概率统计第八次作业

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1.

设 $Z_1 = X + Y$, $Z_2 = XY$

由卷积公式可知

$$f_{z_1}(z) = \int_{-\infty}^{+\infty} f_x(x) f_y(z-x)\mathrm{d}x = \int_0^z \lambda_1 e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2(z-x)}\mathrm{d}x = \int_0^z \lambda_1\lambda_2 e^{-\lambda_1 x - \lambda_2(z-x)}\mathrm{d}x = \lambda_1\lambda_2 e^{-\lambda_2 z}\int_0^z e^{-(\lambda_1-\lambda_2)x}\mathrm{d}x = \begin{cases} \dfrac{\lambda_1\lambda_2\left(e^{-\lambda_2 z} - e^{-\lambda_1 z}\right)}{\lambda_1 - \lambda_2}, & \lambda_1 \neq \lambda_2 \\ \lambda_1\lambda_2 z e^{-\lambda_2 z}, & \lambda_1 = \lambda_2 \end{cases}$$

$$f_{z_2}(z) = \int_{-\infty}^{+\infty} \frac{1}{x} \cdot f(x, \frac{z}{x})\mathrm{d}x = \lambda_1\lambda_2 \int_0^{+\infty} \frac{1}{x} e^{-\lambda_1 x - \lambda_2 \frac{z}{x}}\mathrm{d}x$$

9.

(1)

$$E(X) = \int_0^1 x\mathrm{d}x \int_0^x 12y^2\mathrm{d}y = \int_0^1 4x^4\mathrm{d}x = \frac{4}{5}$$

$$E(Y) = \int_0^1 y\mathrm{d}y \int_y^1 12y^2\mathrm{d}x = \int_0^1 12y^3(1-y)\mathrm{d}y = \frac{3}{5}$$

$$E(XY) = \int_0^1 x\mathrm{d}x \int_0^x 12y^3\mathrm{d}y = \int_0^1 3x^5\mathrm{d}x = \frac{1}{2}$$

$$E(X^2+Y^2) = \int_0^1 \mathrm{d}x \int_0^x 12y^2(x^2+y^2)\mathrm{d}y = \int_0^1 \frac{32x^5}{5}\mathrm{d}x = \frac{16}{15}$$

(2)

$$E(X)=\int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}\frac{x}{y}e^{-(y+\frac{x}{y})}\mathrm{d}x=\int_0^{+\infty}ye^{-y}\mathrm{d}y\int_0^{+\infty}xe^{-x}\mathrm{d}x=\int_0^{+\infty}ye^{-y}\mathrm{d}y=1$$

$$E(Y)=\int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}e^{-(y+\frac{x}{y})}\mathrm{d}x=\int_0^{+\infty}e^{-y}\mathrm{d}y\int_0^{+\infty}e^{-\frac{x}{y}}\mathrm{d}y=\int_0^{+\infty}ye^{-y}\mathrm{d}y=1$$

$$E(XY)=\int_0^{+\infty}\mathrm{d}y\int_0^{+\infty}xy\cdot\frac{1}{y}e^{-(y+\frac{x}{y})}\mathrm{d}x=\int_0^{+\infty}e^{-y}\mathrm{d}y\int_0^{+\infty}xe^{-\frac{x}{y}}\mathrm{d}x=\int_0^{+\infty}y^2e^{-y}\mathrm{d}y=2$$

10.

