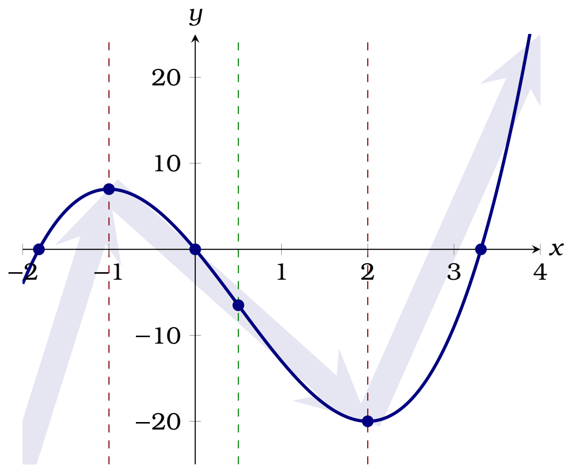


Figure 3.11: We identify the inflection point and note that the curve is concave down when $x < 1/2$ and concave up when $x > 1/2$.

Using the quadratic formula, we see that the x-intercepts of $f(x)$ are

$$x = 0, \qquad x = \frac{3 - \sqrt{105}}{4}, \qquad x = \frac{3 + \sqrt{105}}{4}.$$

Since all of this behavior as described above occurs on the interval $[-2, 4]$, we now have a complete sketch of $f(x)$ on this interval, see the figure below.



Exercises for Section 3.5

Sketch the curves via the procedure outlined in this section. Clearly identify any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts.

(1) $y = x^5 - x$

(2) $y = x(x^2 + 1)$

(3) $y = 2\sqrt{x} - x$

(4) $y = x^3 + 6x^2 + 9x$

(5) $y = x^3 - 3x^2 - 9x + 5$

(6) $y = x^5 - 5x^4 + 5x^3$

(7) $y = x + 1/x$

(8) $y = x^2 + 1/x$

4 The Product Rule and Quotient Rule

4.1 The Product Rule

Consider the product of two simple functions, say

$$f(x) \cdot g(x)$$

where $f(x) = x^2 + 1$ and $g(x) = x^3 - 3x$. An obvious guess for the derivative of $f(x)g(x)$ is the product of the derivatives:

$$\begin{aligned} f'(x)g'(x) &= (2x)(3x^2 - 3) \\ &= 6x^3 - 6x. \end{aligned}$$

Is this guess correct? We can check by rewriting $f(x)$ and $g(x)$ and doing the calculation in a way that is known to work. Write

$$\begin{aligned} f(x)g(x) &= (x^2 + 1)(x^3 - 3x) \\ &= x^5 - 3x^3 + x^3 - 3x \\ &= x^5 - 2x^3 - 3x. \end{aligned}$$

Hence

$$\frac{d}{dx} f(x)g(x) = 5x^4 - 6x^2 - 3,$$

so we see that

$$\frac{d}{dx} f(x)g(x) \neq f'(x)g'(x).$$

So the derivative of $f(x)g(x)$ is **not** as simple as $f'(x)g'(x)$. Never fear, we have a rule for exactly this situation.

Theorem 4.1.1 (The Product Rule) *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Proof *From the limit definition of the derivative, write*

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now we use the exact same trick we used in the proof of Theorem 1.2.2, we add $0 = -f(x+h)g(x) + f(x+h)g(x)$:

$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$
$$= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h}.$$

Now since both $f(x)$ and $g(x)$ are differentiable, they are continuous, see Theorem 2.1.3. Hence

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x)$$
$$= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} g(x)$$
$$= f(x)g'(x) + f'(x)g(x).$$

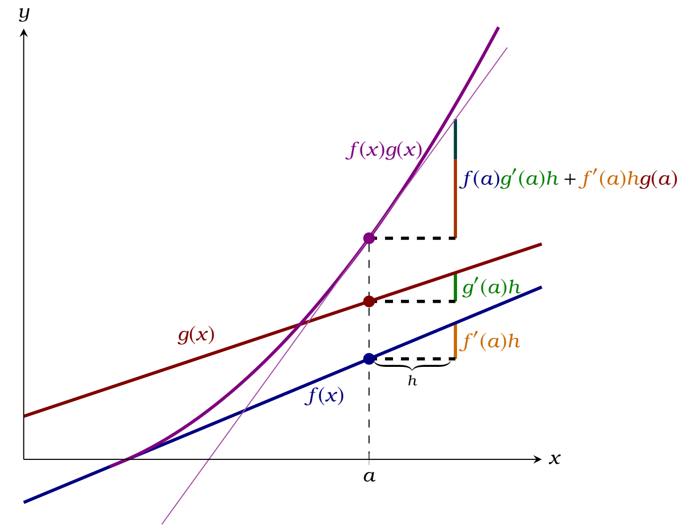


Figure 4.1: A geometric interpretation of the product rule. Since every point on $f(x)g(x)$ is the product of the corresponding points on $f(x)$ and $g(x)$, increasing a by a "small amount" h, increases $f(a)g(a)$ by the sum of $f(a)g'(a)h$ and $f'(a)hg(a)$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f(a)g'(a)h + f'(a)g(a)h}{h} = f(a)g'(a) + f'(a)g(a).$$

Let's return to the example with which we started.

Example 4.1.2 Let $f(x) = (x^2 + 1)$ and $g(x) = (x^3 - 3x)$. Compute:

$$\frac{d}{dx}f(x)g(x).$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx} f(x)g(x) &= f(x)g'(x) + f'(x)g(x) \\ &= (x^2+1)(3x^2-3) + 2x(x^3-3x).\end{aligned}$$

We could stop here—but we should show that expanding this out recovers our previous result. Write

$$\begin{aligned}(x^2+1)(3x^2-3) + 2x(x^3-3x) &= 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2 \\ &= 5x^4 - 6x^2 - 3,\end{aligned}$$

which is precisely what we obtained before.

Exercises for Section 4.1

Compute:

(1) $\dfrac{d}{dx}x^3(x^3-5x+10)$

(2) $\dfrac{d}{dx}(x^2+5x-3)(x^5-6x^3+3x^2-7x+1)$

(3) $\dfrac{d}{dx}e^{2x}=\dfrac{d}{dx}(e^x\cdot e^x)$

(4) $\dfrac{d}{dx}e^{3x}$

(5) $\dfrac{d}{dx}3x^2e^{4x}$

(6) $\dfrac{d}{dx}\dfrac{3e^x}{x^{16}}$

(7) Use the product rule to compute the derivative of $f(x)=(2x-3)^2$ with respect to x. Sketch the function. Find an equation of the tangent line to the curve at $x=2$. Sketch the tangent line at $x=2$.

Use the following table to compute solve the next 4 problems. Note $\left.\dfrac{d}{dx}f(x)\right|_{x=a}$ is the derivative of $f(x)$ evaluated at $x=a$.

x	1	2	3	4
$f(x)$	-2	-3	1	4
$f'(x)$	-1	0	3	5
$g(x)$	1	4	2	-1
$g'(x)$	2	-1	-2	-3

(8) $\left.\dfrac{d}{dx}f(x)g(x)\right|_{x=2}$

(9) $\left.\dfrac{d}{dx}xf(x)\right|_{x=3}$

(10) $\left.\dfrac{d}{dx}xg(x)\right|_{x=4}$

(11) $\left.\dfrac{d}{dx}f(x)g(x)\right|_{x=1}$

(12) Suppose that $f(x)$, $g(x)$, and $h(x)$ are differentiable functions. Show that

$$\frac{d}{dx}f(x)\cdot g(x)\cdot h(x)=f(x)g(x)h'(x)+f(x)g'(x)h(x)+f'(x)g(x)h(x).$$

4.2 The Quotient Rule

We'd like to have a formula to compute

$$\frac{d}{dx}\frac{f(x)}{g(x)}$$

if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x)\cdot(1/g(x))$, that is, this is really a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. This brings us to our next derivative rule.

Theorem 4.2.1 (The Quotient Rule) *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Proof *First note that if we knew how to compute*

$$\frac{d}{dx}\frac{1}{g(x)}$$

then we could use the product rule to complete our proof. Write

$$\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h} \\
&= \lim_{h\to 0}\frac{\frac{g(x)-g(x+h)}{g(x+h)g(x)}}{h} \\
&= \lim_{h\to 0}\frac{g(x) - g(x+h)}{g(x+h)g(x)h} \\
&= \lim_{h\to 0} -\frac{g(x+h) - g(x)}{h}\frac{1}{g(x+h)g(x)} \\
&= -\frac{g'(x)}{g(x)^2}.
\end{aligned}$$

Now we can put this together with the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{f(x)}{g(x)} &= f(x)\frac{-g'(x)}{g(x)^2} + f'(x)\frac{1}{g(x)} \\ &= \frac{-f(x)g'(x) + f'(x)g(x)}{g(x)^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.\end{aligned}$$

Example 4.2.2 Compute:

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x}.$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx}\frac{x^2+1}{x^3-3x} &= \frac{2x(x^3-3x)-(x^2+1)(3x^2-3)}{(x^3-3x)^2} \\ &= \frac{-x^4-6x^2+3}{(x^3-3x)^2}.\end{aligned}$$

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

Example 4.2.3 Compute

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}}$$

in two ways. First using the quotient rule and then using the product rule.

Solution *First, we'll compute the derivative using the quotient rule. Write*

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} = \frac{(-2x)\left(\sqrt{x}\right) - (625-x^2)\left(\frac{1}{2}x^{-1/2}\right)}{x}.$$

Second, we'll compute the derivative using the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} &= \frac{d}{dx}\left(625-x^2\right)x^{-1/2}\\ &= \left(625-x^2\right)\left(\frac{-x^{-3/2}}{2}\right)+(-2x)\left(x^{-1/2}\right).\end{aligned}$$

With a bit of algebra, both of these simplify to

$$-\frac{3x^2+625}{2x^{3/2}}.$$

Exercises for Section 4.2

Find the derivatives of the following functions using the quotient rule.

(1) $\dfrac{x^3}{x^3 - 5x + 10}$

(2) $\dfrac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1}$

(3) $\dfrac{e^x - 4}{2x}$

(4) $\dfrac{2 - x - \sqrt{x}}{x + 2}$

(5) Find an equation for the tangent line to $f(x) = (x^2 - 4)/(5 - x)$ at $x = 3$.

(6) Find an equation for the tangent line to $f(x) = (x - 2)/(x^3 + 4x - 1)$ at $x = 1$.

(7) The curve $y = 1/(1 + x^2)$ is an example of a class of curves each of which is called a *witch of Agnesi*. Find the tangent line to the curve at $x = 5$. Note, the word *witch* here is due to a mistranslation.

Use the following table to compute solve the next 4 problems. Note $\left.\dfrac{d}{dx} f(x)\right|_{x=a}$ is the derivative of $f(x)$ evaluated at $x = a$.

x	1	2	3	4
$f(x)$	-2	-3	1	4
$f'(x)$	-1	0	3	5
$g(x)$	1	4	2	-1
$g'(x)$	2	-1	-2	-3

(8) $\left.\dfrac{d}{dx}\dfrac{f(x)}{g(x)}\right|_{x=2}$

(9) $\left.\dfrac{d}{dx}\dfrac{f(x)}{x}\right|_{x=3}$

(10) $\left.\dfrac{d}{dx}\dfrac{xf(x)}{g(x)}\right|_{x=4}$

(11) $\left.\dfrac{d}{dx}\dfrac{f(x)g(x)}{x}\right|_{x=1}$

(12) If $f'(4) = 5$, $g'(4) = 12$, $f(4)g(4) = 2$, and $g(4) = 6$, compute $f(4)$ and $\dfrac{d}{dx}\dfrac{f(x)}{g(x)}$ at 4.

