第六次作业

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习题 6.1: (A) 5(1, 4)

习题 6.2: (A) 3(2, 6, 7, 10), 5(1, 3), 6(4), 7(2), 8(2), 9(1, 2), 13(1, 3), 14(2) (B) 1(1, 3), 2, 3, 6, 13

6.1 (A)

5.

(1)

当 $x \geq 0, y \geq 0, x + y \leq 1$ 时, 有 $x + y \leq (x+y)^2$, 且并不恒能取等号

$$\therefore \iint_{(\sigma)} (x+y)\mathrm{d}\sigma < \iint_{(\sigma)} (x+y)^2\mathrm{d}\sigma$$

(4)

由区域 $(\sigma_2) = \{(x,y)|x^2+y^2 \leq 1\}$ 和函数 $z(x,y) = x^2y$ 的对称性可知

$$\therefore \iint_{(\sigma_2)} x^2 = 0$$

又因为 x^2y 在 $((\sigma_2) = \{(x,y)|x^2+y^2 \leq 1, y \geq 0\})$ 上大于等于零, 且不恒为零

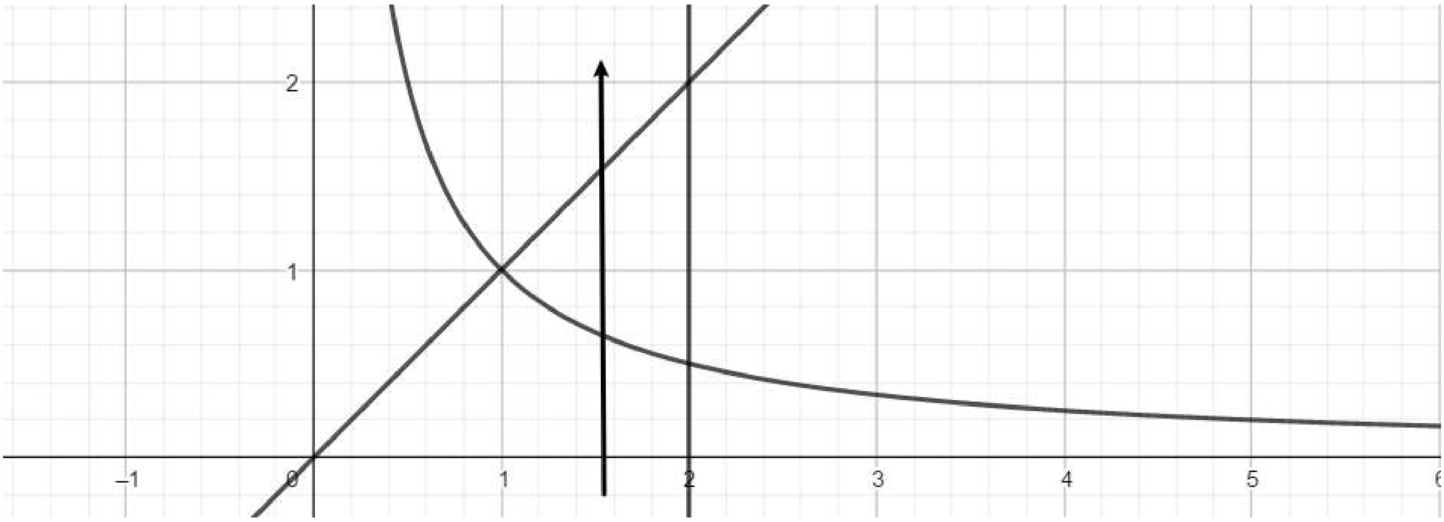
$$\therefore \iint_{(\sigma_1)} x^2 > 0$$

$$\therefore \iint_{(\sigma_1)} x^2 > \iint_{(\sigma_2)} x^2$$

6.2 (A)

3.

(2)



对于 x 型区域:

$$\iint_{(\sigma)} \frac{x^2}{y^2}\mathrm{d}\sigma = \int_1^2 \mathrm{d}x \int_{\frac{1}{x}}^{x} \frac{x^2}{y^2}\mathrm{d}y = \int_1^2 (x^3 - x)\mathrm{d}x = \frac{9}{4}$$

(6)

