习题2.1： （A）1（2、3），2，6，7，8，18， （B）1，3，4，5
习题2.2： （A）1（4、9），3（4、19、13），6（1、3、4、6），8

2.1(A)

1.

(2)

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\ln(x+\Delta x)-\ln x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\ln(1+\frac{\Delta x}{x})}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta x}{x \cdot \Delta x} \\
&= \frac{1}{x}
\end{aligned}
$$

(3)

$$
\begin{aligned}
f'(0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x)-f(0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(0+\Delta x)^2 \sin \frac{1}{0+\Delta x} - 0}{\Delta x} \\
&= \lim_{\Delta x \to 0} \Delta x \sin \frac{1}{\Delta x} \\
&= 0
\end{aligned}
$$

2.

(1)

$$
\begin{aligned}
&\lim_{\Delta x \to 0} \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x} \\
=& - \lim_{\Delta x \to 0} \frac{f(x_0 + (-\Delta x)) - f(x_0)}{-\Delta x} \\
=& - f'(x_0)
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0 - h)}{h} \\
&\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0) + f(x_0) - f(x_0 - h)}{h} \\
=& \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} + \lim_{h \to 0} \frac{f(x_0 - h) - f(x_0)}{-h} \\
=& 2f'(x_0)
\end{aligned}
$$

(3)

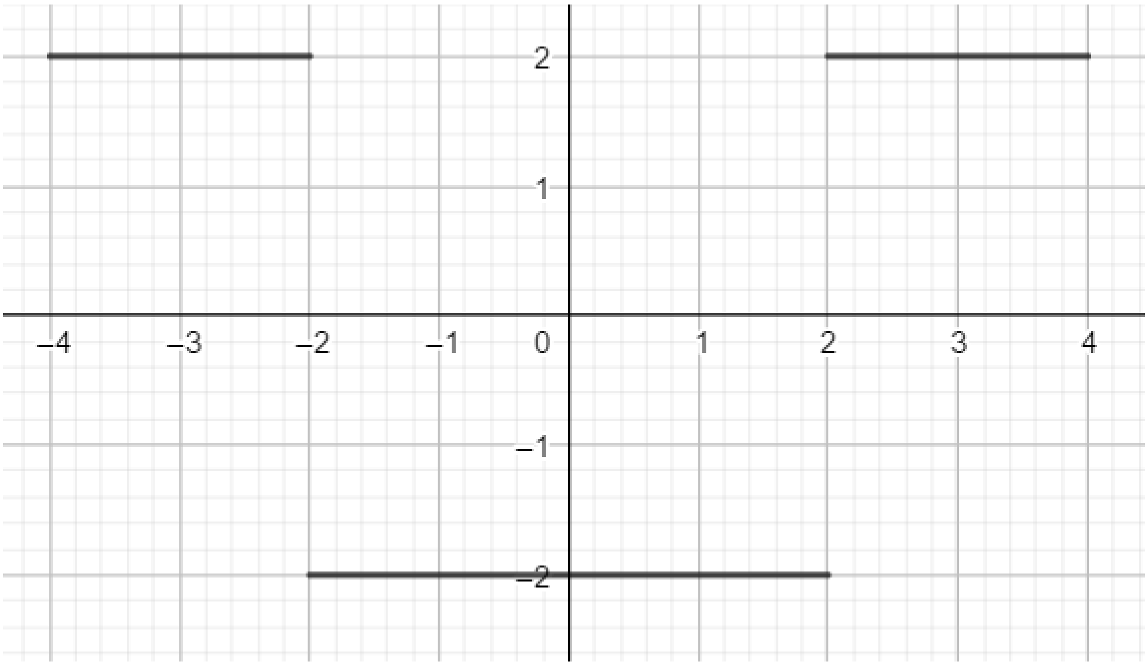
$$
\begin{aligned}
&\lim_{n \to +\infty} n[f(x_0 + \frac{1}{n}) - f(x_0)] \\
=& \lim_{n \to +\infty} \frac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}} \\
=& f'(x_0)
\end{aligned}
$$

