MOOCULUS
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CALCULUS

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This text is based on David Guichard's open-source calculus text which in turn is a modification and expansion of notes written by Neal Koblitz at the University of Washington. David Guichard's text is available at http://www.whitman.edu/mathematics/calculus/ under a Creative Commons license.

The book includes some exercises and examples from *Elementary Calculus: An Approach Using Infinitesimals*, by H. Jerome Keisler, available at http://www.math.wisc.edu/~keisler/calc.html under a Creative Commons license. In addition, the chapter on differential equations and the section on numerical integration are largely derived from the corresponding portions of Keisler's book. Albert Schueller, Barry Balof, and Mike Wills have contributed additional material.

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We will be glad to receive corrections and suggestions for improvement at fowler@math.osu.edu or snapp@math.osu.edu.

Contents

List of Main Theorems

How to Read Mathematics

Reading mathematics is **not** the same as reading a novel. To read mathematics you need:

(a) A pen.

(b) Plenty of blank paper.

(c) A willingness to write things down.

As you read mathematics, you must work along side of the text itself. You must **write** down each expression, **sketch** each graph, and **think** about what you are doing. You should work examples and fill-in the details. This is not an easy task, it is in fact **hard** work. However, mathematics is not a passive endeavor. You, the reader, must become a doer of mathematics.

1 Limits

1.1 The Basic Ideas of Limits

Consider the function:

$$f(x) = \frac{x^2 - 3x + 2}{x - 2}$$

While $f(x)$ is undefined at $x = 2$, we can still plot $f(x)$ at other values, see Figure 1.1. Examining Table 1.1, we see that as x approaches 2, $f(x)$ approaches 1. We write this:

$$\text{As } x \to 2, f(x) \to 1 \qquad \text{or} \qquad \lim_{x\to 2} f(x) = 1.$$

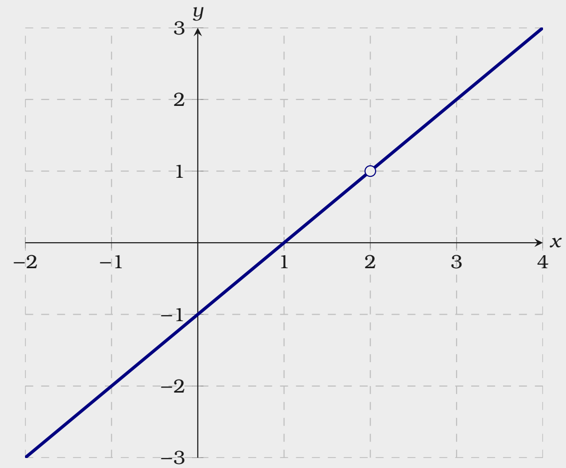


Figure 1.1: A plot of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

x	$f(x)$
1.7	0.7
1.9	0.9
1.99	0.99
1.999	0.999
2	undefined

x	$f(x)$
2	undefined
2.001	1.001
2.01	1.01
2.1	1.1
2.3	1.3

Table 1.1: Values of $f(x) = \dfrac{x^2 - 3x + 2}{x - 2}$.

Intuitively, $\lim_{x\to a} f(x) = L$ when the value of $f(x)$ can be made arbitrarily close to L by making x sufficiently close, but not equal to, a. This leads us to the formal definition of a *limit*.

Definition The **limit** of $f(x)$ as x goes to a is L,

$$\lim_{x\to a} f(x) = L,$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever

$$0 < |x - a| < \delta, \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

If no such value of L can be found, then we say that $\lim_{x\to a} f(x)$ **does not exist**.

Equivalently, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x \neq a$ and $a - \delta < x < a + \delta$, we have $L - \varepsilon < f(x) < L + \varepsilon$.

In Figure 1.2, we see a geometric interpretation of this definition.

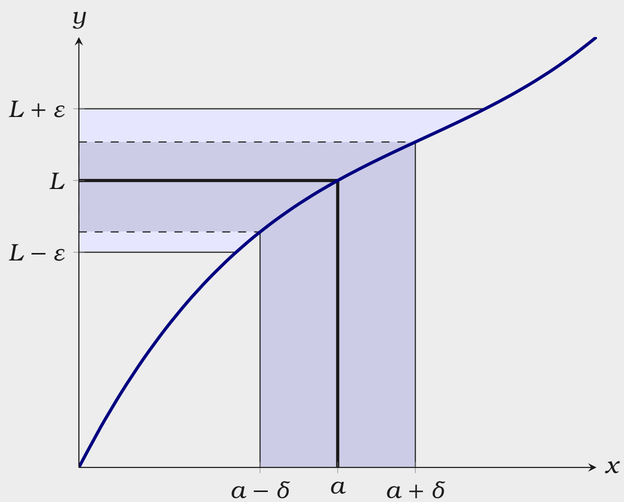


Figure 1.2: A geometric interpretation of the (ε, δ)-criterion for limits. If $0 < |x - a| < \delta$, then we have that $a - \delta < x < a + \delta$. In our diagram, we see that for all such x we are sure to have $L - \varepsilon < f(x) < L + \varepsilon$, and hence $|f(x) - L| < \varepsilon$.

Limits need not exist, let's examine two cases of this.

Example 1.1.1 Let $f(x) = \lfloor x \rfloor$. Explain why the limit

$$\lim_{x \to 2} f(x)$$

does not exist.

Solution *The function $\lfloor x \rfloor$ is the function that returns the greatest integer less than or equal to x. Since $f(x)$ is defined for all real numbers, one might be tempted to think that the limit above is simply $f(2) = 2$. However, this is not the case. If $x < 2$, then $f(x) = 1$. Hence if $\varepsilon = .5$, we can* ***always*** *find a value for x (just to the left of 2) such that*

$$0 < |x - 2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x) - 2|.$$

On the other hand, $\lim_{x \to 2} f(x) \neq 1$, as in this case if $\varepsilon = .5$, we can ***always*** *find a value for x (just to the right of 2) such that*

$$0 < |x - 2| < \delta, \qquad \text{where} \qquad \varepsilon < |f(x) - 1|.$$

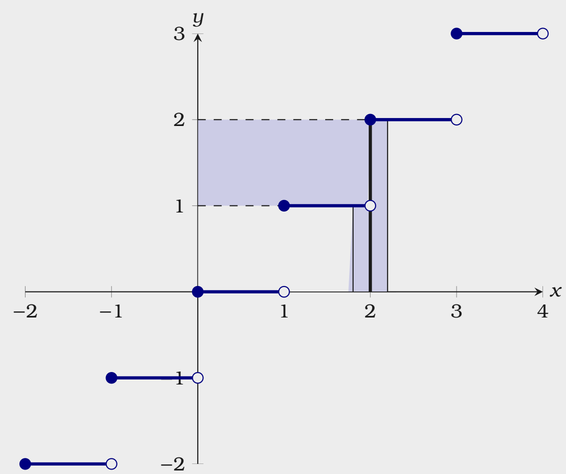


Figure 1.3: A plot of $f(x) = \lfloor x \rfloor$. Note, no matter which $\delta > 0$ is chosen, we can only at best bound $f(x)$ in the interval $[1, 2]$.

With the example of $f(x) = \lfloor x \rfloor$, we see that taking limits is truly different from evaluating functions.

We've illustrated this in Figure 1.3. Moreover, no matter what value one chooses for $\lim_{x\to 2} f(x)$, we will always have a similar issue.

Limits may not exist even if the formula for the function looks innocent.

Example 1.1.2 Let $f(x) = \sin\left(\frac{1}{x}\right)$. Explain why the limit

$$\lim_{x\to 0} f(x)$$

does not exist.

Solution *In this case $f(x)$ oscillates "wildly" as x approaches 0, see Figure 1.4. In fact, one can show that for any given δ, There is a value for x in the interval*

$$0 - \delta < x < 0 + \delta$$

such that $f(x)$ is ***any*** *value in the interval $[-1, 1]$. Hence the limit does not exist.*

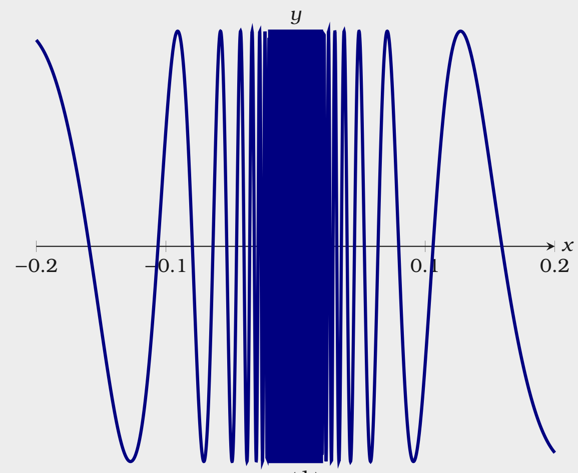


Figure 1.4: A plot of $f(x) = \sin\left(\frac{1}{x}\right)$.

Sometimes the limit of a function exists from one side or the other (or both) even though the limit does not exist. Since it is useful to be able to talk about this situation, we introduce the concept of a *one-sided limit*:

Definition We say that the **limit** of $f(x)$ as x goes to a from the **left** is L,

$$\lim_{x\to a-} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x < a$ and

$$a - \delta < x \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

We say that the **limit** of $f(x)$ as x goes to a from the **right** is L,

$$\lim_{x\to a+} f(x) = L$$

if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $x > a$ and

$$x < a + \delta \qquad \text{we have} \qquad |f(x) - L| < \varepsilon.$$

Limits from the left, or from the right, are collectively called **one-sided limits**.

Example 1.1.3 Let $f(x) = \lfloor x \rfloor$. Discuss

$$\lim_{x\to 2-} f(x), \qquad \lim_{x\to 2+} f(x), \qquad \text{and} \qquad \lim_{x\to 2} f(x).$$

Solution *From the plot of $f(x)$, see Figure 1.3, we see that*

$$\lim_{x\to 2-} f(x) = 1, \qquad \textit{and} \quad \lim_{x\to 2+} f(x) = 2.$$

Since these limits are different, $\lim_{x\to 2} f(x)$ does not exist.

Exercises for Section 1.1

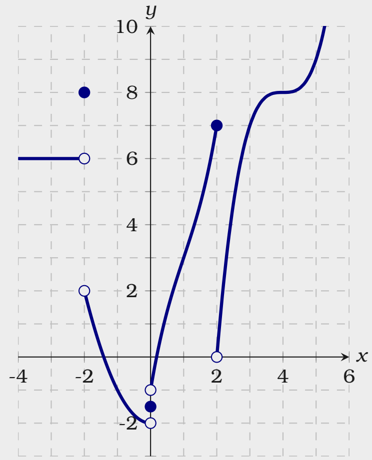


Figure 1.5: A plot of $f(x)$, a piecewise defined function.

(1) Evaluate the expressions by reference to the plot in Figure 1.5.

(a) $\lim\limits_{x\to 4} f(x)$ (e) $\lim\limits_{x\to 0+} f(x)$ (i) $\lim\limits_{x\to 0} f(x+1)$

(b) $\lim\limits_{x\to -3} f(x)$ (f) $f(-2)$ (j) $f(0)$

(c) $\lim\limits_{x\to 0} f(x)$ (g) $\lim\limits_{x\to 2-} f(x)$ (k) $\lim\limits_{x\to 1-} f(x-4)$

(d) $\lim\limits_{x\to 0-} f(x)$ (h) $\lim\limits_{x\to -2-} f(x)$ (l) $\lim\limits_{x\to 0+} f(x-2)$

(2) Use a table and a calculator to estimate $\lim\limits_{x\to 0} \dfrac{\sin(x)}{x}$.

(3) Use a table and a calculator to estimate $\lim\limits_{x\to 0} \dfrac{\sin(2x)}{x}$.

(4) Use a table and a calculator to estimate $\lim\limits_{x\to 0} \dfrac{x}{\sin\left(\frac{x}{3}\right)}$.

(5) Use a table and a calculator to estimate $\lim\limits_{x\to 0} \dfrac{\tan(3x)}{\tan(5x)}$.

(6) Use a table and a calculator to estimate $\lim\limits_{x\to 0} \dfrac{2^x - 1}{x}$.

(7) Use a table and a calculator to estimate $\lim\limits_{x\to 0} (1+x)^{1/x}$.

(8) Sketch a plot of $f(x) = \dfrac{x}{|x|}$ and explain why $\lim\limits_{x\to 0} \dfrac{x}{|x|}$ does not exist.

(9) Let $f(x) = \sin\left(\dfrac{\pi}{x}\right)$. Construct three tables of the following form

x	$f(x)$
$0.d$	
$0.0d$	
$0.00d$	
$0.000d$	

where $d = 1, 3, 7$. What do you notice? How do you reconcile the entries in your tables with the value of $\lim_{x\to 0} f(x)$?

(10) In the theory of special relativity, a moving clock ticks slower than a stationary observer's clock. If the stationary observer records that t_s seconds have passed, then the clock moving at velocity v has recorded that

$$t_v = t_s \sqrt{1 - v^2/c^2}$$

seconds have passed, where c is the speed of light. What happens as $v \to c$ from below?

1.2 Limits by the Definition

Now we are going to get our hands dirty, and really use the definition of a limit.

Recall, $\lim_{x\to a} f(x) = L$, if for every $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$.

Example 1.2.1 Show that $\lim_{x\to 2} x^2 = 4$.

Solution *We want to show that for any given $\varepsilon > 0$, we can find a $\delta > 0$ such that*

$$|x^2 - 4| < \varepsilon$$

whenever $0 < |x - 2| < \delta$. Start by factoring the left-hand side of the inequality above

$$|x + 2||x - 2| < \varepsilon.$$

Since we are going to assume that $0 < |x - 2| < \delta$, we will focus on the factor $|x + 2|$. Since x is assumed to be close to 2, suppose that $x \in [1, 3]$. In this case

$$|x + 2| \leq 3 + 2 = 5,$$

and so we want

$$5 \cdot |x - 2| < \varepsilon$$
$$|x - 2| < \frac{\varepsilon}{5}$$

Recall, we assumed that $x \in [1, 3]$, which is equivalent to $|x - 2| < 1$. Hence we must set $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

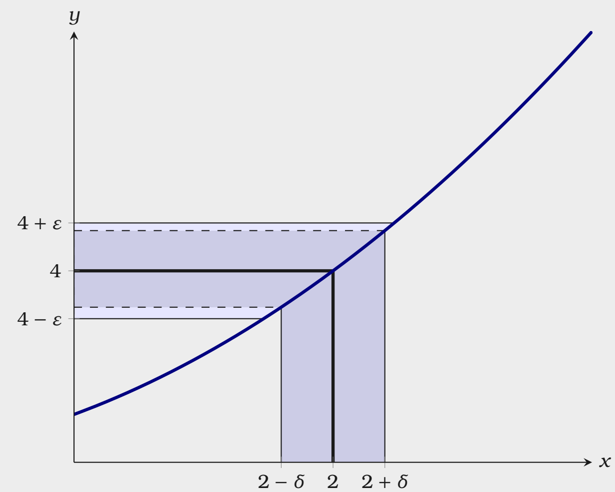


Figure 1.6: The (ε, δ)-criterion for $\lim_{x\to 2} x^2 = 4$. Here $\delta = \min\left(\frac{\varepsilon}{5}, 1\right)$.

When dealing with limits of polynomials, the general strategy is always the same. Let $p(x)$ be a polynomial. If showing

$$\lim_{x\to a} p(x) = L,$$

one must first factor out $|x - a|$ from $|p(x) - L|$. Next bound $x \in [a - 1, a + 1]$ and estimate the largest possible value of

$$\left|\frac{p(x) - L}{x - a}\right|$$

for $x \in [a-1, a+1]$. Call this estimation M. Finally, one must set $\delta = \min\left(\frac{\varepsilon}{M}, 1\right)$.

As you work with limits, you find that you need to do the same procedures again and again. The next theorems will expedite this process.

Theorem 1.2.2 (Limit Product Law) *Suppose* $\lim_{x\to a} f(x) = L$ *and* $\lim_{x\to a} g(x) = M$. *Then*

$$\lim_{x\to a} f(x)g(x) = LM.$$

Proof *Given any* ε *we need to find a* δ *such that*

$$0 < |x-a| < \delta$$

implies

$$|f(x)g(x) - LM| < \varepsilon.$$

Here we use an algebraic trick, add $0 = -f(x)M + f(x)M$*:*

$$\begin{aligned} |f(x)g(x) - LM| &= |f(x)g(x) - f(x)M + f(x)M - LM| \\ &= |f(x)(g(x) - M) + (f(x) - L)M| \\ &\leq |f(x)(g(x) - M)| + |(f(x) - L)M| \\ &= |f(x)||g(x) - M| + |f(x) - L||M|. \end{aligned}$$

Since $\lim_{x\to a} f(x) = L$, *there is a value* δ_1 *so that* $0 < |x-a| < \delta_1$ *implies* $|f(x) - L| < |\varepsilon/(2M)|$. *This means that* $0 < |x-a| < \delta_1$ *implies* $|f(x) - L||M| < \varepsilon/2$.

$$|f(x)g(x) - LM| \leq |f(x)||g(x) - M| + \underbrace{|f(x) - L||M|}_{<\frac{\varepsilon}{2}}.$$

If we can make $|f(x)||g(x) - M| < \varepsilon/2$, *then we'll be done. We can make* $|g(x) - M|$ *smaller than any fixed number by making* x *close enough to* a. *Unfortunately,* $\varepsilon/(2f(x))$ *is not a fixed number since* x *is a variable.*

We will use this same trick again of "adding 0" in the proof of Theorem 4.1.1.

This is all straightforward except perhaps for the "$\leq$". This follows from the *Triangle Inequality*. The **Triangle Inequality** states: If a and b are any real numbers then $|a+b| \leq |a| + |b|$.

Here we need another trick. We can find a δ_2 so that $|x-a| < \delta_2$ implies that $|f(x)-L| < 1$, meaning that $L-1 < f(x) < L+1$. This means that $|f(x)| < N$,

where N is either $|L-1|$ or $|L+1|$, depending on whether L is negative or positive. The important point is that N doesn't depend on x. Finally, we know that there is a δ_3 so that $0 < |x-a| < \delta_3$ implies $|g(x)-M| < \varepsilon/(2N)$. Now we're ready to put everything together. Let δ be the smallest of δ_1, δ_2, and δ_3. Then $|x-a| < \delta$ implies that

$$|f(x)g(x) - LM| \le \underbrace{|f(x)|}_{<N}\underbrace{|g(x)-M|}_{<\frac{\varepsilon}{2N}} + \underbrace{|f(x)-L||M|}_{<\frac{\varepsilon}{2}}.$$

so

$$\begin{aligned} |f(x)g(x) - LM| &\le |f(x)||g(x)-M| + |f(x)-L||M| \\ &< N\frac{\varepsilon}{2N} + \left|\frac{\varepsilon}{2M}\right||M| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This is just what we needed, so by the definition of a limit, $\lim_{x\to a} f(x)g(x) = LM$.

Another useful way to put functions together is composition. If $f(x)$ and $g(x)$ are functions, we can form two functions by composition: $f(g(x))$ and $g(f(x))$. For example, if $f(x) = \sqrt{x}$ and $g(x) = x^2+5$, then $f(g(x)) = \sqrt{x^2+5}$ and $g(f(x)) = (\sqrt{x})^2 + 5 = x+5$. This brings us to our next theorem.

Theorem 1.2.3 (Limit Composition Law) *Suppose that $\lim_{x\to a} g(x) = M$ and $\lim_{x\to M} f(x) = f(M)$. Then*

$$\lim_{x\to a} f(g(x)) = f(M).$$

This is sometimes written as

$$\lim_{x\to a} f(g(x)) = \lim_{g(x)\to M} f(g(x)).$$

Note the special form of the condition on $f(x)$: it is not enough to know that $\lim_{x\to L} f(x)$ exists, though it is a bit tricky to see why. Consider

$$f(x) = \begin{cases} 3 & \text{if } x = 2, \\ 4 & \text{if } x \neq 2. \end{cases}$$

and $g(x) = 2$. Now the conditions of Theorem 1.2.3 are not satisfied, and

$$\lim_{x\to 1} f(g(x)) = 3 \qquad \textit{but} \qquad \lim_{x\to 2} f(x) = 4.$$

Many of the most familiar functions do satisfy the conditions of Theorem 1.2.3. For example:

Theorem 1.2.4 (Limit Root Law) *Suppose that n is a positive integer. Then*

$$\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a},$$

provided that a is positive if n is even.

This theorem is not too difficult to prove from the definition of limit.

Exercises for Section 1.2

(1) For each of the following limits, $\lim_{x\to a} f(x) = L$, use a graphing device to find δ such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \varepsilon$ where $\varepsilon = .1$.

(a) $\lim_{x\to 2}(3x + 1) = 7$

(b) $\lim_{x\to 1}(x^2 + 2) = 3$

(c) $\lim_{x\to \pi} \sin(x) = 0$

(d) $\lim_{x\to 0} \tan(x) = 0$

(e) $\lim_{x\to 1} \sqrt{3x+1} = 2$

(f) $\lim_{x\to -2} \sqrt{1-4x} = 3$

The next set of exercises are for advanced students and can be skipped on first reading.

(2) Use the definition of limits to explain why $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0$. Hint: Use the fact that $|\sin(a)| < 1$ for any real number a.

(3) Use the definition of limits to explain why $\lim_{x\to 4}(2x - 5) = 3$.

(4) Use the definition of limits to explain why $\lim_{x\to -3}(-4x - 11) = 1$.

(5) Use the definition of limits to explain why $\lim_{x\to -2} \pi = \pi$.

(6) Use the definition of limits to explain why $\lim_{x\to -2} \frac{x^2 - 4}{x + 2} = -4$.

(7) Use the definition of limits to explain why $\lim_{x\to 4} x^3 = 64$.

(8) Use the definition of limits to explain why $\lim_{x\to 1}(x^2 + 3x - 1) = 3$.

(9) Use the definition of limits to explain why $\lim_{x\to 9} \frac{x - 9}{\sqrt{x} - 3} = 6$.

(10) Use the definition of limits to explain why $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

1.3 Limit Laws

In this section, we present a handful of tools to compute many limits without explicitly working with the definition of limit. Each of these could be proved directly as we did in the previous section.

Theorem 1.3.1 (Limit Laws) *Suppose that $\lim_{x\to a} f(x) = L$, $\lim_{x\to a} g(x) = M$, k is some constant, and n is a positive integer.*

Constant Law $\lim_{x\to a} kf(x) = k \lim_{x\to a} f(x) = kL$.

Sum Law $\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L + M$.

Product Law $\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x) = LM$.

Quotient Law $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L}{M}$, *if* $M \neq 0$.

Power Law $\lim_{x\to a} f(x)^n = \left(\lim_{x\to a} f(x)\right)^n = L^n$.

Root Law $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$ *provided if n is even, then $f(x) \geq 0$ near a.*

Composition Law *If $\lim_{x\to a} g(x) = M$ and $\lim_{x\to M} f(x) = f(M)$, then $\lim_{x\to a} f(g(x)) = f(M)$.*

Roughly speaking, these rules say that to compute the limit of an algebraic expression, it is enough to compute the limits of the “innermost bits” and then combine these limits. This often means that it is possible to simply plug in a value for the variable, since $\lim_{x\to a} x = a$.

Example 1.3.2 Compute $\lim_{x\to 1} \dfrac{x^2 - 3x + 5}{x - 2}$.

Solution *Using limit laws,*

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2-3x+5}{x-2} &= \frac{\lim_{x\to 1} x^2-3x+5}{\lim_{x\to 1}(x-2)} \\
&= \frac{\lim_{x\to 1} x^2-\lim_{x\to 1} 3x+\lim_{x\to 1} 5}{\lim_{x\to 1} x-\lim_{x\to 1} 2} \\
&= \frac{(\lim_{x\to 1} x)^2-3\lim_{x\to 1} x+5}{\lim_{x\to 1} x-2} \\
&= \frac{1^2-3\cdot 1+5}{1-2} \\
&= \frac{1-3+5}{-1} = -3.
\end{aligned}$$

It is worth commenting on the trivial limit $\lim_{x\to 1} 5$. From one point of view this might seem meaningless, as the number 5 can't "approach" any value, since it is simply a fixed number. But 5 can, and should, be interpreted here as the function that has value 5 everywhere, $f(x)=5$, with graph a horizontal line. From this point of view it makes sense to ask what happens to the height of the function as x approaches 1.

We're primarily interested in limits that aren't so easy, namely limits in which a denominator approaches zero. The basic idea is to "divide out" by the offending factor. This is often easier said than done—here we give two examples of algebraic tricks that work on many of these limits.

Example 1.3.3 Compute $\lim_{x\to 1}\dfrac{x^2+2x-3}{x-1}$.

Solution *We can't simply plug in $x=1$ because that makes the denominator zero. However, when taking limits we assume $x\neq 1$:*

$$\begin{aligned}
\lim_{x\to 1}\frac{x^2+2x-3}{x-1} &= \lim_{x\to 1}\frac{(x-1)(x+3)}{x-1} \\
&= \lim_{x\to 1}(x+3) = 4
\end{aligned}$$

Limits allow us to examine functions where they are not defined.

Example 1.3.4 Compute $\lim\limits_{x\to -1} \dfrac{\sqrt{x+5}-2}{x+1}$.

Solution *Using limit laws,*

$$\begin{aligned}\lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1} &= \lim_{x\to -1} \frac{\sqrt{x+5}-2}{x+1}\,\frac{\sqrt{x+5}+2}{\sqrt{x+5}+2} \\ &= \lim_{x\to -1} \frac{x+5-4}{(x+1)(\sqrt{x+5}+2)} \\ &= \lim_{x\to -1} \frac{x+1}{(x+1)(\sqrt{x+5}+2)} \\ &= \lim_{x\to -1} \frac{1}{\sqrt{x+5}+2} = \frac{1}{4}.\end{aligned}$$

Here we are rationalizing the numerator by multiplying by the conjugate.

We'll conclude with one more theorem that will allow us to compute more difficult limits.

Theorem 1.3.5 (Squeeze Theorem) *Suppose that $g(x) \le f(x) \le h(x)$ for all x close to a but not necessarily equal to a. If*

$$\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x),$$

then $\lim\limits_{x\to a} f(x) = L$.

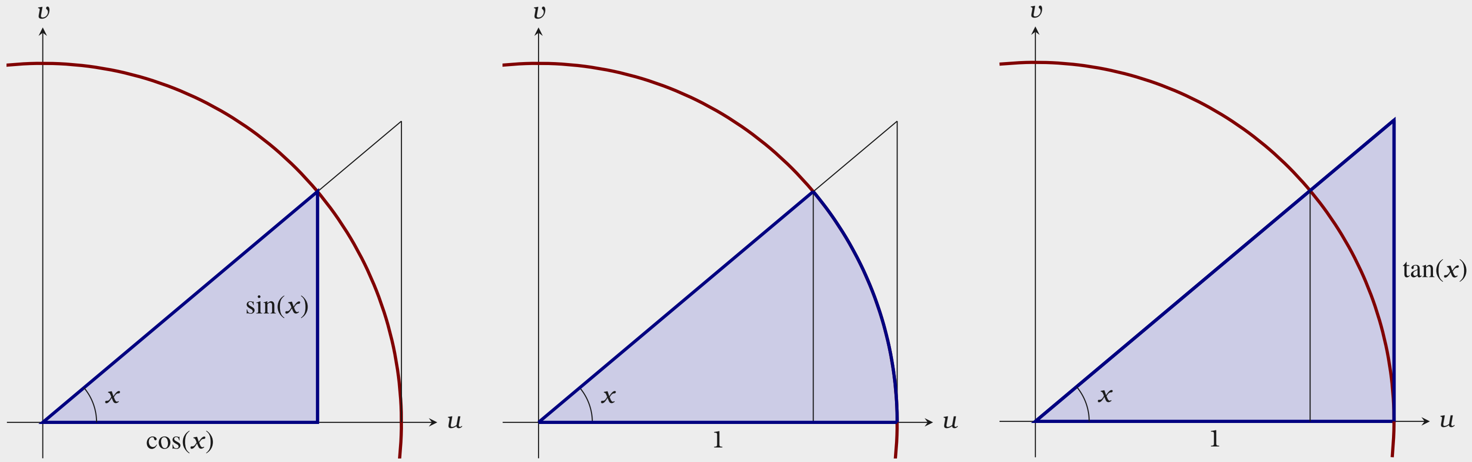
Example 1.3.6 Compute

$$\lim_{x\to 0} \frac{\sin(x)}{x}.$$

For a nice discussion of this limit, see: Richman, Fred. *A circular argument*. College Math. J. 24 (1993), no. 2, 160–162.

The limit in this example will be used in Theorem 6.1.1, and we will give another derivation of this limit in Example 7.1.2.

Solution *To compute this limit, use the Squeeze Theorem, Theorem 1.3.5. First note that we only need to examine* $x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ *and for the present time, we'll assume that* x *is positive—consider the diagrams below:*



Triangle A *Sector* *Triangle B*

From our diagrams above we see that

$$\text{Area of Triangle A} \leq \text{Area of Sector} \leq \text{Area of Triangle B}$$

and computing these areas we find

$$\frac{\cos(x)\sin(x)}{2} \leq \left(\frac{x}{2\pi}\right) \cdot \pi \leq \frac{\tan(x)}{2}.$$

Multiplying through by 2*, and recalling that* $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$ *we obtain*

$$\cos(x)\sin(x) \leq x \leq \frac{\sin(x)}{\cos(x)}.$$

Dividing through by $\sin(x)$ *and taking the reciprocals, we find*

$$\cos(x) \leq \frac{\sin(x)}{x} \leq \frac{1}{\cos(x)}.$$

Note, $\cos(-x) = \cos(x)$ *and* $\dfrac{\sin(-x)}{-x} = \dfrac{\sin(x)}{x}$*, so these inequalities hold for all* $x \in \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$*. Additionally, we know*

$$\lim_{x\to 0} \cos(x) = 1 = \lim_{x\to 0} \frac{1}{\cos(x)},$$

and so we conclude by the Squeeze Theorem, Theorem 1.3.5, $\lim_{x\to 0} \dfrac{\sin(x)}{x} = 1$.

Exercises for Section 1.3

Compute the limits. If a limit does not exist, explain why.

(1) $\lim_{x\to 3} \dfrac{x^2+x-12}{x-3}$

(2) $\lim_{x\to 1} \dfrac{x^2+x-12}{x-3}$

(3) $\lim_{x\to -4} \dfrac{x^2+x-12}{x-3}$

(4) $\lim_{x\to 2} \dfrac{x^2+x-12}{x-2}$

(5) $\lim_{x\to 1} \dfrac{\sqrt{x+8}-3}{x-1}$

(6) $\lim_{x\to 0+} \sqrt{\dfrac{1}{x}+2} - \sqrt{\dfrac{1}{x}}$

(7) $\lim_{x\to 2} 3$

(8) $\lim_{x\to 4} 3x^3-5x$

(9) $\lim_{x\to 0} \dfrac{4x-5x^2}{x-1}$

(10) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

(11) $\lim_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x}$

(12) $\lim_{x\to 0+} \dfrac{\sqrt{2-x^2}}{x+1}$

(13) $\lim_{x\to a} \dfrac{x^3-a^3}{x-a}$

(14) $\lim_{x\to 2} (x^2+4)^3$

(15) $\lim_{x\to 1} \begin{cases} x-5 & \text{if } x \neq 1, \\ 7 & \text{if } x = 1. \end{cases}$

1.4 Infinite Limits

Consider the function

$$f(x) = \frac{1}{(x+1)^2}$$

While the $\lim_{x \to -1} f(x)$ does not exist, see Figure 1.7, something can still be said.

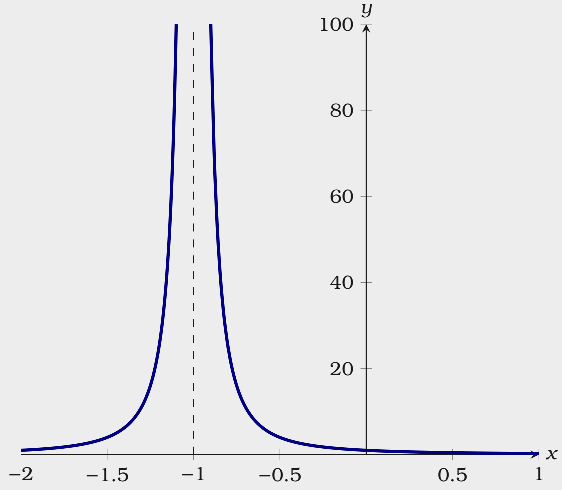


Figure 1.7: A plot of $f(x) = \dfrac{1}{(x+1)^2}$.

Definition If $f(x)$ grows arbitrarily large as x approaches a, we write

$$\lim_{x \to a} f(x) = \infty$$

and say that the limit of $f(x)$ **approaches infinity** as x goes to a.

If $|f(x)|$ grows arbitrarily large as x approaches a and $f(x)$ is negative, we write

$$\lim_{x \to a} f(x) = -\infty$$

and say that the limit of $f(x)$ **approaches negative infinity** as x goes to a.

On the other hand, if we consider the function

$$f(x) = \frac{1}{(x-1)}$$

While we have $\lim_{x \to 1} f(x) \neq \pm\infty$, we do have one-sided limits, $\lim_{x \to 1+} f(x) = \infty$ and $\lim_{x \to 1-} f(x) = -\infty$, see Figure 1.8.

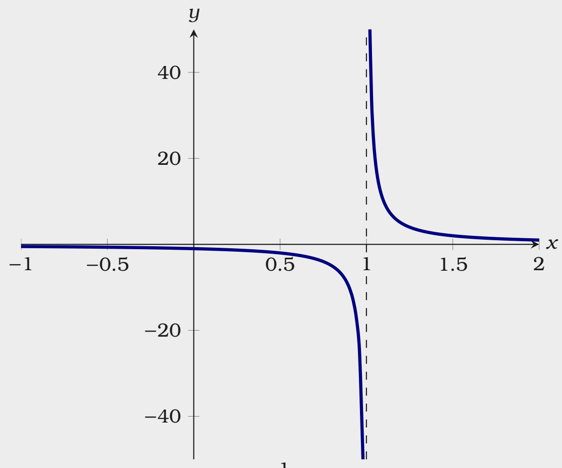


Figure 1.8: A plot of $f(x) = \dfrac{1}{x-1}$.

Definition If

$$\lim_{x \to a} f(x) = \pm\infty, \qquad \lim_{x \to a+} f(x) = \pm\infty, \qquad \text{or} \qquad \lim_{x \to a-} f(x) = \pm\infty,$$

then the line $x = a$ is a **vertical asymptote** of $f(x)$.

Example 1.4.1 Find the vertical asymptotes of

$$f(x) = \frac{x^2 - 9x + 14}{x^2 - 5x + 6}.$$

Solution *Start by factoring both the numerator and the denominator:*

$$\frac{x^2 - 9x + 14}{x^2 - 5x + 6} = \frac{(x-2)(x-7)}{(x-2)(x-3)}$$

Using limits, we must investigate when $x \to 2$ *and* $x \to 3$. *Write*

$$\begin{aligned}\lim_{x\to 2} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 2} \frac{(x-7)}{(x-3)}\\ &= \frac{-5}{-1}\\ &= 5.\end{aligned}$$

Now write

$$\begin{aligned}\lim_{x\to 3} \frac{(x-2)(x-7)}{(x-2)(x-3)} &= \lim_{x\to 3} \frac{(x-7)}{(x-3)}\\ &= \lim_{x\to 3} \frac{-4}{x-3}.\end{aligned}$$

Since $\lim_{x\to 3+} x - 3$ *approaches* 0 *from the right and the numerator is negative,* $\lim_{x\to 3+} f(x) = -\infty$. *Since* $\lim_{x\to 3-} x - 3$ *approaches* 0 *from the left and the numerator is negative,* $\lim_{x\to 3-} f(x) = \infty$. *Hence we have a vertical asymptote at* $x = 3$, *see Figure 1.9.*

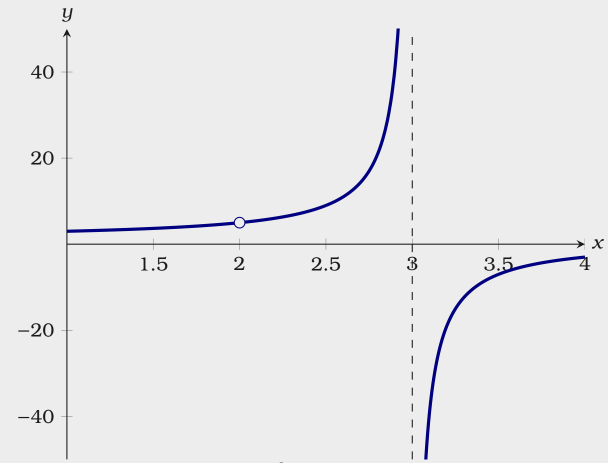


Figure 1.9: A plot of $f(x) = \dfrac{x^2 - 9x + 14}{x^2 - 5 + 6}$.

Exercises for Section 1.4

Compute the limits. If a limit does not exist, explain why.

(1) $\lim_{x\to 1-} \dfrac{1}{x^2-1}$

(2) $\lim_{x\to 4-} \dfrac{3}{x^2-2}$

(3) $\lim_{x\to -1+} \dfrac{1+2x}{x^3-1}$

(4) $\lim_{x\to 3+} \dfrac{x-9}{x^2-6x+9}$

(5) $\lim_{x\to 5} \dfrac{1}{(x-5)^4}$

(6) $\lim_{x\to -2} \dfrac{1}{(x^2+3x+2)^2}$

(7) $\lim_{x\to 0} \dfrac{1}{\frac{x}{x^5}-\cos(x)}$

(8) $\lim_{x\to 0+} \dfrac{x-11}{\sin(x)}$

(9) Find the vertical asymptotes of

$$f(x) = \frac{x-3}{x^2+2x-3}.$$

(10) Find the vertical asymptotes of

$$f(x) = \frac{x^2-x-6}{x+4}.$$

1.5 Limits at Infinity

Consider the function:

$$f(x) = \frac{6x-9}{x-1}$$

As x approaches infinity, it seems like $f(x)$ approaches a specific value. This is a *limit at infinity*.

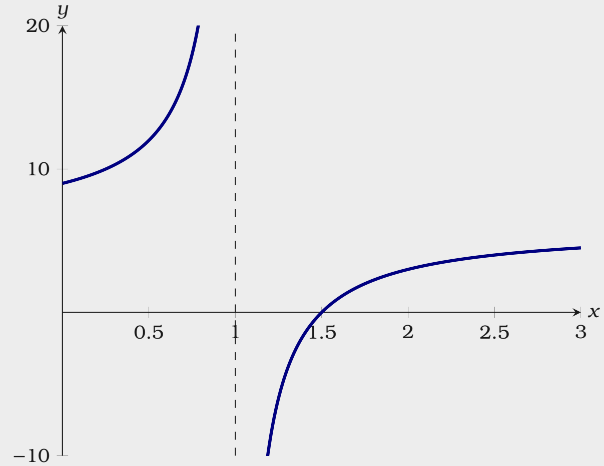


Figure 1.10: A plot of $f(x) = \dfrac{6x-9}{x-1}$.

Definition If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large, we write

$$\lim_{x\to\infty} f(x) = L$$

and we say, the **limit at infinity** of $f(x)$ is L.

If $f(x)$ becomes arbitrarily close to a specific value L by making x sufficiently large and negative, we write

$$\lim_{x\to-\infty} f(x) = L$$

and we say, the **limit at negative infinity** of $f(x)$ is L.