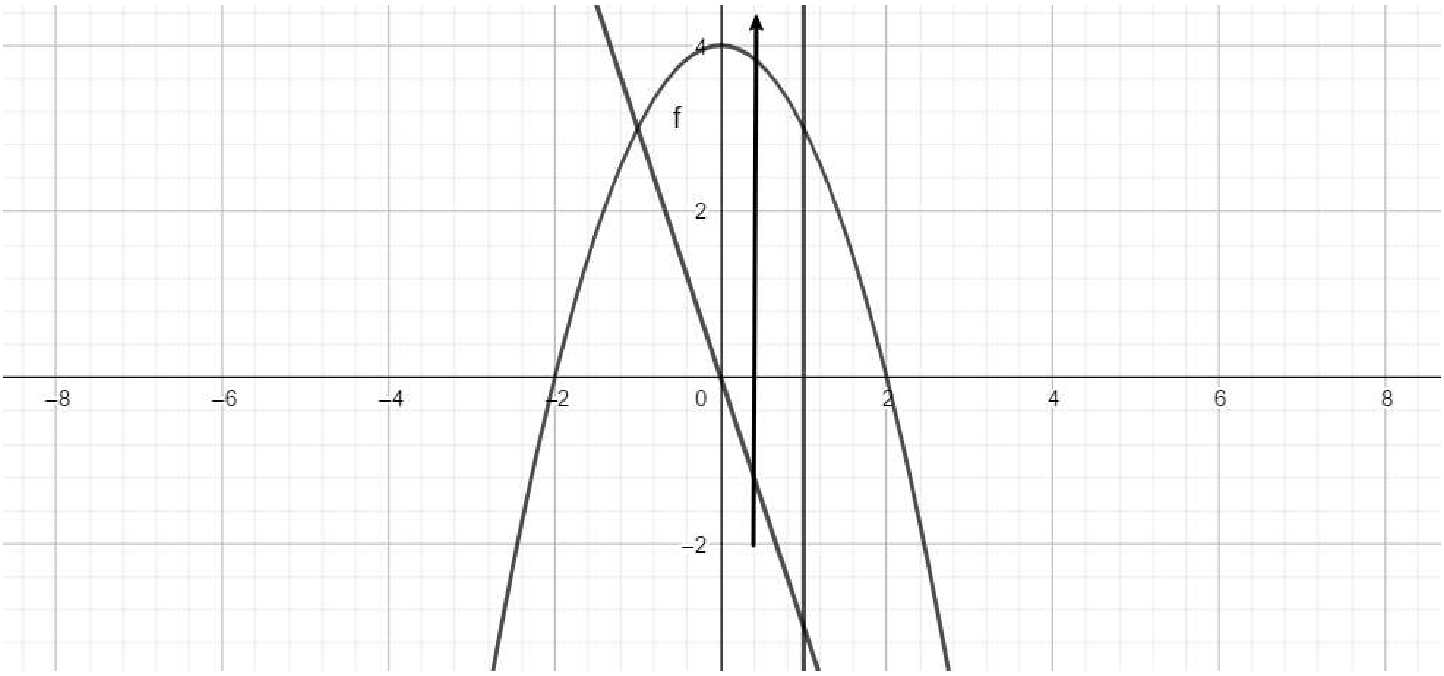
$$\iint_{(\sigma)} e^{-y^2}\mathrm{d}\sigma = \int_0^1 \mathrm{d}y \int_0^y e^{-y^2}\mathrm{d}x = \int_0^1 ye^{-y^2}\mathrm{d}y = \frac{1}{2}\int_0^1 e^{-y^2}\mathrm{d}y^2 = \frac{1}{2} - \frac{1}{2}e$$

(7)

由对称性

$$\begin{aligned}\iint_{(\sigma)} (y + xf(x^2+y^2))\mathrm{d}\sigma &= \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{\sqrt{y}} xf(x^2+y^2)\mathrm{d}x + \int_{-1}^1 \mathrm{d}x \int_{x^2}^1 y\mathrm{d}y \\ &= \frac{1}{2}\int_0^1 \mathrm{d}y \int_y^y f(x^2+y^2)\mathrm{d}x^2 + \int_{-1}^1 (\frac{1}{2} - \frac{1}{2}x^4)\mathrm{d}x \\ &= 0 + \int_{-1}^1 (\frac{1}{2} - \frac{1}{2}x^4)\mathrm{d}x \\ &= \frac{4}{5}\end{aligned}$$

(10)



将 $y=-3x$ 带入 $y=4-x^2$ 解 $x^2-3x-4=(x-4)(x+1)=0$

得 $x=-1,x=4$ (舍去)

