第九次作业

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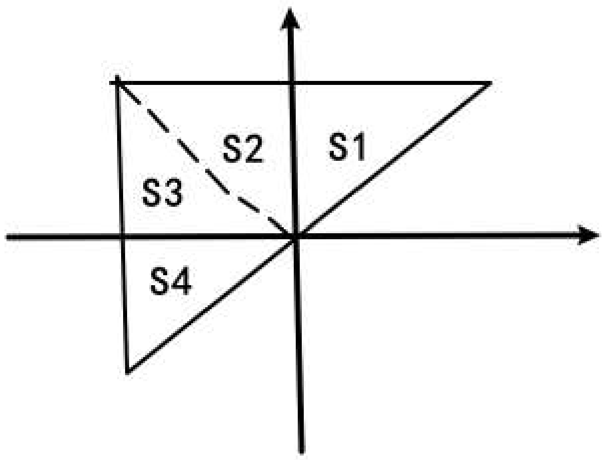
P264 第六章习题: 1, 2, 10, 14, 18, 21, 28, 29, 习题7.1: (A)3(2, 3), 11, 12(3, 6, 9, 11, 14), 14(3, 4), 15(1, 2), 18(2, 4), (B)1, 2, 3, 4(3, 4)

第六章习题

1.

(1) 答案为 (A)



其中 $D_1 = S_1$, 由对称性可知

$$\iint_{S_1} xy\mathrm{d}x\mathrm{d}y + \iint_{S_2} xy\mathrm{d}x\mathrm{d}y = 0, \iint_{S_3} xy\mathrm{d}x\mathrm{d}y + \iint_{S_4} xy\mathrm{d}x\mathrm{d}y = 0,$$
$$\iint_{S_1} \cos x \sin y\mathrm{d}x\mathrm{d}y = \iint_{S_2} \cos x \sin y\mathrm{d}x\mathrm{d}y, \iint_{S_3} \cos x \sin y\mathrm{d}x\mathrm{d}y + \iint_{S_4} \cos x \sin y\mathrm{d}x\mathrm{d}y = 0$$

所以有 $\iint_{(D)} (xy + \cos x \sin y)\mathrm{d}x\mathrm{d}y = 2\iint_{(D_1)} \cos x \sin y\mathrm{d}x\mathrm{d}y$

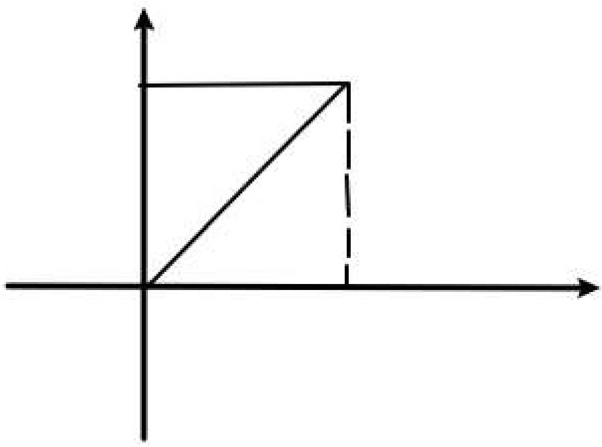
(2) 答案为 (B)

$\because F(t) = \int_1^t \mathrm{d}y \int_y^t f(x)\mathrm{d}x$

$$
\begin{aligned}
\therefore F'(t) &= (\int_1^t \mathrm{d}y \int_y^t f(x)\mathrm{d}x)' \\
&= (\int_1^t (\int_y^t f(x)\mathrm{d}x)\mathrm{d}y)' \\
&= \int_1^t (\int_y^t f(x)\mathrm{d}x)'\mathrm{d}y + \int_t^t f(x)\mathrm{d}x - 0 \\
&= \int_1^t f(t)\mathrm{d}y \\
&= (t-1)f(t)
\end{aligned}
$$

$\therefore F'(2) = f(2)$

(3) 答案为 (D)



$$\because \int_0^1 \mathrm{d}x \int_x^1 f(x)f(y)\mathrm{d}y = \int_0^1 \mathrm{d}y \int_y^1 f(x)f(y)\mathrm{d}x$$

$$\therefore \int_0^1 \mathrm{d}x \int_x^1 f(x)f(y)\mathrm{d}y = \frac{1}{2}\int_0^1 \mathrm{d}x \int_0^1 f(x)f(y)\mathrm{d}y = \frac{1}{2}(\int_0^1 f(x)\mathrm{d}x)^2 = \frac{1}{2}A^2$$

(4) 答案为 (C)

因为对 Ω_1, Ω_2 均有 $z > 0$, 不会因为奇函数的积分特性被消去.

且由对称性可知,

$$\iiint_{(\Omega_1)} z\mathrm{d}V = 4\iiint_{\Omega_2} z\mathrm{d}V$$

(5) 答案为 (A)

$\Omega : (x+1)^2 + (y-1)^2 + z^2 \leqslant 2$

进行换元 $\begin{cases} x=\rho\sin\varphi\cos\theta-1 \\ y=\rho\sin\varphi\sin\theta+1 \\ z=\rho\cos\varphi \end{cases}$

则有 $J=\rho^2\sin\varphi$, 其中 $0\leqslant\varphi\leqslant\pi, 0\leqslant\theta\leqslant 2\pi$

