第五次作业

习题 5.6: (A) 1.(3), 3, 4(2, 3, 6), 5(3), 6, 10(3), 12, 15, 16(2),19, 20

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1. (3)

$\because F(x,y,z)=x^2+y^2-1=0, G(x,y,z)=y^2+z^2-1=0$

$\therefore F_x=2x, F_y=2y, F_z=0, G_x=0, G_y=2y, G_z=2z$

将 y, z 看作 x 的函数 $y=y(x), z=z(x)$, 对 x 求导可得

$$\begin{cases} F_x\cdot 1+F_y\cdot y'+F_z\cdot z'=0 \\ G_x\cdot 1+G_y\cdot y'+G_z\cdot z'=0 \end{cases}$$

$$\therefore y'=\frac{\begin{vmatrix} -F_x & F_z \\ -G_x & G_z \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}}=-\frac{x}{y}, z'=\frac{\begin{vmatrix} F_y & -F_x \\ G_y & -G_x \end{vmatrix}}{\begin{vmatrix} F_y & F_z \\ G_y & G_z \end{vmatrix}}=\frac{x}{z}$$

切线方向为 $\{1,-\frac{x_0}{y_0},\frac{x_0}{z_0}\}$

切线 $\vec{l}: x-1=-\frac{y_0}{x_0}y=\frac{z_0}{x_0}(z-1)$

法平面 $S: x-1-\frac{x_0}{y_0}y+\frac{z_0}{x_0}(z-1)=0$

3.

切线方向 $\boldsymbol{l}=(x'(\theta),y'(\theta),z'(\theta))=(-a\sin\theta,a\cos\theta,k)$

取 Oz 轴正方向的单位向量 $\boldsymbol{k}=(0,0,1)$

$$\therefore \cos\theta=\frac{\boldsymbol{l}\cdot\boldsymbol{k}}{|\boldsymbol{l}|}=\frac{k}{\sqrt{a^2\sin^2\theta+a^2\cos^2\theta+k^2}}=\frac{k}{\sqrt{a^2+k^2}}$$

$\because \cos\theta$ 恒定

$\therefore$ 螺线 $\boldsymbol{r}$ 任意一点切线与 Oz 轴成定角

4.

(2)

$\therefore x'(t) = 2t\sqrt{1+t^2}, y'(t) = 2t\sqrt{1-t^2}$

$\therefore s = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\mathrm{d}t = \int_0^1 \sqrt{8t^2}\mathrm{d}t = \sqrt{2}$

(3)

令 $x = 0$ 得 $y = \pm a$, 令 $y = 0$ 得 $x = \pm a$.

可以看出, $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ 的全长是四倍的第一象限内的弧长.

$\therefore y(x) = (a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{3}{2}}, y'(x) = x^{-\frac{1}{3}}(a^{\frac{2}{3}} - x^{\frac{2}{3}})^{\frac{1}{2}}$

$$
\begin{aligned}
\therefore s &= 4\int_0^a \sqrt{1 + y'(x)^2}\mathrm{d}x \\
&= 4\int_0^a \sqrt{1 + x^{-\frac{2}{3}}(a^{\frac{2}{3}} - x^{\frac{2}{3}})}\mathrm{d}x \\
&= 4a^{\frac{1}{3}}\int_0^a x^{-\frac{1}{3}}\mathrm{d}x \\
&= 4a^{\frac{1}{3}}(\frac{3}{2}x^{\frac{2}{3}})|_0^a \\
&= 6a
\end{aligned}
$$

所以全长为 $6a$

(6)

$\therefore \rho'(\theta) = -a\sin\theta$

$$
\begin{aligned}
\therefore s &= 2\int_0^{\pi} \sqrt{(-a\sin\theta)^2 + a^2(1+\cos\theta)^2}\mathrm{d}\theta \\
&= 2\sqrt{2}a\int_0^{\pi} \sqrt{1+\cos\theta}\mathrm{d}\theta \\
&= 2\sqrt{2}a\int_0^{\pi} \sqrt{2\cos^2\frac{\theta}{2}}\mathrm{d}\theta \\
&= 8a\int_0^{\frac{\pi}{2}} \cos t\mathrm{d}t \\
&= 8a\sin\frac{\pi}{2} \\
&= 8a
\end{aligned}
$$

$\therefore$ 全长为 $8a$.