$$
\begin{aligned}
&\iint_{(\sigma)} x\ln(y+\sqrt{1+y^2})\mathrm{d}\sigma\\
&=\int_3^4\mathrm{d}y\int_{-\sqrt{4-y}}^{\sqrt{4-y}}x\ln(y+\sqrt{1+y^2})\mathrm{d}x+\int_{-1}^{1}x\mathrm{d}x\int_{-3x}^{3}\ln(y+\sqrt{1+y^2})\mathrm{d}y\\
&=\int_{-1}^{1}x\mathrm{d}x\int_{-3x}^{3}\ln(y+\sqrt{1+y^2})\mathrm{d}y\\
&=\int_{-1}^{1}x\mathrm{d}x[y\ln(y+\sqrt{1+y^2})|_{-3x}^{3}-\int_{-3x}^{3}y\mathrm{d}\ln(y+\sqrt{1+y^2})]\\
&=\int_{-1}^{1}x\mathrm{d}x[y\ln(y+\sqrt{1+y^2})|_{-3x}^{3}-\int_{-3x}^{3}y\frac{1+\frac{y}{\sqrt{1+y^2}}}{y+\sqrt{1+y^2}}\mathrm{d}y]\\
&=\int_{-1}^{1}x\mathrm{d}x[y\ln(y+\sqrt{1+y^2})|_{-3x}^{3}-\int_{-3x}^{3}y\frac{\frac{y+\sqrt{1+y^2}}{\sqrt{1+y^2}}}{y+\sqrt{1+y^2}}\mathrm{d}y]\\
&=\int_{-1}^{1}x\mathrm{d}x[y\ln(y+\sqrt{1+y^2})|_{-3x}^{3}-\int_{-3x}^{3}\frac{y}{\sqrt{1+y^2}}\mathrm{d}y]\\
&=\int_{-1}^{1}x\mathrm{d}x(y\ln(y+\sqrt{1+y^2})-\sqrt{1+y^2})|_{-3x}^{3}\\
&=\int_{-1}^{1}x(C+3x\ln(-3x+\sqrt{1+9x^2})+\sqrt{1+9x^2})\mathrm{d}x
\end{aligned}
$$

其中 $C=3\ln(3+\sqrt{10})-\sqrt{10}$, 是个常数

$\ln(-3x+\sqrt{1+9x^2})+\ln(3x+\sqrt{1+9x^2})=\ln(1+9x^2-9x^2)=0$

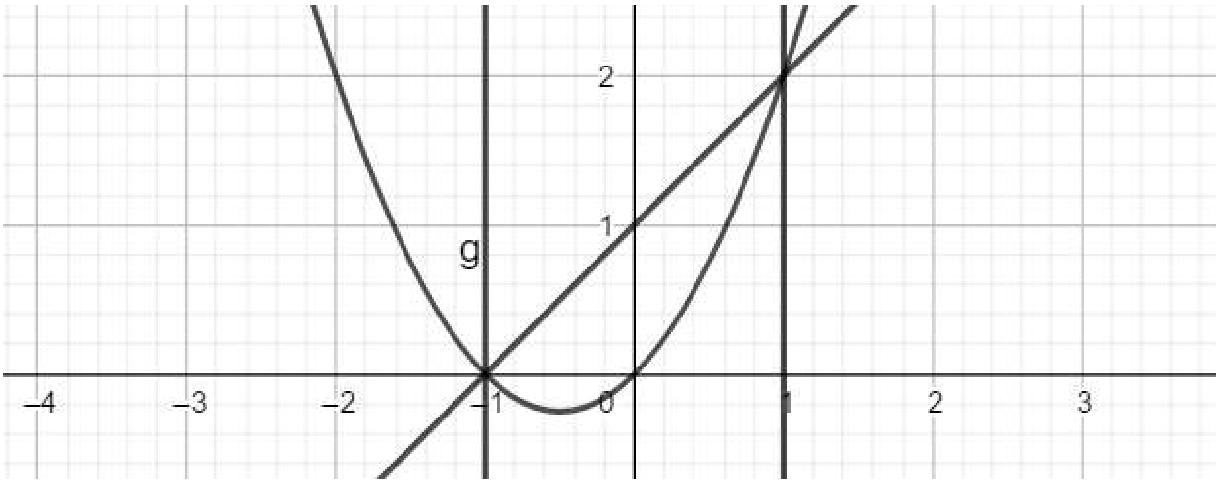
说明 $\ln(-3x+\sqrt{1+9x^2})$ 是奇函数

所以 $x(C+3x\ln(-3x+\sqrt{1+9x^2})+\sqrt{1+9x^2})$ 也是奇函数

$$\therefore \iint_{(\sigma)} x\ln(y+\sqrt{1+y^2})\mathrm{d}\sigma = 0$$

5.

