高等代数作业

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12.

假设这个与 V 上全体线性变换可以交换的线性变换为 σ 对应的矩阵为 A, 而 V 上任意一个线性变换 τ 对应的矩阵为 B.

$\because \sigma$ 与 τ 可交换

$\therefore (\sigma\tau)(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)AB=(\tau\sigma)(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)BA$

$\because \varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 为一组基, 线性无关

$\therefore AB=BA$, 其中 B 为任意一个 $n\times n$ 的矩阵

首先假设 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, B=\begin{pmatrix} b_{11} & 0 & \cdots & 0 \\ 0 & b_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$, 其中 $b_{11}\neq b_{22}\neq\cdots\neq b_{nn}$

$$\therefore AB=\begin{pmatrix} b_{11}a_{11} & b_{22}a_{12} & \cdots & b_{nn}a_{1n} \\ b_{11}a_{21} & b_{22}a_{22} & \cdots & b_{nn}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{11}a_{n1} & b_{22}a_{n2} & \cdots & b_{nn}a_{nn} \end{pmatrix}=BA=\begin{pmatrix} b_{11}a_{11} & b_{11}a_{12} & \cdots & b_{11}a_{1n} \\ b_{22}a_{21} & b_{22}a_{22} & \cdots & b_{22}a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{nn}a_{n1} & b_{nn}a_{n2} & \cdots & b_{nn}a_{nn} \end{pmatrix}$$

$$\therefore AB-BA=\begin{pmatrix} (b_{11}-b_{11})a_{11} & (b_{22}-b_{11})a_{12} & \cdots & (b_{nn}-b_{11})a_{1n} \\ (b_{11}-b_{22})a_{21} & (b_{22}-b_{22})a_{22} & \cdots & (b_{nn}-b_{22})a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ (b_{11}-b_{nn})a_{n1} & (b_{22}-b_{nn})a_{n2} & \cdots & (b_{nn}-b_{nn})a_{nn} \end{pmatrix}=O$$

$\because b_{11}\neq b_{22}\neq\cdots\neq b_{nn}$

$\therefore a_{ij}=0, i\neq j$

再假设 $A=\begin{pmatrix}a_{11}&0&\cdots&0\\0&a_{22}&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&a_{nn}\end{pmatrix},B=\begin{pmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\b_{n1}&b_{n2}&\cdots&b_{nn}\end{pmatrix}$, 其中 $b_{ij}\neq 0$

$\therefore BA-AB=\begin{pmatrix}(a_{11}-a_{11})b_{11}&(a_{22}-a_{11})b_{12}&\cdots&(a_{nn}-a_{11})b_{1n}\\(a_{11}-a_{22})b_{21}&(a_{22}-a_{22})b_{22}&\cdots&(a_{nn}-a_{22})b_{2n}\\\vdots&\vdots&\ddots&\vdots\\(a_{11}-a_{nn})b_{n1}&(a_{22}-a_{nn})b_{n2}&\cdots&(a_{nn}-a_{nn})b_{nn}\end{pmatrix}=O$

$\therefore a_{ii}=a_{jj},i\neq j$

$\therefore A$ 是数量矩阵

$\therefore \sigma$ 是数乘变换

$\therefore V$ 的与全体线性变换可以交换的线性变换是数乘变换

14.(1)

$\because \mathcal{A}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A,$

$\mathcal{A}(\eta_1,\eta_2,\cdots,\eta_n)=(\eta_1,\eta_2,\cdots,\eta_n)B$

$(\eta_1,\eta_2,\cdots,\eta_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)X$

$\therefore \mathcal{A}(\eta_1,\eta_2,\cdots,\eta_n)=\mathcal{A}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)X=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)AX=(\eta_1,\eta_2,\cdots,\eta_n)X^{-1}AX$

$\therefore B=X^{-1}AX=\begin{pmatrix}1&0&0&0\\-2&3&0&0\\0&-1&1&0\\1&-1&1&2\end{pmatrix}^{-1}\begin{pmatrix}1&0&2&1\\-1&2&1&3\\1&2&5&5\\2&-2&1&-2\end{pmatrix}\begin{pmatrix}1&0&0&0\\-2&3&0&0\\0&-1&1&0\\1&-1&1&2\end{pmatrix}=\frac{1}{3}\begin{pmatrix}6&-9&9&6\\2&-4&10&10\\8&-16&40&40\\0&3&-21&-24\end{pmatrix}$

15.(2)

$\therefore \mathcal{A}(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A=(\eta_1,\eta_2,\cdots,\eta_n)$

即求出 $(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)$ 到 $(\eta_1,\eta_2,\cdots,\eta_n)$ 的过渡矩阵

$$\because (\eta_1, \eta_2, \cdots, \eta_n) = \left((\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}^{-1} \right) \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & -1 \\ -1 & -1 & -1 \end{pmatrix} =$$

$$(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) \begin{pmatrix} -2 & -\frac{3}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$

$$\therefore A = \begin{pmatrix} -2 & -\frac{3}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{1}{2} & -\frac{5}{2} \end{pmatrix}$$