5. (3)

$$\therefore y(x) = \frac{x^2}{3}, z(x) = \frac{2xy}{9} = \frac{2x^3}{27}$$

$$\therefore y'(x) = \frac{2x}{3}, z'(x) = \frac{2x^2}{9}$$

$$
\begin{aligned}
\therefore s &= \int_0^3 \sqrt{1 + (\frac{2x}{3})^2 + (\frac{2x^2}{9})^2}\mathrm{d}x \\
&= \int_0^3 \sqrt{(\frac{2x^2}{9}+1)^2}\mathrm{d}x \\
&= \frac{2}{9}\int_0^3 x^2\mathrm{d}x + \int_0^3 \mathrm{d}x \\
&= \frac{2}{9}(\frac{1}{3}x^3)|_0^3 + 3 \\
&= 5
\end{aligned}
$$

6.

圆锥面 $x^2+y^2=z^2$ 的母线为面上一点与原点的连线

在圆锥面上取 $x = ae^t\cos t, y = ae^t\sin t$, 则有 $z = ae^t$, 与曲线 $\boldsymbol{r}$ 相交

此时母线的切线是母线自身, 母线方向即 $\vec{l} = (\cos t, \sin t, 1)$

对曲线 $\boldsymbol{r}$ 的切线方向 $\vec{r} = (ae^t\cos t - ae^t\sin t, ae^t\sin t + ae^t\cos t, ae^t)$

可以去掉 ae^t 得 $\vec{r} = (\cos t - \sin t, \sin t + \cos t, 1)$

$\therefore \cos\theta = \dfrac{\vec{l}\cdot\vec{r}}{|\vec{l}||\vec{r}|} = \dfrac{\cos^2 t - \sin t\cos t + \sin^2 t + \sin t\cos t + 1}{\sqrt{2}\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2 + 1}} = \dfrac{\sqrt{6}}{3}$

$\therefore \cos\theta$ 恒定, 相交角度相同

10. (3)

令 $F(x,y,z) = x^3 + y^3 + z^3 + xyz - 6$

$\therefore F_x = 3x^2 + yz, F_y = 3y^2 + xz, F_z = 3z^2 + xy$

带入点 $(1,2,-1)$ 得

$\therefore F_x = 3 - 2 = 1, F_y = 3\times 2^2 - 1 = 11, F_z = 3 + 2 = 5$

法平面 $F_x(x-x_0) + F_y(y-y_0) + F_z(z-z_0) = 0$

即 $x + 11y + 5z - 18 = 0$

法线 $x - 1 = \dfrac{y-2}{11} = \dfrac{z+1}{5}$

12.

(1)

令 $F(x,y,z) = x^2 - x + y^2 + z^2$

$\therefore F_x = 2x - 1, F_y = 2y, F_z = 2z$

我们可知切平面的法线方向为 $\boldsymbol{n}_0 = (2x-1, 2y, 2z)$

同理可有平面 $x - y - \frac{1}{2}z = 2$ 的法线 $\boldsymbol{n}_1 = (1,-1,-\frac{1}{2})$

平面 $x - y - z = 2$ 的法线 $\boldsymbol{n}_1 = (1,-1,-1)$

要使切平面垂直于这两个平面

$$\therefore \begin{cases} \boldsymbol{n}_0\cdot\boldsymbol{n}_1 = 2x - 1 - 2y - z = 0 \\ \boldsymbol{n}_0\cdot\boldsymbol{n}_2 = 2x - 1 - 2y - 2z = 0 \\ x^2 + y^2 + z^2 = x \end{cases}$$

$$\therefore \begin{cases} x = \frac{1}{2} \pm \frac{\sqrt{2}}{4} \\ y = \pm\frac{\sqrt{2}}{4} \\ z = 0 \end{cases}$$

$\therefore \boldsymbol{n}_0 = (\pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{2}}{2}, 0)$

切平面为 $x + y - \frac{1}{2} - \frac{\sqrt{2}}{2} = 0$ 和 $x + y - \frac{1}{2} + \frac{\sqrt{2}}{2} = 0$

(2)

令 $F(x, y, z) = 3x^2 + y^2 - z^2 - 27$

$\therefore F_x = 6x, F_y = 2y, F_z = -2z$

我们可知切平面的法线方向为 $\boldsymbol{n}_0 = (6x, 2y, -2z)$

联解 $\begin{cases} 10x + 2y - 2z = 27 \\ x + y - z = 0 \\ 3x^2 + y^2 - z^2 = 27 \end{cases}$

解得 $\begin{cases} x = \frac{27}{8} \\ y = \frac{5}{8} \\ z = 4 \end{cases}$

切平面方程为 $6 \cdot \frac{27}{8}(x - \frac{27}{8}) + 2 \cdot \frac{5}{8}(y - \frac{5}{8}) + 4(z - 4) = 0$

即 $81(x - \frac{27}{8}) + 10(y - \frac{5}{8}) + 16(z - 4) = 0$

15.

