第六次作业

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习题 6.1: (A) 5(1, 4)

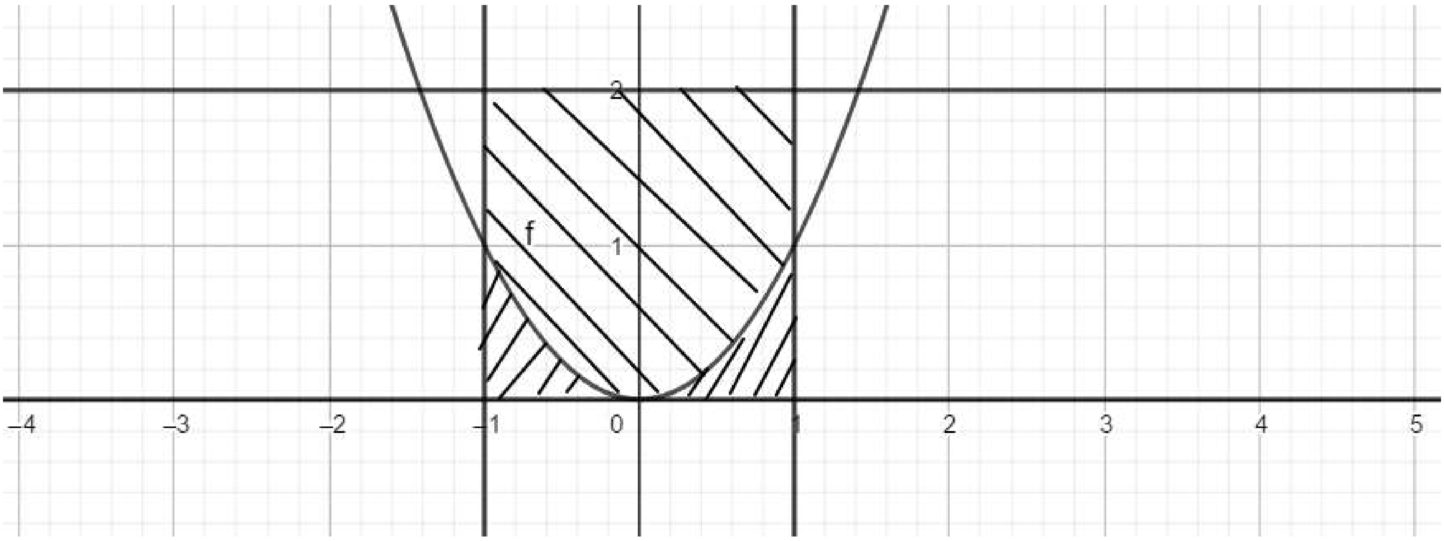
习题 6.2: (A) 3(2, 6, 7, 10), 5(1, 3), 6(4), 7(2), 8(2), 9(1, 2), 13(1, 3), 14(2) (B) 1(1, 3), 2, 3, 6, 13

6.2 (B)

1.

