

习题2.1: (A) 1 (2、3) , 2, 6, 7, 8, 18, (B) 1, 3, 4, 5

习题2.2: (A) 1 (4、9) , 3 (4、19、13) , 6 (1、3、4、6) , 8

2.1(A)

1.

(2)

$$\begin{aligned}f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{\ln(x + \Delta x) - \ln x}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{\ln(1 + \frac{\Delta x}{x})}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{\Delta x}{x \cdot \Delta x} \\&= \frac{1}{x}\end{aligned}$$

(3)

$$\begin{aligned}f'(0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \frac{(0 + \Delta x)^2 \sin \frac{1}{0 + \Delta x} - 0}{\Delta x} \\&= \lim_{\Delta x \rightarrow 0} \Delta x \sin \frac{1}{\Delta x} \\&= 0\end{aligned}$$

2.

(1)

$$\begin{aligned}
& \lim_{\Delta x \rightarrow 0} \frac{f(x_0 - \Delta x) - f(x_0)}{\Delta x} \\
&= - \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + (-\Delta x)) - f(x_0)}{-\Delta x} \\
&= -f'(x_0)
\end{aligned}$$

(2)

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{h} \\
& \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0) + f(x_0) - f(x_0 - h)}{h} \\
&= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} + \lim_{h \rightarrow 0} \frac{f(x_0) - f(x_0 - h)}{-h} \\
&= 2f'(x_0)
\end{aligned}$$

(3)

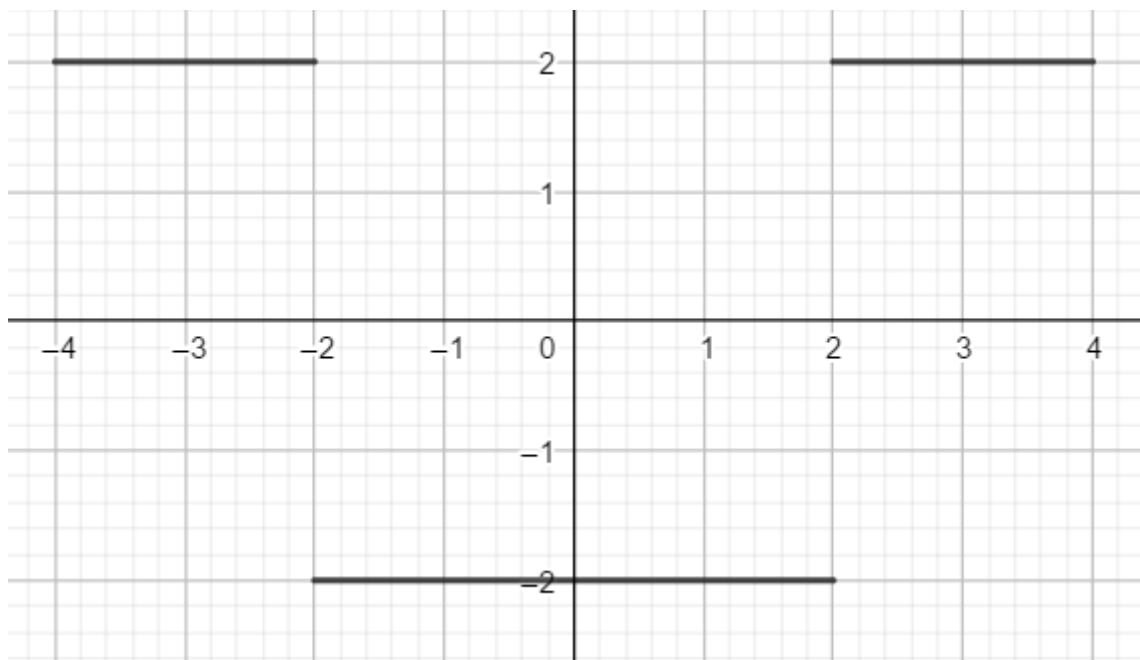
$$\begin{aligned}
& \lim_{n \rightarrow +\infty} n[f(x_0 + \frac{1}{n}) - f(x_0)] \\
&= \lim_{n \rightarrow +\infty} \frac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}} \\
&= f'(x_0)
\end{aligned}$$

(4)

