MOOCULUS
massive open online calculus

CALCULUS

This text is based on David Guichard's open-source calculus text which in turn is a modification and expansion of notes written by Neal Koblitz at the University of Washington. New material has been added, and old material has been modified, so some portions now bear little resemblance to the original.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at `http://www.math.wisc.edu/~keisler/calc.html` under a Creative Commons license. In addition, the chapter on differential equations and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

This book is typeset in the Kerkis font, Kerkis © Department of Mathematics, University of the Aegean.

We will be glad to receive corrections and suggestions for improvement at `fowler@math.osu.edu` or `snapp@math.osu.edu`.

Contents

List of Main Theorems

1 Limits

1.1 The Basic Ideas of Limits

Consider the function:

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

While $f(x)$ is undefined at $x = 2$, we can still plot $f(x)$ at other values, see Figure **??**. Examining Table **??**, we see that as x approaches 2, $f(x)$ approaches 1. We write this:

$$\text{As } x \to 2, f(x) \to 1 \qquad \text{or} \qquad \lim_{x\to 2} f(x) = 1.$$

Intuitively, $\lim\limits_{x\to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. This leads us to the formal definition of a *limit*.

Definition The **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x\to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta, \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

If no such value of L can be found, then we say that $f(x)$ **does not exist** at $x = a$.

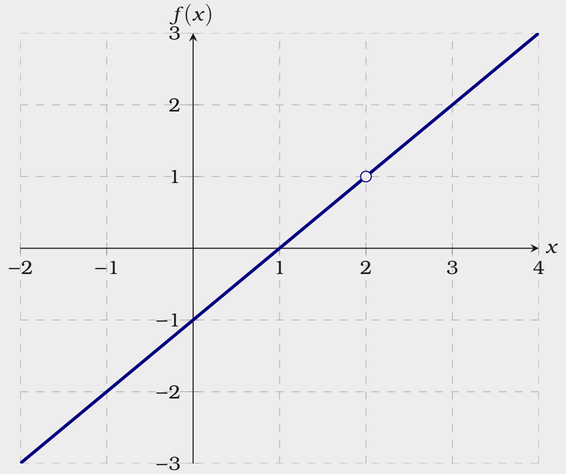


Figure 1.1: A plot of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

x	$f(x)$	x	$f(x)$
1.7	0.7	2	undefined
1.9	0.9	2.001	1.001
1.99	0.99	2.01	1.01
1.999	0.999	2.1	1.1
2	undefined	2.3	1.3

Table 1.1: Values of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

Equivalently, $\lim\limits_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and $a - \delta < x < a + \delta$, we have $L - \varepsilon < f(x) < L + \varepsilon$.

In Figure **??**, we see a geometric interpretation of this definition.

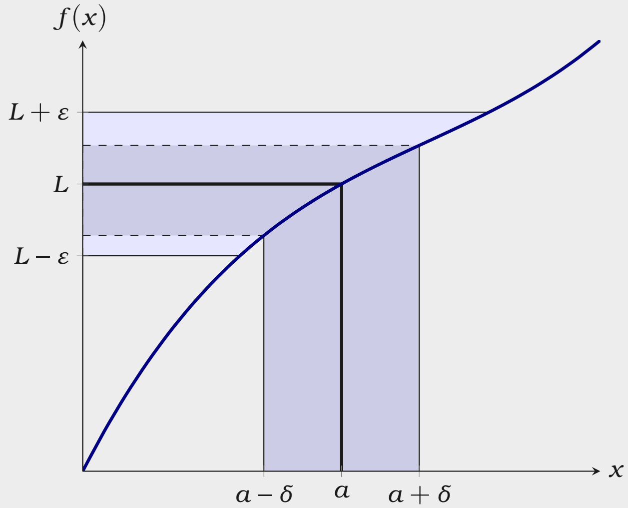


Figure 1.2: A geometric interpretation of the (ε, δ)-criterion for limits. If $0 < |x-a| < \delta$, then we have that $a-\delta < x < a+\delta$. In our diagram, we see that for all such x we are sure to have $L-\varepsilon < f(x) < L+\varepsilon$, and hence $|f(x)-L| < \varepsilon$.

Limits need not exist, let's examine two cases of this.

Example 1.1.1 Let $f(x) = \lfloor x \rfloor$. Explain why the limit

$$\lim_{x \to 2} f(x)$$

does not exist.

Solution *This is the function that returns the greatest integer less than or equal to x. Since $f(x)$ is defined for all real numbers, one might be tempted to think that the limit above is simply $f(2) = 2$. However, this is not the case. If $x < 2$, then $f(x) = 1$. Hence if $\varepsilon = .5$, we can* ***always*** *find a value for x (just to the left of 2) such that*

$$0 < |x-2| < \delta, \qquad \textit{where} \qquad \varepsilon < |f(x)-2|.$$

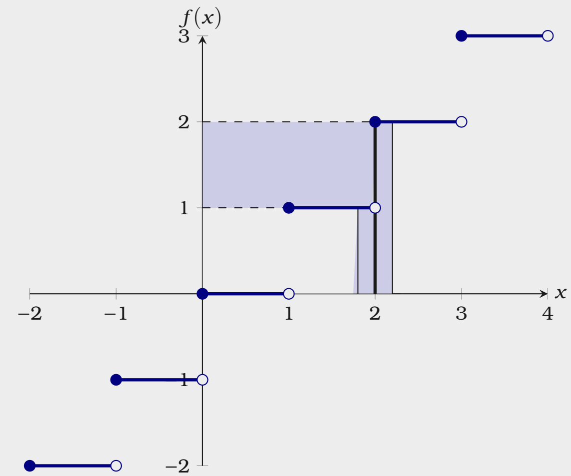


Figure 1.3: A plot of $f(x) = \lfloor x \rfloor$. Note, no matter which $\delta > 0$ is chosen, we can only at best bound $f(x)$ in the interval $[1, 2]$.

On the other hand, $\lim_{x\to 2} f(x) \neq 1$, as in this case if $\varepsilon = .5$, we can ***always*** *find a value for x (just to the right of 2) such that*

$$0 < |x-2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x)-1|.$$

We've illustrated this in Figure **??**. *Moreover, no matter what value one chooses for $\lim_{x\to 2} f(x)$, we will always have a similar issue.*

With the example of $f(x) = \lfloor x \rfloor$, we see that taking limits is truly different from evaluating functions.

Limits may not exist even if the formula for the function looks innocent.

Example 1.1.2 Let $f(x) = \sin\left(\frac{1}{x}\right)$. Explain why the limit

$$\lim_{x\to 0} f(x)$$

does not exist.

Solution *In this case $f(x)$ oscillates "wildly" as x approaches 0, see Figure* **??**. *In fact, one can show that for any given δ, There is a value for x in the interval*

$$0-\delta < x < 0+\delta$$

such that $f(x)$ is ***any*** *value in the interval $[-1,1]$. Hence the limit does not exist.*

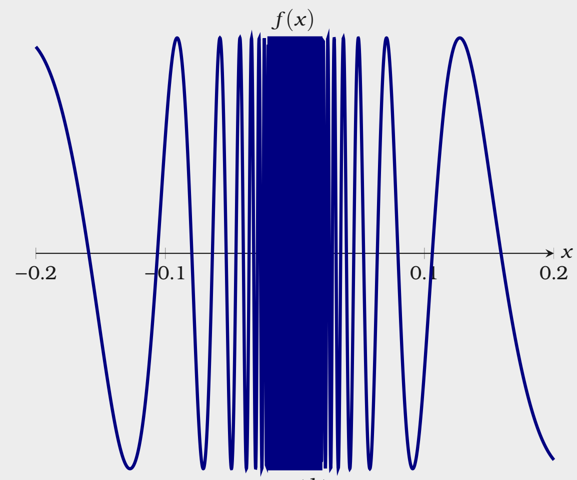


Figure 1.4: A plot of $f(x) = \sin\left(\frac{1}{x}\right)$.

Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of a *one-sided limit*:

Definition We say that the **limit** of $f(x)$ as x goes to a from the **left** is L,

$$\lim_{x\to a-} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and

$$a-\delta < x \qquad \text{we have} \qquad |f(x)-L| < \varepsilon.$$

We say that the **limit** of $f(x)$ as x goes to a from the **right** is L,

$$\lim_{x \to a+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and

$$x < a + \delta \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

Limits from the left, or from the right, are collectively called **one-sided limits**.

Example 1.1.3 Let $f(x) = \lfloor x \rfloor$. Discuss

$$\lim_{x \to 2-} f(x), \qquad \lim_{x \to 2+} f(x), \qquad \text{and} \qquad \lim_{x \to 2+} f(x).$$

Solution *From the plot of $f(x)$, see Figure* **??**, *we see that*

$$\lim_{x \to 2-} f(x) = 1, \qquad \textit{and} \qquad \lim_{x \to 2+} f(x) = 2.$$

Since these limits are different, $\lim_{x \to 2} f(x)$ *does not exist.*

Exercises for Section 1.1

(1) Evaluate the expressions by reference to the plot in Figure **??**.

(a) $\lim_{x\to 4} f(x)$
(b) $\lim_{x\to -3} f(x)$
(c) $\lim_{x\to 0} f(x)$
(d) $\lim_{x\to 0-} f(x)$
(e) $\lim_{x\to 0+} f(x)$
(f) $f(-2)$
(g) $\lim_{x\to 2-} f(x)$
(h) $\lim_{x\to -2-} f(x)$
(i) $\lim_{x\to 0} f(x+1)$
(j) $f(0)$
(k) $\lim_{x\to 1-} f(x-4)$
(l) $\lim_{x\to 0+} f(x-2)$

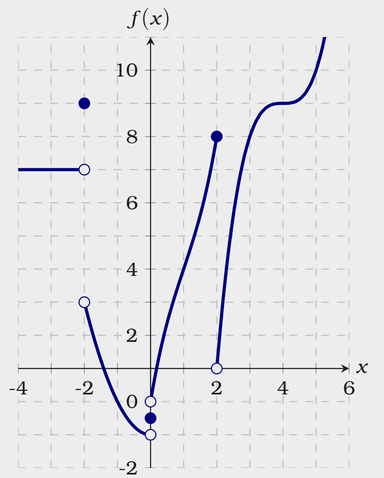


Figure 1.5: A piecewise defined function.

(2) Use a table and a calculator to estimate $\lim_{x\to 0} \dfrac{\sin x}{x}$.

(3) Use a table and a calculator to estimate $\lim_{x\to 0} \dfrac{\tan(3x)}{\tan(5x)}$.

(4) Use a table and a calculator to estimate $\lim_{x\to 0} \dfrac{2^x - 1}{x}$.

(5) Sketch a plot of $f(x) = \dfrac{x}{|x|}$ and explain why $\lim_{x\to 0} \dfrac{x}{|x|}$ does not exist.

(6) Let $f(x) = \sin\left(\dfrac{\pi}{x}\right)$. Construct three tables of the following form

x	$f(x)$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	

where $d = 1, 3, 7$. What do you notice? How do you reconcile the entries in your tables with the value of $\lim_{x\to 0} f(x)$?

1.2 Limits by the Definition

Now we are going to get our hands dirty, and really use the definition of a limit.

Recall, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x-a| + \delta$, we have $|f(x) - L| < \varepsilon$.

Example 1.2.1 Show that $\lim_{x\to 2} x^2 = 4$.

Solution *We want to show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that*

$$|x^2 - 4| < \varepsilon$$

whenever $0 < |x-2| < \delta$. Start by factoring the LHS of the inequality above

$$|x+2||x-2| < \varepsilon.$$

Since we are going to assume that $0 < |x-2| < \delta$, we will focus on the factor $|x+2|$. Since x is assumed to be close to 2, suppose that $x \in [1,3]$. In this case

$$|x+2| \leq 3 + 2 = 5,$$

and so we want

$$5 \cdot |x-2| < \varepsilon$$
$$|x-2| < \frac{\varepsilon}{5}$$

Recall, we assumed that $x \in [1,3]$, which is equivalent to $|x-2| < 1$. Hence we must set $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

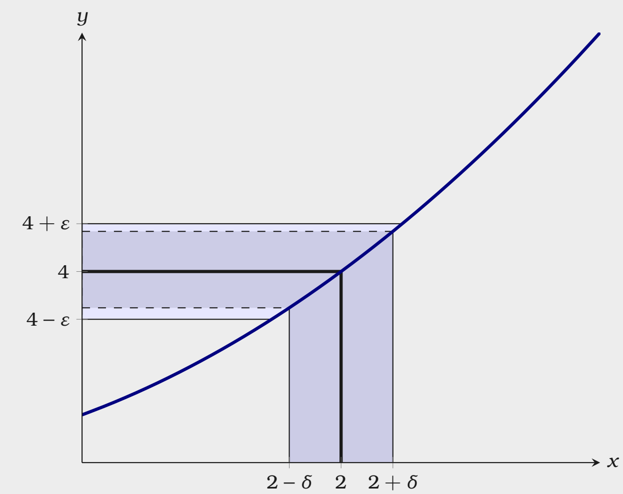


Figure 1.6: The (ε, δ)-criterion for $\lim_{x\to 2} x^2 = 4$. Here $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

When dealing with limits of polynomials, the general strategy is always the same. Let $p(x)$ be a polynomial. If showing

$$\lim_{x\to a} p(x) = L,$$

one must first factor out $|x-a|$ from $|p(x) - L|$. Next bound $x \in [a-1, a+1]$ and estimate the largest possible value of

$$\left|\frac{p(x) - L}{x - a}\right|$$

for $x \in [a-1, a+1]$. Call this estimation M. Finally, one must set $\delta = \min\left(\frac{\varepsilon}{M}, 1\right)$.

As you work with limits, you find that you need to do the same procedures again and again. The next theorems will expedite this process.

Theorem 1.2.2 *Suppose* $\lim\limits_{x\to a} f(x) = L$ *and* $\lim\limits_{x\to a} g(x) = M$. *Then*

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof *Given any* ε *we need to find a* δ *such that*

$$0 < |x < a| < \delta$$

implies

$$|f(x)g(x) - LM| < \varepsilon.$$

Here we use an algebraic trick, add $0 = -f(x)M + f(x)M$*:*

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\le |f(x)(g(x) - M)| + |(f(x) - L)M| \\ &= |f(x)||g(x) - M| + |f(x) - L||M|. \end{aligned}$$

This is all straightforward except perhaps for the "$\le$". This follows from the *Triangle Inequality*. The **Triangle Inequality** states: If a and b are any real numbers then $|a + b| \le |a| + |b|$.