(1)

$$E(\frac{X^2}{X^2+Y^2})=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{x^2}{x^2+y^2}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\cdot\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\mathrm{d}x\mathrm{d}y=\frac{1}{4\pi}\int_0^{2\pi}\cos^2\theta\mathrm{d}\theta\int_0^{+\infty}e^{-\frac{\rho^2}{2}}\mathrm{d}\rho^2=\frac{1}{2}$$

(2)

$$E(\sqrt{X^2+Y^2})=\frac{1}{2\pi\sigma^2}\int_0^{2\pi}\mathrm{d}\theta\int_0^{+\infty}\rho\cdot e^{-\frac{\rho^2}{2\sigma^2}}\cdot\rho\mathrm{d}\rho=\frac{1}{2\pi\sigma^2}\cdot 2\pi\cdot\int_0^{+\infty}\rho^2e^{-\frac{\rho^2}{2\sigma^2}}\mathrm{d}\rho=-\int_0^{+\infty}\rho\mathrm{d}e^{-\frac{\rho^2}{2\sigma^2}}=-\rho e^{-\frac{\rho^2}{2\sigma^2}}|_0^{+\infty}+\int_0^{+\infty}e^{-\frac{\rho^2}{2\sigma^2}}\mathrm{d}\rho=\frac{1}{2}\int_{-\infty}^{+\infty}e^{-\frac{\rho^2}{2\sigma^2}}\mathrm{d}\rho=\frac{1}{2}\cdot 1\cdot\sqrt{2\pi}\sigma=\frac{\sqrt{2\pi}\sigma}{2}$$

14.

(1)

$$E(X_1)=\int_0^{+\infty}x\cdot 2e^{-2x}\mathrm{d}x=-\int_0^{+\infty}x\mathrm{d}e^{-2x}=-xe^{-2x}|_0^{+\infty}+\int_0^{+\infty}e^{-2x}\mathrm{d}x=\frac{1}{2}$$

$$E(X_2)=\int_0^{+\infty}x\cdot 4e^{-4x}\mathrm{d}x=\frac{1}{4}$$

$$E(X_2^2)=\int_0^{+\infty}x^2\cdot 4e^{-4x}\mathrm{d}x=-\int_0^{+\infty}x^2\mathrm{d}e^{-4x}=\int_0^{+\infty}2xe^{-4x}\mathrm{d}x=\frac{1}{2}\int_0^{+\infty}4xe^{-4x}\mathrm{d}x=\frac{1}{8}$$

$$\therefore E(X_1+X_2)=E(X_1)+E(X_2)=\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$

$$E(2X_1-3X_2^2)=2E(X_1)-3E(X_2^2)=1-\frac{3}{8}=\frac{5}{8}$$

(2)

$\because X_1,X_2$ 相互独立

$$\therefore E(X_1X_2)=E(X_1)E(X_2)=\frac{1}{2}\cdot\frac{1}{4}=\frac{1}{8}$$

16.

(1)

$$P(X=k)=\frac{n-1}{n}\cdot\frac{n-2}{n-1}\cdot\cdots\cdot\frac{n-k+1}{n-k+2}\cdot\frac{1}{n-k+1}=\frac{1}{n}$$

所以分布律为 $P(X=k)=\dfrac{1}{n}$

$$\therefore E(X)=\sum_{k=1}^{n}k\cdot\frac{1}{n}=\frac{n+1}{2}$$

(2)

$$E(X)=\sum_{k=1}^{n}P(X\geqslant k)=\sum_{k=1}^{n}\frac{n-k+1}{n}=\frac{n+1}{2}$$

22.

(1)

$$E(Y)=2E(X_1)-E(X_2)+3E(X_3)-\frac{1}{2}E(X_4)=2-2+3\times 3-\frac{1}{2}\times 4=7$$

$D(Y)=4D(X_1)-D(X_2)+9D(X_3)-\frac{1}{4}D(X_4)=4\times(5-1)-(5-2)+9\times(5-3)-\frac{1}{4}\times(5-4)=\frac{123}{4}$

(2)

因为 X,Y 为正态分布且相互独立, 因此 $Z_1=2X+Y$ 和 $Z_2=X-Y$ 也都是正态分布.

