Assignment 2

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Question 1. Some interesting properties of $\mathcal{EL}$

(1)

Let $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ be an interpretation and $\Delta^{\mathcal{I}} = \{a\}$, $A^{\mathcal{I}} = \{a\}$ for all concept name A, $r^{\mathcal{I}} = \{(a, a)\}$ for all role r.

By induction on the structure of $\mathcal{EL}$-concept C:

- Assume that $C = \top$, then $C^{\mathcal{I}} = \Delta^{\mathcal{I}} = \{a\}$.
- Assume that $C = A \in \mathbf{C}$, then $C^{\mathcal{I}} = A^{\mathcal{I}} = \{a\}$ by definition.
- Assume that $C = D \sqcap E$, then $C^{\mathcal{I}} = D^{\mathcal{I}} \cap E^{\mathcal{I}} = \{a\} \cap \{a\} = \{a\}$.
- Assume that $C = \exists r.D$, then $C^{\mathcal{I}} = \{a\}$ by the semantics of existential restriction.

So there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.

(2)

We use the same interpretation $\mathcal{I}$ in (1).

For any $\mathcal{EL}$ concept inclusion $C \sqsubseteq D$ in $\mathcal{EL}$-TBox $\mathcal{T}$ (replace $C \equiv D$ with $C \sqsubseteq D$ and $D \sqsubseteq C$), we can know that $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ as $C^{\mathcal{I}} = D^{\mathcal{I}} = \{a\}$ by (1).

So there exists an intterpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$.

Question 2. Reasoning in $\mathcal{EL}$

(1)

Consider $\mathcal{T}$:

$$\begin{aligned}
\text{Bird} &\equiv \text{Vertebrate} \sqcap \exists \text{has_part.Wing} \\
\text{Reptile} &\sqsubseteq \text{Vertebrate} \sqcap \exists \text{lays.Egg}
\end{aligned}$$

Step 1 gives:

$$\begin{aligned}
\text{Bird} &\sqsubseteq \text{Vertebrate} \sqcap \exists \text{has_part.Wing} \\
\text{Vertebrate} \sqcap \exists \text{has_part.Wing} &\sqsubseteq \text{Bird} \\
\text{Reptile} &\sqsubseteq \text{Vertebrate} \sqcap \exists \text{lays.Egg}
\end{aligned}$$

Step 2 gives:

$$\begin{aligned}
\text{Bird} &\sqsubseteq \text{Vertebrate} \\
\text{Bird} &\sqsubseteq \exists \text{has_part.Wing} \\
\text{Vertebrate} \sqcap \exists \text{has_part.Wing} &\sqsubseteq \text{Bird} \\
\text{Reptile} &\sqsubseteq \text{Vertebrate} \\
\text{Reptile} &\sqsubseteq \exists \text{lays.Egg}
\end{aligned}$$

Step 4 gives:

$$\begin{aligned}
\text{Bird} &\sqsubseteq \text{Vertebrate} \\
\text{Bird} &\sqsubseteq \exists \text{has_part.Wing} \\
X &\sqsubseteq \exists \text{has_part.Wing} \\
\exists \text{has_part.Wing} &\sqsubseteq X \\
\text{Vertebrate} \sqcap X &\sqsubseteq \text{Bird} \\
\text{Reptile} &\sqsubseteq \text{Vertebrate} \\
\text{Reptile} &\sqsubseteq \exists \text{lays.Egg}
\end{aligned}$$

So it is the $\mathcal{T}'$.

(2)

Initalise:

$$
\begin{aligned}
S(\text{Bird}) &= \{\text{Bird}\} \\
S(\text{Vertebrate}) &= \{\text{Vertebrate}\} \\
S(\text{Wing}) &= \{\text{Wing}\} \\
S(X) &= \{X\} \\
S(\text{Reptile}) &= \{\text{Reptile}\} \\
S(\text{Egg}) &= \{\text{Egg}\} \\
R(\text{has\_part}) &= \emptyset \\
R(\text{lays}) &= \emptyset
\end{aligned}
$$

Application of (simpleR) and axiom 1, 6 gives:

$$
\begin{aligned}
S(\text{Bird}) &= \{\text{Bird}, \text{Vertebrate}\} \\
S(\text{Reptile}) &= \{\text{Reptile}, \text{Vertebrate}\}
\end{aligned}
$$

Application of (rightR) and axiom 2, 3, 7 gives:

$$
\begin{aligned}
R(\text{has\_part}) &= \{(\text{Bird}, \text{Wing}), (X, \text{Wing})\} \\
R(\text{lays}) &= \{(\text{Reptile}, \text{Egg})\}
\end{aligned}
$$

Application of (leftR) and axiom 4 gives:

$$
S(\text{Bird}) = \{\text{Bird}, \text{Vertebrate}, X\}
$$

No more rules are applicable.

So the final result is:

$$
\begin{aligned}
S(\text{Bird}) &= \{\text{Bird}, \text{Vertebrate}, X\} \\
S(\text{Vertebrate}) &= \{\text{Vertebrate}\} \\
S(\text{Wing}) &= \{\text{Wing}\} \\
S(X) &= \{X\} \\
S(\text{Reptile}) &= \{\text{Reptile}, \text{Vertebrate}\} \\
S(\text{Egg}) &= \{\text{Egg}\} \\
R(\text{has\_part}) &= \{(\text{Bird}, \text{Wing}), (X, \text{Wing})\} \\
R(\text{lays}) &= \{(\text{Reptile}, \text{Egg})\}
\end{aligned}
$$

(3)

Use the result of (2) and $A \sqsubseteq_{\mathcal{T}'} B$ if and only if $B \in S(A)$, we can obtain that

- $\text{Reptile} \sqsubseteq_{\mathcal{T}'} \text{Vertebrate}$ is true
- $\text{Vertebrate} \sqsubseteq_{\mathcal{T}'} \text{Bird}$ is false

Question 3. Bisimulation & bisimulation invariance

(1)

We extend the notion of bisimulation relation to $\mathcal{ALCN}$ firstly.

Let $\mathcal{I}$ and $\mathcal{J}$ be interpretations. The relation $\rho \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{J}}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$ if

(i) $d\rho e$ implies $d \in A^{\mathcal{I}}$ if and only if $e \in A^{\mathcal{J}}$ for all $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{J}}$, and $I \in \mathbf{C}$.