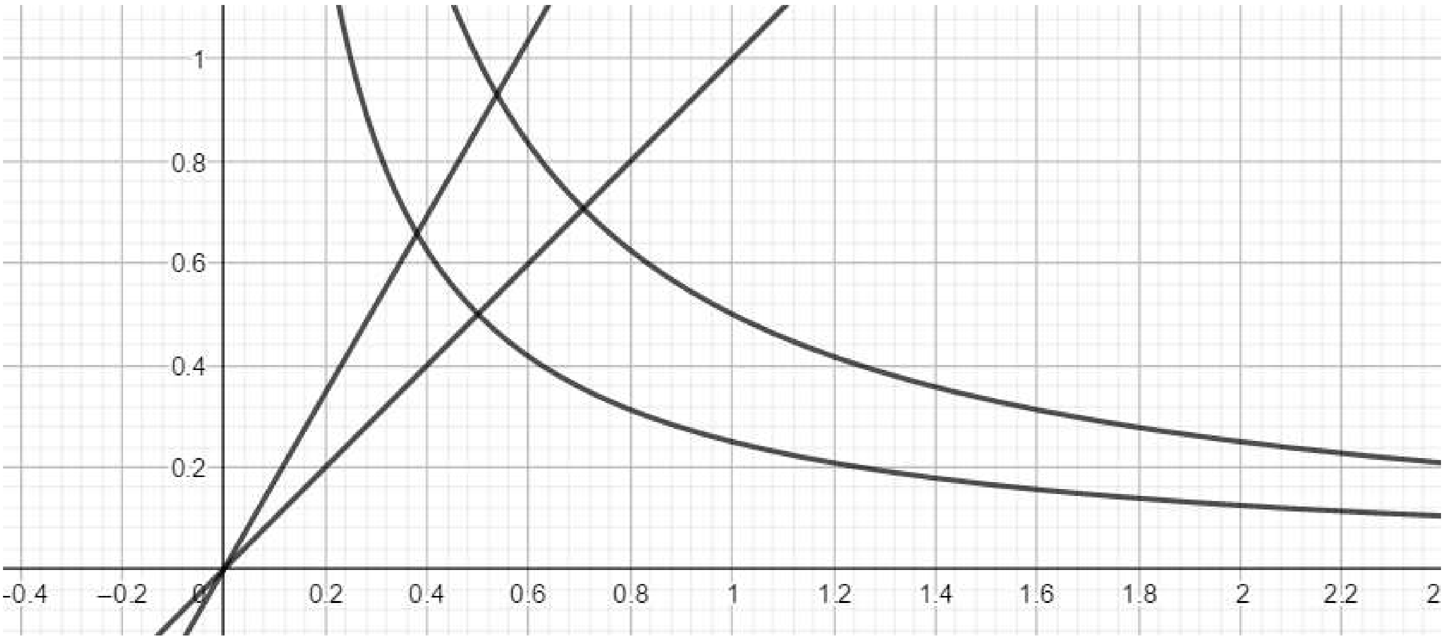
$$
\begin{aligned}
&\therefore \iiint_{\Omega}(x+y+z)\mathrm{d}V \\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^{\sqrt{2}}(\rho\sin\varphi\cos\theta+\rho\sin\varphi\sin\theta+\rho\cos\varphi)\rho^2\sin\varphi\mathrm{d}\rho \\
&=\int_0^{2\pi}\frac{\sqrt{2}\pi\sin\left(\theta+\frac{\pi}{4}\right)}{2}\mathrm{d}\theta \\
&=0
\end{aligned}
$$

(6) 答案为 (C)

因为对于 S, S_1 均有 $z>0$, 根据对称性以及恒正性我们可知

$$\iint_{(S)} z\mathrm{d}S=4\iint_{(S_1)} z\mathrm{d}S$$

(7) 答案为 (B)



$$2xy=1\Rightarrow 2r^2\cos\theta\sin\theta=1\Rightarrow r=\sqrt{\frac{1}{\sin 2\theta}}$$

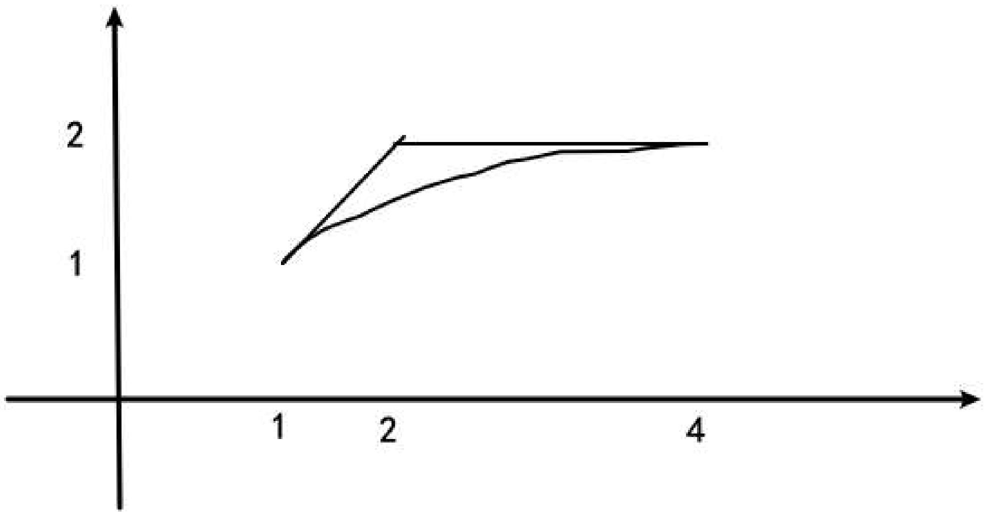
$$4xy=1\Rightarrow 4r^2\cos\theta\sin\theta=1\Rightarrow r=\sqrt{\frac{1}{2\sin 2\theta}}$$