(4)

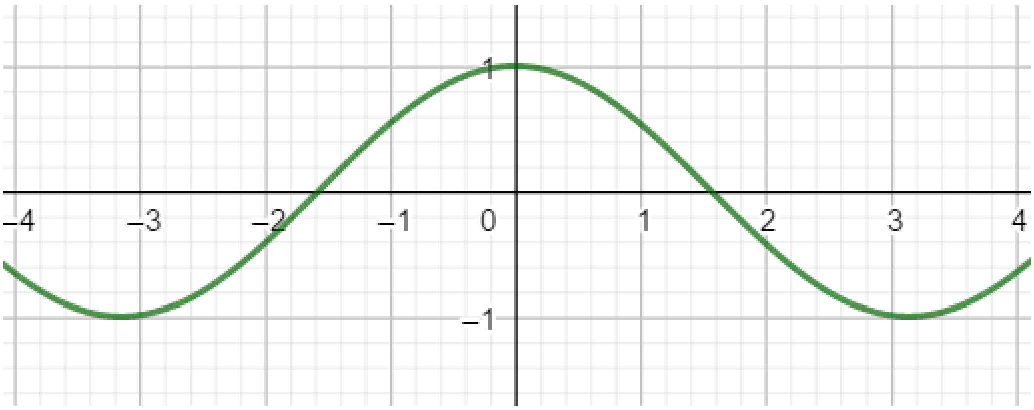
$$
\begin{aligned}
&\lim_{x \to x_0} \frac{x_0 f(x) - x f(x_0)}{x - x_0} \\
=& \lim_{x \to x_0} \frac{x_0 f(x_0 + (x - x_0)) - (x_0 + x - x_0) f(x_0)}{x - x_0} \\
=& \lim_{x \to x_0} \frac{x_0 f(x_0 + (x - x_0)) - x_0 f(x_0)}{x - x_0} - \lim_{x - x_0} \frac{(x - x_0) f(x_0)}{x - x_0} \\
=& x_0 f'(x_0) - f(x_0)
\end{aligned}
$$

6.

(a)



(b)



7.

$\because f$是偶函数, $f'(0)$存在

$\therefore f(x) = f(-x)$

$$\begin{aligned}\therefore f'(0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \\ &= -\lim_{\Delta x \to 0} \frac{f(0-\Delta x) - f(0)}{-\Delta x} \\ &= -f'(0)\end{aligned}$$

$\therefore 2f'(0) = 0$

$\therefore f'(0) = 0$

8.

(1)

$\because \varphi$在$x=a$处连续

$\therefore \lim\limits_{x\to a}\varphi(x)=\varphi(a)$

$$\begin{aligned}\therefore f'_-(a)&=\lim_{\Delta x\to 0^-}\frac{(x-a+\Delta x)\varphi(x+\Delta x)-(x-a)\varphi(x)}{\Delta x}\\&=\lim_{\Delta x\to 0^-}\frac{(a-a+\Delta x)\varphi(a+\Delta x)-(a-a)\varphi(a)}{\Delta x}\\&=\lim_{\Delta x\to 0^-}\frac{\Delta x\cdot\varphi(a+\Delta x)}{\Delta x}\\&=\varphi(a)\end{aligned}$$

$\therefore$ 同理$f'_+(a)=\varphi(a)$

$\therefore f'_-(a)=f'_+(a)$

$\therefore f$在$x=a$处可导

(2)

$$\begin{aligned}\therefore g'_-(a)&=\lim_{\Delta x\to 0}\frac{|x-a+\Delta x|\varphi(x+\Delta x)-|x-a|\varphi(x)}{\Delta x}\\&=\lim_{\Delta x\to 0}\frac{|a-a+\Delta x|\varphi(a+\Delta x)-|a-a|\varphi(a)}{\Delta x}\\&=\lim_{\Delta x\to 0}\frac{|\Delta x|\cdot\varphi(a+\Delta x)}{\Delta x}\\&=-\varphi(a)\end{aligned}$$

$\therefore$ 同理$g'_+(a)=\varphi(a)$

当$\varphi(a)=0$时$,g'_-(a)=g'_+(a)$

$\therefore g$在$x=a$处可导

当$\varphi(a)\neq 0$时$,g'_-(a)\neq g'_+(a)$

$\therefore g$在$x=a$处不可导

18.

