概率统计第四次作业

201300035 方盛俊

3.1

证明 $E(X) = np$:

$\because X \sim B(n, p)$

$\therefore P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\therefore E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k} = (1-p)^n \sum_{k=1}^{\infty} k \binom{n}{k} \left(\frac{p}{1-p} \right)^k$

令 $x = \frac{p}{1-p}$, 则

$\because \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$

两边求导得

$\therefore \sum_{k=1}^{\infty} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$

$\therefore \sum_{k=1}^{\infty} k \binom{n}{k} x^k = nx(1+x)^{n-1}$

$\therefore E(X) = (1-p)^n \cdot n \cdot \frac{p}{1-p} \cdot \left(1 + \frac{p}{1-p}\right)^{n-1} = np$

证明 $\mathrm{Var}(X) = np(1-p)$:

$$
\begin{aligned}
\therefore \mathrm{E}(X^2) &= \sum_{k=0}^{\infty} k^2 \cdot P(X=k) \\
&= \sum_{k=0}^{\infty} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{\infty} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= (1-p)^n \sum_{k=1}^{\infty} k(k-1) \binom{n}{k} \left(\frac{p}{1-p} \right)^k + (1-p)^n \sum_{k=1}^{\infty} k \binom{n}{k} \left(\frac{p}{1-p} \right)^k
\end{aligned}
$$

$\because \sum_{k=1}^{\infty} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$

$\therefore \sum_{k=2}^{\infty} k(k-1) \binom{n}{k} x^{k-2} = n(n-1)(1+x)^{n-2}$

$\therefore \sum_{k=2}^{\infty} k(k-1) \binom{n}{k} x^k = n(n-1)x^2(1+x)^{n-2}$

$\therefore E(X^2) = (1-p)^n \cdot n(n-1)\left(\frac{p}{1-p}\right)^2\left(1+\frac{p}{1-p}\right)^{n-2} + np = n(n-1)p^2 + np = np(1-p) + n^2p^2$

$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = np(1-p)$

3.2

证明 $E(X) = \frac{1}{p}$**:**

$\because P(X=k) = (1-p)^{k-1}p$

$\therefore E(X) = \sum_{k=1}^{\infty} k \cdot P(X=k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p\sum_{k=1}^{\infty} k(1-p)^{k-1}$

$\because \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$

$\therefore \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$

$\therefore \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}$

$\therefore E(X) = p \cdot \frac{1}{p^2} = \frac{1}{p}$

证明 $\mathrm{Var}(X) = \frac{1-p}{p^2}$**:**

$\therefore E(X^2) = \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p = p\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-1} + p\sum_{k=1}^{\infty} k(1-p)^{k-1}$

$\because \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$

$\therefore \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{2}{(1-x)^3}$

$\therefore \sum_{k=2}^{\infty} k(k-1)x^{k-1} = \frac{2x}{(1-x)^3}$

$\therefore E(X^2) = p \cdot \frac{2(1-p)}{p^3} + \frac{1}{p} = \frac{1}{p} + \frac{2(1-p)}{p^2} = \frac{2-p}{p^2}$

$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$

3.3

证明 $E(X) = \frac{r}{p}$**:**

$\because P(X=k) = \binom{k-1}{r-1}p^r(1-p)^{k-r}$

$\therefore E(X) = \sum_{k=r}^{\infty} k \cdot \binom{k-1}{r-1}p^r(1-p)^{k-r} = rp^r\sum_{k=r}^{\infty}\binom{k}{r}(1-p)^{k-r}$

$\because \sum_{k=r}^{\infty} \binom{k}{r} (1-p)^{k-r} = \sum_{k=r}^{\infty} \binom{k+1-1}{r+1-1} (1-p)^{k-r} = p^{-(r+1)}$

$\therefore E(X) = rp^r \cdot p^{-(r+1)} = \frac{r}{p}$

证明 $\mathrm{Var}(X) = \frac{r(1-p)}{p^2}$**:**

$\therefore E(X^2) = \sum_{k=r}^{\infty} k^2 \cdot \binom{k-1}{r-1} p^r (1-p)^{k-r} = r(r+1)p^r \sum_{k=r}^{\infty} \binom{k+1}{r+1} (1-p)^{k-r} - rp^r \sum_{k=r}^{\infty} \binom{k}{r} (1-p)^{k-r}$