$$y=x\Rightarrow\sin\theta=\cos\theta\Rightarrow\theta=\frac{\pi}{4}$$

$$y=\sqrt{3}x \Rightarrow \sin\theta=\sqrt{3}\cos\theta \Rightarrow \theta=\frac{\pi}{3}$$

$$\therefore \iint_D f(x,y)\mathrm{d}x\mathrm{d}y=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\mathrm{d}\theta\int_{\sqrt{\frac{1}{2\sin 2\theta}}}^{\sqrt{\frac{1}{\sin 2\theta}}} f(r\cos\theta, r\sin\theta)r\mathrm{d}r$$

2.



$$\int_1^2\mathrm{d}x\int_{\sqrt{x}}^{x}\sin\frac{\pi x}{2y}+\int_2^4\mathrm{d}x\int_{\sqrt{x}}^{2}\sin\frac{\pi x}{2y}\mathrm{d}y$$
$$=\int_1^2\mathrm{d}y\int_y^{y^2}\sin\frac{\pi x}{2y}\mathrm{d}x$$
$$=\int_1^2(-\frac{2y}{\pi}\cos\frac{\pi y}{2})\mathrm{d}y$$
$$=-\frac{4}{\pi^2}\int_1^2 y\mathrm{d}\sin\frac{\pi y}{2}$$
$$=-\frac{4}{\pi^2}(y\sin\frac{\pi y}{2}|_1^2-\int_1^2\sin\frac{\pi y}{2}\mathrm{d}y)$$
$$=-\frac{4}{\pi^2}(-1-\frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi}\sin t\mathrm{d}t)$$
$$=\frac{4}{\pi^2}+\frac{8}{\pi^3}$$

10.

(1)

对 $\Omega(t)$ 进行球面坐标变换 $\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases}$, 则有 $\Omega(t): 0\leqslant\rho\leqslant t, J=\rho^2\sin\varphi$

则 $\iiint_{\Omega(t)} f(x^2+y^2+z^2)\mathrm{d}V=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\mathrm{d}\varphi\int_0^t f(\rho^2)\rho^2\sin\varphi\mathrm{d}\rho=4\pi\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho$

对 $D(t)$ 极坐标变换 $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \end{cases}$, 则有 $D(t): 0\leqslant\rho\leqslant t, J=\rho$

则 $\iint_{D(t)} f(x^2+y^2)\mathrm{d}\sigma=\int_0^{2\pi}\mathrm{d}\theta\int_0^t f(\rho^2)\rho\mathrm{d}\rho=2\pi\int_0^t f(\rho^2)\rho\mathrm{d}\rho$

当 $t>0$ 时, 我们有 $f(t^2)>0, \int_0^t f(\rho^2)\rho\mathrm{d}\rho>0$

$$\therefore F(t)=\frac{4\pi\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho}{2\pi\int_0^t f(\rho^2)\rho\mathrm{d}\rho}=\frac{2\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho}{\int_0^t f(\rho^2)\rho\mathrm{d}\rho}$$

$$\begin{aligned}\therefore F'(t)&=\frac{2f(t^2)t^2\left(\int_0^t f(\rho^2)\rho\mathrm{d}\rho\right)-2f(t^2)t\left(\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho\right)}{\left(\int_0^t f(\rho^2)\rho\mathrm{d}\rho\right)^2}\\ &=\frac{2f(t^2)t\int_0^t f(\rho^2)\rho(t-\rho)\mathrm{d}\rho}{\left(\int_0^t f(\rho^2)\rho\mathrm{d}\rho\right)^2}\\ &>0\end{aligned}$$

所以 $F(t)$ 在 $(0,+\infty)$ 内单调递增.

(2)

$$\because \int_{-t}^t f(x^2)\mathrm{d}x=2\int_0^t f(\rho^2)\mathrm{d}\rho$$

$$\therefore \frac{2}{\pi}G(t)=\frac{2}{\pi}\cdot\frac{2\pi\int_0^t f(\rho^2)\rho\mathrm{d}\rho}{2\int_0^t f(\rho^2)\mathrm{d}\rho}=\frac{2\int_0^t f(\rho^2)\rho\mathrm{d}\rho}{\int_0^t f(\rho^2)\mathrm{d}\rho}$$

要证 $F(t) > \frac{2}{\pi}G(t)$

即证 $\dfrac{2\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho}{\int_0^t f(\rho^2)\rho\mathrm{d}\rho} > \dfrac{2\int_0^t f(\rho^2)\rho\mathrm{d}\rho}{\int_0^t f(\rho^2)\mathrm{d}\rho}$

即证 $H(t) = \left(\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho\right)\left(\int_0^t f(\rho^2)\mathrm{d}\rho\right) - \left(\int_0^t f(\rho^2)\rho\mathrm{d}\rho\right)^2 > 0$

$$
\begin{aligned}
\because H'(t) &= f(t^2)t^2\int_0^t f(\rho^2)\mathrm{d}\rho + f(t^2)\int_0^t f(\rho^2)\rho^2\mathrm{d}\rho - 2f(t^2)t\int_0^t f(\rho^2)\rho\mathrm{d}\rho \\
&= f(t^2)\int_0^t f(\rho^2)(t^2+\rho^2-2t\rho)\mathrm{d}\rho \\
&= f(t^2)\int_0^t f(\rho^2)(t-\rho)^2\mathrm{d}\rho \\
&> 0
\end{aligned}
$$

$\therefore H(t)$ 是单调递增的, 对于 $t > 0$, 满足 $H(t) > H(0) = 0$

$\therefore F(t) > \frac{2}{\pi}G(t)$ 成立.