Since $\lim\limits_{x\to a} f(x) = L$, *there is a value* δ_1 *so that* $0 < |x - a| < \delta_1$ *implies* $|f(x) - L| < |\varepsilon/(2M)|$. *This means that* $0 < |x - a| < \delta_1$ *implies* $|f(x) - L||M| < \varepsilon/2$.

$$|f(x)g(x) - LM| \le |f(x)||g(x) - M| + \underbrace{|f(x) - L||M|}_{<\frac{\varepsilon}{2}}.$$

If we can make $|f(x)||g(x) - M| < \varepsilon/2$, *then we'll be done. We can make* $|g(x) - M|$ *smaller than any fixed number by making* x *close enough to* a. *Unfortunately,* $\varepsilon/(2f(x))$ *is not a fixed number since* x *is a variable.*

Here we need another trick. We can find a δ_2 *so that* $|x - a| < \delta_2$ *implies that* $|f(x) - L| < 1$, *meaning that* $L - 1 < f(x) < L + 1$. *This means that* $|f(x)| < N$, *where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x)-M| < \varepsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that*

$$|f(x)g(x)-LM| \leq \underbrace{|f(x)|}_{<N}\underbrace{|g(x)-M|}_{<\frac{\varepsilon}{2N}} + \underbrace{|f(x)-L||M|}_{<\frac{\varepsilon}{2}}.$$

so

$$\begin{aligned} |f(x)g(x)-LM| &\leq |f(x)||g(x)-M| + |f(x)-L||M| \\ &< N\frac{\varepsilon}{2N} + \left|\frac{\varepsilon}{2M}\right||M| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This is just what we needed, so by the definition of a limit, $\lim_{x\to a} f(x)g(x) = LM$.

Another useful way to put functions together is composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2+5$, then $f(g(x)) = \sqrt{x^2+5}$ and $g(f(x)) = (\sqrt{x})^2 + 5 = x+5$. Here is a companion to theorem **??** for composition:

Theorem 1.2.3 *Suppose that $\lim_{x\to a} g(x) = L$ and $\lim_{x\to L} f(x) = f(L)$. Then*

$$\lim_{x\to a} f(g(x)) = f(L).$$

Note the special form of the condition on f: it is not enough to know that $\lim_{x\to L} f(x) = M$, though it is a bit tricky to see why. Many of the most familiar functions do have this property, and this theorem can therefore be applied. For example:

Theorem 1.2.4 *Suppose that n is a positive integer. Then*

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Exercises for Section 1.2

(1) Use the definition of limits to explain why $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. Hint: Use the fact that $|\sin a| < 1$ for any real number a.

(2) Use the definition of limits to explain why $\lim_{x\to 4}(2x-5) = 3$.

(3) For each of the following limits, $\lim_{x\to a} f(x) = L$, use a graphing device to find δ such that $0 < |x-a| < \delta$ implies that $|f(x)-L| < \varepsilon$ where $\varepsilon = .1$.

(a) $\lim_{x\to 2}(3x+1) = 7$

(b) $\lim_{x\to 1}(x^2+2) = 3$

(c) $\lim_{x\to \pi}\sin(x) = 0$

(d) $\lim_{x\to 0}\tan(x) = 0$

(e) $\lim_{x\to 1}\sqrt{3x+1} = 2$

(f) $\lim_{x\to -2}\sqrt[3]{1-4x} = 3$

1.3 Limit Laws

In this section, we present a handful of tools to compute many limits without explicitly working with the definition of limit. Each of these could be proved directly as we did in the previous section.

Theorem 1.3.1 (Limit Laws) *Suppose that $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, k is some constant, and n is a positive integer. Then*

- $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = kL$
- $\lim_{x\to a} (f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$
- $\lim_{x\to a} (f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$
- $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$
- $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$
- $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ *provided if n is even, then $f(x) \geq 0$ near a.*

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the "innermost bits" and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 1.3.2 Compute $\lim_{x\to 1} \dfrac{x^2 - 3x + 5}{x - 2}$.

Solution *Using limit laws,*

$$\begin{aligned}
\lim_{x\to 1} \frac{x^2-3x+5}{x-2} &= \frac{\lim\limits_{x\to 1} x^2-3x+5}{\lim\limits_{x\to 1}(x-2)} \\
&= \frac{\lim\limits_{x\to 1} x^2 - \lim\limits_{x\to 1} 3x + \lim\limits_{x\to 1} 5}{\lim\limits_{x\to 1} x - \lim\limits_{x\to 1} 2} \\
&= \frac{\left(\lim\limits_{x\to 1} x\right)^2 - 3\lim\limits_{x\to 1} x + 5}{\lim\limits_{x\to 1} x - 2} \\
&= \frac{1^2 - 3\cdot 1 + 5}{1-2} \\
&= \frac{1-3+5}{-1} = -3.
\end{aligned}$$

It is worth commenting on the trivial limit $\lim\limits_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x) = 5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

We're primarily interested in limits that aren't so easy, namely limits in which a denominator approaches zero. The basic idea is to "divide out" by the offending factor. This is often easier said than done—here we give two examples of algebraic tricks that work on many of these limits.

Example 1.3.3 Compute $\lim\limits_{x\to 1} \dfrac{x^2+2x-3}{x-1}$.

Solution *We can't simply plug in $x = 1$ because that makes the denominator zero. However, when taking limits we assume $x \neq 1$:*

$$\begin{aligned} \lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1} &= \lim_{x \to 1} \frac{(x-1)(x+3)}{x-1} \\ &= \lim_{x \to 1} (x+3) = 4 \end{aligned}$$

Limits allow us to examine functions where they are not defined.

Example 1.3.4 Compute $\lim_{x \to -1} \frac{\sqrt{x+5} - 2}{x+1}$.

Solution *Using limit laws,*

$$\begin{aligned} \lim_{x \to -1} \frac{\sqrt{x+5} - 2}{x+1} &= \lim_{x \to -1} \frac{\sqrt{x+5} - 2}{x+1} \frac{\sqrt{x+5} + 2}{\sqrt{x+5} + 2} \\ &= \lim_{x \to -1} \frac{x + 5 - 4}{(x+1)(\sqrt{x+5} + 2)} \\ &= \lim_{x \to -1} \frac{x+1}{(x+1)(\sqrt{x+5} + 2)} \\ &= \lim_{x \to -1} \frac{1}{\sqrt{x+5} + 2} = \frac{1}{4}. \end{aligned}$$

Here we are rationalizing the numerator by multiplying by the conjugate.

We'll conclude with one more theorem that will allow us to compute more difficult limits.

Theorem 1.3.5 (Squeeze Theorem) *Suppose that $g(x) \le f(x) \le h(x)$ for all x close to a but not necessarily equal to a. If*