因为任意一点 $M(x, y, z)$ 所对应的 $M'(\pm\sqrt{x^2 + z^2}, y, 0)$

曲面方程为 $3(\pm\sqrt{x^2 + z^2})^2 + 2y^2 = 12$

即 $3x^2 + 2y^2 + 3z^2 = 12$

令 $F(x, y, z) = 3x^2 + 2y^2 + 3z^2 - 12$

$\therefore F_x = 6x = 0, F_y = 4y = 4\sqrt{3}, F_z = 6z = 6\sqrt{2}$

由内部指向外部的法向量为 $\boldsymbol{n} = (0, 2\sqrt{3}, 3\sqrt{2})$

16. (2)

切向量 $\boldsymbol{l}' = (1, 4t, -8t^3) = (1, 4, -8)$

归一化得 $\boldsymbol{l} = (\frac{1}{9}, \frac{4}{9}, -\frac{8}{9})$

$$\because \frac{\partial u}{\partial x}=\frac{\sqrt{x^2+y^2+z^2}-\frac{x^2}{\sqrt{x^2+y^2+z^2}}}{x^2+y^2+z^2}=\frac{8}{27}$$

$$\frac{\partial u}{\partial y}=-xy(x^2+y^2+z^2)^{-\frac{3}{2}}=-\frac{2}{27}$$

$$\frac{\partial u}{\partial z}=-xz(x^2+y^2+z^2)^{-\frac{3}{2}}=\frac{2}{27}$$

$$\therefore \frac{\partial u}{\partial \boldsymbol{l}}=\frac{1}{9}\times\frac{8}{27}+\frac{4}{9}\times(-\frac{2}{27})+(-\frac{8}{9})\times\frac{2}{27}=-\frac{16}{243}$$

19.

$$\therefore \frac{\partial F}{\partial x}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=\frac{F_u}{z-c}$$

$$\frac{\partial F}{\partial y}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=\frac{F_v}{z-c}$$

$$\frac{\partial F}{\partial z}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial z}=-\frac{F_u(x-a)}{(z-c)^2}-\frac{F_v(y-b)}{(z-c)^2}$$

切平面为

$$\therefore \frac{F_u}{z_0-c}(x-x_0)+\frac{F_v}{z_0-c}(y-y_0)-[\frac{F_u(x_0-a)}{(z_0-c)^2}+\frac{F_v(y_0-b)}{(z_0-c)^2}](z-z_0)=0$$

$$\therefore (z_0-c)F_u(x-x_0)+(z_0-c)F_v(y-y_0)-[(x_0-a)F_u+(y_0-b)F_v](z-z_0)=0$$

$$\therefore [(z_0-c)(x-x_0)-(x_0-a)(z-z_0)]F_u+[(z_0-c)(y-y_0)-(y_0-b)(z-z_0)]F_v=0$$

其中 x_0,y_0,z_0 可以取到曲面上任意一点.

带入点 (a,b,c) 可得切平面方程恒等于 0,

说明曲面 F 上任意一点的切平面均经过定点 (a,b,c).

20.

令 $u=x-az, v=y-bz$

$$\therefore \frac{\partial F}{\partial x}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=F_u$$

$$\frac{\partial F}{\partial y}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=F_v$$

$$\frac{\partial F}{\partial z}=\frac{\partial F}{\partial u}\frac{\partial u}{\partial z}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial z}=-aF_u-bF_v$$

切平面法向量为 $\boldsymbol{n}=(F_u,F_v,-aF_u-bF_v)$

令直线方向向量为 $\boldsymbol{l}=(a,b,1)$

$\therefore \boldsymbol{n}\cdot\boldsymbol{l}=aF_u+bF_v-aF_u-bF_v=0$

$\therefore$ 任意一点切平面平行于向量 $\boldsymbol{l}=(a,b,1)$

$\therefore$ 法平面平行于直线 $\dfrac{x}{a}=\dfrac{y}{b}=z$