5 The Chain Rule

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine functions: composition. The *chain rule* allows us to deal with this case.

5.1 The Chain Rule

Consider

$$h(x) = (1+2x)^5.$$

While there are several different ways to differentiate this function, if we let $f(x) = x^5$ and $g(x) = 1+2x$, then we can express $h(x) = f(g(x))$. The question is, can we compute the derivative of a composition of functions using the derivatives of the constituents $f(x)$ and $g(x)$? To do so, we need the *chain rule*.

Theorem 5.1.1 (Chain Rule) *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

Proof *Let g_0 be some x-value and consider the following:*

$$f'(g_0) = \lim_{h\to 0}\frac{f(g_0+h)-f(g_0)}{h}.$$

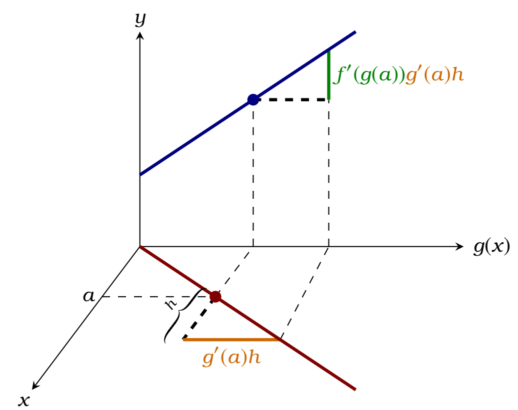


Figure 5.1: A geometric interpretation of the chain rule. Increasing a by a "small amount" h, increases $f(g(a))$ by $f'(g(a))g'(a)h$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f'(g(a))g'(a)h}{h} = f'(g(a))g'(a).$$

Set $h = g - g_0$ *and we have*

$$f'(g_0) = \lim_{g \to g_0} \frac{f(g) - f(g_0)}{g - g_0}. \tag{5.1}$$

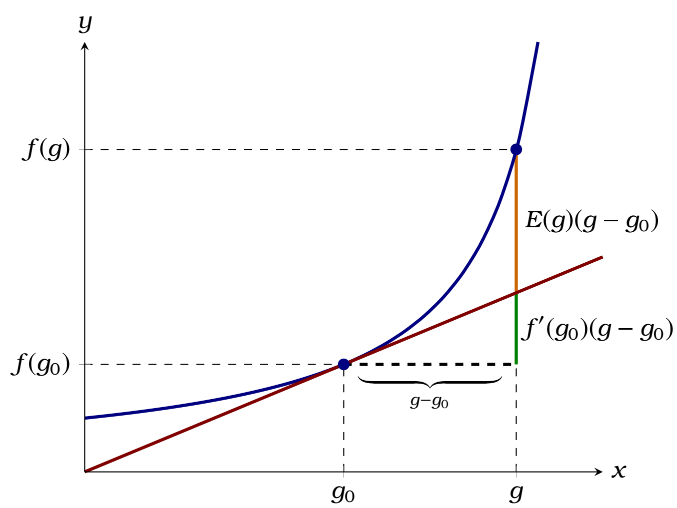
At this point, we might like to set $g = g(x+h)$ *and* $g_0 = g(x)$*; however, we cannot as we cannot be sure that*

$$g(x+h) - g(x) \neq 0 \qquad \text{when } h \neq 0.$$

To overcome this difficulty, let $E(g)$ *be the "error term" that gives the difference between the slope of the secant line from* $f(g_0)$ *to* $f(g)$ *and* $f'(g_0)$*,*

$$E(g) = \frac{f(g) - f(g_0)}{g - g_0} - f'(g_0).$$

In particular, $E(g)(g - g_0)$ *is the difference between* $f(g)$ *and the tangent line of* $f(x)$ *at* $x = g$*, see the figure below:*



Hence we see that

$$f(g) - f(g_0) = \left(f'(g_0) + E(g)\right)(g - g_0), \tag{5.2}$$

and so

$$\frac{f(g) - f(g_0)}{g - g_0} = f'(g_0) + E(g).$$

Combining this with Equation 5.1, we have that

$$\begin{aligned} f'(g_0) &= \lim_{g \to g_0} \frac{f(g) - f(g_0)}{g - g_0} \\ &= \lim_{g \to g_0} f'(g_0) + E(g) \\ &= f'(g_0) + \lim_{g \to g_0} E(g), \end{aligned}$$

and hence it follows that $\lim_{g \to g_0} E(g) = 0$. *At this point, we may return to the "well-worn path." Starting with Equation 5.2, divide both sides by* h *and set* $g = g(x+h)$ *and* $g_0 = g(x)$

$$\frac{f(g(x+h)) - f(g(x))}{h} = \left(f'(g(x)) + E(g(x))\right)\frac{g(x+h) - g(x)}{h}.$$

Taking the limit as h *approaches* 0, *we see*

$$\begin{aligned} \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} &= \lim_{h \to 0} \left(f'(g(x)) + E(g(x))\right)\frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \left(f'(g(x)) + E(g(x))\right) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(g(x))g'(x). \end{aligned}$$

Hence, $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen. Let's return to our motivating example.

Example 5.1.2 Compute:

$$\frac{d}{dx}(1+2x)^5$$

Solution *Set* $f(x) = x^5$ *and* $g(x) = 1 + 2x$, *now*

$$f'(x) = 5x^4 \qquad \textit{and} \qquad g'(x) = 2.$$

Hence

$$\begin{aligned}\frac{d}{dx}(1+2x)^5 &= \frac{d}{dx}f(g(x)) \\ &= f'(g(x))g'(x) \\ &= 5(1+2x)^4 \cdot 2 \\ &= 10(1+2x)^4.\end{aligned}$$

Let's see a more complicated chain of compositions.

Example 5.1.3 Compute:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}}$$

Solution *Set* $f(x) = \sqrt{x}$ *and* $g(x) = 1 + x$. *Hence,*

$$\sqrt{1+\sqrt{x}} = f(g(f(x))) \qquad \textit{and} \qquad \frac{d}{dx}f(g(f(x))) = f'(g(f(x)))g'(f(x))f'(x).$$

Since

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad \textit{and} \qquad g'(x) = 1$$

We have that

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot 1 \cdot \frac{1}{2\sqrt{x}}.$$

Using the chain rule, the power rule, and the product rule it is possible to avoid using the quotient rule entirely.

Example 5.1.4 Compute:

$$\frac{d}{dx}\frac{x^3}{x^2+1}$$

Solution *Rewriting this as*

$$\frac{d}{dx}x^3(x^2+1)^{-1},$$

set $f(x) = x^{-1}$ *and* $g(x) = x^2 + 1$. *Now*

$$x^3(x^2+1)^{-1} = x^3 f(g(x)) \qquad \text{and} \qquad \frac{d}{dx}x^3 f(g(x)) = 3x^2 f(g(x)) + x^3 f'(g(x))g'(x).$$

Since $f'(x) = \dfrac{-1}{x^2}$ *and* $g'(x) = 2x$, *write*

$$\frac{d}{dx}\frac{x^3}{x^2+1} = \frac{3x^2}{x^2+1} - \frac{2x^4}{(x^2+1)^2}.$$

Exercises for Section 5.1

Compute the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

(1) $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$

(2) $x^3 - 2x^2 + 4\sqrt{x}$

(3) $(x^2 + 1)^3$

(4) $x\sqrt{169 - x^2}$

(5) $(x^2 - 4x + 5)\sqrt{25 - x^2}$

(6) $\sqrt{r^2 - x^2}$, r is a constant

(7) $\sqrt{1 + x^4}$

(8) $\dfrac{1}{\sqrt{5 - \sqrt{x}}}$.

(9) $(1 + 3x)^2$

(10) $\dfrac{(x^2 + x + 1)}{(1 - x)}$

(11) $\dfrac{\sqrt{25 - x^2}}{x}$

(12) $\sqrt{\dfrac{169}{x} - x}$

(13) $\sqrt{x^3 - x^2 - (1/x)}$

(14) $100/(100 - x^2)^{3/2}$

(15) $\sqrt[3]{x + x^3}$

(16) $\sqrt{(x^2 + 1)^2 + \sqrt{1 + (x^2 + 1)^2}}$

(17) $(x + 8)^5$

(18) $(4 - x)^3$

(19) $(x^2 + 5)^3$

(20) $(6 - 2x^2)^3$

(21) $(1 - 4x^3)^{-2}$

(22) $5(x + 1 - 1/x)$

(23) $4(2x^2 - x + 3)^{-2}$

(24) $\dfrac{1}{1 + 1/x}$

(25) $\dfrac{-3}{4x^2 - 2x + 1}$

(26) $(x^2 + 1)(5 - 2x)/2$

(27) $(3x^2 + 1)(2x - 4)^3$

(28) $\dfrac{x + 1}{x - 1}$

(29) $\dfrac{x^2 - 1}{x^2 + 1}$

(30) $\dfrac{(x - 1)(x - 2)}{x - 3}$

(31) $\dfrac{2x^{-1} - x^{-2}}{3x^{-1} - 4x^{-2}}$

(32) $3(x^2 + 1)(2x^2 - 1)(2x + 3)$

(33) $\dfrac{1}{(2x + 1)(x - 3)}$

(34) $((2x + 1)^{-1} + 3)^{-1}$

(35) $(2x + 1)^3(x^2 + 1)^2$

(36) Find an equation for the tangent line to $f(x) = (x-2)^{1/3}/(x^3+4x-1)^2$ at $x = 1$.

(37) Find an equation for the tangent line to $y = 9x^{-2}$ at $(3, 1)$.

(38) Find an equation for the tangent line to $(x^2-4x+5)\sqrt{25-x^2}$ at $(3, 8)$.

(39) Find an equation for the tangent line to $\dfrac{(x^2+x+1)}{(1-x)}$ at $(2, -7)$.

(40) Find an equation for the tangent line to $\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ at $(1, \sqrt{4+\sqrt{5}})$.

5.2 Implicit Differentiation

The functions we've been dealing with so far have been *explicit functions*, meaning that the dependent variable is written in terms of the independent variable. For example:

$$y = 3x^2 - 2x + 1, \qquad y = e^{3x}, \qquad y = \frac{x-2}{x^2-3x+2}.$$

However, there are another type of functions, called *implicit functions*. In this case, the dependent variable is not stated explicitly in terms of the independent variable. For example:

$$x^2 + y^2 = 4, \qquad x^3 + y^3 = 9xy, \qquad x^4 + 3x^2 = x^{2/3} + y^{2/3} = 1.$$

Your inclination might be simply to solve each of these for y and go merrily on your way. However this can be difficult and it may require two *branches*, for example to explicitly plot $x^2 + y^2 = 4$, one needs both $y = \sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$. Moreover, it may not even be possible to solve for y. To deal with such situations, we use *implicit differentiation*. Let's see an illustrative example:

Example 5.2.1 Consider the curve defined by

$$x^3 + y^3 = 9xy.$$

(a) Compute $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line at $(4, 2)$.

