Assignment 2

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⋆ **This assignment, due on 29th April, contributes to 10% of the total mark of the course.**

Question 1. Some interesting properties of $\mathcal{EL}$

- Show that every $\mathcal{EL}$-concept is satisfiable (regardless of the presence of an $\mathcal{EL}$-TBox). That is, for every $\mathcal{EL}$-concept C there exists an interpretation $\mathcal{I}$ such that $C^{\mathcal{I}} \neq \emptyset$.
- Show that every $\mathcal{EL}$-TBox is consistent. That is, for every $\mathcal{EL}$-TBox $\mathcal{T}$ there exists an interpretation $\mathcal{I}$ such that $\mathcal{I} \models \mathcal{T}$.

Question 2. Reasoning in $\mathcal{EL}$

Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox containing the following (primitive) concept definitions:

$$\mathsf{Bird} \equiv \mathsf{Vertebrate} \sqcap \exists \mathsf{has\_part}.\mathsf{Wing}$$

$$\mathsf{Reptile} \sqsubseteq \mathsf{Vertebrate} \sqcap \exists \mathsf{lays}.\mathsf{Egg}$$

- Compute an $\mathcal{EL}$-TBox $\mathcal{T}'$ in normal form using the pre-processing algorithm given in the lecture.
- Apply the algorithm from the lecture slides deciding whether $A \sqsubseteq_{\mathcal{T}'} B$, where A, B are concept names. Using the normalized TBox $\mathcal{T}'$ as input and explain step-by-step which rules are applied.
- Using the output of the algorithm, decide whether

$$\mathsf{Reptile} \sqsubseteq_{\mathcal{T}'} \mathsf{Vertebrate}$$

$$\mathsf{Vertebrate} \sqsubseteq_{\mathcal{T}'} \mathsf{Bird}$$

Question 3. Bisimulation & bisimulation invariance

In the lecture we defined bisimulation for $\mathcal{ALC}$ and showed bisimulation invariance of $\mathcal{ALC}$ (Theorem 3.2).

- Define a notion of "$\mathcal{ALCN}$-bisimulation" that is appropriate for $\mathcal{ALCN}$ in the sense that bisimilar elements satisfy the same $\mathcal{ALCN}$-concepts.
- Use the definition to show that $\mathcal{ALCQ}$ is more expressive than $\mathcal{ALCN}$.

Question 4. Closure under Disjoint Union

Recall Theorem 3.8 from the lecture, which says that the disjoint union of a family of models of an $\mathcal{ALC}$-TBox $\mathcal{T}$ is again a model of $\mathcal{T}$. Note that the disjoint union is only defined for concept and role names.

- Extend the notion of disjoint union to individual names such that the following holds: for any family $(\mathcal{I}_\nu)_{\nu \in \Omega}$ of models of an $\mathcal{ALC}$-knowledge base $\mathcal{K}$, the disjoint union $\biguplus_{\nu \in \Omega} \mathcal{I}_\nu$ is also a model of $\mathcal{K}$.

Question 5. Closure under Disjoint Union
Let $\mathcal{K} = \{\mathcal{T}, \mathcal{A}\}$ be a consistent $\mathcal{ALC}$-KB. We write $C \sqsubseteq_{\mathcal{K}} D$ if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$ holds for every model $\mathcal{I}$ of $\mathcal{K}$.

- Prove that for all $\mathcal{ALC}$-concepts C and D we have $C \sqsubseteq_{\mathcal{K}} D$ iff $C \sqsubseteq_{\mathcal{T}} D$. Hint: Use the modified definition of disjoint union from the previous exercise.

Question 6. Finite model property
Let C be an $\mathcal{ALC}$-concept that is satisfiable w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$. Show truth or falsity of the following statement:

- for all $m \geq 1$ there is a finite model $\mathcal{I}_m$ of $\mathcal{T}$ such that $|C^{\mathcal{I}_m}| \geq m$.
- Does it hold if the condition "$|C^{\mathcal{I}_m}| \geq m$" is replaced by "$|C^{\mathcal{I}_m}| = m$"?

Question 7. Bisimulation over filtration
Let C be an $\mathcal{ALC}$-concept, $\mathcal{T}$ an $\mathcal{ALC}$-TBox, $\mathcal{I}$ an interpretation and $\mathcal{J}$ its filtration w.r.t. $\mathsf{sub}(C) \cup \mathsf{sub}(\mathcal{T})$ (see Definition 3.14 for the definition of filtration). Show truth or falsity of the following statement:

- the relation $\rho = \{(\mathsf{d}, [\mathsf{d}]) \mid \mathsf{d} \in \Delta^{\mathcal{I}}\}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.

