习题5.3 (A) 1.(2. 6. 9), 2(2), 4, 6, 13(2), 18, 22, 23(1), 24(4)

习题 5.3 (A)

1.

(2)

$$\frac{\partial z}{\partial x}=\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\cdot\frac{\sqrt{x^2+y^2}-x\cdot\frac{x}{\sqrt{x^2+y^2}}}{x^2+y^2}=\frac{1}{|y|}-\frac{x^2}{|y|(x^2+y^2)}$$

$$\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-\frac{x^2}{x^2+y^2}}}\cdot\frac{-x\cdot\frac{y}{\sqrt{x^2+y^2}}}{x^2+y^2}=-\frac{xy}{|y|(x^2+y^2)}$$

(6)

$$\frac{\partial u}{\partial x}=\frac{zx^{z-1}}{y^z}$$

$$\frac{\partial u}{\partial y}=-\frac{zx^z}{y^{z-1}}$$

$$\frac{\partial u}{\partial z}=(\frac{x}{y})^z\ln\frac{x}{y}$$

(9)

$u=xze^{\sin(yz)}$

$$\frac{\partial u}{\partial x}=ze^{\sin(yz)}$$

$$\frac{\partial u}{\partial y}=xz^2e^{sin(yz)}\cos(yz)$$

$$\frac{\partial u}{\partial z}=xe^{\sin(yz)}+xyze^{\sin(yz)}\cos(yz)$$

2. (2)

$$
\begin{aligned}
f_y(\pi, \frac{\pi}{4}) &= \frac{2\sin(x-2y)\cos(x+y)+\sin(x+y)\cos(x-2y)}{\cos^2(x+y)} \\
&= \frac{2\sin\frac{\pi}{2}\cos\frac{5\pi}{4}+\sin\frac{5\pi}{4}\cos\frac{\pi}{2}}{\cos^2\frac{5\pi}{4}} \\
&= -2\sqrt{2}
\end{aligned}
$$

4.

(1)

$$\therefore f_x(0,0) = \lim_{x\to 0}\frac{x\sin\dfrac{1}{x^2+0^2}-0}{x} = \sin\frac{1}{x^2},\ f_x(0,0)\text{ 不存在.}$$

$$\therefore f_y(0,0) = \lim_{y\to 0}\frac{0\times\sin\dfrac{1}{0^2+y^2}-0}{y} = 0,\ f_y(0,0)\text{ 存在.}$$

(2)

$$
\begin{aligned}
\therefore f_x^+(0,0) &= \lim_{x\to 0^+}\frac{|x|g(x,0)-0}{x} = g(0,0) \\
f_x^-(0,0) &= \lim_{x\to 0^-}\frac{|x|g(x,0)-0}{x} = -g(0,0)
\end{aligned}
$$

$$
\begin{aligned}
\therefore f_y^+(0,0) &= \lim_{y\to 0^+}\frac{|y|g(0,y)-0}{y} = g(0,0) \\
f_y^-(0,0) &= \lim_{y\to 0^+}\frac{|y|g(0,y)-0}{y} = -g(0,0)
\end{aligned}
$$

若偏导数 $f_x(0,0), f_y(0,0)$ 存在, 那么 $g(0,0) = -g(0,0)$

即 $g(0,0)=0$ 时两个偏导数均存在.

假设 f 在 $(0,0)$ 可微

$\therefore \Delta f = f_x(0,0)\Delta x + f_y(0,0)\Delta y + o(\rho)$

$\because \Delta f = f(0+\Delta x, 0+\Delta y) - f(0,0) = |\Delta x - \Delta y| g(\Delta x, \Delta y)$

$$\therefore \lim_{(\Delta x,\Delta y)\to(0,0)}\frac{\Delta f}{\rho} = \frac{|\Delta x-\Delta y|g(\Delta x,\Delta y)}{\sqrt{\Delta x^2+\Delta y^2}} = 0$$

$\therefore g(0,0)=0$ 时 f 在 $(0,0)$ 处可微

6.

(1)

设 $e_l = (\cos\theta, \sin\theta)$

$$\therefore \frac{\partial f(0,0)}{\partial \boldsymbol{l}} = \lim_{t\to 0}\frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t}$$
$$= \lim_{t\to 0}\frac{(t^2\cos\theta\sin\theta)^{\frac{1}{3}}}{t}$$
$$= \lim_{t\to 0}\frac{(\frac{1}{2}\sin 2\theta)^{\frac{1}{3}}}{t^{\frac{1}{3}}}$$

当 $\theta = \frac{k\pi}{2}$, 即沿着两个坐标轴正负方向时, 有 $\frac{\partial f(0,0)}{\partial \boldsymbol{l}} = 0$

即沿着两个坐标轴正负方向存在方向导数.