Solution *Starting with*

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\dfrac{d}{dx}$ *to both sides of the equation to obtain*

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}9xy.$$

Applying the sum rule we see

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

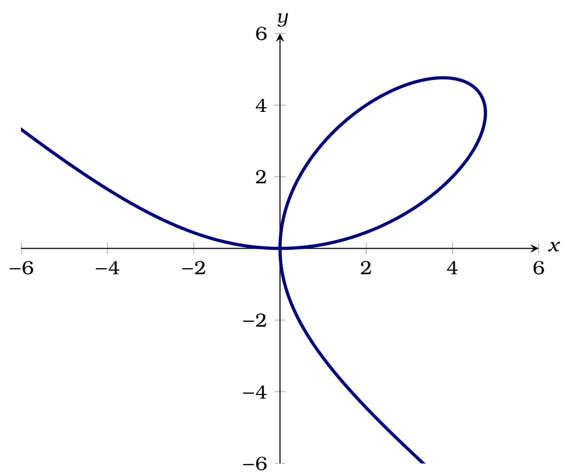


Figure 5.2: A plot of $x^3 + y^3 = 9xy$. While this is not a function of y in terms of x, the equation still defines a relation between x and y.

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand $\frac{d}{dx}y^3$ *is somewhat different. Here you imagine that* $y = y(x)$, *and hence by the chain rule*

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3\\ &= 3(y(x))^2 \cdot y'(x)\\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Considering the final term $\frac{d}{dx}9xy$, *we again imagine that* $y = y(x)$. *Hence*

$$\begin{aligned}\frac{d}{dx}9xy &= 9\frac{d}{dx}x \cdot y(x)\\ &= 9\left(x \cdot y'(x) + y(x)\right)\\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

Putting this all together we are left with the equation

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

At this point, we solve for $\frac{dy}{dx}$. *Write*

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 9x\frac{dy}{dx} + 9y\\ 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} &= 9y - 3x^2\\ \frac{dy}{dx}\left(3y^2 - 9x\right) &= 9y - 3x^2\\ \frac{dy}{dx} &= \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}.\end{aligned}$$

For the second part of the problem, we simply plug $x = 4$ *and* $y = 2$ *into the formula above, hence the slope of the tangent line at* $(4, 2)$ *is* $\frac{5}{4}$, *see Figure 5.3.*

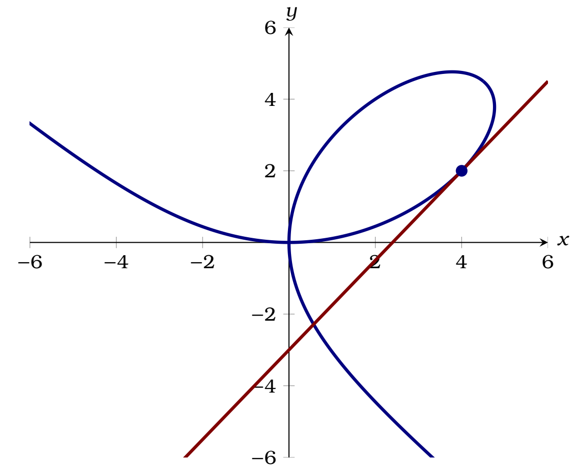


Figure 5.3: A plot of $x^3 + y^3 = 9xy$ along with the tangent line at $(4, 2)$.

You might think that the step in which we solve for $\frac{dy}{dx}$ could sometimes be difficult—after all, we're using implicit differentiation here instead of the more difficult task of solving the equation $x^3 + y^3 = 9xy$ for y, so maybe there are functions where after taking the derivative we obtain something where it is hard to solve for $\frac{dy}{dx}$. In fact, *this never happens*. All occurrences $\frac{dy}{dx}$ arise from applying the chain rule, and whenever the chain rule is used it deposits a single $\frac{dy}{dx}$ multiplied by some other expression. Hence our expression is linear in $\frac{dy}{dx}$, it will always be possible to group the terms containing $\frac{dy}{dx}$ together and factor out the $\frac{dy}{dx}$, just as in the previous example.

The Derivative of Inverse Functions

Geometrically, there is a close relationship between the plots of e^x and $\ln(x)$, they are reflections of each other over the line $y = x$, see Figure 5.4. One may suspect that we can use the fact that $\frac{d}{dx}e^x = e^x$, to deduce the derivative of $\ln(x)$. We will use implicit differentiation to exploit this relationship computationally.

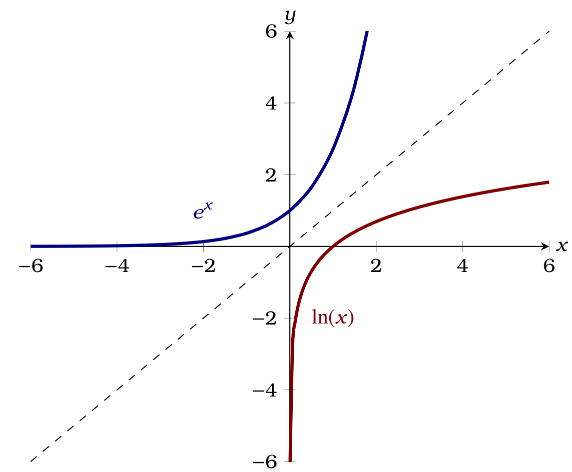


Figure 5.4: A plot of e^x and $\ln(x)$. Since they are inverse functions, they are reflections of each other across the line $y = x$.

Theorem 5.2.2 (The Derivative of the Natural Logrithm)

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Proof *Recall*

$$\ln(x) = y \qquad \Leftrightarrow \qquad e^y = x.$$

Hence

$$
\begin{aligned}
e^y &= x \\
\frac{d}{dx}e^y &= \frac{d}{dx}x && \textit{Differentiate both sides.} \\
e^y\frac{dy}{dx} &= 1 && \textit{Implicit differentiation.} \\
\frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}.
\end{aligned}
$$

Since $y = \ln(x)$, $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.

There is one catch to the proof given above. To write $\dfrac{d}{dx}(e^y) = e^y\dfrac{dy}{dx}$ we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $1/x$. The *Inverse Function Theorem* guarantees this.

Theorem 5.2.3 (Inverse Function Theorem) *If $f(x)$ is a differentiable function, and $f'(x)$ is continuous, and $f'(a) \neq 0$, then*

(a) *$f^{-1}(y)$ is defined for y near $f(a)$,*

(b) *$f^{-1}(y)$ is differentiable near $f(a)$,*

(c) *$\dfrac{d}{dy}f^{-1}(y)$ is continuous near $f(a)$, and*

(d) $\dfrac{d}{dy}f^{-1}(y) = \dfrac{1}{f'(f^{-1}(y))}$.

Exercises for Section 5.2

Compute $\dfrac{dy}{dx}$:

(1) $x^2 + y^2 = 4$

(2) $y^2 = 1 + x^2$

(3) $x^2 + xy + y^2 = 7$

(4) $x^3 + xy^2 = y^3 + yx^2$

(5) $x^2 y - y^3 = 6$

(6) $\sqrt{x} + \sqrt{y} = 9$

(7) $xy^{3/2} + 4 = 2x + y$

(8) $\dfrac{1}{x} + \dfrac{1}{y} = 7$

(9) A hyperbola passing through $(8, 6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8, 6)$.

(10) The graph of the equation $x^2 - xy + y^2 = 9$ is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel.

(11) Repeat the previous problem for the points at which the ellipse intersects the y-axis.

(12) Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.

(13) Find an equation for the tangent line to $x^4 = y^2 + x^2$ at $(2, \sqrt{12})$. This curve is the *kampyle of Eudoxus*.

(14) Find an equation for the tangent line to $x^{2/3} + y^{2/3} = a^{2/3}$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0$ and $y_1 \neq 0$. This curve is an *astroid*.

(15) Find an equation for the tangent line to $(x^2 + y^2)^2 = x^2 - y^2$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0, -1, 1$. This curve is a *lemniscate*.

5.3 Logarithmic Differentiation

Logarithms were originally developed as a computational tool. The key fact that made this possible is that

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

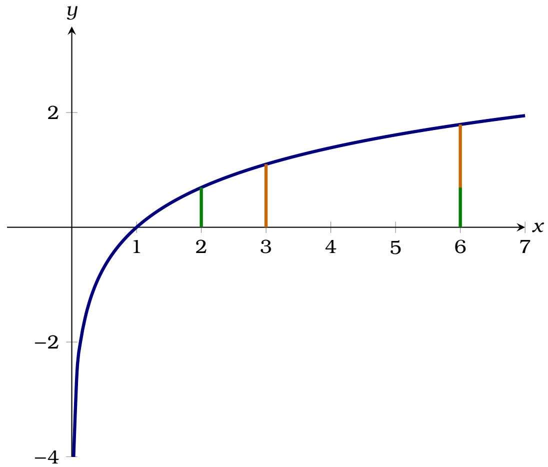


Figure 5.5: A plot of $\ln(x)$. Here we see that

$$\ln(2 \cdot 3) = \ln(2) + \ln(3).$$

While this may seem quite abstract, before the days of calculators and computers, this was critical knowledge for anyone in a computational discipline. Suppose you wanted to compute

$$138 \cdot 23.4$$

You would start by writing both in scientific notation

$$\left(1.38 \cdot 10^2\right) \cdot \left(2.34 \cdot 10^1\right).$$

Next you would use a log-table, which gives $\log_{10}(N)$ for values of N ranging between 0 and 9. We've reproduced part of such a table below.

N	0	1	2	3	4	5	6	7	8	9
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430
			...	...	...	...	...	...		
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
			...	...	...	...	...	...		
3.2	0.5052	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172

Figure 5.6: Part of a base-10 logarithm table.

From the table, we see that

$$\log_{10}(1.38) \approx 0.1399 \qquad \text{and} \qquad \log_{10}(2.34) \approx 0.3692$$

Add these numbers together to get 0.5091. Essentially, we know the following at this point:

$$\begin{array}{ccccc} \log_{10}(?) & = & \log_{10}(1.38) & + & \log_{10}(2.34) \\ \wr & & \wr & & \wr \\ 0.5091 & = & 0.1399 & + & 0.3692 \end{array}$$

Using the table again, we see that $\log_{10}(3.23) \approx 0.5091$. Since we were working in scientific notation, we need to multiply this by 10^3. Our final answer is

$$3230 \approx 138 \cdot 23.4$$

Since $138 \cdot 23.4 = 3229.2$, this is a good approximation. The moral is:

Logarithms allow us to use addition in place of of multiplication.

When taking derivatives, both the product rule and the quotient rule can be cumbersome to use. Logarithms will save the day. A key point is the following

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

which follows from the chain rule. Let's look at an illustrative example to see how this is actually used.

