Assignment 3

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⋆ **This assignment, due on 20th May, contributes to 10% of the total mark of the course. This time, up to 3 bonus marks are awardable, in the case of the successful completion of the last three questions. By awarding students these "blessings", an increase of their final marks may occur but will be capped at the mark total, which is 100 in this course.**

Question 1. Termination

Let E be an $\mathcal{ALC}$-concept. By $\#E$ we denote the number of occurrences of the constructors $\sqcap$, $\sqcup$, $\exists$, $\forall$ in E. The multiset $M(E)$ contains, for each occurrence of a subconcept of the form $\neg F$ in E, the number $\#F$.

- Following this representation, prove that exhaustively applying the transformations below to an $\mathcal{ALC}$ concept always terminates, regardless of the order of rule application:

$$\begin{aligned}
\neg(E \sqcap F) &\to \neg\neg\neg E \sqcup \neg\neg\neg F \\
\neg(E \sqcup F) &\to \neg\neg\neg E \sqcap \neg\neg\neg F \\
\neg\neg E &\to E \\
\neg(\exists r.E) &\to \forall r.\neg E \\
\neg(\forall r.E) &\to \exists r.\neg E
\end{aligned}$$

Question 2. Normal form

Let $\mathcal{T}$ be an acyclic TBox in NNF. $\mathcal{T}^{\sqsubseteq}$ is obtained from $\mathcal{T}$ by replacing each concept definition $A \equiv C$ with the concept inclusion $A \sqsubseteq C$.

- Prove that every concept name is satisfiable w.r.t. $\mathcal{T}$ iff it is satisfiable w.r.t. $\mathcal{T}^{\sqsubseteq}$. Does this also hold for the acyclic TBox $\{A \equiv C \sqcap \neg B, B \equiv P, C \equiv P\}$?

Question 3. $\mathcal{ALC}$-Worlds algorithm

Use the $\mathcal{ALC}$-Worlds algorithm taught in lectures to decide the satisfiability of the concept name B_0 w.r.t. the following simple TBox $\mathcal{T}$:

$$\begin{aligned}
\{&B_0 \equiv B_1 \sqcap B_2, \quad B_1 \equiv \exists r.B_3, \quad B_2 \equiv B_4 \sqcap B_5, \quad B_3 \equiv P \\
&B_4 \equiv \exists r.B_6, \quad B_5 \equiv B_6 \sqcap B_7, \quad B_6 \equiv Q, \quad B_7 \equiv \forall r.B_4 \\
&B_8 \equiv \forall r.B_9, \quad B_9 \equiv \forall r.B_{10}, \quad B_{10} \equiv \neg P\}
\end{aligned}$$

- Draw the recursion tree of a successful run and of an unsuccessful run. Does the algorithm return a positive or negative result on this input?

Question 4. $\mathcal{ALC}$-Elim algorithm

Use the $\mathcal{ALC}$-Elim algorithm given in the reference book (Page 118) to decide the satisfiability of

- (a) the concept name B w.r.t. $\mathcal{T} := \{B \sqsubseteq \exists r.B, \top \sqsubseteq B, \forall r.B \sqsubseteq \exists r.B\}$
- (b) the concept $\forall r.\forall r.\neg B$ w.r.t. $\mathcal{T} := \{\neg A \sqsubseteq B, A \sqsubseteq \neg B, \top \sqsubseteq \neg \forall r.A\}$.

Give the constructed type sequence Γ_0, Γ_1,.... In case of satisfiability, also give the satisfying model constructed in the proof of Lemma 5.10.

Question 5. Finite Boolean games

Let $\Gamma_1 := \{q_1, q_3\}$ and $\Gamma_2 := \{q_2, q_4\}$. Determine whether Player 1 has a winning strategy in the following finite Boolean games.

- (a) $((q_1 \wedge q_3) \rightarrow \neg q_2) \wedge (\neg q_1 \rightarrow q_1) \wedge (\neg q_2 \rightarrow (q_3 \vee q_4))$
- (b) $(q_1 \vee \neg q_2) \wedge (q_2 \vee q_3) \wedge (\neg q_3 \vee \neg q_4) \wedge (\neg q_1 \vee \neg q_2 \vee q_3 \vee q_4)$

Question 6. Infinite Boolean games

Determine whether Player 2 has a winning strategy in the following infinite Boolean games where the initial configuration t_0 assigns *false* to all variables.

- $\phi := (x_1 \wedge x_2 \wedge \neg y_1) \vee (x_3 \wedge x_4 \wedge \neg y_2) \vee (\neg(x_1 \vee x_4) \wedge y_1 \wedge y_2)$

  provided that: $\Gamma_1 := \{x_1, x_2, x_3, x_4\}$ and $\Gamma_2 := \{y_1, y_2\}$
- $\phi := ((x_1 \leftrightarrow \neg y_1) \wedge (x_2 \leftrightarrow \neg y_2) \wedge (x_1 \wedge y_2)) \vee ((x_1 \leftrightarrow y_1) \wedge (x_2 \leftrightarrow y_2) \wedge (x_1 \leftrightarrow \neg x_2))$

  provided that: $\Gamma_1 := \{x_1, x_2\}$ and $\Gamma_2 := \{y_1, y_2\}$

Question 7. Infinite Boolean games

Are the following variations of infinite Boolean games also EXPTIME-hard?

