概率统计第九次作业

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第九次作业: 117页: 28,30,32,34,35; 86页:13,14,15; 87页:20; 88页:29.

28.

$E(X) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} xf(x,y)\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \rho\cos\theta \cdot \rho \mathrm{d}\rho = \int_0^{2\pi} \cos\theta \mathrm{d}\theta \int_0^1 \rho^2 \mathrm{d}\rho = 0$

$E(Y) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} yf(x,y)\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \rho\sin\theta \cdot \rho \mathrm{d}\rho = \int_0^{2\pi} \sin\theta \mathrm{d}\theta \int_0^1 \rho^2 \mathrm{d}\rho = 0$

$E(XY) = \int_{-\infty}^{+\infty} \mathrm{d}x \int_{-\infty}^{+\infty} xyf(x,y)\mathrm{d}y = \int_0^{2\pi} \mathrm{d}\theta \int_0^1 \rho^2\sin\theta\cos\theta \cdot \rho \mathrm{d}\rho = \int_0^{2\pi} \sin\theta\cos\theta \mathrm{d}\theta \int_0^1 \rho^3 \mathrm{d}\rho = 0$

因此 $E(XY) = E(X)E(Y)$, 即 X 和 Y 是不相关的.

$f_x(x) = \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}y = \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \frac{1}{\pi}\mathrm{d}y = \frac{2}{\pi}\sqrt{1-x^2}, -1 \leqslant x \leqslant 1$

$f_y(y) = \int_{-\infty}^{+\infty} f(x,y)\mathrm{d}x = \frac{2}{\pi}\sqrt{1-y^2}, -1 \leqslant y \leqslant 1$

显然我们可以看出 $f(x,y) \neq f_x(x)f_y(y)$, 即 X 和 Y 不是互相独立的.

30.

$\because \rho_{XY} = 0$

$\therefore E(XY) = E(X)E(Y)$

$\because E(X)=P(A), E(Y)=P(B), E(XY)=P(AB)$

$\therefore P(AB)=P(A)P(B)$

$\therefore A, B$ 是相互独立的

$\therefore X, Y$ 是相互独立的

32.

$$E(X)=\int_0^2 \mathrm{d}x \int_0^2 x\cdot\frac{1}{8}(x+y)\mathrm{d}y=\int_0^2 \frac{x\,(x+1)}{4}\mathrm{d}x=\frac{7}{6}$$

$$E(Y)=\frac{7}{6}$$

$$E(XY)=\int_0^2 \mathrm{d}x \int_0^2 xy\cdot\frac{1}{8}(x+y)\mathrm{d}y=\int_0^2 \frac{x\,(3x+4)}{12}\mathrm{d}x=\frac{4}{3}$$

$$E(X^2)=\int_0^2 \mathrm{d}x \int_0^2 x^2\cdot\frac{1}{8}(x+y)\mathrm{d}y=\int_0^2 \frac{x^2\,(x+1)}{4}\mathrm{d}x=\frac{5}{3}$$

$$E(Y^2)=\frac{5}{3}$$

$$\operatorname{Cov}(X,Y)=E(XY)-E(X)E(Y)=\frac{4}{3}-\frac{7}{6}\times\frac{7}{6}=-\frac{1}{36}$$

$$\rho_{XY}=\frac{\operatorname{Cov}(X,Y)}{\sqrt{\operatorname{Var}(X)}\sqrt{\operatorname{Var}(Y)}}=\frac{-\frac{1}{36}}{(\sqrt{\frac{5}{3}-\frac{7}{6}\times\frac{7}{6}})^2}=-\frac{1}{11}$$

$$D(X+Y)=\operatorname{Var}(X)+\operatorname{Var}(Y)+2\operatorname{Cov}(X,Y)=2\times(\frac{5}{3}-\frac{7}{6}\times\frac{7}{6})+2\times(-\frac{1}{36})=\frac{5}{9}$$

34.