Example 5.3.1 Compute:

$$\frac{d}{dx}\frac{x^9 e^{4x}}{\sqrt{x^2+4}}$$

Recall the properties of logarithms:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $\log_b(x^y) = y\log_b(x)$

Solution *While we could use the product and quotient rule to solve this problem, it would be tedious. Start by taking the logarithm of the function to be differentiated.*

$$\begin{aligned}\ln\left(\frac{x^9 e^{4x}}{\sqrt{x^2+4}}\right) &= \ln\left(x^9 e^{4x}\right) - \ln\left(\sqrt{x^2+4}\right)\\ &= \ln\left(x^9\right) + \ln\left(e^{4x}\right) - \ln\left((x^2+4)^{1/2}\right)\\ &= 9\ln(x) + 4x - \frac{1}{2}\ln(x^2+4).\end{aligned}$$

Setting $f(x) = \dfrac{x^9 e^{4x}}{\sqrt{x^2+4}}$, *we can write*

$$\ln(f(x)) = 9\ln(x) + 4x - \frac{1}{2}\ln(x^2+4).$$

Differentiating both sides, we find

$$\frac{f'(x)}{f(x)} = \frac{9}{x} + 4 - \frac{x}{x^2+4}.$$

Finally we solve for $f'(x)$, write

$$f'(x) = \left(\frac{9}{x} + 4 - \frac{x}{x^2+4}\right)\left(\frac{x^9 e^{4x}}{\sqrt{x^2+4}}\right).$$

The process above is called *logarithmic differentiation*. Logarithmic differentiation allows us to compute new derivatives too.

Example 5.3.2 Compute:

$$\frac{d}{dx}x^x$$

Solution *The function x^x is tricky to differentiate. We cannot use the power rule, as the exponent is not a constant. However, if we set $f(x) = x^x$ we can write*

$$\begin{aligned}\ln(f(x)) &= \ln\left(x^x\right)\\ &= x\ln(x).\end{aligned}$$

Differentiating both sides, we find

$$\begin{aligned}\frac{f'(x)}{f(x)} &= x \cdot \frac{1}{x} + \ln(x)\\ &= 1 + \ln(x).\end{aligned}$$

Now we can solve for $f'(x)$,

$$f'(x) = x^x + x^x\ln(x).$$

Finally recall that previously we only proved the power rule, Theorem 2.2.2, for positive exponents. Now we'll use logarithmic differentiation to give a proof for all real-valued exponents. We restate the power rule for convenience sake:

Theorem 5.3.3 (Power Rule) *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof *We will use logarithmic differentiation. Set* $f(x) = x^n$. *Write*

$$\begin{aligned} \ln(f(x)) &= \ln\left(x^n\right) \\ &= n\ln(x). \end{aligned}$$

Now differentiate both sides, and solve for $f'(x)$

$$\begin{aligned} \frac{f'(x)}{f(x)} &= \frac{n}{x} \\ f'(x) &= \frac{nf(x)}{x} \\ &= nx^{n-1}. \end{aligned}$$

Thus we see that the power rule holds for all real-valued exponents.

While logarithmic differentiation might seem strange and new at first, with a little practice it will seem much more natural to you.

Exercises for Section 5.3

Use logarithmic differentiation to compute the following:

(1) $\dfrac{d}{dx}(x+1)^3\sqrt{x^4+5}$

(2) $\dfrac{d}{dx}x^2e^{5x}$

(3) $\dfrac{d}{dx}x^{\ln(x)}$

(4) $\dfrac{d}{dx}x^{100x}$

(5) $\dfrac{d}{dx}\left((3x)^{4x}\right)$

(6) $\dfrac{d}{dx}x^{(e^x)}$

(7) $\dfrac{d}{dx}x^{\pi}+\pi^{x}$

(8) $\dfrac{d}{dx}\left(1+\dfrac{1}{x}\right)^x$

(9) $\dfrac{d}{dx}(\ln(x))^x$

(10) $\dfrac{d}{dx}\left(f(x)g(x)h(x)\right)$

6 The Derivatives of Trigonometric Functions and their Inverses

6.1 The Derivative of Trigonometric Functions

Up until this point of the course we have been largely ignoring a large class of functions—those involving $\sin(x)$ and $\cos(x)$. It is now time to visit our two friends who concern themselves periodically with triangles and circles.

Theorem 6.1.1 (The Derivative of $\sin(x)$)

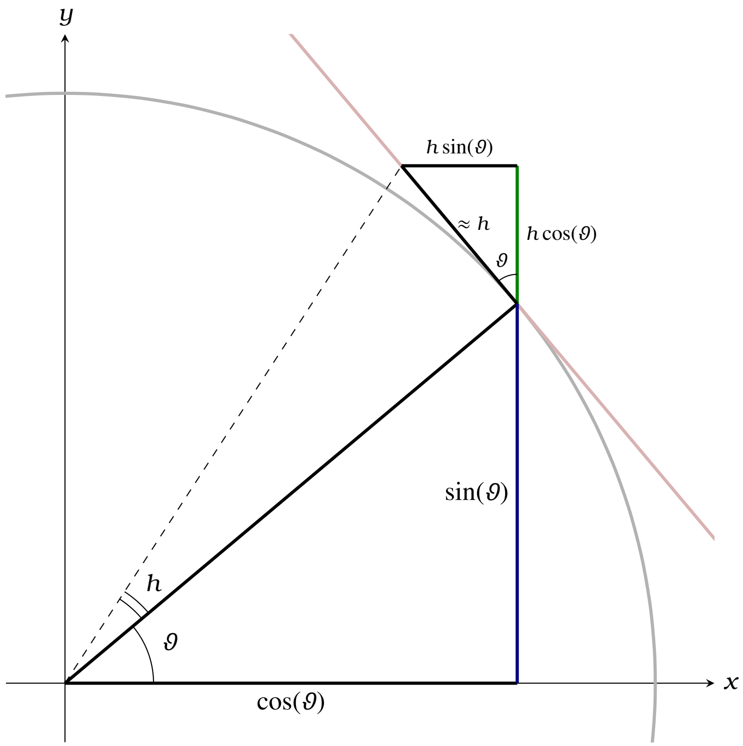
$$\frac{d}{dx}\sin(x) = \cos(x).$$

Proof *Using the definition of the derivative, write*

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} && \textit{Trig Identity.} \\
&= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)-\sin(x)}{h}+\frac{\sin(h)\cos x}{h}\right) \\
&= \lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 = \cos x. && \textit{See Example 1.3.6.}
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(h)-1}{h} &= \lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\cdot\frac{\cos(h)+1}{\cos(h)+1}\right) \\
&= \lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(h)+1)} \\
&= \lim_{h\to 0}\frac{-\sin^2(h)}{h(\cos(h)+1)} \\
&= -\lim_{h\to 0}\left(\frac{\sin(h)}{h}\cdot\frac{\sin(h)}{(\cos(h)+1)}\right) \\
&= -1\cdot\frac{0}{2} = 0.
\end{aligned}$$

Consider the following geometric interpretation of the derivative of $\sin(\vartheta)$.



Here we see that Increasing ϑ by a "small amount" h, increases $\sin(\vartheta)$ by $h\cos(\vartheta)$. Hence,

$$\frac{\Delta y}{\Delta \vartheta} \approx \frac{h\cos(\vartheta)}{h} = \cos(\vartheta).$$

Since the tangent line to the circle is locally a good approximation for the circle and radians measure the arc length of the unit circle, the hypotenuse of the small triangle in the figure is approximately h.

The derivative of a function measures the slope of the plot of a function. If we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true, see Figure 6.1.

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

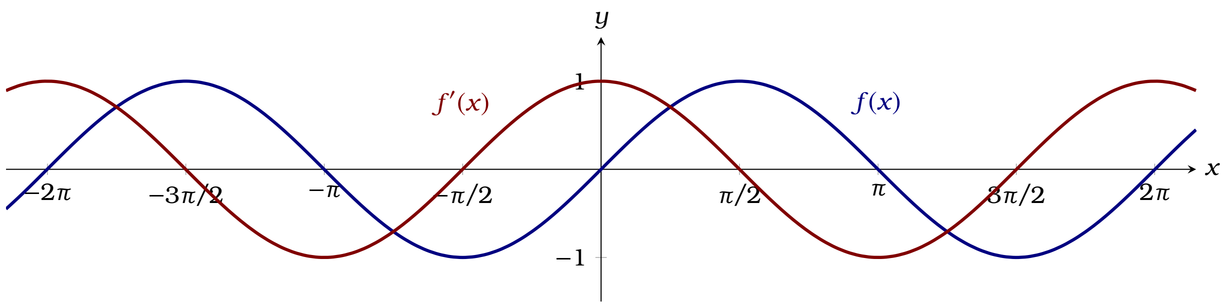


Figure 6.1: Here we see a plot of $f(x) = \sin(x)$ and its derivative $f'(x) = \cos(x)$. One can readily see that $\cos(x)$ is positive when $\sin(x)$ is increasing, and that $\cos(x)$ is negative when $\sin(x)$ is decreasing.

Theorem 6.1.2 (The Derivative of cos(x))

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Proof *Recall that*

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right),$$
$$\sin(x) = -\cos\left(x + \frac{\pi}{2}\right).$$

Now:

$$\begin{aligned}\frac{d}{dx}\cos(x) &= \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right)\\ &= \cos\left(x + \frac{\pi}{2}\right)\cdot 1\\ &= -\sin(x).\end{aligned}$$

Next we have:

Theorem 6.1.3 (The Derivative of tan(x))

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

Proof *We'll rewrite* $\tan(x)$ *as* $\dfrac{\sin(x)}{\cos(x)}$ *and use the quotient rule. Write*

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)} \\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x).\end{aligned}$$

Finally, we have

Theorem 6.1.4 (The Derivative of sec(x))

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

Proof *We'll rewrite* $\sec(x)$ *as* $(\cos(x))^{-1}$ *and use the power rule and the chain rule. Write*

$$\begin{aligned}\frac{d}{dx}\sec(x) &= \frac{d}{dx}(\cos(x))^{-1} \\ &= -1(\cos(x))^{-2}(-\sin(x)) \\ &= \frac{\sin(x)}{\cos^2(x)} \\ &= \sec(x)\tan(x).\end{aligned}$$

The derivatives of the cotangent and cosecant are similar and left as exercises. Putting this all together, we have:

Theorem 6.1.5 (The Derivatives of Trigonometric Functions)

- $\dfrac{d}{dx}\sin(x) = \cos(x)$.
- $\dfrac{d}{dx}\cos(x) = -\sin(x)$.
- $\dfrac{d}{dx}\tan(x) = \sec^2(x)$.
- $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$.
- $\dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x)$.
- $\dfrac{d}{dx}\cot(x) = -\csc^2(x)$.

Warning When working with derivatives of trigonometric functions, we suggest you use **radians** for angle measure. For example, while

$$\sin\left((90^\circ)^2\right) = \sin\left(\left(\frac{\pi}{2}\right)^2\right),$$

one must be careful with derivatives as

$$\frac{d}{dx}\sin\left(x^2\right)\Big|_{x=90^\circ} \neq \underbrace{2\cdot 90\cdot\cos(90^2)}_{\text{incorrect}}$$

Alternatively, one could think of x° as meaning $\dfrac{x\cdot\pi}{180}$, as then $90^\circ = \dfrac{90\cdot\pi}{180} = \dfrac{\pi}{2}$. In this case

$$2\cdot 90^\circ\cdot\cos((90^\circ)^2) = 2\cdot\frac{\pi}{2}\cdot\cos\left(\left(\frac{\pi}{2}\right)^2\right).$$

Exercises for Section 6.1

Find the derivatives of the following functions.