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

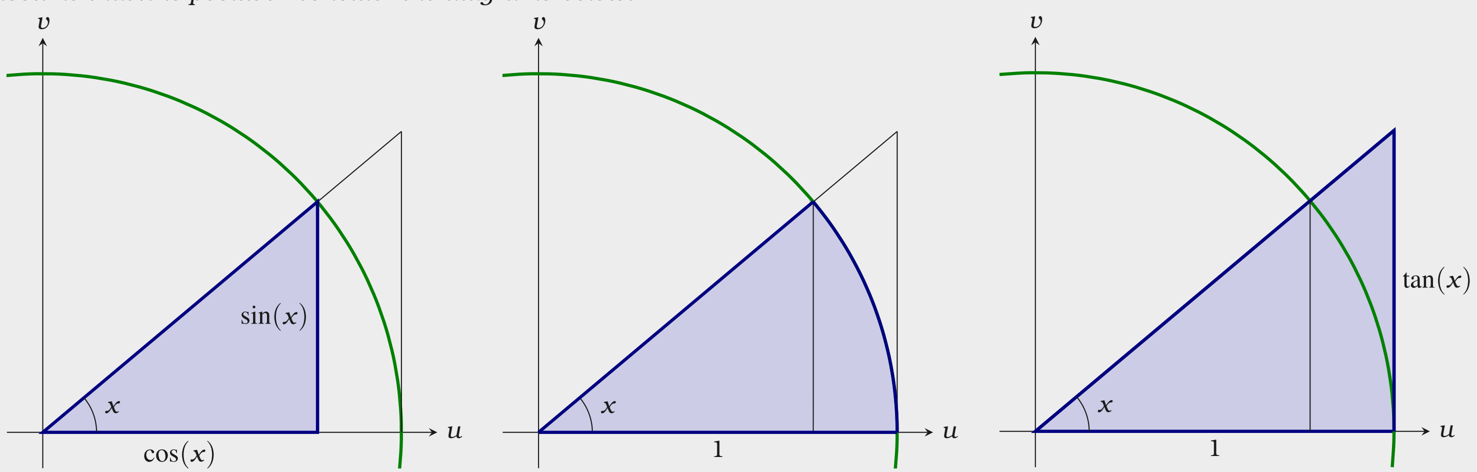
then $\lim_{x\to a} f(x) = L$.

For a nice discussion of this limit, see: Richman, Fred. *A circular argument*. College Math. J. 24 (1993), no. 2, 160–162.

Example 1.3.6 Compute

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

Solution *To compute this limit, use the Squeeze Theorem, Theorem* **??**. *First note that we only need to examine $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ and for the present time, we'll assume that x is positive—consider the diagrams below:*



Triangle A *Sector* *Triangle B*

From our diagrams above we see that

$$\text{Area of Triangle A} \le \text{Area of Sector} \le \text{Area of Triangle B}$$

and computing these areas we find

$$\frac{\cos(x)\sin(x)}{2} \leq \left(\frac{x}{2\pi}\right)\cdot\pi \leq \frac{\tan(x)}{2}.$$

Multiplying through by 2*, and recalling that* $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ *we obtain*

$$\cos(x)\sin(x) \leq x \leq \frac{\sin(x)}{\cos(x)}.$$

Dividing through by $\sin(x)$ *and taking the reciprocals, we find*

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}.$$

Note, $\cos(-x) = \cos(x)$ *and* $\dfrac{\sin(-x)}{-x} = \dfrac{\sin(x)}{x}$*, so these inequalities hold for all* $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$*. Additionally, we know*

$$\lim_{x\to 0}\cos(x) = 1 = \lim_{x\to 0}\frac{1}{\cos(x)},$$

and so we conclude by the Squeeze Theorem, Theorem **??**, $\displaystyle\lim_{x\to 0}\frac{\sin(x)}{x} = 1.$

Exercises for Section 1.3

Compute the limits. If a limit does not exist, explain why.

(1) $\lim_{x\to 3} \dfrac{x^2+x-12}{x-3}$

(2) $\lim_{x\to 1} \dfrac{x^2+x-12}{x-3}$

(3) $\lim_{x\to -4} \dfrac{x^2+x-12}{x-3}$

(4) $\lim_{x\to 2} \dfrac{x^2+x-12}{x-2}$

(5) $\lim_{x\to 1} \dfrac{\sqrt{x+8}-3}{x-1}$

(6) $\lim_{x\to 0+} \sqrt{\dfrac{1}{x}+2} - \sqrt{\dfrac{1}{x}}.$

(7) $\lim_{x\to 2} 3$

(8) $\lim_{x\to 4} 3x^3-5x$

(9) $\lim_{x\to 0} \dfrac{4x-5x^2}{x-1}$

(10) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

(11) $\lim_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x}$

(12) $\lim_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x+1}$

(13) $\lim_{x\to a} \dfrac{x^3-a^3}{x-a}$

(14) $\lim_{x\to 2} (x^2+4)^3$

(15) $\lim_{x\to 1} \begin{cases} x-5 & x\neq 1, \\ 7 & x=1. \end{cases}$

1.4 Infinite Limits

Consider the function

$$f(x) = \frac{1}{(x+1)^2}$$

While the $\lim_{x\to -1} f(x)$ does not exist, see Figure **??**, something can still be said.

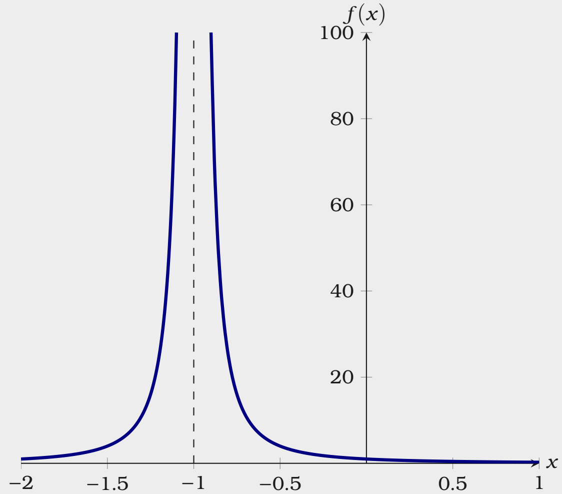


Figure 1.7: A plot of $f(x) = \dfrac{1}{(x+1)^2}$.

Definition If $f(x)$ grows arbitrarily large as x approaches a, we write

$$\lim_{x\to a} f(x) = \infty$$

and say that the limit of $f(x)$ **approaches infinity** as x goes to a.

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x\to a} f(x) = -\infty$$

and say that the limit of $f(x)$ **approaches negative infinity** as x goes to a.

On the other hand, if we consider the function

$$f(x) = \frac{1}{(x-1)}$$

While we have $\lim_{x\to 1} f(x) \neq \pm\infty$, we do have one-sided limits, $\lim_{x\to 1+} f(x) = \infty$ and $\lim_{x\to 1-} f(x) = -\infty$, see Figure **??**.

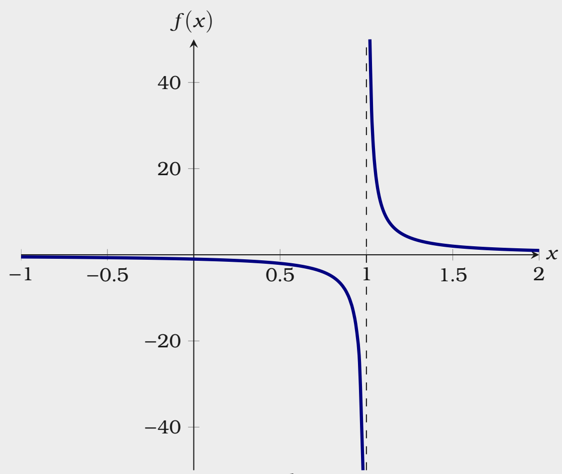


Figure 1.8: A plot of $f(x) = \dfrac{1}{x-1}$.

Definition If

$$\lim_{x\to a} f(x) = \pm\infty, \qquad \lim_{x\to a+} f(x) = \pm\infty, \qquad \text{or} \qquad \lim_{x\to a-} f(x) = \pm\infty,$$

then the line $x = a$ is a **vertical asymptote** of $f(x)$.

