概率统计第十一次作业

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6.9

令 $\bar{X} = \sum_{i=1}^{k} X_i^2$

则根据 Chernoff 方法有

$P[\bar{X} \geqslant (1+\epsilon)k] = P[e^{t\bar{X}} \geqslant e^{t(1+\epsilon)k}] \leqslant e^{-t(1+\epsilon)k} E[e^{t\bar{X}}]$

而

$$
\begin{aligned}
E[e^{t\bar{X}}] = E[e^{\sum_{i=1}^{k} tX_i^2}] = \prod_{i=1}^{k} E[e^{tX_i^2}] \\
&= \prod_{i=1}^{k} \int_{-\infty}^{+\infty} e^{tx^2} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x \\
&= \prod_{i=1}^{k} \sqrt{\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{(t-\frac{1}{2})(x^2+y^2)} \mathrm{d}x\mathrm{d}y} \\
&= \prod_{i=1}^{k} \sqrt{\int_{0}^{+\infty} \frac{1}{2} e^{(t-\frac{1}{2})\rho^2} \mathrm{d}\rho^2} \\
&= \prod_{i=1}^{k} \sqrt{\int_{0}^{+\infty} \frac{1}{2} e^{(t-\frac{1}{2})x} \mathrm{d}x} \\
&= \frac{1}{(1-2t)^{\frac{k}{2}}}
\end{aligned}
$$

其中 $0 < t < \frac{1}{2}$

因此 $P[\bar{X} \geqslant (1+\epsilon)k] = P[e^{t\bar{X}} \geqslant e^{t(1+\epsilon)k}] \leqslant \dfrac{e^{-t(1+\epsilon)k}}{(1-2t)^{\frac{k}{2}}}$

令 $f(t)=\dfrac{e^{-t(1+\epsilon)k}}{(1-2t)^{\frac{k}{2}}}$, 则 $f'(t)=k\left(1-2t\right)^{-\frac{k}{2}-1}\left(-\epsilon\left(1-2t\right)+2t\right)e^{-kt(\epsilon+1)}$

令 $f'(t)=0$ 得 $t=\dfrac{\epsilon}{2+2\epsilon}$

带入可得 $f(t)=\dfrac{e^{-t(1+\epsilon)k}}{(1-2t)^{\frac{k}{2}}}=\dfrac{e^{-\frac{\epsilon}{2+2\epsilon}(1+\epsilon)k}}{(1-\frac{\epsilon}{1+\epsilon})^{\frac{k}{2}}}=(\epsilon+1)^{\frac{k}{2}}\,e^{-\frac{\epsilon k}{2}}$

只需证明 $(\epsilon+1)^{\frac{k}{2}}\,e^{-\frac{k\epsilon}{2}}\leqslant e^{-\frac{k\epsilon}{2}\frac{\epsilon-\epsilon^2}{2}}$, 即证 $\epsilon^3-\epsilon^2+2\epsilon-2\ln(\epsilon+1)\geqslant 0$

令 $h(\epsilon)=\epsilon^3-\epsilon^2+2\epsilon-2\ln(\epsilon+1), h'(\epsilon)=3\epsilon^2-2\epsilon+2-\dfrac{2}{\epsilon+1}=\dfrac{\epsilon^2\left(3\epsilon+1\right)}{\epsilon+1}>0$

因此 $h(\epsilon)>h(0)=0$

因此 $P[\bar{X}\geqslant(1+\epsilon)k]=P[e^{t\bar{X}}\geqslant e^{t(1+\epsilon)k}]\leqslant e^{-\frac{k(\epsilon^2-\epsilon^3)}{4}}$

6.10

由 Chernoff 方法有

$$P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)\geqslant\epsilon]\leqslant e^{nt\epsilon}E[\exp(\sum_{i=1}^{n}t(X_i-\mu))]=e^{-nt\epsilon}(E[e^{t(X_1-\mu)}])^n$$

设 $Y=X_1-\mu$, 使用公式 $\ln z\leqslant z-1$ 有

$$\begin{aligned}\ln E[e^{t(X_1-\mu)}]=\ln E[e^{tY}]\leqslant E[e^{tY}]-1=t^2E[\frac{e^{tY}-tY-1}{t^2Y^2}Y^2]\\=t^2E[\frac{e^t-t-1}{t^2}Y^2]=(e^t-t-1)\sigma^2\end{aligned}$$