(1)

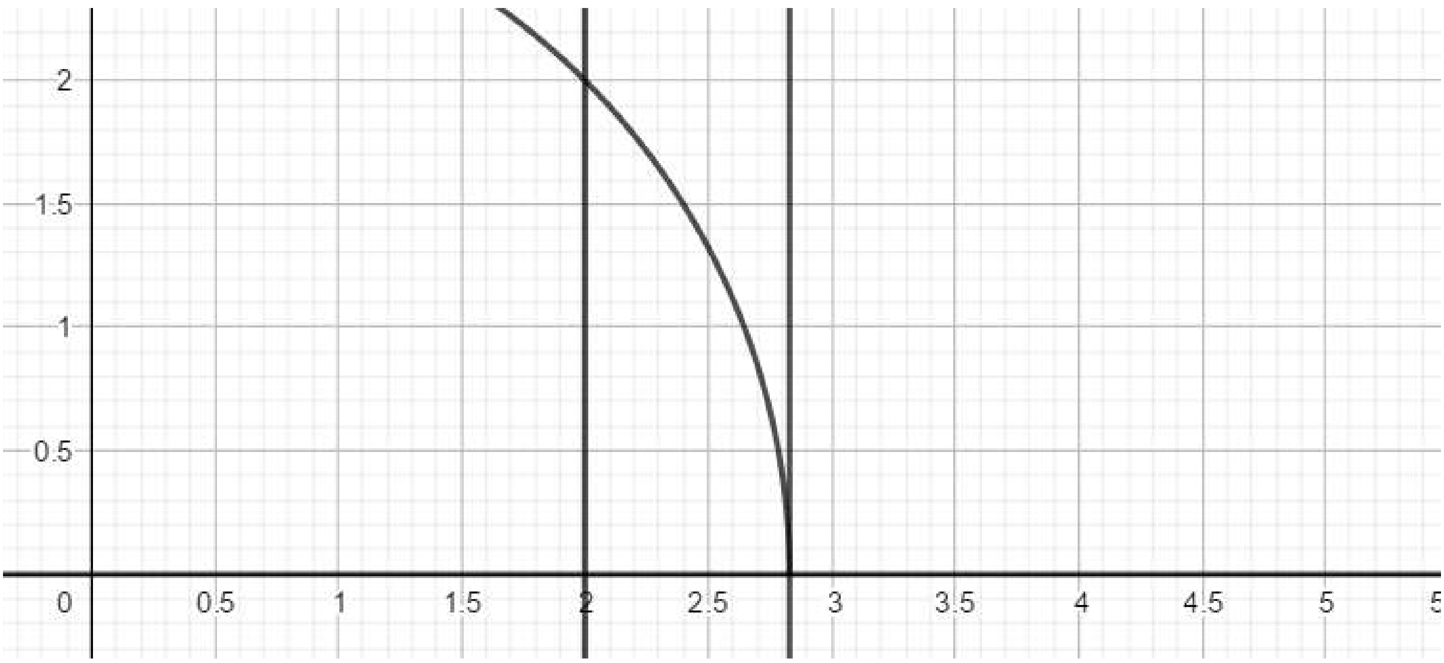


对于 x^2+x 有最小值 $(-\frac{1}{2})^2-\frac{1}{2}=-\frac{1}{4}$

反解 $y=x^2+x$ 得 $x=\pm\sqrt{y+\frac{1}{4}}-\frac{1}{2}$

$$\int_{-1}^{1}\mathrm{d}x\int_{x^2+x}^{x+1}f(x,y)\mathrm{d}y=\int_{-\frac{1}{4}}^{0}\mathrm{d}y\int_{-\sqrt{y+\frac{1}{4}}-\frac{1}{2}}^{\sqrt{y+\frac{1}{4}}-\frac{1}{2}}f(x,y)\mathrm{d}x+\int_{0}^{2}\mathrm{d}y\int_{y-1}^{\sqrt{y+\frac{1}{4}}-\frac{1}{2}}f(x,y)\mathrm{d}x$$

(3)