(ii) if $d_1, \cdots, d_n$ are all the distinct elements of $\Delta^{\mathcal{I}}$ such that $(d, d_i) \in R^{\mathcal{I}}$ for $1 \leq i \leq n$, then there are exactly n distinct elements $e_1, \cdots, e_n$ of $\Delta^{\mathcal{J}}$ such that $(e, e_i) \in R^{\mathcal{J}}$ for all $1 \leq i \leq n$.

(iii) if $e_1, \cdots, e_n$ are all the distinct elements of $\Delta^{\mathcal{J}}$ such that $(e, e_i) \in R^{\mathcal{J}}$ for $1 \leq i \leq n$, then there are exactly n distinct elements $d_1, \cdots, d_n$ of $\Delta^{\mathcal{I}}$ such that $(d, d_i) \in R^{\mathcal{J}}$ for all $1 \leq i \leq n$.

Then we prove that $\mathcal{ALCN}$ is bisimulation invariant for the bisimulation relation.

We omit the part of original part and add new step:

Assumed that $C = (\leq nR)$. Then $d \in (\leq nR)^{\mathcal{I}}$

if and only if exists all $m \leq n$ elements $d_1, \cdots, d_m$ with $(d, d_i) \in R^{\mathcal{I}}$ (semantics of $\leq nR$)

if and only if exists exactly $m \leq n$ elements $e_1, \cdots, e_m$ with $(e, e_i) \in R^{\mathcal{I}}$ (hypothesis and $d\rho e$)

if and only if $d_2 \in (\leq nR)^{\mathcal{I}_2}$.

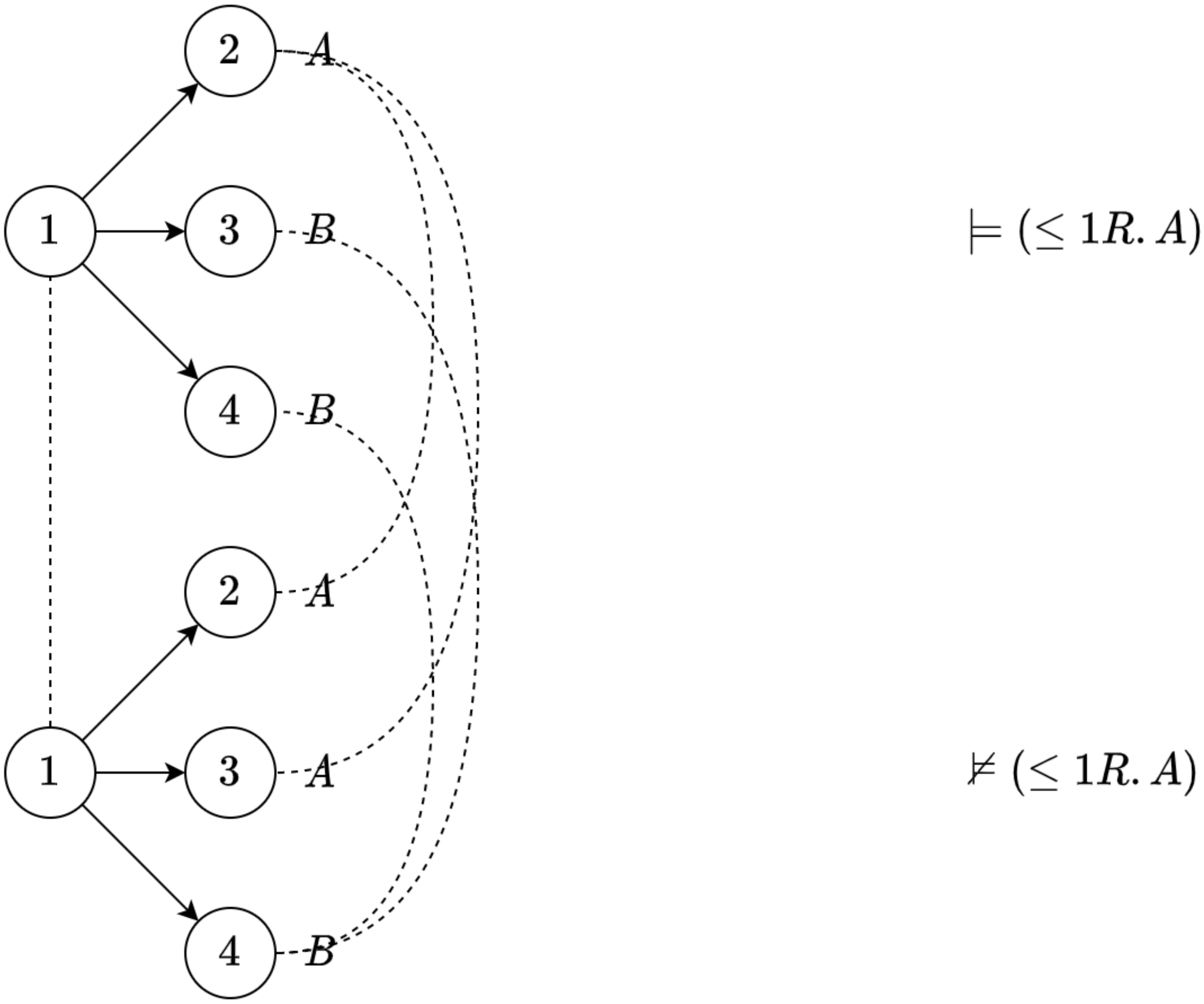
Assumed that $C = (\geq nR)$. Then $d \in (\geq nR)^{\mathcal{I}}$

if and only if exists all $m \geq n$ elements $d_1, \cdots, d_m$ with $(d, d_i) \in R^{\mathcal{I}}$ (semantics of $\geq nR$)

if and only if exists exactly $m \geq n$ elements $e_1, \cdots, e_m$ with $(e, e_i) \in R^{\mathcal{I}}$ (hypothesis and $d\rho e$)

if and only if $d_2 \in (\geq nR)^{\mathcal{I}_2}$.

(2)



As the image, there is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$, so $\mathcal{ALC}$ cannot distinguish the interpretations $\mathcal{I}$ and $\mathcal{J}$ because of (1).

But $\mathcal{ALCQ}$ can distinguish them by $(\leq 1R.A)$.

So $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.