(1)

$\because D(X)=E(X^2)-E(X)^2=E(X^2)=4$

$\quad D(Y)=E(Y^2)-E(Y)^2=E(Y^2)=16$

$$\rho_{XY}=\frac{\operatorname{Cov}(X,Y)}{\sqrt{D(X)}\sqrt{D(Y)}}=\frac{E(XY)-E(X)E(Y)}{\sqrt{4}\cdot\sqrt{16}}=\frac{E(XY)}{8}=-0.5$$

$\therefore E(XY)=-4$

$\therefore E(W)=a^2E(X^2)+6aE(XY)+9E(Y^2)=4a^2-24a+144$

$\therefore a=-\dfrac{-24}{2\times 4}=3$ 时有最小值 $E(W)=4\times 3^2-24\times 3+144=108$

(2)

$$\begin{aligned}\therefore \operatorname{Cov}(W,V)&=E(WV)-E(W)E(V)\\&=E[(X-aY)(X+aY)]-E(X-aY)E(X+aY)\\&=E(X^2-a^2Y^2)-(E(X)-aE(Y))(E(X)+aE(Y))\\&=[E(X^2)-E(X)^2]-a^2[E(Y^2)-E(Y)^2]\\&=\sigma_X^2-a^2\sigma_Y^2\\&=0\end{aligned}$$

$\because W,V$ 也是正态分布, 它们不相关.

$\therefore W,V$ 互相独立.

35.

$$\therefore \mu=\begin{pmatrix}\mu_x\\ \mu_x\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},\Sigma=\begin{pmatrix}\sigma_x^2 & \rho_{XY}\sigma_x\sigma_y\\ \rho_{XY}\sigma_x\sigma_y & \sigma_y^2\end{pmatrix}=\begin{pmatrix}3 & -\frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & 4\end{pmatrix}$$

$$\therefore f(x,y)=\\ \frac{1}{2\pi\sqrt{1-(-\frac{1}{4})^2}\sqrt{3}\cdot 2}\exp\left(-\frac{1}{2(1-\frac{1}{16})}\left[\frac{x^2}{3}+\frac{y^2}{4}-\frac{2\cdot(-\frac{1}{4})xy}{2\sqrt{3}}\right]\right)=\\ \frac{1}{3\sqrt{5}\pi}\exp\left(-\frac{8}{15}\left(\frac{x^2}{3}+\frac{y^2}{4}+\frac{\sqrt{3}xy}{12}\right)\right)$$

13.

(1)

$$\because \int_{-1}^{1}\mathrm{d}x\int_{x^2}^{1}cx^2y\mathrm{d}y=c\int_{-1}^{1}x^2(\frac{1}{2}-\frac{x^4}{2})\mathrm{d}x=\frac{4c}{21}=1$$

$$\therefore c = \frac{21}{4}$$

$$\therefore f_Y(y) = \int_{-\sqrt{y}}^{\sqrt{y}} cx^2y\mathrm{d}x = \frac{2}{3}cy^{\frac{5}{2}}$$

$$\therefore f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)} = \frac{x^2y}{\frac{2}{3}y^{\frac{5}{2}}} = \frac{3}{2}x^2y^{-\frac{3}{2}}$$

$$\therefore f_{X|Y}(x|y=\frac{1}{2}) = \frac{3}{2}x^2 \cdot (\frac{1}{2})^{-\frac{3}{2}} = 3\sqrt{2}x^2, x^2 \leqslant \frac{1}{2}$$

(2)

$$\therefore f_X(x) = \int_{x^2}^{1} cx^2y\mathrm{d}y = \frac{1}{2}cx^2(1-x^4)$$

$$\therefore f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{x^2y}{\frac{1}{2}x^2(1-x^4)} = -\frac{2y}{x^4-1}$$

$$\therefore f_{Y|X}(y|x=\frac{1}{3}) = -\frac{2y}{(\frac{1}{3})^4-1} = \frac{81}{40}y, \frac{1}{9} \leqslant y \leqslant 1$$

$$\therefore f_{Y|X}(y|x=\frac{1}{2}) = -\frac{2y}{(\frac{1}{2})^4-1} = \frac{32}{15}y, \frac{1}{4} \leqslant y \leqslant 1$$