- (a) Player 1 wins if the constructed truth assignment falsifies the formula ϕ instead of satisfying it.
- (b) Player 2 starts instead of Player 1.
- (c) The variables are not assigned to a specific player; instead, the active player can choose any variable and assign to it a new truth value; variables can be chosen multiple times.
- (d) The two players must always flip the assignment of a variable, i.e., the truth assignment cannot be left unchanged.

Question 8. Complexity of concept satisfiability in $\mathcal{ALC}$ extensions

The universal role is a role u such that its extension is fixed as $\Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$ in any interpretation $\mathcal{I}$. Let $\mathcal{ALC}^u$ be a DL extending $\mathcal{ALC}$ with the universal role.

- Show that concept satisfiability in $\mathcal{ALC}^u$ without TBoxes is EXPTIME-complete.

Question 9. Tree model property in $\mathcal{ALC}$ extensions

A role complement is defined as a role of the form $\neg r$, where r is a role name. The semantics of role complements is defined as follows.

$$(\neg r)^{\mathcal{I}} := \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \backslash r^{\mathcal{I}}$$

The description logic $\mathcal{ALC}^{\neg}$ extends $\mathcal{ALC}$ by role complements, i.e., role complements are allowed to occur in existential restrictions, value restrictions, and role assertions.

- Show that $\mathcal{ALC}^{\neg}$ does not have the tree model property.

~~Question 10~~. A complex in $\mathcal{ALC}$ extensions

The DL $\mathcal{S}$ extends $\mathcal{ALC}$ with *transitivity axioms* $\mathsf{trans}(r)$ for role names $r \in \mathsf{R}$. Their semantics is defined as follows: $\mathcal{I} \models \mathsf{trans}(r)$ iff $r^{\mathcal{I}}$ is transitive. Furthermore, an $\mathcal{S}$ knowledge base $\mathcal{K} := (\mathcal{T}, \mathcal{A}, \mathcal{R})$ consists of an $\mathcal{ALC}$ knowledge base $(\mathcal{T}, \mathcal{A})$, and an RBox $\mathcal{R}$ of transitivity axioms. Prove the following statements:

- (a) For an arbitrary TBox $\mathcal{T}$, the concept $C_{\mathcal{T}}$ is defined as $\bigsqcap_{C \sqsubseteq D \in \mathcal{T}} \neg C \sqcup D$. Then $\mathcal{T}$ and $\mathcal{T}' = \{\top \sqsubseteq C_{\mathcal{T}}\}$ have the same models.

- (b) Let $\mathcal{K} := \{\mathcal{T}, \mathcal{A}, \mathcal{R}\}$ be an knowledge base such that, without loss of generality, $\mathcal{T}$ consists of a single GCI $\top \sqsubseteq C_{\mathcal{T}}$, and $C_{\mathcal{T}}$ is in NNF. Define the $\mathcal{ALC}$ knowledge base $\mathcal{K}^{+} := (\mathcal{T}^{+}, \mathcal{A})$ where

$$\mathcal{T}^{+} := \mathcal{T} \cup \{\forall r.C \sqsubseteq \forall r.\forall r.C \mid \mathsf{trans}(r) \in \mathcal{R} \text{ and } \forall r.C \in \mathsf{Sub}(C_{\mathcal{T}})\}.$$

Then $\mathcal{K}$ is consistent iff $\mathcal{K}^{+}$ is consistent. Consequently, the tableau algorithm for $\mathcal{ALC}$ can also be used for $\mathcal{S}$.

- (c) Deciding the consistency of an $\mathcal{S}$ knowledge base (with a general TBox) is EXPTIME-complete.

Question 11 (with 1 bonus mark). Pushdown automata

Let a k-PDA be a pushdown automaton that has k stacks. Thus a 0-PDA is an NFA and a 1-PDA is a conventional PDA. You already know that 1-PDAs are more powerful (recognize a larger class of languages) than 0-PDAs.

- (a) Show that 2-PDAs are more powerful than 1-PDAs.

- (b) Show that 3-PDAs are *not* more powerful than 2-PDAs.

Question 12 (with 1 bonus mark). Turing machine

- Define a standard TM that decides the following language:

$$L = \{m \in \{0,1\}^* \mid |m|_0 = |m|_1\}$$

where $|m|_0$ and $|m|_1$ denote respectively the number of 0's and 1's in m.

Question 13 (with 1 bonus mark). Decidability

Let A be the language containing only the single string s, where

$$\mathrm{s} = \begin{cases} 0 & \text{if reasonably-priced yummies will never be found in the canteens of NJU} \\ 1 & \text{if reasonably-priced yummies will be found in the canteens of NJU someday} \end{cases}$$

- (a) Is A decidable? Why or why not? For the sake of this question, assume that whether reasonably-priced yummies will be found in the canteens of NJU has an unambiguous YES or NO answer.