$\because \sum_{k=r}^{\infty} \binom{k+1}{r+1} (1-p)^{k-r} = \sum_{k=r}^{\infty} \binom{k+2-1}{r+2-1} (1-p)^{k-r} = p^{-(r+2)}$

$\therefore E(X^2) = r(r+1)p^r \cdot p^{-(r+2)} - \frac{r}{p}$

$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = r(r+1)p^r \cdot p^{-(r+2)} - \frac{r}{p} - \frac{r^2}{p^2} = \frac{r\,(1-p)}{p^2}$

3.4

证明 $E(X) = \lambda$**:**

$\because P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}$

$\therefore E(X) = \sum_{k=0}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$

证明 $\mathrm{Var}(X) = \lambda$**:**

$\therefore E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=2}^{\infty} k(k-1) \cdot \frac{\lambda^k}{k!} e^{-\lambda} + \sum_{k=1}^{\infty} k \cdot \frac{\lambda^k}{k!} e^{-\lambda} = \lambda^2 e^{-\lambda} \cdot e^{\lambda} + \lambda = \lambda^2 + \lambda$

$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = \lambda$

3.5

设 Y 表示一个的叶节点的高度.

设 X_i 表示第 i 轮该叶节点是否被选中, 选中时 $X_i = 1$, 未选中时 $X_i = 0$.

因为在第 i 轮一共有 i 个节点, 因此该节点被选中的概率 $P(X_i = 1) = \frac{1}{i}$, 则 $E(X_i) = 1 \cdot P(X_i = 1) = \frac{1}{i}$

$\therefore E(Y) = E\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} E(X_i) = \sum_{i=1}^{k} \frac{1}{i} \approx \ln k$

3.6

有放回:

$\because P(X=k) = \frac{\sum_{i=1}^{5} \binom{5}{i} (k-1)^{5-i}}{10^5}$

$\therefore P(X=1) = 0.00001, P(X=2) = 0.00031, P(X=3) = 0.00211, P(X=4) = 0.00781, P(X=5) = 0.02101, P(X=6) = 0.04651, P(X=7) = 0.09031, P(X=8) = 0.15961, P(X=9) = 0.26281, P(X =$

$10) = 0.40951$

X	1	2	3	4	5	6	7	8	9	10
P	0.00001	0.00031	0.00211	0.00781	0.02101	0.04651	0.09031	0.15961	0.26281	0.40951

无放回:

$$\because P(X=k)=\frac{\binom{k-1}{4}}{\binom{10}{5}}$$

X	5	6	7	8	9	10
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P	$\frac{1}{252}$	$\frac{5}{252}$	$\frac{5}{84}$	$\frac{5}{36}$	$\frac{5}{18}$	$\frac{1}{2}$

3.7

令 $X \sim B(n, 0.99)$, 则 X 服从参数为 n 和 0.99 的二项分布.

$$\therefore P(X \geqslant k)=\sum_{k=k}^{n}\binom{n}{k}\times 0.99^{k}\times 0.01^{n-k}$$

令 $n=102$, 则

$$\therefore P(X \geqslant 100)=\sum_{k=100}^{102}\binom{102}{k}\times 0.99^{k}\times 0.01^{102-k}=0.916911014889440$$

令 $n=103$, 则

$$\therefore P(X=100)=\sum_{k=100}^{103}\binom{103}{k}\times 0.99^{k}\times 0.01^{103-k}=0.979758767886053$$

所以 $x=103-100=3$. 即 x 最小值是 3.

3.8

2. (1)

$$\because P(X=k)=\frac{\binom{k-1}{2}}{\binom{5}{3}}$$

$$\therefore P(X=3)=0.1, P(X=4)=0.3, P(X=5)=0.6$$

X	3	4	5
P	0.1	0.3	0.6

2. (2)

$$\because P(X=k)=\frac{\binom{6-k}{1}}{\binom{6}{2}}$$

$$\therefore P(X=1)=\frac{1}{3}, P(X=2)=\frac{4}{15}, P(X=3)=\frac{1}{5}, P(X=4)=\frac{2}{15}, P(X=5)=\frac{1}{15}$$