14.

$\because L : \frac{x^2}{4} + \frac{y^2}{3} = 1 \Rightarrow 3x^2 + 4y^2 = 12$

$$
\begin{aligned}
&\therefore \oint_{(L)}(2xy+3x^2+4y^2)\mathrm{d}s \\
&= \oint_{(L)}(2xy+12)\mathrm{d}s \\
&= \oint_{(L)}2xy\mathrm{d}s + 12\oint_{(L)}\mathrm{d}s \\
&= 0 + 12a \\
&= 12a
\end{aligned}
$$

18.

当 L 不包围点 $(0,0)$ 时,

使用 Green 公式:

$$I=\oint_L \frac{x\mathrm{d}y-y\mathrm{d}x}{4x^2+y^2}=\iint_{L-\Gamma}\frac{(4x^2+y^2)-x\cdot 8x+(4x^2+y^2)-y\cdot 2y}{(4x^2+y^2)^2}\mathrm{d}x\mathrm{d}y=0$$

当 L 包围点 $(0,0)$ 时,

令 $\Gamma: 4x^2+y^2=\delta^2$, 沿正向. 并作 $\begin{cases}x=\frac{1}{2}\delta\cos\theta\\ y=\delta\sin\theta\end{cases}$, 则有 $L-\Gamma$ 不包围点 $(0,0)$

$$\begin{aligned}I&=\oint_L \frac{x\mathrm{d}y-y\mathrm{d}x}{4x^2+y^2}\\&=\oint_{L-\Gamma}\frac{x\mathrm{d}y-y\mathrm{d}x}{4x^2+y^2}+\oint_\Gamma \frac{x\mathrm{d}y-y\mathrm{d}x}{4x^2+y^2}\\&=0+\oint_\Gamma \frac{\frac{1}{2}\delta\cos\theta\mathrm{d}\sin\theta-\delta\sin\theta\mathrm{d}\frac{1}{2}\cos\theta}{\delta^2}\\&=\pi\end{aligned}$$

21.

令 $\Gamma: x=0$, 从 $(0,-R)$ 到 $(0,R)$, 使用 Green 公式

$$\begin{aligned}&\int_L \frac{y^2}{\sqrt{a^2+x^2}}\mathrm{d}x+(ax+2y\ln(x+\sqrt{a^2+x^2}))\mathrm{d}y\\&=\oint_{L+\Gamma}\frac{y^2}{\sqrt{a^2+x^2}}\mathrm{d}x+(ax+2y\ln(x+\sqrt{a^2+x^2}))\mathrm{d}y-\int_{-R}^{R}2y\ln a\mathrm{d}y\\&=\iint_{L+\Gamma}\left(-\frac{2y}{\sqrt{a^2+x^2}}+a+\frac{2y}{\sqrt{a^2+x^2}}\right)\mathrm{d}x\mathrm{d}y\\&=a\iint_{L+\Gamma}\mathrm{d}x\mathrm{d}y\\&=\frac{1}{2}a\pi R^2\end{aligned}$$

28.

$\because z=1-x^2-y^2$

$\therefore \dfrac{\partial(y,z)}{\partial(x,y)}=\begin{vmatrix}0&1\\-2x&-2y\end{vmatrix}=2x, \dfrac{\partial(z,x)}{\partial(x,y)}=\begin{vmatrix}-2x&-2y\\1&0\end{vmatrix}=2y, \dfrac{\partial(x,y)}{\partial(x,y)}=1$

$$
\begin{aligned}
\therefore I &= \iint_{\Sigma} 2x^3 \mathrm{d}y \wedge \mathrm{d}z + 2y^3 \mathrm{d}z \wedge x + 3(z^2-1)\mathrm{d}x \wedge \mathrm{d}y \\
&= \iint_{S} (2x^3 \cdot 2x + 2y^3 \cdot 2y + 3((1-x^2-y^2)^2-1))\mathrm{d}x \wedge \mathrm{d}y \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 (4\rho^4(\cos^4\theta + \sin^4\theta) + 3(-2\rho^2+\rho^4))\rho \mathrm{d}\rho \\
&= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 (2t^2(\cos^4\theta + \sin^4\theta) + \frac{3}{2}(-2t+t^2))\mathrm{d}t \\
&= \int_0^{2\pi} (\frac{2}{3}(\cos^4\theta + \sin^4\theta) - 1)\mathrm{d}\theta \\
&= -\pi
\end{aligned}
$$

29.