(3)

$$\therefore P(Y \geqslant \frac{1}{4}|X=\frac{1}{2}) = \int_{\frac{1}{4}}^{1} \frac{32}{15}y\mathrm{d}y = 1$$

$$\therefore P(Y \geqslant \frac{3}{4}|X=\frac{1}{2}) = \int_{\frac{3}{4}}^{1} \frac{32}{15}y\mathrm{d}y = \frac{7}{15}$$

14.

$$\because f_X(x) = \int_{-x}^{x} 1\mathrm{d}y = 2x$$

$$\therefore f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1}{2x}$$

$$\therefore f_{Y|X}(y|x) = \begin{cases} \frac{1}{2x}, & |y| < x, 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\because f_Y(y)=\begin{cases}\displaystyle\int_y^1 1\mathrm{d}x=1-y, & 0<y<1\\ \displaystyle\int_{-y}^1 1\mathrm{d}x=1+y, & -1<y\leqslant 0\\ 0, & \text{otherwise}\end{cases}$$

$$\therefore f_{X|Y}(x|y)=\frac{f(x,y)}{f_Y(y)}=\begin{cases}\dfrac{1}{1-y}, & 0<y<1, |y|<x, 0<x<1\\ \dfrac{1}{1+y}, & -1<y<0, |y|<x, 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

15.

(1)

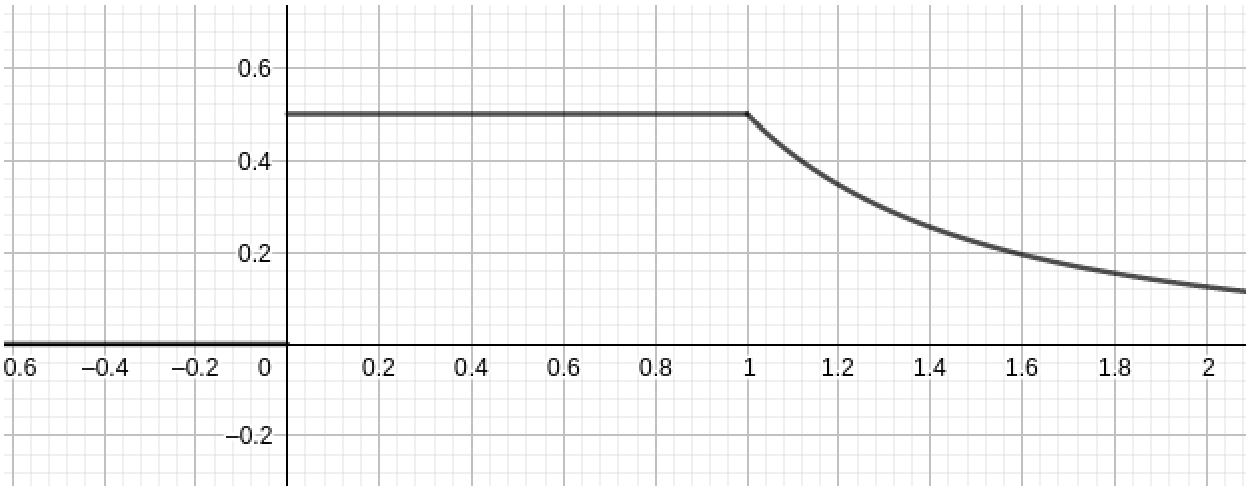
$$\because X\sim U(0,1)$$

$$\therefore f_X(x)=\begin{cases}1, & 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

$$\therefore f(x,y)=f_{Y|X}(y|x)f_X(x)=\begin{cases}x, & 0<y<\dfrac{1}{x}, 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