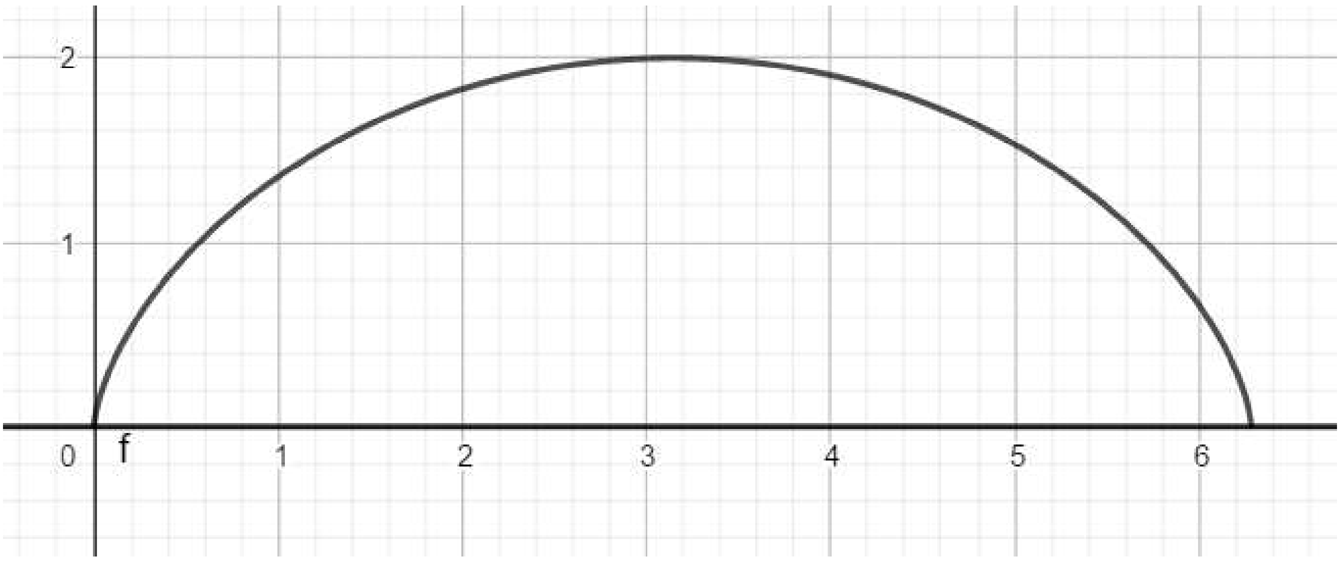
(1)

令 $y - x^2 \geq 0$ 可得 $y \geq x^2$



$$
\begin{aligned}
\iint_{(\sigma)} \sqrt{|y-x^2|}\mathrm{d}\sigma &= 2\int_0^1 \mathrm{d}x \int_0^{x^2} \sqrt{x^2-y}\mathrm{d}y + 2\int_0^1 \mathrm{d}x \int_{x^2}^{2} \sqrt{y-x^2}\mathrm{d}y \\
&= -2\int_0^1 \mathrm{d}x \int_0^{x^2} \sqrt{x^2-y}\mathrm{d}(x^2-y) + 2\int_0^1 \mathrm{d}x \int_{x^2}^{2} \sqrt{y-x^2}\mathrm{d}(y-x^2) \\
&= -2\int_0^1 \mathrm{d}x \int_{x^2}^{0} \sqrt{t}\mathrm{d}t + 2\int_0^1 \mathrm{d}x \int_0^{2-x^2} \sqrt{t}\mathrm{d}t \\
&= \frac{4}{3}\int_0^1 x^3\mathrm{d}x + \frac{4}{3}\int_0^1 (2-x^2)^{\frac{3}{2}}\mathrm{d}x \\
&= \frac{1}{3} + \frac{4}{3}x(2-x^2)^{\frac{3}{2}}|_0^1 - \frac{4}{3}\int_0^1 x\mathrm{d}(2-x^2)^{\frac{3}{2}} \\
&= \frac{5}{3} + 4\int_0^1 x^2(2-x^2)^{\frac{1}{2}}\mathrm{d}x \\
&= \frac{5}{3} + 2\int_0^1 (2x^2-x^4)^{\frac{1}{2}}\mathrm{d}x^2 \\
&= \frac{5}{3} + 2\int_0^1 (1-(t-1)^2)^{\frac{1}{2}}\mathrm{d}t \\
&= \frac{5}{3} + 2\int_{-1}^0 (1-u^2)^{\frac{1}{2}}\mathrm{d}u \\
&= \frac{5}{3} + \frac{\pi}{2}
\end{aligned}
$$

(3)



$$\iint_{(\sigma)} y^2 \mathrm{d}\sigma = \int_0^{a(2\pi-\sin 2\pi)} \mathrm{d}x \int_0^{a(1-\cos t)} y^2 \mathrm{d}y$$
$$= \int_0^{2\pi} \frac{1}{3}(a(1-\cos t))^3 \mathrm{d}a(t-\sin t)$$
$$= \frac{1}{3}a^4 \int_0^{2\pi} (1-\cos t)^4 \mathrm{d}t$$
$$= \frac{32}{3}a^4 \int_0^{\pi} \sin^8 u \mathrm{d}u$$