X	1	2	3	4	5
P	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

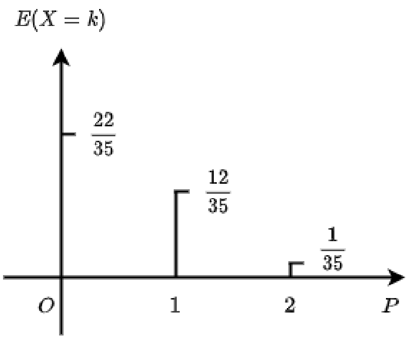
3. (1)

$\because P(X=k)=\frac{\binom{2}{k}\binom{13}{3-k}}{\binom{15}{3}}$

$\therefore P(X=0)=\frac{22}{35}, P(X=1)=\frac{12}{35}, P(X=2)=\frac{1}{35}$

X	0	1	2
P	$\frac{22}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

3. (2)



3.9

2.

设 Y 为每次抽取 10 件产品进行检验时发现的次品数. 则 $Y \sim B(10, 0.1)$

$\therefore P(Y=0)=\binom{10}{0}\times 0.1^0\times 0.9^{10}=0.9^{10}$

$\therefore P(Y=1)=\binom{10}{1}\times 0.1^1\times 0.9^{9}=0.9^{9}$

$\therefore P(Y>1)=1-0.9^{10}-0.9^9=1-1.9\times 0.9^9$

即发现超过 1 件次品的概率为 $p=1-1.9\times 0.9^9$

则我们有 $X\sim(4,p)$, 即 $X\sim(4,1-1.9\times 0.9^9)$

$\therefore E(X)=4p=4-7.6\times 0.9^9$

3.

对于 $X=1$ 的所有情况:

$(3,0,0,0),(2,1,0,0),(2,0,1,0),(2,0,0,1),(1,2,0,0),(1,0,2,0),(1,0,0,2),(1,1,1,0),(1,0,1,1),(1,1,0,1)$

共 10 种.

对于 $X=2$ 的所有情况:

$(0,3,0,0),(0,2,1,0),(0,2,0,1),(0,1,2,0),(0,1,0,2),(0,1,1,1)$

共 6 种.

对于 $X=3$ 的所有情况:

$(0,0,3,0),(0,0,2,1),(0,0,1,2)$

共 3 种.

对于 $X=4$ 的所有情况:

$(0,0,0,3)$

共 1 种.

$\therefore P(X=1)=\dfrac{10}{10+6+3+1}=0.5, P(X=2)=\dfrac{6}{20}=0.3, P(X=3)=\dfrac{3}{20}=0.15, P(X=3)=\dfrac{1}{20}=0.05$

X	**1**	**2**	**3**	**4**
P	0.5	0.3	0.15	0.05

$\therefore E(X)=1\times 0.5+2\times 0.3+3\times 0.15+4\times 0.05=1.75$

3.10

4. (1)

因为 $\sum_{j=1}^{\infty}\left|(-1)^{j+1}\dfrac{3^j}{j}\cdot\dfrac{2}{3^j}\right|=\sum_{j=1}^{\infty}\dfrac{2}{j}$ 调和级数发散, 即 $E(X)$ 并不绝对收敛.

所以 X 的数学期望并不存在.

4. (2)

$$\because P(X=k)=\left(\prod_{i=1}^{k-1}\frac{i}{i+1}\right)\cdot\frac{1}{k+1}$$

$\therefore E(X)=\sum_{k=1}^{\infty}k\cdot\left(\prod_{i=1}^{k-1}\dfrac{i}{i+1}\right)\cdot\dfrac{1}{k+1}=\sum_{k=1}^{\infty}\dfrac{1}{k+1}$ 调和级数发散.

6. (1)

$E(X)=-2\times 0.4+0\times 0.3+2\times 0.3=-0.2$

$E(X^2)=4\times 0.4+0\times 0.3+4\times 0.3=2.8$

$E(3X^2+5)=3E(X^2)+5=3\times 2.8+5=13.4$

6. (2)

$\because X\sim\pi(\lambda)$

$$\therefore P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$$

$$\therefore E\left(\frac{1}{X+1}\right)=\sum_{k=0}^{\infty}\frac{1}{k+1}\cdot\frac{\lambda^k}{k!}e^{-\lambda}=\frac{e^{-\lambda}}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!}=\frac{\left(e^{\lambda}-1\right)e^{-\lambda}}{\lambda}$$