(1) $\sin^2(\sqrt{x})$

(2) $\sqrt{x}\sin(x)$

(3) $\dfrac{1}{\sin(x)}$

(4) $\dfrac{x^2+x}{\sin(x)}$

(5) $\sqrt{1-\sin^2(x)}$

(6) $\sin(x)\cos(x)$

(7) $\sin(\cos(x))$

(8) $\sqrt{x\tan(x)}$

(9) $\tan(x)/(1+\sin(x))$

(10) $\cot(x)$

(11) $\csc(x)$

(12) $x^3\sin(23x^2)$

(13) $\sin^2(x)+\cos^2(x)$

(14) $\sin(\cos(6x))$

(15) Compute $\dfrac{d}{d\vartheta}\dfrac{\sec(\vartheta)}{1+\sec(\vartheta)}$.

(16) Compute $\dfrac{d}{dt}t^5\cos(6t)$.

(17) Compute $\dfrac{d}{dt}\dfrac{t^3\sin(3t)}{\cos(2t)}$.

(18) Find all points on the graph of $f(x)=\sin^2(x)$ at which the tangent line is horizontal.

(19) Find all points on the graph of $f(x)=2\sin(x)-\sin^2(x)$ at which the tangent line is horizontal.

(20) Find an equation for the tangent line to $\sin^2(x)$ at $x=\pi/3$.

(21) Find an equation for the tangent line to $\sec^2(x)$ at $x=\pi/3$.

(22) Find an equation for the tangent line to $\cos^2(x)-\sin^2(4x)$ at $x=\pi/6$.

(23) Find the points on the curve $y=x+2\cos(x)$ that have a horizontal tangent line.

6.2 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often we are interested in finding specific angles, say ϑ such that

$$\sin(\vartheta) = .7$$

Hence we want to be able to invert functions like $\sin(\vartheta)$ and $\cos(\vartheta)$.

However, since these functions are not one-to-one, meaning there are are infinitely many angles with $\sin(\vartheta) = .7$, it is impossible to find a true inverse function for $\sin(\vartheta)$. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function has a proper inverse.

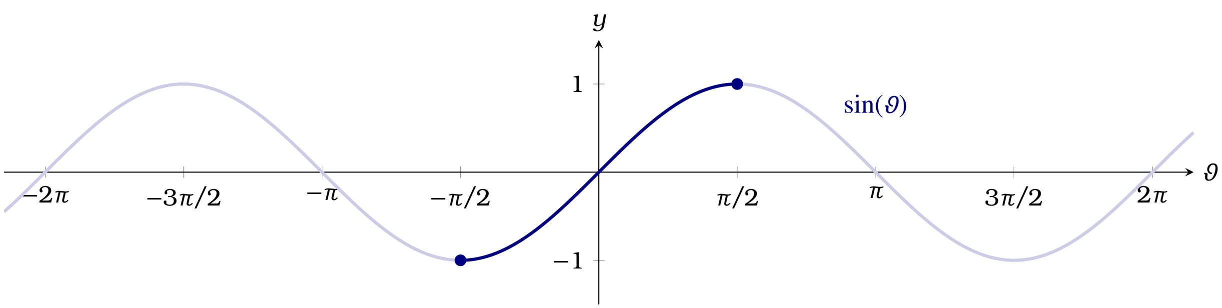


Figure 6.2: The function $\sin(\vartheta)$ takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$. If we restrict $\sin(\vartheta)$ to this interval, then this restricted function has an inverse.

In a similar fashion, we need to restrict cosine to be able to take an inverse.

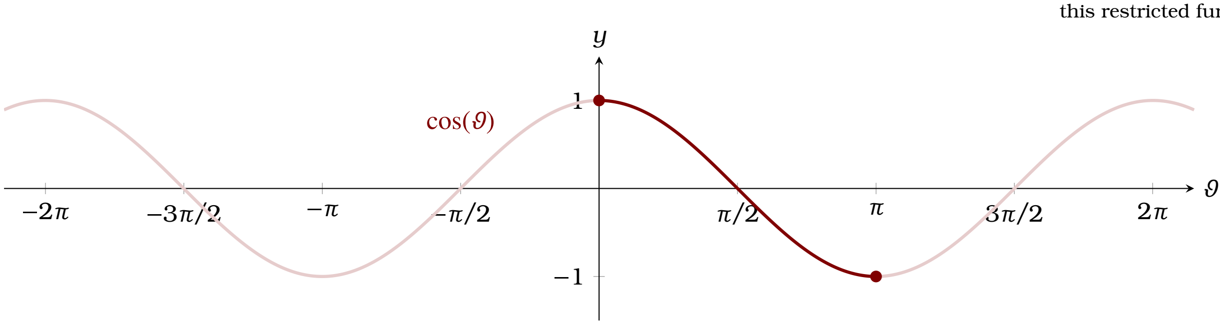
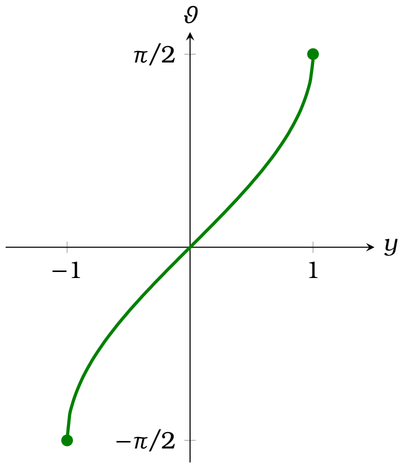
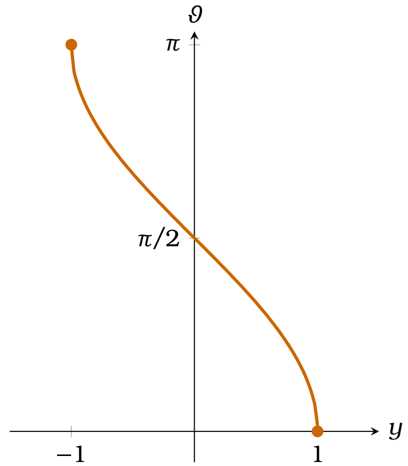


Figure 6.3: The function $\cos(\vartheta)$ takes on all values between -1 and 1 exactly once on the interval $[0, \pi]$. If we restrict $\cos(\vartheta)$ to this interval, then this restricted function has an inverse.

By examining both sine and cosine on restricted domains, we can now produce functions arcsine and arccosine:



Here we see a plot of $\arcsin(y)$, the inverse function of $\sin(\vartheta)$ when it is restricted to the interval $[-\pi/2, \pi/2]$.



Here we see a plot of $\arccos(y)$, the inverse function of $\cos(\vartheta)$ when it is restricted to the interval $[0, \pi]$.

Recall that a function and its inverse undo each other in either order, for example,

$$\sqrt[3]{x^3} = x \qquad \text{and} \qquad \sqrt[3]{x}^3 = x.$$

However, since arcsine is the inverse of sine restricted to the interval $[-\pi/2, \pi/2]$, this does not work with sine and arcsine, for example

$$\arcsin(\sin(\pi)) = 0.$$

Moreover, there is a similar situation for cosine and arccosine as

$$\arccos(\cos(2\pi)) = 0.$$

Once you get a feel for how $\arcsin(y)$ and $\arccos(y)$ behave, let's examine tangent.

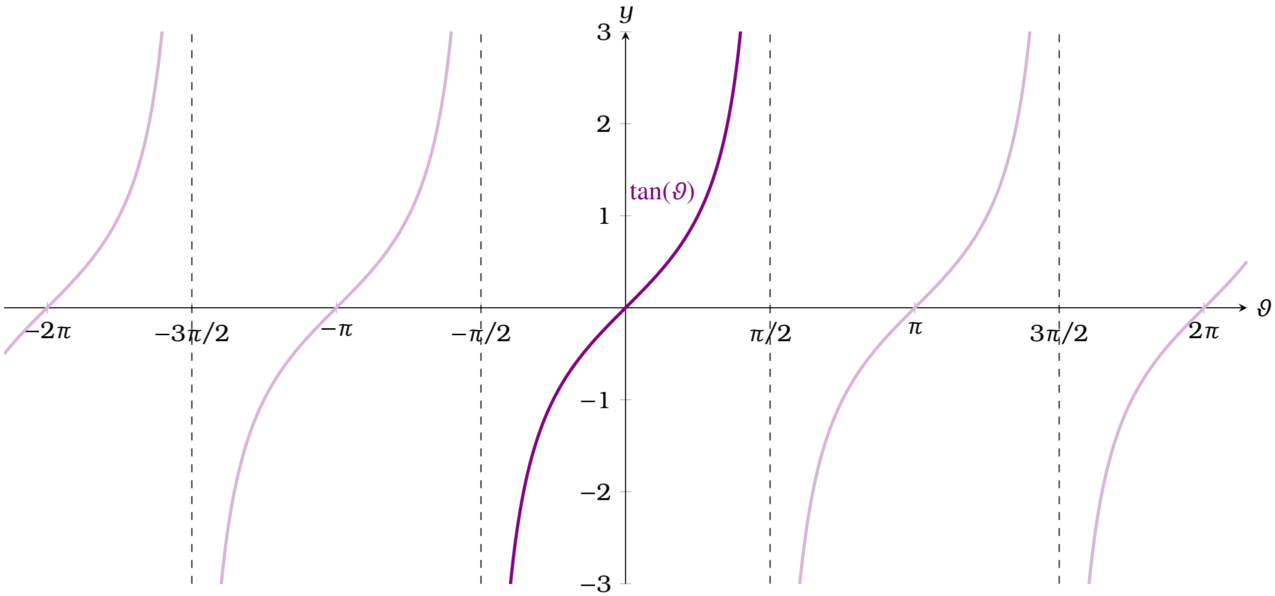


Figure 6.4: The function $\tan(\vartheta)$ takes on all values in $\mathbb{R}$ exactly once on the open interval $(-\pi/2, \pi/2)$. If we restrict $\tan(\vartheta)$ to this interval, then this restricted function has an inverse.

Again, only working on a restricted domain of tangent, we can produce an inverse function, arctangent.

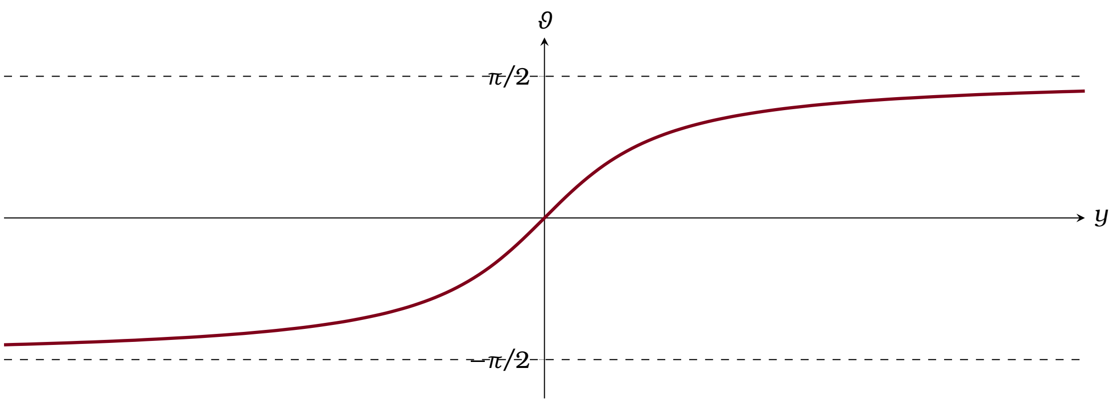


Figure 6.5: Here we see a plot of $\arctan(y)$, the inverse function of $\tan(\vartheta)$ when it is restricted to the interval $(-\pi/2, \pi/2)$.