$\because f(0)=0, f'(0)=2$

$$\therefore \lim_{x\to 0}\frac{f(x)}{\sin 2x}=\lim_{x\to 0}\frac{f(0+x)-f(0)}{x\cdot\frac{2\sin 2x}{2x}}$$
$$=\frac{f'(0)}{2}$$
$$=1$$

2.1(B)

1.

$\because f$在$x=0$处连续

$\therefore \lim\limits_{x\to 0} f(x)=f(0)$

$\because \lim\limits_{x\to 0}\dfrac{f(x)}{x}$存在

$\therefore \lim\limits_{x\to 0} f(x)=\dfrac{f(x)}{x}\cdot x=f(0)$

$\therefore f(0)=0$

$\therefore f'(0)=\lim\limits_{\Delta x\to 0}\dfrac{f(0+\Delta x)-f(0)}{\Delta x}=\dfrac{f(\Delta x)}{\Delta x}$

$\therefore f'(0)$存在

$\therefore f$在$x=0$处可导

3.

$\therefore f'(a)=\lim\limits_{\Delta x\to 0}\dfrac{f(a+\Delta x)-f(a)}{\Delta x}$存在,且$f(a)\neq 0$

令$f'(a)=\lim\limits_{\Delta x\to 0}\dfrac{f(a+\Delta x)-f(a)}{\Delta x}=A$

$$\therefore \lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{A\cdot \Delta x}=1$$

令$\Delta x=\dfrac{1}{n}$,则

$$\begin{aligned}\therefore \lim_{n\to\infty}\left[\frac{f(a+\frac{1}{n})}{f(a)}\right]^n &= \lim_{\Delta x\to 0}\left[\frac{f(a+\Delta x)}{f(a)}\right]^{\frac{1}{\Delta x}} \\ &= \lim_{\Delta x\to 0}\left[1+\frac{f(a+\Delta x)-f(a)}{f(a)}\right]^{\frac{1}{\Delta x}} \\ &= \lim_{\Delta x\to 0}\left[1+\frac{A\cdot\Delta x}{f(a)}\right]^{\frac{f(a)}{A\cdot\Delta x}\cdot\frac{A}{f(a)}} \\ &= e^{\frac{f'(a)}{f(a)}}\end{aligned}$$

4.

$$\therefore f'(0)=(\sin x)'|_{x=0}=\cos 0=1, f(0)=0$$

$$\therefore f'(0)=\lim_{\Delta x\to 0}\frac{f(0+\Delta x)-f(0)}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(\Delta x)}{\Delta x}=1$$

令$\Delta x=\dfrac{2}{n}$,则

$$\therefore \lim_{n\to\infty} n^{\frac{1}{2}}\sqrt{f(\frac{2}{n})}=\lim_{\Delta x\to 0}\sqrt{2}\cdot\sqrt{\frac{f(\Delta x)}{\Delta x}}=\sqrt{2}$$

5.

(1)

$$\because \lim_{x\to 0} f(x)=x^n\sin\frac{1}{x}=0$$

$\therefore f(x)$在$x=0$处连续

(2)

当$n\geq 2$时,

$$\begin{aligned}
\because f'(0) &= \lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(0+\Delta x)^n \sin \frac{1}{0+\Delta x} - 0}{\Delta x} \\
&= \lim_{\Delta x \to 0} \Delta x^{n-1} \sin \frac{1}{\Delta x} \\
&= 0
\end{aligned}$$

$\therefore f(x)$在$x=0$处可导

当$n=1$时,

$\because f'(0) = \lim_{\Delta x \to 0} \sin \frac{1}{\Delta x}$不存在

$\therefore f(x)$在$x=0$处不可导

(3)