Example 1.5.1 Compute

$$\lim_{x\to\infty} \frac{6x-9}{x-1}.$$

Solution *Write*

$$\begin{aligned}
\lim_{x\to\infty} \frac{6x-9}{x-1} &= \lim_{x\to\infty} \frac{6x-9}{x-1}\,\frac{1/x}{1/x} \\
&= \lim_{x\to\infty} \frac{\frac{6x}{x}-\frac{9}{x}}{\frac{x}{x}-\frac{1}{x}} \\
&= \lim_{x\to\infty} \frac{6}{1} \\
&= 6.
\end{aligned}$$

Here is a somewhat different example of a limit at infinity.

Example 1.5.2 Compute
$$\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4.$$

Solution *We can bound our function*
$$-1/x + 4 \leq \frac{\sin(7x)}{x} + 4 \leq 1/x + 4.$$
Since
$$\lim_{x\to\infty} -1/x + 4 = 4 = \lim_{x\to\infty} 1/x + 4$$
we conclude by the Squeeze Theorem, Theorem 1.3.5, $\lim_{x\to\infty} \frac{\sin(7x)}{x} + 4 = 4$.

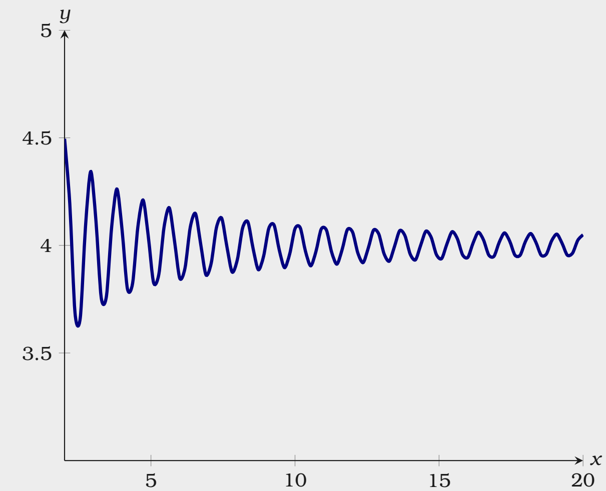


Figure 1.11: A plot of $f(x) = \frac{\sin(7x)}{x} + 4$.

Definition If
$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L,$$
then the line $y = L$ is a **horizontal asymptote** of $f(x)$.

Example 1.5.3 Give the horizontal asymptotes of
$$f(x) = \frac{6x-9}{x-1}$$

Solution *From our previous work, we see that* $\lim_{x\to\infty} f(x) = 6$, *and upon further inspection, we see that* $\lim_{x\to-\infty} f(x) = 6$. *Hence the horizontal asymptote of* $f(x)$ *is the line* $y = 6$.

It is a common misconception that a function cannot cross an asymptote. As the next example shows, a function can cross an asymptote, and in this case this occurs an infinite number of times!

Example 1.5.4 Give a horizontal asymptote of
$$f(x) = \frac{\sin(7x)}{x} + 4.$$

Solution *Again from previous work, we see that* $\lim_{x\to\infty} f(x) = 4$. *Hence* $y = 4$ *is a horizontal asymptote of* $f(x)$.

We conclude with an infinite limit at infinity.

Example 1.5.5 Compute

$$\lim_{x\to\infty} \ln(x)$$

Solution *The function* $\ln(x)$ *grows very slowly, and seems like it may have a horizontal asymptote, see Figure 1.12. However, if we consider the definition of the natural log*

$$\ln(x) = y \qquad \Leftrightarrow \qquad e^y = x$$

Since we need to raise e to higher and higher values to obtain larger numbers, we see that $\ln(x)$ *is unbounded, and hence* $\lim_{x\to\infty} \ln(x) = \infty$.

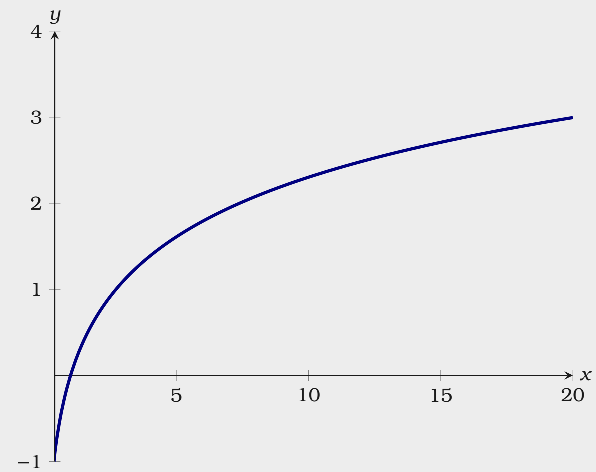


Figure 1.12: A plot of $f(x) = \ln(x)$.

Exercises for Section 1.5

Compute the limits.

(1) $\lim_{x\to\infty} \frac{1}{x}$

(2) $\lim_{x\to\infty} \frac{-x}{\sqrt{4+x^2}}$

(3) $\lim_{x\to\infty} \frac{2x^2-x+1}{4x^2-3x-1}$

(4) $\lim_{x\to-\infty} \frac{x^3-4}{3x^2+4x-1}$

(5) $\lim_{x\to\infty} \left(\frac{4}{x}+\pi\right)$

(6) $\lim_{x\to\infty} \frac{\cos(x)}{\ln(x)}$

(7) $\lim_{x\to\infty} \frac{\sin\left(x^7\right)}{\sqrt{x}}$

(8) $\lim_{x\to\infty} \left(17+\frac{32}{x}-\frac{(\sin(x/2))^2}{x^3}\right)$

(9) Suppose a population of feral cats on a certain college campus t years from now is approximated by

$$p(t) = \frac{1000}{5+2e^{-0.1t}}.$$

Approximately how many feral cats are on campus 10 years from now? 50 years from now? 100 years from now? 1000 years from now? What do you notice about the prediction—is this realistic?

(10) The amplitude of an oscillating spring is given by

$$a(t) = \frac{\sin(t)}{t}.$$

What happens to the amplitude of the oscillation over a long period of time?

1.6 Continuity

Informally, a function is continuous if you can "draw it" without "lifting your pencil." We need a formal definition.

Definition A function f is **continuous at a point** a if $\lim_{x\to a} f(x) = f(a)$.

Example 1.6.1 Find the discontinuities (the values for x where a function is not continuous) for the function given in Figure 1.13.

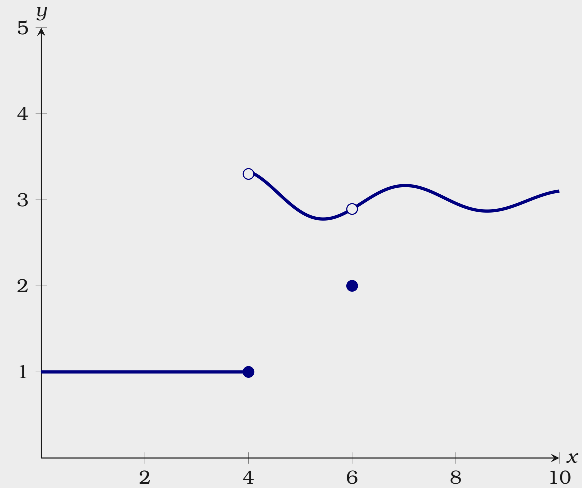


Figure 1.13: A plot of a function with discontinuities at $x = 4$ and $x = 6$.

Solution *From Figure 1.13 we see that* $\lim_{x\to 4} f(x)$ *does not exist as*

$$\lim_{x\to 4-} f(x) = 1 \qquad \textit{and} \qquad \lim_{x\to 4+} f(x) \approx 3.5$$

Hence $\lim_{x\to 4} f(x) \neq f(4)$, *and so* $f(x)$ *is not continuous at* $x = 4$.

We also see that $\lim_{x\to 6} f(x) \approx 3$ *while* $f(6) = 2$. *Hence* $\lim_{x\to 6} f(x) \neq f(6)$, *and so* $f(x)$ *is not continuous at* $x = 6$.

Building from the definition of *continuous at a point*, we can now define what it means for a function to be *continuous* on an interval.

Definition A function f is **continuous on an interval** if it is continuous at every point in the interval.

In particular, we should note that if a function is not defined on an interval, then it **cannot** be continuous on that interval.

Example 1.6.2 Consider the function

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

see Figure 1.14. Is this function continuous?

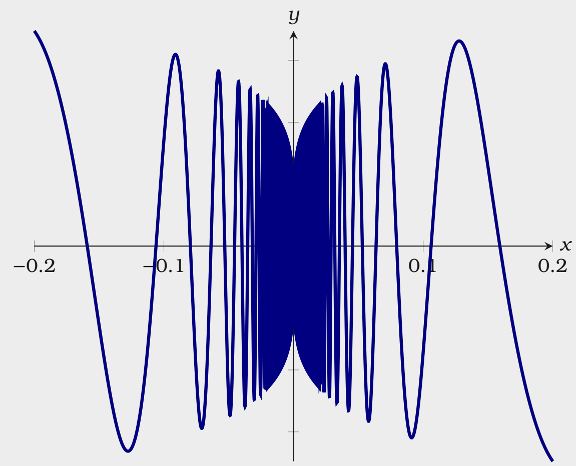


Figure 1.14: A plot of

$$f(x) = \begin{cases} \sqrt[5]{x}\sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Solution *Considering $f(x)$, the only issue is when $x = 0$. We must show that $\lim_{x\to 0} f(x) = 0$. Note*

$$-|\sqrt[5]{x}| \leq f(x) \leq |\sqrt[5]{x}|.$$

Since

$$\lim_{x\to 0} -|\sqrt[5]{x}| = 0 = \lim_{x\to 0} |\sqrt[5]{x}|,$$

we see by the Squeeze Theorem, Theorem 1.3.5, that $\lim_{x\to 0} f(x) = 0$. Hence $f(x)$ is continuous.

Here we see how the informal definition of continuity being that you can "draw it" without "lifting your pencil" differs from the formal definition.

We close with a useful theorem about continuous functions:

The Intermediate Value Theorem is most frequently used when $d = 0$.

For a nice proof of this theorem, see: Walk, Stephen M. *The intermediate value theorem is NOT obvious—and I am going to prove it to you.* College Math. J. 42 (2011), no. 4, 254–259.

Theorem 1.6.3 (Intermediate Value Theorem) *If $f(x)$ is a continuous function for all x in the closed interval $[a, b]$ and d is between $f(a)$ and $f(b)$, then there is a number c in $[a, b]$ such that $f(c) = d$.*

In Figure 1.15, we see a geometric interpretation of this theorem.

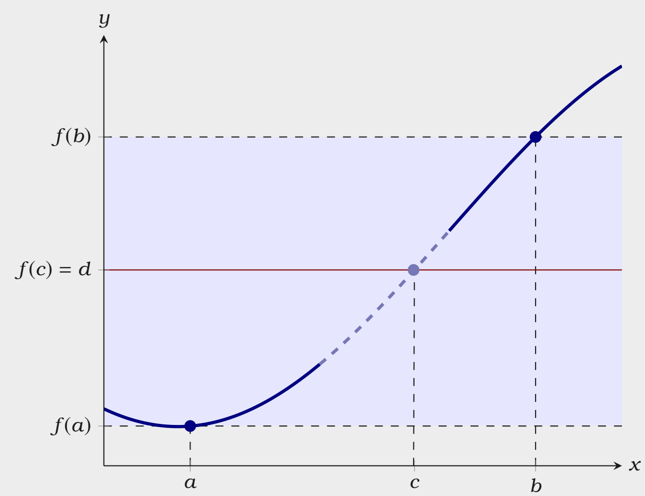


Figure 1.15: A geometric interpretation of the Intermediate Value Theorem. The function $f(x)$ is continuous on the interval $[a, b]$. Since d is in the interval $[f(a), f(b)]$, there exists a value c in $[a, b]$ such that $f(c) = d$.

Example 1.6.4 Explain why the function $f(x) = x^3 + 3x^2 + x - 2$ has a root between 0 and 1.

Solution *By Theorem 1.3.1, $\lim_{x\to a} f(x) = f(a)$, for all real values of a, and hence f is continuous. Since $f(0) = -2$ and $f(1) = 3$, and 0 is between -2 and 3, by the Intermediate Value Theorem, Theorem 1.6.3, there is a $c \in [0, 1]$ such that $f(c) = 0$.*

This example also points the way to a simple method for approximating roots.

Example 1.6.5 Approximate a root of $f(x) = x^3 + 3x^2 + x - 2$ to one decimal place.

Solution *If we compute* $f(0.1)$, $f(0.2)$, *and so on, we find that* $f(0.6) < 0$ *and* $f(0.7) > 0$, *so by the Intermediate Value Theorem,* f *has a root between* 0.6 *and* 0.7. *Repeating the process with* $f(0.61)$, $f(0.62)$, *and so on, we find that* $f(0.61) < 0$ *and* $f(0.62) > 0$, *so by the Intermediate Value Theorem, Theorem 1.6.3,* $f(x)$ *has a root between* 0.61 *and* 0.62, *and the root is* 0.6 *rounded to one decimal place.*

Exercises for Section 1.6

(1) Consider the function

$$f(x) = \sqrt{x-4}$$

Is $f(x)$ continuous at the point $x = 4$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(2) Consider the function

$$f(x) = \frac{1}{x+3}$$

Is $f(x)$ continuous at the point $x = 3$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(3) Consider the function

$$f(x) = \begin{cases} 2x - 3 & \text{if } x < 1, \\ 0 & \text{if } x \geq 1. \end{cases}$$

Is $f(x)$ continuous at the point $x = 1$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(4) Consider the function

$$f(x) = \begin{cases} \dfrac{x^2 + 10x + 25}{x - 5} & \text{if } x \neq 5, \\ 10 & \text{if } x = 5. \end{cases}$$

Is $f(x)$ continuous at the point $x = 5$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(5) Consider the function

$$f(x) = \begin{cases} \dfrac{x^2 + 10x + 25}{x + 5} & \text{if } x \neq -5, \\ 0 & \text{if } x = -5. \end{cases}$$

Is $f(x)$ continuous at the point $x = -5$? Is $f(x)$ a continuous function on $\mathbb{R}$?

(6) Determine the interval(s) on which the function $f(x) = x^7 + 3x^5 - 2x + 4$ is continuous.

(7) Determine the interval(s) on which the function $f(x) = \dfrac{x^2 - 2x + 1}{x + 4}$ is continuous.

(8) Determine the interval(s) on which the function $f(x) = \dfrac{1}{x^2 - 9}$ is continuous.

(9) Approximate a root of $f(x) = x^3 - 4x^2 + 2x + 2$ to two decimal places.

(10) Approximate a root of $f(x) = x^4 + x^3 - 5x + 1$ to two decimal places.

2 Basics of Derivatives

2.1 Slopes of Tangent Lines via Limits

Suppose that $f(x)$ is a function. It is often useful to know how sensitive the value of $f(x)$ is to small changes in x. To give you a feeling why this is true, consider the following:

- If $p(t)$ represents the position of an object with respect to time, the rate of change gives the velocity of the object.
- If $v(t)$ represents the velocity of an object with respect to time, the rate of change gives the acceleration of the object.
- The rate of change of a function can help us approximate a complicated function with a simple function.
- The rate of change of a function can be used to help us solve equations that we would not be able to solve via other methods.

The rate of change of a function is the slope of the tangent line. For now, consider the following informal definition of a *tangent line*:

> Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

We illustrate this informal definition with Figure 2.1.

The *derivative* of a function $f(x)$ at x, is the slope of the tangent line at x. To find the slope of this line, we consider *secant* lines, lines that locally intersect the curve

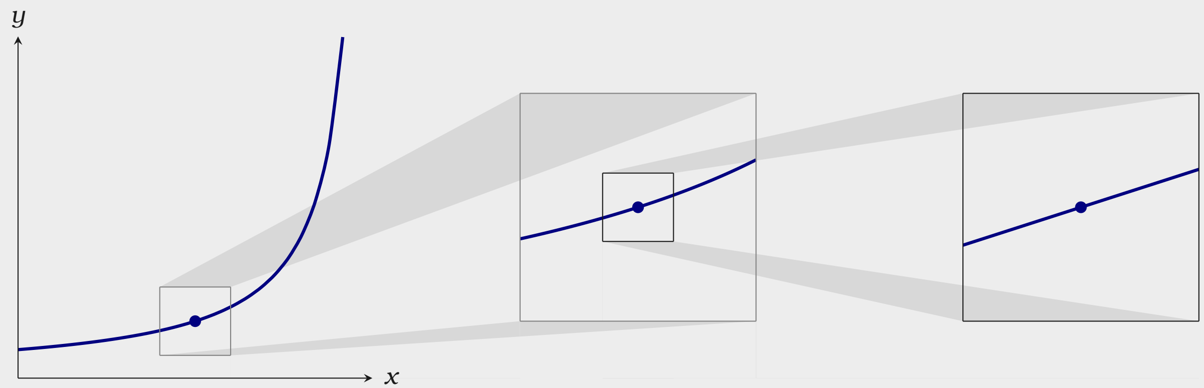


Figure 2.1: Given a function $f(x)$, if one can "zoom in" on $f(x)$ sufficiently so that $f(x)$ seems to be a straight line, then that line is the **tangent line** to $f(x)$ at the point determined by x.

at two points. The slope of any secant line that passes through the points $(x, f(x))$ and $(x+h, f(x+h))$ is given by

$$\frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h},$$

see Figure 2.2. This leads to the *limit definition of the derivative*:

Definition of the Derivative The **derivative** of $f(x)$ is the function

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If this limit does not exist for a given value of x, then $f(x)$ is not **differentiable** at x.

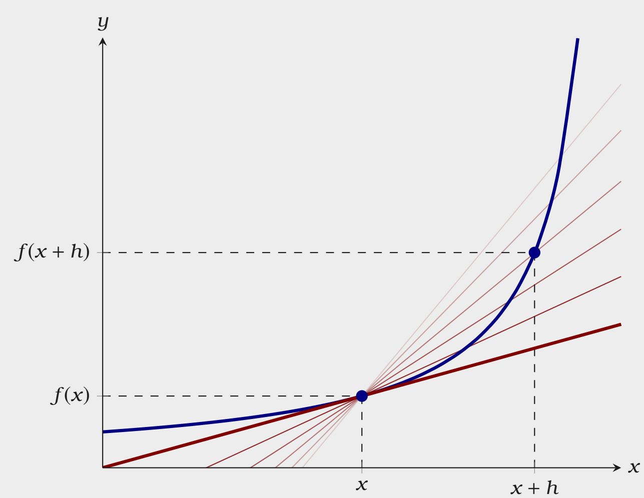


Figure 2.2: Tangent lines can be found as the limit of secant lines. The slope of the tangent line is given by $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Definition There are several different notations for the derivative, we'll mainly use

$$\frac{d}{dx}f(x) = f'(x).$$

If one is working with a function of a variable other than x, say t we write

$$\frac{d}{dt}f(t) = f'(t).$$

However, if $y = f(x)$, $\dfrac{dy}{dx}$, $\dot{y}$, and $D_x f(x)$ are also used.

Now we will give a number of examples, starting with a basic example.

Example 2.1.1 Compute

$$\frac{d}{dx}(x^3 + 1).$$

Solution *Using the definition of the derivative,*

$$\begin{aligned}
\frac{d}{dx}f(x) &= \lim_{h\to 0} \frac{(x+h)^3 + 1 - (x^3+1)}{h} \\
&= \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 + 1 - x^3 - 1}{h} \\
&= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\
&= \lim_{h\to 0}(3x^2 + 3xh + h^2) \\
&= 3x^2.
\end{aligned}$$

See Figure 2.3.

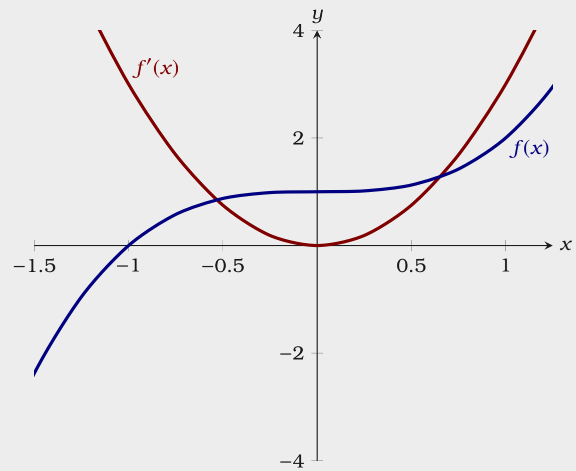


Figure 2.3: A plot of $f(x) = x^3 + 1$ and $f'(x) = 3x^2$.

Next we will consider the derivative a function that is not continuous on $\mathbb{R}$.

Example 2.1.2 Compute

$$\frac{d}{dt}\frac{1}{t}.$$

Solution *Using the definition of the derivative,*

$$\begin{aligned}
\frac{d}{dt}\frac{1}{t} &= \lim_{h\to 0}\frac{\frac{1}{t+h}-\frac{1}{t}}{h}\\
&= \lim_{h\to 0}\frac{\frac{t}{t(t+h)}-\frac{t+h}{t(t+h)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{t-(t+h)}{t(t+h)}}{h}\\
&= \lim_{h\to 0}\frac{t-t-h}{t(t+h)h}\\
&= \lim_{h\to 0}\frac{-h}{t(t+h)h}\\
&= \lim_{h\to 0}\frac{-1}{t(t+h)}\\
&= \frac{-1}{t^2}.
\end{aligned}$$

This function is differentiable at all real numbers except for $t = 0$, see Figure 2.4.

As you may have guessed, there is some connection to continuity and differentiability.

Theorem 2.1.3 (Differentiability Implies Continuity) *If $f(x)$ is a differentiable function at $x = a$, then $f(x)$ is continuous at $x = a$.*

Proof *We want to show that $f(x)$ is continuous at $x = a$, hence we must show that*

$$\lim_{x\to a} f(x) = f(a).$$

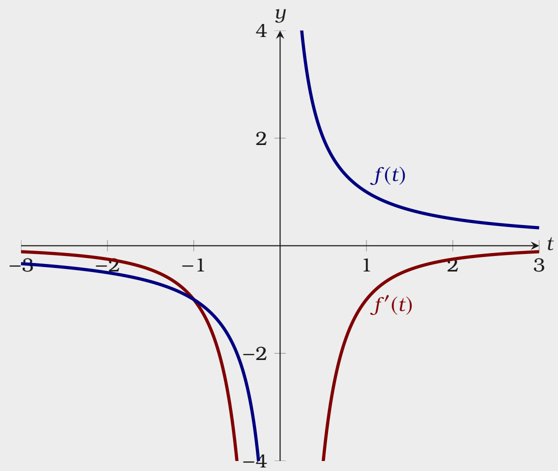


Figure 2.4: A plot of $f(t) = \frac{1}{t}$ and $f'(t) = \frac{-1}{t^2}$.

Consider

$$\begin{aligned}
\lim_{x\to a}(f(x)-f(a)) &= \lim_{x\to a}\left((x-a)\frac{f(x)-f(a)}{x-a}\right) && \textit{Multiply and divide by } (x-a).\\
&= \lim_{h\to 0} h\cdot\frac{f(a+h)-f(a)}{h} && \textit{Set } x=a+h.\\
&= \left(\lim_{h\to 0} h\right)\left(\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}\right) && \textit{Limit Law.}\\
&= 0\cdot f'(a) = 0.
\end{aligned}$$

Since

$$\lim_{x\to a}(f(x)-f(a)) = 0$$

we see that $\lim_{x\to a} f(x) = f(a)$*, and so* $f(x)$ *is continuous.*

This theorem is often written as its contrapositive:

If $f(x)$ is not continuous at $x = a$, then $f(x)$ is not differentiable at $x = a$.

Let's see a function that is continuous whose derivative does not exist everywhere.

Example 2.1.4 Compute

$$\frac{d}{dx}|x|.$$

Solution *Using the definition of the derivative,*

$$\frac{d}{dx}|x| = \lim_{h\to 0}\frac{|x+h|-|x|}{h}.$$

If x *is positive we may assume that* x *is larger than* h*, as we are taking the limit as* h *goes to* 0*,*

$$\begin{aligned}
\lim_{h\to 0}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0}\frac{x+h-x}{h}\\
&= \lim_{h\to 0}\frac{h}{h}\\
&= 1.
\end{aligned}$$

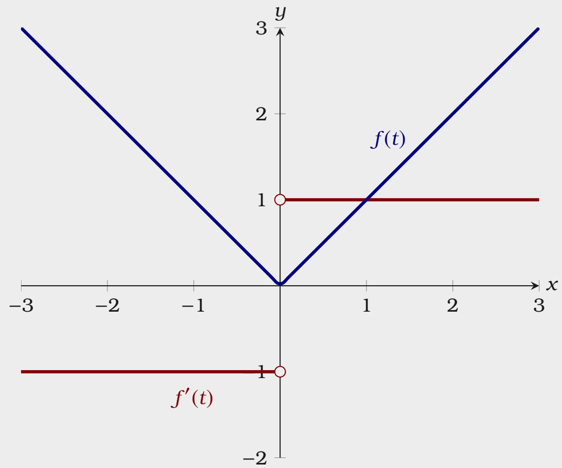


Figure 2.5: A plot of $f(x) = |x|$ and

$$f'(x) = \begin{cases} 1 & \text{if } x > 0,\\ -1 & \text{if } x < 0.\end{cases}$$

If x is negative we may assume that $|x|$ is larger than h, as we are taking the limit as h goes to 0,

$$\begin{aligned}\lim_{h\to 0}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0}\frac{x-h-x}{h}\\ &= \lim_{h\to 0+}\frac{-h}{h}\\ &= -1.\end{aligned}$$

However we still have one case left, when $x = 0$. In this situation, we must consider the one-sided limits:

$$\lim_{h\to 0+}\frac{|x+h|-|x|}{h} \qquad \textit{and} \qquad \lim_{h\to 0-}\frac{|x+h|-|x|}{h}.$$

In the first case,

$$\begin{aligned}\lim_{h\to 0+}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0+}\frac{0+h-0}{h}\\ &= \lim_{h\to 0+}\frac{h}{h}\\ &= 1.\end{aligned}$$

On the other hand

$$\begin{aligned}\lim_{h\to 0-}\frac{|x+h|-|x|}{h} &= \lim_{h\to 0-}\frac{|0+h|-0}{h}\\ &= \lim_{h\to 0-}\frac{|h|}{h}\\ &= -1.\end{aligned}$$

Hence we see that the derivative is

$$f'(x) = \begin{cases} 1 & \textit{if } x > 0,\\ -1 & \textit{if } x < 0.\end{cases}$$

Note this function is undefined at 0, *see Figure 2.5.*

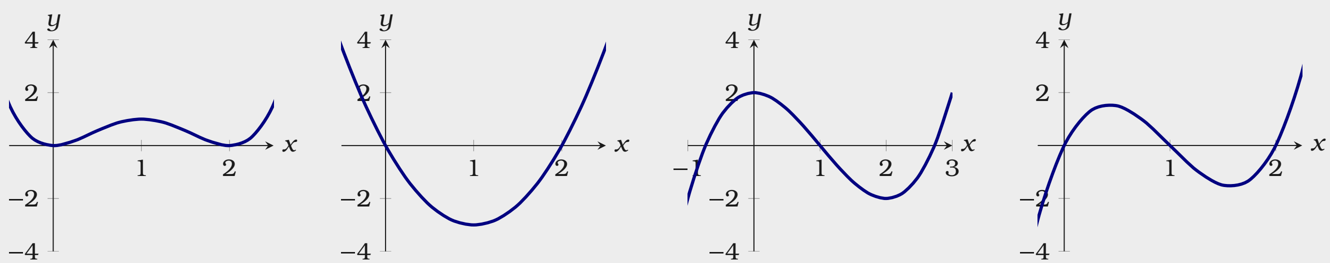
Thus from Theorem 2.1.3, we see that all differentiable functions on $\mathbb{R}$ are continuous on $\mathbb{R}$. Nevertheless as the previous example shows, there are continuous functions on $\mathbb{R}$ that are not differentiable on $\mathbb{R}$.

Exercises for Section 2.1

These exercises are conceptual in nature and require one to think about what the derivative means.

(1) If the line $y = 7x - 4$ is tangent to $f(x)$ at $x = 2$, find $f(2)$ and $f'(2)$.

(2) Here are plots of four functions.



$p(x)$ $q(x)$ $r(x)$ $s(x)$

Two of these functions are the derivatives of the other two, identify which functions are the derivatives of the others.

(3) If $f(3) = 6$ and $f(3.1) = 6.4$, estimate $f'(3)$.

(4) If $f(-2) = 4$ and $f(-2 + h) = (h + 2)^2$, compute $f'(-2)$.

(5) If $f'(x) = x^3$ and $f(1) = 2$, approximate $f(1.2)$.

(6) Consider the plot of $f(x)$ in Figure 2.6.

(a) On which subinterval(s) of $[0, 6]$ is $f(x)$ continuous?

(b) On which subinterval(s) of $[0, 6]$ is $f(x)$ differentiable?

(c) Sketch a plot of $f'(x)$.

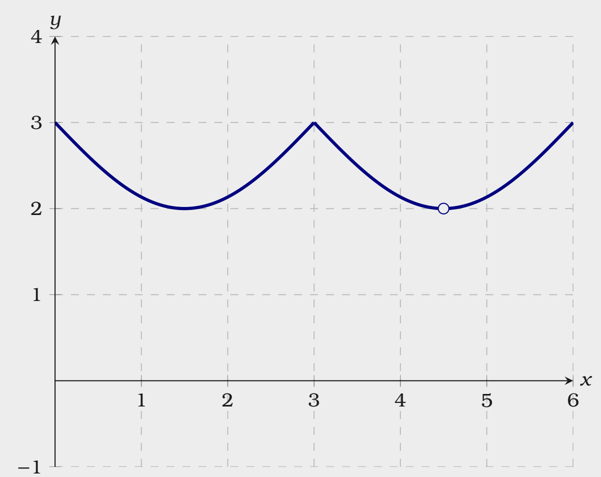


Figure 2.6: A plot of $f(x)$.

These exercises are computational in nature.

(7) Let $f(x) = x^2 - 4$. Use the definition of the derivative to compute $f'(-3)$ and find the equation of the tangent line to the curve at $x = -3$.

(8) Let $f(x) = \dfrac{1}{x+2}$. Use the definition of the derivative to compute $f'(1)$ and find the equation of the tangent line to the curve at $x = 1$.

(9) Let $f(x) = \sqrt{x-3}$. Use the definition of the derivative to compute $f'(5)$ and find the equation of the tangent line to the curve at $x = 5$.

(10) Let $f(x) = \dfrac{1}{\sqrt{x}}$. Use the definition of the derivative to compute $f'(4)$ and find the equation of the tangent line to the curve at $x = 4$.

2.2 Basic Derivative Rules

It is tedious to compute a limit every time we need to know the derivative of a function. Fortunately, we can develop a small collection of examples and rules that allow us to compute the derivative of almost any function we are likely to encounter. We will start simply and build-up to more complicated examples.

The Constant Rule

The simplest function is a constant function. Recall that derivatives measure the rate of change of a function at a given point. Hence, the derivative of a constant function is zero. For example:

- The constant function plots a horizontal line—so the slope of the tangent line is 0.
- If $p(t)$ represents the position of an object with respect to time and $p(t)$ is constant, then the object is not moving, so its velocity is zero. Hence $\frac{d}{dt}p(t) = 0$.
- If $v(t)$ represents the velocity of an object with respect to time and $v(t)$ is constant, then the object's acceleration is zero. Hence $\frac{d}{dt}v(t) = 0$.

The examples above lead us to our next theorem.

To gain intuition, you should compute the derivative of $f(x) = 6$ using the limit definition of the derivative.

Theorem 2.2.1 (The Constant Rule) *Given a constant c,*

$$\frac{d}{dx}c = 0.$$

Proof *From the limit definition of the derivative, write*

$$\begin{aligned}\frac{d}{dx}c &= \lim_{h\to 0}\frac{c-c}{h}\\ &= \lim_{h\to 0}\frac{0}{h}\\ &= \lim_{h\to 0}0 = 0.\end{aligned}$$

The Power Rule

Now let's examine derivatives of powers of a single variable. Here we have a nice rule.

To gain intuition, you should compute the derivative of $f(x) = x^3$ using the limit definition of the derivative.

Theorem 2.2.2 (The Power Rule) *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof *At this point we will only prove this theorem for n being a positive integer. Later in Section 5.3, we will give the complete proof. From the limit definition of the derivative, write*

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

Start by expanding the term $(x+h)^n$

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{x^n + \binom{n}{1}x^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n - x^n}{h}$$

Note, by the Binomial Theorem, we write $\binom{n}{k}$ for the coefficients. Canceling the terms x^n and $-x^n$, and noting $\binom{n}{1} = \binom{n}{n-1} = n$, write

$$\begin{aligned}\frac{d}{dx}x^n &= \lim_{h\to 0}\frac{nx^{n-1}h + \binom{n}{2}x^{n-2}h^2 + \cdots + \binom{n}{n-1}xh^{n-1} + h^n}{h}\\ &= \lim_{h\to 0} nx^{n-1} + \binom{n}{2}x^{n-2}h + \cdots + \binom{n}{n-1}xh^{n-2} + h^{n-1}.\end{aligned}$$

Since every term but the first has a factor of h, we see

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = nx^{n-1}.$$

Recall, the **Binomial Theorem** states that if n is a nonnegative integer, then

$$(a+b)^n = a^n b^0 + \binom{n}{1}a^{n-1}b^1 + \cdots + \binom{n}{n-1}a^1 b^{n-1} + a^0 b^n$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Now we will show you several examples. We begin with something basic.

Example 2.2.3 Compute

$$\frac{d}{dx}x^{13}.$$

Solution *Applying the power rule, we write*

$$\frac{d}{dx}x^{13} = 13x^{12}.$$

Sometimes, it is not as obvious that one should apply the power rule.

Example 2.2.4 Compute

$$\frac{d}{dx}\frac{1}{x^4}.$$

Solution *Applying the power rule, we write*

$$\frac{d}{dx}\frac{1}{x^4} = \frac{d}{dx}x^{-4} = -4x^{-5}.$$

The power rule also applies to radicals once we rewrite them as exponents.

Example 2.2.5 Compute

$$\frac{d}{dx}\sqrt[5]{x}.$$

Solution *Applying the power rule, we write*

$$\frac{d}{dx}\sqrt[5]{x} = \frac{d}{dx}x^{1/5} = \frac{x^{-4/5}}{5}.$$

The Sum Rule

We want to be able to take derivatives of functions "one piece at a time." The *sum rule* allows us to do this. The sum rule says that we can add the rates of change of two functions to obtain the rate of change of the sum of both functions. For example, viewing the derivative as the velocity of an object, the sum rule states that the velocity of the person walking on a moving bus is the sum of the velocity of the bus and the walking person.

Theorem 2.2.6 (The Sum Rule) *If $f(x)$ and $g(x)$ are differentiable and c is a constant, then*

(a) $\dfrac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$,

(b) $\dfrac{d}{dx}(f(x)-g(x)) = f'(x)-g'(x)$,

(c) $\dfrac{d}{dx}(c\cdot f(x)) = c\cdot f'(x)$.

Proof *We will only prove part (a) above, the rest are similar. Write*

$$\begin{aligned}
\frac{d}{dx}(f(x)+g(x)) &= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-(f(x)+g(x))}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)+g(x+h)-f(x)-g(x)}{h}\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)+g(x+h)-g(x)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}+\frac{g(x+h)-g(x)}{h}\right)\\
&= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}+\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}\\
&= f'(x)+g'(x).
\end{aligned}$$

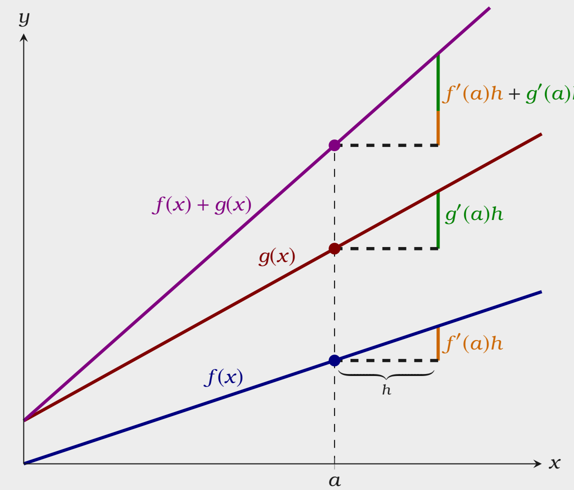


Figure 2.7: A geometric interpretation of the sum rule. Since every point on $f(x)+g(x)$ is the sum of the corresponding points on $f(x)$ and $g(x)$, increasing a by a "small amount" h, increases $f(a)+g(a)$ by the sum of $f'(a)h$ and $g'(a)h$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f'(a)h+g'(a)h}{h} = f'(a)+g'(a).$$

Example 2.2.7 Compute

$$\frac{d}{dx}\left(x^5+\frac{1}{x}\right).$$

Solution *Write*

$$\begin{aligned}
\frac{d}{dx}\left(x^5+\frac{1}{x}\right) &= \frac{d}{dx}x^5+\frac{d}{dx}x^{-1}\\
&= 5x^4 - x^{-2}.
\end{aligned}$$

Example 2.2.8 Compute

$$\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right).$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx}\left(\frac{3}{\sqrt[3]{x}} - 2\sqrt{x} + \frac{1}{x^7}\right) &= 3\frac{d}{dx}x^{-1/3} - 2\frac{d}{dx}x^{1/2} + \frac{d}{dx}x^{-7}\\ &= -x^{-4/3} - x^{-1/2} - 7x^{-8}.\end{aligned}$$

The Derivative of e^x

We don't know anything about derivatives that allows us to compute the derivatives of exponential functions without getting our hands dirty. Let's do a little work with the definition of the derivative:

$$\begin{aligned}\frac{d}{dx}a^x &= \lim_{h\to 0}\frac{a^{x+h} - a^x}{h}\\ &= \lim_{h\to 0}\frac{a^x a^h - a^x}{h}\\ &= \lim_{h\to 0} a^x\frac{a^h - 1}{h}\\ &= a^x \lim_{h\to 0}\frac{a^h - 1}{h}\\ &= a^x \cdot \underbrace{(\text{constant})}_{\lim_{h\to 0}\frac{a^h-1}{h}}\end{aligned}$$

There are two interesting things to note here: We are left with a limit that involves h but not x, which means that whatever $\lim_{h\to 0}(a^h - 1)/h$ is, we know that it is a number, that is, a constant. This means that a^x has a remarkable property: Its derivative is a constant times itself. Unfortunately it is beyond the scope of this text to compute the limit

$$\lim_{h\to 0}\frac{a^h - 1}{h}.$$

However, we can look at some examples. Consider $(2^h - 1)/h$ and $(3^h - 1)/h$:

h	$(2^h-1)/h$
-1	$.5$
-0.1	≈ 0.6700
-0.01	≈ 0.6910
-0.001	≈ 0.6929
-0.0001	≈ 0.6931
-0.00001	≈ 0.6932

h	$(2^h-1)/h$
1	1
0.1	≈ 0.7177
0.01	≈ 0.6956
0.001	≈ 0.6834
0.0001	≈ 0.6932
0.00001	≈ 0.6932

h	$(3^h-1)/h$
-1	≈ 0.6667
-0.1	≈ 1.0404
-0.01	≈ 1.0926
-0.001	≈ 1.0980
-0.0001	≈ 1.0986
-0.00001	≈ 1.0986

h	$(3^h-1)/h$
1	2
0.1	≈ 1.1612
0.01	≈ 1.1047
0.001	≈ 1.0992
0.0001	≈ 1.0987
0.00001	≈ 1.0986

While these tables don't prove a pattern, it turns out that

$$\lim_{h\to 0}\frac{2^h-1}{h}\approx .7 \qquad \text{and} \qquad \lim_{h\to 0}\frac{3^h-1}{h}\approx 1.1.$$

Moreover, if you do more examples you will find that the limit varies directly with the value of a: bigger a, bigger limit; smaller a, smaller limit. As we can already see, some of these limits will be less than 1 and some larger than 1. Somewhere between $a = 2$ and $a = 3$ the limit will be exactly 1. This happens when

$$a = e = 2.718281828459045\ldots.$$

This brings us to our next definition.