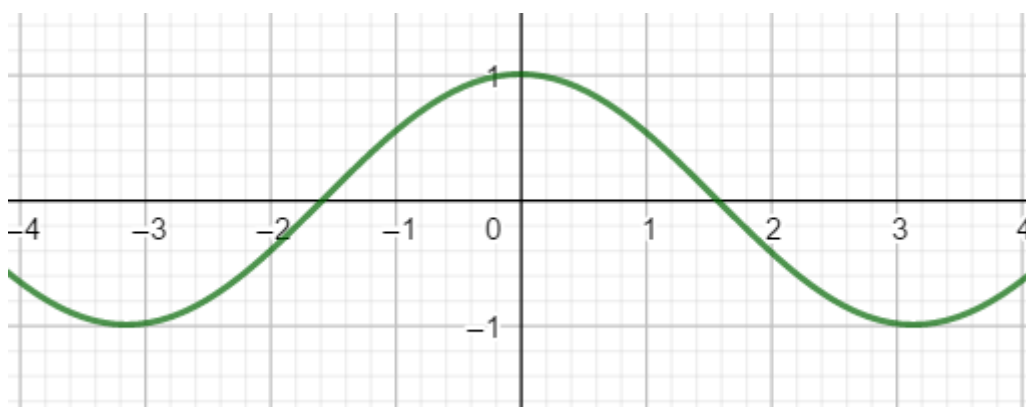
$$\begin{aligned}
& \lim_{x \rightarrow x_0} \frac{x_0 f(x) - x f(x_0)}{x - x_0} \\
&= \lim_{x \rightarrow x_0} \frac{x_0 f(x_0 + (x - x_0)) - (x_0 + x - x_0) f(x_0)}{x - x_0} \\
&= \lim_{x \rightarrow x_0} \frac{x_0 f(x_0 + (x - x_0)) - x_0 f(x_0)}{x - x_0} - \lim_{x \rightarrow x_0} \frac{(x - x_0) f(x_0)}{x - x_0} \\
&= x_0 f'(x_0) - f(x_0)
\end{aligned}$$

6.

(a)



(b)



7.

$\because f$ 是偶函数, $f'(0)$ 存在

$$\therefore f(x) = f(-x)$$

$$\begin{aligned} \therefore f'(0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} \\ &= - \lim_{\Delta x \rightarrow 0} \frac{f(0 - \Delta x) - f(0)}{-\Delta x} \\ &= -f'(0) \end{aligned}$$

$$\therefore 2f'(0) = 0$$

$$\therefore f'(0) = 0$$

8.

(1)

$\because \varphi$ 在 $x = a$ 处连续

$$\therefore \lim_{x \rightarrow a} \varphi(x) = \varphi(a)$$

$$\begin{aligned}\therefore f'_-(a) &= \lim_{\Delta x \rightarrow 0^-} \frac{(x - a + \Delta x)\varphi(x + \Delta x) - (x - a)\varphi(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{(a - a + \Delta x)\varphi(a + \Delta x) - (a - a)\varphi(a)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0^-} \frac{\Delta x \cdot \varphi(a + \Delta x)}{\Delta x} \\ &= \varphi(a)\end{aligned}$$

$$\therefore \text{同理 } f'_+(a) = \varphi(a)$$

$$\therefore f'_-(a) = f'_+(a)$$

$\therefore f$ 在 $x = a$ 处可导

(2)

$$\begin{aligned}\therefore g'_-(a) &= \lim_{\Delta x \rightarrow 0} \frac{|x - a + \Delta x|\varphi(x + \Delta x) - |x - a|\varphi(x)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{|a - a + \Delta x|\varphi(a + \Delta x) - |a - a|\varphi(a)}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{|\Delta x| \cdot \varphi(a + \Delta x)}{\Delta x} \\ &= -\varphi(a)\end{aligned}$$

$$\therefore \text{同理 } g'_+(a) = \varphi(a)$$

当 $\varphi(a) = 0$ 时, $g'_-(a) = g'_+(a)$

$\therefore g$ 在 $x = a$ 处可导

当 $\varphi(a) \neq 0$ 时, $g'_-(a) \neq g'_+(a)$

$\therefore g$ 在 $x = a$ 处不可导

18.

$$\because f(0) = 0, f'(0) = 2$$

$$\begin{aligned}\therefore \lim_{x \rightarrow 0} \frac{f(x)}{\sin 2x} &= \lim_{x \rightarrow 0} \frac{f(0+x) - f(0)}{x \cdot \frac{2 \sin 2x}{2x}} \\ &= \frac{f'(0)}{2} \\ &= 1\end{aligned}$$

2.1(B)

1.