Question 4. Closure under Disjoint Union

Let $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ be an ALC-knowledge base and $(\mathcal{I}_v)_{v\in\Omega}$ a family of models of $\mathcal{K}$.

We extend the notion of disjoint union to individual names.

- $\Delta^{\mathcal{J}} = \{(d,v) | v \in \Omega \text{ and } d \in \Delta^{\mathcal{I}_v}\}$
- $A^{\mathcal{J}} = \{(d,v) | v \in \Omega \text{ and } d \in A^{\mathcal{I}_v}\}$ for all $A \in \mathbf{C}$
- $r^{\mathcal{J}} = \{((d,v),(e,v)) | v \in \Omega \text{ and } (d,e) \in r^{\mathcal{I}_v}\}$ for all $r \in \mathbf{R}$
- $a^{\mathcal{J}} = (a^{\mathcal{I}_{v0}}, v_0)$ for all individual names a occurring in $\mathcal{A}$ and $v_0 \in \Omega$ is a singe index picked up previously and arbitrarily.

Then we prove that its disjoint union $\mathcal{J} = \biguplus_{v \in \Omega}$ is also a model of $\mathcal{K}$.

Assume that $\mathcal{J}$ is not a model of $\mathcal{T}$. Then there is a GCI $C \sqsubseteq D$ in $\mathcal{T}$ and an element $(d,v) \in \Delta^{\mathcal{J}}$ such that $(d,v) \in C^{\mathcal{J}}$, but $(d,v) \notin D^{\mathcal{J}}$. By the bisimulation between $\mathcal{I}_v$ and $\mathcal{J}$, this implies $d \in C^{\mathcal{I}_v}$ and $d \notin D^{\mathcal{I}_v}$, which contradicts our assumption that $\mathcal{I}_v$ is a model of $\mathcal{K}$.

Assume that $\mathcal{J}$ is not a model of $\mathcal{A}$. And we assume that there is assertion $a : C$ in $\mathcal{A}$ and the element $(a^{\mathcal{I}_{v0}}, v_0) \notin C^{\mathcal{J}}$. By the bisimulation between $\mathcal{I}_{v0}$ and $\mathcal{J}$, this implies $a^{\mathcal{I}_{v0}} \notin C^{\mathcal{I}_{v0}}$, which contradicts our assumption that $\mathcal{I}_{v0}$ is a model of $\mathcal{K}$. Then we assume that there is assertion $(a,b) : r$ in $\mathcal{A}$ and $((a^{\mathcal{I}_{v0}}, v_0),(b^{\mathcal{I}_{v0}}, v_0)) \notin r^{\mathcal{J}}$. By the bisimulation between $\mathcal{I}_{v0}$ and $\mathcal{J}$, this implies $(a^{\mathcal{I}_{v0}}, b^{\mathcal{I}_{v0}}) \notin r^{\mathcal{I}_{v0}}$, which contradicts our assumption that $\mathcal{I}_{v0}$ is a model of $\mathcal{K}$.

Question 5. Closure under Disjoint Union

$\Leftarrow$:

We have $C \sqsubseteq_{\mathcal{T}} D$, so $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every model $\mathcal{I}$ of $\mathcal{T}$.

Because each model J of $\mathcal{K}$ is must be a model of $\mathcal{T}$, so $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$ holds for every model $\mathcal{J}$ of $\mathcal{K}$.

So we know $C \sqsubseteq_{\mathcal{K}} D$.

$\Rightarrow$:

We have $C \sqsubseteq_{\mathcal{K}} D$, so $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every model $\mathcal{I}$ of $\mathcal{K}$. And because $\mathcal{K}$ is a consisten $\mathcal{ALC}$-KB, so there is a model $\mathcal{I}_1$ of $\mathcal{K}$ satisfying $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1}$, which is also a model of $\mathcal{T}$.

Assumed $C \not\sqsubseteq_{\mathcal{T}} D$, then there is an model $\mathcal{I}_2$ of $\mathcal{T}$ and $C^{\mathcal{I}_2} \not\subseteq D^{\mathcal{I}_2}$.

We can get the disjoint union $\mathcal{J}$ of $\mathcal{I}_1$ and $\mathcal{I}_2$. By the previous exercise we attain that there is a bisimulation between $\mathcal{I}_1$ and $\mathcal{J}$. We need to prove $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Assumed $C^{\mathcal{J}} \not\subseteq D^{\mathcal{J}}$, then there is an element $(d,v) \in C^{\mathcal{J}}$ but $(d,v) \notin D^{\mathcal{J}}$. By bisimulation between $\mathcal{I}_1$ and $\mathcal{J}$, this implies $d \in C^{\mathcal{I}_1}$ but $d \notin D^{\mathcal{I}_1}$, which contradicts the former conclusion $C^{\mathcal{I}_1} \subseteq D^{\mathcal{I}_1}$. So we know $C^{\mathcal{J}} \subseteq D^{\mathcal{J}}$. Using the bisimulation between $\mathcal{J}$ and $\mathcal{I}_2$, and the same steps, we could attain $C^{\mathcal{I}_2} \subseteq D^{\mathcal{I}_2}$, which contradicts the former assumption $C^{\mathcal{I}_2} \not\subseteq D^{\mathcal{I}_2}$.

So we know $C \sqsubseteq_{\mathcal{T}} D$.

Question 6. Finite model property

(1)

Because C is a satisfiable $\mathcal{ALC}$-concept with respect to $\mathcal{T}$. So by the finite model property, there is a finite model $\mathcal{I}$ such that $|C^{\mathcal{I}}| \geq 1$.

Let $\mathcal{I}_m = \biguplus_{v \in \{1, \cdots, m\}} \mathcal{I}$, e.t. the m-fold disjoint union of $\mathcal{I}$ itself. So $|C^{I_m}| = m|C^{\mathcal{I}}| \geq m$.

So for all $m \geq 1$ there is a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_m}| \geq m$.

(2)

It doesn't hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$".

Let $C = \top$, $\mathcal{T} = \{A \sqsubseteq \exists r.\neg A, \neg A \sqsubseteq \exists r.A\}$ and $m = 1$.

For any model $\mathcal{I}$ of $\mathcal{T}$, because $\Delta^{\mathcal{I}} \neq \emptyset$ and $(A \sqcup \neg A)^{\mathcal{I}} = \top^{\mathcal{I}} = \Delta^{\mathcal{I}}$, so $A^{\mathcal{I}} \neq \emptyset$ or $(\neg A)^{\mathcal{I}} \neq \emptyset$.