Definition Euler's number is defined to be the number e such that

$$\lim_{h\to 0}\frac{e^h-1}{h}=1.$$

Now we see that the function e^x has a truly remarkable property:

Theorem 2.2.9 (The Derivative of e^x)

$$\frac{d}{dx}e^x = e^x.$$

Proof *From the limit definition of the derivative, write*

$$\begin{aligned}\frac{d}{dx}e^x &= \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to 0}\frac{e^xe^h-e^x}{h}\\ &= \lim_{h\to 0}e^x\frac{e^h-1}{h}\\ &= e^x\lim_{h\to 0}\frac{e^h-1}{h}\\ &= e^x.\end{aligned}$$

Hence e^x is its own derivative. In other words, the slope of the plot of e^x is the same as its height, or the same as its second coordinate: The function $f(x) = e^x$ goes through the point (a, e^a) and has slope e^a there, no matter what a is.

Example 2.2.10 Compute:

$$\frac{d}{dx}\left(8\sqrt{x}+7e^x\right)$$

Solution *Write:*

$$\begin{aligned}\frac{d}{dx}\left(8\sqrt{x}+7e^x\right) &= 8\frac{d}{dx}x^{1/2}+7\frac{d}{dx}e^x\\ &= 4x^{-1/2}+7e^x.\end{aligned}$$

Exercises for Section 2.2

Compute:

(1) $\dfrac{d}{dx}5$

(2) $\dfrac{d}{dx}-7$

(3) $\dfrac{d}{dx}e^7$

(4) $\dfrac{d}{dx}\dfrac{1}{\sqrt{2}}$

(5) $\dfrac{d}{dx}x^{100}$

(6) $\dfrac{d}{dx}x^{-100}$

(7) $\dfrac{d}{dx}\dfrac{1}{x^5}$

(8) $\dfrac{d}{dx}x^{\pi}$

(9) $\dfrac{d}{dx}x^{3/4}$

(10) $\dfrac{d}{dx}\dfrac{1}{(\sqrt[7]{x})^9}$

(11) $\dfrac{d}{dx}\left(5x^3+12x^2-15\right)$

(12) $\dfrac{d}{dx}\left(-4x^5+3x^2-\dfrac{5}{x^2}\right)$

(13) $\dfrac{d}{dx}5(-3x^2+5x+1)$

(14) $\dfrac{d}{dx}\left(3\sqrt{x}+\dfrac{1}{x}-x^e\right)$

(15) $\dfrac{d}{dx}\left(\dfrac{x^2}{x^7}+\dfrac{\sqrt{x}}{x}\right)$

Expand or simplify to compute the following:

(16) $\dfrac{d}{dx}\left((x+1)(x^2+2x-3)\right)$

(17) $\dfrac{d}{dx}\dfrac{x^3-2x^2-5x+6}{(x-1)}$

(18) $\dfrac{d}{dx}\dfrac{x-5}{\sqrt{x}-\sqrt{5}}$

(19) $\dfrac{d}{dx}((x+1)(x+1)(x-1)(x-1))$

(20) Suppose the position of an object at time t is given by $f(t) = -49t^2/10 + 5t + 10$. Find a function giving the velocity of the object at time t. The acceleration of an object is the rate at which its velocity is changing, which means it is given by the derivative of the velocity function. Find the acceleration of the object at time t.

(21) Let $f(x) = x^3$ and $c = 3$. Sketch the graphs of $f(x)$, $cf(x)$, $f'(x)$, and $(cf(x))'$ on the same diagram.

(22) Find a cubic polynomial whose graph has horizontal tangents at $(-2, 5)$ and $(2, 3)$.

(23) Find an equation for the tangent line to $f(x) = x^3/4 - 1/x$ at $x = -2$.

(24) Find an equation for the tangent line to $f(x) = 3x^2 - \pi^3$ at $x = 4$.

(25) Prove that $\frac{d}{dx}(cf(x)) = cf'(x)$ using the definition of the derivative.

3 Curve Sketching

Whether we are interested in a function as a purely mathematical object or in connection with some application to the real world, it is often useful to know what the graph of the function looks like. We can obtain a good picture of the graph using certain crucial information provided by derivatives of the function and certain limits.

3.1 Extrema

Local *extrema* on a function are points on the graph where the y coordinate is larger (or smaller) than all other y coordinates on the graph at points "close to" (x, y).

Definition

(a) A point $(x, f(x))$ is a **local maximum** if there is an interval $a < x < b$ with $f(x) \geq f(z)$ for every z in (a, b).

(b) A point $(x, f(x))$ is a **local minimum** if there is an interval $a < x < b$ with $f(x) \leq f(z)$ for every z in (a, b).

A **local extremum** is either a local maximum or a local minimum.

Local maximum and minimum points are quite distinctive on the graph of a function, and are therefore useful in understanding the shape of the graph. In many applied problems we want to find the largest or smallest value that a function achieves (for example, we might want to find the minimum cost at which some task can be performed) and so identifying maximum and minimum points will be useful for applied problems as well.

If $(x, f(x))$ is a point where $f(x)$ reaches a local maximum or minimum, and if the derivative of f exists at x, then the graph has a tangent line and the tangent line must be horizontal. This is important enough to state as a theorem, though we will not prove it.

Theorem 3.1.1 (Fermat's Theorem) *If $f(x)$ has a local extremum at $x = a$ and $f(x)$ is differentiable at a, then $f'(a) = 0$.*

Thus, the only points at which a function can have a local maximum or minimum are points at which the derivative is zero, see Figure 3.1, or the derivative is undefined, as in Figure 3.2. This brings us to our next definition.

Definition Any value of x for which $f'(x)$ is zero or undefined is called a **critical point** for $f(x)$.

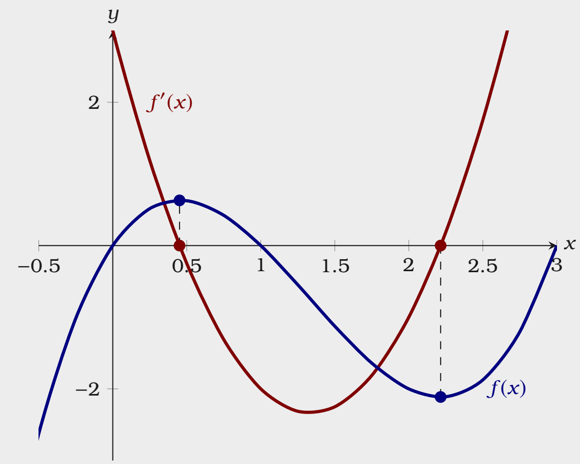


Figure 3.1: A plot of $f(x) = x^3 - 4x^2 + 3x$ and $f'(x) = 3x^2 - 8x + 3$.

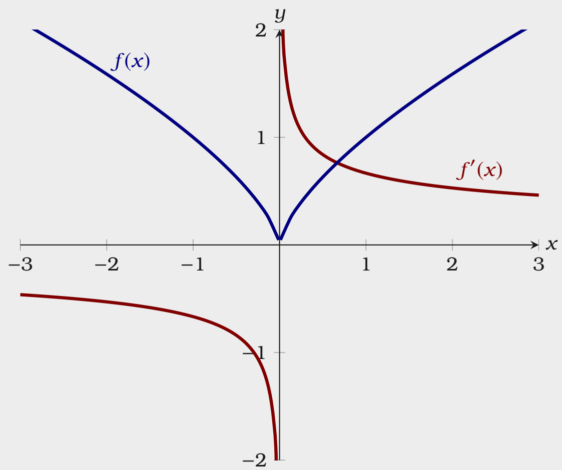


Figure 3.2: A plot of $f(x) = x^{2/3}$ and $f'(x) = \dfrac{2}{3x^{1/3}}$.

Warning When looking for local maximum and minimum points, you are likely to make two sorts of mistakes:

- You may forget that a maximum or minimum can occur where the derivative does not exist, and so forget to check whether the derivative exists everywhere.
- You might assume that any place that the derivative is zero is a local maximum or minimum point, but this is not true, see Figure 3.3.

Since the derivative is zero or undefined at both local maximum and local minimum points, we need a way to determine which, if either, actually occurs. The most elementary approach is to test directly whether the y coordinates near the potential maximum or minimum are above or below the y coordinate at the point of interest.

It is not always easy to compute the value of a function at a particular point. The task is made easier by the availability of calculators and computers, but they have their own drawbacks—they do not always allow us to distinguish between values that are very close together. Nevertheless, because this method is conceptually simple and sometimes easy to perform, you should always consider it.

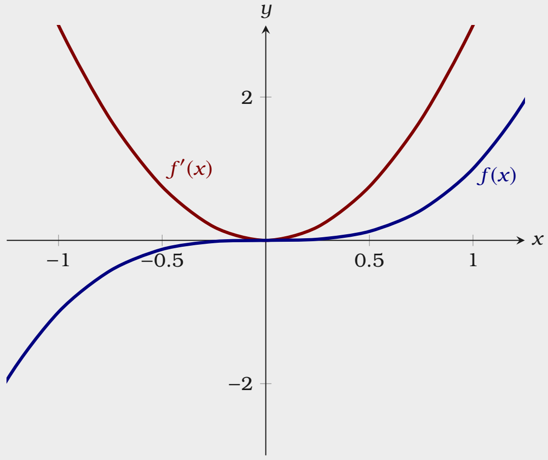


Figure 3.3: A plot of $f(x) = x^3$ and $f'(x) = 3x^2$. While $f'(0) = 0$, there is neither a maximum nor minimum at $(0, f(0))$.

Example 3.1.2 Find all local maximum and minimum points for the function $f(x) = x^3 - x$.

Solution *Write*

$$\frac{d}{dx} f(x) = 3x^2 - 1.$$

This is defined everywhere and is zero at $x = \pm\sqrt{3}/3$. *Looking first at* $x = \sqrt{3}/3$, *we see that*

$$f(\sqrt{3}/3) = -2\sqrt{3}/9.$$

Now we test two points on either side of $x = \sqrt{3}/3$, *making sure that neither is farther away than the nearest critical point; since* $\sqrt{3} < 3$, $\sqrt{3}/3 < 1$ *and we can use* $x = 0$ *and* $x = 1$. *Since*

$$f(0) = 0 > -2\sqrt{3}/9 \qquad \text{and} \qquad f(1) = 0 > -2\sqrt{3}/9,$$

there must be a local minimum at $x = \sqrt{3}/3$.

For $x = -\sqrt{3}/3$, *we see that* $f(-\sqrt{3}/3) = 2\sqrt{3}/9$. *This time we can use* $x = 0$ *and* $x = -1$, *and we find that* $f(-1) = f(0) = 0 < 2\sqrt{3}/9$, *so there must be a local maximum at* $x = -\sqrt{3}/3$, *see Figure 3.4.*

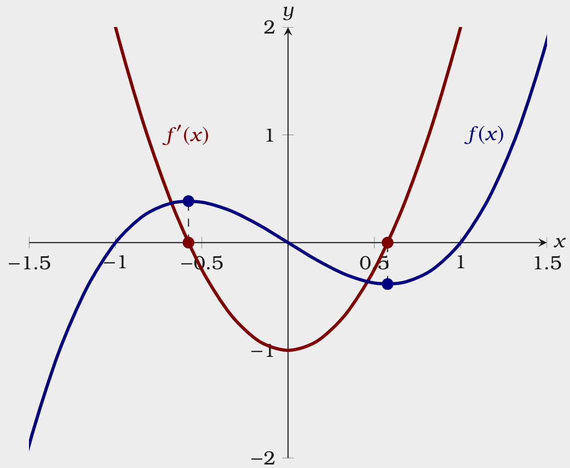


Figure 3.4: A plot of $f(x) = x^3 - x$ and $f'(x) = 3x^2 - 1$.

Exercises for Section 3.1

In the following problems, find the x values for local maximum and minimum points by the method of this section.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = -\dfrac{x^4}{4} + x^3 + x^2$

(8) $f(x) = \begin{cases} x - 1 & x < 2 \\ x^2 & x \geq 2 \end{cases}$

(9) $f(x) = \begin{cases} x - 3 & x < 3 \\ x^3 & 3 \leq x \leq 5 \\ 1/x & x > 5 \end{cases}$

(10) $f(x) = x^2 - 98x + 4$

(11) $f(x) = \begin{cases} -2 & x = 0 \\ 1/x^2 & x \neq 0 \end{cases}$

(12) How many critical points can a quadratic polynomial function have?

(13) Explore the family of functions $f(x) = x^3 + cx + 1$ where c is a constant. How many and what types of local extrema are there? Your answer should depend on the value of c, that is, different values of c will give different answers.

3.2 The First Derivative Test

The method of the previous section for deciding whether there is a local maximum or minimum at a critical point by testing "near-by" points is not always convenient. Instead, since we have already had to compute the derivative to find the critical points, we can use information about the derivative to decide. Recall that

- If $f'(x) > 0$ on an interval, then $f(x)$ is increasing on that interval.
- If $f'(x) < 0$ on an interval, then $f(x)$ is decreasing on that interval.

So how exactly does the derivative tell us whether there is a maximum, minimum, or neither at a point? Use the *first derivative test*.

Theorem 3.2.1 (First Derivative Test) *Suppose that $f(x)$ is continuous on an interval, and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.*
- *If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.*
- *If $f'(x)$ has the same sign to the left and right of $f'(a)$, then $f'(a)$ is not a local extremum.*

Example 3.2.2 Consider the function

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Find the intervals on which $f(x)$ is increasing and decreasing and identify the local extrema of $f(x)$.

Solution *Start by computing*

$$\frac{d}{dx} f(x) = x^3 + x^2 - 2x.$$

Now we need to find when this function is positive and when it is negative. To do this, solve

$$f'(x) = x^3 + x^2 - 2x = 0.$$

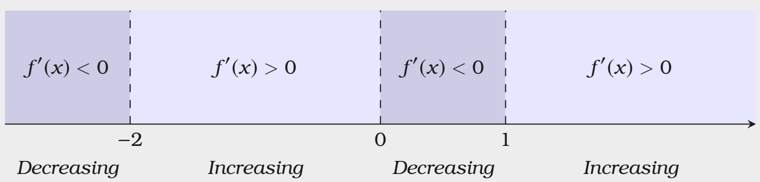
Factor $f'(x)$

$$\begin{aligned} f'(x) &= x^3 + x^2 - 2x \\ &= x(x^2 + x - 2) \\ &= x(x+2)(x-1). \end{aligned}$$

So the critical points (when $f'(x) = 0$*) are when* $x = -2$*,* $x = 0$*, and* $x = 1$*. Now we can check points* ***between*** *the critical points to find when* $f'(x)$ *is increasing and decreasing:*

$$f'(-3) = -12 \qquad f'(.5) = -0.625 \qquad f'(-1) = 2 \qquad f'(2) = 8$$

From this we can make a sign table:



Hence $f(x)$ *is increasing on* $(-2, 0) \cup (1, \infty)$ *and* $f(x)$ *is decreasing on* $(-\infty, -2) \cup (0, 1)$*. Moreover, from the first derivative test, Theorem 3.2.1, the local maximum is at* $x = 0$ *while the local minima are at* $x = -2$ *and* $x = 1$*, see Figure 3.5.*

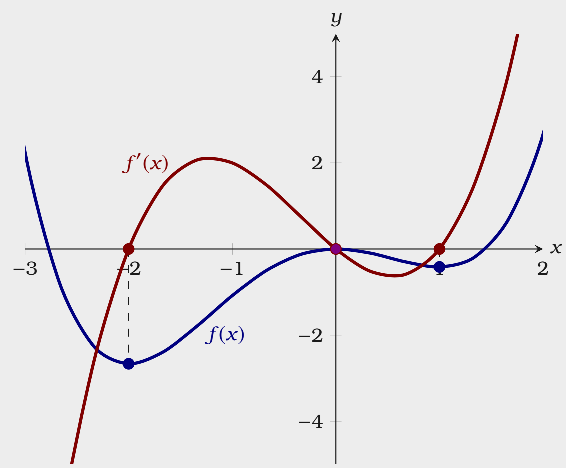


Figure 3.5: A plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f'(x) = x^3 + x^2 - 2x$.

Hence we have seen that if $f'(x)$ is zero and increasing at a point, then $f(x)$ has a local minimum at the point. If $f'(x)$ is zero and decreasing at a point then $f(x)$ has a local maximum at the point. Thus, we see that we can gain information about $f(x)$ by studying how $f'(x)$ changes. This leads us to our next section.

Exercises for Section 3.2

In the following exercises, find all critical points and identify them as local maximum points, local minimum points, or neither.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

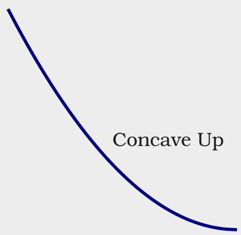
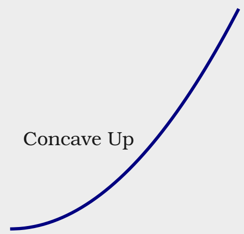
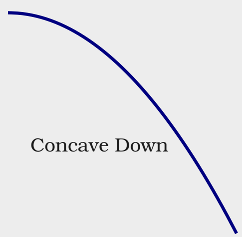
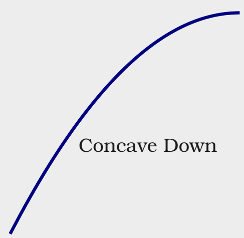
(6) $y = (x^2 - 1)/x$

(7) $f(x) = |x^2 - 121|$

(8) Let $f(x) = ax^2 + bx + c$ with $a \neq 0$. Show that $f(x)$ has exactly one critical point using the first derivative test. Give conditions on a and b which guarantee that the critical point will be a maximum.

3.3 Concavity and Inflection Points

We know that the sign of the derivative tells us whether a function is increasing or decreasing. Likewise, the sign of the second derivative $f''(x)$ tells us whether $f'(x)$ is increasing or decreasing. We summarize this in the table below:

	$f'(x) < 0$	$f'(x) > 0$
$f''(x) > 0$	 Here $f'(x) < 0$ and $f''(x) > 0$. This means that $f(x)$ slopes down and is getting *less steep*. In this case the curve is **concave up**.	 Here $f'(x) > 0$ and $f''(x) > 0$. This means that $f(x)$ slopes up and is getting *steeper*. In this case the curve is **concave up**.
$f''(x) < 0$	 Here $f'(x) < 0$ and $f''(x) < 0$. This means that $f(x)$ slopes down and is getting *steeper*. In this case the curve is **concave down**.	 Here $f'(x) > 0$ and $f''(x) < 0$. This means that $f(x)$ slopes up and is getting less *steep*. In this case the curve is **concave down**.

If we are trying to understand the shape of the graph of a function, knowing where it is concave up and concave down helps us to get a more accurate picture. It is worth summarizing what we have seen already in to a single theorem.

Theorem 3.3.1 (Test for Concavity) *Suppose that $f''(x)$ exists on an interval.*

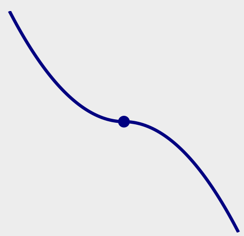
(a) *If $f''(x) > 0$ on an interval, then $f(x)$ is concave up on that interval.*

(b) *If $f''(x) < 0$ on an interval, then $f(x)$ is concave down on that interval.*

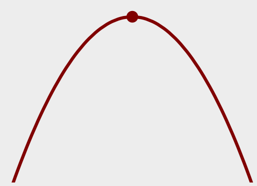
Of particular interest are points at which the concavity changes from up to down or down to up.

Definition If $f(x)$ is continuous and its concavity changes either from up to down or down to up at $x = a$, then $f(x)$ has an **inflection point** at $x = a$.

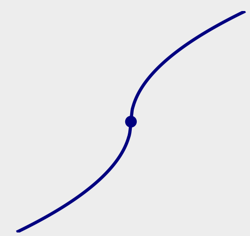
It is instructive to see some examples and nonexamples of inflection points.



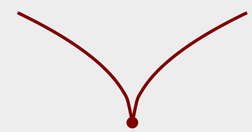
This is an inflection point. The concavity changes from concave up to concave down.



This is **not** an inflection point. The curve is concave down on either side of the point.



This is an inflection point. The concavity changes from concave up to concave down.



This is **not** an inflection point. The curve is concave down on either side of the point.

We identify inflection points by first finding where $f''(x)$ is zero and then checking to see whether $f''(x)$ does in fact go from positive to negative or negative to positive at these points.

Warning Even if $f''(a) = 0$, the point determined by $x = a$ might **not** be an inflection point.

Example 3.3.2 Describe the concavity of $f(x) = x^3 - x$.

Solution *To start, compute the first and second derivative of $f(x)$ with respect to x,*

$$f'(x) = 3x^2 - 1 \qquad \textit{and} \qquad f''(x) = 6x.$$

Since $f''(0) = 0$, there is potentially an inflection point at zero. Since $f''(x) > 0$ when $x > 0$ and $f''(x) < 0$ when $x < 0$ the concavity does change from down to up at zero—there is an inflection point at $x = 0$. The curve is concave down for all $x < 0$ and concave up for all $x > 0$, see Figure 3.6.

Note that we need to compute and analyze the second derivative to understand concavity, so we may as well try to use the second derivative test for maxima and minima. If for some reason this fails we can then try one of the other tests.

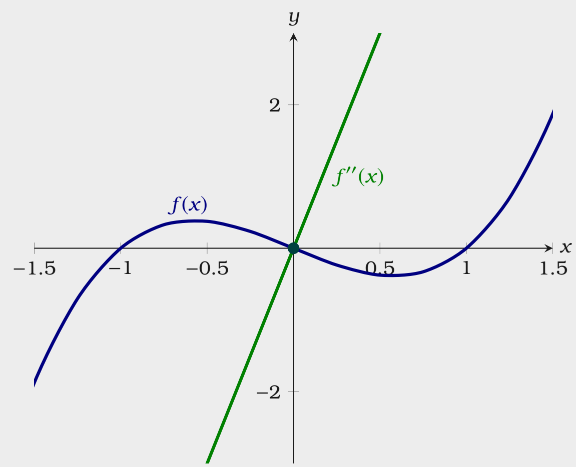


Figure 3.6: A plot of $f(x) = x^3 - x$ and $f''(x) = 6x$. We can see that the concavity change at $x = 0$.

Exercises for Section 3.3

In the following exercises, describe the concavity of the functions.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - \dfrac{1}{x^2}$

(8) $y = x^5 - x$

(9) $y = x + 1/x$

(10) $y = x^2 + 1/x$

(11) Identify the intervals on which the graph of the function $f(x) = x^4 - 4x^3 + 10$ is of one of these four shapes: concave up and increasing; concave up and decreasing; concave down and increasing; concave down and decreasing.

3.4 The Second Derivative Test

Recall the first derivative test, Theorem 3.2.1:

- If $f'(x) > 0$ to the left of a and $f'(x) < 0$ to the right of a, then $f(a)$ is a local maximum.
- If $f'(x) < 0$ to the left of a and $f'(x) > 0$ to the right of a, then $f(a)$ is a local minimum.

If $f'(x)$ changes from positive to negative it is decreasing. In this case, $f''(x)$ might be negative, and if in fact $f''(x)$ is negative then $f'(x)$ is definitely decreasing, so there is a local maximum at the point in question. On the other hand, if $f'(x)$ changes from negative to positive it is increasing. Again, this means that $f''(x)$ might be positive, and if in fact $f''(x)$ is positive then $f'(x)$ is definitely increasing, so there is a local minimum at the point in question. We summarize this as the *second derivative test*.

Theorem 3.4.1 (Second Derivative Test) *Suppose that $f''(x)$ is continuous on an open interval and that $f'(a) = 0$ for some value of a in that interval.*

- *If $f''(a) < 0$, then $f(x)$ has a local maximum at a.*
- *If $f''(a) > 0$, then $f(x)$ has a local minimum at a.*
- *If $f''(a) = 0$, then the test is inconclusive. In this case, $f(x)$ may or may not have a local extremum at $x = a$.*

The second derivative test is often the easiest way to identify local maximum and minimum points. Sometimes the test fails and sometimes the second derivative is quite difficult to evaluate. In such cases we must fall back on one of the previous tests.

Example 3.4.2 Once again, consider the function

$$f(x) = \frac{x^4}{4} + \frac{x^3}{3} - x^2$$

Use the second derivative test, Theorem 3.4.1, to locate the local extrema of $f(x)$.

Solution *Start by computing*

$$f'(x) = x^3 + x^2 - 2x \qquad \text{and} \qquad f''(x) = 3x^2 + 2x - 2.$$

Using the same technique as used in the solution of Example 3.2.2, we find that

$$f'(-2) = 0, \qquad f'(0) = 0, \qquad f'(1) = 0.$$

Now we'll attempt to use the second derivative test, Theorem 3.4.1,

$$f''(-2) = 6, \qquad f''(0) = -2, \qquad f''(1) = 3.$$

Hence we see that $f(x)$ has a local minimum at $x = -2$, a local maximum at $x = 0$, and a local minimum at $x = 1$, see Figure 3.7.

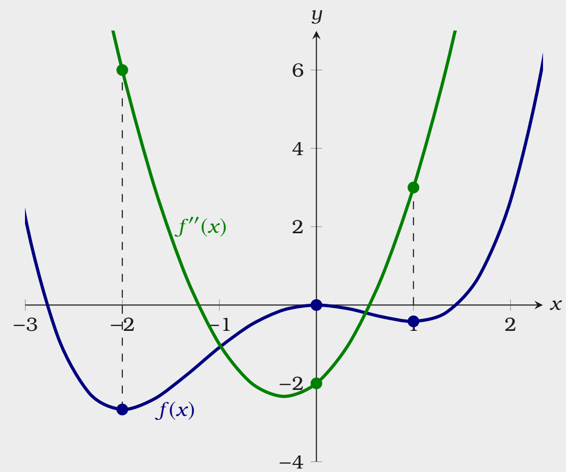


Figure 3.7: A plot of $f(x) = x^4/4 + x^3/3 - x^2$ and $f''(x) = 3x^2 + 2x - 2$.

Warning If $f''(a) = 0$, then the second derivative test gives no information on whether $x = a$ is a local extremum.

Exercises for Section 3.4

Find all local maximum and minimum points by the second derivative test.

(1) $y = x^2 - x$

(2) $y = 2 + 3x - x^3$

(3) $y = x^3 - 9x^2 + 24x$

(4) $y = x^4 - 2x^2 + 3$

(5) $y = 3x^4 - 4x^3$

(6) $y = (x^2 - 1)/x$

(7) $y = 3x^2 - \dfrac{1}{x^2}$

(8) $y = x^5 - x$

(9) $y = x + 1/x$

(10) $y = x^2 + 1/x$

3.5 Sketching the Plot of a Function

In this section, we will give some general guidelines for sketching the plot of a function.

Procedure for Sketching the Plots of Functions

- Find the y-intercept, this is the point $(0, f(0))$. Place this point on your graph.
- Find candidates for vertical asymptotes, these are points where $f(x)$ is undefined.
- Compute $f'(x)$ and $f''(x)$.
- Find the critical points, the points where $f'(x) = 0$.
- Use the second derivative test to identify local extrema and/or find the intervals where your function is increasing/decreasing.
- Find the candidates for inflection points, the points where $f''(x) = 0$.
- Identify inflection points and concavity.
- If possible find the x-intercepts, the points where $f(x) = 0$. Place these points on your graph.
- Find horizontal asymptotes.
- Determine an interval that shows all relevant behavior.

At this point you should be able to sketch the plot of your function.

Let's see this procedure in action. We'll sketch the plot of $2x^3 - 3x^2 - 12x$. Following our guidelines above, we start by computing $f(0) = 0$. Hence we see that the y-intercept is $(0, 0)$. Place this point on your plot, see Figure 3.8.

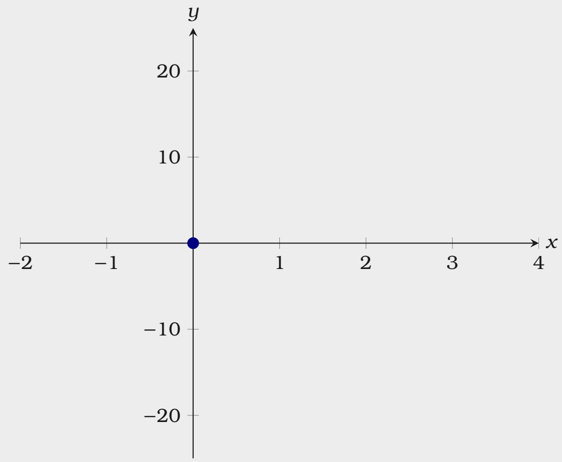


Figure 3.8: We start by placing the point $(0, 0)$.

Note that there are no vertical asymptotes as our function is defined for all real numbers. Now compute $f'(x)$ and $f''(x)$,

$$f'(x) = 6x^2 - 6x - 12 \qquad \text{and} \qquad f''(x) = 12x - 6.$$

The critical points are where $f'(x) = 0$, thus we need to solve $6x^2 - 6x - 12 = 0$ for x. Write

$$\begin{aligned} 6x^2 - 6x - 12 &= 0 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0. \end{aligned}$$

Thus

$$f'(2) = 0 \qquad \text{and} \qquad f'(-1) = 0.$$

Mark the critical points $x = 2$ and $x = -1$ on your plot, see Figure 3.9.

Check the second derivative evaluated at the critical points. In this case,

$$f''(-1) = -18 \qquad \text{and} \qquad f''(2) = 18,$$

hence $x = -1$, corresponding to the point $(-1, 7)$ is a local maximum and $x = 2$, corresponding to the point $(2, -20)$ is local minimum of $f(x)$. Moreover, this tells us that our function is increasing on $[-2, -1)$, decreasing on $(-1, 2)$, and increasing on $(2, 4]$. Identify this on your plot, see Figure 3.10.

The candidates for the inflection points are where $f''(x) = 0$, thus we need to solve $12x - 6 = 0$ for x. Write

$$\begin{aligned} 12x - 6 &= 0 \\ x - 1/2 &= 0 \\ x &= 1/2. \end{aligned}$$

Thus $f''(1/2) = 0$. Checking points, $f''(0) = -6$ and $f''(1) = 6$. Hence $x = 1/2$ is an inflection point, with $f(x)$ concave down to the left of $x = 1/2$ and $f(x)$ concave up to the right of $x = 1/2$. We can add this information to our plot, see Figure 3.11.

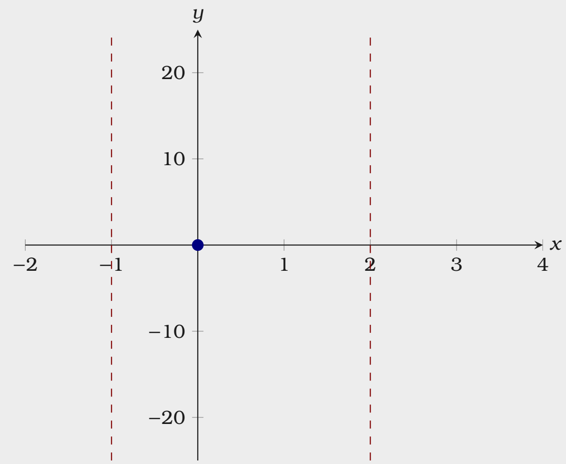


Figure 3.9: Now we add the critical points $x = -1$ and $x = 2$.

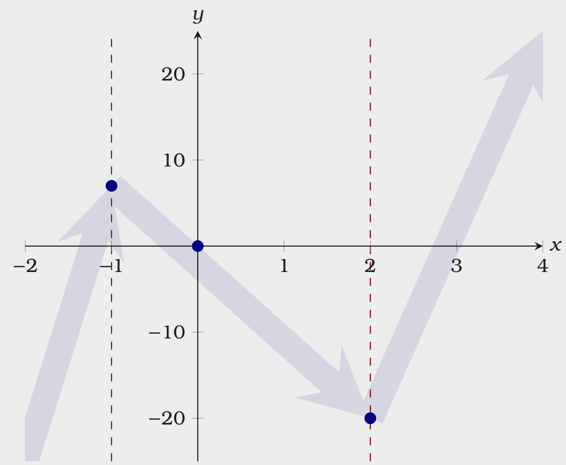


Figure 3.10: We have identified the local extrema of $f(x)$ and where this function is increasing and decreasing.

Finally, in this case, $f(x) = 2x^3 - 3x^2 - 12x$, we can find the x-intercepts. Write

$$2x^3 - 3x^2 - 12x = 0$$
$$x(2x^2 - 3x - 12) = 0.$$

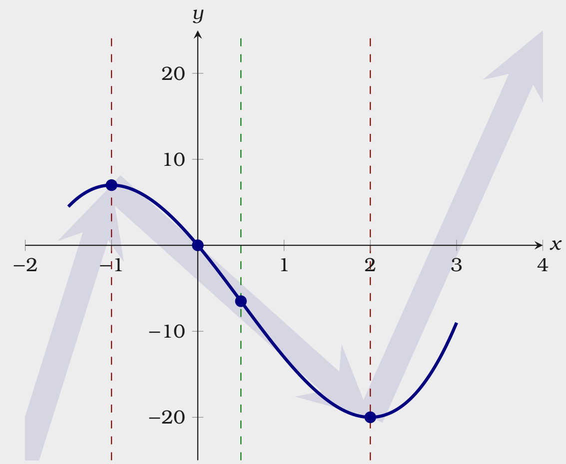
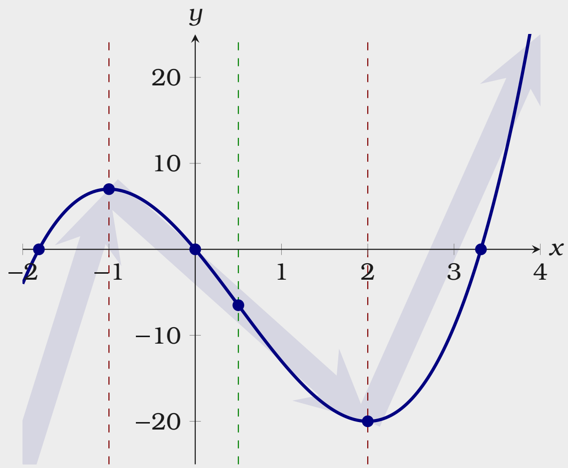


Figure 3.11: We identify the inflection point and note that the curve is concave down when $x < 1/2$ and concave up when $x > 1/2$.

Using the quadratic formula, we see that the x-intercepts of $f(x)$ are

$$x = 0, \qquad x = \frac{3 - \sqrt{105}}{4}, \qquad x = \frac{3 + \sqrt{105}}{4}.$$

Since all of this behavior as described above occurs on the interval $[-2, 4]$, we now have a complete sketch of $f(x)$ on this interval, see the figure below.



Exercises for Section 3.5

Sketch the curves via the procedure outlined in this section. Clearly identify any interesting features, including local maximum and minimum points, inflection points, asymptotes, and intercepts.

(1) $y = x^5 - x$

(2) $y = x(x^2 + 1)$

(3) $y = 2\sqrt{x} - x$

(4) $y = x^3 + 6x^2 + 9x$

(5) $y = x^3 - 3x^2 - 9x + 5$

(6) $y = x^5 - 5x^4 + 5x^3$

(7) $y = x + 1/x$

(8) $y = x^2 + 1/x$

4 The Product Rule and Quotient Rule

4.1 The Product Rule

Consider the product of two simple functions, say

$$f(x) \cdot g(x)$$

where $f(x) = x^2 + 1$ and $g(x) = x^3 - 3x$. An obvious guess for the derivative of $f(x)g(x)$ is the product of the derivatives:

$$\begin{aligned} f'(x)g'(x) &= (2x)(3x^2 - 3) \\ &= 6x^3 - 6x. \end{aligned}$$

Is this guess correct? We can check by rewriting $f(x)$ and $g(x)$ and doing the calculation in a way that is known to work. Write

$$\begin{aligned} f(x)g(x) &= (x^2 + 1)(x^3 - 3x) \\ &= x^5 - 3x^3 + x^3 - 3x \\ &= x^5 - 2x^3 - 3x. \end{aligned}$$

Hence

$$\frac{d}{dx} f(x)g(x) = 5x^4 - 6x^2 - 3,$$

so we see that

$$\frac{d}{dx} f(x)g(x) \neq f'(x)g'(x).$$

So the derivative of $f(x)g(x)$ is **not** as simple as $f'(x)g'(x)$. Never fear, we have a rule for exactly this situation.

Theorem 4.1.1 (The Product Rule) *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx}f(x)g(x) = f(x)g'(x) + f'(x)g(x).$$

Proof *From the limit definition of the derivative, write*

$$\frac{d}{dx}(f(x)g(x)) = \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

Now we use the exact same trick we used in the proof of Theorem 1.2.2, we add $0 = -f(x+h)g(x) + f(x+h)g(x)$:

$$\begin{aligned}
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0}\frac{f(x+h)g(x+h) - f(x+h)g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)g(x) - f(x)g(x)}{h}.
\end{aligned}$$

Now since both $f(x)$ and $g(x)$ are differentiable, they are continuous, see Theorem 2.1.3. Hence

$$\begin{aligned}
&= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}g(x) \\
&= \lim_{h\to 0} f(x+h)\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} + \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\lim_{h\to 0} g(x) \\
&= f(x)g'(x) + f'(x)g(x).
\end{aligned}$$

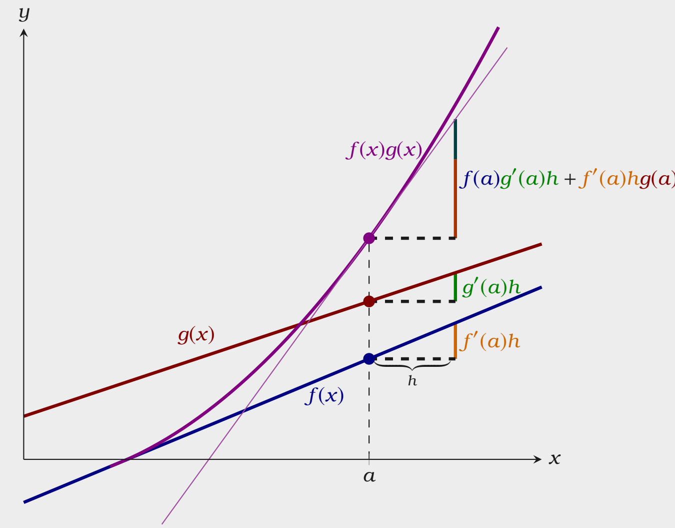


Figure 4.1: A geometric interpretation of the product rule. Since every point on $f(x)g(x)$ is the product of the corresponding points on $f(x)$ and $g(x)$, increasing a by a "small amount" h, increases $f(a)g(a)$ by the sum of $f(a)g'(a)h$ and $f'(a)hg(a)$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f(a)g'(a)h + f'(a)g(a)h}{h} = f(a)g'(a) + f'(a)g(a).$$

Let's return to the example with which we started.

Example 4.1.2 Let $f(x) = (x^2+1)$ and $g(x) = (x^3 - 3x)$. Compute:

$$\frac{d}{dx}f(x)g(x).$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx}f(x)g(x) &= f(x)g'(x) + f'(x)g(x)\\ &= (x^2+1)(3x^2-3) + 2x(x^3-3x).\end{aligned}$$

We could stop here—but we should show that expanding this out recovers our previous result. Write

$$\begin{aligned}(x^2+1)(3x^2-3) + 2x(x^3-3x) &= 3x^4 - 3x^2 + 3x^2 - 3 + 2x^4 - 6x^2\\ &= 5x^4 - 6x^2 - 3,\end{aligned}$$

which is precisely what we obtained before.