We leave it to you, the reader, to investigate the functions arcsecant, arccosecant, and arccotangent.

The Derivatives of Inverse Trigonometric Functions

What is the derivative of the arcsine? Since this is an inverse function, we can find its derivative by using implicit differentiation and the Inverse Function Theorem, Theorem 5.2.3.

Theorem 6.2.1 (The Derivative of arcsin(y))

$$\frac{d}{dy}\arcsin(y) = \frac{1}{\sqrt{1-y^2}}.$$

Proof *To start, note that the Inverse Function Theorem, Theorem 5.2.3 assures us that this derivative actually exists. Recall*

$$\arcsin(y) = \vartheta \qquad \Rightarrow \qquad \sin(\vartheta) = y.$$

Hence

$$\begin{aligned}
\sin(\vartheta) &= y \\
\frac{d}{dy}\sin(\vartheta) &= \frac{d}{dy}y \\
\cos(\vartheta)\frac{d\vartheta}{dy} &= 1 \\
\frac{d\vartheta}{dy} &= \frac{1}{\cos(\vartheta)}.
\end{aligned}$$

At this point, we would like $\cos(\vartheta)$ *written in terms of* y*. Since*

$$\cos^2(\vartheta) + \sin^2(\vartheta) = 1$$

and $\sin(\vartheta) = y$*, we may write*

$$\begin{aligned}
\cos^2(\vartheta) + y^2 &= 1 \\
\cos^2(\vartheta) &= 1 - y^2 \\
\cos(\vartheta) &= \pm\sqrt{1-y^2}.
\end{aligned}$$

Since $\vartheta = \arcsin(y)$ *we know that* $-\pi/2 \le \vartheta \le \pi/2$*, and the cosine of an angle in this interval is always positive. Thus* $\cos(\vartheta) = \sqrt{1-y^2}$ *and*

$$\frac{d}{dy}\arcsin(y) = \frac{1}{\sqrt{1-y^2}}.$$

We can do something similar with arccosine.

Theorem 6.2.2 (The Derivative of arccos(y))

$$\frac{d}{dy}\arccos(y) = \frac{-1}{\sqrt{1-y^2}}.$$

Proof *To start, note that the Inverse Function Theorem, Theorem 5.2.3 assures us that this derivative actually exists. Recall*

$$\arccos(y) = \vartheta \qquad \Rightarrow \qquad \cos(\vartheta) = y.$$

Hence

$$\begin{aligned} \cos(\vartheta) &= y \\ \frac{d}{dy}\cos(\vartheta) &= \frac{d}{dy}y \\ -\sin(\vartheta)\frac{d\vartheta}{dy} &= 1 \\ \frac{d\vartheta}{dy} &= \frac{-1}{\sin(\vartheta)}. \end{aligned}$$

At this point, we would like $\sin(\vartheta)$ *written in terms of* y*. Since*

$$\cos^2(\vartheta) + \sin^2(\vartheta) = 1$$

and $\cos(\vartheta) = y$, *we may write*

$$\begin{aligned} y^2 + \sin^2(\vartheta) &= 1 \\ \sin^2(\vartheta) &= 1 - y^2 \\ \sin(\vartheta) &= \pm\sqrt{1 - y^2}. \end{aligned}$$

Since $\vartheta = \arccos(y)$ *we know that* $0 \le \vartheta \le \pi$, *and the sine of an angle in this interval is always positive. Thus* $\sin(\vartheta) = \sqrt{1 - y^2}$ *and*

$$\frac{d}{dy}\arccos(y) = \frac{-1}{\sqrt{1 - y^2}}.$$

Finally, let's look at arctangent.

Theorem 6.2.3 (The Derivative of arctan(y))

$$\frac{d}{dy}\arctan(y) = \frac{1}{1 + y^2}.$$

Proof *To start, note that the Inverse Function Theorem, Theorem 5.2.3 assures us that this derivative actually exists. Recall*

$$\arctan(y) = \vartheta \qquad \Rightarrow \qquad \tan(\vartheta) = y.$$

Hence

$$\begin{aligned} \tan(\vartheta) &= y \\ \frac{d}{dy}\tan(\vartheta) &= \frac{d}{dy}y \\ \sec^2(\vartheta)\frac{d\vartheta}{dy} &= 1 \\ \frac{d\vartheta}{dy} &= \frac{1}{\sec^2(\vartheta)}. \end{aligned}$$

At this point, we would like $\sec^2(\vartheta)$ *written in terms of* y. *Recall*

$$\sec^2(\vartheta) = 1 + \tan^2(\vartheta)$$

and $\tan(\vartheta) = y$, *we may write* $\sec^2(\vartheta) = 1 + y^2$. *Hence*

$$\frac{d}{dy}\arctan(y) = \frac{1}{1+y^2}.$$

We leave it to you, the reader, to investigate the derivatives of arcsecant, arccosecant, and arccotangent. However, as a gesture of friendship, we now present you with a list of derivative formulas for inverse trigonometric functions.

Theorem 6.2.4 (The Derivatives of Inverse Trigonometric Functions)

- $\dfrac{d}{dy}\arcsin(y) = \dfrac{1}{\sqrt{1-y^2}}$.
- $\dfrac{d}{dy}\arccos(y) = \dfrac{-1}{\sqrt{1-y^2}}$.
- $\dfrac{d}{dy}\arctan(y) = \dfrac{1}{1+y^2}$.
- $\dfrac{d}{dy}\operatorname{arcsec}(y) = \dfrac{1}{|y|\sqrt{y^2-1}}$ *for* $|y| > 1$.
- $\dfrac{d}{dy}\operatorname{arccsc}(y) = \dfrac{-1}{|y|\sqrt{y^2-1}}$ *for* $|y| > 1$.
- $\dfrac{d}{dy}\operatorname{arccot}(y) = \dfrac{-1}{1+y^2}$.

Exercises for Section 6.2

(1) The inverse of cot is usually defined so that the range of arccotangent is $(0, \pi)$. Sketch the graph of $y = \operatorname{arccot}(x)$. In the process you will make it clear what the domain of arccotangent is. Find the derivative of the arccotangent.

(2) Find the derivative of $\arcsin(x^2)$.

(3) Find the derivative of $\arctan(e^x)$.

(4) Find the derivative of $\arccos(\sin x^3)$

(5) Find the derivative of $\ln((\arcsin(x))^2)$

(6) Find the derivative of $\arccos(e^x)$

(7) Find the derivative of $\arcsin(x) + \arccos(x)$

(8) Find the derivative of $\log_5(\arctan(x^x))$

Answers to Exercises

Answers for 1.1

1. (a) 8, (b) 6, (c) DNE, (d) -2, (e) -1, (f) 8, (g) 7, (h) 6, (i) 3, (j) $-3/2$, (k) 6, (l) 2 **2.** 1 **3.** 2 **4.** 3 **5.** $3/5$ **6.** $0.6931 \approx \ln(2)$ **7.** $2.718 \approx e$ **8.** Consider what happens when x is near zero and positive, as compared to when x is near zero and negative. **9.** The limit does not exist, so it is not surprising that the resulting values are so different. **10.** When v approaches c from below, then t_v approaches zero—meaning that one second to the stationary observations seems like very little time at all for our traveler.

Answers for 1.2

1. For these problems, there are many possible values of δ, so we provide an inequality that δ must satisfy when $\varepsilon = 0.1$. (a) $\delta < 1/30$, (b) $\delta < \dfrac{\sqrt{110}}{10} - 1 \approx 0.0488$, (c) $\delta < \arcsin(1/10) \approx 0.1002$, (d) $\delta < \arctan(1/10) \approx 0.0997$ (e) $\delta < 13/100$, (f) $\delta < 59/400$ **2.** Let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $0 < |x - 0| < \delta$, then $|x \cdot 1| < \varepsilon$, since $\sin\left(\dfrac{1}{x}\right) \leq 1$, $\left|x \sin\left(\dfrac{1}{x}\right) - 0\right| < \varepsilon$. **3.** Let $\varepsilon > 0$. Set $\delta = \varepsilon/2$. If $0 < |x - 4| < \delta$, then $|2x - 8| < 2\delta = \varepsilon$, and then because $|2x - 8| = |(2x - 5) - 3|$, we conclude $|(2x - 5) - 3| < \varepsilon$. **4.** Let $\varepsilon > 0$. Set $\delta = \varepsilon/4$. If $0 < |x - (-3)| < \delta$, then $|-4x - 12| < 4\delta = \varepsilon$, and then because $|-4x - 12| = |(-4x - 11) - 1|$, we conclude $|(-4x - 11) - 1| < \varepsilon$. **5.** Let $\varepsilon > 0$. No matter what I choose for δ, if x is within δ of -2, then π is within ε of π. **6.** As long as $x \neq -2$, we have $\dfrac{x^2 - 4}{x + 2} = x - 2$, and the limit is not sensitive to the value of the function at the point -2; the limit only depends on nearby values, so we really want to compute $\lim_{x \to -2} (x-2)$. Let $\varepsilon > 0$. Set $\delta = \varepsilon$. Then if $0 < |x-(-2)| < \delta$, we have $|(x-2)-(-4)| < \varepsilon$. **7.** Let $\varepsilon > 0$. Pick δ so that $\delta < 1$ and $\delta < \dfrac{\varepsilon}{61}$. Suppose $0 < |x-4| < \delta$. Then $4-\delta < x < 4+\delta$. Cube to get $(4-\delta)^3 < x^3 < (4+\delta)^3$. Expanding the right-side inequality, we get $x^3 < \delta^3 + 12 \cdot \delta^2 + 48 \cdot \delta + 64 < \delta + 12\delta + 48\delta + 64 = 64 + \varepsilon$. The other inequality is similar. **8.** Let $\varepsilon > 0$. Pick δ small enough so that $\delta < \varepsilon/6$ and $\delta < 1$. Assume $|x-1| < \delta$, so $6 \cdot |x-1| < \varepsilon$. Since x is within $\delta < 1$ of 1, we know $0 < x < 2$. So $|x+4| < 6$. Putting it together, $|x+4| \cdot |x-1| < \varepsilon$, so $|x^2+3x-4| < \varepsilon$, and therefore $|(x^2+3x-1)-3| < \varepsilon$. **9.** Let $\varepsilon > 0$. Set $\delta = 3\varepsilon$. Assume $0 < |x-9| < \delta$. Divide both sides by 3 to get $\dfrac{|x-9|}{3} < \varepsilon$. Note that $\sqrt{x}+3 > 3$, so $\dfrac{|x-9|}{\sqrt{x}+3} < \varepsilon$. This can be rearranged to conclude $\left|\dfrac{x-9}{\sqrt{x}-3} - 6\right| < \varepsilon$. **10.** Let $\varepsilon > 0$. Set δ to be the minimum of 2ε and 1. Assume x is within δ of 2, so $|x-2| < 2\varepsilon$ and $1 < x < 3$. So $\left|\dfrac{x-2}{2}\right| < \varepsilon$. Since $1 < x < 3$, we also have $2x > 2$, so $\left|\dfrac{x-2}{2x}\right| < \varepsilon$. Simplifying, $\left|\dfrac{1}{2} - \dfrac{1}{x}\right| < \varepsilon$, which is what we wanted.