对于 $\int \sin^n x \mathrm{d}x$:

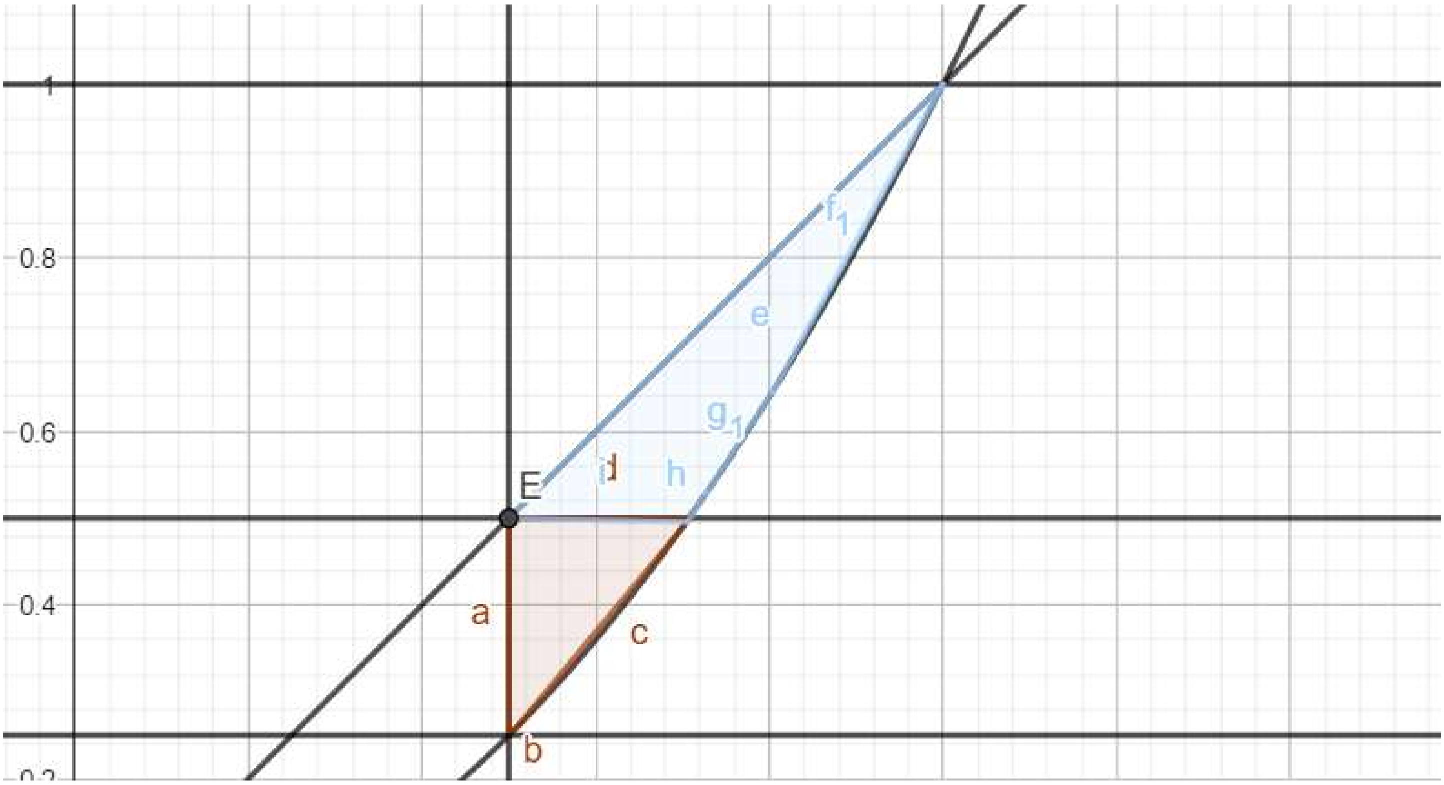
$$\begin{aligned}
\therefore I_n &= \int \sin^n x \mathrm{d}x \\
&= \int \sin^{n-1} x \mathrm{d}(-\cos x) \\
&= -\cos x \sin^{n-1} + \int \cos x \mathrm{d} \sin^{n-1} x \\
&= -\cos x \sin^{n-1} + \int \cos^2 x (n-1) \sin^{n-2} x \mathrm{d}x \\
&= -\cos x \sin^{n-1} + (n-1) \int (1-\sin^2 x) \sin^{n-2} x \mathrm{d}x \\
&= -\cos x \sin^{n-1} + (n-1)(I_{n-2} - I_n)
\end{aligned}$$