Exercises for Section 4.1

Compute:

(1) $\dfrac{d}{dx}x^3(x^3-5x+10)$

(2) $\dfrac{d}{dx}(x^2+5x-3)(x^5-6x^3+3x^2-7x+1)$

(3) $\dfrac{d}{dx}e^{2x}=\dfrac{d}{dx}(e^x\cdot e^x)$

(4) $\dfrac{d}{dx}e^{3x}$

(5) $\dfrac{d}{dx}3x^2e^{4x}$

(6) $\dfrac{d}{dx}\dfrac{3e^x}{x^{16}}$

(7) Use the product rule to compute the derivative of $f(x)=(2x-3)^2$ with respect to x. Sketch the function. Find an equation of the tangent line to the curve at $x=2$. Sketch the tangent line at $x=2$.

Use the following table to compute solve the next 4 problems. Note $\dfrac{d}{dx}f(x)\Big|_{x=a}$ is the derivative of $f(x)$ evaluated at $x=a$.

x	1	2	3	4
$f(x)$	-2	-3	1	4
$f'(x)$	-1	0	3	5
$g(x)$	1	4	2	-1
$g'(x)$	2	-1	-2	-3

(8) $\dfrac{d}{dx}f(x)g(x)\Big|_{x=2}$

(9) $\dfrac{d}{dx}xf(x)\Big|_{x=3}$

(10) $\dfrac{d}{dx}xg(x)\Big|_{x=4}$

(11) $\dfrac{d}{dx}f(x)g(x)\Big|_{x=1}$

(12) Suppose that $f(x)$, $g(x)$, and $h(x)$ are differentiable functions. Show that

$$\frac{d}{dx}f(x)\cdot g(x)\cdot h(x)=f(x)g(x)h'(x)+f(x)g'(x)h(x)+f'(x)g(x)h(x).$$

4.2 The Quotient Rule

We'd like to have a formula to compute

$$\frac{d}{dx}\frac{f(x)}{g(x)}$$

if we already know $f'(x)$ and $g'(x)$. Instead of attacking this problem head-on, let's notice that we've already done part of the problem: $f(x)/g(x) = f(x)\cdot(1/g(x))$, that is, this is really a product, and we can compute the derivative if we know $f'(x)$ and $(1/g(x))'$. This brings us to our next derivative rule.

Theorem 4.2.1 (The Quotient Rule) *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.$$

Proof *First note that if we knew how to compute*

$$\frac{d}{dx}\frac{1}{g(x)}$$

then we could use the product rule to complete our proof. Write

$$\begin{aligned}
\frac{d}{dx}\frac{1}{g(x)} &= \lim_{h\to 0}\frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}\\
&= \lim_{h\to 0}\frac{\frac{g(x)-g(x+h)}{g(x+h)g(x)}}{h}\\
&= \lim_{h\to 0}\frac{g(x)-g(x+h)}{g(x+h)g(x)h}\\
&= \lim_{h\to 0} -\frac{g(x+h)-g(x)}{h}\frac{1}{g(x+h)g(x)}\\
&= -\frac{g'(x)}{g(x)^2}.
\end{aligned}$$

Now we can put this together with the product rule:

$$\begin{aligned}\frac{d}{dx}\frac{f(x)}{g(x)} &= f(x)\frac{-g'(x)}{g(x)^2} + f'(x)\frac{1}{g(x)}\\ &= \frac{-f(x)g'(x) + f'(x)g(x)}{g(x)^2}\\ &= \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.\end{aligned}$$

Example 4.2.2 Compute:

$$\frac{d}{dx}\frac{x^2+1}{x^3-3x}.$$

Solution *Write*

$$\begin{aligned}\frac{d}{dx}\frac{x^2+1}{x^3-3x} &= \frac{2x(x^3-3x) - (x^2+1)(3x^2-3)}{(x^3-3x)^2}\\ &= \frac{-x^4-6x^2+3}{(x^3-3x)^2}.\end{aligned}$$

It is often possible to calculate derivatives in more than one way, as we have already seen. Since every quotient can be written as a product, it is always possible to use the product rule to compute the derivative, though it is not always simpler.

Example 4.2.3 Compute

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}}$$

in two ways. First using the quotient rule and then using the product rule.

Solution *First, we'll compute the derivative using the quotient rule. Write*

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} = \frac{(-2x)\left(\sqrt{x}\right) - (625-x^2)\left(\frac{1}{2}x^{-1/2}\right)}{x}.$$

Second, we'll compute the derivative using the product rule:

$$\frac{d}{dx}\frac{625-x^2}{\sqrt{x}} = \frac{d}{dx}\left(625-x^2\right)x^{-1/2}$$
$$= \left(625-x^2\right)\left(\frac{-x^{-3/2}}{2}\right) + (-2x)\left(x^{-1/2}\right).$$

With a bit of algebra, both of these simplify to

$$-\frac{3x^2+625}{2x^{3/2}}.$$

Exercises for Section 4.2

Find the derivatives of the following functions using the quotient rule.

(1) $\dfrac{x^3}{x^3 - 5x + 10}$

(2) $\dfrac{x^2 + 5x - 3}{x^5 - 6x^3 + 3x^2 - 7x + 1}$

(3) $\dfrac{e^x - 4}{2x}$

(4) $\dfrac{2 - x - \sqrt{x}}{x + 2}$

(5) Find an equation for the tangent line to $f(x) = (x^2 - 4)/(5 - x)$ at $x = 3$.

(6) Find an equation for the tangent line to $f(x) = (x - 2)/(x^3 + 4x - 1)$ at $x = 1$.

(7) The curve $y = 1/(1 + x^2)$ is an example of a class of curves each of which is called a *witch of Agnesi*. Find the tangent line to the curve at $x = 5$. Note, the word *witch* here is due to a mistranslation.

Use the following table to compute solve the next 4 problems. Note $\left.\dfrac{d}{dx} f(x)\right|_{x=a}$ is the derivative of $f(x)$ evaluated at $x = a$.

x	1	2	3	4
$f(x)$	−2	−3	1	4
$f'(x)$	−1	0	3	5
$g(x)$	1	4	2	−1
$g'(x)$	2	−1	−2	−3

(8) $\left.\dfrac{d}{dx} \dfrac{f(x)}{g(x)}\right|_{x=2}$

(9) $\left.\dfrac{d}{dx} \dfrac{f(x)}{x}\right|_{x=3}$

(10) $\left.\dfrac{d}{dx} \dfrac{xf(x)}{g(x)}\right|_{x=4}$

(11) $\left.\dfrac{d}{dx} \dfrac{f(x)g(x)}{x}\right|_{x=1}$

(12) If $f'(4) = 5$, $g'(4) = 12$, $f(4)g(4) = 2$, and $g(4) = 6$, compute $f(4)$ and $\dfrac{d}{dx} \dfrac{f(x)}{g(x)}$ at 4.

5 The Chain Rule

So far we have seen how to compute the derivative of a function built up from other functions by addition, subtraction, multiplication and division. There is another very important way that we combine functions: composition. The *chain rule* allows us to deal with this case.

5.1 The Chain Rule

Consider

$$h(x) = (1 + 2x)^5.$$

While there are several different ways to differentiate this function, if we let $f(x) = x^5$ and $g(x) = 1 + 2x$, then we can express $h(x) = f(g(x))$. The question is, can we compute the derivative of a composition of functions using the derivatives of the constituents $f(x)$ and $g(x)$? To do so, we need the *chain rule*.

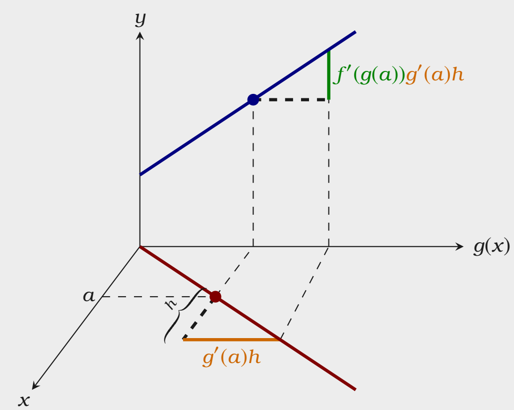


Figure 5.1: A geometric interpretation of the chain rule. Increasing a by a "small amount" h, increases $f(g(a))$ by $f'(g(a))g'(a)h$. Hence,

$$\frac{\Delta y}{\Delta x} \approx \frac{f'(g(a))g'(a)h}{h} = f'(g(a))g'(a).$$

Theorem 5.1.1 (Chain Rule) *If $f(x)$ and $g(x)$ are differentiable, then*

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

Proof *Let g_0 be some x-value and consider the following:*

$$f'(g_0) = \lim_{h \to 0} \frac{f(g_0 + h) - f(g_0)}{h}.$$

Set $h = g - g_0$ and we have

$$f'(g_0) = \lim_{g \to g_0} \frac{f(g) - f(g_0)}{g - g_0}. \tag{5.1}$$

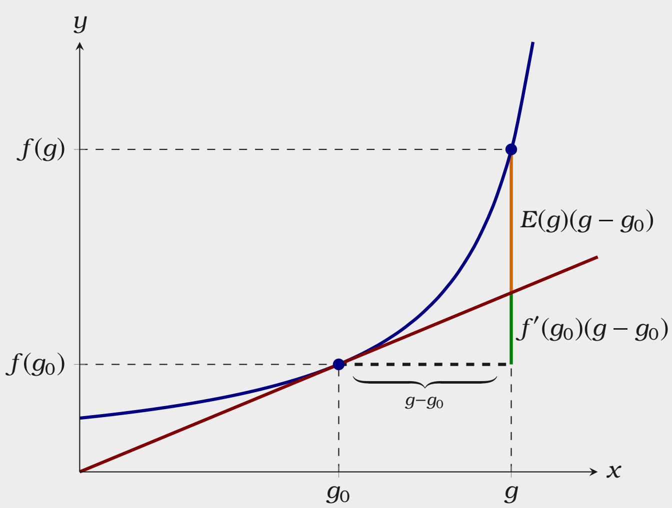
At this point, we might like to set $g = g(x+h)$ and $g_0 = g(x)$; however, we cannot as we cannot be sure that

$$g(x+h) - g(x) \neq 0 \qquad \text{when } h \neq 0.$$

To overcome this difficulty, let $E(g)$ be the "error term" that gives the difference between the slope of the secant line from $f(g_0)$ to $f(g)$ and $f'(g_0)$,

$$E(g) = \frac{f(g) - f(g_0)}{g - g_0} - f'(g_0).$$

In particular, $E(g)(g - g_0)$ is the difference between $f(g)$ and the tangent line of $f(x)$ at $x = g$, see the figure below:



Hence we see that

$$f(g) - f(g_0) = \left(f'(g_0) + E(g)\right)(g - g_0), \tag{5.2}$$

and so

$$\frac{f(g) - f(g_0)}{g - g_0} = f'(g_0) + E(g).$$

Combining this with Equation 5.1, we have that

$$\begin{aligned} f'(g_0) &= \lim_{g \to g_0} \frac{f(g) - f(g_0)}{g - g_0} \\ &= \lim_{g \to g_0} f'(g_0) + E(g) \\ &= f'(g_0) + \lim_{g \to g_0} E(g), \end{aligned}$$

and hence it follows that $\lim_{g \to g_0} E(g) = 0$. *At this point, we may return to the "well-worn path." Starting with Equation 5.2, divide both sides by* h *and set* $g = g(x+h)$ *and* $g_0 = g(x)$

$$\frac{f(g(x+h)) - f(g(x))}{h} = \left(f'(g(x)) + E(g(x))\right) \frac{g(x+h) - g(x)}{h}.$$

Taking the limit as h *approaches* 0, *we see*

$$\begin{aligned} \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h} &= \lim_{h \to 0} \left(f'(g(x)) + E(g(x))\right) \frac{g(x+h) - g(x)}{h} \\ &= \lim_{h \to 0} \left(f'(g(x)) + E(g(x))\right) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(g(x)) g'(x). \end{aligned}$$

Hence, $\dfrac{d}{dx} f(g(x)) = f'(g(x)) g'(x)$.

It will take a bit of practice to make the use of the chain rule come naturally—it is more complicated than the earlier differentiation rules we have seen. Let's return to our motivating example.

Example 5.1.2 Compute:

$$\frac{d}{dx}(1+2x)^5$$

Solution *Set* $f(x) = x^5$ *and* $g(x) = 1 + 2x$*, now*

$$f'(x) = 5x^4 \qquad \text{and} \qquad g'(x) = 2.$$

Hence

$$\begin{aligned}\frac{d}{dx}(1+2x)^5 &= \frac{d}{dx}f(g(x))\\ &= f'(g(x))g'(x)\\ &= 5(1+2x)^4 \cdot 2\\ &= 10(1+2x)^4.\end{aligned}$$

Let's see a more complicated chain of compositions.

Example 5.1.3 Compute:

$$\frac{d}{dx}\sqrt{1+\sqrt{x}}$$

Solution *Set* $f(x) = \sqrt{x}$ *and* $g(x) = 1 + x$. *Hence,*

$$\sqrt{1+\sqrt{x}} = f(g(f(x))) \qquad \text{and} \qquad \frac{d}{dx}f(g(f(x))) = f'(g(f(x)))g'(f(x))f'(x).$$

Since

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad \text{and} \qquad g'(x) = 1$$

We have that

$$\frac{d}{dx}\sqrt{1+\sqrt{x}} = \frac{1}{2\sqrt{1+\sqrt{x}}} \cdot 1 \cdot \frac{1}{2\sqrt{x}}.$$

Using the chain rule, the power rule, and the product rule it is possible to avoid using the quotient rule entirely.

Example 5.1.4 Compute:

$$\frac{d}{dx}\frac{x^3}{x^2+1}$$

Solution *Rewriting this as*

$$\frac{d}{dx}x^3(x^2+1)^{-1},$$

set $f(x) = x^{-1}$ *and* $g(x) = x^2 + 1$. *Now*

$$x^3(x^2+1)^{-1} = x^3 f(g(x)) \qquad \text{and} \qquad \frac{d}{dx}x^3 f(g(x)) = 3x^2 f(g(x)) + x^3 f'(g(x))g'(x).$$

Since $f'(x) = \dfrac{-1}{x^2}$ *and* $g'(x) = 2x$, *write*

$$\frac{d}{dx}\frac{x^3}{x^2+1} = \frac{3x^2}{x^2+1} - \frac{2x^4}{(x^2+1)^2}.$$

Exercises for Section 5.1

Compute the derivatives of the functions. For extra practice, and to check your answers, do some of these in more than one way if possible.

(1) $x^4 - 3x^3 + (1/2)x^2 + 7x - \pi$

(2) $x^3 - 2x^2 + 4\sqrt{x}$

(3) $(x^2 + 1)^3$

(4) $x\sqrt{169 - x^2}$

(5) $(x^2 - 4x + 5)\sqrt{25 - x^2}$

(6) $\sqrt{r^2 - x^2}$, r is a constant

(7) $\sqrt{1 + x^4}$

(8) $\dfrac{1}{\sqrt{5 - \sqrt{x}}}$.

(9) $(1 + 3x)^2$

(10) $\dfrac{(x^2 + x + 1)}{(1 - x)}$

(11) $\dfrac{\sqrt{25 - x^2}}{x}$

(12) $\sqrt{\dfrac{169}{x} - x}$

(13) $\sqrt{x^3 - x^2 - (1/x)}$

(14) $100/(100 - x^2)^{3/2}$

(15) $\sqrt[3]{x + x^3}$

(16) $\sqrt{(x^2 + 1)^2 + \sqrt{1 + (x^2 + 1)^2}}$

(17) $(x + 8)^5$

(18) $(4 - x)^3$

(19) $(x^2 + 5)^3$

(20) $(6 - 2x^2)^3$

(21) $(1 - 4x^3)^{-2}$

(22) $5(x + 1 - 1/x)$

(23) $4(2x^2 - x + 3)^{-2}$

(24) $\dfrac{1}{1 + 1/x}$

(25) $\dfrac{-3}{4x^2 - 2x + 1}$

(26) $(x^2 + 1)(5 - 2x)/2$

(27) $(3x^2 + 1)(2x - 4)^3$

(28) $\dfrac{x + 1}{x - 1}$

(29) $\dfrac{x^2 - 1}{x^2 + 1}$

(30) $\dfrac{(x - 1)(x - 2)}{x - 3}$

(31) $\dfrac{2x^{-1} - x^{-2}}{3x^{-1} - 4x^{-2}}$

(32) $3(x^2 + 1)(2x^2 - 1)(2x + 3)$

(33) $\dfrac{1}{(2x + 1)(x - 3)}$

(34) $((2x + 1)^{-1} + 3)^{-1}$

(35) $(2x + 1)^3(x^2 + 1)^2$

(36) Find an equation for the tangent line to $f(x) = (x-2)^{1/3}/(x^3+4x-1)^2$ at $x = 1$.

(37) Find an equation for the tangent line to $y = 9x^{-2}$ at $(3, 1)$.

(38) Find an equation for the tangent line to $(x^2 - 4x + 5)\sqrt{25 - x^2}$ at $(3, 8)$.

(39) Find an equation for the tangent line to $\dfrac{(x^2 + x + 1)}{(1 - x)}$ at $(2, -7)$.

(40) Find an equation for the tangent line to $\sqrt{(x^2+1)^2 + \sqrt{1 + (x^2+1)^2}}$ at $(1, \sqrt{4 + \sqrt{5}})$.

5.2 Implicit Differentiation

The functions we've been dealing with so far have been *explicit functions*, meaning that the dependent variable is written in terms of the independent variable. For example:

$$y = 3x^2 - 2x + 1, \qquad y = e^{3x}, \qquad y = \frac{x-2}{x^2-3x+2}.$$

However, there are another type of functions, called *implicit functions*. In this case, the dependent variable is not stated explicitly in terms of the independent variable. For example:

$$x^2 + y^2 = 4, \qquad x^3 + y^3 = 9xy, \qquad x^4 + 3x^2 = x^{2/3} + y^{2/3} = 1.$$

Your inclination might be simply to solve each of these for y and go merrily on your way. However this can be difficult and it may require two *branches*, for example to explicitly plot $x^2 + y^2 = 4$, one needs both $y = \sqrt{4-x^2}$ and $y = -\sqrt{4-x^2}$. Moreover, it may not even be possible to solve for y. To deal with such situations, we use *implicit differentiation*. Let's see an illustrative example:

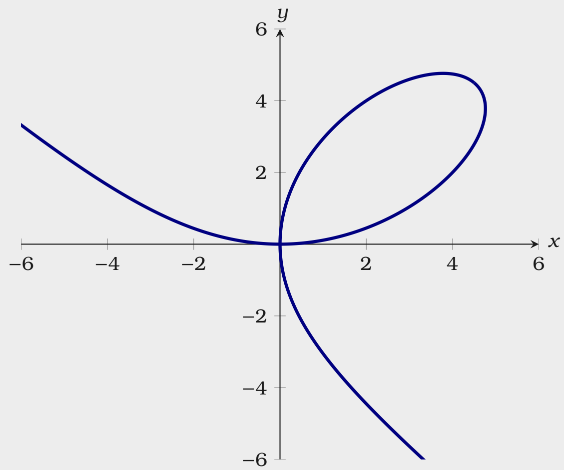


Figure 5.2: A plot of $x^3 + y^3 = 9xy$. While this is not a function of y in terms of x, the equation still defines a relation between x and y.

Example 5.2.1 Consider the curve defined by

$$x^3 + y^3 = 9xy.$$

(a) Compute $\dfrac{dy}{dx}$.

(b) Find the slope of the tangent line at $(4, 2)$.

Solution *Starting with*

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\dfrac{d}{dx}$ *to both sides of the equation to obtain*

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}9xy.$$

Applying the sum rule we see

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of these terms in turn. To start

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand $\frac{d}{dx}y^3$ *is somewhat different. Here you imagine that* $y = y(x)$, *and hence by the chain rule*

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3\\ &= 3(y(x))^2 \cdot y'(x)\\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

Considering the final term $\frac{d}{dx}9xy$, *we again imagine that* $y = y(x)$. *Hence*

$$\begin{aligned}\frac{d}{dx}9xy &= 9\frac{d}{dx}x \cdot y(x)\\ &= 9\left(x \cdot y'(x) + y(x)\right)\\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

Putting this all together we are left with the equation

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

At this point, we solve for $\frac{dy}{dx}$. *Write*

$$\begin{aligned}3x^2 + 3y^2\frac{dy}{dx} &= 9x\frac{dy}{dx} + 9y\\ 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} &= 9y - 3x^2\\ \frac{dy}{dx}\left(3y^2 - 9x\right) &= 9y - 3x^2\\ \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} &= \frac{3y - x^2}{y^2 - 3x}.\end{aligned}$$

For the second part of the problem, we simply plug $x = 4$ *and* $y = 2$ *into the formula above, hence the slope of the tangent line at* $(4, 2)$ *is* $\frac{5}{4}$, *see Figure 5.3.*

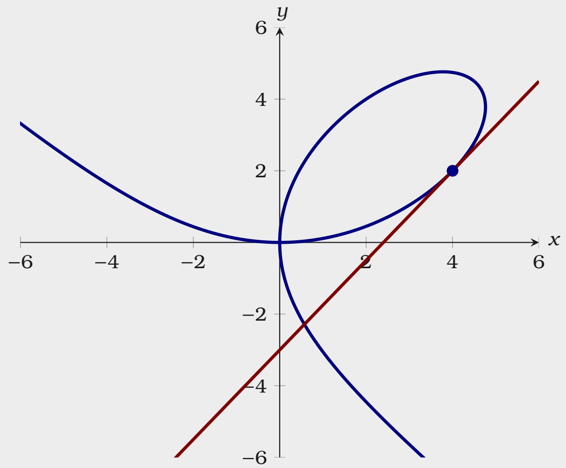


Figure 5.3: A plot of $x^3 + y^3 = 9xy$ along with the tangent line at $(4, 2)$.

You might think that the step in which we solve for $\dfrac{dy}{dx}$ could sometimes be difficult—after all, we're using implicit differentiation here instead of the more difficult task of solving the equation $x^3 + y^3 = 9xy$ for y, so maybe there are functions where after taking the derivative we obtain something where it is hard to solve for $\dfrac{dy}{dx}$. In fact, *this never happens*. All occurrences $\dfrac{dy}{dx}$ arise from applying the chain rule, and whenever the chain rule is used it deposits a single $\dfrac{dy}{dx}$ multiplied by some other expression. Hence our expression is linear in $\dfrac{dy}{dx}$, it will always be possible to group the terms containing $\dfrac{dy}{dx}$ together and factor out the $\dfrac{dy}{dx}$, just as in the previous example.

The Derivative of Inverse Functions

Geometrically, there is a close relationship between the plots of e^x and $\ln(x)$, they are reflections of each other over the line $y = x$, see Figure 5.4. One may suspect that we can use the fact that $\dfrac{d}{dx}e^x = e^x$, to deduce the derivative of $\ln(x)$. We will use implicit differentiation to exploit this relationship computationally.

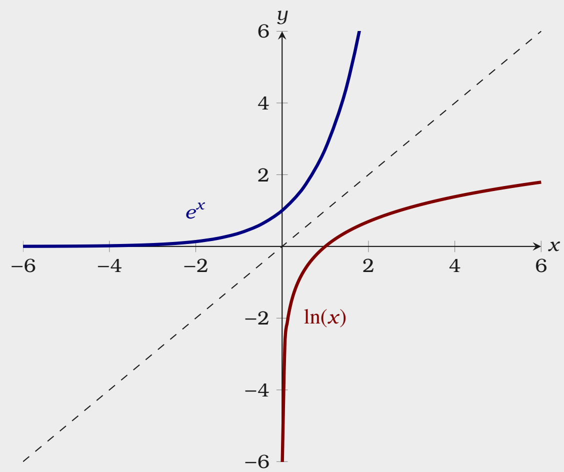


Figure 5.4: A plot of e^x and $\ln(x)$. Since they are inverse functions, they are reflections of each other across the line $y = x$.

Theorem 5.2.2 (The Derivative of the Natural Logrithm)

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

Proof *Recall*

$$\ln(x) = y \qquad \Leftrightarrow \qquad e^y = x.$$

Hence

$$\begin{aligned} e^y &= x \\ \frac{d}{dx}e^y &= \frac{d}{dx}x && \textit{Differentiate both sides.} \\ e^y\frac{dy}{dx} &= 1 && \textit{Implicit differentiation.} \\ \frac{dy}{dx} &= \frac{1}{e^y} = \frac{1}{x}. \end{aligned}$$

Since $y = \ln(x)$, $\dfrac{d}{dx}\ln(x) = \dfrac{1}{x}$.

There is one catch to the proof given above. To write $\frac{d}{dx}(e^y) = e^y\frac{dy}{dx}$ we need to know that the function y *has* a derivative. All we have shown is that *if* it has a derivative then that derivative must be $1/x$. The *Inverse Function Theorem* guarantees this.

Theorem 5.2.3 (Inverse Function Theorem) *If $f(x)$ is a differentiable function, and $f'(x)$ is continuous, and $f'(a) \neq 0$, then*

(a) *$f^{-1}(y)$ is defined for y near $f(a)$,*

(b) *$f^{-1}(y)$ is differentiable near $f(a)$,*

(c) *$\dfrac{d}{dy}f^{-1}(y)$ is continuous near $f(a)$, and*

(d) $\dfrac{d}{dy}f^{-1}(y) = \dfrac{1}{f'(f^{-1}(y))}$.

Exercises for Section 5.2

Compute $\dfrac{dy}{dx}$:

(1) $x^2 + y^2 = 4$

(2) $y^2 = 1 + x^2$

(3) $x^2 + xy + y^2 = 7$

(4) $x^3 + xy^2 = y^3 + yx^2$

(5) $x^2 y - y^3 = 6$

(6) $\sqrt{x} + \sqrt{y} = 9$

(7) $xy^{3/2} + 4 = 2x + y$

(8) $\dfrac{1}{x} + \dfrac{1}{y} = 7$

(9) A hyperbola passing through $(8,6)$ consists of all points whose distance from the origin is a constant more than its distance from the point (5,2). Find the slope of the tangent line to the hyperbola at $(8,6)$.

(10) The graph of the equation $x^2 - xy + y^2 = 9$ is an ellipse. Find the lines tangent to this curve at the two points where it intersects the x-axis. Show that these lines are parallel.

(11) Repeat the previous problem for the points at which the ellipse intersects the y-axis.

(12) Find the points on the ellipse from the previous two problems where the slope is horizontal and where it is vertical.

(13) Find an equation for the tangent line to $x^4 = y^2 + x^2$ at $(2, \sqrt{12})$. This curve is the *kampyle of Eudoxus*.

(14) Find an equation for the tangent line to $x^{2/3} + y^{2/3} = a^{2/3}$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0$ and $y_1 \neq 0$. This curve is an *astroid*.

(15) Find an equation for the tangent line to $(x^2 + y^2)^2 = x^2 - y^2$ at a point (x_1, y_1) on the curve, with $x_1 \neq 0, -1, 1$. This curve is a *lemniscate*.

5.3 Logarithmic Differentiation

Logarithms were originally developed as a computational tool. The key fact that made this possible is that

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

While this may seem quite abstract, before the days of calculators and computers, this was critical knowledge for anyone in a computational discipline. Suppose you wanted to compute

$$138 \cdot 23.4$$

You would start by writing both in scientific notation

$$\left(1.38 \cdot 10^2\right) \cdot \left(2.34 \cdot 10^1\right).$$

Next you would use a log-table, which gives $\log_{10}(N)$ for values of N ranging between 0 and 9. We've reproduced part of such a table below.

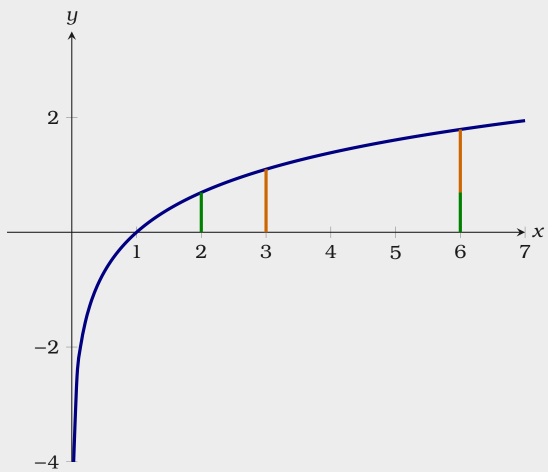


Figure 5.5: A plot of $\ln(x)$. Here we see that

$$\ln(2 \cdot 3) = \ln(2) + \ln(3).$$

N	0	1	2	3	4	5	6	7	8	9
1.3	0.1139	0.1173	0.1206	0.1239	0.1271	0.1303	0.1335	0.1367	0.1399	0.1430
			...							
2.3	0.3617	0.3636	0.3655	0.3674	0.3692	0.3711	0.3729	0.3747	0.3766	0.3784
			...							
3.2	0.5052	0.5065	0.5079	0.5092	0.5105	0.5119	0.5132	0.5145	0.5159	0.5172

Figure 5.6: Part of a base-10 logarithm table.

From the table, we see that

$$\log_{10}(1.38) \approx 0.1399 \qquad \text{and} \qquad \log_{10}(2.34) \approx 0.3692$$

Add these numbers together to get 0.5091. Essentially, we know the following at this point:

$$\begin{array}{ccccc} \log_{10}(?) & = & \log_{10}(1.38) & + & \log_{10}(2.34) \\ \wr\wr & & \wr\wr & & \wr\wr \\ 0.5091 & = & 0.1399 & + & 0.3692 \end{array}$$

Using the table again, we see that $\log_{10}(3.23) \approx 0.5091$. Since we were working in scientific notation, we need to multiply this by 10^3. Our final answer is

$$3230 \approx 138 \cdot 23.4$$

Since $138 \cdot 23.4 = 3229.2$, this is a good approximation. The moral is:

Logarithms allow us to use addition in place of multiplication.

When taking derivatives, both the product rule and the quotient rule can be cumbersome to use. Logarithms will save the day. A key point is the following

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{f(x)} \cdot f'(x) = \frac{f'(x)}{f(x)}$$

which follows from the chain rule. Let's look at an illustrative example to see how this is actually used.

Example 5.3.1 Compute:

$$\frac{d}{dx}\frac{x^9 e^{4x}}{\sqrt{x^2+4}}$$

Recall the properties of logarithms:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b(x/y) = \log_b(x) - \log_b(y)$
- $\log_b(x^y) = y\log_b(x)$

Solution *While we could use the product and quotient rule to solve this problem, it would be tedious. Start by taking the logarithm of the function to be differentiated.*

$$\begin{aligned}\ln\left(\frac{x^9 e^{4x}}{\sqrt{x^2+4}}\right) &= \ln\left(x^9 e^{4x}\right) - \ln\left(\sqrt{x^2+4}\right)\\ &= \ln\left(x^9\right) + \ln\left(e^{4x}\right) - \ln\left((x^2+4)^{1/2}\right)\\ &= 9\ln(x) + 4x - \frac{1}{2}\ln(x^2+4).\end{aligned}$$

Setting $f(x) = \frac{x^9 e^{4x}}{\sqrt{x^2+4}}$*, we can write*

$$\ln(f(x)) = 9\ln(x) + 4x - \frac{1}{2}\ln(x^2+4).$$

Differentiating both sides, we find

$$\frac{f'(x)}{f(x)} = \frac{9}{x} + 4 - \frac{x}{x^2+4}.$$

Finally we solve for $f'(x)$, write

$$f'(x) = \left(\frac{9}{x} + 4 - \frac{x}{x^2+4}\right)\left(\frac{x^9 e^{4x}}{\sqrt{x^2+4}}\right).$$

The process above is called *logarithmic differentiation*. Logarithmic differentiation allows us to compute new derivatives too.

Example 5.3.2 Compute:

$$\frac{d}{dx}x^x$$

Solution *The function x^x is tricky to differentiate. We cannot use the power rule, as the exponent is not a constant. However, if we set $f(x) = x^x$ we can write*

$$\begin{aligned}\ln(f(x)) &= \ln\left(x^x\right)\\ &= x\ln(x).\end{aligned}$$

Differentiating both sides, we find

$$\begin{aligned}\frac{f'(x)}{f(x)} &= x \cdot \frac{1}{x} + \ln(x)\\ &= 1 + \ln(x).\end{aligned}$$

Now we can solve for $f'(x)$,

$$f'(x) = x^x + x^x\ln(x).$$

Finally recall that previously we only proved the power rule, Theorem 2.2.2, for positive exponents. Now we'll use logarithmic differentiation to give a proof for all real-valued exponents. We restate the power rule for convenience sake:

Theorem 5.3.3 (Power Rule) *For any real number n,*

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Proof *We will use logarithmic differentiation. Set $f(x) = x^n$. Write*

$$\begin{aligned}\ln(f(x)) &= \ln\left(x^n\right)\\ &= n\ln(x).\end{aligned}$$

Now differentiate both sides, and solve for $f'(x)$

$$\begin{aligned}\frac{f'(x)}{f(x)} &= \frac{n}{x}\\ f'(x) &= \frac{nf(x)}{x}\\ &= nx^{n-1}.\end{aligned}$$

Thus we see that the power rule holds for all real-valued exponents.

While logarithmic differentiation might seem strange and new at first, with a little practice it will seem much more natural to you.

Exercises for Section 5.3

Use logarithmic differentiation to compute the following:

(1) $\dfrac{d}{dx}(x+1)^3\sqrt{x^4+5}$

(2) $\dfrac{d}{dx}x^2e^{5x}$

(3) $\dfrac{d}{dx}x^{\ln(x)}$

(4) $\dfrac{d}{dx}x^{100x}$

(5) $\dfrac{d}{dx}\left((3x)^{4x}\right)$

(6) $\dfrac{d}{dx}x^{(e^x)}$

(7) $\dfrac{d}{dx}x^{\pi}+\pi^{x}$

(8) $\dfrac{d}{dx}\left(1+\dfrac{1}{x}\right)^x$

(9) $\dfrac{d}{dx}(\ln(x))^x$

(10) $\dfrac{d}{dx}\left(f(x)g(x)h(x)\right)$

6 The Derivatives of Trigonometric Functions and their Inverses

6.1 The Derivatives of Trigonometric Functions

Up until this point of the course we have been largely ignoring a large class of functions—those involving $\sin(x)$ and $\cos(x)$. It is now time to visit our two friends who concern themselves periodically with triangles and circles.

Theorem 6.1.1 (The Derivative of sin(*x*))

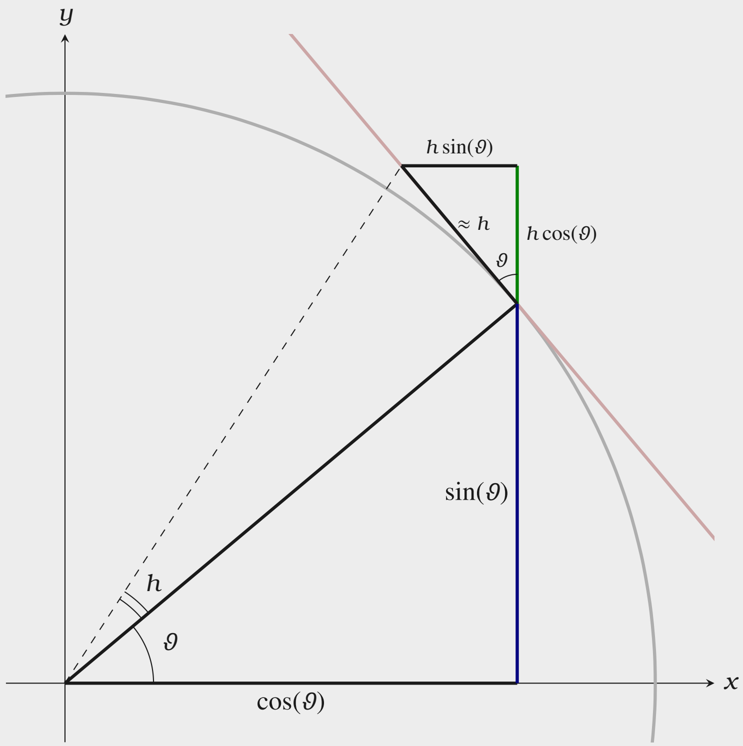
$$\frac{d}{dx}\sin(x) = \cos(x).$$

Proof *Using the definition of the derivative, write*

$$\begin{aligned}
\frac{d}{dx}\sin(x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h} \\
&= \lim_{h\to 0}\frac{\sin(x)\cos(h)+\sin(h)\cos(x)-\sin(x)}{h} && \textit{Trig Identity.} \\
&= \lim_{h\to 0}\left(\frac{\sin(x)\cos(h)-\sin(x)}{h}+\frac{\sin(h)\cos x}{h}\right) \\
&= \lim_{h\to 0}\left(\sin(x)\frac{\cos(h)-1}{h}+\cos(x)\frac{\sin(h)}{h}\right) \\
&= \sin(x)\cdot 0+\cos(x)\cdot 1 = \cos x. && \textit{See Example 1.3.6.}
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0}\frac{\cos(h)-1}{h} &= \lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\cdot\frac{\cos(h)+1}{\cos(h)+1}\right) \\
&= \lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(h)+1)} \\
&= \lim_{h\to 0}\frac{-\sin^2(h)}{h(\cos(h)+1)} \\
&= -\lim_{h\to 0}\left(\frac{\sin(h)}{h}\cdot\frac{\sin(h)}{(\cos(h)+1)}\right) \\
&= -1\cdot\frac{0}{2} = 0.
\end{aligned}$$

Consider the following geometric interpretation of the derivative of $\sin(\vartheta)$.



Here we see that Increasing ϑ by a "small amount" h, increases $\sin(\vartheta)$ by $h\cos(\vartheta)$. Hence,

$$\frac{\Delta y}{\Delta\vartheta} \approx \frac{h\cos(\vartheta)}{h} = \cos(\vartheta).$$

Since the tangent line to the circle is locally a good approximation for the circle and radians measure the arc length of the unit circle, the hypotenuse of the small triangle in the figure is approximately h.

The derivative of a function measures the slope of the plot of a function. If we examine the graphs of the sine and cosine side by side, it should be that the latter appears to accurately describe the slope of the former, and indeed this is true, see Figure 6.1.

Of course, now that we know the derivative of the sine, we can compute derivatives of more complicated functions involving the sine.

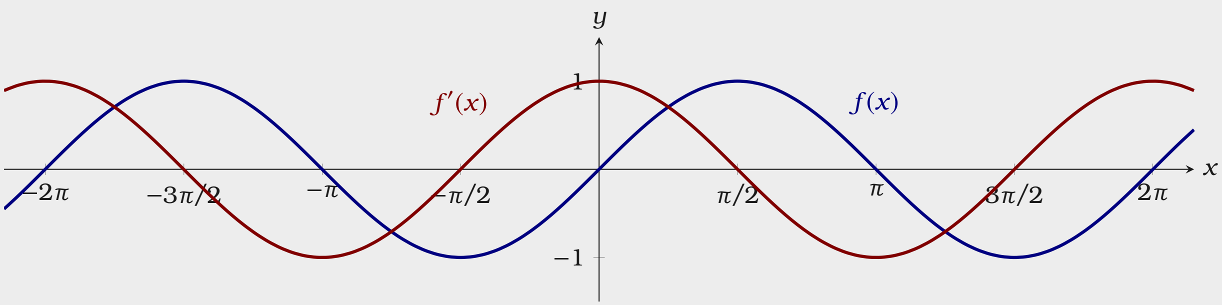


Figure 6.1: Here we see a plot of $f(x) = \sin(x)$ and its derivative $f'(x) = \cos(x)$. One can readily see that $\cos(x)$ is positive when $\sin(x)$ is increasing, and that $\cos(x)$ is negative when $\sin(x)$ is decreasing.