Answers for 1.3

1. 7 **2.** 5 **3.** 0 **4.** DNE **5.** 1/6 **6.** 0 **7.** 3 **8.** 172 **9.** 0 **10.** 2 **11.** DNE **12.** $\sqrt{2}$ **13.** $3a^2$ **14.** 512 **15.** -4

Answers for 1.4

1. $-\infty$ **2.** 3/14 **3.** 1/2 **4.** $-\infty$ **5.** ∞ **6.** ∞ **7.** 0 **8.** $-\infty$ **9.** $x = 1$ and $x = -3$ **10.** $x = -4$

Answers for 1.5

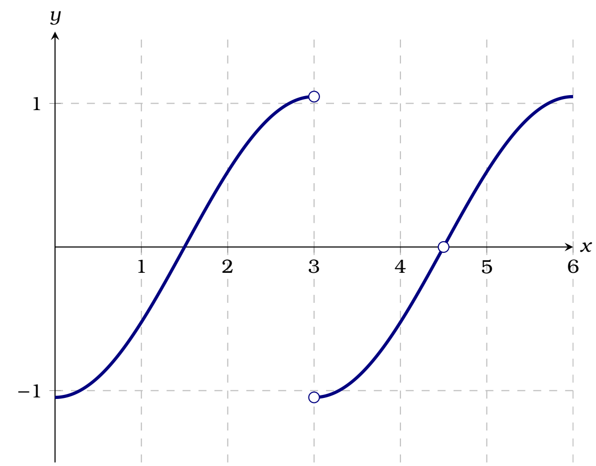
1. 0 **2.** -1 **3.** $\dfrac{1}{2}$ **4.** $-\infty$ **5.** π **6.** 0 **7.** 0 **8.** 17 **9.** After 10 years, $\approx$ 174 cats; after 50 years, $\approx$ 199 cats; after 100 years, $\approx$ 200 cats; after 1000 years, $\approx$ 200 cats; in the sense that the population of cats cannot grow indefinitely this is somewhat realistic. **10.** The amplitude goes to zero.

Answers for 1.6

1. $f(x)$ is continuous at $x = 4$ but it is not continuous on $\mathbb{R}$. **2.** $f(x)$ is continuous at $x = 3$ but it is not continuous on $\mathbb{R}$. **3.** $f(x)$ is not continuous at $x = 1$ and it is not continuous on $\mathbb{R}$. **4.** $f(x)$ is not continuous at $x = 5$ and it is not continuous on $\mathbb{R}$. **5.** $f(x)$ is continuous at $x = -5$ and it is also continuous on $\mathbb{R}$. **6.** $\mathbb{R}$ **7.** $(-\infty, -4) \cup (-4, \infty)$ **8.** $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **9.** $x = -0.48$, $x = 1.31$, or $x = 3.17$ **10.** $x = 0.20$, or $x = 1.35$

Answers for 2.1

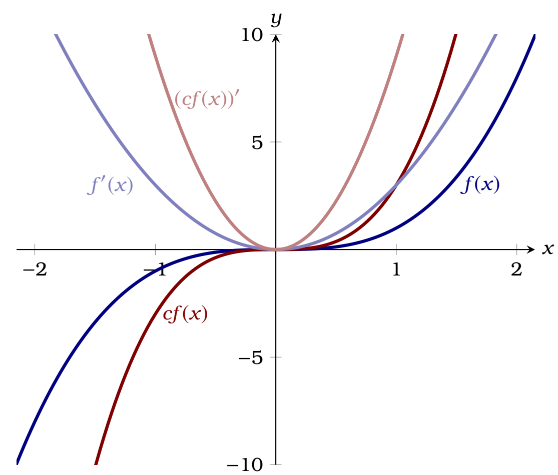
1. $f(2) = 10$ and $f'(2) = 7$ **2.** $p'(x) = s(x)$ and $r'(x) = q(x)$ **3.** $f'(3) \approx 4$ **4.** $f'(-2) = 4$ **5.** $f(1.2) \approx 2.2$ **6.** (a) $[0, 4.5) \cup (4.5, 6]$, (b) $[0, 3) \cup (3, 6]$, (c) See Figure **7.** $f'(-3) = -6$ with tangent line $y = -6x - 13$ **8.** $f'(1) = -1/9$ with tangent line $y = \dfrac{-1}{9}x + \dfrac{4}{9}$ **9.** $f'(5) = \dfrac{1}{2\sqrt{2}}$ with tangent line $y = \dfrac{1}{2\sqrt{2}}x - \dfrac{1}{2\sqrt{2}}$ **10.** $f'(4) = \dfrac{-1}{16}$ with tangent line $y = \dfrac{-1}{16}x + \dfrac{3}{4}$



Answer 2.1.6: (c) a sketch of $f'(x)$.

Answers for 2.2

1. 0 **2.** 0 **3.** 0 **4.** 0 **5.** $100x^{99}$ **6.** $-100x^{-101}$ **7.** $-5x^{-6}$ **8.** $\pi x^{\pi-1}$ **9.** $(3/4)x^{-1/4}$ **10.** $-(9/7)x^{-16/7}$ **11.** $15x^2 + 24x$ **12.** $-20x^4 + 6x + 10/x^3$ **13.** $-30x + 25$ **14.** $\dfrac{3}{2}x^{-1/2} - x^{-2} - ex^{e-1}$ **15.** $-5x^{-6} - x^{-3/2}/2$ **16.** $3x^2 + 6x - 1$ **17.** $2x - 1$ **18.** $x^{-1/2}/2$ **19.** $4x^3 - 4x$ **20.** $-49t/5 + 5$, $-49/5$ **21.** See Figure **22.** $x^3/16 - 3x/4 + 4$ **23.** $y = 13x/4 + 5$ **24.** $y = 24x - 48 - \pi^3$ **25.** $\dfrac{d}{dx}cf(x) = \lim_{h\to 0}\dfrac{cf(x+h) - cf(x)}{h} = c\lim_{h\to 0}\dfrac{f(x+h) - f(x)}{h} = cf'(x)$.



Answer 2.2.21.

Answers for 3.1

1. min at $x = 1/2$ **2.** min at $x = -1$, max at $x = 1$ **3.** max at $x = 2$, min at $x = 4$ **4.** min at $x = \pm 1$, max at $x = 0$. **5.** min at $x = 1$ **6.** none **7.** min at $x = 0$, max at $x = \dfrac{3 \pm \sqrt{17}}{2}$ **8.** none **9.** local max at $x = 5$ **10.** local min at $x = 49$ **11.** local min at $x = 0$ **12.** one **13.** if $c \geq 0$, then there are no local extrema; if $c < 0$ then there is a local max at $x = -\sqrt{\frac{|c|}{3}}$ and a local min at $x = \sqrt{\frac{|c|}{3}}$

Answers for 3.2

1. min at $x = 1/2$ **2.** min at $x = -1$, max at $x = 1$ **3.** max at $x = 2$, min at $x = 4$ **4.** min at $x = \pm 1$, max at $x = 0$. **5.** min at $x = 1$ **6.** none **7.** max at $x = 0$, min at $x = \pm 11$ **8.** $f'(x) = 2ax + b$, this has only one root and hence one critical point; $a < 0$ to guarantee a maximum.

Answers for 3.3

1. concave up everywhere **2.** concave up when $x < 0$, concave down when $x > 0$ **3.** concave down when $x < 3$, concave up when $x > 3$ **4.** concave up when $x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$, concave down when $-1/\sqrt{3} < x < 1/\sqrt{3}$ **5.** concave up when $x < 0$ or $x > 2/3$, concave down when $0 < x < 2/3$ **6.** concave up when $x < 0$, concave down when $x > 0$ **7.** concave up when $x < -1$ or $x > 1$, concave down when $-1 < x < 0$ or $0 < x < 1$ **8.** concave up on $(0, \infty)$ **9.** concave up on $(0, \infty)$ **10.** concave up on $(-\infty, -1)$ and $(0, \infty)$ **11.** up/incr: $(3, \infty)$, up/decr: $(-\infty, 0)$, $(2, 3)$, down/decr: $(0, 2)$

Answers for 3.4

1. min at $x = 1/2$ **2.** min at $x = -1$, max at $x = 1$ **3.** max at $x = 2$, min at $x = 4$ **4.** min at $x = \pm 1$, max at $x = 0$. **5.** min at $x = 1$ **6.** none **7.** none **8.** max at $-5^{-1/4}$, min at $5^{-1/4}$ **9.** max at -1, min at 1 **10.** min at $2^{-1/3}$

Answers for 3.5

1. y-intercept at $(0, 0)$; no vertical asymptotes; critical points: $x = \pm 1/\sqrt[4]{5}$; local max at $x = -1/\sqrt[4]{5}$, local min at $x = -1/\sqrt[4]{5}$; increasing on $(-\infty, -1/\sqrt[4]{5})$, decreasing on $(-1/\sqrt[4]{5}, 1/\sqrt[4]{5})$, increasing on $(1/\sqrt[4]{5}, \infty)$; concave down on $(-\infty, 0)$, concave up on $(0, \infty)$; root at $x = 0$; no horizontal asymptotes; interval for sketch: $[-1.2, 1.2]$