$E(Z_1)=2E(X)+E(Y)=2\times720+640=2080$

$D(Z_1)=4D(X)+D(Y)=4\times30^2+25^2=4225$

$E(Z_2)=E(X)-E(Y)=720-640=80$

$D(Z_2)=D(X)+D(Y)=30^2+25^2=1525$

因此 $Z_1\sim N(2080,4225)$, $Z_2\sim N(80,1525)$

$E(Z_3)=E(X)+E(Y)=720+640=1360$

$D(Z_3)=D(X)+D(Y)=30^2+25^2=1525$

$\therefore P(X>Y)=P(X-Y>0)=1-P(Z_2\leqslant0)=1-P(\frac{Z_2-80}{\sqrt{1525}}\leqslant\frac{0-80}{\sqrt{1525}})=1-\Phi(\frac{-80}{\sqrt{1525}})=\Phi(\frac{80}{\sqrt{1525}})$

$\therefore P(X+Y)=1-P(Z_3\leqslant1400)=1-P(\frac{Z_3-1360}{\sqrt{1525}}\leqslant\frac{1400-1360}{\sqrt{1525}})=1-\Phi(\frac{40}{\sqrt{1525}})$

25.

(1)

$E(XY)=\int_0^1\mathrm{d}x\int_0^1 xy\mathrm{d}y=\frac{1}{4}$

$E(X/Y)=\int_0^1\mathrm{d}x\int_0^1\frac{x}{y}\mathrm{d}y=\int_0^1 x\mathrm{d}x\int_0^1\frac{1}{y}\mathrm{d}y=\frac{1}{2}\int_0^1\frac{1}{y}\mathrm{d}y$ 发散

$E(\ln(XY))=\int_0^1\mathrm{d}x\int_0^1\ln(xy)\mathrm{d}y=\int_0^1\ln(x)\mathrm{d}x\int_0^1\ln(y)\mathrm{d}y=1$

$$E(|Y-X|)=\int_0^1 \mathrm{d}x\int_0^1 |y-x|\mathrm{d}y=2\int_0^1 \mathrm{d}x\int_x^1 (y-x)\mathrm{d}y=\frac{1}{3}$$

(2)

$$A=XY, C=2X+2Y$$

$$E(A)=E(XY)=\frac{1}{4}$$

$$E(C)=2E(X)+2E(Y)=2$$

$$E(X^2)=E(Y^2)=\int_0^1 x^2\mathrm{d}x=\frac{1}{3}$$

$$\mathrm{Cov}(A,C)=E(AC)-E(A)E(C)=2E(X^2)E(Y)+2E(X)E(Y^2)-\frac{1}{4}\times 2=2\times\frac{1}{3}\times\frac{1}{2}\times 2-\frac{1}{4}\times 2=\frac{1}{6}$$

$$\mathrm{Var}(A)=E(A^2)-E(A)^2=E(X^2)E(Y^2)-\frac{1}{16}=\frac{1}{9}-\frac{1}{16}=\frac{7}{144}$$

$$\mathrm{Var}(C)=E(C^2)-E(C)^2=4[E(X^2)+2E(XY)+E(Y^2)]-4=4\times(\frac{1}{3}+2\times\frac{1}{4}+\frac{1}{3})-4=\frac{2}{3}$$

$$\rho_{AC}=\frac{\mathrm{Cov}(A,C)}{\sqrt{\mathrm{Var}(A)\mathrm{Var}(C)}}=\frac{\frac{1}{6}}{\sqrt{\frac{7}{144}\cdot\frac{2}{3}}}=\frac{\sqrt{42}}{7}$$

27.

(1)

$$\mathrm{Cov}(X,Y)=\int_0^1 x^3\mathrm{d}x-\int_0^1 x^2\mathrm{d}x\int_0^1 x\mathrm{d}x=\frac{1}{4}-\frac{1}{3}\times\frac{1}{2}=\frac{1}{12}\neq 0$$

因此既不独立, 也不是不相关.

(2)

$$\mathrm{Cov}(X,Y)=\int_{-1}^1 \frac{1}{2}x^3\mathrm{d}x-\int_{-1}^1 \frac{1}{2}x^2\mathrm{d}x\int_{-1}^1 \frac{1}{2}x\mathrm{d}x=0-\frac{1}{3}\times 0=0$$

因此虽不独立, 但是不相关.