Example 1.4.1 Find the vertical asymptotes of

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

Solution *Start by factoring both the numerator and the denominator:*

$$\frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

Using limits, we must investigate when $x \to 2$ and $x \to 3$. Write

$$\begin{aligned} \lim_{x \to 2} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x \to 2} \frac{(x-7)}{(x-3)} \\ &= \frac{-5}{-1} \\ &= 5. \end{aligned}$$

Now write

$$\begin{aligned} \lim_{x \to 3} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x \to 3} \frac{(x-7)}{(x-3)} \\ &= \lim_{x \to 3} \frac{-4}{x-3}. \end{aligned}$$

Hence $\lim_{x \to 3+} f(x) = -\infty$, $\lim_{x \to 3-} f(x) = \infty$, and we have a vertical asymptote at $x = 3$, see Figure **??**.

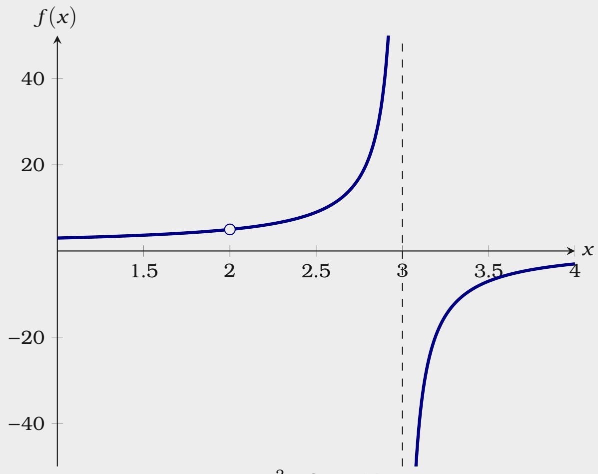


Figure 1.9: A plot of $f(x) = \dfrac{x^2 - 9x + 14}{x^2 - 5 + 6}$.

Exercises for Section 1.4

FILL IN

1.5 Limits at Infinity

Consider the function:

$$f(x) = \frac{6x-9}{x-1}$$

As x approaches infinity, it seems like $f(x)$ approaches a specific value. This is a *limit at infinity*.

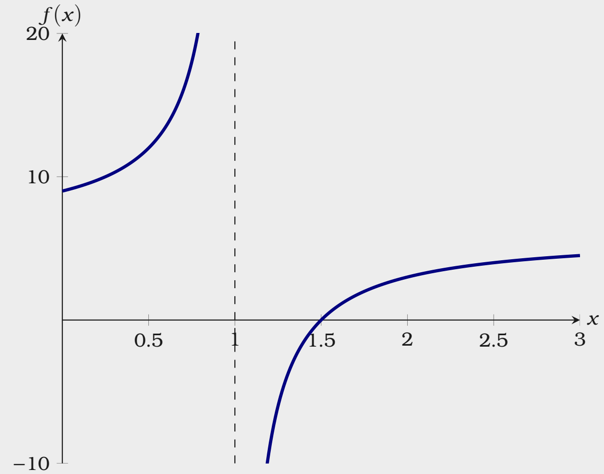


Figure 1.10: A plot of $f(x) = \frac{6x-9}{x-1}$.

Definition If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

$$\lim_{x\to\infty} f(x) = L$$

and we say, the **limit at infinity** of $f(x)$ is L.

If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x\to-\infty} f(x) = L$$

and we say, the **limit at negative infinity** of $f(x)$ is L.

Example 1.5.1 Compute

$$\lim_{x\to\infty} \frac{6x-9}{x-1}.$$

Solution *Write*

$$\begin{aligned}
\lim_{x\to\infty} \frac{6x-9}{x-1} &= \lim_{x\to\infty} \frac{6x-9}{x-1}\frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{\frac{6x}{x}-\frac{9}{x}}{\frac{x}{x}-\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{6}{1} \\
&= 6.
\end{aligned}$$

Here is a somewhat different example of a limit at infinity.

Example 1.5.2 Compute
$$\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4.$$

Solution *We can bound our function*
$$-1/x + 4 \le \frac{\sin(7x)}{x} + 4 \le 1/x + 4.$$
Since
$$\lim_{x\to\infty} -1/x + 4 = 4 = \lim_{x\to\infty} 1/x + 4$$
we conclude by the Squeeze Theorem, Theorem **??**, $\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4 = 4.$

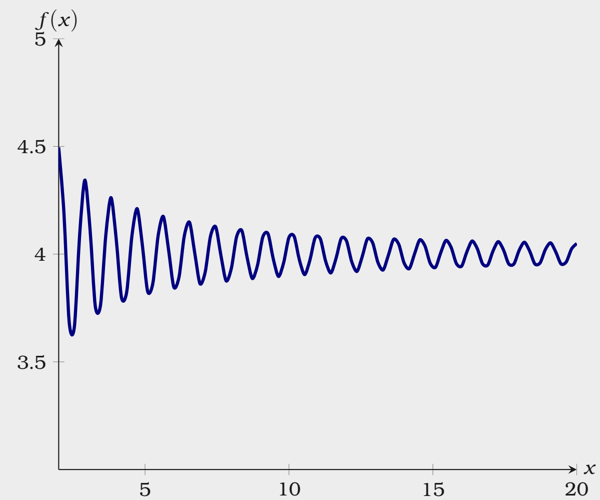


Figure 1.11: A plot of $f(x) = \frac{\sin(7x)}{x} + 4.$.

Definition If
$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$
then the line $y = L$ is a **horizontal asymptote** of $f(x)$.

Example 1.5.3 Give the horizontal asymptotes of
$$f(x) = \frac{6x-9}{x-1}$$

Solution *From our previous work, we see that* $\lim_{x\to\infty} f(x) = 6$, *and upon further inspection, we see that* $\lim_{x\to-\infty} f(x) = 6$. *Hence the horizontal asymptote of* $f(x)$ *is the line* $y = 6$.

It is a common misconception that a function cannot cross an asymptote. As the next example shows, a function can cross an asymptote, and in this case this occurs an infinite number of times!

Example 1.5.4 Give a horizontal asymptote of

$$f(x) = \frac{\sin(7x)}{x} + 4.$$

Solution *Again from previous work, we see that* $\lim_{x\to\infty} f(x) = 4$. *Hence* $y = 4$ *is a horizontal asymptote of* $f(x)$.

We conclude with an infinite limit at infinity.

Example 1.5.5 Compute

$$\lim_{x\to\infty} \ln(x)$$

Solution *The function* $\ln(x)$ *grows very slowly, and seems like it may have a horizontal asymptote, see Figure* **??**. *However, if we consider the definition of the natural log*

$$\ln(x) = y \qquad \Leftrightarrow \qquad e^y = x$$

Since we need to raise e to higher and higher values to obtain larger numbers, we see that $\ln(x)$ *is unbounded, and hence* $\lim_{x\to\infty} \ln(x) = \infty$.

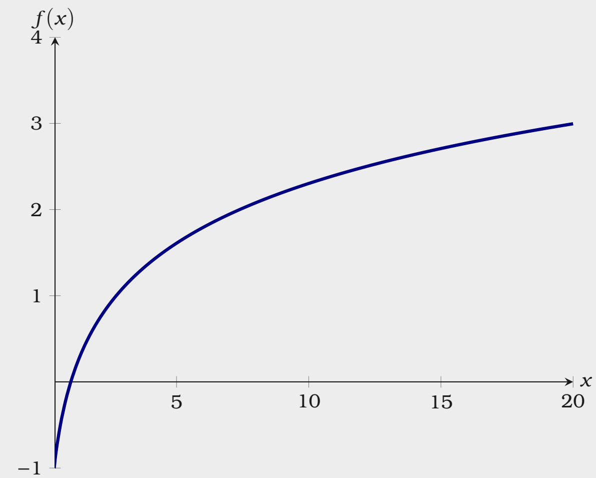


Figure 1.12: A plot of $f(x) = \ln(x)$.