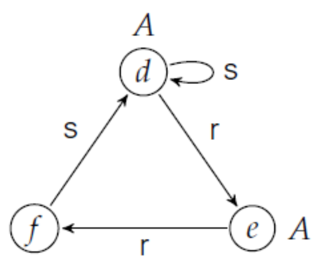
Question 8. Bisimulation within the same interpretation
We define "*bisimulations on* $\mathcal{I}$" as bisimulations between an interpretation $\mathcal{I}$ and itself. Let $\mathsf{d}, \mathsf{e} \in \Delta^{\mathcal{I}}$ be two elements. We write $\mathsf{d} \approx_{\mathcal{I}} \mathsf{e}$ if they are bisimilar, i.e., if there is a bisimulation ρ on $\mathcal{I}$ such that $\mathsf{d}\ \rho\ \mathsf{e}$.

- Show that $\approx_{\mathcal{I}}$ is a bisimulation on $\mathcal{I}$.

Consider the interpretation $\mathcal{J}$ defined like the filtration, but with $\approx_{\mathcal{I}}$ instead of $\simeq$.

- Show that $\rho = \{(\mathsf{d}, [\mathsf{d}]_{\approx_{\mathcal{I}}})\} \mid \mathsf{d} \in \Delta^{\mathcal{I}}$ is a bisimulation between $\mathcal{I}$ and $\mathcal{J}$.
- Show that, if $\mathcal{I}$ is a model of an $\mathcal{ALC}$-concept C w.r.t. an $\mathcal{ALC}$-TBox $\mathcal{T}$, then so is $\mathcal{J}$.

Question 9. Unravelling
Draw the unravelling of the following interpretation $\mathcal{I}$ at d up to depth 5, i.e., restricted to d-paths of length at most 5 (see Definition 3.21):



Question 10. Tree model property

- Show the truth or falsity of the following statement: if $\mathcal{K}$ is an $\mathcal{ALC}$-KB and C an $\mathcal{ALC}$-concept such that C is satisfiable w.r.t. $\mathcal{K}$, then C has a tree model w.r.t. $\mathcal{K}$.

Question 11. Tableau algorithm

- Apply the Tableau algorithm consistent$(\mathcal{A})$ to the following ABox:

$$\mathcal{A} = \{(b,a):r, (a,b):r, (a,c):s, (c,b):s, a:\exists s.A, b:\forall r.((\forall s.\neg A)\sqcup(\exists r.B)), c:\forall s.(B\sqcap(\forall s.\bot))\}.$$

If $\mathcal{A}$ is consistent, draw the model generated by the algorithm.

Question 12. Extension of Tableau algorithm

We consider the concept constructor $\rightarrow$ (implication) with the following semantics:

$$(C \rightarrow D)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid x \in C^{\mathcal{I}} \text{ implies } x \in D^{\mathcal{I}}\}.$$

To extend consistent($\mathcal{A}$) to this constructor, we propose two alternatives new expansion rules:

The deterministic $\rightarrow$-rule	
Condition:	$\mathcal{A}$ contains $a : C \rightarrow D$ and $a : C$, but not $a : D$
Action:	$\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : D\}$

The nondeterministic $\rightarrow$-rule	
Condition:	$\mathcal{A}$ contains $a : C \rightarrow D$, but neither $a : \dot{\neg} C$ nor $a : D$
Action:	$\mathcal{A} \longrightarrow \mathcal{A} \cup \{a : X\}$ for some $X \in \{\dot{\neg} C, D\}$

For each rule, determine whether the extended algorithm remains terminating, sound, and complete.

Question 13. Modification of Tableau algorithm

We consider an $\mathcal{ALC}$ TBox $\mathcal{T}$ consisting only of the following two kinds of axioms:

- role inclusions of the form $r \sqsubseteq s$, and
- role disjointness constraints of the form disjoint(r, s).

where r and s are role names. An interpretation $\mathcal{I}$ satisfies these axioms if

- $r^{\mathcal{I}} \subseteq s^{\mathcal{I}}$, and
- $r^{\mathcal{I}} \cap s^{\mathcal{I}} = \emptyset$, respectively.

Modify the Tableau algorithm consistent($\mathcal{A}$) to decide consistency of $(\mathcal{T}, \mathcal{A})$, where $\mathcal{A}$ is an ABox and $\mathcal{T}$ an TBox containing only role inclusions and role disjointness constraints. Show that the algorithm remains terminating, sound, and complete.