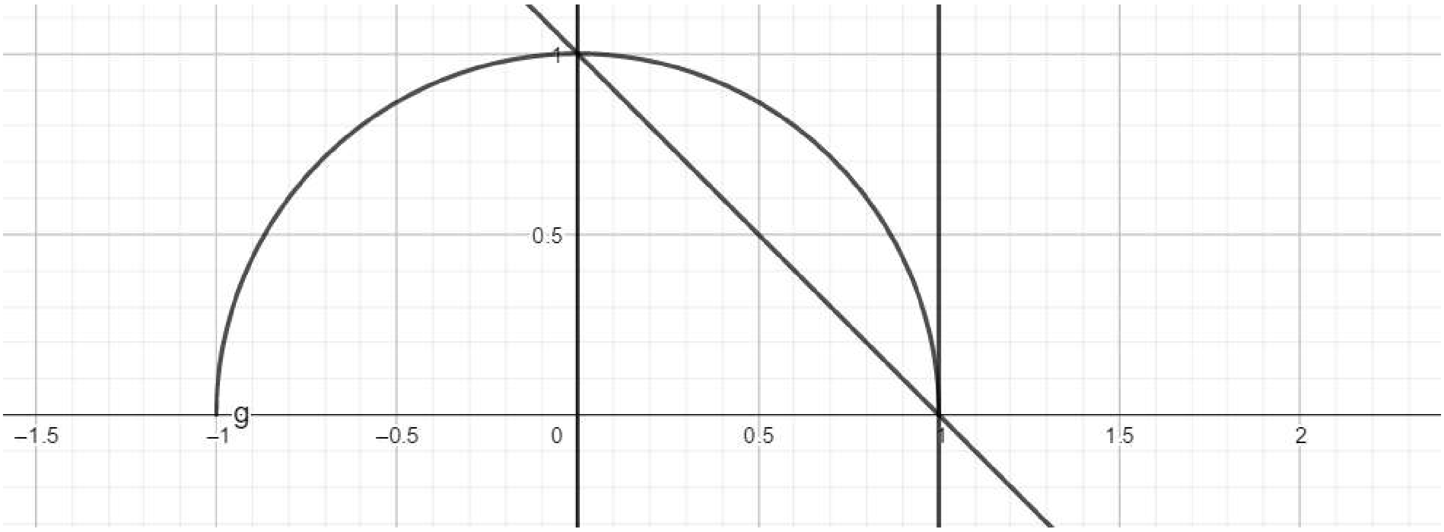


$$
\int_0^2 \mathrm{d}x \int_0^x f(x,y)\mathrm{d}y + \int_2^{\sqrt{8}} \mathrm{d}x \int_0^{\sqrt{8-x^2}} f(x,y)\mathrm{d}y
$$
$$
= \int_0^2 \mathrm{d}y \int_y^2 f(x,y)\mathrm{d}x + \int_0^2 \mathrm{d}y \int_2^{\sqrt{8-y^2}} f(x,y)\mathrm{d}x
$$

6. (4)

$$
\iint_{(\sigma)} \arctan\frac{y}{x}\mathrm{d}\sigma = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^1 \arctan\frac{r\sin\theta}{r\cos\theta}\cdot r\mathrm{d}r = \int_0^{\frac{\pi}{2}} \frac{1}{2}\theta\mathrm{d}\theta = \frac{\pi^2}{16}
$$

7. (2)