Exercises for Section 1.5

FILL IN

1.6 Continuity

Informally, a function is continuous if you can "draw it" without "lifting your pencil." We need a formal definition.

Definition A function f is **continuous at a point** a if $\lim_{x \to a} f(x) = f(a)$.

Example 1.6.1 Find the discontinuities (the values for x where a function is not continuous) for the function given in Figure **??**.

Solution *From Figure* **??** *we see that* $\lim_{x \to 4} f(x)$ *does not exist as*

$$\lim_{x \to 4-} f(x) = 1 \qquad \text{and} \qquad \lim_{x \to 4+} f(x) \approx 3.5$$

Hence $\lim_{x \to 4} f(x) \neq f(4)$, *and so* $f(x)$ *is not continuous at* $x = 4$.

We also see that $\lim_{x \to 6} f(x) \approx 3$ *while* $f(6) = 2$. *Hence* $\lim_{x \to 6} f(x) \neq f(6)$, *and so* $f(x)$ *is not continuous at* $x = 6$.

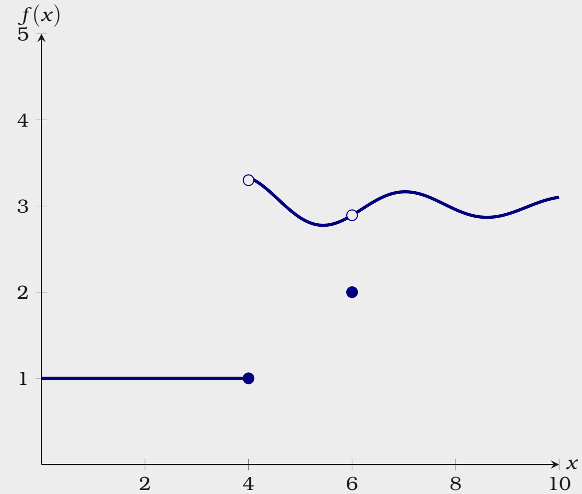


Figure 1.13: A plot of a function with discontinuities at $x = 4$ and $x = 6$.

Building from the definition of *continuous at a point*, we can now define what it means for a function to be *continuous* on an interval.

Definition A function f is **continuous on an interval** if it is continuous at every point in the interval.

In particular, we should note that if a function is not defined on an interval, then it **cannot** be continuous on that interval.

Example 1.6.2 Consider the function

$$f(x) = \begin{cases} \sqrt[5]{x} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

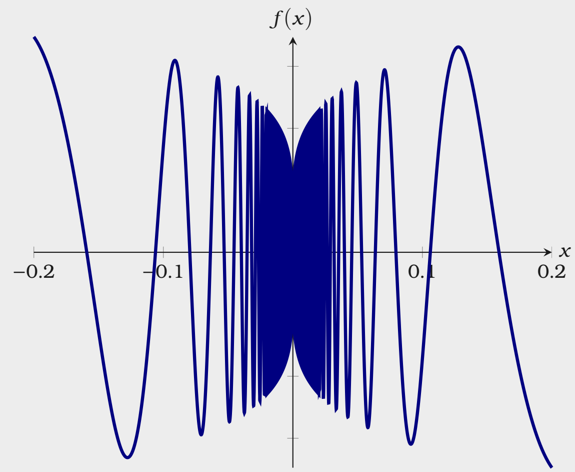


Figure 1.14: A plot of

$$f(x) = \begin{cases} \sqrt[5]{x} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

see Figure **??**. Is this function continuous?

Solution *Considering $f(x)$, the only issue is when $x = 0$. We must show that* $\lim_{x\to 0} f(x) = 0$.

PROOF

This function is continuous. Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.

We close with a useful theorem about continuous functions:

Theorem 1.6.3 (Intermediate Value Theorem) *If $f(x)$ is a function that is continuous for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*

The Intermediate Value Theorem is most frequently used when $d = 0$.

For a nice proof of this theorem, see: Walk, Stephen M. *The intermediate value theorem is NOT obvious—and I am going to prove it to you.* College Math. J. 42 (2011), no. 4, 254–259.

In Figure **??**, we see a geometric interpretation of this theorem.

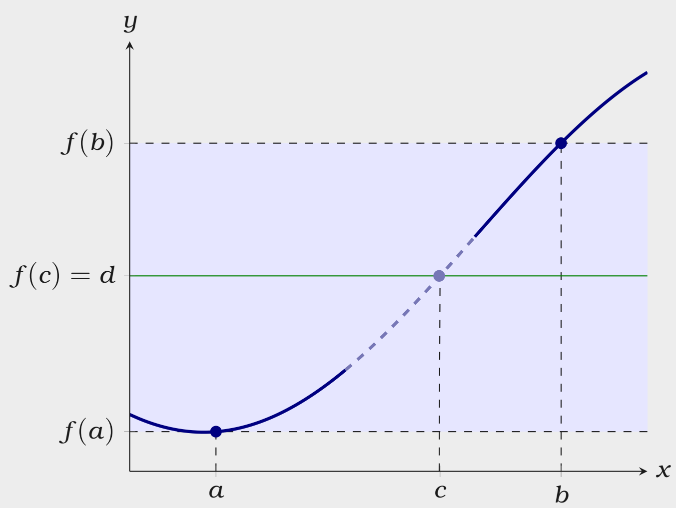


Figure 1.15: A geometric interpretation of the Intermediate Value Theorem. The function $f(x)$ is continuous on the interval $[a, b]$. Since d is in the interval $[f(a), f(b)]$, there exists a value c in $[a, b]$ such that $f(c) = d$.

Example 1.6.4 Explain why the function $f(x) = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

Solution *By Theorem* **??**, $\lim_{x \to a} f(x) = f(a)$, *for all real values of* a, *and hence* f *is continuous. Since* $f(0) = -2$ *and* $f(1) = 3$, *and* 0 *is between* -2 *and* 3, *there is a* $c \in [0, 1]$ *such that* $f(c) = 0$.

This example also points the way to a simple method for approximating roots.

Example 1.6.5 Approximate a root of $f(x) = x^3 + 3x^2 + x - 2$ to one decimal place.

Solution *If we compute* $f(0.1)$, $f(0.2)$, *and so on, we find that* $f(0.6) < 0$ *and* $f(0.7) > 0$, *so by the Intermediate Value Theorem,* f *has a root between* 0.6 *and* 0.7. *Repeating the process with* $f(0.61)$, $f(0.62)$, *and so on, we find that* $f(0.61) < 0$ *and* $f(0.62) > 0$, *so* f *has a root between* 0.61 *and* 0.62, *and the root is* 0.6 *rounded to one decimal place.*

Exercises for Section 1.6

(1) Consider the function

$$h(x) = \begin{cases} 2x - 3, & \text{if } x < 1, \\ 0, & \text{if } x \geq 1. \end{cases}$$

Show that it is continuous at the point $x = 0$. Is h a continuous function?

(2) Approximate a root of $f = x^3 - 4x^2 + 2x + 2$ to one decimal place.

(3) Approximate a root of $f = x^4 + x^3 - 5x + 1$ to one decimal place.

2 Differentiation

2.1 Slopes of Tangent Lines via Limits

Suppose that $f(x)$ is a function. It is often useful to know how sensitive the value of $f(x)$ is to small changes in x. To give you a feeling why this is true, consider the following:

- If the change is zero, then x gives a local maximal or minimal values for $f(x)$.
- If $p(t)$ determines the position of an object with respect to time, the change gives the velocity of the object.
- If $v(t)$ determines the velocity of an object with respect to time, the change gives the acceleration of the object.
- The change can help us approximate a complicated function with a simple function.
- The change can be used to help us solve equations that we would not be able to solve via other methods.