当 $\theta = \frac{k\pi}{2}$ 时, $\lim_{t\to 0}\frac{(\frac{1}{2}\sin 2\theta)^{\frac{1}{3}}}{t^{\frac{1}{3}}}$ 极限不存在, 不存在方向导数.

$\therefore f(x,y)$ 在点 $(0,0)$ 只沿着两个坐标轴的正负方向存在方向导数.

(2)

令 $(x,y) = (t\cos\theta, t\sin\theta)$

$$\lim_{(x,y)\to(0,0)} f(x,y) - f(0,0) = f(t\cos\theta, t\sin\theta) = (\frac{1}{2}t^2\sin 2\theta)^{\frac{1}{3}} = 0$$

$\therefore f(x,y)$ 在点 $(0,0)$ 连续.

13. (2)

令 $f(x,y) = x^y, (x_0, y_0) = (1,1), \Delta x = -0.03, \Delta y = 0.05$, 则

$f_x(1,1) = yx^{y-1}|_{(1,1)} = 1$, $f_y(1,1) = x^y\ln x|_{(1,1)} = 0$

$\therefore 0.97^{1.05} \approx f(1,1) + f_x(1,1)\Delta x + f_y(1,1)\Delta y = 1 - 0.03 = 0.97$

18.

$\because \boldsymbol{l} = (3-1, -2, 2-1) = (2, -2, 1)$

$\therefore \boldsymbol{e}_l = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$

$$\because \frac{\partial u}{\partial x} = \frac{\sqrt{y^2+z^2}}{(x+\sqrt{y^2+z^2})\sqrt{y^2+z^2}},$$
$$\frac{\partial u}{\partial y} = \frac{y}{(x+\sqrt{y^2+z^2})\sqrt{y^2+z^2}},$$
$$\frac{\partial u}{\partial y} = \frac{z}{(x+\sqrt{y^2+z^2})\sqrt{y^2+z^2}}$$

$$\therefore \frac{\partial u}{\partial \boldsymbol{l}} = \frac{2\sqrt{y^2+z^2}-2y+z}{3(x+\sqrt{y^2+z^2})\sqrt{y^2+z^2}}$$

22.

$$\because \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{\partial r}{\partial z} = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\therefore \nabla r = (\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}})$$

$$\because \frac{\partial \frac{1}{r}}{\partial x} = -\frac{x}{(x^2+y^2+x^2)^{\frac{3}{2}}}, \frac{\partial \frac{1}{r}}{\partial y} = -\frac{y}{(x^2+y^2+x^2)^{\frac{3}{2}}}, \frac{\partial \frac{1}{r}}{\partial z} = -\frac{z}{(x^2+y^2+x^2)^{\frac{3}{2}}}$$

$$\therefore \nabla \frac{1}{r} = (-\frac{x}{(x^2+y^2+x^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2+x^2)^{\frac{3}{2}}}, -\frac{z}{(x^2+y^2+x^2)^{\frac{3}{2}}})$$

23. (1)

$$\frac{\partial z}{\partial x} = f_1 + 2f_2 + 2xf_3$$
$$\frac{\partial z}{\partial y} = f_2 - 3yf_3$$

24. (4)

$$\because T = \frac{1}{2a\sqrt{\pi t}} \exp(-\frac{(x-a)^2}{4a^2t})$$

$$\therefore \frac{\partial T}{\partial t} = -\frac{1}{4at\sqrt{\pi t}} \exp(-\frac{(x-a)^2}{4a^2t}) + \frac{(x-a)^2}{8a^3t^2\sqrt{\pi t}} \exp(-\frac{(x-a)^2}{4a^2t})$$

$$\therefore a^2\frac{\partial^2 T}{\partial x^2} = a^2\frac{\partial}{\partial x}(-\frac{2(x-a)}{8a^3t\sqrt{\pi t}}\exp(-\frac{(x-a)^2}{4a^2t}))$$

$$= -\frac{1}{4at\sqrt{\pi t}}\exp(-\frac{(x-a)^2}{4a^2t}) + \frac{(x-a)^2}{8a^3t^2\sqrt{\pi t}}\exp(-\frac{(x-a)^2}{4a^2t})$$

$$\therefore \frac{\partial T}{\partial t} = a^2\frac{\partial^2 T}{\partial x^2}$$