由 Stokes 公式可知

$\because z = 2 - x - y$

$$
\therefore \frac{\partial(y,z)}{\partial(x,y)} = \begin{vmatrix} 0 & 1 \\ -1 & -1 \end{vmatrix} = 1, \frac{\partial(z,x)}{\partial(x,y)} = \begin{vmatrix} -1 & -1 \\ 1 & 0 \end{vmatrix} = 1, \frac{\partial(x,y)}{\partial(x,y)} = 1
$$

$$
\begin{aligned}
&\oint_{(L)} (y^2-z^2)\mathrm{d}x + (2z^2-x^2)\mathrm{d}y + (3x^2-y^2)\mathrm{d}z \\
&= \iint_{\Sigma} (-2y-4z)\mathrm{d}y \wedge \mathrm{d}z + (-2z-6x)\mathrm{d}z \wedge \mathrm{d}x + (-2x-2y)\mathrm{d}x \wedge \mathrm{d}y \\
&= \iint_{\Sigma} (-2y-4z-2z-6x-2x-2y)\mathrm{d}x \wedge \mathrm{d}y \\
&= \iint_{\Sigma} (-8x-4y-6(2-x-y))\mathrm{d}x \wedge \mathrm{d}y \\
&= \int_{-1}^{0} \mathrm{d}x \int_{-x-1}^{x+1} (-2x+2y-12)\mathrm{d}y + \int_0^1 \mathrm{d}x \int_{x-1}^{-x+1} (-2x+2y-12)\mathrm{d}y \\
&= \int_{-1}^{0} -4\left(x+1\right)\left(x+6\right)\mathrm{d}x + \int_0^1 4\left(x-1\right)\left(x+6\right)\mathrm{d}x \\
&= -24
\end{aligned}
$$

7.1 (A)

3.

(2)

因为级数的部分和数列为

$$\begin{aligned}S_n &= \frac{1}{4\cdot 7}+\frac{1}{7\cdot 10}+\cdots+\frac{1}{(3n+1)(3n+4)}\\&=\frac{1}{3\cdot 4}-\frac{1}{3\cdot 7}+\frac{1}{3\cdot 7}-\frac{1}{3\cdot 10}+\cdots+\frac{1}{3(3n+1)}-\frac{1}{3(3n+4)}\\&=\frac{1}{12}-\frac{1}{3(3n+4)}\end{aligned}$$

$$\lim_{n\to\infty} S_n=\frac{1}{12}$$

所以该级数收敛, 且其和为 $\frac{1}{12}$

(3)

因为级数的部分和数列为

$$\begin{aligned}S_n &= (\sqrt{3}-2\sqrt{2}+\sqrt{1})+(\sqrt{4}-2\sqrt{3}+\sqrt{2})+\cdots+(\sqrt{n+2}-2\sqrt{n+1}+\sqrt{n})\\&=-\sqrt{2}+\sqrt{1}+\sqrt{n+2}-\sqrt{n+1}\\&=1-\sqrt{2}+\sqrt{n+2}-\sqrt{n+1}\\&=1-\sqrt{2}+\frac{(n+2)-(n+1)}{\sqrt{n+2}+\sqrt{n+1}}\\&=1-\sqrt{2}+\frac{1}{\sqrt{n+2}+\sqrt{n+1}}\end{aligned}$$

$$\lim_{n\to\infty} S_n=1-\sqrt{2}$$

所以该级数收敛, 且其和为 $1-\sqrt{2}$

11.

(1)

不正确.

令 $a_n=-1, b_n=\frac{1}{n(n+1)}$

其中 $\lim\limits_{n\to\infty} S_{bn}=\lim\limits_{n\to\infty}(1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1})=\lim\limits_{n\to\infty}(1-\frac{1}{n+1})=1$

即满足 $\sum\limits_{n=1}^{\infty} b_n$ 是收敛的. 并且有 $a_n=-1\leqslant b_n=\frac{1}{n(n+1)}$

但是 $\lim\limits_{n\to\infty} S_{an}=1+1+\cdots+1=n\to-\infty$, 是发散的

所以该命题不正确.

(2)

对于 $a_n=\frac{(-1)^n}{\sqrt{n}}, b_n=\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}$

易知 $\sum_{n=1}^{\infty} a_n$ 是交错级数, 所以收敛.

并且满足 $\lim_{n\to\infty}\frac{b_n}{a_n}=\lim_{n\to\infty}\frac{\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}}{\frac{(-1)^n}{\sqrt{n}}}=\lim_{n\to\infty}\left(1+\frac{1}{(-1)^n\sqrt{n}}\right)=1$

假设 $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{\sqrt{n}}+\frac{1}{n}\right)$ 收敛.

则 $\sum_{n=1}^{\infty} b_n=\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}+\sum_{n=1}^{\infty}\frac{1}{n}$, 即后面两个收敛数列的和.

但是这里仅有 $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ 是收敛的, 而 $\sum_{n=1}^{\infty}\frac{1}{n}$ 是发散的.

所以假设不成立, $\sum_{n=1}^{\infty} b_n$ 只能是发散的.

所以该命题不正确.

(3)

令 $a_n=\frac{1}{n(n+1)}$, 易知 $\sum_{n=1}^{\infty} a_n=1$ 收敛.

此时有 $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{n(n+1)}{(n+1)(n+2)}=\lim_{n\to\infty}\frac{n}{n+2}=1$

不满足 $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lambda<1$

所以该命题不正确.

(4)

不正确.

对于调和数列 $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\cdots$, 其中 $a_n=\frac{1}{n}\to 0\ (n\to\infty)$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n}&=1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})+\cdots\\&>1+\frac{1}{2}+(\frac{1}{4}+\frac{1}{4})+(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8})+\cdots\\&=1+\frac{1}{2}+\frac{1}{2}+\cdots\\&=1+\frac{1}{2}k\to\infty\end{aligned}$$

是发散的.

所以该命题不正确.

(5)

不正确.