The rate of change of a function is the slope of the tangent line. Part of our goal will be to give a formal definition of a tangent line. For now, consider the following informal definition:

> Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

While this is merely an informal definition of a tangent line, it contains the essence of how the formal definition will be eventually constructed. We illustrate this informal definition with Figure **??**.

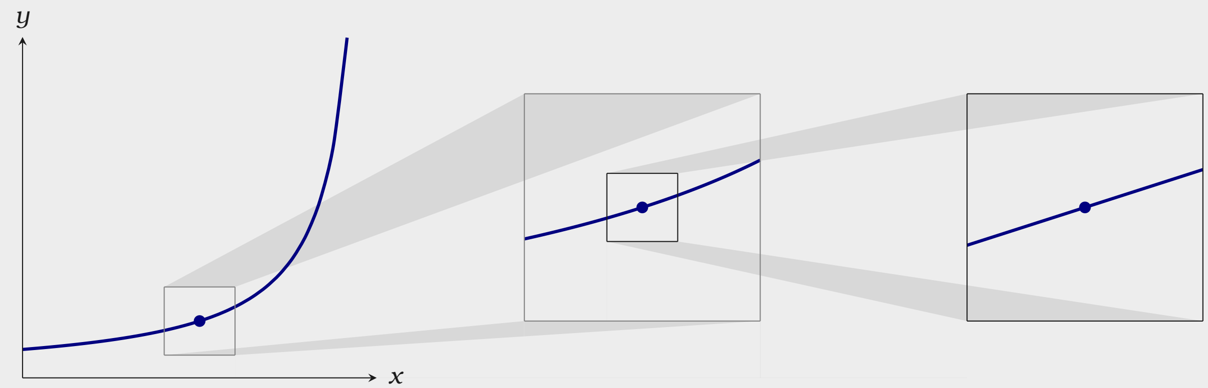


Figure 2.1: Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

Example 2.1.1 $|x|$

Example 2.1.2 Discuss the derivative of the function $y = x^{2/3}$, shown in figure **??**. We will later see how to compute this derivative; for now we use the fact that $y' = (2/3)x^{-1/3}$. Visually this looks much like the absolute value function, but it technically has a cusp, not a corner. The absolute value function has no tangent line at 0 because there are (at least) two obvious contenders—the tangent line of the left side of the curve and the tangent line of the right side. The function $y = x^{2/3}$ does not have a tangent line at 0, but unlike the absolute value function it can be said to have a single direction: as we approach 0 from either side the tangent line becomes closer and closer to a vertical line; the curve is vertical at 0. But as before, if you imagine traveling along the curve, an abrupt change in direction is required at 0: a full 180 degree turn.

differentialble => continuous

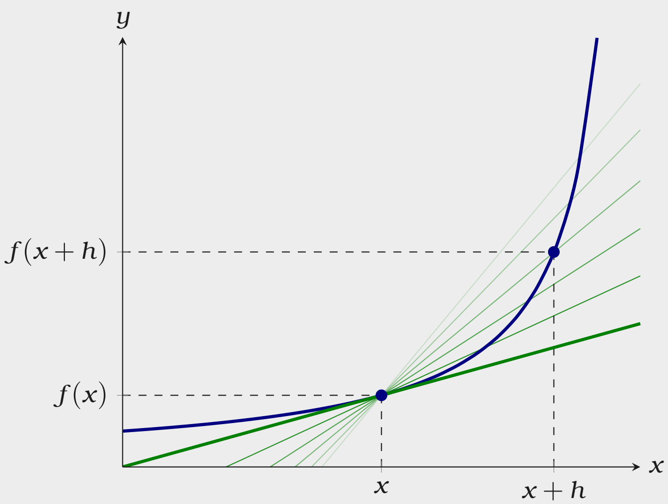


Figure 2.2: Tangent lines can be found as the limit of secant lines. The slope of the tangent line is given by $\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.

not continuous => not diff

Now some examples of of computing derivatives via limits.

Example 2.1.3 Find the derivative of $y = f(t) = t^2$.

We compute

$$\begin{aligned} y' = \lim_{h\to 0} \frac{\Delta y}{h} &= \lim_{h\to 0} \frac{(t+h)^2 - t^2}{h} \\ &= \lim_{h\to 0} \frac{t^2 + 2th + h^2 - t^2}{h} \\ &= \lim_{h\to 0} \frac{2th + h^2}{h} \\ &= \lim_{h\to 0} 2t + h = 2t. \end{aligned}$$

Example 2.1.4

Find the derivative of $y = f(x) = 1/x$.

The computation:

$$\begin{aligned} y' = \lim_{h\to 0} \frac{\Delta y}{h} &= \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{\Delta x} \\ &= \lim_{h\to 0} \frac{\frac{x}{x(x+h)} - \frac{x+h}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h} \\ &= \lim_{h\to 0} \frac{x-x-h}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-h}{x(x+h)h} \\ &= \lim_{h\to 0} \frac{-1}{x(x+h)} = \frac{-1}{x^2} \end{aligned}$$

Take, for example, $y = f(x) = \sqrt{625 - x^2}$ (the upper semicircle of radius 25 centered at the origin). When $x = 7$, we find that $y = \sqrt{625 - 49} = 24$. Suppose we want to know how much y changes when x increases a little, say to 7.1 or 7.01.

What if we try to do all the algebra without using a specific value for x? Let's copy from above, replacing 7 by x. We'll have to do a bit more than that—for example, the "24" in the calculation came from $\sqrt{625-7^2}$, so we'll need to fix that too.

$$
\begin{aligned}
&\frac{\sqrt{625-(x+h)^2}-\sqrt{625-x^2}}{h} = \\
&= \frac{\sqrt{625-(x+h)^2}-\sqrt{625-x^2}}{h}\,\frac{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}{\sqrt{625-(x+h)^2}+\sqrt{625-x^2}} \\
&= \frac{625-(x+h)^2-625+x^2}{h(\sqrt{625-(x+h)^2}+\sqrt{625-x^2})} \\
&= \frac{625-x^2-2xh-h^2-625+x^2}{h(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{h(-2x-h)}{h(\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2})} \\
&= \frac{-2x-h}{\sqrt{625-(x+\Delta x)^2}+\sqrt{625-x^2}}
\end{aligned}
$$

Now what happens when h is very close to zero? Again it seems apparent that the quotient will be very close to

$$\frac{-2x}{\sqrt{625-x^2}+\sqrt{625-x^2}} = \frac{-2x}{2\sqrt{625-x^2}} = \frac{-x}{\sqrt{625-x^2}}.$$

Replacing x by 7 gives $-7/24$, as before, and now we can easily do the computation for 12 or any other value of x between -25 and 25.

So now we have a single, simple formula, $-x/\sqrt{625-x^2}$, that tells us the slope of the tangent line for any value of x. This slope, in turn, tells us how sensitive the value of y is to changes in the value of x.

What do we call such a formula? That is, a formula with one variable, so that substituting an "input" value for the variable produces a new "output" value? This is a function. Starting with one function, $\sqrt{625-x^2}$, we have derived, by means of some slightly nasty algebra, a new function, $-x/\sqrt{625-x^2}$, that gives us important information about the original function. This new function in fact is called the **derivative** of the original function. If the original is referred to as f or y then the derivative is often written f' or y' and pronounced "f prime" or "y prime", so in this case we might write $f'(x) = -x/\sqrt{625-x^2}$. At a particular point, say $x = 7$, we say that $f'(7) = -7/24$ or "f prime of 7 is $-7/24$" or "the derivative of f at 7 is $-7/24$."