(2)

$$\therefore f_Y(y)=\begin{cases}\displaystyle\int_0^{\frac{1}{y}} x\mathrm{d}x=\frac{1}{2y^2}, & y\geqslant 1\\ \displaystyle\int_0^1 x\mathrm{d}x=\frac{1}{2}, & 0<y<1\\ 0, & \text{otherwise}\end{cases}$$



(3)

$$P(X > Y) = \int_0^1 \mathrm{d}y \int_y^1 x \mathrm{d}x = \int_0^1 (\frac{1}{2} - \frac{y^2}{2}) \mathrm{d}y = \frac{1}{3}$$

20.

(1)

$$\therefore f(x, y) = f_X(x) f_Y(y) = \begin{cases} \lambda\mu e^{-\lambda x} e^{-\mu y}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore f_{X|Y}(x|y) = \frac{f(x, y)}{f_Y(y)} = \begin{cases} \lambda e^{-\lambda x}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(2)

$$\therefore P(X \leqslant Y) = \int_0^{+\infty} \mathrm{d}y \int_0^y \lambda\mu e^{-\lambda x} e^{-\mu y} \mathrm{d}x = \int_0^{+\infty} \mu e^{-\mu y} (1 - e^{-\lambda y}) \mathrm{d}y = \int_0^{+\infty} \mu e^{-\mu y} \mathrm{d}y - \int_0^{+\infty} \mu e^{-(\mu+\lambda) y} \mathrm{d}y = 1 - \frac{\mu}{\lambda + \mu} = \frac{\lambda}{\lambda + \mu}$$

$$\therefore P(X > Y) = \frac{\mu}{\lambda + \mu}$$

分布律为

Z	0	1
P	$\frac{\mu}{\lambda+\mu}$	$\frac{\lambda}{\lambda+\mu}$

分布函数为

$$F_Z(z)=\begin{cases}0, & z<0\\ \dfrac{\mu}{\lambda+\mu}, & 0\leqslant z<1\\ 1, & z\geqslant 1\end{cases}$$

29.

(1)

$$\because 1=\int_0^1\mathrm{d}x\int_0^{+\infty}be^{-(x+y)}\mathrm{d}y=b\int_0^1e^{-x}\mathrm{d}x\int_0^{+\infty}e^{-y}\mathrm{d}y=b(1-e^{-1})$$

$$\therefore b=\frac{1}{1-e^{-1}}$$

(2)

$$\therefore f_X(x)=\int_0^{+\infty}\frac{1}{1-e^{-1}}e^{-(x+y)}\mathrm{d}y=\frac{e^{-x}}{1-e^{-1}},0<x<1$$

$$f_Y(y)=\int_0^1\frac{1}{1-e^{-1}}e^{-(x+y)}\mathrm{d}x=e^{-y},0<y<+\infty$$

(3)

$$\therefore f(x,y)=f_X(x)f_Y(y)$$

$\therefore X,Y$ 相互独立.

$$\because F_X(x)=\int_0^x\frac{e^{-x}}{1-e^{-1}}\mathrm{d}x=\frac{1}{1-e^{-1}}\int_0^xe^{-x}\mathrm{d}x=\frac{1-e^{-x}}{1-e^{-1}},0<x<1$$

$$F_Y(y)=\int_0^ye^{-y}\mathrm{d}y=1-e^{-y},y>0$$

$$\therefore F_U(u)=F_X(u)F_Y(u)=\frac{(1-e^{-u})^2}{1-e^{-1}},0<u<1$$

$$\therefore F_U(u)=\begin{cases}0, & u\leqslant 0\\ \dfrac{(1-e^{-u})^2}{1-e^{-1}}, & 0<u<1\\ 1-e^{-u}, & u\geqslant 1\end{cases}$$