(3)

$\mathrm{Cov}(X,Y)=\int_0^{2\pi}\frac{1}{2\pi}\sin x\cos x\mathrm{d}x-\int_0^{2\pi}\frac{1}{2\pi}\sin x\mathrm{d}x\int_0^{2\pi}\frac{1}{2\pi}\cos x\mathrm{d}x=0-0\times 0=0$

因此虽不独立, 但是不相关.

(4)

$\mathrm{Cov}(X,Y)=\int_0^1\mathrm{d}x\int_0^1 xy(x+y)\mathrm{d}y-\int_0^1\mathrm{d}x\int_0^1 x(x+y)\mathrm{d}y\int_0^1\mathrm{d}x\int_0^1 y(x+y)\mathrm{d}y=\int_0^1\frac{x\left(3x+2\right)}{6}\mathrm{d}x-\int_0^1 x\left(x+\frac{1}{2}\right)\mathrm{d}x\int_0^1(\frac{x}{2}+\frac{1}{3})\mathrm{d}x=\frac{1}{3}-\frac{7}{12}\times\frac{7}{12}=-\frac{1}{144}\neq 0$

因此既不独立, 也不是不相关.

(5)

$\mathrm{Cov}(X,Y)=\int_0^1\mathrm{d}x\int_0^1 2xy^2\mathrm{d}y-\int_0^1\mathrm{d}x\int_0^1 2xy\mathrm{d}y\int_0^1\mathrm{d}x\int_0^1 2y^2\mathrm{d}y=\int_0^1\frac{2x}{3}\mathrm{d}x-\int_0^1 x\mathrm{d}x\int_0^1\frac{2}{3}\mathrm{d}x=\frac{1}{3}-\frac{1}{2}\times\frac{2}{3}=0$

而我们又知道 $f_x(x)=\int_0^1 2y\mathrm{d}y=1, f_y(y)=\int_0^1 2y\mathrm{d}x=2y$, 则 $f(x,y)=f_x(x)f_y(y)$ 恒成立.

因此既独立, 也不相关.

31.

$E(X)=\int_0^1\mathrm{d}x\int_{-x}^{x} x\mathrm{d}y=\int_0^1 2x^2\mathrm{d}x=\frac{2}{3}$

$E(Y)=\int_0^1\mathrm{d}x\int_{-x}^{x} y\mathrm{d}y=0$

$E(XY)=\int_0^1\mathrm{d}x\int_{-x}^{x} xy\mathrm{d}y=0$

$\mathrm{Cov}(X,Y)=E(XY)-E(X)E(Y)=0$

21.

(1)

令 $0 < z - x < 1$ 可得 $z - 1 < x < z$, 且有 $0 < x < 1$, 而 z 的取值范围是 $0 < z < 2$.

当 $0 < z \leqslant 1$ 时,

$$f_z(z) = \int_{-\infty}^{+\infty} f(x, z-x)\mathrm{d}x = \int_0^z [x + (z-x)]\mathrm{d}x = z^2$$

当 $1 < z < 2$ 时,

$$f_z(z) = \int_{-\infty}^{+\infty} f(x, z-x)\mathrm{d}x = \int_z^1 [x + (z-x)]\mathrm{d}x = z\,(1-z)$$

则 $f_z(z) = \begin{cases} z^2, & 0 < z \leqslant 1 \\ z(1-z), & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases}$

(2)

令 $0 < \dfrac{z}{x} < 1$ 得 $z < x$, 而 z 的取值范围是 $0 < z < 1$

$$f_z(z) = \int_0^1 \frac{1}{x} f(x, \frac{z}{x})\mathrm{d}x = \int_z^1 \frac{1}{x}\cdot(x + \frac{z}{x})\mathrm{d}x = 2 - 2z$$

因此 $f_z(z) = \begin{cases} 2-2z, & 0 < z < 1 \\ 0, & \text{otherwise} \end{cases}$

22.