Theorem 6.1.2 (The Derivative of cos(x))

$$\frac{d}{dx}\cos(x) = -\sin(x).$$

Proof *Recall that*

$$\cos(x) = \sin\left(x + \frac{\pi}{2}\right),$$
$$\sin(x) = -\cos\left(x + \frac{\pi}{2}\right).$$

Now:

$$\begin{aligned} \frac{d}{dx}\cos(x) &= \frac{d}{dx}\sin\left(x + \frac{\pi}{2}\right) \\ &= \cos\left(x + \frac{\pi}{2}\right)\cdot 1 \\ &= -\sin(x). \end{aligned}$$

Next we have:

Theorem 6.1.3 (The Derivative of tan(x))

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

Proof *We'll rewrite* $\tan(x)$ *as* $\dfrac{\sin(x)}{\cos(x)}$ *and use the quotient rule. Write*

$$\begin{aligned}\frac{d}{dx}\tan(x) &= \frac{d}{dx}\frac{\sin(x)}{\cos(x)}\\ &= \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}\\ &= \frac{1}{\cos^2(x)}\\ &= \sec^2(x).\end{aligned}$$

Finally, we have

Theorem 6.1.4 (The Derivative of sec(x))

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x).$$

Proof *We'll rewrite* $\sec(x)$ *as* $(\cos(x))^{-1}$ *and use the power rule and the chain rule. Write*

$$\begin{aligned}\frac{d}{dx}\sec(x) &= \frac{d}{dx}(\cos(x))^{-1}\\ &= -1(\cos(x))^{-2}(-\sin(x))\\ &= \frac{\sin(x)}{\cos^2(x)}\\ &= \sec(x)\tan(x).\end{aligned}$$

The derivatives of the cotangent and cosecant are similar and left as exercises. Putting this all together, we have:

Theorem 6.1.5 (The Derivatives of Trigonometric Functions)

- $\dfrac{d}{dx}\sin(x) = \cos(x)$.
- $\dfrac{d}{dx}\cos(x) = -\sin(x)$.
- $\dfrac{d}{dx}\tan(x) = \sec^2(x)$.
- $\dfrac{d}{dx}\sec(x) = \sec(x)\tan(x)$.
- $\dfrac{d}{dx}\csc(x) = -\csc(x)\cot(x)$.
- $\dfrac{d}{dx}\cot(x) = -\csc^2(x)$.

Warning When working with derivatives of trigonometric functions, we suggest you use **radians** for angle measure. For example, while

$$\sin\left((90^\circ)^2\right) = \sin\left(\left(\frac{\pi}{2}\right)^2\right),$$

one must be careful with derivatives as

$$\frac{d}{dx}\sin\left(x^2\right)\Big|_{x=90^\circ} \neq \underbrace{2\cdot 90\cdot\cos(90^2)}_{\text{incorrect}}$$

Alternatively, one could think of x° as meaning $\dfrac{x\cdot\pi}{180}$, as then $90^\circ = \dfrac{90\cdot\pi}{180} = \dfrac{\pi}{2}$. In this case

$$2\cdot 90^\circ\cdot\cos((90^\circ)^2) = 2\cdot\frac{\pi}{2}\cdot\cos\left(\left(\frac{\pi}{2}\right)^2\right).$$

Exercises for Section 6.1

Find the derivatives of the following functions.

(1) $\sin^2(\sqrt{x})$

(2) $\sqrt{x}\sin(x)$

(3) $\dfrac{1}{\sin(x)}$

(4) $\dfrac{x^2+x}{\sin(x)}$

(5) $\sqrt{1-\sin^2(x)}$

(6) $\sin(x)\cos(x)$

(7) $\sin(\cos(x))$

(8) $\sqrt{x\tan(x)}$

(9) $\tan(x)/(1+\sin(x))$

(10) $\cot(x)$

(11) $\csc(x)$

(12) $x^3\sin(23x^2)$

(13) $\sin^2(x)+\cos^2(x)$

(14) $\sin(\cos(6x))$

(15) Compute $\dfrac{d}{d\vartheta}\dfrac{\sec(\vartheta)}{1+\sec(\vartheta)}$.

(16) Compute $\dfrac{d}{dt}t^5\cos(6t)$.

(17) Compute $\dfrac{d}{dt}\dfrac{t^3\sin(3t)}{\cos(2t)}$.

(18) Find all points on the graph of $f(x) = \sin^2(x)$ at which the tangent line is horizontal.

(19) Find all points on the graph of $f(x) = 2\sin(x) - \sin^2(x)$ at which the tangent line is horizontal.

(20) Find an equation for the tangent line to $\sin^2(x)$ at $x = \pi/3$.

(21) Find an equation for the tangent line to $\sec^2(x)$ at $x = \pi/3$.

(22) Find an equation for the tangent line to $\cos^2(x) - \sin^2(4x)$ at $x = \pi/6$.

(23) Find the points on the curve $y = x + 2\cos(x)$ that have a horizontal tangent line.

6.2 Inverse Trigonometric Functions

The trigonometric functions frequently arise in problems, and often we are interested in finding specific angles, say ϑ such that

$$\sin(\vartheta) = .7$$

Hence we want to be able to invert functions like $\sin(\vartheta)$ and $\cos(\vartheta)$.

However, since these functions are not one-to-one, meaning there are are infinitely many angles with $\sin(\vartheta) = .7$, it is impossible to find a true inverse function for $\sin(\vartheta)$. Nevertheless, it is useful to have something like an inverse to the sine, however imperfect. The usual approach is to pick out some collection of angles that produce all possible values of the sine exactly once. If we "discard" all other angles, the resulting function has a proper inverse.

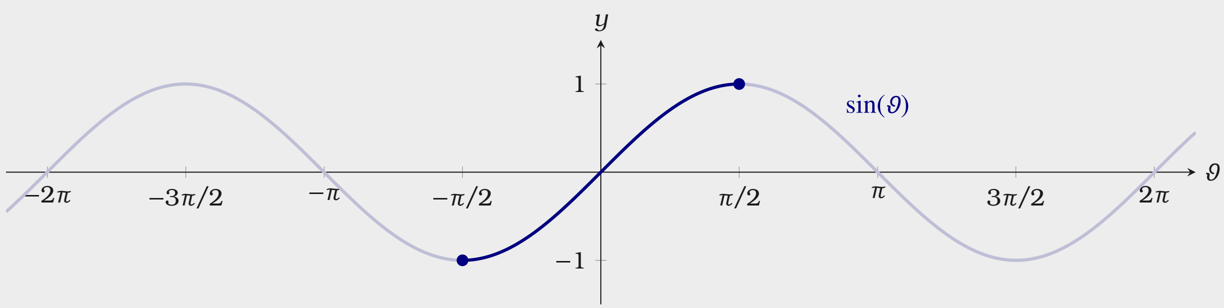


Figure 6.2: The function $\sin(\vartheta)$ takes on all values between -1 and 1 exactly once on the interval $[-\pi/2, \pi/2]$. If we restrict $\sin(\vartheta)$ to this interval, then this restricted function has an inverse.

In a similar fashion, we need to restrict cosine to be able to take an inverse.

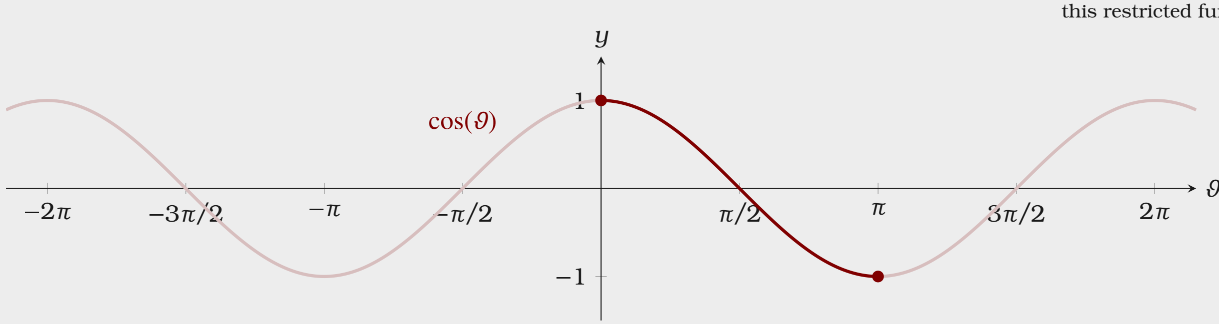
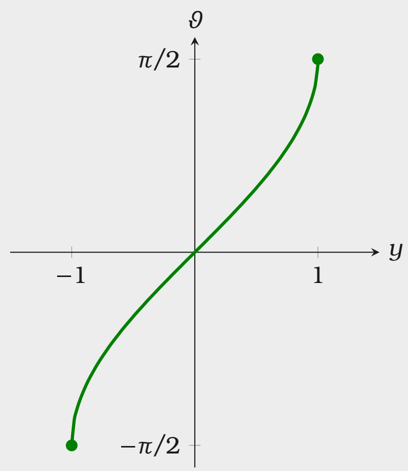
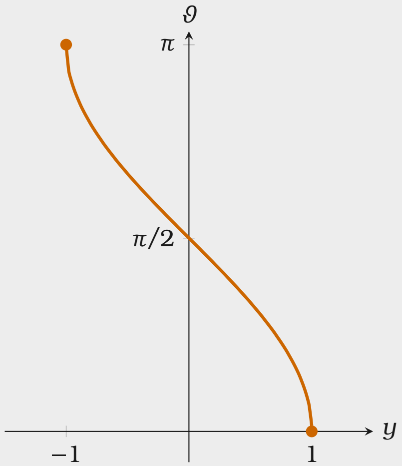


Figure 6.3: The function $\cos(\vartheta)$ takes on all values between -1 and 1 exactly once on the interval $[0, \pi]$. If we restrict $\cos(\vartheta)$ to this interval, then this restricted function has an inverse.

By examining both sine and cosine on restricted domains, we can now produce functions arcsine and arccosine:



Here we see a plot of $\arcsin(y)$, the inverse function of $\sin(\vartheta)$ when it is restricted to the interval $[-\pi/2, \pi/2]$.



Here we see a plot of $\arccos(y)$, the inverse function of $\cos(\vartheta)$ when it is restricted to the interval $[0, \pi]$.

Recall that a function and its inverse undo each other in either order, for example,

$$\sqrt[3]{x^3} = x \qquad \text{and} \qquad \left(\sqrt[3]{x}\right)^3 = x.$$

Compare this with the fact that while $\left(\sqrt{x}\right)^2 = x$, we have that $\sqrt{x^2} = |x|$.

However, since arcsine is the inverse of sine restricted to the interval $[-\pi/2, \pi/2]$, this does not work with sine and arcsine, for example

$$\arcsin(\sin(\pi)) = 0.$$

Moreover, there is a similar situation for cosine and arccosine as

$$\arccos(\cos(2\pi)) = 0.$$

Once you get a feel for how $\arcsin(y)$ and $\arccos(y)$ behave, let's examine tangent.

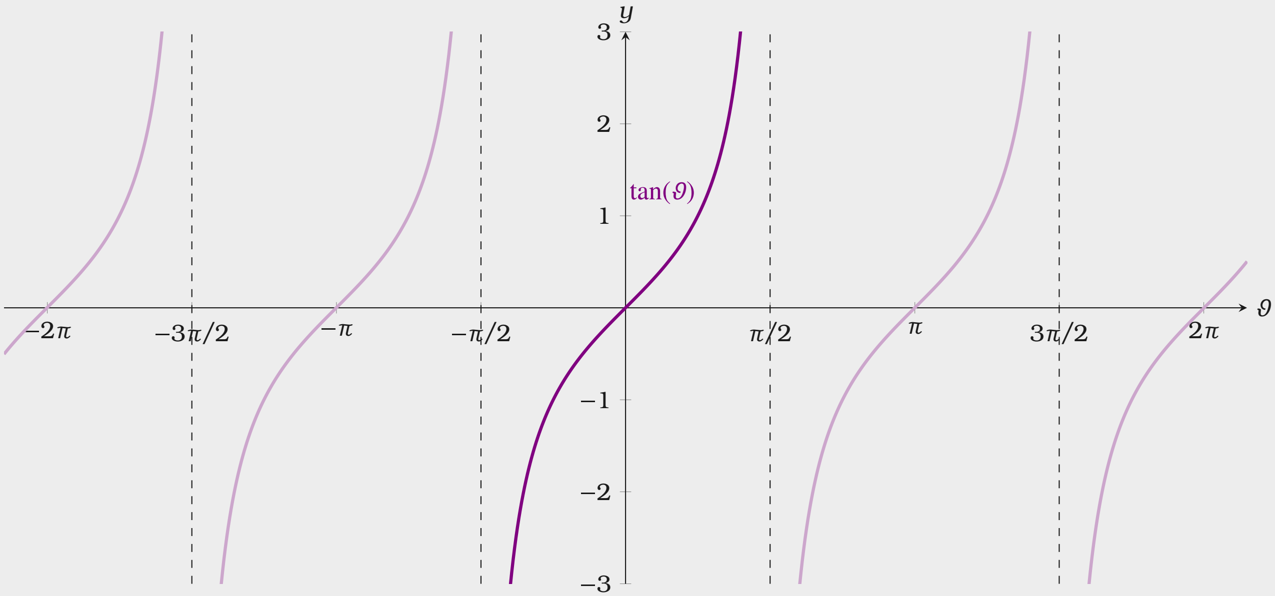


Figure 6.4: The function $\tan(\vartheta)$ takes on all values in $\mathbb{R}$ exactly once on the open interval $(-\pi/2, \pi/2)$. If we restrict $\tan(\vartheta)$ to this interval, then this restricted function has an inverse.

Again, only working on a restricted domain of tangent, we can produce an inverse function, arctangent.

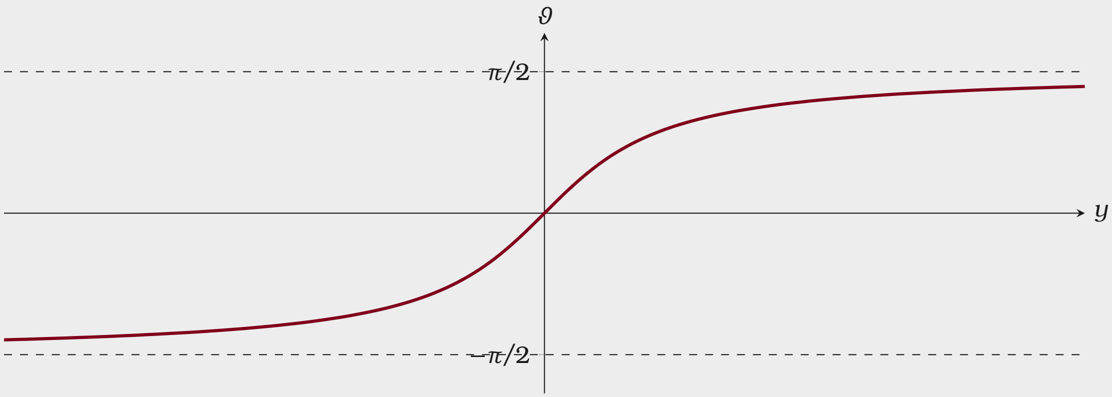


Figure 6.5: Here we see a plot of $\arctan(y)$, the inverse function of $\tan(\vartheta)$ when it is restricted to the interval $(-\pi/2, \pi/2)$.

We leave it to you, the reader, to investigate the functions arcsecant, arccosecant, and arccotangent.

The Derivatives of Inverse Trigonometric Functions

What is the derivative of the arcsine? Since this is an inverse function, we can find its derivative by using implicit differentiation and the Inverse Function Theorem, Theorem 5.2.3.

Theorem 6.2.1 (The Derivative of arcsin(y))

$$\frac{d}{dy}\arcsin(y) = \frac{1}{\sqrt{1-y^2}}.$$

Proof *To start, note that the Inverse Function Theorem, Theorem 5.2.3 assures us that this derivative actually exists. Recall*

$$\arcsin(y) = \vartheta \qquad \Rightarrow \qquad \sin(\vartheta) = y.$$

Hence

$$\begin{aligned} \sin(\vartheta) &= y \\ \frac{d}{dy}\sin(\vartheta) &= \frac{d}{dy}y \\ \cos(\vartheta)\frac{d\vartheta}{dy} &= 1 \\ \frac{d\vartheta}{dy} &= \frac{1}{\cos(\vartheta)}. \end{aligned}$$

At this point, we would like $\cos(\vartheta)$ *written in terms of* y*. Since*

$$\cos^2(\vartheta) + \sin^2(\vartheta) = 1$$

and $\sin(\vartheta) = y$*, we may write*

$$\begin{aligned} \cos^2(\vartheta) + y^2 &= 1 \\ \cos^2(\vartheta) &= 1 - y^2 \\ \cos(\vartheta) &= \pm\sqrt{1-y^2}. \end{aligned}$$

Since $\vartheta = \arcsin(y)$ *we know that* $-\pi/2 \le \vartheta \le \pi/2$, *and the cosine of an angle in this interval is always positive. Thus* $\cos(\vartheta) = \sqrt{1-y^2}$ *and*

$$\frac{d}{dy}\arcsin(y) = \frac{1}{\sqrt{1-y^2}}.$$

We can do something similar with arccosine.

Theorem 6.2.2 (The Derivative of arccos(y))

$$\frac{d}{dy}\arccos(y) = \frac{-1}{\sqrt{1-y^2}}.$$

Proof *To start, note that the Inverse Function Theorem, Theorem 5.2.3 assures us that this derivative actually exists. Recall*

$$\arccos(y) = \vartheta \qquad \Rightarrow \qquad \cos(\vartheta) = y.$$

Hence

$$\begin{aligned} \cos(\vartheta) &= y \\ \frac{d}{dy}\cos(\vartheta) &= \frac{d}{dy}y \\ -\sin(\vartheta)\frac{d\vartheta}{dy} &= 1 \\ \frac{d\vartheta}{dy} &= \frac{-1}{\sin(\vartheta)}. \end{aligned}$$

At this point, we would like $\sin(\vartheta)$ *written in terms of* y. *Since*

$$\cos^2(\vartheta) + \sin^2(\vartheta) = 1$$

and $\cos(\vartheta) = y$*, we may write*

$$\begin{aligned} y^2 + \sin^2(\vartheta) &= 1 \\ \sin^2(\vartheta) &= 1 - y^2 \\ \sin(\vartheta) &= \pm\sqrt{1 - y^2}. \end{aligned}$$

Since $\vartheta = \arccos(y)$ *we know that* $0 \le \vartheta \le \pi$*, and the sine of an angle in this interval is always positive. Thus* $\sin(\vartheta) = \sqrt{1 - y^2}$ *and*

$$\frac{d}{dy}\arccos(y) = \frac{-1}{\sqrt{1 - y^2}}.$$

Finally, let's look at arctangent.

Theorem 6.2.3 (The Derivative of arctan(y))

$$\frac{d}{dy}\arctan(y) = \frac{1}{1 + y^2}.$$

Proof *To start, note that the Inverse Function Theorem, Theorem 5.2.3 assures us that this derivative actually exists. Recall*

$$\arctan(y) = \vartheta \qquad \Rightarrow \qquad \tan(\vartheta) = y.$$

Hence

$$\begin{aligned} \tan(\vartheta) &= y \\ \frac{d}{dy}\tan(\vartheta) &= \frac{d}{dy}y \\ \sec^2(\vartheta)\frac{d\vartheta}{dy} &= 1 \\ \frac{d\vartheta}{dy} &= \frac{1}{\sec^2(\vartheta)}. \end{aligned}$$

At this point, we would like $\sec^2(\vartheta)$ *written in terms of* y. *Recall*

$$\sec^2(\vartheta) = 1 + \tan^2(\vartheta)$$

and $\tan(\vartheta) = y$, *we may write* $\sec^2(\vartheta) = 1 + y^2$. *Hence*

$$\frac{d}{dy}\arctan(y) = \frac{1}{1+y^2}.$$

We leave it to you, the reader, to investigate the derivatives of arcsecant, arccosecant, and arccotangent. However, as a gesture of friendship, we now present you with a list of derivative formulas for inverse trigonometric functions.

Theorem 6.2.4 (The Derivatives of Inverse Trigonometric Functions)

- $\dfrac{d}{dy}\arcsin(y) = \dfrac{1}{\sqrt{1-y^2}}$.
- $\dfrac{d}{dy}\arccos(y) = \dfrac{-1}{\sqrt{1-y^2}}$.
- $\dfrac{d}{dy}\arctan(y) = \dfrac{1}{1+y^2}$.
- $\dfrac{d}{dy}\operatorname{arcsec}(y) = \dfrac{1}{|y|\sqrt{y^2-1}}$ *for* $|y| > 1$.
- $\dfrac{d}{dy}\operatorname{arccsc}(y) = \dfrac{-1}{|y|\sqrt{y^2-1}}$ *for* $|y| > 1$.
- $\dfrac{d}{dy}\operatorname{arccot}(y) = \dfrac{-1}{1+y^2}$.

Exercises for Section 6.2

(1) The inverse of cot is usually defined so that the range of arccotangent is $(0, \pi)$. Sketch the graph of $y = \operatorname{arccot}(x)$. In the process you will make it clear what the domain of arccotangent is. Find the derivative of the arccotangent.

(2) Find the derivative of $\arcsin(x^2)$.

(3) Find the derivative of $\arctan(e^x)$.

(4) Find the derivative of $\arccos(\sin x^3)$

(5) Find the derivative of $\ln((\arcsin(x))^2)$

(6) Find the derivative of $\arccos(e^x)$

(7) Find the derivative of $\arcsin(x) + \arccos(x)$

(8) Find the derivative of $\log_5(\arctan(x^x))$

7 Applications of Differentiation

7.1 L'Hôpital's Rule

Derivatives allow us to take problems that were once difficult to solve and convert them to problems that are easier to solve. Let us consider l'Hôpital's rule:

Theorem 7.1.1 (L'Hôpital's Rule) *Let $f(x)$ and $g(x)$ be functions that are differentiable near a. If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \qquad \text{or} \pm\infty,$$

and $\lim_{x\to a} \dfrac{f'(x)}{g'(x)}$ exists, and $g'(x) \neq 0$ for all x near a, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

L'Hôpital's rule applies even when $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \mp\infty$. See Example 7.1.4.

This theorem is somewhat difficult to prove, in part because it incorporates so many different possibilities, so we will not prove it here.

L'Hôpital's rule allows us to investigate limits of *indeterminate form*.

Definition (List of Indeterminate Forms)

0/0 This refers to a limit of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$ where $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$.

∞/∞ This refers to a limit of the form $\lim_{x\to a}\frac{f(x)}{g(x)}$ where $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to a$.

0·∞ This refers to a limit of the form $\lim_{x\to a}(f(x)\cdot g(x))$ where $f(x)\to 0$ and $g(x)\to\infty$ as $x\to a$.

∞−∞ This refers to a limit of the form $\lim_{x\to a}(f(x)-g(x))$ where $f(x)\to\infty$ and $g(x)\to\infty$ as $x\to a$.

1^∞ This refers to a limit of the form $\lim_{x\to a}f(x)^{g(x)}$ where $f(x)\to 1$ and $g(x)\to\infty$ as $x\to a$.

0^0 This refers to a limit of the form $\lim_{x\to a}f(x)^{g(x)}$ where $f(x)\to 0$ and $g(x)\to 0$ as $x\to a$.

∞^0 This refers to a limit of the form $\lim_{x\to a}f(x)^{g(x)}$ where $f(x)\to\infty$ and $g(x)\to 0$ as $x\to a$.

In each of these cases, the value of the limit is **not** immediately obvious. Hence, a careful analysis is required!

Our first example is the computation of a limit that was somewhat difficult before, see Example 1.3.6. Note, this is an example of the indeterminate form $0/0$.

Example 7.1.2 (0/0) Compute

$$\lim_{x\to 0}\frac{\sin(x)}{x}.$$

Solution *Set* $f(x) = \sin(x)$ *and* $g(x) = x$. *Since both* $f(x)$ *and* $g(x)$ *are differentiable functions at* 0, *and*

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0,$$

so this situation is ripe for l'Hôpital's Rule. Now

$$f'(x) = \cos(x) \qquad \textit{and} \qquad g'(x) = 1.$$

L'Hôpital's rule tells us that

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1.$$

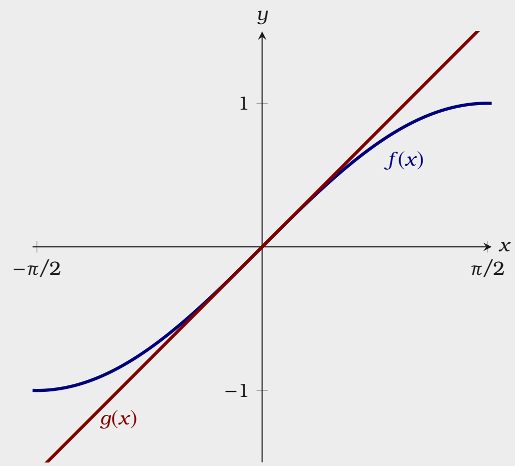


Figure 7.1: A plot of $f(x) = \sin(x)$ and $g(x) = x$. Note how the tangent lines for each curve are coincide at $x = 0$.

From this example, we gain an intuitive feeling for why l'Hôpital's rule is true: If two functions are both 0 when $x = a$, and if their tangent lines have the same slope, then the functions coincide as x approaches a. See Figure 7.1.

Our next set of examples will run through the remaining indeterminate forms one is likely to encounter.

Example 7.1.3 (∞/∞) Compute

$$\lim_{x\to \pi/2+} \frac{\sec(x)}{\tan(x)}.$$

Solution *Set* $f(x) = \sec(x)$ *and* $g(x) = \tan(x)$. *Both* $f(x)$ *and* $g(x)$ *are differentiable near* $\pi/2$. *Additionally,*

$$\lim_{x\to \pi/2+} f(x) = \lim_{x\to \pi/2+} g(x) = -\infty.$$

This situation is ripe for l'Hôpital's Rule. Now

$$f'(x) = \sec(x)\tan(x) \qquad \textit{and} \qquad g'(x) = \sec^2(x),$$

L'Hôpital's rule tells us that

$$\lim_{x\to \pi/2+} \frac{\sec(x)}{\tan(x)} = \lim_{x\to \pi/2+} \frac{\sec(x)\tan(x)}{\sec^2(x)} = \lim_{x\to \pi/2+} \sin(x) = 1.$$

Example 7.1.4 ($0 \cdot \infty$) Compute

$$\lim_{x \to 0+} x \ln x.$$

Solution *This doesn't appear to be suitable for l'Hôpital's Rule. As x approaches zero, $\ln x$ goes to $-\infty$, so the product looks like*

$$(\textit{something very small}) \cdot (\textit{something very large and negative}).$$

This product could be anything—a careful analysis is required. Write

$$x \ln x = \frac{\ln x}{x^{-1}}.$$

Set $f(x) = \ln(x)$ and $g(x) = x^{-1}$. Since both functions are differentiable near zero and

$$\lim_{x \to 0+} \ln(x) = -\infty \qquad \textit{and} \qquad \lim_{x \to 0+} x^{-1} = \infty,$$

we may apply l'Hôpital's rule. Write

$$f'(x) = x^{-1} \qquad \textit{and} \qquad g'(x) = -x^{-2},$$

so

$$\lim_{x \to 0+} x \ln x = \lim_{x \to 0+} \frac{\ln x}{x^{-1}} = \lim_{x \to 0+} \frac{x^{-1}}{-x^{-2}} = \lim_{x \to 0+} -x = 0.$$

One way to interpret this is that since $\lim_{x \to 0^+} x \ln x = 0$, the function x approaches zero much faster than $\ln x$ approaches $-\infty$.

Indeterminate Forms Involving Subtraction

There are two basic cases here, we'll do an example of each.

Example 7.1.5 ($\infty - \infty$) Compute

$$\lim_{x \to 0} \left(\cot(x) - \csc(x)\right).$$

Solution *Here we simply need to write each term as a fraction,*

$$\begin{aligned}\lim_{x\to 0}(\cot(x)-\csc(x)) &= \lim_{x\to 0}\left(\frac{\cos(x)}{\sin(x)}-\frac{1}{\sin(x)}\right)\\ &= \lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)}\end{aligned}$$

Setting $f(x)=\cos(x)-1$ *and* $g(x)=\sin(x)$*, both functions are differentiable near zero and*

$$\lim_{x\to 0}(\cos(x)-1)=\lim_{x\to 0}\sin(x)=0.$$

We may now apply l'Hôpital's rule. Write

$$f'(x)=-\sin(x) \qquad \textit{and} \qquad g'(x)=\cos(x),$$

so

$$\lim_{x\to 0}(\cot(x)-\csc(x))=\lim_{x\to 0}\frac{\cos(x)-1}{\sin(x)}=\lim_{x\to 0}\frac{-\sin(x)}{\cos(x)}=0.$$

Sometimes one must be slightly more clever.

Example 7.1.6 ($\infty-\infty$) Compute

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right).$$

Solution *Again, this doesn't appear to be suitable for l'Hôpital's Rule. A bit of algebraic manipulation will help. Write*

$$\begin{aligned}\lim_{x\to\infty}\left(\sqrt{x^2+x}-x\right) &= \lim_{x\to\infty}\left(x\left(\sqrt{1+1/x}-1\right)\right)\\ &= \lim_{x\to\infty}\frac{\sqrt{1+1/x}-1}{x^{-1}}\end{aligned}$$

Now set $f(x)=\sqrt{1+1/x}-1$, $g(x)=x^{-1}$*. Since both functions are differentiable for large values of* x *and*

$$\lim_{x\to\infty}(\sqrt{1+1/x}-1)=\lim_{x\to\infty}x^{-1}=0,$$

we may apply l'Hôpital's rule. Write

$$f'(x) = (1/2)(1 + 1/x)^{-1/2} \cdot (-x^{-2}) \qquad \textit{and} \qquad g'(x) = -x^{-2}$$

so

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right) &= \lim_{x\to\infty} \frac{\sqrt{1+1/x} - 1}{x^{-1}} \\
&= \lim_{x\to\infty} \frac{(1/2)(1+1/x)^{-1/2} \cdot (-x^{-2})}{-x^{-2}} \\
&= \lim_{x\to\infty} \frac{1}{2\sqrt{1+1/x}} \\
&= \frac{1}{2}.
\end{aligned}$$

Exponential Indeterminate Forms

There is a standard trick for dealing with the indeterminate forms

$$1^\infty, \qquad 0^0, \qquad \infty^0.$$

Given $u(x)$ and $v(x)$ such that

$$\lim_{x\to a} u(x)^{v(x)}$$

falls into one of the categories described above, rewrite as

$$\lim_{x\to a} e^{v(x)\ln(u(x))}$$

and then examine

$$\lim_{x\to a} \frac{\ln(u(x))}{v(x)^{-1}}$$

using l'Hôpital's rule. Since these forms are all very similar, we will only give a single example.

Example 7.1.7 (1^∞) Compute

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x.$$

Solution *Write*

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)}.$$

So now look at

$$\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}}$$

Setting $f(x) = \ln\left(1+\frac{1}{x}\right)$ *and* $g(x) = x^{-1}$*, both functions are differentiable for large values of* x *and*

$$\lim_{x\to\infty}\ln\left(1+\frac{1}{x}\right) = \lim_{x\to\infty} x^{-1} = 0.$$

We may now apply l'Hôpital's rule. Write

$$f'(x) = \frac{-x^{-2}}{1+\frac{1}{x}} \qquad \text{and} \qquad g'(x) = -x^{-2},$$

so

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln\left(1+\frac{1}{x}\right)}{x^{-1}} &= \lim_{x\to\infty}\frac{\frac{-x^{-2}}{1+\frac{1}{x}}}{-x^{-2}}\\ &= \lim_{x\to\infty}\frac{1}{1+\frac{1}{x}}\\ &= 1.\end{aligned}$$

Hence,

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} e^{x\ln\left(1+\frac{1}{x}\right)} = e^1 = e.$$

Exercises for Section 7.1

Compute the limits.

(1) $\displaystyle\lim_{x\to 0} \frac{\cos x - 1}{\sin x}$

(2) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^3}$

(3) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+x} - \sqrt{x^2-x}$

(4) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{x}$

(5) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$

(6) $\displaystyle\lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}}$

(7) $\displaystyle\lim_{x\to 0} \frac{\sqrt{9+x} - 3}{x}$

(8) $\displaystyle\lim_{t\to 1^+} \frac{(1/t) - 1}{t^2 - 2t + 1}$

(9) $\displaystyle\lim_{x\to 2} \frac{2 - \sqrt{x+2}}{4 - x^2}$

(10) $\displaystyle\lim_{t\to\infty} \frac{t + 5 - 2/t - 1/t^3}{3t + 12 - 1/t^2}$

(11) $\displaystyle\lim_{y\to\infty} \frac{\sqrt{y+1} + \sqrt{y-1}}{y}$

(12) $\displaystyle\lim_{x\to 1} \frac{\sqrt{x} - 1}{\sqrt[3]{x} - 1}$

(13) $\displaystyle\lim_{x\to 0} \frac{(1-x)^{1/4} - 1}{x}$

(14) $\displaystyle\lim_{t\to 0} \left(t + \frac{1}{t}\right)((4-t)^{3/2} - 8)$

(15) $\displaystyle\lim_{t\to 0^+} \left(\frac{1}{t} + \frac{1}{\sqrt{t}}\right)(\sqrt{t+1} - 1)$

(16) $\displaystyle\lim_{x\to 0} \frac{x^2}{\sqrt{2x+1} - 1}$

(17) $\displaystyle\lim_{u\to 1} \frac{(u-1)^3}{(1/u) - u^2 + 3/u - 3}$

(18) $\displaystyle\lim_{x\to 0} \frac{2 + (1/x)}{3 - (2/x)}$

(19) $\displaystyle\lim_{x\to 0^+} \frac{1 + 5/\sqrt{x}}{2 + 1/\sqrt{x}}$

(20) $\displaystyle\lim_{x\to 0^+} \frac{3 + x^{-1/2} + x^{-1}}{2 + 4x^{-1/2}}$

(21) $\displaystyle\lim_{x\to\infty} \frac{x + x^{1/2} + x^{1/3}}{x^{2/3} + x^{1/4}}$

(22) $\displaystyle\lim_{t\to\infty} \frac{1 - \sqrt{\frac{t}{t+1}}}{2 - \sqrt{\frac{4t+1}{t+2}}}$

(23) $\displaystyle\lim_{t\to\infty} \frac{1 - \frac{t}{t-1}}{1 - \sqrt{\frac{t}{t-1}}}$

(24) $\displaystyle\lim_{x\to -\infty} \frac{x + x^{-1}}{1 + \sqrt{1-x}}$

(25) $\displaystyle\lim_{x\to\pi/2} \frac{\cos x}{(\pi/2) - x}$

(26) $\lim_{x\to 0} \dfrac{e^x - 1}{x}$

(27) $\lim_{x\to 0} \dfrac{x^2}{e^x - x - 1}$

(28) $\lim_{x\to 1} \dfrac{\ln x}{x - 1}$

(29) $\lim_{x\to 0} \dfrac{\ln(x^2 + 1)}{x}$

(30) $\lim_{x\to 1} \dfrac{x \ln x}{x^2 - 1}$

(31) $\lim_{x\to 0} \dfrac{\sin(2x)}{\ln(x + 1)}$

(32) $\lim_{x\to 1} \dfrac{x^{1/4} - 1}{x}$

(33) $\lim_{x\to 1^+} \dfrac{\sqrt{x}}{x - 1}$

(34) $\lim_{x\to 1} \dfrac{\sqrt{x} - 1}{x - 1}$

(35) $\lim_{x\to \infty} \dfrac{x^{-1} + x^{-1/2}}{x + x^{-1/2}}$

(36) $\lim_{x\to \infty} \dfrac{x + x^{-2}}{2x + x^{-2}}$

(37) $\lim_{x\to \infty} \dfrac{5 + x^{-1}}{1 + 2x^{-1}}$

(38) $\lim_{x\to \infty} \dfrac{4x}{\sqrt{2x^2 + 1}}$

(39) $\lim_{x\to 0} \dfrac{3x^2 + x + 2}{x - 4}$

(40) $\lim_{x\to 0} \dfrac{\sqrt{x + 1} - 1}{\sqrt{x + 4} - 2}$

(41) $\lim_{x\to 0} \dfrac{\sqrt{x + 1} - 1}{\sqrt{x + 2} - 2}$

(42) $\lim_{x\to 0^+} \dfrac{\sqrt{x + 1} + 1}{\sqrt{x + 1} - 1}$

(43) $\lim_{x\to 0} \dfrac{\sqrt{x^2 + 1} - 1}{\sqrt{x + 1} - 1}$

(44) $\lim_{x\to \infty} (x + 5)\left(\dfrac{1}{2x} + \dfrac{1}{x + 2}\right)$

(45) $\lim_{x\to 0^+} (x + 5)\left(\dfrac{1}{2x} + \dfrac{1}{x + 2}\right)$

(46) $\lim_{x\to 1} (x + 5)\left(\dfrac{1}{2x} + \dfrac{1}{x + 2}\right)$

(47) $\lim_{x\to 2} \dfrac{x^3 - 6x - 2}{x^3 + 4}$

(48) $\lim_{x\to 2} \dfrac{x^3 - 6x - 2}{x^3 - 4x}$

(49) $\lim_{x\to 1+} \dfrac{x^3 + 4x + 8}{2x^3 - 2}$

7.2 The Derivative as a Rate

The world is constantly changing around us. To simplify matters we will only consider change in one dimension. This means that if we think of a ball being tossed in the air, we will consider its vertical movement separately from its lateral and forward movement. To understand how things change, we need to understand the *rate* of change. Let's start out with some rather basic ideas.

Definition Given a function $f(x)$, the **average rate of change** over the interval $[a, a+\Delta x]$ is given by

$$\frac{f(a+\Delta x)-f(a)}{\Delta x}.$$

Example 7.2.1 Suppose you drive a car on a 600 mile road trip. Your distance from home is recorded by the plot shown in Figure 7.2. What was your average velocity during hours 4–8 of your trip?

Solution *Examining Figure 7.2, we see that we were around* 240 *miles from home at hour* 4*, and* 360 *miles from home at hour* 8*. Hence our average velocity was*

$$\frac{360-240}{8-4} = \frac{120}{4} = 30 \textit{ miles per hour.}$$

Of course if you look at Figure 7.2 closely, you see that sometimes we were driving faster and other times we were driving slower. To get more information, we need to know the *instantaneous rate of change*.

Definition Given a function, the **instantaneous rate of change** at $x = a$ is given by

$$\left.\frac{d}{dx}f(x)\right|_{x=a}.$$

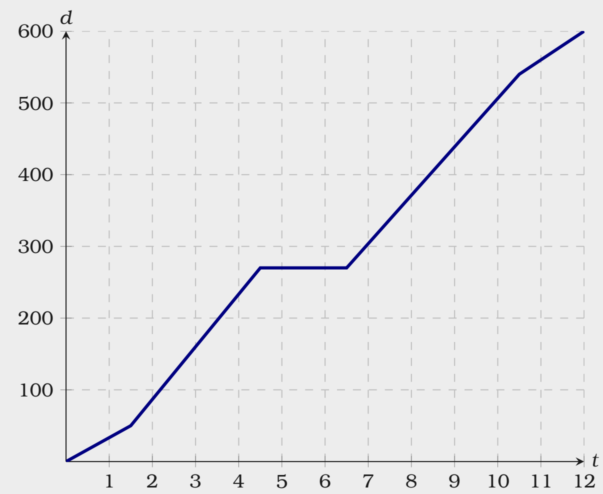


Figure 7.2: Here we see a plot of the distance traveled on a 600 mile road trip.