We assume $A^{\mathcal{I}} = \emptyset$, so $(\exists r.A)^{\mathcal{I}} = \emptyset$. By the CGI $\neg A \sqsubseteq \exists r.A$, we can know $(\neg A)^{\mathcal{I}} \subseteq (\exists r.A)^{\mathcal{I}}$ then $(\neg A)^{\mathcal{I}} = \emptyset$, which contradicts the former conclusion $A^{\mathcal{I}} \neq \emptyset$ or $(\neg A)^{\mathcal{I}} \neq \emptyset$.

We assume $(\neg A)^{\mathcal{I}} = \emptyset$, so $(\exists r.\neg A)^{\mathcal{I}} = \emptyset$. By the CGI $A \sqsubseteq \exists r.\neg A$, we can know $A^{\mathcal{I}} \subseteq (\exists r.\neg A)^{\mathcal{I}}$ then $A^{\mathcal{I}} = \emptyset$, which contradicts the former conclusion $A^{\mathcal{I}} \neq \emptyset$ or $(\neg A)^{\mathcal{I}} \neq \emptyset$.

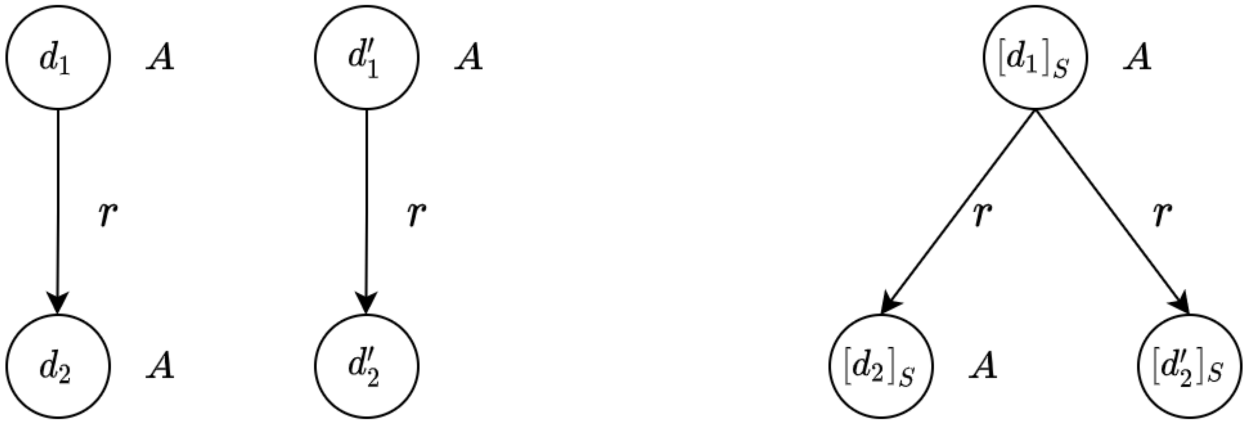
Then we can know $|C^{\mathcal{I}}| = |\top^{\mathcal{I}}| = |A^{\mathcal{I}}| + |(\neg A)^{\mathcal{I}}| \geq 1 + 1 = 2$, which contradicts $|C^{\mathcal{I}}| = m = 1$.

So it doesn't hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$".

Question 7. Bisimulation over filtration

The statement "the relation $\rho = \{(d, [d]) | d \in \Delta^{\mathcal{I}}\}$ is a bisimulation between I and J" is false.

Let $C = A$ and $\mathcal{T} = \{\exists r.\top \sqsubseteq \top\}$, so $S = \text{sub}(C) \cup \text{sub}(\mathcal{T}) = \{\top, A, \exists r.\top\}$.



We can see that the right interpretation $\mathcal{J}$ is the $\mathcal{S}$-filtration of the left interpretation $\mathcal{I}$ with respect to $\mathrm{sub}(C) \cup \mathrm{sub}(\mathcal{T})$.

But relation $\rho = \{(d, [d]) | d \in \Delta^{\mathcal{I}}\}$ is not a bisimmulation between $\mathcal{I}$ and $\mathcal{J}$.

Question 8. Bisimulation within the same interpretation

(1)

We need to prove that $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

(i) $d \approx_{\mathcal{I}} e$ implies there is a bisimulation ρ on $\mathcal{I}$ such that $d\rho e$, which imlies

$d \in A^{\mathcal{I}}$ if and only if $e \in A^{\mathcal{I}}$

for all $d \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{I}}$, and $A \in \mathbf{C}$.

(ii) $d \approx_{\mathcal{I}} e$ and $(d, d') \in r^{\mathcal{I}}$ implies there is a bisimulation ρ on $\mathcal{I}$ such that $d\rho e$ and $(d, d') \in r^{\mathcal{I}}$, which implies the existence of $e' \in \Delta^{\mathcal{I}}$ such that

$d' \rho e'$ and $(e, e') \in r^{\mathcal{I}}$, and then

$d' \approx_{\mathcal{I}} e'$ and $(e, e') \in r^{\mathcal{I}}$ because of the definition of $d' \approx_{\mathcal{I}} e'$,

for all $d, d' \in \Delta^{\mathcal{I}}$, $e \in \Delta^{\mathcal{I}}$, and $r \in \mathbf{R}$.

(iii) Same property in the opposite direction with same method.

So $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

(2)

By the definition like filtration, we can know

$[d]_{\approx_\mathcal{I}} = \{e \in \Delta^\mathcal{I} | d \approx_\mathcal{I} e\}$

And the $\mathcal{J}$ is defined as follow:

$\Delta^\mathcal{J} = \{[d]_{\approx_\mathcal{I}} | d \in \Delta^\mathcal{I}\}$

$A^\mathcal{J} = \{[d]_{\approx_\mathcal{I}} | \text{there is } d' \in [d]_{\approx_\mathcal{I}} \text{ with } d' \in A^\mathcal{I}\}$ for all $A \in \mathbf{C}$

$r^\mathcal{J} = \{([d]_{\approx_\mathcal{I}}, [e]_{\approx_\mathcal{I}}) | \text{there is } d' \in [d]_{\approx_\mathcal{I}}, e' \in [e]_{\approx_\mathcal{I}} \text{ with } (d', e') \in r^\mathcal{I}\}$ for all $r \in \mathbf{R}$

Now we show that $\rho = \{(d, [d]_{\approx_\mathcal{I}}) | d \in \Delta^\mathcal{I}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

(i) $(d, [d]_{\approx_\mathcal{I}}) \in \rho$ implies

$\Rightarrow$:

Assume $d \in A^\mathcal{I}$. Because there is $d \in [d]_{\approx_\mathcal{I}}$ as $d \approx_\mathcal{I} d$ with $d \in A^\mathcal{I}$, we can know $[d]_{\approx_\mathcal{I}} \in A^\mathcal{J}$ by the definition of $A^\mathcal{J}$.