$$\therefore I_n = \frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}, \quad n \geq 2$$

$$\begin{aligned}
\therefore I_8 &= \frac{1}{8} \cos x \sin^7 x + \frac{7}{8}(\frac{1}{6} \cos x \sin^5 x + \frac{5}{6}(\frac{1}{4} \cos x \sin^3 x + \frac{3}{4}(\frac{1}{2} \cos x \sin x + \frac{1}{2}x))) + C \\
&= \frac{1}{8} \cos x \sin^7 x + \frac{7}{48} \cos x \sin^5 x + \frac{35}{192} \cos x \sin^3 x + \frac{105}{384} \cos x \sin x + \frac{105}{384} x + C
\end{aligned}$$

$$\therefore \iint_{(\sigma)} y^2 \mathrm{d}\sigma = \frac{32}{3}a^4 \int_0^{\pi} \sin^8 u \mathrm{d}u = \frac{32}{3}a^4 \cdot \frac{105}{384}\pi = \frac{35}{12}\pi a^4$$

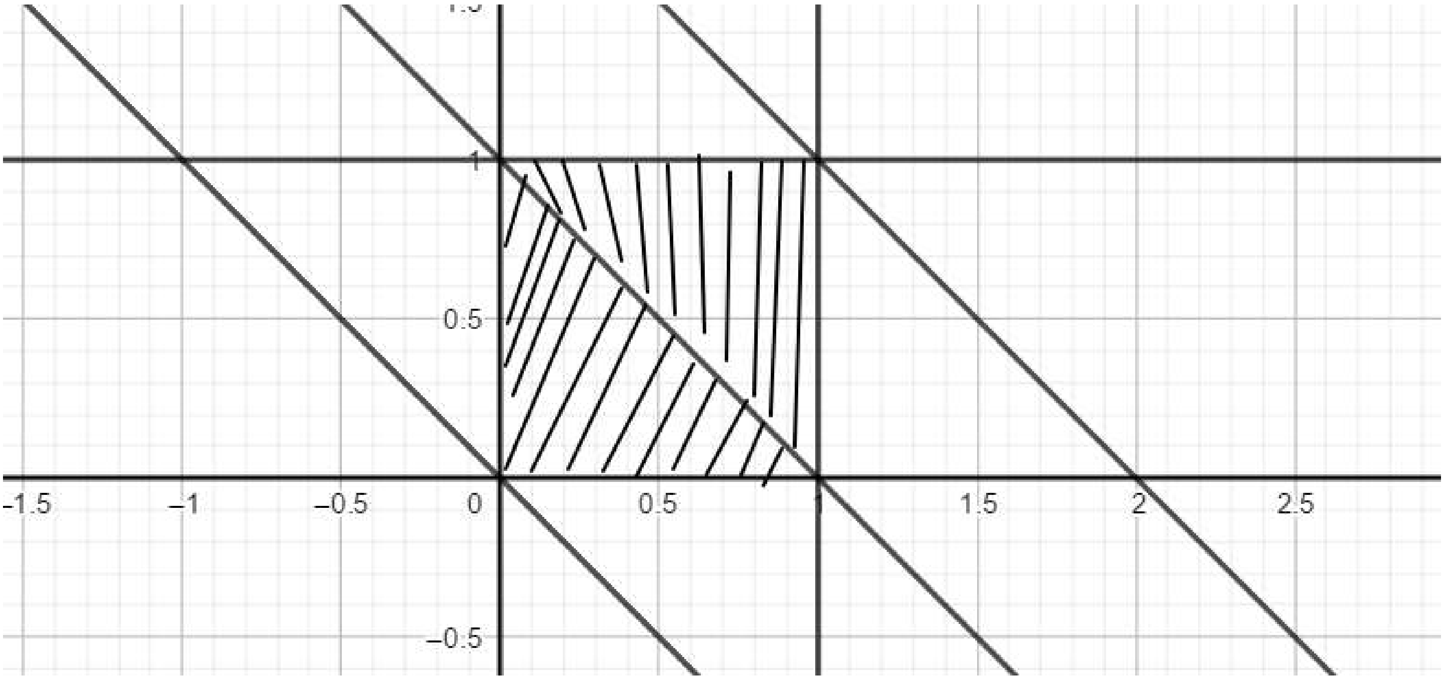
2.