令 $0 \leqslant z - y \leqslant 1$ 得 $z - 1 \leqslant y \leqslant z$, 且有 $y > 0$, 而 z 的取值范围是 $z > 0$

当 $0 < z < 1$ 时,

$$f_z(z) = \int_0^z e^{-y}\cdot 1\mathrm{d}y = 1 - e^{-z}$$

当 $z \geqslant 1$ 时,

$$f_z(z)=\int_{z-1}^{z} e^{-y}\mathrm{d}y=(e-1)e^{-z}$$

因此 $f_z(z)=\begin{cases}1-e^{-z}, & 0<z<1\\ (e-1)e^{-z}, & z\geqslant 1\\ 0, & \text{otherwise}\end{cases}$

24.

(1)

$$f_x(x)=\int_{0}^{+\infty}\frac{1}{2}(x+y)e^{-(x+y)}\mathrm{d}y=\frac{(x+1)\,e^{-x}}{2}$$

$$f_y(y)=\int_{0}^{+\infty}\frac{1}{2}(x+y)e^{-(x+y)}\mathrm{d}x=\frac{(y+1)\,e^{-y}}{2}$$

因此 $f_x(x)f_y(y)=\dfrac{1}{4}(xy+x+y+1)e^{-(x+y)}\neq f(x,y)$

因此并不独立.

(2)

令 $z-x>0$ 得 $x<z$, 且有 $x>0$, 而 z 的取值范围为 $z>0$

$$f_z(z)=\int_{0}^{z}\frac{1}{2}(x+z-x)e^{-(x+z-x)}\mathrm{d}x=\frac{1}{2}z^2e^{-z}$$

因此 $f_z(z)=\begin{cases}\frac{1}{2}z^2e^{-z}, & z>0\\ 0, & \text{otherwise}\end{cases}$

25.

令 $z-x>1$ 得 $x<z-1$, 且有 $x>1$, 而 z 的取值范围为 $z>2$

$$f_z(z)=\int_{1}^{z-1} e^{1-x}\cdot e^{1-(z-x)}\mathrm{d}x=\int_{1}^{z-1} e^{2-z}\mathrm{d}x=(z-2)\,e^{2-z}$$

因此 $f_z(z)=\begin{cases}(z-2)e^{2-z}, & z>2\\ 0, & \text{otherwise}\end{cases}$

26.

令 $zx > 0$ 得 $x > 0$, 且有 $x > 0$, 而 z 的取值范围为 $z > 0$

$$f_z(x) = \int_0^{+\infty} xe^{-x} \cdot e^{-zx} \mathrm{d}x = \int_0^{+\infty} xe^{-(z+1)x} \mathrm{d}x = \frac{1}{(z+1)^2}$$

因此 $f_z(z) = \begin{cases} \dfrac{1}{(z+1)^2}, & z > 0 \\ 0, & \text{otherwise} \end{cases}$

27.

因此 $A = XY$

令 $0 < \dfrac{z}{x} < 1$ 得 $x > z$, 且有 $0 < x < 1$, 而 z 的取值范围为 $0 < z < 1$

$$f_A(z) = \int_z^1 \frac{1}{x} \cdot 1 \cdot 1 \mathrm{d}x = -\ln z$$

因此 $f_A(z) = \begin{cases} -\ln z, & 0 < z < 1 \\ 0, & \text{otherwise} \end{cases}$

32.

对于极小分布, 我们有

$$F_z(z) = 1 - P(X > z)P(Y > z)$$

因为 X, Y 独立且服从同一分布, 则

$$F_z(z) = 1 - P(X > z)^2$$

所以我们可得

$$P(a < \min\{X, Y\} \leqslant b) = F_z(b) - F_z(a) = P(X > a)^2 - P(X > b)^2$$