调和数列 $\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\cdots$ 是发散的, 其中 $a_n=\frac{1}{n}\to 0\ (n\to\infty)$

但是对于 $b_n=a_n^2=\frac{1}{n^2}$

使用 Cauchy 收敛准则:

对于 $\sum_{n=1}^{\infty}\frac{1}{n^2}, \forall\varepsilon>0, \exists N, n>N$ 时, 对 $\forall p\in\mathbb{N}$

$$\begin{aligned}&|a_n+a_{n+1}+\cdots+a_{n+p}|\\=&\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(n+p)^2}\\<&\frac{1}{(n-1)n}+\frac{1}{n(n+1)}+\cdots+\frac{1}{(n+p-1)(n+p)}\\=&\frac{1}{n-1}-\frac{1}{n+p}\\<&\frac{1}{n-1}\\<&\varepsilon\end{aligned}$$

因此 $\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\cdots$ 是收敛的.

所以该命题不正确.

(6)

$\because \sum_{n=1}^{\infty} a_n^2$ 是收敛的.

由阶估法可知

令 $\lim_{n \to \infty} n^p a_n^2 = \lambda, 0 < \lambda < +\infty$, 则有 $p > 1$.

$\because \lim_{n \to \infty} n^p a_n^2 = (\lim_{n \to \infty} n^{\frac{p}{2}} |a_n|)^2 = \lambda$

$\therefore \lim_{n \to \infty} n^{\frac{p}{2}} |a_n| = \lim_{n \to \infty} n^{\frac{p}{2}+1} \frac{|a_n|}{n} = \sqrt{\lambda}$

由阶估法可知

$\because \frac{p}{2} + 1 > \frac{1}{2} + 1 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{|a_n|}{n}$ 收敛

由绝对收敛可以推出收敛可知

$\therefore \sum_{n=1}^{\infty} \frac{a_n}{n}$ 收敛

12.

(3)

$\because \lim_{n \to \infty} n^{\alpha+\frac{1}{2}} \frac{\sqrt{n+2} - \sqrt{n-2}}{n^\alpha} = \lim_{n \to \infty} \sqrt{n} \cdot \frac{(n+2)-(n-2)}{\sqrt{n+2} + \sqrt{n-2}} = 2$

由阶估法可知

当 $\alpha + \frac{1}{2} > 1$ 即 $\alpha > \frac{1}{2}$ 时, $\sum_{n=1}^{\infty} \frac{\sqrt{n+2} - \sqrt{n-2}}{n^\alpha}$ 收敛.

当 $\alpha + \frac{1}{2} \leqslant 1$ 即 $\alpha \leqslant \frac{1}{2}$ 时, $\sum_{n=1}^{\infty} \frac{\sqrt{n+2} - \sqrt{n-2}}{n^\alpha}$ 发散.

(6)

$$a_n=\frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n}\leqslant\frac{n^3(\sqrt{2}+1)^n}{3^n}<\frac{n^3(\frac{3}{2}+1)^n}{3^n}=n^3\left(\frac{5}{6}\right)^n$$

令 $b_n=n^3\left(\frac{5}{6}\right)^n$, 使用 D' Alembert 比值法:

$$\because\lim_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim_{n\to\infty}\frac{(n+1)^3\left(\frac{5}{6}\right)^{n+1}}{n^3\left(\frac{5}{6}\right)^n}=\frac{5}{6}<1$$

$\therefore\sum_{n=1}^{\infty}b_n$ 是收敛的

$\because a_n<b_n$, 由比较判别法可知

$\therefore\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\frac{n^3[\sqrt{2}+(-1)^n]^n}{3^n}$ 是收敛的.

(9)

使用极小值替换和阶估法:

$$\lim_{n\to\infty}n^2\cdot n\ln\left(1+\frac{2}{n^3}\right)=\lim_{n\to\infty}n^2\cdot n\cdot\frac{2}{n^3}=2$$

因为 $2>1$, 使用阶估法可知

$\sum_{n=1}^{\infty}n\ln\left(1+\frac{2}{n^3}\right)$ 收敛.

(11)

令 $a_n=n!\left(\frac{x}{n}\right)^n$, 使用 D' Alembert 比值法:

$$\therefore\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!\left(\frac{x}{n+1}\right)^{n+1}}{n!\left(\frac{x}{n}\right)^n}=\lim_{n\to\infty}x\cdot(1-\frac{1}{n+1})^n=\frac{x}{e}$$

当 $0<\frac{x}{e}<1$ 即 $0<x<e$ 时, $\sum_{n=1}^{\infty}n!\left(\frac{x}{n}\right)^n$ 收敛.

当 $\frac{x}{e}>1$ 即 $x>e$ 时, $\sum_{n=1}^{\infty}n!\left(\frac{x}{n}\right)^n$ 发散.

当 $\frac{x}{e}=1$ 即 $x=e$ 时, $\sum_{n=1}^{\infty}n!\left(\frac{e}{n}\right)^n$ 敛散性暂时无法判断.

(14)

当 $\alpha=1$ 时, $\sum_{n=1}^{\infty}\frac{\alpha^n}{1+\alpha^{2n}}=\sum_{n=1}^{\infty}\frac{1}{2}$ 发散.

当 $\alpha>1$ 时,

令 $a_n=\frac{\alpha^n}{1+\alpha^{2n}}$, 使用 D' Alembert 比值法:

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{\alpha^{n+1}}{1+\alpha^{2n+2}}}{\frac{\alpha^n}{1+\alpha^{2n}}}=\lim_{n\to\infty}\alpha\cdot\frac{1+\alpha^{2n}}{1+\alpha^{2n+2}}=\frac{1}{\alpha}<1$$

$\therefore \sum_{n=1}^{\infty}\frac{\alpha^n}{1+\alpha^{2n}}$ 收敛.