当$n=1$时, $f'(x)$在$x=0$处不存在, 则也不连续

当$n=2$时,

$\therefore f'(x) = nx^{n-1} \sin \frac{1}{x} + x^n \cos \frac{1}{x} \cdot (-\frac{1}{x^2}) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$

$\therefore \lim\limits_{x \to 0} = \lim\limits_{x \to 0} (2x \sin \frac{1}{x} - \cos \frac{1}{x})$

此时$\cos \frac{1}{x}$不定, 极限不存在

$\therefore f'(x)$在$x=0$处不连续

当$n \geq 3$时,

$\therefore \lim\limits_{x \to 0} = \lim\limits_{x \to 0} (nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x}) = 0$

$\therefore f'(x)$在$x=0$处连续

2.2(A)

1.

(4)

$$
\begin{aligned}
y' &= (\frac{\sin x}{\cos x})' \sec x + \tan x (\frac{1}{\cos x})' \\
&= (\frac{1}{\cos^2 x}) \sec x + \tan x (\frac{\sin x}{\cos^2 x}) \\
&= \frac{\sin^2 x + 1}{\cos^3 x}
\end{aligned}
$$

(9)

$$
\begin{aligned}
y' &= \frac{1 + (\sqrt{x + \sqrt{x}})'}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \\
&= \frac{1 + \frac{1 + (\sqrt{x})'}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \\
&= \frac{1 + \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}
\end{aligned}
$$

3.

(4)

$\because y = \sqrt[3]{\frac{1+x}{1-x}}$

$\therefore \ln y = \frac{1}{3} \ln(1+x) - \frac{1}{3} \ln(1-x)$

$\therefore \frac{y'}{y} = \frac{1}{3+3x} + \frac{1}{3-3x}$

$\therefore y' = (\frac{1}{3+3x} + \frac{1}{3-3x}) \sqrt[3]{\frac{1+x}{1-x}}$

(13)

$$
\begin{aligned}
y' &= a^a x^{a^a-1} + a^{x^a}(ax^{a-1})\ln a + a^{a^x}(a^x \ln a)\ln a \\
&= a^a x^{a^a-1} + a^{x^a+1}x^{a-1}\ln a + a^{a^x+x}\ln^2 a
\end{aligned}
$$

(19)

$\because y = \sqrt[3]{\dfrac{1-\sin 2x}{1+\sin 2x}}$

$\therefore \ln y = \dfrac{1}{3}\ln(1-\sin 2x) - \dfrac{1}{3}\ln(1+\sin 2x)$

$\therefore \dfrac{y'}{y} = -\dfrac{2\cos 2x}{3-3\sin 2x} - \dfrac{2\cos 2x}{3+3\sin 2x} = -\dfrac{4}{3\cos 2x}$

$\therefore y' = \dfrac{4}{3\cos 2x}\sqrt[3]{\dfrac{1-\sin 2x}{1+\sin 2x}}$

6.

(1)

$y' = 2xf'(x^2)$

(3)

$y' = \sin 2x[f'(\sin^2 x) - f'(\cos^2 x)]$

(4)

$y' = e^x f'(e^x)e^{g(x)} + f(e^x)g'(x)e^{g(x)}$

(6)

$$
\begin{aligned}
y' &= \frac{1+e^{\frac{1}{x}} - x(1+e^{\frac{1}{x}})'}{(1+e^{\frac{1}{x}})^2} \\
&= \frac{1+e^{\frac{1}{x}} + \frac{1}{x}e^{\frac{1}{x}}}{(1+e^{\frac{1}{x}})^2}
\end{aligned}
$$

8.

设该点为(x_0, y_0),设切线斜率为k

题目意思可转化为$y_0 = -kx_0$

即证$k = -\frac{y_0}{x_0}$

$\because xy = a$

$\therefore y = \frac{a}{x}$

由导数几何意义可知$k = y' = -\frac{a}{x_0^2} = -\frac{y_0}{x_0}$

$\therefore$ 可知命题成立