Example 7.2.2 Again suppose, you drive a car 600 mile road trip. Your distance from home is recorded by the plot shown in Figure 7.2. What was your instantaneous velocity 8 hours into your trip?

Solution *Since the instantaneous rate of change is measured by the derivative, we need to find the slope of the tangent line to the curve. At 7 hours, the curve is growing at an essentially constant rate. In fact, the growth rate seems to be constant from* $(7, 300)$ *to* $(10, 500)$. *This gives us an instantaneous growth rate at hour 8 of about* $200/3 \approx 67$ *miles per hour.*

Physical Applications

In physical applications, we are often concerned about *position*, *velocity*, *speed*, *acceleration*.

$$\begin{aligned} p(t) &= \text{position with respect to time.} \\ v(t) &= p'(t) = \text{velocity with respect to time.} \\ s(t) &= |v(t)| = \text{speed, the absolute value of velocity.} \\ a(t) &= v'(t) = \text{acceleration with respect to time.} \end{aligned}$$

Let's see an example.

Example 7.2.3 The Mostar bridge in Bosnia is 25 meters above the river Neretva. For fun, you decided to dive off the bridge. Your position t seconds after jumping off is

$$p(t) = -4.9t^2 + 25.$$

When do you hit the water? What is your instantaneous velocity as you enter the water? What is your average velocity during your dive?

Solution *To find when you hit the water, you must solve*

$$-4.9t^2 + 25 = 0$$

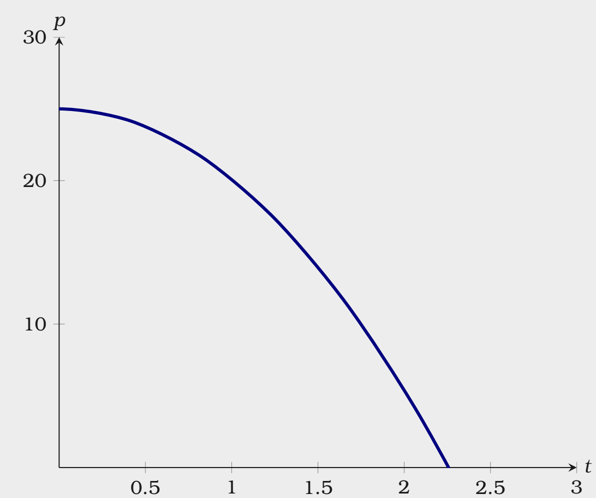


Figure 7.3: Here we see a plot of $p(t) = -4.9t^2 + 25$. Note, time is on the t-axis and vertical height is on the p-axis.

Write

$$\begin{aligned} -4.9t^2 &= -25 \\ t^2 &\approx 5.1 \\ t &\approx 2.26. \end{aligned}$$

Hence after approximately 2.26 *seconds, you gracefully enter the river.*

Your instantaneous velocity is given by $p'(t)$. *Write*

$$p'(t) = -9.8t,$$

so your instantaneous velocity when you enter the water is approximately $-9.8 \cdot 2.26 \approx -22$ *meters per second.*

Finally, your average velocity during your dive is given by

$$\frac{p(2.26) - p(0)}{2.26} \approx \frac{0 - 25}{2.26} = -11.06 \text{ meters per second.}$$

Biological Applications

In biological applications, we are often concerned with how animals and plants grow, though there are numerous other applications too.

Example 7.2.4 A certain bacterium divides into two cells every 20 minutes. The initial population of a culture is 120 cells. Find a formula for the population. What is the average growth rate during the first 4 hours? What is the instantaneous growth rate of the population at 4 hours? What rate is the population growing at 20 hours?

Solution *Since we start with* 120 *cells, and this population doubles every* 20 *minutes, then the population doubles three times an hour. So the formula for the population is*

$$p(t) = 120 \cdot 2^{3t}$$

where t is time measured in hours.

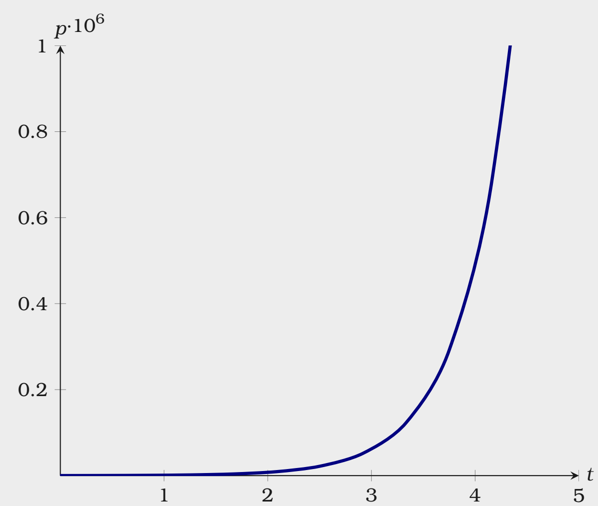


Figure 7.4: Here we see a plot of $p(t) = 120 \cdot 2^{3x}$. Note, time is on the t-axis and population is on the p-axis.

Now, the average growth rate during the first 4 hours is given by

$$\frac{p(4)-p(0)}{4} = \frac{491520-120}{4} = 122850 \text{ cells per hour.}$$

We compute the instantaneous growth rate of the population with

$$p'(t) = \ln(2)\cdot 360\cdot 2^{3t}.$$

So $p'(4) \approx 1022087$ *cells per hour. Note how fast* $p(t)$ *is growing, this is why it is important to stop bacterial infections fast!*

Exercises for Section 7.2

Exercises related to physical applications:

(1) The position of a particle in meters is given by $1/t^3$ where is t is measured in seconds. What is the acceleration of the particle after 4 seconds?

(2) On the Earth, the position of a ball dropped from a height of 100 meters is given by

$$-4.9t^2 + 100, \qquad \text{(ignoring air resistance)}$$

where time is in seconds. On the Moon, the position of a ball dropped from a height of 100 meters is given by

$$-0.8t^2 + 100,$$

where time is in seconds. How long does it take the ball to hit the ground on the Earth? What is the speed immediately before it hits the ground? How long does it take the ball to hit the ground on the Moon? What is the speed immediately before it hits the ground?

(3) A 10 gallon jug is filled with water. If a valve can drain the jug in 15 minutes, Torricelli's Law tells us that the volume of water in the jug is given by

$$V(t) = 10\,(1 - t/15)^2 \qquad \text{where} \qquad 0 \le t \le 14.$$

What is the average rate that water flows out (change in volume) from 5 to 10 minutes? What is the instantaneous rate that water flows out at 7 minutes?

(4) Starting at rest, the position of a car is given by $p(t) = 1.4t^2$ m, where t is time in seconds. How many seconds does it take the car to reach 96 km/hr? What is the car's average velocity (in km/h) on that time period?

Exercises related to biological applications:

(5) A certain bacterium triples its population every 15 minutes. The initial population of a culture is 300 cells. Find a formula for the population after t hours.

(6) The blood alcohol content of man starts at 0.18 mg/ml. It is metabolized by the body over time, and after t hours, it is given by

$$c(t) = .18e^{-0.15t}.$$

What rate is the man metabolizing alcohol at after 2 hours?

(7) The area of mold on a square piece of bread that is 10 cm per side is modeled by

$$a(t) = \frac{90}{1 + 150e^{-1.8t}} \text{ cm}^2$$

where t is time measured in days. What rate is the mold growing after 3 days? After 10 days?

7.3 Related Rates Problems

Suppose we have two variables x and y which are both changing with respect to time. A *related rates* problem is a problem where we know one rate at a given instant, and wish to find the other. If y is written in terms of x, and we are given $\frac{dx}{dt}$, then it is easy to find $\frac{dy}{dt}$ using the chain rule:

$$\frac{dy}{dt} = y'(x(t)) \cdot x'(t).$$

In many cases, particularly the interesting ones, our functions will be related in some other way. Nevertheless, in each case we'll use the same strategy:

Guidelines for Related Rates Problems

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Find an equation. We want an equation that relates all relevant functions.

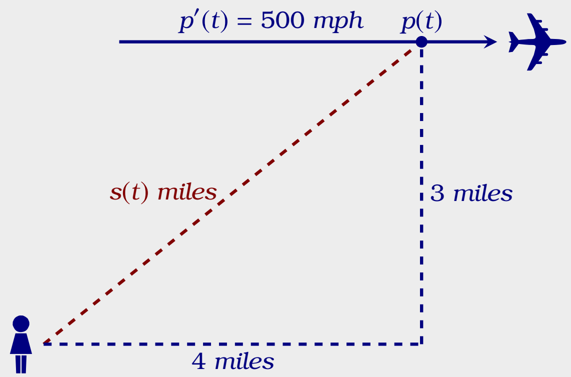
Differentiate the equation. Here we will often use implicit differentiation.

Evaluate the equation at the desired values. The known values should let you solve for the relevant rate.

Let's see a concrete example.

Example 7.3.1 A plane is flying directly away from you at 500 mph at an altitude of 3 miles. How fast is the plane's distance from you increasing at the moment when the plane is flying over a point on the ground 4 miles from you?

Solution *We'll use our general strategy to solve this problem. To start, **draw a picture**.*



Next we need to ***find an equation****. By the Pythagorean Theorem we know that*

$$p^2 + 3^2 = s^2.$$

Now we ***differentiate the equation****. Write*

$$2p(t)p'(t) = 2s(t)s'(t).$$

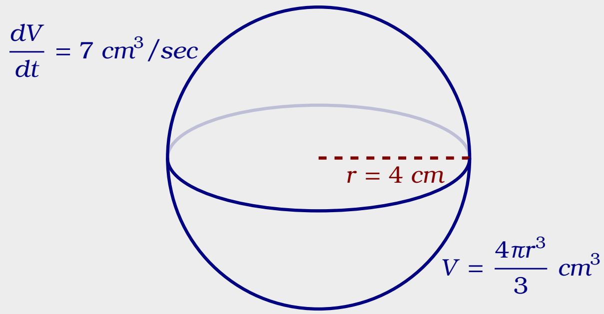
Now we'll ***evaluate the equation at the desired values****. We are interested in the time at which* $p(t) = 4$ *and* $p'(t) = 500$*. Additionally, at this time we know that* $4^2 + 9 = s^2$*, so* $s(t) = 5$*. Putting together all the information we get*

$$2(4)(500) = 2(5)s'(t),$$

thus $s'(t) = 400$ *mph.*

Example 7.3.2 You are inflating a spherical balloon at the rate of 7 cm^3/sec. How fast is its radius increasing when the radius is 4 cm?

Solution *To start,* ***draw a picture****.*



Next we need to ***find an equation****. Thinking of the variables r and V as functions of time, they are related by the equation*

$$V(t) = \frac{4\pi(r(t))^3}{3}.$$

Now we need to ***differentiate the equation****. Taking the derivative of both sides gives*

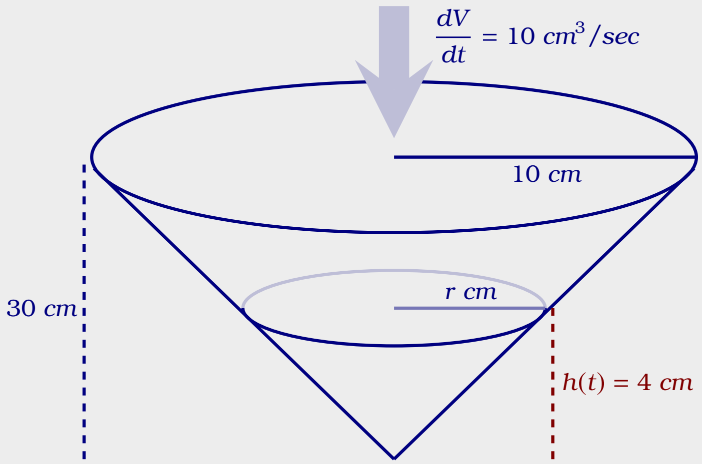
$$\frac{dV}{dt} = 4\pi(r(t))^2 \cdot r'(t).$$

Finally we ***evaluate the equation at the desired values****. Set* $r(t) = 4$ *cm and* $\frac{dV}{dt} = 7$ *cm*^{3}*/sec. Write*

$$7 = 4\pi 4^2 r'(t),$$
$$r'(t) = 7/(64\pi)\ cm/sec.$$

Example 7.3.3 Water is poured into a conical container at the rate of 10 cm^3/sec. The cone points directly down, and it has a height of 30 cm and a base radius of 10 cm. How fast is the water level rising when the water is 4 cm deep?

Solution *To start,* ***draw a picture****.*



Note, no attempt was made to draw this picture to scale, rather we want all of the relevant information to be available to the mathematician.

Now we need to ***find an equation****. The formula for the volume of a cone tells us that*

$$V = \frac{\pi}{3} r^2 h.$$

Now we must ***differentiate the equation****. We should use implicit differentiation, and treat each of the variables as functions of t. Write*

$$\frac{dV}{dt} = \frac{\pi}{3}\left(2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right). \tag{7.1}$$

At this point we ***evaluate the equation at the desired values****. At first something seems to be wrong, we do not know $\frac{dr}{dt}$. However, the dimensions of the cone of water must have the same proportions as those of the container. That is, because of similar triangles,*

$$\frac{r}{h} = \frac{10}{30} \qquad so \qquad r = h/3.$$

In particular, we see that when $h = 4$, $r = 4/3$ and

$$\frac{dr}{dt} = \frac{1}{3}\cdot\frac{dh}{dt}.$$

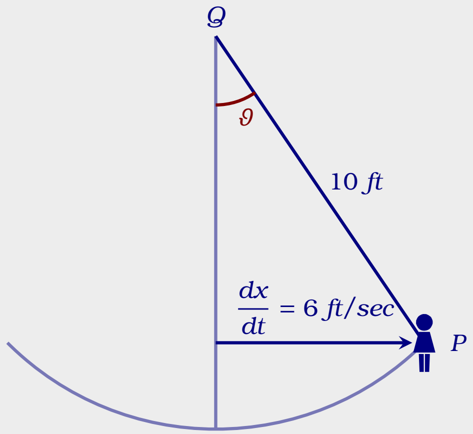
Now we can ***evaluate the equation at the desired values****. Starting with Equation 7.1, we plug in* $\frac{dV}{dt} = 10$, $r = 4/3$, $\frac{dr}{dt} = \frac{1}{3} \cdot \frac{dh}{dt}$ *and* $h = 4$. *Write*

$$\begin{aligned} 10 &= \frac{\pi}{3}\left(2 \cdot \frac{4}{3} \cdot 4 \cdot \frac{1}{3} \cdot \frac{dh}{dt} + \left(\frac{4}{3}\right)^2 \frac{dh}{dt}\right) \\ 10 &= \frac{\pi}{3}\left(\frac{32}{9}\frac{dh}{dt} + \frac{16}{9}\frac{dh}{dt}\right) \\ 10 &= \frac{16\pi}{9}\frac{dh}{dt} \\ \frac{90}{16\pi} &= \frac{dh}{dt}. \end{aligned}$$

Thus, $\frac{dh}{dt} = \frac{90}{16\pi}$ *cm/sec.*

Example 7.3.4 A swing consists of a board at the end of a 10 ft long rope. Think of the board as a point P at the end of the rope, and let Q be the point of attachment at the other end. Suppose that the swing is directly below Q at time $t = 0$, and is being pushed by someone who walks at 6 ft/sec from left to right. What is the angular speed of the rope in deg/sec after 1 sec?

Solution *To start,* ***draw a picture****.*



*Now we must **find an equation**. From the right triangle in our picture, we see*

$$\sin(\vartheta) = x/10.$$

*We can now **differentiate the equation**. Taking derivatives we obtain*

$$\cos(\vartheta) \cdot \vartheta'(t) = 0.1x'(t).$$

*Now we can **evaluate the equation at the desired values**. When $t = 1$ sec, the person was pushed by someone who walks 6 ft/sec. Hence we have a $6-8-10$ right triangle, with $x'(t) = 6$, and $\cos\vartheta = 8/10$. Thus*

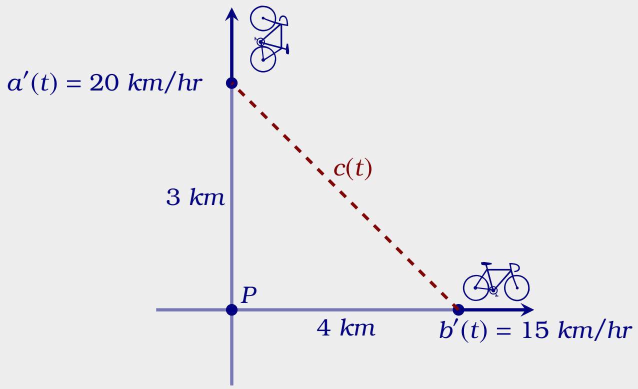
$$(8/10)\vartheta'(t) = 6/10,$$

and so $\vartheta'(t) = 6/8 = 3/4$ rad/sec, or approximately 43 deg/sec.

We have seen that sometimes there are apparently more than two variables that change with time, but as long as you know the rates of change of all but one of them you can find the rate of change of the remaining one. As in the case when there are just two variables, take the derivative of both sides of the equation relating all of the variables, and then substitute all of the known values and solve for the unknown rate.

Example 7.3.5 A road running north to south crosses a road going east to west at the point P. Cyclist A is riding north along the first road, and cyclist B is riding east along the second road. At a particular time, cyclist A is 3 kilometers to the north of P and traveling at 20 km/hr, while cyclist B is 4 kilometers to the east of P and traveling at 15 km/hr. How fast is the distance between the two cyclists changing?

Solution *We start the same way we always do, we **draw a picture**.*



Here $a(t)$ is the distance of cyclist A north of P at time t, and $b(t)$ the distance of cyclist B east of P at time t, and $c(t)$ is the distance from cyclist A to cyclist B at time t.

*We must **find an equation**. By the Pythagorean Theorem,*

$$c(t)^2 = a(t)^2 + b(t)^2.$$

*Now we can **differentiate the equation**. Taking derivatives we get*

$$2c(t)c'(t) = 2a(t)a'(t) + 2b(t)b'(t).$$

*Now we can **evaluate the equation at the desired values**. We know that $a(t) = 3$, $a'(t) = 20$, $b(t) = 4$ and $b'(t) = 15$. Hence by the Pythagorean Theorem, $c(t) = 5$. So*

$$2 \cdot 5 \cdot c'(t) = 2 \cdot 3 \cdot 20 + 2 \cdot 4 \cdot 15$$

solving for $c'(t)$ we find $c'(t) = 24$ km/hr.

Exercises for Section 7.3

(1) A cylindrical tank standing upright (with one circular base on the ground) has radius 20 cm. How fast does the water level in the tank drop when the water is being drained at 25 cm^3/sec?

(2) A cylindrical tank standing upright (with one circular base on the ground) has radius 1 meter. How fast does the water level in the tank drop when the water is being drained at 3 liters per second?

(3) A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The foot of the ladder is pulled away from the wall at the rate of 0.6 m/sec. How fast is the top sliding down the wall when the foot of the ladder is 5 m from the wall?

(4) A ladder 13 meters long rests on horizontal ground and leans against a vertical wall. The top of the ladder is being pulled up the wall at 0.1 meters per second. How fast is the foot of the ladder approaching the wall when the foot of the ladder is 5 m from the wall?

(5) A rotating beacon is located 2 miles out in the water. Let A be the point on the shore that is closest to the beacon. As the beacon rotates at 10 rev/min, the beam of light sweeps down the shore once each time it revolves. Assume that the shore is straight. How fast is the point where the beam hits the shore moving at an instant when the beam is lighting up a point 2 miles along the shore from the point A?

(6) A baseball diamond is a square 90 ft on a side. A player runs from first base to second base at 15 ft/sec. At what rate is the player's distance from third base decreasing when she is half way from first to second base?

(7) Sand is poured onto a surface at 15 cm^3/sec, forming a conical pile whose base diameter is always equal to its altitude. How fast is the altitude of the pile increasing when the pile is 3 cm high?

(8) A boat is pulled in to a dock by a rope with one end attached to the front of the boat and the other end passing through a ring attached to the dock at a point 5 ft higher than the front of the boat. The rope is being pulled through the ring at the rate of 0.6 ft/sec. How fast is the boat approaching the dock when 13 ft of rope are out?

(9) A balloon is at a height of 50 meters, and is rising at the constant rate of 5 m/sec. A bicyclist passes beneath it, traveling in a straight line at the constant speed of 10 m/sec. How fast is the distance between the bicyclist and the balloon increasing 2 seconds later?

(10) A pyramid-shaped vat has square cross-section and stands on its tip. The dimensions at the top are 2 m $\times$ 2 m, and the depth is 5 m. If water is flowing into the vat at 3 m^3/min, how fast is the water level rising when the depth of water (at the deepest point) is 4 m? Note: the volume of any "conical" shape (including pyramids) is (1/3)(height)(area of base).

(11) A woman 5 ft tall walks at the rate of 3.5 ft/sec away from a streetlight that is 12 ft above the ground. At what rate is the tip of her shadow moving? At what rate is her shadow lengthening?

(12) A man 1.8 meters tall walks at the rate of 1 meter per second toward a streetlight that is 4 meters above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow shortening?

(13) A police helicopter is flying at 150 mph at a constant altitude of 0.5 mile above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 1 mile from the helicopter, and that this distance is decreasing at 190 mph. Find the speed of the car.

(14) A police helicopter is flying at 200 kilometers per hour at a constant altitude of 1 km above a straight road. The pilot uses radar to determine that an oncoming car is at a distance of exactly 2 kilometers from the helicopter, and that this distance is decreasing at 250 kph. Find the speed of the car.

(15) A road running in a northwest direction crosses a road going east to west at a 120° at a point P. Car A is driving northwesterly along the first road, and car B

is driving east along the second road. At a particular time car A is 10 kilometers to the northwest of P and traveling at 80 km/hr, while car B is 15 kilometers to the east of P and traveling at 100 km/hr. How fast is the distance between the two cars changing? Hint, recall the law of cosines: $c^2 = a^2 + b^2 - 2ab\cos\vartheta$.

(16) A road running north to south crosses a road going east to west at the point P. Car A is 300 meters north of P, car B is 400 meters east of P, both cars are going at constant speed toward P, and the two cars will collide in 10 seconds. How fast is the distance between the two cars changing?

(17) A road running north to south crosses a road going east to west at the point P. Eight seconds ago car A started from rest at P and has been driving north, picking up speed at the steady rate of 5 m/sec^2. Six seconds after car A started, car B passed P moving east at constant speed 60 m/sec. How fast is the distance between the two cars changing?

(18) Suppose a car is driving north along a road at 80 km/hr and an airplane is flying east at speed 200 km/hr. Their paths crossed at a point P. At a certain time, the car is 10 kilometers north of P and the airplane is 15 kilometers to the east of P at an altitude of 2 km. How fast is the distance between car and airplane changing?

(19) Suppose a car is driving north along a road at 80 km/hr and an airplane is flying east at speed 200 km/hr. Their paths crossed at a point P. At a certain time, the car is 10 kilometers north of P and the airplane is 15 kilometers to the east of P at an altitude of 2 km—gaining altitude at 10 km/hr. How fast is the distance between car and airplane changing?

(20) A light shines from the top of a pole 20 m high. An object is dropped from the same height from a point 10 m away, so that its height at time t seconds is $h(t) = 20 - 9.8t^2/2$. How fast is the object's shadow moving on the ground one second later?

8 Optimization

Many important applied problems involve finding the best way to accomplish some task. Often this involves finding the maximum or minimum value of some function: The minimum time to make a certain journey, the minimum cost for doing a task, the maximum power that can be generated by a device, and so on. Many of these problems can be solved by finding the appropriate function and then using techniques of calculus to find the maximum or the minimum value required.

8.1 Maximum and Minimum Values of Curves

We already know how to find local extrema. We wish to find *absolute extrema*.

It is common to leave off the word "absolute" when asking for absolute extrema. Hence a "maximum" or a "minimum" refers to an absolute extremum. On the other hand, local extrema are always specified as such.

Definition

(a) A point $(x, f(x))$ is an **absolute maximum** on an interval if $f(x) \geq f(z)$ for every z in that interval.

(b) A point $(x, f(x))$ is an **absolute minimum** on an interval if $f(x) \leq f(z)$ for every z in that interval.

A **absolute extremum** is either an absolute maximum or an absolute minimum.

If we are working on an finite closed interval, then we have the following theorem.

Theorem 8.1.1 (Extreme Value Theorem) *If $f(x)$ is a continuous function for all x in the closed interval $[a, b]$, then there are points c and d in $[a, b]$, such that $(c, f(c))$ is an absolute maximum and $(d, f(d))$ is an absolute minimum on $[a, b]$.*

In Figure 8.1, we see a geometric interpretation of this theorem.

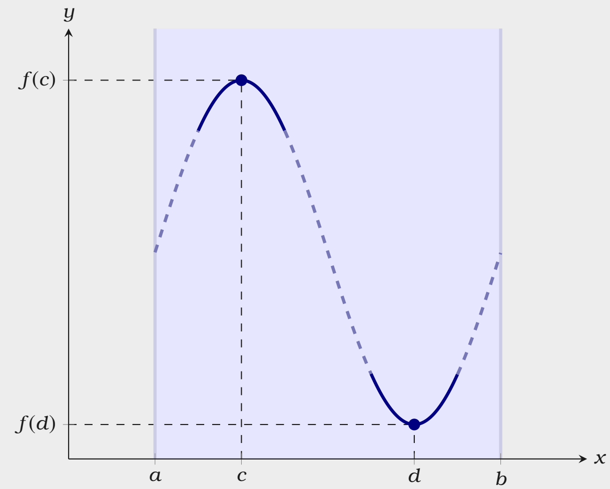


Figure 8.1: A geometric interpretation of the Extreme Value Theorem. A continuous function $f(x)$ attains both an absolute maximum and an absolute minimum on an interval $[a, b]$. Note, it may be the case that $a = c$, $b = d$, or that $d < c$.

Example 8.1.2 Find the (absolute) maximum and minimum values of $f(x) = x^2$ on the interval $[-2, 1]$.

Solution *To start, write*

$$\frac{d}{dx}x^2 = 2x.$$

The critical point is at $x = 0$. By the Extreme Value Theorem, Theorem 8.1.1, we must also consider the endpoints of the closed interval, $x = -2$ and $x = 1$. Check

$$f(-2) = 4, \qquad f(0) = 0, \qquad f(1) = 1.$$

So on the interval $[-2, 1]$, the absolute maximum of $f(x)$ is 4 at $x = -2$ and the absolute minimum is 0 at $x = 0$, see Figure 8.2.

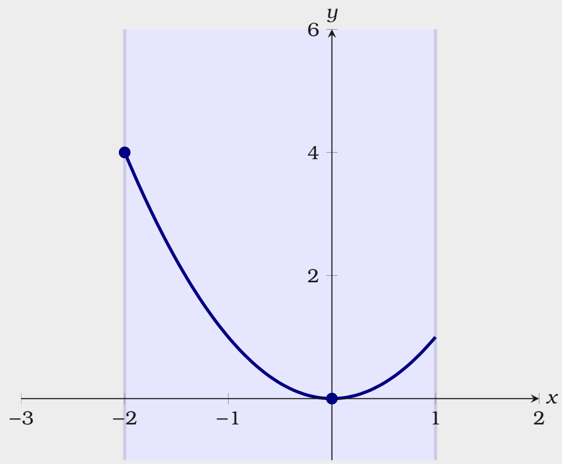


Figure 8.2: A plot of the function $f(x) = x^2$ on the interval $[-2, 1]$.

It is possible that there is no global maximum or minimum. It is difficult, and not particularly useful, to express a complete procedure for determining whether this is the case. Generally, the best approach is to gain enough understanding of the shape of the graph to decide.

Example 8.1.3 Find the (absolute) maximum and minimum values of the function $f(x) = |x - 2|$ on the interval $[1, 4]$.

Solution *To start, rewrite $f(x)$ as*

$$f(x) = \sqrt{(x-2)^2},$$

now

$$\frac{d}{dx}f(x) = \frac{2(x-2)}{2\sqrt{(x-2)^2}} = \frac{x-2}{|x-2|}.$$

The derivative $f'(x)$ is never zero, but $f'(x)$ is undefined at $x = 2$, so we have a critical point at $x = 2$. Compute $f(2) = 0$. Checking the endpoints we get $f(1) = 1$ and $f(4) = 2$. The smallest of these numbers is $f(2) = 0$, which is, therefore, the minimum value of $f(x)$ on the interval and the maximum is $f(4) = 2$, see Figure 8.3.

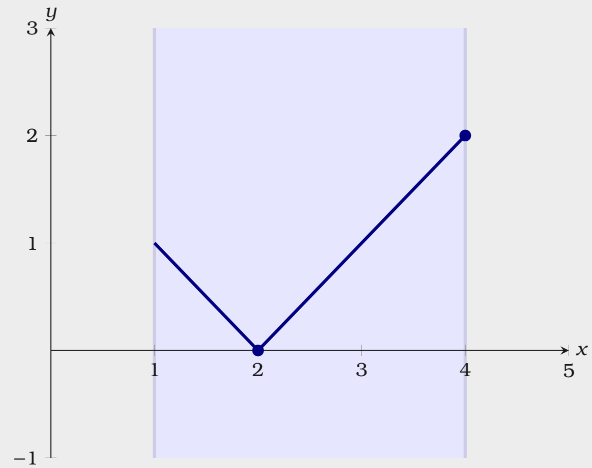


Figure 8.3: A plot of the function $f(x) = |x - 2|$ on the interval $[1, 4]$.

Warning The Extreme Value Theorem, Theorem 8.1.1, requires that the function in question be **continuous** on a **closed** interval. For example consider $f(x) = \tan(x)$ on $(-\pi/2, \pi/2)$. In this case, the function is continuous on $(-\pi/2, \pi/2)$, but the interval is not closed. Hence, the Extreme Value Theorem **does not apply**, see Figure 8.4.

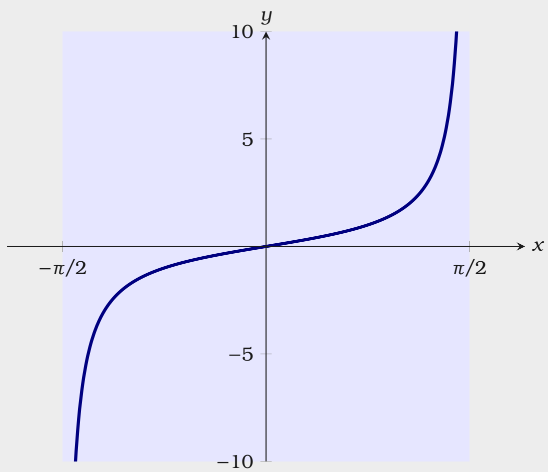


Figure 8.4: A plot of the function $f(x) = \tan(x)$ on the interval $(-\pi/2, \pi/2)$. Here the Extreme Value Theorem does not apply.

Finally, if there are several critical points in the interval, then the mathematician might want to use the second derivative test, Theorem 3.4.1, to identify if the critical points are local maxima or minima, rather than simply evaluating the function at these points. Regardless, it depends on the situation, and we will leave it up to you—our capable reader.

Exercises for Section 8.1

Find the maximum value and minimum values of $f(x)$ for x on the given interval.

(1) $f(x) = x - 2x^2$ on the interval $[0, 1]$

(2) $f(x) = x - 2x^3$ on the interval $[-1, 1]$

(3) $f(x) = x^3 - 6x^2 + 12x - 8$ on the interval $[1, 3]$

(4) $f(x) = -x^3 - 3x^2 - 2x$ on the interval $[-2, 0]$

(5) $f(x) = \sin^2(x)$ on the interval $[\pi/4, 5\pi/3]$

(6) $f(x) = \arctan(x)$ on the interval $[-1, 1]$

(7) $f(x) = e^{\sin(x)}$ on the interval $[-\pi, \pi]$

(8) $f(x) = \ln(\cos(x))$ on the interval $[-\pi/6, \pi/3]$

(9) $f(x) = \begin{cases} 1 + 4x - x^2 & \text{if } x \leq 3, \\ (x+5)/2 & \text{if } x > 3, \end{cases}$ on the interval $[0, 4]$

(10) $f(x) = \begin{cases} (x+5)/2 & \text{if } x < 3, \\ 1 + 4x - x^2 & \text{if } x \geq 3, \end{cases}$ on the interval $[0, 4]$

8.2 Basic Optimization Problems

In this section, we will present several worked examples of optimization problems. Our method for solving these problems is essentially the following:

Guidelines for Optimization

Draw a picture. If possible, draw a schematic picture with all the relevant information.

Determine your goal. We need identify what needs to be optimized.

Find constraints. What limitations are set on our optimization?

Solve for a single variable. Now you should have a function to optimize.

Use calculus to find the extreme values. Be sure to check your answer!

Example 8.2.1 Of all rectangles of area 100 cm^2, which has the smallest perimeter?

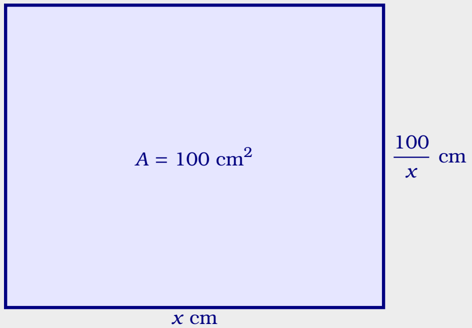


Figure 8.5: A rectangle with an area of 100 cm^2.

Solution *First we draw a picture, see Figure 8.5. If x denotes one of the sides of the rectangle, then the adjacent side must be $100/x$.*

The perimeter of this rectangle is given by

$$p(x) = 2x + 2\frac{100}{x}.$$

We wish to minimize $p(x)$. Note, not all values of x make sense in this problem: lengths of sides of rectangles must be positive, so $x > 0$. If $x > 0$ then so is $100/x$, so we need no second condition on x.

We next find $p'(x)$ and set it equal to zero. Write

$$p'(x) = 2 - 200/x^2 = 0.$$

Solving for x gives us $x = \pm 10$. We are interested only in $x > 0$, so only the value $x = 10$ is of interest. Since $p'(x)$ is defined everywhere on the interval $(0, \infty)$, there are no more critical values, and there are no endpoints. Is there a local maximum, minimum, or neither at $x = 10$? The second derivative is $p''(x) = 400/x^3$, and $f''(10) > 0$, so there is a local minimum. Since there is only one critical value, this is also the global minimum, so the rectangle with smallest perimeter is the 10 cm×10 cm square.

Example 8.2.2 You want to sell a certain number n of items in order to maximize your profit. Market research tells you that if you set the price at \$1.50, you will be able to sell 5000 items, and for every 10 cents you lower the price below \$1.50 you will be able to sell another 1000 items. Suppose that your fixed costs ("start-up costs") total \$2000, and the per item cost of production ("marginal cost") is \$0.50. Find the price to set per item and the number of items sold in order to maximize profit, and also determine the maximum profit you can get.

Solution *The first step is to convert the problem into a function maximization problem. The revenue for selling n items at x dollars is given by*

$$r(x) = nx$$

and the cost of producing n items is given by

$$c(x) = 2000 + 0.5n.$$

However, from the problem we see that the number of items sold is itself a function of x,

$$n(x) = 5000 + 1000(1.5 - x)/0.10$$

So profit is give by:

$$\begin{aligned}P(x) &= r(x) - c(x)\\&= nx - (2000 + 0.5n)\\&= (5000 + 1000(1.5 - x)/0.10)x - 2000 - 0.05(5000 + 1000(1.5 - x)/0.10)\\&= -10000x^2 + 25000x - 12000.\end{aligned}$$

We want to know the maximum value of this function when x is between 0 and 1.5. The derivative is $P'(x) = -20000x + 25000$, which is zero when $x = 1.25$. Since $P''(x) = -20000 < 0$, there must be a local maximum at $x = 1.25$, and since this is the only critical value it must be a global maximum as well. Alternately, we could compute $P(0) = -12000$, $P(1.25) = 3625$, and $P(1.5) = 3000$ and note that $P(1.25)$ is the maximum of these. Thus the maximum profit is \$3625, attained when we set the price at \$1.25 and sell 7500 items.

Example 8.2.3 Find the rectangle with largest area that fits inside the graph of the parabola $y = x^2$ below the line $y = a$, where a is an unspecified constant value, with the top side of the rectangle on the horizontal line $y = a$. See Figure 8.6.

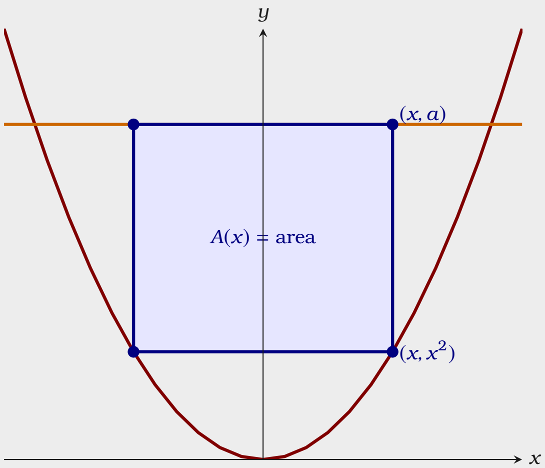


Figure 8.6: A plot of the parabola $y = x^2$ along with the line $y = a$ and the rectangle in question.

Solution *We want to maximize value of $A(x)$. The lower right corner of the rectangle is at (x, x^2), and once this is chosen the rectangle is completely determined. Then the area is*

$$A(x) = (2x)(a - x^2) = -2x^3 + 2ax.$$

We want the maximum value of $A(x)$ when x is in $[0, \sqrt{a}]$. You might object to allowing $x = 0$ or $x = \sqrt{a}$, since then the "rectangle" has either no width or no height, so is not "really" a rectangle. But the problem is somewhat easier if we simply allow such rectangles, which have zero area as we may then apply the Extreme Value Theorem, Theorem 8.1.1.

Setting $0 = A'(x) = -6x^2 + 2a$ we find $x = \sqrt{a/3}$ as the only critical point. Testing this and the two endpoints, we have $A(0) = A(\sqrt{a}) = 0$ and $A(\sqrt{a/3}) = (4/9)\sqrt{3}a^{3/2}$. Hence, the maximum area thus occurs when the *rectangle has dimensions $2\sqrt{a/3} \times (2/3)a$.*

Example 8.2.4 If you fit the largest possible cone inside a sphere, what fraction of the volume of the sphere is occupied by the cone? (Here by "cone" we mean a right circular cone, i.e., a cone for which the base is perpendicular to the axis of symmetry, and for which the cross-section cut perpendicular to the axis of symmetry at any point is a circle.)

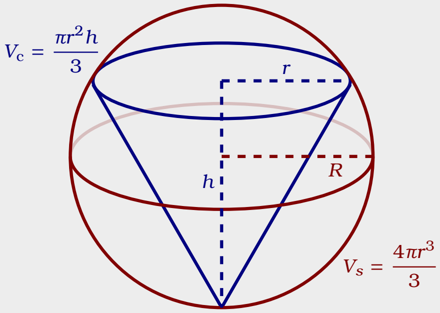


Figure 8.7: A cone inside a sphere.