当 $0<\alpha<1$ 时,

令 $a_n=\frac{\alpha^n}{1+\alpha^{2n}}$, 使用 D' Alembert 比值法:

$$\therefore \lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{\alpha^{n+1}}{1+\alpha^{2n+2}}}{\frac{\alpha^n}{1+\alpha^{2n}}}=\lim_{n\to\infty}\alpha\cdot\frac{1+\alpha^{2n}}{1+\alpha^{2n+2}}=\alpha<1$$

$\therefore \sum_{n=1}^{\infty}\frac{\alpha^n}{1+\alpha^{2n}}$ 收敛.

14.

(3)

令 $a_n=\frac{1}{n-\ln n}, f(x)=\frac{1}{x-\ln x}, x\geqslant 1$

易知 $x-\ln x>0$ 在 $x\geqslant 1$ 时均成立.

则 $f'(x)=-\frac{1-\frac{1}{x}}{(x-\ln x)^2}\leqslant 0$, $f(x)$ 大于零且单调递减, 并且 $\lim_{n\to\infty}\frac{1}{x-\ln x}=0.$

由莱布尼茨定理可知

$\sum_{n=1}^{\infty}(-1)^{n-1}a_n$ 收敛.

$\because a_n = \dfrac{1}{n - \ln n} \geqslant \dfrac{1}{n}, \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ 发散.

$\therefore \sum\limits_{n=1}^{\infty} a_n$ 也发散.

$\therefore \sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ 条件收敛.

(4)

当 $a > 1$ 时,

令 $a_n = \sqrt[n]{a} - 1$, 易知 a_n 单调递减且趋于 0

由莱布尼茨定理可知

$\sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ 收敛.

$\because \lim\limits_{n\to\infty} n(\sqrt[n]{a} - 1) = \lim\limits_{n\to\infty} n(e^{\frac{1}{n}\ln a} - 1) = \lim\limits_{n\to\infty} n \cdot \dfrac{1}{n} \ln a = \ln a$

$\because 1 \leqslant 1$, 由阶估法可知

$\therefore \sum\limits_{n=1}^{\infty} a_n$ 发散.

$\therefore \sum\limits_{n=1}^{\infty} (-1)^{n-1} a_n$ 条件收敛.

15.

(1)

$\because a_n = \dfrac{1}{\sqrt{n} - 1} - \dfrac{1}{\sqrt{n} + 1} = \dfrac{\sqrt{n} + 1 - \sqrt{n} + 1}{n - 1} = \dfrac{2}{n - 1} > 0$

所以不是交错级数, 不满足 Leibniz 准则的条件, 发散.

(2)

$\because [1 + (-1)^n] \dfrac{1}{n} \sin \dfrac{1}{n} \geqslant 0$

所以不是交错级数, 不满足 Leibniz 准则的条件.

令 $b_n=\frac{2}{n}\sin\frac{1}{n}$, 则有 $[1+(-1)^n]\frac{1}{n}\sin\frac{1}{n}\leqslant b_n$

$\because \lim\limits_{n\to\infty} n^2 b_n=\lim\limits_{n\to\infty} n^2\cdot\frac{2}{n}\sin\frac{1}{n}=\lim\limits_{n\to\infty} n^2\cdot\frac{2}{n}\cdot\frac{1}{n}=2$

由阶估法可知

$\therefore 2>1, \sum\limits_{n=1}^{\infty} b_n$ 收敛.

由比较判别法和 $[1+(-1)^n]\frac{1}{n}\sin\frac{1}{n}\leqslant b_n$ 可知

$\therefore \sum\limits_{n=1}^{\infty}[1+(-1)^n]\frac{1}{n}\sin\frac{1}{n}$ 收敛

18.

(2)

令 $b_n=\frac{n}{2^n}$

$\because \lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n}=\lim\limits_{n\to\infty}\frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}}=\lim\limits_{n\to\infty}\frac{1}{2}\cdot\frac{n+1}{n}=\frac{1}{2}$

由 D' Alembert 可知

$\therefore \sum\limits_{n=1}^{\infty} b_n$ 是收敛的, 即 $\sum\limits_{n=1}^{\infty}(-1)^{\frac{n(n+1)}{2}}\frac{n}{2^n}$ 是绝对收敛的.

(4)

令 $a_n=\frac{\ln\left(2+\frac{1}{n}\right)}{\sqrt{9n^2-4}}, f(x)=\frac{\ln\left(2+\frac{1}{x}\right)}{\sqrt{9x^2-4}}, x\geqslant 1$

$\therefore f'(x)=\frac{-9x^2(2x+1)\ln\left(\frac{2x+1}{x}\right)-9x^2+4}{x(2x+1)\left(9x^2-4\right)^{\frac{3}{2}}}<0, f(x)\to 0, x\to+\infty$

由莱布尼茨定理可知 $\sum\limits_{n=1}^{\infty}(-1)^{n+1}a_n$ 即 $\sum\limits_{n=1}^{\infty}(-1)^{n+1}\frac{\ln\left(2+\frac{1}{n}\right)}{\sqrt{9n^2-4}}$ 收敛.

$\because \lim\limits_{n\to\infty} n a_n=\lim\limits_{n\to\infty} n\cdot\frac{\ln\left(2+\frac{1}{n}\right)}{\sqrt{9n^2-4}}=\frac{\ln 2}{3}$

由阶估法可知

$1 \leqslant 1, \sum_{n=1}^{\infty} a_n$ 发散, 即 $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln(2+\frac{1}{n})}{9n^2-4}$ 条件收敛.