$\Leftarrow$:

Assume $[d]_{\approx_\mathcal{I}} \in A^\mathcal{J}$. There is $d' \in [d]_{\approx_\mathcal{I}}$ with $d' \in A^\mathcal{I}$. Because $d \in [d]_{\approx_\mathcal{I}}$ as $d \approx_\mathcal{I} d$, we can know that $d \approx_\mathcal{I} d'$. And $\approx_\mathcal{I}$ is a bisimulation on $\mathcal{I}$, which implies $d' \in A^\mathcal{I}$ if and only if $d \in A^\mathcal{I}$.

So $d \in A^\mathcal{I}$ if and only if $[d]_{\approx_\mathcal{I}} \in A^\mathcal{J}$

for all $d \in \Delta^\mathcal{I}$, $[d]_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$, and $A \in \mathbf{C}$.

(ii) $(d, [d]_{\approx_\mathcal{I}}) \in \rho$ and $(d, e) \in r^\mathcal{I}$ implies there is $d \in [d]_{\approx_\mathcal{I}}, e \in [e]_{\approx_\mathcal{I}}$ with $(d, e) \in r^\mathcal{I}$, which implies the existence of $[e]_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$ such that

$(e, [e]_{\approx_\mathcal{I}}) \in \rho$ and $([d]_{\approx_\mathcal{I}}, [e]_{\approx_\mathcal{I}}) \in r^\mathcal{J}$

for all $d, e \in \Delta^\mathcal{I}$, $[d]_{\approx_\mathcal{I}} \in \Delta^\mathcal{J}$, and $r \in \mathbf{R}$.

(iii) $(d, [d]_{\approx_\mathcal{I}}) \in \rho$ and $([d]_{\approx_\mathcal{I}}, [e]_{\approx_\mathcal{I}}) \in r^\mathcal{J}$ implies there is $d' \in [d]_{\approx_\mathcal{I}}, e' \in [e]_{\approx_\mathcal{I}}$ with $(d', e') \in r^\mathcal{I}$. Because $d \in [d]_{\approx_\mathcal{I}}$, we can know $d \approx_\mathcal{I} d'$. And $\approx_\mathcal{I}$ is a bisimulation on $\mathcal{I}$, which implies the existence of $e \in \Delta^\mathcal{I}$ such that

$e \approx_{\mathcal{I}} e'$ and $(d, e) \in r^{\mathcal{I}}$

So we can know

$(e, [e]_{\approx_{\mathcal{I}}}) \in \rho$ and $(d, e) \in r^{\mathcal{I}}$

for all $d \in \Delta^{\mathcal{I}}$, $[d]_{\approx_{\mathcal{I}}}, [e]_{\approx_{\mathcal{I}}} \in \Delta^{\mathcal{J}}$, and $r \in \mathbf{R}$.

So we show that $\rho = \{(d, [d]_{\approx_{\mathcal{I}}}) | d \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

(3)

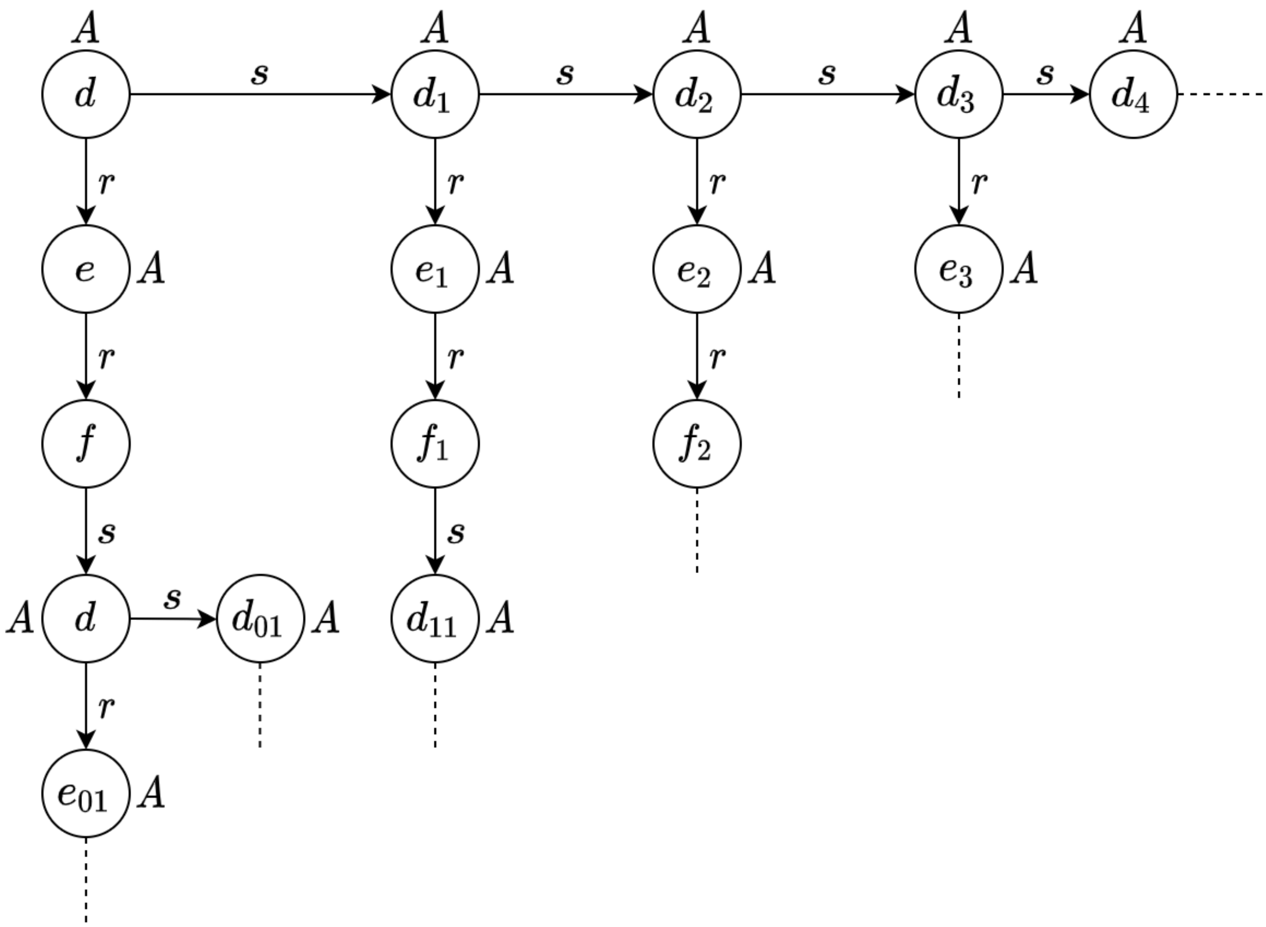
Because $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C with respect to an $\mathcal{ALC}$-TBox $\mathcal{T}$, then $C^{\mathcal{I}} \neq \emptyset$.