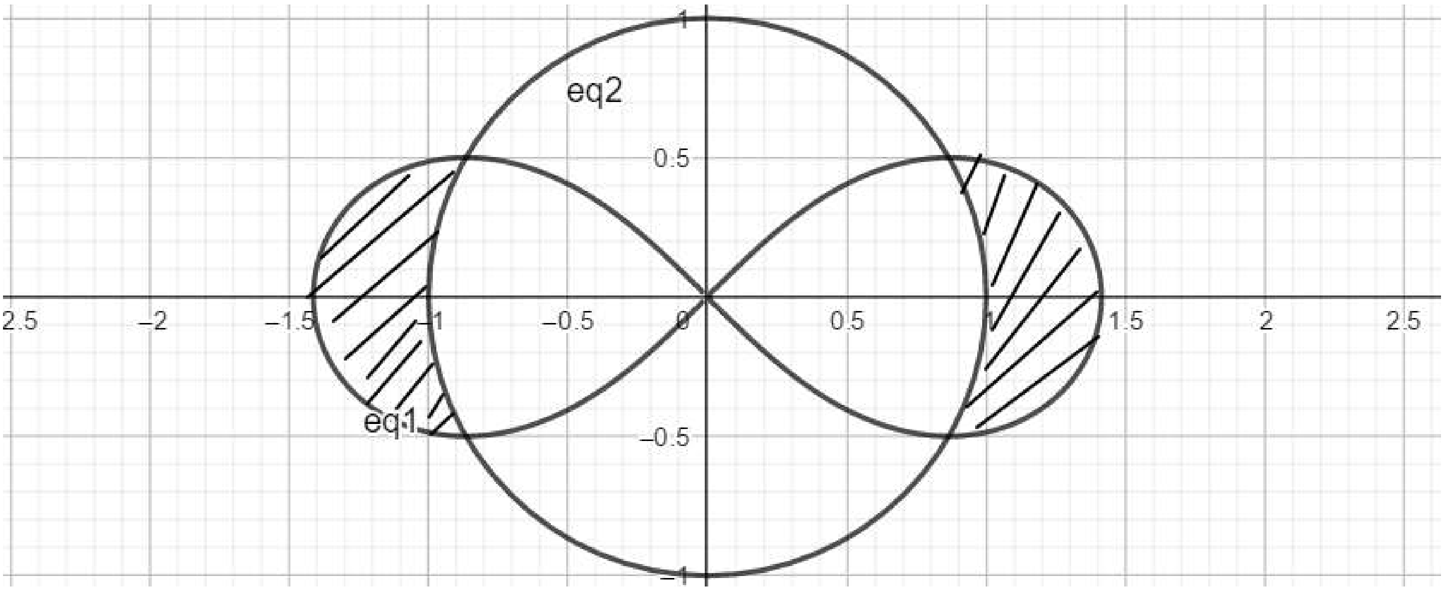


对于 $y=1-x$, 带入 $x=r\cos\theta, y=r\sin\theta$

可得 $r=\dfrac{1}{\sin\theta+\cos\theta}$

$$
\int_0^1 \mathrm{d}x \int_{1-x}^{\sqrt{1-x^2}} (x^2+y^2)^{-\frac{3}{2}}\mathrm{d}y = \int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_{\frac{1}{\sin\theta+\cos\theta}}^1 r^{-3}\cdot r\mathrm{d}r = \int_0^{\frac{\pi}{2}} (\sin\theta+\cos\theta)\mathrm{d}\theta = 2
$$

8. (2)



将 $(x^2+y^2)^2=2a^2(x^2-y^2)$ 化为极坐标方程得 $r^2=2a^2(\cos^2\theta-\sin^2\theta)$

将 $x^2+y^2=a^2$ 化为极坐标方程得 $r^2=a^2$

联立两个方程得 $\cos^2\theta-\sin^2\theta=2\cos^2\theta-1=\frac{1}{2}$

解得 $\cos\theta=\pm\frac{\sqrt{3}}{2}$, 对应 $\theta=\pm\frac{\pi}{6},\pm\frac{5\pi}{6}$

$$
\begin{aligned}
S&=4\iint_{(\sigma)}=4\int_0^{\frac{\pi}{6}}\mathrm{d}\theta\int_a^{\sqrt{2}a\sqrt{\cos^2\theta-\sin^2\theta}}r\mathrm{d}r\\
&=4\int_0^{\frac{\pi}{6}}(a^2(\cos^2\theta-\sin^2\theta)-\frac{1}{2}a^2)\mathrm{d}\theta\\
&=4a^2\int_0^{\frac{\pi}{6}}(\cos^2\theta-\sin^2\theta)\mathrm{d}\theta-\frac{\pi}{3}a^2\\
&=2a^2\sin(2x)|_0^{\frac{\pi}{6}}-\frac{\pi}{3}a^2\\
&=(\sqrt{3}-\frac{\pi}{3})a^2
\end{aligned}
$$

9.

(1)

$$V=\iint_{(\sigma)}(x^2+y^2)\mathrm{d}\sigma=\int_0^4\mathrm{d}x\int_0^{4-x}(x^2+y^2)\mathrm{d}y=\frac{88}{3}$$

(2)

将 $x^2+y^2=2ax$ 换成极坐标方程得 $r=2a\cos\theta$

$$\therefore V=\iint_{(\sigma)}\sqrt{x^2+y^2}\mathrm{d}\sigma=2\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{2a\cos\theta}r^2\mathrm{d}r=\frac{16a^3}{3}\int_0^1(1-\sin^2\theta)\mathrm{d}\sin\theta=\frac{32a^3}{9}$$

13.

(1)

令 $x=ar\cos\theta, y=br\sin\theta, \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}\le 1$ 可变为 $r\le 1$

$$\therefore J=\begin{vmatrix} a\cos\theta & b\sin\theta \\ -ar\sin\theta & br\cos\theta \end{vmatrix}=abr\cos^2\theta+abr\sin^2\theta=abr$$

$$\begin{aligned}\therefore I&=\iint_{D_{r\theta}}\sqrt{1-r^2}\cdot abr\mathrm{d}\sigma\\&=4ab\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^1 r\sqrt{1-r^2}\mathrm{d}r\\&=4ab\int_0^{\frac{\pi}{2}}\frac{1}{3}\mathrm{d}\theta\\&=\frac{2\pi}{3}ab\end{aligned}$$

(3)

令 $u=xy, v=\dfrac{y}{x}$, 即 $x=\sqrt{\dfrac{u}{v}}=u^{\frac{1}{2}}v^{-\frac{1}{2}}, y=\sqrt{uv}=u^{\frac{1}{2}}v^{\frac{1}{2}}$

$$\therefore J=\begin{vmatrix} \frac{1}{2}u^{-\frac{1}{2}}v^{-\frac{1}{2}} & \frac{1}{2}u^{-\frac{1}{2}}v^{\frac{1}{2}} \\ -\frac{1}{2}u^{\frac{1}{2}}v^{-\frac{3}{2}} & \frac{1}{2}u^{\frac{1}{2}}v^{-\frac{1}{2}} \end{vmatrix}=\frac{1}{2v}$$

$$\begin{aligned}\therefore I&=\iint_{D_{uv}}u\cdot\frac{1}{2v}\mathrm{d}\sigma\\&=\frac{1}{2}\int_1^4\mathrm{d}v\int_1^2\frac{u}{v}\mathrm{d}u\\&=\frac{3}{4}\int_1^4\frac{1}{v}\mathrm{d}v\\&=\frac{3}{2}\ln 2\end{aligned}$$