7.1 (B)

1.

$\because \frac{a_{n+1}}{a_n} \leqslant \frac{b_{n+1}}{b_n}$

$\therefore \frac{a_{n+1}}{b_{n+1}} \leqslant \frac{a_n}{b_n}$

$\therefore \frac{a_{n+1}}{b_{n+1}} \leqslant \frac{a_n}{b_n} \leqslant \cdots \leqslant \frac{a_1}{b_1}$

$\therefore a_{n+1} \leqslant \frac{a_1}{b_1} b_{n+1}$

由比较判别法可知, 因为 $\sum_{n=1}^{\infty} b_n$ 是收敛的

$\therefore \sum_{n=1}^{\infty} a_n$ 也是收敛的.

2.

(1)

$\because \lim_{n\to\infty}\frac{-\ln a_n}{\ln n} = \lim_{n\to\infty}\frac{\ln a_n}{\ln\frac{1}{n}} = q$

$\forall \varepsilon > 0, \exists N > 0, n > N$, 有

$\therefore -\varepsilon < \frac{\ln a_n}{\ln\frac{1}{n}} - q < \varepsilon$

$\therefore q-\varepsilon < \frac{\ln a_n}{\ln\frac{1}{n}} < q+\varepsilon$

$\therefore (q+\varepsilon)\ln\frac{1}{n} < \ln a_n < (q-\varepsilon)\ln\frac{1}{n}$

$\therefore \ln a_n - \ln\frac{1}{n^{q-\varepsilon}} = \ln\frac{a_n}{\frac{1}{n^{q-\varepsilon}}} < 0$

$\therefore 0 < \dfrac{a_n}{\frac{1}{n^{q-\varepsilon}}} = n^{q-\varepsilon} a_n < 1$

令 $\varepsilon = \frac{q-1}{2} > 0$, 即 $q - \varepsilon = \frac{q+1}{2}$

$\therefore 0 < n^{\frac{q+1}{2}} a_n < 1$

$\therefore 0 < a_n < \dfrac{1}{n^{\frac{q+1}{2}}}$

由 $\dfrac{1}{n^{\frac{q+1}{2}}}$ 收敛可知, $\displaystyle\sum_{n=1}^{\infty} a_n$ 收敛.

(2)

同 (1), 可推出

$\forall \varepsilon > 0, \exists N > 0, n > N$, 有

$\therefore q - \varepsilon < \dfrac{\ln a_n}{\ln \frac{1}{n}} < q + \varepsilon$

令 $\varepsilon = 1 - q > 0$, 即 $q + \varepsilon = 1$

$\therefore \dfrac{\ln a_n}{\ln \frac{1}{n}} < 1$

$\therefore \ln a_n > \ln \dfrac{1}{n}$

$\therefore a_n > \dfrac{1}{n}$

由 $\dfrac{1}{n}$ 发散可知, $\displaystyle\sum_{n=1}^{\infty} a_n$ 发散.

3.

$\because f(x)$ 在 $x = 0$ 某一领域有二阶连续导数, 且 $\displaystyle\lim_{n \to \infty} \frac{f(x)}{x} = 0$

$\therefore f(0) = 0, f'(0) = 0,$

由 Taylor 展开可得

$\therefore f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2}\xi^2 = \dfrac{f''(\xi)}{2}x^2$, 其中 $0 < \xi < x$

$\because f''(x)$ 在 $x = 0$ 的某一邻域连续

$\therefore f''(\xi)$ 有界, 不妨令 $|f''(\xi)| \leqslant M$

$\therefore f(\frac{1}{n}) = \frac{f''(\xi)}{2} \cdot \frac{1}{n^2} \leqslant \frac{M}{2} \cdot \frac{1}{n^2}$

$\because \sum\limits_{n=1}^{\infty} \frac{M}{2} \cdot \frac{1}{n^2}$ 是收敛的

$\therefore \sum\limits_{n=1}^{\infty} f\left(\frac{1}{n}\right)$ 是收敛的

4.

(3)

令 $a_n = \left(\frac{\alpha^n}{n+1}\right)^n$, 使用 Cauchy 根值法

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \frac{\alpha^n}{n+1} = \begin{cases} +\infty, & \alpha > 1 \\ 0, & \alpha \leqslant 1 \end{cases}$$

当 $\alpha \leqslant 1$, 即 $\frac{\alpha^n}{n+1} < 1$ 时, 有 $\sum\limits_{n=1}^{\infty} \left(\frac{\alpha^n}{n+1}\right)^n$ 收敛

当 $\alpha > 1$, 即 $\frac{\alpha^n}{n+1} > 1$ 时, 有 $\sum\limits_{n=1}^{\infty} \left(\frac{\alpha^n}{n+1}\right)^n$ 发散

(4)

$\because \tan(\sqrt{n^2+1}\pi) = \tan((\sqrt{n^2+1}-n)\pi) = \tan\frac{\pi}{\sqrt{n^2+1}+n}$

$\therefore \lim\limits_{n\to\infty} n \cdot \tan\frac{\pi}{\sqrt{n^2+1}+n} = \lim\limits_{n\to\infty} n \cdot \frac{\pi}{\sqrt{n^2+1}+n} = \frac{\pi}{2}$

由阶估法可知

$1 \leqslant 1, \sum\limits_{n=1}^{\infty} \tan(\sqrt{n^2+1}\pi)$ 发散.