这里使用了 $tY\leqslant t$ 以及 $\dfrac{e^z-z-1}{z^2}$ 非单调递减. 因而有

$$e^t-t-1\leqslant\frac{t^2}{2}\sum_{k=0}^{+\infty}(\frac{t}{3})^k=\frac{t^2}{2(1-\frac{t}{3})}$$

因此有 $P[\dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(X_i-\mu)\geqslant\epsilon]\leqslant\exp(-nt\epsilon+\dfrac{nt^2\sigma^2}{2(1-\frac{t}{3})})$

带入 $t=\frac{\epsilon}{\sigma^2+\frac{\epsilon}{3}}$ 最后可得

$$P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)\geqslant\epsilon]\leqslant\exp(-\frac{n\epsilon^2}{2\sigma^2+2b\epsilon})$$

6.11

对任意 $t>0$, 根据 Chernoff 有

$$P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)\geqslant\epsilon]\leqslant e^{-nt\epsilon}E[\exp(\sum_{i=1}^{n}(X_i-\mu))]=e^{-nt\epsilon-n\mu t}(E[e^{tX_1}])^n$$

使用公式 $\ln z\leqslant z-1$ 有

$$\ln E[e^{tX_1}]\leqslant E[e^{tX}]-1=\sum_{m=1}^{+\infty}E[X^m]\frac{t^m}{m!}\leqslant t\mu+\frac{t^2\sigma^2}{2}\sum_{m=2}^{+\infty}(bt)^{m-2}=t\mu+\frac{t^2\sigma^2}{2(1-bt)}$$

由此得

$$P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)\geqslant\epsilon]\leqslant\exp(-nt\epsilon+\frac{nt^2\sigma^2}{2(1-bt)})$$

取 $t=\frac{\epsilon}{\sigma^2+b\epsilon}$ 代入可得

$$P[\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)\geqslant\epsilon]\leqslant\exp(-\frac{n\epsilon^2}{2\sigma^2+2b\epsilon})$$

6.12

令 $\delta=\exp(\frac{-n\epsilon^2}{2\sigma^2+2b\epsilon})$ 可得 $n\epsilon^2-2b\ln\frac{1}{\delta}\cdot\epsilon-2\sigma^2\ln\frac{1}{\delta}=0$

解得 $\epsilon=\frac{b\ln\frac{1}{\delta}+\sqrt{b^2\ln^2\frac{1}{\delta}+n\cdot2\sigma^2\ln\frac{1}{\delta}}}{n}$

因此 $\frac{1}{n}\sum_{i=1}^{n}X_i\leqslant\mu+\frac{b}{n}\ln\frac{1}{\delta}+\sqrt{\frac{b^2}{n^2}\ln^2\frac{1}{\delta}+\frac{2\sigma^2}{n}\ln\frac{1}{\delta}}$

6.13

首先证明高斯随机变量是一种亚高斯随机变量.

$$E[e^{(X-\mu)t}]=\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}\sigma}e^{xt}e^{-\frac{x^2}{2\sigma^2}}\,\mathrm{d}x=e^{\frac{\sigma^2t^2}{2}}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{(t\sigma-\frac{x}{\sigma})^2}{2}}\,\mathrm{d}\frac{x}{\sigma}=e^{\frac{\sigma^2t^2}{2}}$$

因此高斯随机变量是参数为 σ^2 的亚高斯随机变量.

下面证明 n 个相互独立, 参数为 b 的亚高斯随机变量, 且满足 $E[X_i]=0$ 时有 $E[\max\limits_{i\in[n]}X_i]\leqslant\sqrt{2b\ln n}$

根据琴生不等式有

$$\exp(tE[\max_{i\in[n]}X_i])\leqslant E[\exp(t\max_{i\in[n]}X_i)]=E[\max_{i\in[n]}e^{tX_i}]\leqslant\sum_{i=1}^{n}E[e^{tX_i}]\leqslant ne^{\frac{t^2b}{2}}$$

两边取对数可得

$$E[\max_{i\in[n]}X_i]\leqslant\frac{\ln n}{t}+\frac{bt}{2}$$

令 $f(t)=\dfrac{\ln n}{t}+\dfrac{bt}{2},f'(t)=\dfrac{b}{2}-\dfrac{\ln n}{t^2}$

可得 $t_{\min}=\sqrt{\dfrac{2\ln n}{b}}$, 代入可得

$$E[\max_{i\in[n]}X_i]\leqslant\sqrt{2b\ln n}$$

综合可得

$$E[\max_{i\in[n]}X_i]=E[\max_{i\in[n]}(X_i-\mu)]+\mu\leqslant\mu+\sqrt{2\sigma^2\ln n}$$