$\because f$ 在 $x = 0$ 处连续

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

$\because \lim_{x \rightarrow 0} \frac{f(x)}{x}$ 存在

$$\therefore \lim_{x \rightarrow 0} f(x) = \frac{f(x)}{x} \cdot x = f(0)$$

$$\therefore f(0) = 0$$

$$\therefore f'(0) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \frac{f(\Delta x)}{\Delta x}$$

$\therefore f'(0)$ 存在

$\therefore f$ 在 $x = 0$ 处可导

3.

$$\therefore f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \text{ 存在, 且 } f(a) \neq 0$$

$$\text{令 } f'(a) = \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} = A$$

$$\therefore \lim_{\Delta x \rightarrow 0} \frac{f(a + \Delta x) - f(a)}{A \cdot \Delta x} = 1$$

$$\text{令 } \Delta x = \frac{1}{n}, \text{ 则}$$

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} \left[\frac{f(a + \frac{1}{n})}{f(a)} \right]^n &= \lim_{\Delta x \rightarrow 0} \left[\frac{f(a + \Delta x)}{f(a)} \right]^{\frac{1}{\Delta x}} \\ &= \lim_{\Delta x \rightarrow 0} \left[1 + \frac{f(a + \Delta x) - f(a)}{f(a)} \right]^{\frac{1}{\Delta x}} \\ &= \lim_{\Delta x \rightarrow 0} \left[1 + \frac{A \cdot \Delta x}{f(a)} \right]^{\frac{f(a)}{A \cdot \Delta x} \cdot \frac{A}{f(a)}} \\ &= e^{\frac{f'(a)}{f(a)}} \end{aligned}$$

4.

$$\therefore f'(0) = (\sin x)'|_{x=0} = \cos 0 = 1, f(0) = 0$$

$$\therefore f'(0) = \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{f(\Delta x)}{\Delta x} = 1$$

$$\text{令 } \Delta x = \frac{2}{n}, \text{ 则}$$

$$\therefore \lim_{n \rightarrow \infty} n^{\frac{1}{2}} \sqrt{f\left(\frac{2}{n}\right)} = \lim_{\Delta x \rightarrow 0} \sqrt{2} \cdot \sqrt{\frac{f(\Delta x)}{\Delta x}} = \sqrt{2}$$

5.

(1)

$$\therefore \lim_{x \rightarrow 0} f(x) = x^n \sin \frac{1}{x} = 0$$

$\therefore f(x)$ 在 $x = 0$ 处连续

(2)

当 $n \geq 2$ 时,

$$\begin{aligned}
 \because f'(0) &= \lim_{\Delta x \rightarrow 0} \frac{f(0 + \Delta x) - f(0)}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \frac{(0 + \Delta x)^n \sin \frac{1}{0 + \Delta x} - 0}{\Delta x} \\
 &= \lim_{\Delta x \rightarrow 0} \Delta x^{n-1} \sin \frac{1}{\Delta x} \\
 &= 0
 \end{aligned}$$

$\therefore f(x)$ 在 $x = 0$ 处可导

当 $n = 1$ 时,

$$\because f'(0) = \lim_{\Delta x \rightarrow 0} \sin \frac{1}{\Delta x} \text{ 不存在}$$

$\therefore f(x)$ 在 $x = 0$ 处不可导

(3)

当 $n = 1$ 时, $f'(x)$ 在 $x = 0$ 处不存在, 则也不连续

当 $n = 2$ 时,

$$\therefore f'(x) = nx^{n-1} \sin \frac{1}{x} + x^n \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

$$\therefore \lim_{x \rightarrow 0} = \lim_{x \rightarrow 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right)$$

此时 $\cos \frac{1}{x}$ 不定, 极限不存在

$\therefore f'(x)$ 在 $x = 0$ 处不连续

当 $n \geq 3$ 时,

$$\therefore \lim_{x \rightarrow 0} = \lim_{x \rightarrow 0} \left(nx^{n-1} \sin \frac{1}{x} - x^{n-2} \cos \frac{1}{x}\right) = 0$$

$\therefore f'(x)$ 在 $x = 0$ 处连续

2.2(A)

1.