联解 $x=\sqrt{y}$ 和 $y=\frac{1}{2}$ 得 $x=\frac{\sqrt{2}}{2}$

$$
\begin{aligned}
&\int_{\frac{1}{4}}^{\frac{1}{2}} \mathrm{d}y \int_{\frac{1}{2}}^{\sqrt{y}} e^{\frac{y}{x}} \mathrm{d}x+\int_{\frac{1}{2}}^{1} \mathrm{d}y \int_{y}^{\sqrt{y}} e^{\frac{y}{x}} \mathrm{d}x \\
=&\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}x \int_{x^2}^{\frac{1}{2}} e^{\frac{y}{x}} \mathrm{d}y+\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}x \int_{\frac{1}{2}}^{x} e^{\frac{y}{x}} \mathrm{d}y+\int_{\frac{\sqrt{2}}{2}}^{1} \mathrm{d}x \int_{x^2}^{x} e^{\frac{y}{x}} \mathrm{d}y \\
=&\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}}(ex-xe^{x})\mathrm{d}x+\int_{\frac{\sqrt{2}}{2}}^{1}(ex-xe^{x})\mathrm{d}x \\
=&\int_{\frac{1}{2}}^{1} ex\mathrm{d}x-\int_{\frac{1}{2}}^{1} xe^{x}\mathrm{d}x \\
=&\frac{3}{8}e-\frac{1}{2}e^{\frac{1}{2}}
\end{aligned}
$$

3.



当 $t \le 0$ 时,

易知 $f(x,y)=0$ 在 $x+y\le t$ 恒成立

$$\therefore F(t)=\iint_{x+y\le t} f(x,y)\mathrm{d}\sigma=0$$

当 $0<t\le 1$ 时,

$$\therefore F(t)=\iint_{x+y\le t} f(x,y)\mathrm{d}\sigma=\int_0^t \mathrm{d}x\int_0^{t-x}2x\mathrm{d}y=\frac{1}{3}t^3$$