14. (2)

令 $u=x+y, v=\dfrac{y}{x}$, 即 $x=\dfrac{u}{v+1}, y=\dfrac{uv}{v+1}$

$$\therefore J=\begin{vmatrix} \frac{1}{v+1} & \frac{v}{v+1} \\ -\frac{u}{(v+1)^2} & \frac{u}{(v+1)^2} \end{vmatrix}=\frac{u}{(v+1)^2}$$

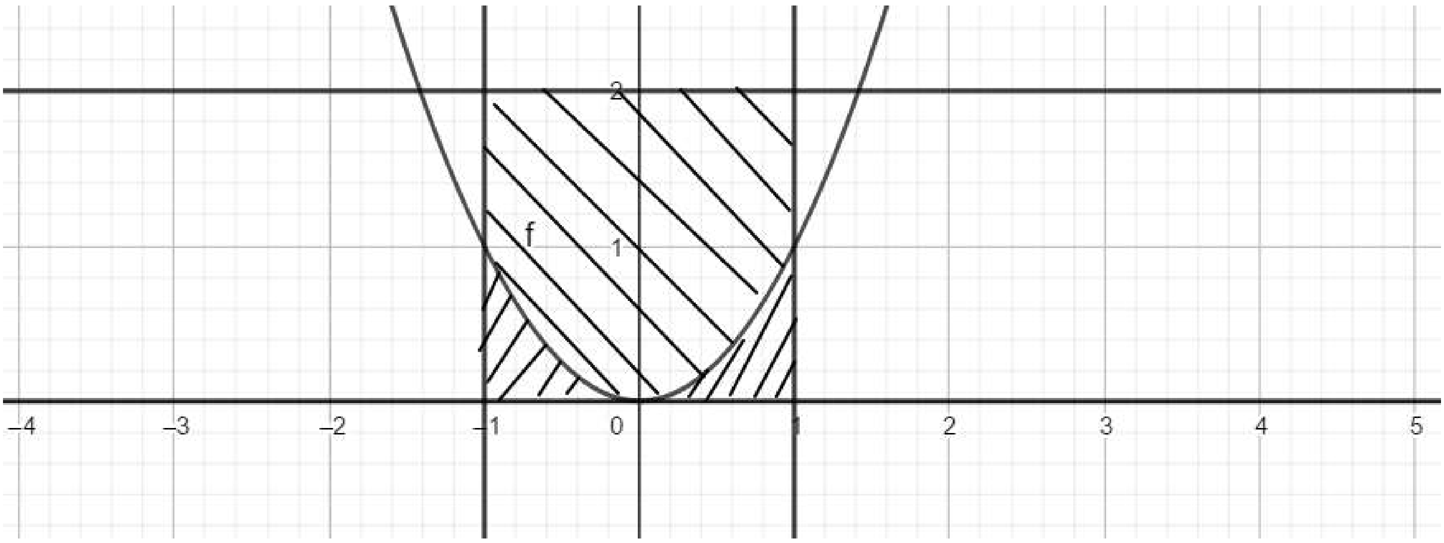
$$
\begin{aligned}
\therefore I &= \iint_{D_{uv}} \frac{u}{(v+1)^2}\mathrm{d}\sigma \\
&= \frac{1}{2}\int_{\alpha}^{\beta}\mathrm{d}v\int_{a}^{b}\frac{u}{(v+1)^2}\mathrm{d}u \\
&= \frac{b^2-a^2}{4}\int_{\alpha}^{\beta}\frac{1}{(v+1)^2}\mathrm{d}v \\
&= \frac{b^2-a^2}{4}\int_{(\alpha+1)}^{(\beta+1)}\frac{1}{t^2}\mathrm{d}t \\
&= \frac{b^2-a^2}{4}(\frac{1}{\alpha+1}-\frac{1}{\beta+1})
\end{aligned}
$$

6.2 (B)

1.