Solution *Let R be the radius of the sphere, and let r and h be the base radius and height of the cone inside the sphere. Our goal is to maximize the volume of the cone: $V_c = \pi r^2 h/3$. The largest r could be is R and the largest h could be is $2R$.*

Notice that the function we want to maximize, $\pi r^2 h/3$, depends on two variables. Our next step is to find the relationship and use it to solve for one of the variables in terms of the other, so as to have a function of only one variable to maximize. In this problem, the condition is apparent in the figure, as the upper corner of the triangle, whose coordinates are $(h-R, r)$, must be on the circle of radius R. Write

$$(h-R)^2 + r^2 = R^2.$$

Solving for r^2, since r^2 is found in the formula for the volume of the cone, we find

$$r^2 = R^2 - (h-R)^2.$$

Substitute this into the formula for the volume of the cone to find

$$\begin{aligned} V_c(h) &= \pi(R^2 - (h-R)^2)h/3 \\ &= -\frac{\pi}{3}h^3 + \frac{2}{3}\pi h^2 R \end{aligned}$$

We want to maximize $V_c(h)$ when h is between 0 and $2R$. We solve

$$V_c'(h) = -\pi h^2 + (4/3)\pi h R = 0,$$

finding $h = 0$ or $h = 4R/3$. We compute

$$V_c(0) = V_c(2R) = 0 \qquad \text{and} \qquad V_c(4R/3) = (32/81)\pi R^3.$$

The maximum is the latter. Since the volume of the sphere is $(4/3)\pi R^3$, the fraction of the sphere occupied by the cone is

$$\frac{(32/81)\pi R^3}{(4/3)\pi R^3} = \frac{8}{27} \approx 30\%.$$

Example 8.2.5 You are making cylindrical containers to contain a given volume. Suppose that the top and bottom are made of a material that is N times as expensive (cost per unit area) as the material used for the lateral side of the cylinder. Find (in terms of N) the ratio of height to base radius of the cylinder that minimizes the cost of making the containers.

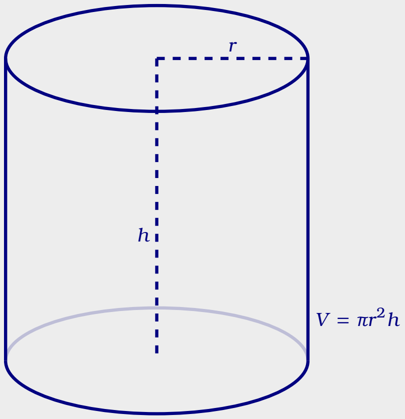


Figure 8.8: A cylinder with radius r, height h, volume V, c for the cost per unit area of the lateral side of the cylinder.

Solution *First we draw a picture, see Figure 8.8. Now we can write an expression for the cost of materials:*

$$C = 2\pi crh + 2\pi r^2 Nc.$$

Since we know that $V = \pi r^2 h$, we can use this relationship to eliminate h (we could eliminate r, but it's a little easier if we eliminate h, which appears in only one place in the above formula for cost). We find

$$\begin{aligned} C(r) &= 2c\pi r\frac{V}{\pi r^2} + 2Nc\pi r^2 \\ &= \frac{2cV}{r} + 2Nc\pi r^2. \end{aligned}$$

We want to know the minimum value of this function when r is in $(0, \infty)$. Setting

$$C'(r) = -2cV/r^2 + 4Nc\pi r = 0$$

we find $r = \sqrt[3]{V/(2N\pi)}$. Since $C''(r) = 4cV/r^3 + 4Nc\pi$ is positive when r is positive, there is a local minimum at the critical value, and hence a global minimum since there is only one critical value.

Finally, since $h = V/(\pi r^2)$,

$$\begin{aligned}\frac{h}{r} &= \frac{V}{\pi r^3}\\ &= \frac{V}{\pi(V/(2N\pi))}\\ &= 2N,\end{aligned}$$

so the minimum cost occurs when the height h is $2N$ times the radius. If, for example, there is no difference in the cost of materials, the height is twice the radius.

Example 8.2.6 Suppose you want to reach a point A that is located across the sand from a nearby road, see Figure 8.9. Suppose that the road is straight, and b is the distance from A to the closest point C on the road. Let v be your speed on the road, and let w, which is less than v, be your speed on the sand. Right now you are at the point D, which is a distance a from C. At what point B should you turn off the road and head across the sand in order to minimize your travel time to A?

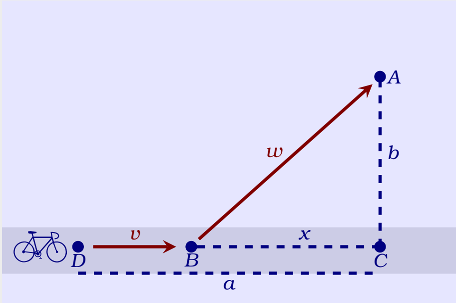


Figure 8.9: A road where one travels at rate v, with sand where one travels at rate w. Where should one turn off of the road to minimize total travel time from D to A?

Solution *Let x be the distance short of C where you turn off, the distance from B to C. We want to minimize the total travel time. Recall that when traveling at constant velocity, time is distance divided by velocity.*

You travel the distance from D to B at speed v, and then the distance from B to A at speed w. The distance from D to B is $a - x$. By the Pythagorean theorem, the distance from B to A is

$$\sqrt{x^2 + b^2}.$$

Hence the total time for the trip is

$$T(x) = \frac{a-x}{v} + \frac{\sqrt{x^2+b^2}}{w}.$$

We want to find the minimum value of T when x is between 0 and a. As usual *we set $T'(x) = 0$ and solve for x. Write*

$$T'(x) = -\frac{1}{v} + \frac{x}{w\sqrt{x^2 + b^2}} = 0.$$

We find that

$$x = \frac{wb}{\sqrt{v^2 - w^2}}$$

Notice that a does not appear in the last expression, but a is not irrelevant, since we are interested only in critical values that are in $[0, a]$, and $wb/\sqrt{v^2 - w^2}$ is either in this interval or not. If it is, we can use the second derivative to test it:

$$T''(x) = \frac{b^2}{(x^2 + b^2)^{3/2} w}.$$

Since this is always positive there is a local minimum at the critical point, and so it is a global minimum as well.

If the critical value is not in $[0, a]$ it is larger than a. In this case the minimum must occur at one of the endpoints. We can compute

$$T(0) = \frac{a}{v} + \frac{b}{w}$$
$$T(a) = \frac{\sqrt{a^2 + b^2}}{w}$$

but it is difficult to determine which of these is smaller by direct comparison. If, as is likely in practice, we know the values of v, w, a, and b, then it is easy to determine this. With a little cleverness, however, we can determine the minimum in general. We have seen that $T''(x)$ is always positive, so the derivative $T'(x)$ is always increasing. We know that at $wb/\sqrt{v^2 - w^2}$ the derivative is zero, so for values of x less than that critical value, the derivative is negative. This means that $T(0) > T(a)$, so the minimum occurs when $x = a$.

So the upshot is this: If you start farther away from C than $wb/\sqrt{v^2 - w^2}$ then you always want to cut across the sand when you are a distance $wb/\sqrt{v^2 - w^2}$ from point C. If you start closer than this to C, you should cut directly across the sand.

Exercises for Section 8.2

(1) Find the dimensions of the rectangle of largest area having fixed perimeter 100.

(2) Find the dimensions of the rectangle of largest area having fixed perimeter P.

(3) A box with square base and no top is to hold a volume 100. Find the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base.

(4) A box with square base is to hold a volume 200. The bottom and top are formed by folding in flaps from all four sides, so that the bottom and top consist of two layers of cardboard. Find the dimensions of the box that requires the least material. Also find the ratio of height to side of the base.

(5) A box with square base and no top is to hold a volume V. Find (in terms of V) the dimensions of the box that requires the least material for the five sides. Also find the ratio of height to side of the base. (This ratio will not involve V.)

(6) You have 100 feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area?

(7) You have l feet of fence to make a rectangular play area alongside the wall of your house. The wall of the house bounds one side. What is the largest size possible (in square feet) for the play area?

(8) Marketing tells you that if you set the price of an item at \$10 then you will be unable to sell it, but that you can sell 500 items for each dollar below \$10 that you set the price. Suppose your fixed costs total \$3000, and your marginal cost is \$2 per item. What is the most profit you can make?

(9) Find the area of the largest rectangle that fits inside a semicircle of radius 10 (one side of the rectangle is along the diameter of the semicircle).

(10) Find the area of the largest rectangle that fits inside a semicircle of radius r (one side of the rectangle is along the diameter of the semicircle).

(11) For a cylinder with surface area 50, including the top and the bottom, find the ratio of height to base radius that maximizes the volume.

(12) For a cylinder with given surface area S, including the top and the bottom, find the ratio of height to base radius that maximizes the volume.

(13) You want to make cylindrical containers to hold 1 liter using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the dimensions of the container using the least amount of material, and also find the ratio of height to radius for this container.

(14) You want to make cylindrical containers of a given volume V using the least amount of construction material. The side is made from a rectangular piece of material, and this can be done with no material wasted. However, the top and bottom are cut from squares of side $2r$, so that $2(2r)^2 = 8r^2$ of material is needed (rather than $2\pi r^2$, which is the total area of the top and bottom). Find the optimal ratio of height to radius.

(15) Given a right circular cone, you put an upside-down cone inside it so that its vertex is at the center of the base of the larger cone and its base is parallel to the base of the larger cone. If you choose the upside-down cone to have the largest possible volume, what fraction of the volume of the larger cone does it occupy? (Let H and R be the height and base radius of the larger cone, and let h and r be the height and base radius of the smaller cone. Hint: Use similar triangles to get an equation relating h and r.)

(16) In Example 8.2.6, what happens if $w \geq v$ (i.e., your speed on sand is at least your speed on the road)?

(17) A container holding a fixed volume is being made in the shape of a cylinder with a hemispherical top. (The hemispherical top has the same radius as the

cylinder.) Find the ratio of height to radius of the cylinder which minimizes the cost of the container if (a) the cost per unit area of the top is twice as great as the cost per unit area of the side, and the container is made with no bottom; (b) the same as in (a), except that the container is made with a circular bottom, for which the cost per unit area is 1.5 times the cost per unit area of the side.

(18) A piece of cardboard is 1 meter by 1/2 meter. A square is to be cut from each corner and the sides folded up to make an open-top box. What are the dimensions of the box with maximum possible volume?

(19) (a) A square piece of cardboard of side a is used to make an open-top box by cutting out a small square from each corner and bending up the sides. How large a square should be cut from each corner in order that the box have maximum volume? (b) What if the piece of cardboard used to make the box is a rectangle of sides a and b?

(20) A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top; the colored glass transmits only 1/2 as much light per unit area as the the clear glass. If the distance from top to bottom (across both the rectangle and the semicircle) is 2 meters and the window may be no more than 1.5 meters wide, find the dimensions of the rectangular portion of the window that lets through the most light.

(21) A window consists of a rectangular piece of clear glass with a semicircular piece of colored glass on top. Suppose that the colored glass transmits only k times as much light per unit area as the clear glass (k is between 0 and 1). If the distance from top to bottom (across both the rectangle and the semicircle) is a fixed distance H, find (in terms of k) the ratio of vertical side to horizontal side of the rectangle for which the window lets through the most light.

(22) You are designing a poster to contain a fixed amount A of printing (measured in square centimeters) and have margins of a centimeters at the top and bottom and b centimeters at the sides. Find the ratio of vertical dimension to horizontal dimension of the printed area on the poster if you want to minimize the amount of posterboard needed.

(23) The strength of a rectangular beam is proportional to the product of its width w times the square of its depth d. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r.

(24) What fraction of the volume of a sphere is taken up by the largest cylinder that can be fit inside the sphere?

(25) The U.S. post office will accept a box for shipment only if the sum of the length and girth (distance around) is at most 108 in. Find the dimensions of the largest acceptable box with square front and back.

(26) Find the dimensions of the lightest cylindrical can containing 0.25 liter (=250 cm^3) if the top and bottom are made of a material that is twice as heavy (per unit area) as the material used for the side.

(27) A conical paper cup is to hold 1/4 of a liter. Find the height and radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone.

(28) A conical paper cup is to hold a fixed volume of water. Find the ratio of height to base radius of the cone which minimizes the amount of paper needed to make the cup. Use the formula $\pi r\sqrt{r^2+h^2}$ for the area of the side of a cone, called the **lateral area** of the cone.

(29) If you fit the cone with the largest possible surface area (lateral area plus area of base) into a sphere, what percent of the volume of the sphere is occupied by the cone?

(30) Two electrical charges, one a positive charge A of magnitude a and the other a negative charge B of magnitude b, are located a distance c apart. A positively charged particle P is situated on the line between A and B. Find where P should be put so that the pull away from A towards B is minimal. Here assume that the force from each charge is proportional to the strength of the source and inversely proportional to the square of the distance from the source.

(31) Find the fraction of the area of a triangle that is occupied by the largest rectangle that can be drawn in the triangle (with one of its sides along a side of the

triangle). Show that this fraction does not depend on the dimensions of the given triangle.

(32) How are your answers to Problem 8 affected if the cost per item for the x items, instead of being simply \$2, decreases below \$2 in proportion to x (because of economy of scale and volume discounts) by 1 cent for each 25 items produced?

9 Linear Approximation

9.1 Linear Approximation and Differentials

Given a function, a *linear approximation* is a fancy phrase for something you already know.

Definition If $f(x)$ is a differentiable function at $x = a$, then a **linear approximation** for $f(x)$ at $x = a$ is given by

$$\ell(x) = f'(a)(x - a) + f(a).$$

A linear approximation of $f(x)$ is a good approximation of $f(x)$ as long as x is "not too far" from a. As we see from Figure 2.1, if one can "zoom in" on $f(x)$ sufficiently, then $f(x)$ and the linear approximation are nearly indistinguishable. Linear approximations allow us to make approximate "difficult" computations.

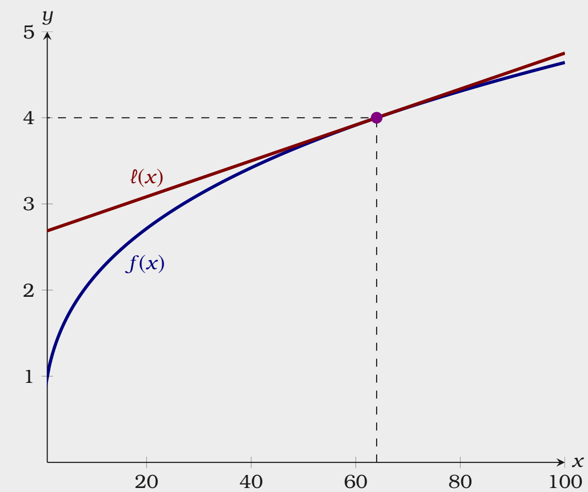


Figure 9.1: A linear approximation of $f(x) = \sqrt[3]{x}$ at $x = 4$.

Example 9.1.1 Use a linear approximation of $f(x) = \sqrt[3]{x}$ at $x = 64$ to approximate $\sqrt[3]{50}$.

Solution *To start, write*

$$\frac{d}{dx}f(x) = \frac{d}{dx}x^{1/3} = \frac{1}{3x^{2/3}}.$$

so our linear approximation is

$$\begin{aligned}\ell(x) &= \frac{1}{3 \cdot 64^{2/3}}(x-64)+4 \\ &= \frac{1}{48}(x-64)+4 \\ &= \frac{x}{48}+\frac{8}{3}.\end{aligned}$$

Now we evaluate $\ell(50) \approx 3.71$ *and compare it to* $\sqrt[3]{50} \approx 3.68$, *see Figure 9.1. From this we see that the linear approximation, while perhaps inexact, is computationally* ***easier*** *than computing the cube root.*

With modern calculators and computing software it may not appear necessary to use linear approximations. But in fact they are quite useful. In cases requiring an explicit numerical approximation, they allow us to get a quick rough estimate which can be used as a "reality check" on a more complex calculation. In some complex calculations involving functions, the linear approximation makes an otherwise intractable calculation possible, without serious loss of accuracy.

Example 9.1.2 Use a linear approximation of $f(x) = \sin(x)$ at $x = 0$ to approximate $\sin(0.3)$.

Solution *To start, write*

$$\frac{d}{dx}f(x) = \cos(x),$$

so our linear approximation is

$$\begin{aligned}\ell(x) &= \cos(0)\cdot(x-0)+0 \\ &= x.\end{aligned}$$

Hence a linear approximation for $\sin(x)$ *at* $x = 0$ *is* $\ell(x) = x$, *and so* $\ell(0.3) = 0.3$. *Comparing this to* $\sin(.3) \approx 0.295$. *As we see the approximation is quite good. For this reason, it is common to approximate* $\sin(x)$ *with its linear approximation* $\ell(x) = x$ *when* x *is near zero, see Figure 9.2.*

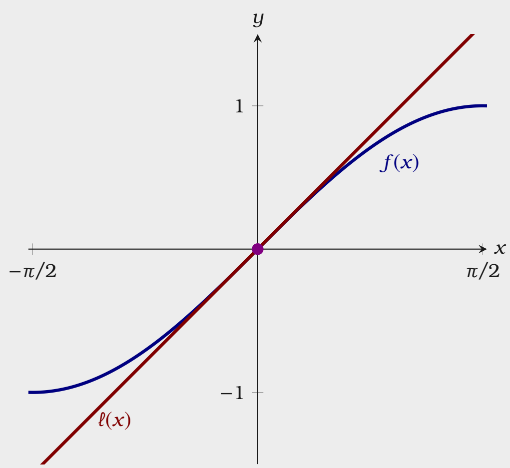


Figure 9.2: A linear approximation of $f(x) = \sin(x)$ at $x = 0$.

Differentials

The notion of a *differential* goes back to the origins of calculus, though our modern conceptualization of a differential is somewhat different than how they were initially understood.

Definition Let $f(x)$ be a differentiable function. We define a new independent variable dx, and a new dependent variable

$$dy = f'(x) \cdot dx.$$

The variables dx and dy are called **differentials**, see Figure 9.3.

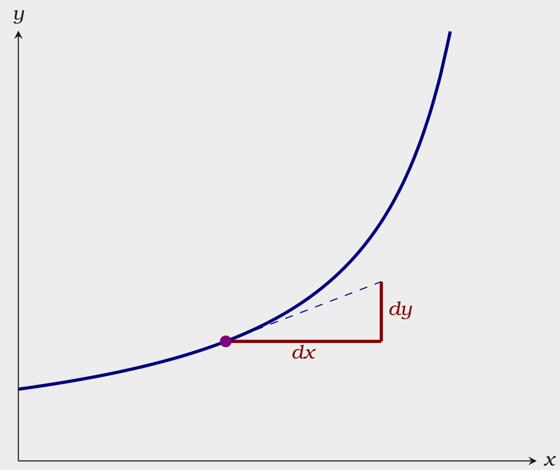


Figure 9.3: While dy and dx are both variables, dy depends on dx, and approximates how much a function grows after a change of size dx from a given point.

Note, it is now the case (by definition!) that

$$\frac{dy}{dx} = f'(x).$$

Essentially, differentials allow us to solve the problems presented in the previous examples from a slightly different point of view. Recall, when h is near but not equal zero,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

hence,

$$f'(x)h \approx f(x+h) - f(x)$$

since h is simply a variable, and dx is simply a variable, we can replace h with dx to write

$$\begin{aligned} f'(x) \cdot dx &\approx f(x+dx) - f(x) \\ dy &\approx f(x+dx) - f(x). \end{aligned}$$

From this we see that

$$f(x+dx) \approx dy + f(x).$$

While this is something of a “sleight of hand” with variables, there are contexts where the language of differentials is common. We will repeat our previous examples using differentials.

Example 9.1.3 Use differentials to approximate $\sqrt[3]{50}$.

Solution *Since* $4^3 = 64$ *is a perfect cube near* 50*, we will set* $dx = -14$*. In this case*

$$\frac{dy}{dx} = f'(x) = \frac{1}{3x^{2/3}}$$

hence

$$\begin{aligned} dy &= \frac{1}{3x^{2/3}} \cdot dx \\ &= \frac{1}{3 \cdot 64^{2/3}} \cdot (-14) \\ &= \frac{1}{3 \cdot 64^{2/3}} \cdot (-14) \\ &= \frac{-7}{24} \end{aligned}$$

Now $f(50) \approx f(64) + \dfrac{-7}{24} \approx 3.71$.

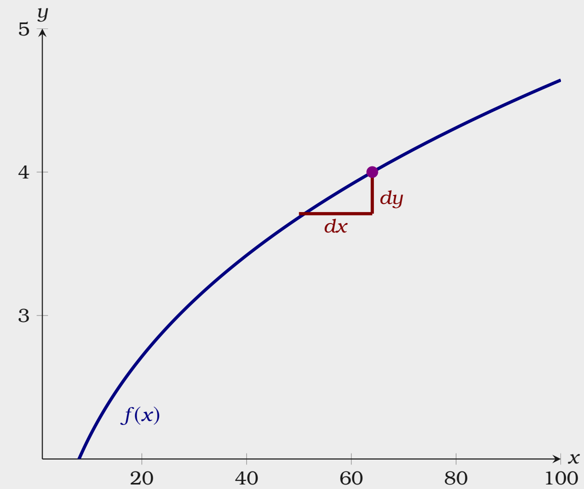


Figure 9.4: A plot of $f(x) = \sqrt[3]{x}$ along with the differentials dx and dy.

Example 9.1.4 Use differentials to approximate $\sin(0.3)$.

Solution *Since* $\sin(0) = 0$*, we will set* $dx = 0.3$*. In this case*

$$\frac{dy}{dx} = f'(x) = \cos(x)$$

hence

$$\begin{aligned} dy &= \cos(0) \cdot dx \\ &= 1 \cdot (0.3) \\ &= 0.3 \end{aligned}$$

Now $f(.3) \approx f(0) + 0.3 \approx 0.3$.

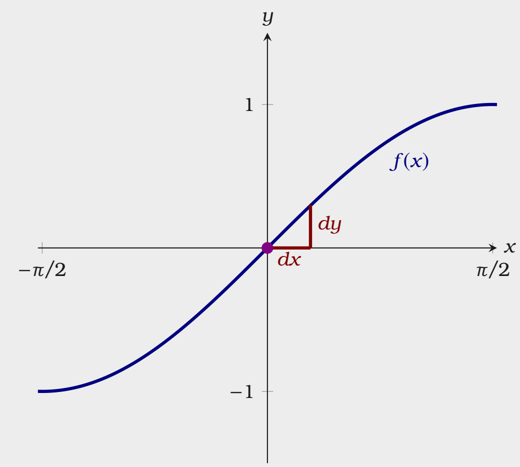


Figure 9.5: A plot of $f(x) = \sin(x)$ along with the differentials dx and dy.

The upshot is that linear approximations and differentials are simply two slightly different ways of doing the exact same thing.

Exercises for Section 9.1

(1) Use a linear approximation of $f(x) = \sin(x/2)$ at $x = 0$ to approximate $f(0.1)$.

(2) Use a linear approximation of $f(x) = \sqrt[3]{x}$ at $x = 8$ to approximate $f(10)$.

(3) Use a linear approximation of $f(x) = \sqrt[5]{x}$ at $x = 243$ to approximate $f(250)$.

(4) Use a linear approximation of $f(x) = \ln(x)$ at $x = 1$ to approximate $f(1.5)$.

(5) Use a linear approximation of $f(x) = \ln(\sqrt{x})$ at $x = 1$ to approximate $f(1.5)$.

(6) Let $f(x) = \sin(x/2)$. If $x = 1$ and $dx = 1/2$, what is dy?

(7) Let $f(x) = \sqrt{x}$. If $x = 1$ and $dx = 1/10$, what is dy?

(8) Let $f(x) = \ln(x)$. If $x = 1$ and $dx = 1/10$, what is dy?

(9) Let $f(x) = \sin(2x)$. If $x = \pi$ and $dx = \pi/100$, what is dy?

(10) Use differentials to estimate the amount of paint needed to apply a coat of paint 0.02 cm thick to a sphere with diameter 40 meters. Hint: Recall that the volume of a sphere of radius r is $V = (4/3)\pi r^3$. Notice that you are given that $dr = 0.02$.

9.2 Iterative Methods

Newton's Method

Suppose you have a function $f(x)$, and you want to solve $f(x) = 0$. Solving equations symbolically, is difficult. However, Newton's method gives us a procedure, for finding a solution to many equations to as many decimal places as you want.

Newton's Method Let $f(x)$ be a differentiable function and let a_0 be a guess for a solution to the equation

$$f(x) = 0.$$

We can produce a sequence of points $x = a_0, a_1, a_2, a_3, \ldots$ via the recursive formula

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$$

that (hopefully!) are successively better approximations of a solution to the equation $f(x) = 0$.

The point

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}$$

is the solution to the equation $\ell_n(x) = 0$, where $\ell_n(x)$ is the linear approximation of $f(x)$ at $x = a_n$.

Let's see if we can explain the logic behind this method. Consider the following cubic function

$$f(x) = x^3 - 4x^2 - 5x - 7.$$

While there is a "cubic formula" for finding roots, it can be difficult to implement. Since it is clear that $f(10)$ is positive, and $f(0)$ is negative, by the Intermediate Value Theorem 1.6.3, there is a solution to the equation $f(x) = 0$ in the interval $[0, 10]$. Let's compute $f'(x) = 3x^2 - 8x - 5$ and guess that $a_0 = 7$ is a solution. We can easily see that

$$f(a_0) = f(7) = 105 \qquad \text{and} \qquad f'(a_0) = f'(7) = 86.$$

This might seem pretty bad, but if we look at the linear approximation of $f(x)$ at $x = 7$, we find

$$\ell_0(x) = 86(x-7) + 105 \qquad \text{which is the same as} \qquad \ell_0(x) = f'(a_0)(x - a_0) + f(a_0).$$

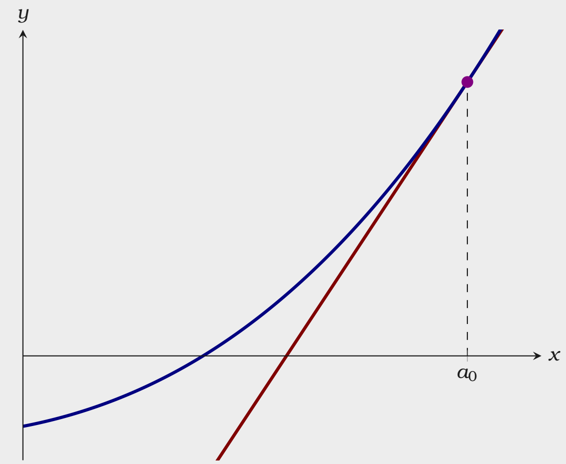


Figure 9.6: Here we see our first guess, along with the linear approximation at that point.

Now $\ell_0(a_1) = 0$ when

$$a_1 = 7 - \frac{105}{86} \qquad \text{which is the same as} \qquad a_1 = a_0 - \frac{f(a_0)}{f'(a_0)}.$$

To remind you what is going on geometrically see Figure 9.6. Now we repeat the procedure letting a_1 be our new guess. Now

$$f(a_1) \approx 23.5.$$

We see our new guess is better than our first. If we look at the linear approximation of $f(x)$ at $x = a_1$, we find

$$\ell_1(x) = f'(a_0)(x - a_0) + f(a_0).$$

Now $\ell_1(a_2) = 0$ when

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}.$$

See Figure 9.7 to see what is going on geometrically. Again, we repeat our procedure letting a_2 be our next guess, note

$$f(a_2) \approx 2.97,$$

we are getting much closer to a root of $f(x)$. Looking at the linear approximation of $f(x)$ at $x = a_2$, we find

$$\ell_2(x) = f'(a_2)(x - a_2) + f(a_2).$$

Setting $a_3 = a_2 - \dfrac{f(a_2)}{f'(a_2)}$, $a_3 \approx 5.22$. We now have $\ell_2(a_3) = 0$. Checking by evaluating $f(x)$ at a_3, we find

$$f(a_3) \approx 0.14.$$

We are now very close to a root of $f(x)$, see Figure 9.8. This process, Newton's Method, could be repeated indefinitely to obtain closer and closer approximations to a root of $f(x)$.

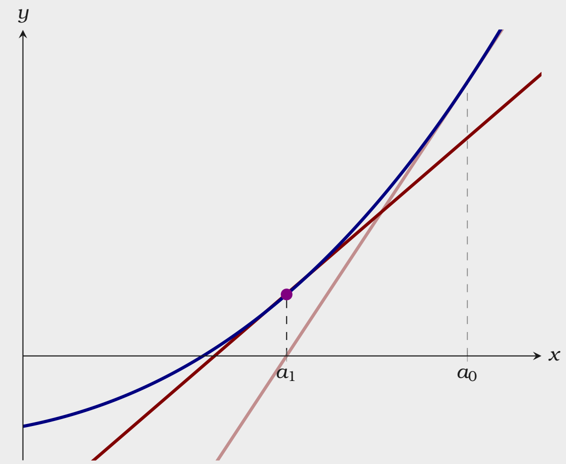


Figure 9.7: Here we see our second guess, along with the linear approximation at that point.

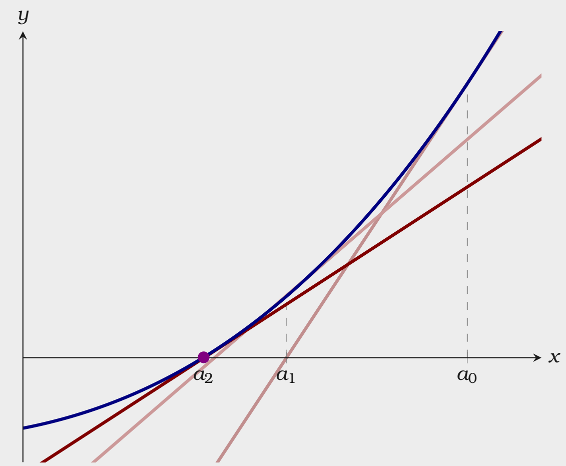


Figure 9.8: Here we see our third guess, along with the linear approximation at that point.

Example 9.2.1 Use Newton's Method to approximate the solution to

$$x^3 = 50$$

to two decimal places.

Solution *To start, set* $f(x) = x^3 - 50$. *We will use Newton's Method to approximate a solution to the equation*

$$f(x) = x^3 - 50 = 0.$$

Let's choose $a_0 = 4$ *as our first guess. Now compute*

$$f'(x) = 3x^2.$$

At this point we can make a table:

n	a_n	$f(a_n)$	$a_n - f(a_n)/f'(a_n)$
0	4	14	≈ 3.708
1	3.708	≈ 0.982	≈ 3.684
2	3.684	≈ -0.001	≈ 3.684

Hence after only two iterations, we have the solution to three (and hence two) decimal places.

In practice, which is to say, if you need to approximate a value in the course of designing a bridge or a building or an airframe, you will need to have some confidence that the approximation you settle on is accurate enough. As a rule of thumb, once a certain number of decimal places stop changing from one approximation to the next it is likely that those decimal places are correct. Still, this may not be enough assurance, in which case we can test the result for accuracy.

Sometimes questions involving Newton's Method do not mention an equation that needs to be solved. Here you must reinterpret the question as one that is asking for a solution to an equation of the form $f(x) = 0$.

Example 9.2.2 Use Newton's Method to approximate $\sqrt[3]{50}$ to two decimal places.

Solution *The* $\sqrt[3]{50}$ *is simply a solution to the equation*

$$x^3 - 50 = 0.$$

Since we did this in the previous example, we have found $\sqrt[3]{50} \approx 3.68$.

Warning Sometimes a bad choice for a_0 will not lead to a root. Consider

$$f(x) = x^3 - 3x^2 - x - 4.$$

If we choose our initial guess to be $a_0 = 1$ and make a table we find:

n	a_n	$f(a_n)$	$a_n - f(a_n)/f'(a_n)$
0	1	-7	-0.75
1	-0.75	≈ -5.359	≈ 0.283
2	0.283	≈ -4.501	≈ -1.548
3	-1.548	≈ -13.350	≈ -0.685
4	-0.685	≈ -5.044	≈ 0.432

. . . .

As you can see, we are not converging to a root, which is approximately $x = 3.589$.

Iterative procedures like Newton's method are well suited for computers. It enables us to solve equations that are otherwise impossible to solve through symbolic methods.

Euler's Method

While Newton's Method allows us to solve equations that are otherwise impossible to solve, and hence is of computational importance, Euler's Method is more of theoretical importance to us.

The name "Euler" is pronounced "Oiler."

Euler's Method Given a function $f(x)$, and an initial value (x_0, y_0) we wish to find a polygonal curve defined by (x_n, y_n) such that this polygonal curve approximates $F(x)$ where $F'(x) = f(x)$, and $F(x_0) = y_0$.

(a) Choose a step size, call it h.

(b) Our polygonal curve defined by connecting the points as described by the iterative process below:

n	x_n	y_n
0	x_0	y_0
1	$x_0 + h$	$y_0 + h \cdot f(x_0)$
2	$x_1 + h$	$y_1 + h \cdot f(x_1)$
3	$x_2 + h$	$y_2 + h \cdot f(x_2)$
4	$x_3 + h$	$y_3 + h \cdot f(x_3)$

. . . .

Let's see an example of Euler's Method in action.

Example 9.2.3 Suppose that the velocity in meters per second of a ball tossed from a height of 1 meter is given by

$$v(t) = -9.8t + 6.$$

Rounding to two decimals at each step, use Euler's Method with $h = 0.2$ to approximate the height of the ball after 1 second.

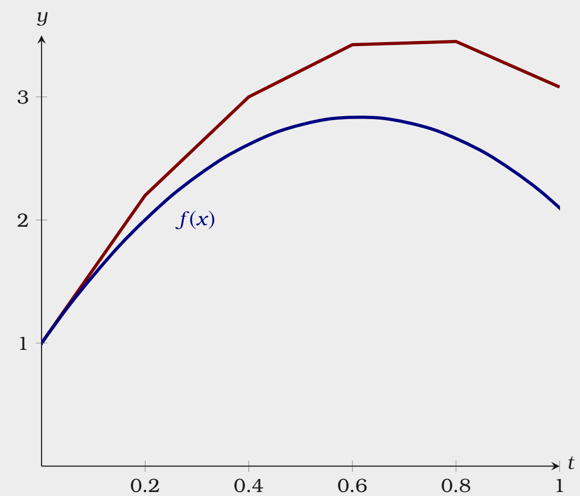


Figure 9.9: Here we see our polygonal curve found via Euler's Method and the (unknown) function $F(x)$. Choosing a smaller step-size h would yield a better approximation.

Solution *We simply need to make a table and use Euler's Method.*

n	t_n	y_n
0	0	1
1	0.2	2.2
2	0.4	3.01
3	0.6	3.42
4	0.8	3.45
5	1	3.08

Hence the ball is at a height of about 3.08 *meters, see Figure 9.9.*

Exercises for Section 9.2

(1) The function $f(x) = x^2 - 2x - 5$ has a root between 3 and 4, because $f(3) = -2$ and $f(4) = 3$. Use Newton's Method to approximate the root to two decimal places.

(2) The function $f(x) = x^3 - 3x^2 - 3x + 6$ has a root between 3 and 4, because $f(3) = -3$ and $f(4) = 10$. Use Newton's Method to approximate the root to two decimal places.

(3) The function $f(x) = x^5 - 2x^3 + 5$ has a root between -2 and -1, because $f(-2) = -11$ and $f(-1) = 6$. Use Newton's Method to approximate the root to two decimal places.

(4) The function $f(x) = x^5 - 5x^4 + 5x^2 - 6$ has a root between 4 and 5, because $f(4) = -182$ and $f(5) = 119$. Use Newton's Method to approximate the root to two decimal places.

(5) Approximate the fifth root of 7, using $x_0 = 1.5$ as a first guess. Use Newton's method to find x_3 as your approximation.

(6) Use Newton's Method to approximate the cube root of 10 to two decimal places.

(7) A rectangular piece of cardboard of dimensions 8×17 is used to make an open-top box by cutting out a small square of side x from each corner and bending up the sides. If $x = 2$, then the volume of the box is $2 \cdot 4 \cdot 13 = 104$. Use Newton's method to find a value of x for which the box has volume 100, accurate to two decimal places.

(8) Given $f(x) = 3x - 4$, use Euler's Method with a step size 0.2 to estimate $F(2)$ where $F'(x) = f(x)$ and $F(1) = 5$, to two decimal places.

(9) Given $f(x) = x^2 + 2x + 1$, use Euler's Method with a step size 0.2 to estimate $F(3)$ where $F'(x) = f(x)$ and $F(2) = 3$, to two decimal places.

(10) Given $f(x) = x^2 - 5x + 7$, use Euler's Method with a step size 0.2 to estimate $F(2)$ where $F'(x) = f(x)$ and $F(1) = -4$, to two decimal places.

9.3 The Mean Value Theorem

Here are some interesting questions involving derivatives:

(a) Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?

(b) Suppose you drive a car from toll booth on a toll road to another toll booth 30 miles away in half of an hour. Must you have been driving at 60 miles per hour at some point?

(c) Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?

While these problems sound very different, it turns out that the problems are very closely related. We'll start simply:

Theorem 9.3.1 (Rolle's Theorem) *Suppose that $f(x)$ is differentiable on the interval (a, b), is continuous on the interval $[a, b]$, and $f(a) = f(b)$. Then*

$$f'(c) = 0$$

for some $a < c < b$.

Proof *By the Extreme Value Theorem, Theorem 8.1.1, we know that $f(x)$ has a maximum and minimum value on $[a, b]$.*

If maximum and minimum both occur at the endpoints, then $f(x) = f(a) = f(b)$ at every point in $[a, b]$. Hence the function is a horizontal line, and it has derivative zero everywhere on (a, b). We may choose any c at all to get $f'(c) = 0$.

If the maximum or minimum occurs at a point c with $a < c < b$, then by Fermat's Theorem, Theorem 3.1.1, $f'(c) = 0$.

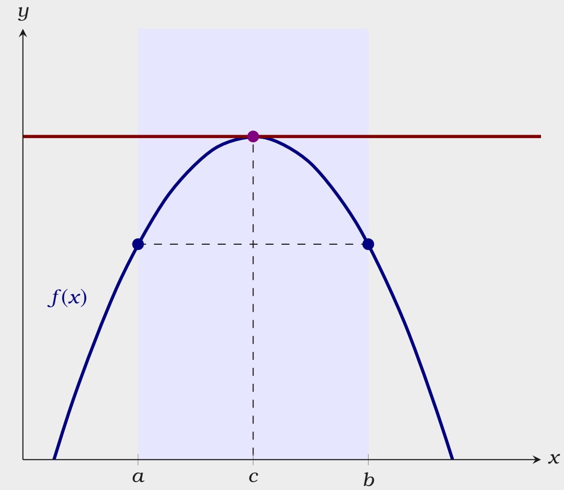


Figure 9.10: A geometric interpretation of Rolle's Theorem.

We can now answer our first question above.

Example 9.3.2 Suppose you toss a ball into the air and then catch it. Must the ball's vertical velocity have been zero at some point?

Solution *If $p(t)$ is the position of the ball at time t, then we may apply Rolle's Theorem to see at some time c, $p'(c) = 0$. Hence the velocity must be zero at some point.*

Rolle's Theorem is a special case of a more general theorem.

Theorem 9.3.3 (Mean Value Theorem) *Suppose that $f(x)$ has a derivative on the interval (a, b) and is continuous on the interval $[a, b]$. Then*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

for some $a < c < b$.