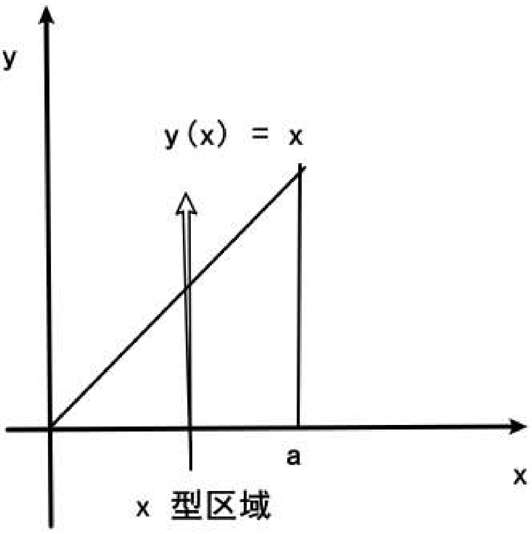
当 $1<t\le 2$ 时,

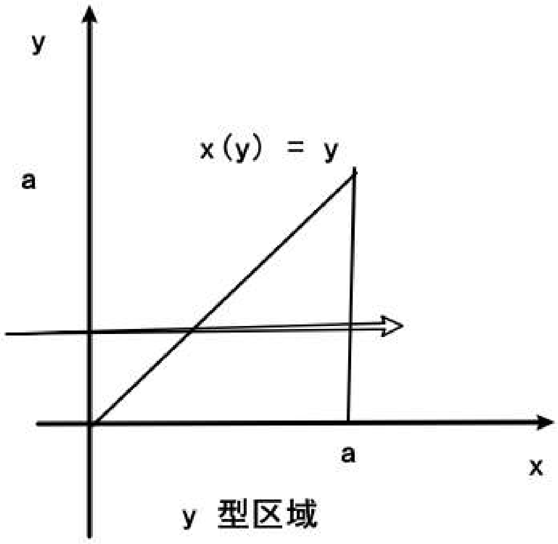
$$\begin{aligned}\therefore F(t)&=\iint_{x+y\le t} f(x,y)\mathrm{d}\sigma\\&=\frac{1}{3}+\int_0^{t-1}\mathrm{d}x\int_{1-x}^{1}2x\mathrm{d}y+\int_{t-1}^{1}\mathrm{d}x\int_{1-x}^{t-x}2x\mathrm{d}y\\&=\frac{1}{3}+\frac{2}{3}(t-1)^3+[(t-1)-(t-1)^3]\\&=-\frac{1}{3}t^3+t^2-\frac{1}{3}\end{aligned}$$

当 $t>2$ 时,

$$\therefore F(t)=F(2)=1$$

6.





如图所示, $\int_0^a \mathrm{d}x \int_0^x f(x,y)\mathrm{d}y$ 对应的 x 型区域为 $y=x, x=0, x=1, y=0$ 围成的区域.

而 $\int_0^a \mathrm{d}x \int_y^a f(x,y)\mathrm{d}y$ 对应的 y 型区域为 $y=x, x=0, x=1, y=0$ 围成的区域.

两者所对应的区域一模一样, 并且我们知道 f 在该区域内连续.

所以我们有 $\int_0^a \mathrm{d}x \int_0^x f(x,y)\mathrm{d}y = \int_0^a \mathrm{d}x \int_y^a f(x,y)\mathrm{d}y.$

同理有 $\int_0^a \mathrm{d}x \int_x^a f(x,y)\mathrm{d}y = \int_0^a \mathrm{d}x \int_0^y f(x,y)\mathrm{d}y.$

$\therefore \int_0^a \mathrm{d}y \int_0^y f(x)\mathrm{d}x = \int_0^a \mathrm{d}x \int_x^a f(x)\mathrm{d}y = \int_0^a \mathrm{d}x[yf(x)]|_x^a = \int_0^a (a-x)f(x)\mathrm{d}x$

13.

$$
\begin{aligned}
\therefore f(t) &= e^{4\pi t^2} + \iint_{x^2+y^2\le 4t^2} f(\frac{1}{2}\sqrt{x^2+y^2})\mathrm{d}\sigma \\
&= e^{4\pi t^2} + 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2t} f(\frac{1}{2}\rho)\rho\mathrm{d}\rho \\
&= e^{4\pi t^2} + 2\pi \int_0^{2t} f(\frac{1}{2}\rho)\rho\mathrm{d}\rho
\end{aligned}
$$

$\therefore f'(t) - 8\pi t f(t) = 8\pi t e^{4\pi t^2}$

将其变为齐次线性微分方程 $f'(t) - 8\pi t f(t) = 0$

对这个方程解得 $y = C_1 e^{\int 8\pi t \mathrm{d}t} = C_1 e^{4\pi t^2}$

对于原方程的解, 解得

$$\therefore f(t)=e^{4\pi t^2}\left(\int(8\pi t e^{4\pi t^2})e^{-4\pi t^2}\mathrm{d}t+C\right)=e^{4\pi t^2}(4\pi t^2+C)$$

当 $t=0$ 时, $f(t)=1$, 带入可得 $f(0)=C=1$

解得 $f(t)=(4\pi t^2+1)e^{4\pi t^2}$