(answers may vary) **2.** y-intercept at $(0,0)$; no vertical asymptotes; no critical points; no local extrema; increasing on $(-\infty,\infty)$; concave down on $(-\infty,0)$, concave up on $(0,\infty)$; roots at $x=0$; no horizontal asymptotes; interval for sketch: $[-3,3]$ (answers may vary) **3.** y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=1$; local max at $x=1$; increasing on $[0,1)$, decreasing on $(1,\infty)$; concave down on $[0,\infty)$; roots at $x=0$, $x=4$; no horizontal asymptotes; interval for sketch: $[0,6]$ (answers may vary) **4.** y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=-3$, $x=-1$; local max at $x=-3$, local min at $x=-1$; increasing on $(-\infty,-3)$, decreasing on $(-3,-1)$, increasing on $(-1,\infty)$; concave down on $(-\infty,-2)$, concave up on $(-2,\infty)$; roots at $x=-3$, $x=0$; no horizontal asymptotes; interval for sketch: $[-5,3]$ (answers may vary) **5.** y-intercept at $(0,5)$; no vertical asymptotes; critical points: $x=-1$, $x=3$; local max at $x=-1$, local min at $x=3$; increasing on $(-\infty,-1)$, decreasing on $(-1,3)$, increasing on $(3,\infty)$; concave down on $(-\infty,1)$, concave up on $(1,\infty)$; roots are too difficult to be determined—cubic formula could be used; no horizontal asymptotes; interval for sketch: $[-2,5]$ (answers may vary) **6.** y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=0$, $x=\frac{10\pm\sqrt{85}}{5}$; local max at $x=\frac{10-\sqrt{85}}{5}$, local min at $x=\frac{10+\sqrt{85}}{5}$; increasing on $(-\infty,\frac{10-\sqrt{85}}{5})$, decreasing on $(\frac{10-\sqrt{85}}{5},\frac{10+\sqrt{85}}{5})$, increasing on $(\frac{10+\sqrt{85}}{5},\infty)$; concave down on $(-\infty,0)$, concave up on $(0,\frac{15-\sqrt{195}}{10})$, concave down on $(\frac{15-\sqrt{195}}{10},\frac{15+\sqrt{195}}{10})$, concave up on $(\frac{15+\sqrt{195}}{10},\infty)$; roots at $x=0$, $x=\frac{5\pm\sqrt{21}}{2}$; no horizontal asymptotes; interval for sketch: $[-1,5]$ (answers may vary) **7.** no y-intercept; vertical asymptote at $x=0$; critical points: $x=0$, $x=\pm1$; local max at $x=-1$, local min at 1; increasing on $(-\infty,-1)$, decreasing on $(-1,0)\cup(0,1)$, increasing on $(1,\infty)$; concave down on $(-\infty,0)$, concave up on $(0,\infty)$; no roots; no horizontal asymptotes; interval for sketch: $[-2,2]$ (answers may vary) **8.** no y-intercept; vertical asymptote at $x=0$; critical points: $x=0$, $x=\frac{1}{\sqrt[3]{2}}$; local min at $x=\frac{1}{\sqrt[3]{2}}$; decreasing on $(-\infty,0)$, decreasing on $(0,\frac{1}{\sqrt[3]{2}})$, increasing on $(\frac{1}{\sqrt[3]{2}},\infty)$; concave up on $(-\infty,-1)$, concave down on $(-1,0)$, concave up on $(0,\infty)$;

root at $x = -1$; no horizontal asymptotes; interval for sketch: $[-3, 2]$ (answers may vary)

Answers for 4.1

1. $3x^2(x^3-5x+10)+x^3(3x^2-5)$ **2.** $(x^2+5x-3)(5x^4-18x^2+6x-7)+(2x+5)(x^5-6x^3+3x^2-7x+1)$ **3.** $2e^{2x}$ **4.** $3e^{3x}$ **5.** $6xe^{4x}+12x^2e^{4x}$ **6.** $\dfrac{-48e^x}{x^{17}}+\dfrac{3e^x}{x^{16}}$ **7.** $f' = 4(2x-3)$, $y = 4x-7$ **8.** 3 **9.** 10 **10.** -13 **11.** -5 **12.** $\dfrac{d}{dx}f(x)g(x)h(x) = \dfrac{d}{dx}f(x)(g(x)h(x)) = f(x)\dfrac{d}{dx}(g(x)h(x))+f'(x)g(x)h(x) = f(x)(g(x)h'(x)+g'(x)h(x))+f'(x)g(x)h(x) = f(x)g(x)h'(x)+f(x)g'(x)h(x))+f'(x)g(x)h(x)$

Answers for 4.2

1. $\dfrac{3x^2}{x^3-5x+10}-\dfrac{x^3(3x^2-5)}{(x^3-5x+10)^2}$ **2.** $\dfrac{2x+5}{x^5-6x^3+3x^2-7x+1}-\dfrac{(x^2+5x-3)(5x^4-18x^2+6x-7)}{(x^5-6x^3+3x^2-7x+1)^2}$ **3.** $\dfrac{2xe^x-(e^x-4)2}{4x^2}$ **4.** $\dfrac{(x+2)(-1-(1/2)x^{-1/2})-(2-x-\sqrt{x})}{(x+2)^2}$ **5.** $y = 17x/4-41/4$ **6.** $y = 11x/16-15/16$ **7.** $y = 19/169-5x/338$ **8.** $-1/4$ **9.** $8/9$ **10.** 24 **11.** -3 **12.** $f(4) = 1/3$, $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = 13/18$

Answers for 5.1

1. $4x^3-9x^2+x+7$ **2.** $3x^2-4x+2/\sqrt{x}$ **3.** $6(x^2+1)^2x$ **4.** $\sqrt{169-x^2}-x^2/\sqrt{169-x^2}$ **5.** $(2x-4)\sqrt{25-x^2}-(x^2-4x+5)x/\sqrt{25-x^2}$ **6.** $-x/\sqrt{r^2-x^2}$ **7.** $2x^3/\sqrt{1+x^4}$ **8.** $\dfrac{1}{4\sqrt{x}(5-\sqrt{x})^{3/2}}$ **9.** $6+18x$ **10.** $\dfrac{2x+1}{1-x}+\dfrac{x^2+x+1}{(1-x)^2}$ **11.** $-1/\sqrt{25-x^2}-\sqrt{25-x^2}/x^2$ **12.** $\dfrac{1}{2}\left(\dfrac{-169}{x^2}-1\right)\Big/\sqrt{\dfrac{169}{x}-x}$ **13.** $\dfrac{3x^2-2x+1/x^2}{2\sqrt{x^3-x^2-(1/x)}}$ **14.** $\dfrac{300x}{(100-x^2)^{5/2}}$ **15.** $\dfrac{1+3x^2}{3(x+x^3)^{2/3}}$ **16.** $\left(4x(x^2+1)+\dfrac{4x^3+4x}{2\sqrt{1+(x^2+1)^2}}\right)\Big/ 2\sqrt{(x^2+1)^2+\sqrt{1+(x^2+1)^2}}$ **17.** $5(x+8)^4$ **18.** $-3(4-x)^2$ **19.** $6x(x^2+5)^2$ **20.** $-12x(6-2x^2)^2$ **21.** $24x^2(1-4x^3)^{-3}$ **22.** $5+5/x^2$ **23.** $-8(4x-$

$1)(2x^2 - x + 3)^{-3}$ **24.** $1/(x+1)^2$ **25.** $3(8x-2)/(4x^2-2x+1)^2$ **26.** $-3x^2+5x-1$ **27.** $6x(2x-4)^3 + 6(3x^2+1)(2x-4)^2$ **28.** $-2/(x-1)^2$ **29.** $4x/(x^2+1)^2$ **30.** $(x^2-6x+7)/(x-3)^2$ **31.** $-5/(3x-4)^2$ **32.** $60x^4+72x^3+18x^2+18x-6$ **33.** $(5-4x)/((2x+1)^2(x-3)^2)$ **34.** $1/(2(2+3x)^2)$ **35.** $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$ **36.** $y = 23x/96 - 29/96$ **37.** $y = 3 - 2x/3$ **38.** $y = 13x/2 - 23/2$ **39.** $y = 2x - 11$ **40.** $y = \dfrac{20+2\sqrt{5}}{5\sqrt{4+\sqrt{5}}}x + \dfrac{3\sqrt{5}}{5\sqrt{4+\sqrt{5}}}$

Answers for 5.2

1. $-x/y$ **2.** x/y **3.** $-(2x+y)/(x+2y)$ **4.** $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$ **5.** $\dfrac{-2xy}{x^2-3y^2}$ **6.** $-\sqrt{y}/\sqrt{x}$ **7.** $\dfrac{y^{3/2}-2}{1-y^{1/2}3x/2}$ **8.** $-y^2/x^2$ **9.** 1 **10.** $y = 2x \pm 6$ **11.** $y = x/2 \pm 3$ **12.** $(\sqrt{3}, 2\sqrt{3})$, $(-\sqrt{3}, -2\sqrt{3})$, $(2\sqrt{3}, \sqrt{3})$, $(-2\sqrt{3}, -\sqrt{3})$ **13.** $y = 7x/\sqrt{3} - 8/\sqrt{3}$ **14.** $y = (-y_1^{1/3}x + y_1^{1/3}x_1 + x_1^{1/3}y_1)/x_1^{1/3}$ **15.** $(y-y_1)/(x-x_1) = (2x_1^3+2x_1y_1^2-x_1)/(2y_1^3+2y_1x_1^2+y_1)$

Answers for 5.3

1. $(x+1)^3\sqrt{x^4+5}(3/(x+1)+2x^3/(x^4+5))$ **2.** $(2/x+5)x^2e^{5x}$ **3.** $2\ln(x)x^{\ln(x)-1}$ **4.** $(100+100\ln(x))x^{100x}$ **5.** $(4+4\ln(3x))(3x)^{4x}$ **6.** $((e^x)/x + e^x\ln(x))x^{e^x}$ **7.** $\pi x^{\pi-1} + \pi^x\ln(\pi)$ **8.** $(\ln(1+1/x) - 1/(x+1))(1+1/x)^x$ **9.** $(1/\ln(x) + \ln(\ln(x)))(\ln(x))^x$ **10.** $(f'(x)/f(x) + g'(x)/g(x) + h'(x)/h(x))f(x)g(x)h(x)$

Answers for 6.1

1. $\sin(\sqrt{x})\cos(\sqrt{x})/\sqrt{x}$ **2.** $\dfrac{\sin(x)}{2\sqrt{x}} + \sqrt{x}\cos(x)$ **3.** $-\dfrac{\cos(x)}{\sin^2(x)}$ **4.** $\dfrac{(2x+1)\sin(x) - (x^2+x)\cos(x)}{\sin^2(x)}$ **5.** $\dfrac{-\sin(x)\cos(x)}{\sqrt{1-\sin^2(x)}}$ **6.** $\cos^2(x) - \sin^2(x)$ **7.** $-\sin(x)\cos(\cos(x))$ **8.** $\dfrac{\tan(x) + x\sec^2(x)}{2\sqrt{x\tan(x)}}$ **9.** $\dfrac{\sec^2(x)(1+\sin(x)) - \tan(x)\cos(x)}{(1+\sin(x))^2}$ **10.** $-\csc^2(x)$ **11.** $-\csc(x)\cot(x)$ **12.** $3x^2\sin(23x^2) + 46x^4\cos(23x^2)$ **13.** 0 **14.** $-6\cos(\cos(6x))\sin(6x)$ **15.** $\sin(\vartheta)/(\cos(\vartheta)+1)^2$ **16.** $5t^4\cos(6t) - 6t^5\sin(6t)$ **17.** $3t^2(\sin(3t) + t\cos(3t))/\cos(2t) + 2t^3\sin(3t)\sin(2t)/\cos^2(2t)$

18. $n\pi/2$, any integer n **19.** $\pi/2+n\pi$, any integer n **20.** $\sqrt{3}x/2+3/4-\sqrt{3}\pi/6$ **21.** $8\sqrt{3}x+4-8\sqrt{3}\pi/3$ **22.** $3\sqrt{3}x/2-\sqrt{3}\pi/4$ **23.** $\pi/6+2n\pi$, $5\pi/6+2n\pi$, any integer n

Answers for 6.2

1. $\dfrac{-1}{1+x^2}$ **2.** $\dfrac{2x}{\sqrt{1-x^4}}$ **3.** $\dfrac{e^x}{1+e^{2x}}$ **4.** $-3x^2\cos(x^3)/\sqrt{1-\sin^2(x^3)}$ **5.** $\dfrac{2}{(\arcsin(x))\sqrt{1-x^2}}$ **6.** $-e^x\sqrt{1-e^{2x}}$ **7.** 0 **8.** $\dfrac{(1+\ln x)x^x}{\ln 5(1+x^{2x})\arctan(x^x)}$

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