(4)

$$\begin{aligned}y' &= \left(\frac{\sin x}{\cos x}\right)' \sec x + \tan x \left(\frac{1}{\cos x}\right)' \\&= \left(\frac{1}{\cos^2 x}\right) \sec x + \tan x \left(\frac{\sin x}{\cos^2 x}\right) \\&= \frac{\sin^2 x + 1}{\cos^3 x}\end{aligned}$$

(9)

$$\begin{aligned}y' &= \frac{1 + (\sqrt{x + \sqrt{x}})'}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \\&= \frac{1 + \frac{1 + (\sqrt{x})'}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \\&= \frac{1 + \frac{1 + \frac{1}{2\sqrt{x}}}{2\sqrt{x + \sqrt{x}}}}{2\sqrt{x + \sqrt{x + \sqrt{x}}}}\end{aligned}$$

3.

(4)

$$\because y = \sqrt[3]{\frac{1+x}{1-x}}$$

$$\therefore \ln y = \frac{1}{3} \ln(1+x) - \frac{1}{3} \ln(1-x)$$

$$\therefore \frac{y'}{y} = \frac{1}{3+3x} + \frac{1}{3-3x}$$

$$\therefore y' = \left(\frac{1}{3+3x} + \frac{1}{3-3x}\right) \sqrt[3]{\frac{1+x}{1-x}}$$

(13)

$$\begin{aligned}
 y' &= a^a x^{a^a-1} + a^{x^a} (ax^{a-1}) \ln a + a^{a^x} (a^x \ln a) \ln a \\
 &= a^a x^{a^a-1} + a^{x^a+1} x^{a-1} \ln a + a^{a^x+x} \ln^2 a
 \end{aligned}$$

(19)

$$\therefore y = \sqrt[3]{\frac{1 - \sin 2x}{1 + \sin 2x}}$$

$$\therefore \ln y = \frac{1}{3} \ln(1 - \sin 2x) - \frac{1}{3} \ln(1 + \sin 2x)$$

$$\therefore \frac{y'}{y} = -\frac{2 \cos 2x}{3 - 3 \sin 2x} - \frac{2 \cos 2x}{3 + 3 \sin 2x} = -\frac{4}{3 \cos 2x}$$

$$\therefore y' = \frac{4}{3 \cos 2x} \sqrt[3]{\frac{1 - \sin 2x}{1 + \sin 2x}}$$

6.

(1)

$$y' = 2x f'(x^2)$$

(3)

$$y' = \sin 2x [f'(\sin^2 x) - f'(\cos^2 x)]$$

(4)

$$y' = e^x f'(e^x) e^{g(x)} + f(e^x) g'(x) e^{g(x)}$$

(6)

当 $x = 0$ 时,

$$y'_- = \lim_{\Delta x \rightarrow 0^-} \frac{\frac{\Delta x}{1+e^{\frac{1}{\Delta x}}} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{1}{1+e^{\frac{1}{\Delta x}}} = 1$$

$$y'_+ = \lim_{\Delta x \rightarrow 0^+} \frac{\frac{\Delta x}{1+e^{\frac{1}{\Delta x}}} - 0}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \frac{1}{1+e^{\frac{1}{\Delta x}}} = 0$$

左导和右导极限不同, 此时导数不存在

当 $x \neq 0$ 时,

$$\begin{aligned} y' &= \frac{1 + e^{\frac{1}{x}} - x(1 + e^{\frac{1}{x}})'}{(1 + e^{\frac{1}{x}})^2} \\ &= \frac{1 + e^{\frac{1}{x}} + \frac{1}{x}e^{\frac{1}{x}}}{(1 + e^{\frac{1}{x}})^2} \end{aligned}$$

8.

设该点为 (x_0, y_0) , 设切线斜率为 k

题目意思可转化为 $y_0 = -kx_0$

$$\text{即证 } k = -\frac{y_0}{x_0}$$

$$\because xy = a$$

$$\therefore y = \frac{a}{x}$$

$$\text{由导数几何意义可知 } k = y' = -\frac{a}{x_0^2} = -\frac{y_0}{x_0}$$

\therefore 可知命题成立