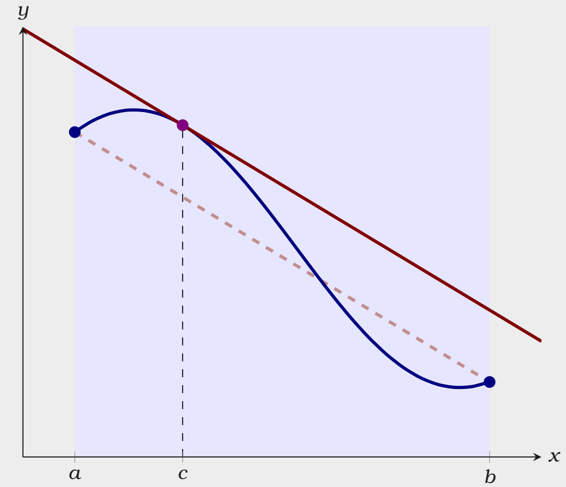


Figure 9.11: A geometric interpretation of the Mean Value Theorem

Proof *Let*

$$m = \frac{f(b) - f(a)}{b - a},$$

and consider a new function $g(x) = f(x) - m(x - a) - f(a)$. We know that $g(x)$ has a derivative on $[a, b]$, since $g'(x) = f'(x) - m$. We can compute $g(a) = f(a) - m(a - a) - f(a) = 0$ and

$$\begin{aligned} g(b) = f(b) - m(b - a) - f(a) &= f(b) - \frac{f(b) - f(a)}{b - a}(b - a) - f(a) \\ &= f(b) - (f(b) - f(a)) - f(a) \\ &= 0. \end{aligned}$$

So $g(a) = g(b) = 0$. Now by Rolle's Theorem, that at some c,

$$g'(c) = 0 \qquad \text{for some } a < c < b.$$

But we know that $g'(c) = f'(c) - m$, so

$$0 = f'(c) - m = f'(c) - \frac{f(b) - f(a)}{b - a}.$$

Hence

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

We can now answer our second question above.

Example 9.3.4 Suppose you drive a car from toll booth on a toll road to another toll booth 30 miles away in half of an hour. Must you have been driving at 60 miles per hour at some point?

Solution *If $p(t)$ is the position of the car at time t, and 0 hours is the starting time with $1/2$ hours being the final time, the Mean Value Theorem states there is a time c*

$$p'(c) = \frac{30 - 0}{1/2} = 60 \qquad \text{where } 0 < c < 1/2.$$

Since the derivative of position is velocity, this says that the car must have been driving at 60 miles per hour at some point.

Now we will address the unthinkable, could there be a function $f(x)$ whose derivative is zero on an interval that is not constant? As we will see, the answer is "no."

Theorem 9.3.5 *If $f'(x) = 0$ for all x in an interval I, then $f(x)$ is constant on I.*

Proof *Let $a < b$ be two points in I. By the Mean Value Theorem we know*

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

for some c in the interval (a, b). Since $f'(c) = 0$ we see that $f(b) = f(a)$. Moreover, since a and b were arbitrarily chosen, $f(x)$ must be the constant function.

Now let's answer our third question.

Example 9.3.6 Suppose two different functions have the same derivative. What can you say about the relationship between the two functions?

Solution *Set* $h'(x) = f'(x) - g'(x)$*. Now* $h'(x) = 0$ *on the interval* (a, b)*. This means that* $h(x) = k$ *where* k *is some constant. Hence*

$$g(x) = f(x) + k.$$

Example 9.3.7 Describe all functions whose derivative is $\sin(x)$.

Solution *One such function is* $-\cos(x)$*, so all such functions have the form* $-\cos(x) + k$*, see Figure* 9.12.

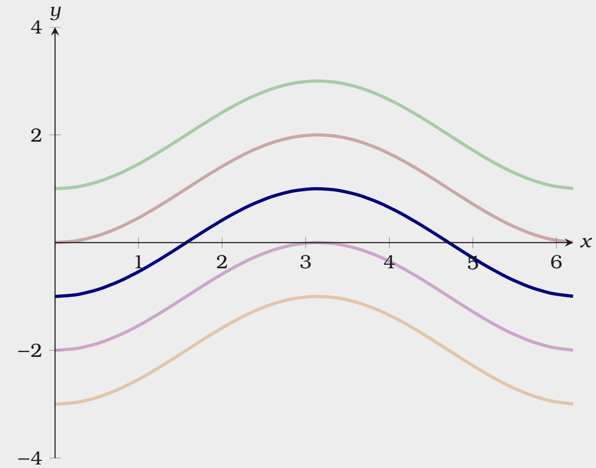


Figure 9.12: Functions of the form $-\cos(x) + k$, each of whose derivative is $\sin(x)$.

Exercises for Section 9.3

(1) Let $f(x) = x^2$. Find a value $c \in (-1, 2)$ so that $f'(c)$ equals the slope between the endpoints of $f(x)$ on $[-1, 2]$.

(2) Verify that $f(x) = x/(x+2)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[1, 4]$ and then find all of the values, c, that satisfy the conclusion of the theorem.

(3) Verify that $f(x) = 3x/(x+7)$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-2, 6]$ and then find all of the values, c, that satisfy the conclusion of the theorem.

(4) Let $f(x) = \tan(x)$. Show that $f(\pi) = f(2\pi) = 0$ but there is no number $c \in (\pi, 2\pi)$ such that $f'(c) = 0$. Why does this not contradict Rolle's theorem?

(5) Let $f(x) = (x-3)^{-2}$. Show that there is no value $c \in (1, 4)$ such that $f'(c) = (f(4) - f(1))/(4-1)$. Why is this not a contradiction of the Mean Value Theorem?

(6) Describe all functions with derivative $x^2 + 47x - 5$.

(7) Describe all functions with derivative $\dfrac{1}{1+x^2}$.

(8) Describe all functions with derivative $x^3 - \dfrac{1}{x}$.

(9) Describe all functions with derivative $\sin(2x)$.

(10) Show that the equation $6x^4 - 7x + 1 = 0$ does not have more than two distinct real roots.

(11) Let $f(x)$ be differentiable on $\mathbb{R}$. Suppose that $f'(x) \neq 0$ for every x. Prove that f has at most one real root.

Answers to Exercises

Answers for 1.1

1. (a) 8, (b) 6, (c) DNE, (d) -2, (e) -1, (f) 8, (g) 7, (h) 6, (i) 3, (j) $-3/2$, (k) 6, (l) 2 **2.** 1 **3.** 2 **4.** 3 **5.** $3/5$ **6.** $0.6931 \approx \ln(2)$ **7.** $2.718 \approx e$ **8.** Consider what happens when x is near zero and positive, as compared to when x is near zero and negative. **9.** The limit does not exist, so it is not surprising that the resulting values are so different. **10.** When v approaches c from below, then t_v approaches zero—meaning that one second to the stationary observations seems like very little time at all for our traveler.

Answers for 1.2

1. For these problems, there are many possible values of δ, so we provide an inequality that δ must satisfy when $\varepsilon = 0.1$. (a) $\delta < 1/30$, (b) $\delta < \dfrac{\sqrt{110}}{10} - 1 \approx 0.0488$, (c) $\delta < \arcsin(1/10) \approx 0.1002$, (d) $\delta < \arctan(1/10) \approx 0.0997$ (e) $\delta < 13/100$, (f) $\delta < 59/400$ **2.** Let $\varepsilon > 0$. Set $\delta = \varepsilon$. If $0 < |x - 0| < \delta$, then $|x \cdot 1| < \varepsilon$, since $\sin\left(\dfrac{1}{x}\right) \leq 1$, $\left|x \sin\left(\dfrac{1}{x}\right) - 0\right| < \varepsilon$. **3.** Let $\varepsilon > 0$. Set $\delta = \varepsilon/2$. If $0 < |x - 4| < \delta$, then $|2x - 8| < 2\delta = \varepsilon$, and then because $|2x - 8| = |(2x - 5) - 3|$, we conclude $|(2x - 5) - 3| < \varepsilon$. **4.** Let $\varepsilon > 0$. Set $\delta = \varepsilon/4$. If $0 < |x - (-3)| < \delta$, then $|-4x - 12| < 4\delta = \varepsilon$, and then because $|-4x - 12| = |(-4x - 11) - 1|$, we conclude $|(-4x - 11) - 1| < \varepsilon$. **5.** Let $\varepsilon > 0$. No matter what I choose for δ, if x is within δ of -2, then π is within ε of π. **6.** As long as $x \neq -2$, we have $\dfrac{x^2 - 4}{x + 2} = x - 2$, and the limit is not sensitive to the value of the function at the point -2; the limit only depends on nearby values, so we really want to compute $\lim_{x\to -2}(x-2)$. Let $\varepsilon > 0$. Set $\delta = \varepsilon$. Then if $0 < |x-(-2)| < \delta$, we have $|(x-2)-(-4)| < \varepsilon$. **7.** Let $\varepsilon > 0$. Pick δ so that $\delta < 1$ and $\delta < \frac{\varepsilon}{61}$. Suppose $0 < |x-4| < \delta$. Then $4-\delta < x < 4+\delta$. Cube to get $(4-\delta)^3 < x^3 < (4+\delta)^3$. Expanding the right-side inequality, we get $x^3 < \delta^3 + 12\cdot\delta^2 + 48\cdot\delta + 64 < \delta + 12\delta + 48\delta + 64 = 64 + \varepsilon$. The other inequality is similar. **8.** Let $\varepsilon > 0$. Pick δ small enough so that $\delta < \varepsilon/6$ and $\delta < 1$. Assume $|x-1| < \delta$, so $6\cdot|x-1| < \varepsilon$. Since x is within $\delta < 1$ of 1, we know $0 < x < 2$. So $|x+4| < 6$. Putting it together, $|x+4|\cdot|x-1| < \varepsilon$, so $|x^2+3x-4| < \varepsilon$, and therefore $|(x^2+3x-1)-3| < \varepsilon$. **9.** Let $\varepsilon > 0$. Set $\delta = 3\varepsilon$. Assume $0 < |x-9| < \delta$. Divide both sides by 3 to get $\frac{|x-9|}{3} < \varepsilon$. Note that $\sqrt{x}+3 > 3$, so $\frac{|x-9|}{\sqrt{x}+3} < \varepsilon$. This can be rearranged to conclude $\left|\frac{x-9}{\sqrt{x}-3} - 6\right| < \varepsilon$. **10.** Let $\varepsilon > 0$. Set δ to be the minimum of 2ε and 1. Assume x is within δ of 2, so $|x-2| < 2\varepsilon$ and $1 < x < 3$. So $\left|\frac{x-2}{2}\right| < \varepsilon$. Since $1 < x < 3$, we also have $2x > 2$, so $\left|\frac{x-2}{2x}\right| < \varepsilon$. Simplifying, $\left|\frac{1}{2} - \frac{1}{x}\right| < \varepsilon$, which is what we wanted.

Answers for 1.3

1. 7 **2.** 5 **3.** 0 **4.** DNE **5.** 1/6 **6.** 0 **7.** 3 **8.** 172 **9.** 0 **10.** 2 **11.** DNE **12.** $\sqrt{2}$ **13.** $3a^2$ **14.** 512 **15.** -4

Answers for 1.4

1. $-\infty$ **2.** 3/14 **3.** 1/2 **4.** $-\infty$ **5.** ∞ **6.** ∞ **7.** 0 **8.** $-\infty$ **9.** $x = 1$ and $x = -3$ **10.** $x = -4$

Answers for 1.5

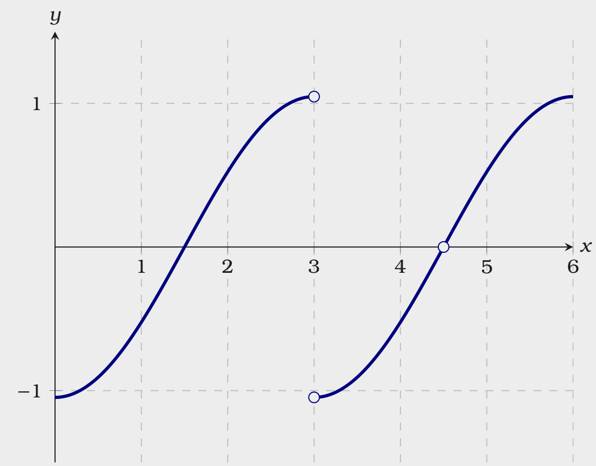
1. 0 **2.** -1 **3.** $\frac{1}{2}$ **4.** $-\infty$ **5.** π **6.** 0 **7.** 0 **8.** 17 **9.** After 10 years, $\approx$ 174 cats; after 50 years, $\approx$ 199 cats; after 100 years, $\approx$ 200 cats; after 1000 years, $\approx$ 200 cats; in the sense that the population of cats cannot grow indefinitely this is somewhat realistic. **10.** The amplitude goes to zero.

Answers for 1.6

1. $f(x)$ is continuous at $x = 4$ but it is not continuous on $\mathbb{R}$. **2.** $f(x)$ is continuous at $x = 3$ but it is not continuous on $\mathbb{R}$. **3.** $f(x)$ is not continuous at $x = 1$ and it is not continuous on $\mathbb{R}$. **4.** $f(x)$ is not continuous at $x = 5$ and it is not continuous on $\mathbb{R}$. **5.** $f(x)$ is continuous at $x = -5$ and it is also continuous on $\mathbb{R}$. **6.** $\mathbb{R}$ **7.** $(-\infty, -4) \cup (-4, \infty)$ **8.** $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ **9.** $x = -0.48$, $x = 1.31$, or $x = 3.17$ **10.** $x = 0.20$, or $x = 1.35$

Answers for 2.1

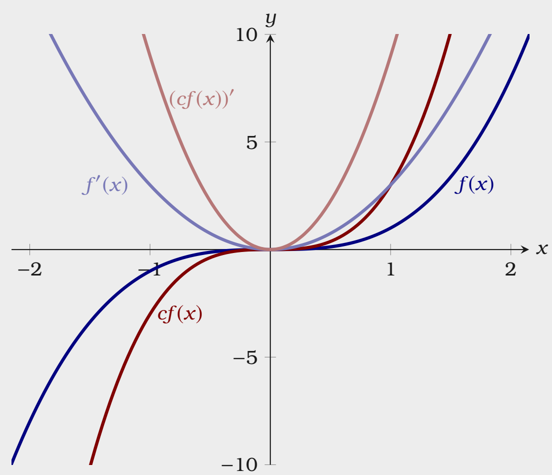
1. $f(2) = 10$ and $f'(2) = 7$ **2.** $p'(x) = s(x)$ and $r'(x) = q(x)$ **3.** $f'(3) \approx 4$ **4.** $f'(-2) = 4$ **5.** $f(1.2) \approx 2.2$ **6.** (a) $[0, 4.5) \cup (4.5, 6]$, (b) $[0, 3) \cup (3, 6]$, (c) See Figure **7.** $f'(-3) = -6$ with tangent line $y = -6x - 13$ **8.** $f'(1) = -1/9$ with tangent line $y = \frac{-1}{9}x + \frac{4}{9}$ **9.** $f'(5) = \frac{1}{2\sqrt{2}}$ with tangent line $y = \frac{1}{2\sqrt{2}}x - \frac{1}{2\sqrt{2}}$ **10.** $f'(4) = \frac{-1}{16}$ with tangent line $y = \frac{-1}{16}x + \frac{3}{4}$



Answer 2.1.6: (c) a sketch of $f'(x)$.

Answers for 2.2

1. 0 **2.** 0 **3.** 0 **4.** 0 **5.** $100x^{99}$ **6.** $-100x^{-101}$ **7.** $-5x^{-6}$ **8.** $\pi x^{\pi-1}$ **9.** $(3/4)x^{-1/4}$ **10.** $-(9/7)x^{-16/7}$ **11.** $15x^2 + 24x$ **12.** $-20x^4 + 6x + 10/x^3$ **13.** $-30x + 25$ **14.** $\frac{3}{2}x^{-1/2} - x^{-2} - ex^{e-1}$ **15.** $-5x^{-6} - x^{-3/2}/2$ **16.** $3x^2 + 6x - 1$ **17.** $2x - 1$ **18.** $x^{-1/2}/2$ **19.** $4x^3 - 4x$ **20.** $-49t/5 + 5$, $-49/5$ **21.** See Figure **22.** $x^3/16 - 3x/4 + 4$ **23.** $y = 13x/4 + 5$ **24.** $y = 24x - 48 - \pi^3$ **25.** $\frac{d}{dx}cf(x) = \lim_{h\to 0}\frac{cf(x+h) - cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = cf'(x)$.



Answer 2.2.21.

Answers for 3.1

1. min at $x = 1/2$ **2.** min at $x = -1$, max at $x = 1$ **3.** max at $x = 2$, min at $x = 4$ **4.** min at $x = \pm 1$, max at $x = 0$. **5.** min at $x = 1$ **6.** none **7.** min at $x = 0$, max at $x = \frac{3 \pm \sqrt{17}}{2}$ **8.** none **9.** local max at $x = 5$ **10.** local min at $x = 49$ **11.** local min at $x = 0$ **12.** one **13.** if $c \geq 0$, then there are no local extrema; if $c < 0$ then there is a local max at $x = -\sqrt{\frac{|c|}{3}}$ and a local min at $x = \sqrt{\frac{|c|}{3}}$

Answers for 3.2

1. min at $x = 1/2$ **2.** min at $x = -1$, max at $x = 1$ **3.** max at $x = 2$, min at $x = 4$ **4.** min at $x = \pm 1$, max at $x = 0$. **5.** min at $x = 1$ **6.** none **7.** max at $x = 0$, min at $x = \pm 11$ **8.** $f'(x) = 2ax + b$, this has only one root and hence one critical point; $a < 0$ to guarantee a maximum.

Answers for 3.3

1. concave up everywhere **2.** concave up when $x < 0$, concave down when $x > 0$ **3.** concave down when $x < 3$, concave up when $x > 3$ **4.** concave up when $x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$, concave down when $-1/\sqrt{3} < x < 1/\sqrt{3}$ **5.** concave up when $x < 0$ or $x > 2/3$, concave down when $0 < x < 2/3$ **6.** concave up when $x < 0$, concave down when $x > 0$ **7.** concave up when $x < -1$ or $x > 1$, concave down when $-1 < x < 0$ or $0 < x < 1$ **8.** concave up on $(0, \infty)$ **9.** concave up on $(0, \infty)$ **10.** concave up on $(-\infty, -1)$ and $(0, \infty)$ **11.** up/incr: $(3, \infty)$, up/decr: $(-\infty, 0)$, $(2, 3)$, down/decr: $(0, 2)$

Answers for 3.4

1. min at $x = 1/2$ **2.** min at $x = -1$, max at $x = 1$ **3.** max at $x = 2$, min at $x = 4$ **4.** min at $x = \pm 1$, max at $x = 0$. **5.** min at $x = 1$ **6.** none **7.** none **8.** max at $-5^{-1/4}$, min at $5^{-1/4}$ **9.** max at -1, min at 1 **10.** min at $2^{-1/3}$

Answers for 3.5

1. y-intercept at $(0, 0)$; no vertical asymptotes; critical points: $x = \pm 1/\sqrt[4]{5}$; local max at $x = -1/\sqrt[4]{5}$, local min at $x = -1/\sqrt[4]{5}$; increasing on $(-\infty, -1/\sqrt[4]{5})$, decreasing on $(-1/\sqrt[4]{5}, 1/\sqrt[4]{5})$, increasing on $(1/\sqrt[4]{5}, \infty)$; concave down on $(-\infty, 0)$, concave up on $(0, \infty)$; root at $x = 0$; no horizontal asymptotes; interval for sketch: $[-1.2, 1.2]$

(answers may vary) **2.** y-intercept at $(0,0)$; no vertical asymptotes; no critical points; no local extrema; increasing on $(-\infty,\infty)$; concave down on $(-\infty,0)$, concave up on $(0,\infty)$; roots at $x=0$; no horizontal asymptotes; interval for sketch: $[-3,3]$ (answers may vary) **3.** y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=1$; local max at $x=1$; increasing on $[0,1)$, decreasing on $(1,\infty)$; concave down on $[0,\infty)$; roots at $x=0$, $x=4$; no horizontal asymptotes; interval for sketch: $[0,6]$ (answers may vary) **4.** y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=-3$, $x=-1$; local max at $x=-3$, local min at $x=-1$; increasing on $(-\infty,-3)$, decreasing on $(-3,-1)$, increasing on $(-1,\infty)$; concave down on $(-\infty,-2)$, concave up on $(-2,\infty)$; roots at $x=-3$, $x=0$; no horizontal asymptotes; interval for sketch: $[-5,3]$ (answers may vary) **5.** y-intercept at $(0,5)$; no vertical asymptotes; critical points: $x=-1$, $x=3$; local max at $x=-1$, local min at $x=3$; increasing on $(-\infty,-1)$, decreasing on $(-1,3)$, increasing on $(3,\infty)$; concave down on $(-\infty,1)$, concave up on $(1,\infty)$; roots are too difficult to be determined—cubic formula could be used; no horizontal asymptotes; interval for sketch: $[-2,5]$ (answers may vary) **6.** y-intercept at $(0,0)$; no vertical asymptotes; critical points: $x=0$, $x=\dfrac{10\pm\sqrt{85}}{5}$; local max at $x=\dfrac{10-\sqrt{85}}{5}$, local min at $x=\dfrac{10+\sqrt{85}}{5}$; increasing on $(-\infty,\dfrac{10-\sqrt{85}}{5})$, decreasing on $(\dfrac{10-\sqrt{85}}{5},\dfrac{10+\sqrt{85}}{5})$, increasing on $(\dfrac{10+\sqrt{85}}{5},\infty)$; concave down on $(-\infty,0)$, concave up on $(0,\dfrac{15-\sqrt{195}}{10})$, concave down on $(\dfrac{15-\sqrt{195}}{10},\dfrac{15+\sqrt{195}}{10})$, concave up on $(\dfrac{15+\sqrt{195}}{10},\infty)$; roots at $x=0$, $x=\dfrac{5\pm\sqrt{21}}{2}$; no horizontal asymptotes; interval for sketch: $[-1,5]$ (answers may vary) **7.** no y-intercept; vertical asymptote at $x=0$; critical points: $x=0$, $x=\pm1$; local max at $x=-1$, local min at 1; increasing on $(-\infty,-1)$, decreasing on $(-1,0)\cup(0,1)$, increasing on $(1,\infty)$; concave down on $(-\infty,0)$, concave up on $(0,\infty)$; no roots; no horizontal asymptotes; interval for sketch: $[-2,2]$ (answers may vary)
8. no y-intercept; vertical asymptote at $x=0$; critical points: $x=0$, $x=\dfrac{1}{\sqrt[3]{2}}$; local min at $x=\dfrac{1}{\sqrt[3]{2}}$; decreasing on $(-\infty,0)$, decreasing on $(0,\dfrac{1}{\sqrt[3]{2}})$, increasing on $(\dfrac{1}{\sqrt[3]{2}},\infty)$; concave up on $(-\infty,-1)$, concave down on $(-1,0)$, concave up on $(0,\infty)$;

root at $x = -1$; no horizontal asymptotes; interval for sketch: $[-3, 2]$ (answers may vary)

Answers for 4.1

1. $3x^2(x^3 - 5x + 10) + x^3(3x^2 - 5)$ **2.** $(x^2 + 5x - 3)(5x^4 - 18x^2 + 6x - 7) + (2x + 5)(x^5 - 6x^3 + 3x^2 - 7x + 1)$ **3.** $2e^{2x}$ **4.** $3e^{3x}$ **5.** $6xe^{4x} + 12x^2e^{4x}$ **6.** $\dfrac{-48e^x}{x^{17}} + \dfrac{3e^x}{x^{16}}$ **7.** $f' = 4(2x - 3)$, $y = 4x - 7$ **8.** 3 **9.** 10 **10.** -13 **11.** -5 **12.** $\dfrac{d}{dx}f(x)g(x)h(x) = \dfrac{d}{dx}f(x)(g(x)h(x)) = f(x)\dfrac{d}{dx}(g(x)h(x)) + f'(x)g(x)h(x) = f(x)(g(x)h'(x)+g'(x)h(x))+f'(x)g(x)h(x) = f(x)g(x)h'(x)+f(x)g'(x)h(x))+f'(x)g(x)h(x)$

Answers for 4.2

1. $\dfrac{3x^2}{x^3 - 5x + 10} - \dfrac{x^3(3x^2 - 5)}{(x^3 - 5x + 10)^2}$ **2.** $\dfrac{2x + 5}{x^5 - 6x^3 + 3x^2 - 7x + 1} - \dfrac{(x^2 + 5x - 3)(5x^4 - 18x^2 + 6x - 7)}{(x^5 - 6x^3 + 3x^2 - 7x + 1)^2}$ **3.** $\dfrac{2xe^x - (e^x - 4)2}{4x^2}$ **4.** $\dfrac{(x + 2)(-1 - (1/2)x^{-1/2}) - (2 - x - \sqrt{x})}{(x + 2)^2}$ **5.** $y = 17x/4 - 41/4$ **6.** $y = 11x/16 - 15/16$ **7.** $y = 19/169 - 5x/338$ **8.** $-1/4$ **9.** $8/9$ **10.** 24 **11.** -3 **12.** $f(4) = 1/3$, $\dfrac{d}{dx}\dfrac{f(x)}{g(x)} = 13/18$

Answers for 5.1

1. $4x^3 - 9x^2 + x + 7$ **2.** $3x^2 - 4x + 2/\sqrt{x}$ **3.** $6(x^2 + 1)^2x$ **4.** $\sqrt{169 - x^2} - x^2/\sqrt{169 - x^2}$ **5.** $(2x - 4)\sqrt{25 - x^2} - (x^2-4x+5)x/\sqrt{25 - x^2}$ **6.** $-x/\sqrt{r^2 - x^2}$ **7.** $2x^3/\sqrt{1 + x^4}$ **8.** $\dfrac{1}{4\sqrt{x}(5 - \sqrt{x})^{3/2}}$ **9.** $6 + 18x$ **10.** $\dfrac{2x + 1}{1 - x} + \dfrac{x^2 + x + 1}{(1 - x)^2}$ **11.** $-1/\sqrt{25 - x^2} - \sqrt{25 - x^2}/x^2$ **12.** $\dfrac{1}{2}\left(\dfrac{-169}{x^2} - 1\right)\Big/\sqrt{\dfrac{169}{x} - x}$ **13.** $\dfrac{3x^2 - 2x + 1/x^2}{2\sqrt{x^3 - x^2 - (1/x)}}$ **14.** $\dfrac{300x}{(100 - x^2)^{5/2}}$ **15.** $\dfrac{1 + 3x^2}{3(x + x^3)^{2/3}}$ **16.** $\left(4x(x^2 + 1) + \dfrac{4x^3 + 4x}{2\sqrt{1 + (x^2 + 1)^2}}\right)\Big/ 2\sqrt{(x^2 + 1)^2 + \sqrt{1 + (x^2 + 1)^2}}$ **17.** $5(x+8)^4$ **18.** $-3(4-x)^2$ **19.** $6x(x^2+5)^2$ **20.** $-12x(6 - 2x^2)^2$ **21.** $24x^2(1 - 4x^3)^{-3}$ **22.** $5 + 5/x^2$ **23.** $-8(4x -$

$1)(2x^2 - x + 3)^{-3}$ **24.** $1/(x+1)^2$ **25.** $3(8x-2)/(4x^2-2x+1)^2$ **26.** $-3x^2+5x-1$ **27.** $6x(2x-4)^3+6(3x^2+1)(2x-4)^2$ **28.** $-2/(x-1)^2$ **29.** $4x/(x^2+1)^2$ **30.** $(x^2-6x+7)/(x-3)^2$ **31.** $-5/(3x-4)^2$ **32.** $60x^4+72x^3+18x^2+18x-6$ **33.** $(5-4x)/((2x+1)^2(x-3)^2)$ **34.** $1/(2(2+3x)^2)$ **35.** $56x^6+72x^5+110x^4+100x^3+60x^2+28x+6$ **36.** $y = 23x/96 - 29/96$ **37.** $y = 3 - 2x/3$ **38.** $y = 13x/2 - 23/2$ **39.** $y = 2x - 11$ **40.** $y = \dfrac{20+2\sqrt{5}}{5\sqrt{4+\sqrt{5}}}x + \dfrac{3\sqrt{5}}{5\sqrt{4+\sqrt{5}}}$

Answers for 5.2

1. $-x/y$ **2.** x/y **3.** $-(2x+y)/(x+2y)$ **4.** $(2xy-3x^2-y^2)/(2xy-3y^2-x^2)$ **5.** $\dfrac{-2xy}{x^2-3y^2}$ **6.** $-\sqrt{y}/\sqrt{x}$ **7.** $\dfrac{y^{3/2}-2}{1-y^{1/2}3x/2}$ **8.** $-y^2/x^2$ **9.** 1 **10.** $y = 2x \pm 6$ **11.** $y = x/2 \pm 3$ **12.** $(\sqrt{3}, 2\sqrt{3})$, $(-\sqrt{3}, -2\sqrt{3})$, $(2\sqrt{3}, \sqrt{3})$, $(-2\sqrt{3}, -\sqrt{3})$ **13.** $y = 7x/\sqrt{3} - 8/\sqrt{3}$ **14.** $y = (-y_1^{1/3}x + y_1^{1/3}x_1 + x_1^{1/3}y_1)/x_1^{1/3}$ **15.** $(y-y_1)/(x-x_1) = (2x_1^3 + 2x_1y_1^2 - x_1)/(2y_1^3 + 2y_1x_1^2 + y_1)$

Answers for 5.3

1. $(x+1)^3\sqrt{x^4+5}(3/(x+1)+2x^3/(x^4+5))$ **2.** $(2/x+5)x^2e^{5x}$ **3.** $2\ln(x)x^{\ln(x)-1}$ **4.** $(100+100\ln(x))x^{100x}$ **5.** $(4+4\ln(3x))(3x)^{4x}$ **6.** $((e^x)/x + e^x\ln(x))x^{e^x}$ **7.** $\pi x^{\pi-1} + \pi^x\ln(\pi)$ **8.** $(\ln(1+1/x) - 1/(x+1))(1+1/x)^x$ **9.** $(1/\ln(x) + \ln(\ln(x)))(\ln(x))^x$ **10.** $(f'(x)/f(x) + g'(x)/g(x) + h'(x)/h(x))f(x)g(x)h(x)$

Answers for 6.1

1. $\sin(\sqrt{x})\cos(\sqrt{x})/\sqrt{x}$ **2.** $\dfrac{\sin(x)}{2\sqrt{x}} + \sqrt{x}\cos(x)$ **3.** $-\dfrac{\cos(x)}{\sin^2(x)}$ **4.** $\dfrac{(2x+1)\sin(x) - (x^2+x)\cos(x)}{\sin^2(x)}$ **5.** $\dfrac{-\sin(x)\cos(x)}{\sqrt{1-\sin^2(x)}}$ **6.** $\cos^2(x) - \sin^2(x)$ **7.** $-\sin(x)\cos(\cos(x))$ **8.** $\dfrac{\tan(x) + x\sec^2(x)}{2\sqrt{x\tan(x)}}$ **9.** $\dfrac{\sec^2(x)(1+\sin(x)) - \tan(x)\cos(x)}{(1+\sin(x))^2}$ **10.** $-\csc^2(x)$ **11.** $-\csc(x)\cot(x)$ **12.** $3x^2\sin(23x^2) + 46x^4\cos(23x^2)$ **13.** 0 **14.** $-6\cos(\cos(6x))\sin(6x)$ **15.** $\sin(\vartheta)/(\cos(\vartheta)+1)^2$ **16.** $5t^4\cos(6t) - 6t^5\sin(6t)$ **17.** $3t^2(\sin(3t) + t\cos(3t))/\cos(2t) + 2t^3\sin(3t)\sin(2t)/\cos^2(2t)$

18. $n\pi/2$, any integer n **19.** $\pi/2+n\pi$, any integer n **20.** $\sqrt{3}x/2+3/4-\sqrt{3}\pi/6$ **21.** $8\sqrt{3}x+4-8\sqrt{3}\pi/3$ **22.** $3\sqrt{3}x/2-\sqrt{3}\pi/4$ **23.** $\pi/6+2n\pi$, $5\pi/6+2n\pi$, any integer n

Answers for 6.2

1. $\dfrac{-1}{1+x^2}$ **2.** $\dfrac{2x}{\sqrt{1-x^4}}$ **3.** $\dfrac{e^x}{1+e^{2x}}$ **4.** $-3x^2\cos(x^3)/\sqrt{1-\sin^2(x^3)}$ **5.** $\dfrac{2}{(\arcsin(x))\sqrt{1-x^2}}$ **6.** $-e^x/\sqrt{1-e^{2x}}$ **7.** 0 **8.** $\dfrac{(1+\ln x)x^x}{\ln 5(1+x^{2x})\arctan(x^x)}$

Answers for 7.1

1. 0 **2.** ∞ **3.** 1 **4.** 0 **5.** 0 **6.** 1 **7.** 1/6 **8.** $-\infty$ **9.** 1/16 **10.** 1/3 **11.** 0 **12.** 3/2 **13.** $-1/4$ **14.** -3 **15.** 1/2 **16.** 0 **17.** 0 **18.** $-1/2$ **19.** 5 **20.** ∞ **21.** ∞ **22.** 2/7 **23.** 2 **24.** $-\infty$ **25.** 1 **26.** 1 **27.** 2 **28.** 1 **29.** 0 **30.** 1/2 **31.** 2 **32.** 0 **33.** ∞ **34.** 1/2 **35.** 0 **36.** 1/2 **37.** 5 **38.** $2\sqrt{2}$ **39.** $-1/2$ **40.** 2 **41.** 0 **42.** ∞ **43.** 0 **44.** 3/2 **45.** ∞ **46.** 5 **47.** $-1/2$ **48.** does not exist **49.** ∞

Answers for 7.2

1. $3/256\ \mathrm{m/s}^2$ **2.** on the Earth: ≈ 4.5 s, ≈ 44 m/s; on the Moon: ≈ 11.2 s, ≈ 18 m/s **3.** average rate: ≈ -0.67 gal/min; instantaneous rate: ≈ -0.71 gal/min **4.** ≈ 9.5 s; ≈ 48 km/h. **5.** $p(t) = 300\cdot 3^{4t}$ **6.** $\approx -.02$ mg/ml per hour **7.** ≈ 39 cm/day; ≈ 0 cm/day

Answers for 7.3

1. $1/(16\pi)$ cm/s **2.** $3/(1000\pi)$ meters/second **3.** 1/4 m/s **4.** $-6/25$ m/s **5.** 80π mi/min **6.** $3\sqrt{5}$ ft/s **7.** $20/(3\pi)$ cm/s **8.** 13/20 ft/s **9.** $5\sqrt{10}/2$ m/s **10.** 75/64 m/min **11.** tip: 6 ft/s, length: 5/2 ft/s **12.** tip: 20/11 m/s, length: 9/11 m/s **13.** $380/\sqrt{3}-150\approx 69.4$ mph **14.** $500/\sqrt{3}-200\approx 88.7$ km/hr **15.** $136\sqrt{475}/19\approx 156$ km/hr **16.** -50 m/s **17.** 68

m/s **18.** $3800/\sqrt{329} \approx 210$ km/hr **19.** $820/\sqrt{329} + 150\sqrt{57}/\sqrt{47} \approx 210$ km/hr **20.** $4000/49$ m/s

Answers for 8.1

1. max at $(1/4, 1/8)$, min at $(1, -1)$ **2.** max at $(-1, 1)$, min at $(1, -1)$ **3.** max at $(3, 1)$, min at $(1, -1)$ **4.** max at $(-1 + 1/\sqrt{3}, 2/(3\sqrt{3}))$, min at $(-1 - 1/\sqrt{3}, -2/(3\sqrt{3}))$ **5.** max at $(\pi/2, 1)$ and $(3\pi/2, 1)$, min at $(\pi, 0)$ **6.** max at $(1, \pi/4)$, min at $(-1, -\pi/4)$ **7.** max at $(\pi/2, e)$, min at $(-\pi/2, 1/e)$ **8.** max at $(0, 0)$, min at $(\pi/3, -\ln(2)$ **9.** max at $(2, 5)$, min at $(0, 1)$ **10.** max at $(3, 4)$, min at $(4, 1)$

Answers for 8.2

1. 25×25 **2.** $P/4 \times P/4$ **3.** $w = l = 2 \cdot 5^{2/3}$, $h = 5^{2/3}$, $h/w = 1/2$ **4.** $\sqrt[3]{100} \times \sqrt[3]{100} \times 2\sqrt[3]{100}$, $h/s = 2$ **5.** $w = l = 2^{1/3}V^{1/3}$, $h = V^{1/3}/2^{2/3}$, $h/w = 1/2$ **6.** 1250 square feet **7.** $l^2/8$ square feet **8.** \$5000 **9.** 100 **10.** r^2 **11.** $h/r = 2$ **12.** $h/r = 2$ **13.** $r = 5$, $h = 40/\pi$, $h/r = 8/\pi$ **14.** $8/\pi$ **15.** $4/27$ **16.** Go direct from A to D. **17.** (a) 2, (b) $7/2$ **18.** $\dfrac{\sqrt{3}}{6} \times \dfrac{\sqrt{3}}{6} + \dfrac{1}{2} \times \dfrac{1}{4} - \dfrac{\sqrt{3}}{12}$ **19.** (a) $a/6$, (b) $(a + b - \sqrt{a^2 - ab + b^2})/6$ **20.** 1.5 meters wide by 1.25 meters tall **21.** If $k \le 2/\pi$ the ratio is $(2 - k\pi)/4$; if $k \ge 2/\pi$, the ratio is zero: the window should be semicircular with no rectangular part. **22.** a/b **23.** $w = 2r/\sqrt{3}$, $h = 2\sqrt{2}r/\sqrt{3}$ **24.** $1/\sqrt{3} \approx 58\%$ **25.** $18 \times 18 \times 36$ **26.** $r = 5/(2\pi)^{1/3} \approx 2.7$ cm, $h = 5 \cdot 2^{5/3}/\pi^{1/3} = 4r \approx 10.8$ cm **27.** $h = \dfrac{750}{\pi}\left(\dfrac{2\pi^2}{750^2}\right)^{1/3}$, $r = \left(\dfrac{750^2}{2\pi^2}\right)^{1/6}$ **28.** $h/r = \sqrt{2}$ **29.** The ratio of the volume of the sphere to the volume of the cone is $1033/4096 + 33/4096\sqrt{17} \approx 0.2854$, so the cone occupies approximately 28.54% of the sphere. **30.** P should be at distance $c\sqrt[3]{a}/(\sqrt[3]{a} + \sqrt[3]{b})$ from charge A. **31.** $1/2$ **32.** \$7000

Answers for 9.1

1. $\sin(0.1/2) \approx 0.05$ **2.** $\sqrt[3]{10} \approx 2.17$ **3.** $\sqrt[5]{250} \approx 3.017$ **4.** $\ln(1.5) \approx 0.5$ **5.** $\ln(\sqrt{1.5}) \approx 0.25$ **6.** $dy = .22$ **7.** $dy = 0.05$ **8.** $dy = 0.1$ **9.** $dy =$

$\pi/50$ **10.** $dV = 8\pi/25$

Answers for 9.2

1. 3.45 **2.** 3.36 **3.** −1.72 **4.** 4.79 **5.** $x_3 = 1.475773162$ **6.** 2.15 **7.** 2.19 or 1.26 **8.** 5.2 **9.** 14.64 **10.** −1.96

Answers for 9.3

1. $c = 1/2$ **2.** $c = \sqrt{18} - 2$ **3.** $c = \sqrt{65} - 7$ **4.** $f(x)$ is not continuous on $[\pi, 2\pi]$ **5.** $f(x)$ is not continuous on $[1, 4]$ **6.** $x^3/3 + 47x^2/2 - 5x + k$ **7.** $\arctan(x) + k$ **8.** $x^4/4 - \ln(x) + k$ **9.** $-\cos(2x)/2 + k$ **10.** Seeking a contradiction, suppose that we have 3 real roots, call them a, b, and c. By Rolle's Theorem, $24x^3 - 7$ must have a root on both (a, b) and (c, d), but this is impossible as $24x^3 - 7$ has only one real root. **11.** Seeking a contradiction, suppose that we have 2 real roots, call them a, b. By Rolle's Theorem, $f'(x)$ must have a root on (a, b), but this is impossible.

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