Let $d \in \Delta^{\mathcal{I}}$ be such that $d \in C^{\mathcal{I}}$. Since there is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$, so $[d]_{\approx_{\mathcal{I}}} \in C^{\mathcal{J}}$ by bisimulation invariance of $\mathcal{ALC}$.

It is also easy to see that $\mathcal{J}$ is a model of $\mathcal{T}$. Let $D \sqsubseteq E$ be a GCI in $\mathcal{T}$, and $[e]_{\approx_{\mathcal{I}}} \in D^{\mathcal{J}}$. We must show $[e]_{\approx_{\mathcal{I}}} \in E^{\mathcal{J}}$. By bisimulation invariance, $e \in D^{\mathcal{I}}$ and thus $e \in E^{\mathcal{J}}$ since $\mathcal{I}$ is a model of $\mathcal{T}$. And then $e \in E^{\mathcal{J}}$ implies $[e]_{\approx_{\mathcal{I}}} \in E^{\mathcal{J}}$.

So $\mathcal{J}$ is a model of an $\mathcal{ALC}$-concept C with respect to an $\mathcal{ALC}$-TBox $\mathcal{T}$.

Question 9. Unravelling



Question 10. Tree model property

The statement "if $\mathcal{K}$ is an $\mathcal{ALC}$-KB and C an $\mathcal{ALC}$-concept such that C is satisfiable w.r.t. $\mathcal{K}$, then C has a tree model w.r.t. $\mathcal{K}$" is false.

Let $C = \top, \mathcal{K} = (\mathcal{T}, \mathcal{A}), \mathcal{T} = \emptyset, \mathcal{A} = \{a : A, b : \neg A, (a, b) : r, (b, a) : r\}$.

For any model $\mathcal{I}$ of such $\mathcal{K}$, we can know that $a^{\mathcal{I}}$ and $b^{\mathcal{I}}$ are two distinct elements, and $(a^{\mathcal{I}}, b^{\mathcal{I}}), (b^{\mathcal{I}}, a^{\mathcal{I}}) \in r^{\mathcal{I}}$ both. So there is a ring "$a \xrightarrow{r} b \xrightarrow{r} a$" for any model $\mathcal{I}$ of such $\mathcal{K}$.

So the statement is false.

Question 11. Tableau algorithm

Init:

$$\mathcal{A}_0 = \mathcal{A} = \{(b, a) : r, (a, b) : r, (a, c) : s, (c, b) : s, a : \exists s.A,\\ b : \forall r.((\forall s.\neg A) \sqcup (\exists r.B)), c : \forall s.(B \sqcap (\forall s.\bot))\}$$

An application of $\rightarrow_\exists$ and $a : \exists s.A$ gives

$$\mathcal{A}_1 = \mathcal{A}_0 \cup \{(a, d) : s, d : A\}$$

An application of $\rightarrow_\forall$ and $b : \forall r.((\forall s.\neg A) \sqcup (\exists r.B))$ gives:

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{a : (\forall s.\neg A) \sqcup (\exists r.B)\}$$

An application of $\rightarrow_\forall$ and $c : \forall s.(B \sqcap (\forall s.\bot))$ gives:

$$\mathcal{A}_3 = \mathcal{A}_2 \cup \{b : B \sqcap (\forall s.\bot)\}$$

An application of $\rightarrow_\sqcap$ and $b : B \sqcap (\forall s.\bot)$ gives:

$$\mathcal{A}_4 = \mathcal{A}_3 \cup \{b : B, b : \forall s.\bot\}$$

An application of $\rightarrow_\sqcup$ and $a : (\forall s.\neg A) \sqcup (\exists r.B)$ gives:

Firstly, we can try

$$\mathcal{A}_5 = \mathcal{A}_4 \cup \{a : \forall s.\neg A\}$$

An application of $\rightarrow_\forall$ and $a : \forall s.\neg A$ gives

$$\mathcal{A}_6 = \mathcal{A}_5 \cup \{c : \neg A, d : \neg A\}$$

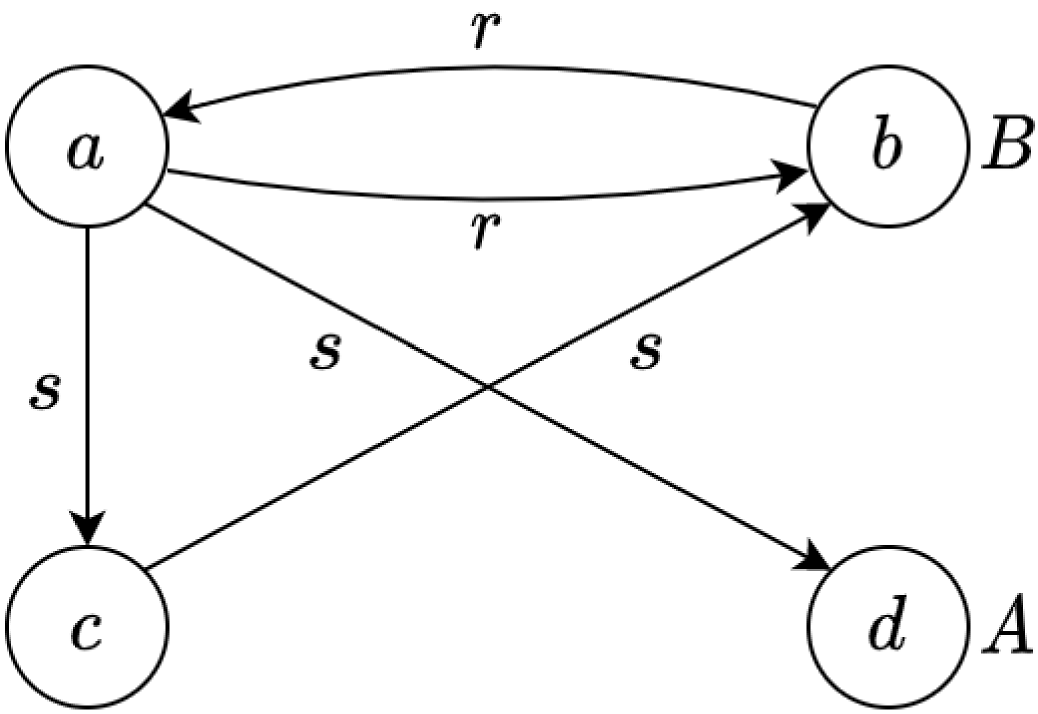
We have abtained a clash because $d : A$ and $d : \neg A$, thus this choice was unsuccessful.

Secondly, we can try

$$\mathcal{A}_5^* = \mathcal{A}_4 \cup \{a : \exists r.B\}$$

No rule is applicable to $\mathcal{A}_5^*$ and it does not contain a clash.

Thus, $\mathcal{A}$ is consistent.



Question 12. Extension of Tableau algorithm

We need to add a new law $\neg(C \to D) \equiv C \sqcap \neg D$ to push negations inwards dealing with new concept constructor $\to$ (implication).

We prove $C \to D \equiv \neg C \sqcup D$ firstly. For any interpretation $\mathcal{I}$, there is

$$\begin{aligned}(C \to D)^{\mathcal{I}} &= \{x \in \Delta^{\mathcal{I}} | x \in C^{\mathcal{I}} \text{ implies } x \in D^{\mathcal{I}}\} \\ &= \{x \in \Delta^{\mathcal{I}} | x \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}} \text{ or } x \in D^{\mathcal{I}}\} \\ &= (\neg C \sqcup D)^{\mathcal{I}}\end{aligned}$$

Then we prove that $\neg(C \to D) \equiv C \sqcap \neg D$. With lemma $\neg D \equiv \neg C \sqcap \neg D$ we can know

$$\neg(C \to D) \equiv \neg(\neg C \sqcup D) \equiv C \sqcap \neg D$$

So we can still get the normalised ABox $\mathcal{A}$ with NNF by preprocessing.

For the deterministic $\to$-rule:

Terminating:

We omit part of original proof and add the new proof for $\rightarrow$.

Let $m = |\operatorname{sub}(\mathcal{A})|$.

- After applying application, it will add a new assertion of the form $a : C$ and $C \in \operatorname{sub}(\mathcal{A})$. So for any individual name a, we have $\operatorname{con}_{\mathcal{A}}(a) \leq m$.
- It is still only $\exists$-rule that adds a new individual name. With the same original proof, a given individual name can cause the addition of at most m new individual names, and the out-degree of each tree in the forest-shaped ABox is thus bounded by m.
- With the same original proof, the depth of each tree in the forest-shaped ABox is bounded by m.

There properties ensure that there is a bound on the size of the ABox that can be constructed via rule applications, and thus a bound on the number of recursive applications of expand.

Soundness:

The rule doesn't remains sound.

Let $\mathcal{A} = \{a : (C \sqcup D) \rightarrow E, a : \neg E, a : C\}$

Because $a : C \sqcup D \notin \mathcal{A}$, so the deterministic $\rightarrow$-rule is not applicable. And so do other rules.

So the $\operatorname{consistent}(\mathcal{A})$ will return "consistent" because no rules are applicable and no clash in it.

But $\mathcal{A}$ is not consistent. We replace $(C \sqcup D) \rightarrow E$ with $\neg(C \sqcup D) \sqcup E$ in preprocessing and call the original $\operatorname{consistent}(\mathcal{A})$, we will find there is a clash $\{a : E, a : \neg E\} \subseteq \mathcal{A}'$.

So the rule doesn't remains sound.

Completeness:

We only modify and add the necessary proof.

Let $\mathcal{A}$ be consistent, and consider a model $\mathcal{I}$ of $\mathcal{A}$. Since $\mathcal{A}$ is consistent, it cannot contain a clash.

If $\mathcal{A}$ is complete, since it does not contain a clash, expand simply returns $\mathcal{A}$ and consistent returns "consistent". If $\mathcal{A}$ is not complete, then expand calls itself recursively until $\mathcal{A}$ is complete; each call selects a rule and applies it.

- The deterministic $\to$-rule: If $a : C \to D \in \mathcal{A}$ and $a : C \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \to D)^{\mathcal{I}}$. Thus $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ or $a^{\mathcal{I}} \in D^{\mathcal{I}}$ by the semantics of $\to$, but $a^{\mathcal{I}} \in C^{\mathcal{I}}$, so $a^{\mathcal{I}} \in D^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup \{a : D\}$, so $\mathcal{A}$ is still consistent after the rule is applied.

For the nondeterministic $\to$-rule:

Terminating:

We omit part of original proof and add the new proof for $\to$.

Let $m = |\operatorname{sub}(\mathcal{A})|$.

- After applying application, it will add a new assertion of the form $a : C$ and $C \in \operatorname{sub}(\mathcal{A})$. So for any individual name a, we have $\operatorname{con}_{\mathcal{A}}(a) \leq m$.
- It is still only $\exists$-rule that adds a new individual name. With the same original proof, a given individual name can cause the addition of at most m new individual names, and the out-degree of each tree in the forest-shaped ABox is thus bounded by m.
- With the same original proof, the depth of each tree in the forest-shaped ABox is bounded by m.

There properties ensure that there is a bound on the size of the ABox that can be constructed via rule applications, and thus a bound on the number of recursive applications of expand.

Soundness:

We only modify and add the necessary proof.