To summarize, we compute the derivative of $f(x)$ by forming the difference quotient

$$\frac{f(x+h)-f(x)}{h},$$

which is the slope of a line, then we figure out what happens when h gets very close to 0.

We should note that in the particular case of a circle, there's a simple way to find the derivative. Since the tangent to a circle at a point is perpendicular to the radius drawn to the point of contact, its slope is the negative reciprocal of the slope of the radius. The radius joining $(0,0)$ to $(7,24)$ has slope 24/7. Hence, the tangent line has slope $-7/24$. In general, a radius to the point $(x, \sqrt{625-x^2})$ has slope $\sqrt{625-x^2}/x$, so the slope of the tangent line is $-x/\sqrt{625-x^2}$, as before. It is **NOT** always true that a tangent line is perpendicular to a line from the origin—don't use this shortcut in any other circumstance.

As above, and as you might expect, for different values of x we generally get different values of the derivative $f'(x)$. Could it be that the derivative always has the same value? This would mean that the slope of f, or the slope of its tangent line, is the same everywhere. One curve that always has the same slope is a line; it seems odd to talk about the tangent line to a line, but if it makes sense at all the tangent line must be the line itself. It is not hard to see that the derivative of $f(x) = mx + b$ is $f'(x) = m$; see exercise **??** .

2.2 Basic Derivative Rules

constant rule

2.2.1 The Sum Rule

We don't want to rely on some more-or-less obvious physical interpretation to determine what is true mathematically, so let's see how to verify these rules by computation. We'll do one and leave the other for the exercises.

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{f(x+\Delta x)+g(x+h)-(f(x)+g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(x+\Delta x)+g(x+h)-f(x)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+\Delta x)-f(x)+g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x+\Delta x)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\frac{f(x+\Delta x)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x)
\end{aligned}$$

This is sometimes called the **sum rule** for derivatives.

Example 2.2.1 Find the derivative of $f(x) = x^5 + 5x^2$. We have to invoke linearity twice here:

$$f'(x) = \frac{d}{dx}(x^5+5x^2) = \frac{d}{dx}x^5 + \frac{d}{dx}(5x^2) = 5x^4 + 5\frac{d}{dx}(x^2) = 5x^4 + 5\cdot 2x^1 = 5$$

Example 2.2.2 Find the derivative of $f(x) = 3/x^4 - 2x^2 + 6x - 7$.

$$f'(x) = \frac{d}{dx}\left(\frac{3}{x^4} - 2x^2 + 6x - 7\right) = \frac{d}{dx}(3x^{-4} - 2x^2 + 6x - 7) = -12x^{-5} - 4x + 6.$$

2.2.2 The Power Rule

Now let's look at the limit:

$$\begin{aligned}
\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{(x+h)^n - x^n}{\Delta x} \\
&= \lim_{h\to 0}\frac{x^n + nx^{n-1}h + a_2x^{n-2}h^2 + \cdots + a_{n-1}xh^{n-1} + h^n - x^n}{h} \\
&= \lim_{h\to 0}\frac{nx^{n-1}h + a_2x^{n-2}h^2 + \cdots + a_{n-1}xh^{n-1} + h^n}{h} \\
&= \lim_{h\to 0} nx^{n-1} + a_2x^{n-2}h + \cdots + a_{n-1}xh^{n-2} + h^{n-1} = nx^{n-1}.
\end{aligned}$$

Now without much trouble we can verify the formula for negative integers. First let's look at an example:

Example 2.2.3 Find the derivative of $y = x^{-3}$. Using the formula, $y' = -3x^{-3-1} = -3x^{-4}$.

Here is the general computation. Suppose n is a negative integer; the algebra is easier to follow if we use $n = -m$ in the computation, where m is a positive integer.

$$\begin{aligned}
\frac{d}{dx}x^n = \frac{d}{dx}x^{-m} &= \lim_{h\to 0}\frac{(x+h)^{-m} - x^{-m}}{\Delta x} \\
&= \lim_{h\to 0}\frac{\frac{1}{(x+h)^m} - \frac{1}{x^m}}{h} \\
&= \lim_{h\to 0}\frac{x^m - (x+h)^m}{(x+h)^m x^m h} \\
&= \lim_{h\to 0}\frac{x^m - (x^m + mx^{m-1}h + a_2x^{m-2}h^2 + \cdots + a_{m-1}xh^{m-1} + h^m)}{(x+h)^m x^m h} \\
&= \lim_{h\to 0}\frac{-mx^{m-1} - a_2x^{m-2}h - \cdots - a_{m-1}xh^{m-2} - h^{m-1})}{(x+h)^m x^m} \\
&= \frac{-mx^{m-1}}{x^m x^m} = \frac{-mx^{m-1}}{x^{2m}} = -mx^{m-1-2m} = nx^{-m-1} = nx^{n-1}.
\end{aligned}$$

We will later see why the other cases of the power rule work, but from now on we will use the power rule whenever n is any real number. Let's note here a simple case in which the power rule applies, or almost applies, but is not really needed. Suppose that $f(x) = 1$; remember that this "1" is a function, not "merely" a number, and that $f(x) = 1$ has a graph that is a horizontal line, with slope zero everywhere. So we know that $f'(x) = 0$. We might also write $f(x) = x^0$, though there is some question about just what this means at $x = 0$. If we apply the power rule, we get $f'(x) = 0x^{-1} = 0/x = 0$, again noting that there is a problem at $x = 0$. So the power rule "works" in this case, but it's really best to just remember that the derivative of any constant function is zero.

exponent rule

dfn e^x

Exercises for Section 2.2

Find the derivatives of the given functions.

(1) x^{100}

(2) x^{-100}

(3) $\frac{1}{x^5}$

(4) x^{π}

(5) $x^{3/4}$

(6) $x^{-9/7}$

Find the derivatives of the functions in 1–6.

(7) $5x^3 + 12x^2 - 15$

(8) $-4x^5 + 3x^2 - 5/x^2$

(9) $5(-3x^2 + 5x + 1)$

(10) $f(x) + g(x)$, where $f(x) = x^2 - 3x + 2$ and $g(x) = 2x^3 - 5x$

(11) $(x+1)(x^2 + 2x - 3)$

(12) $\sqrt{625 - x^2} + 3x^3 + 12$ (See section **??**.)

(13) Find an equation for the tangent line to $f(x) = x^3/4 - 1/x$ at $x = -2$.

(14) Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$.

(15) Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the speed of the object at time t. The acceleration of an object is the rate at which its speed is changing, which means it is given by the derivative of the speed function. Find the acceleration of the object at time t.

(16) Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of f, cf, f', and $(cf)'$ on the same diagram.

(17) The general polynomial P of degree n in the variable x has the form $P(x) = \sum_{k=0}^{n} a_k x^k = a_0 + a_1 x + \ldots + a_n x^n$. What is the derivative (with respect to x) of P?

(18) Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$.

(19) Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

(20) Suppose that f and g are differentiable at x. Show that $f - g$ is differentiable at x using the two linearity properties from this section.

2.3 The Product Rule and Quotient Rule

2.4 The Derivative of Trigonometric Functions

2.5 The Chain Rule

2.6 Rates of Change

2.7 Implicit Differentiation

2.8 Applications

2.8.1 Related Rates

Answers to selected exercises