The construction of $\mathcal{I}$ means that it trivially satisfies all role assertions in $\mathcal{A}'$. By induction on the structure of concepts, we show the following property:

if $a : C \in \mathcal{A}'$, then $a^{\mathcal{I}} \in C^{\mathcal{I}}$

Induction Basis: C is a conceptname: by definition of $\mathcal{I}$, if $a : C \in \mathcal{A}'$, then $a^{\mathcal{I}} \in C^{\mathcal{I}}$ as required.

Induction Steps:

- $C = \neg D$: since $\mathcal{A}'$ is clash-free, $a : \neg D \in \mathcal{A}'$ implies that $a : D \in \mathcal{A}'$. Since all concepts in $\mathcal{A}$ are in NNF, D is a concept name. By definition of $\mathcal{I}$, $a^{\mathcal{I}} \notin D^{\mathcal{I}}$, which implies $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}} = C^{\mathcal{I}}$ as required.

- $C = D \to E$: if $a : D \to E \in \mathcal{A}'$, then completeness of $\mathcal{A}'$ implies that $\{a : E\} \subseteq \mathcal{A}'$ or $\{a : \dot{\neg} D\} \subseteq \mathcal{A}'$ (otherwise the nondeterministic $\to$-rule would be applicable). Thus $a^{\mathcal{I}} \in E^{\mathcal{I}}$ or $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus D^{\mathcal{I}}$ by induction, and hence $a^{\mathcal{I}} \in (\Delta^{\mathcal{I}} \setminus D^{\mathcal{I}}) \cup E^{\mathcal{I}} = (\neg D \sqcup E)^{\mathcal{I}} = (D \to E)^{\mathcal{I}}$ by the semantics of $\to$.

As a consequence, $\mathcal{I}$ satisfies all concept assertions in $\mathcal{A}'$ and thus in $\mathcal{A}$, and it satisfies all role assertions in $\mathcal{A}'$ and thus in $\mathcal{A}$ by definition. Hence $\mathcal{A}$ has a model and thus is consistent.

Completeness:

We only modify and add the necessary proof.

Let $\mathcal{A}$ be consistent, and consider a model $\mathcal{I}$ of $\mathcal{A}$. Since $\mathcal{A}$ is consistent, it cannot contain a clash.

If $\mathcal{A}$ is complete, since it does not contain a clash, expand simply returns $\mathcal{A}$ and consistent returns "consistent". If $\mathcal{A}$ is not complete, then expand calls itself recursively until $\mathcal{A}$ is complete; each call selects a rule and applies it.

- The nondeterministic $\to$-rule: If $a : C \to D \in \mathcal{A}$, then $a^{\mathcal{I}} \in (C \to D)^{\mathcal{I}}$. Thus $a^{\mathcal{I}} \in \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$ or $a^{\mathcal{I}} \in D^{\mathcal{I}}$ by the semantics of $\to$. Therefore, at least one of the ABoxes $\mathcal{A}' \in \exp(\mathcal{A}, \text{nondeterministic } \to \text{-rule}, a : C \to D)$ is consistent. Thus, one of the calls of expand is applied to a consistent ABox.

Question 13. Modification of Tableau algorithm

We modify the notion of a clash to that $\mathcal{A}$ contains a clash if, for some individual name a, and for some concept C, $\{a : C, a : \neg C\} \subseteq \mathcal{A}$, or for some individual names a and b, and for some role names r and s, $\{(a, b) : r, (a, b) : s\} \subseteq \mathcal{A}$ and $\{\text{disjoint}(r, s)\} \subseteq \mathcal{T}$.

And we add a new expansion rule:

$\sqsubseteq$-rule: if $(a, b) : r \in \mathcal{A}$, $r \sqsubseteq s \in \mathcal{T}$ and $(a, b) : s \notin \mathcal{A}$, then $\mathcal{A} \longrightarrow \mathcal{A} \cup \{(a, b) : s\}$

Termination:

We need to prove that the $\sqsubseteq$-rule is terminable. Because the number of individual names in $\mathcal{A}$ is bounded by the original proof, the number of new role assertions is no more than square of the number of individual names, and thus it is bounded.

Soundness:

Let $\mathcal{A}'$ be the set return by $\text{expand}(\mathcal{A})$. Since the algorithm returns "consistent", $\mathcal{A}'$ is a complete and clash-free ABox.

We use same definition of $\mathcal{I}$ from original proof.

We firstly prove that $\mathcal{I}$ satisfies each $\text{disjoint}(r, s) \in \mathcal{T}$. Assume $r^{\mathcal{I}} \cap s^{\mathcal{I}} \neq \emptyset$, thus there are such a and b that $(a, b) \in r^{\mathcal{I}}$ and $(a, b) \in s^{\mathcal{I}}$. So we can know $\{(a, b) : r, (a, b) : s\} \subseteq \mathcal{A}'$, which contradicts $\mathcal{A}'$ is a clash-free ABox.

We secondly prove that $\mathcal{I}$ satisfies each $r \sqsubseteq s \in \mathcal{T}$. Assume there are such a and b that $(a, b) \in r^{\mathcal{I}}$ but $(a, b) \in s^{\mathcal{I}}$. Therefore, $(a, b) : r \in \mathcal{A}'$ but $(a, b) : s \notin \mathcal{A}'$, which contradicts $\mathcal{A}'$ is a complete ABox.

Completeness:

Let $\mathcal{A}$ be consistent, and consider a model $\mathcal{I}$ of $\mathcal{A}$. Since $\mathcal{A}$ is consistent, it cannot contain a clash.

If $\mathcal{A}$ is complete, since it does not contain a clash, expand simply return $\mathcal{A}$ and consistent returns "consistent". If $\mathcal{A}$ is not complete, then expand calls itself recursively until $\mathcal{A}$ is complete; each call selects a rule and applies it.

We omit the original proof, and add a new step to it:

The $\sqsubseteq$-rule: if $(a, b) : r \in \mathcal{A}$ and $r \sqsubseteq s \in \mathcal{T}$, then $(a, b) \in r^{\mathcal{I}}$. As $\mathcal{I}$ is a model of $\mathcal{T}$, $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$, thus $(a, b) \in s^{\mathcal{I}}$. Therefore, $\mathcal{I}$ is still a model of $\mathcal{A} \cup \{(a, b) : s\}$, so $\mathcal{A}$ is still consistent after the rule is applied.