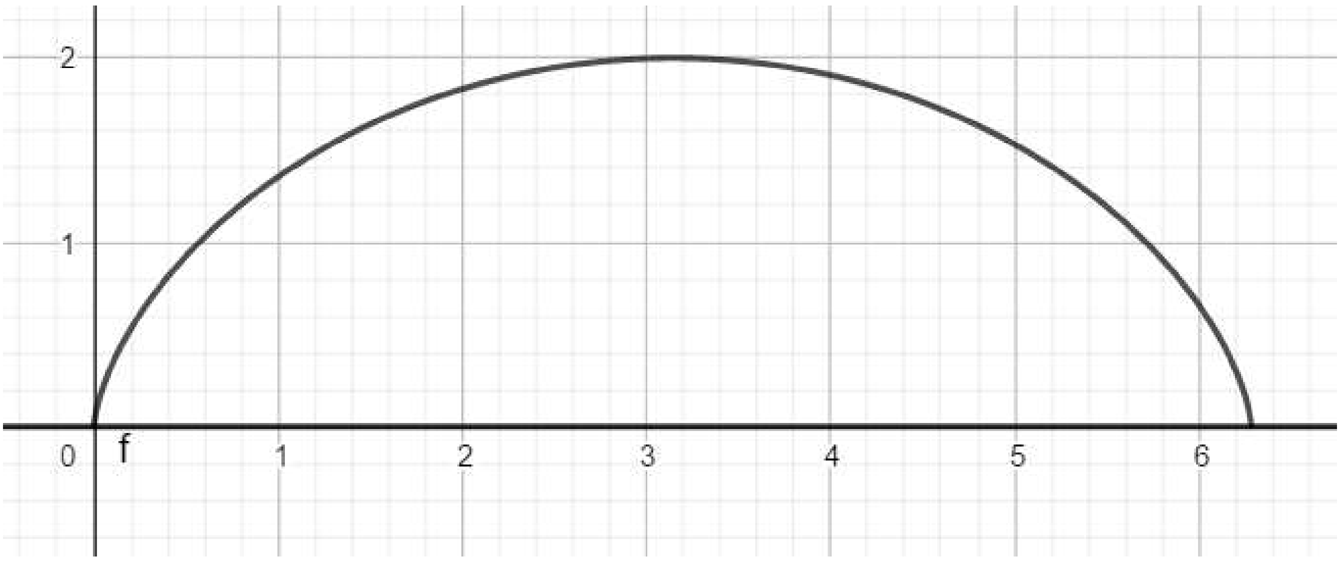
(1)

令 $y-x^2\geq 0$ 可得 $y\geq x^2$



$$\begin{aligned}
\iint_{(\sigma)} \sqrt{|y-x^2|}\mathrm{d}\sigma &= 2\int_0^1 \mathrm{d}x \int_0^{x^2} \sqrt{x^2-y}\mathrm{d}y + 2\int_0^1 \mathrm{d}x \int_{x^2}^{2} \sqrt{y-x^2}\mathrm{d}y \\
&= -2\int_0^1 \mathrm{d}x \int_0^{x^2} \sqrt{x^2-y}\mathrm{d}(x^2-y) + 2\int_0^1 \mathrm{d}x \int_{x^2}^{2} \sqrt{y-x^2}\mathrm{d}(y-x^2) \\
&= -2\int_0^1 \mathrm{d}x \int_{x^2}^{0} \sqrt{t}\mathrm{d}t + 2\int_0^1 \mathrm{d}x \int_0^{2-x^2} \sqrt{t}\mathrm{d}t \\
&= \frac{4}{3}\int_0^1 x^3\mathrm{d}x + \frac{4}{3}\int_0^1 (2-x^2)^{\frac{3}{2}}\mathrm{d}x \\
&= \frac{1}{3} + \frac{4}{3}x(2-x^2)^{\frac{3}{2}}|_0^1 - \frac{4}{3}\int_0^1 x\mathrm{d}(2-x^2)^{\frac{3}{2}} \\
&= \frac{5}{3} + 4\int_0^1 x^2(2-x^2)^{\frac{1}{2}}\mathrm{d}x \\
&= \frac{5}{3} + 2\int_0^1 (2x^2-x^4)^{\frac{1}{2}}\mathrm{d}x^2 \\
&= \frac{5}{3} + 2\int_0^1 (1-(t-1)^2)^{\frac{1}{2}}\mathrm{d}t \\
&= \frac{5}{3} + 2\int_{-1}^0 (1-u^2)^{\frac{1}{2}}\mathrm{d}u \\
&= \frac{5}{3} + \frac{\pi}{2}
\end{aligned}$$

(3)



$$\iint_{(\sigma)} y^2 \mathrm{d}\sigma = \int_0^{a(2\pi - \sin 2\pi)} \mathrm{d}x \int_0^{a(1-\cos t)} y^2 \mathrm{d}y$$
$$= \int_0^{2\pi} \frac{1}{3}(a(1-\cos t))^3 \mathrm{d}a(t-\sin t)$$
$$= \frac{1}{3}a^4 \int_0^{2\pi} (1-\cos t)^4 \mathrm{d}t$$
$$= \frac{32}{3}a^4 \int_0^{\pi} \sin^8 u \mathrm{d}u$$

对于 $\int \sin^n x \mathrm{d}x$:

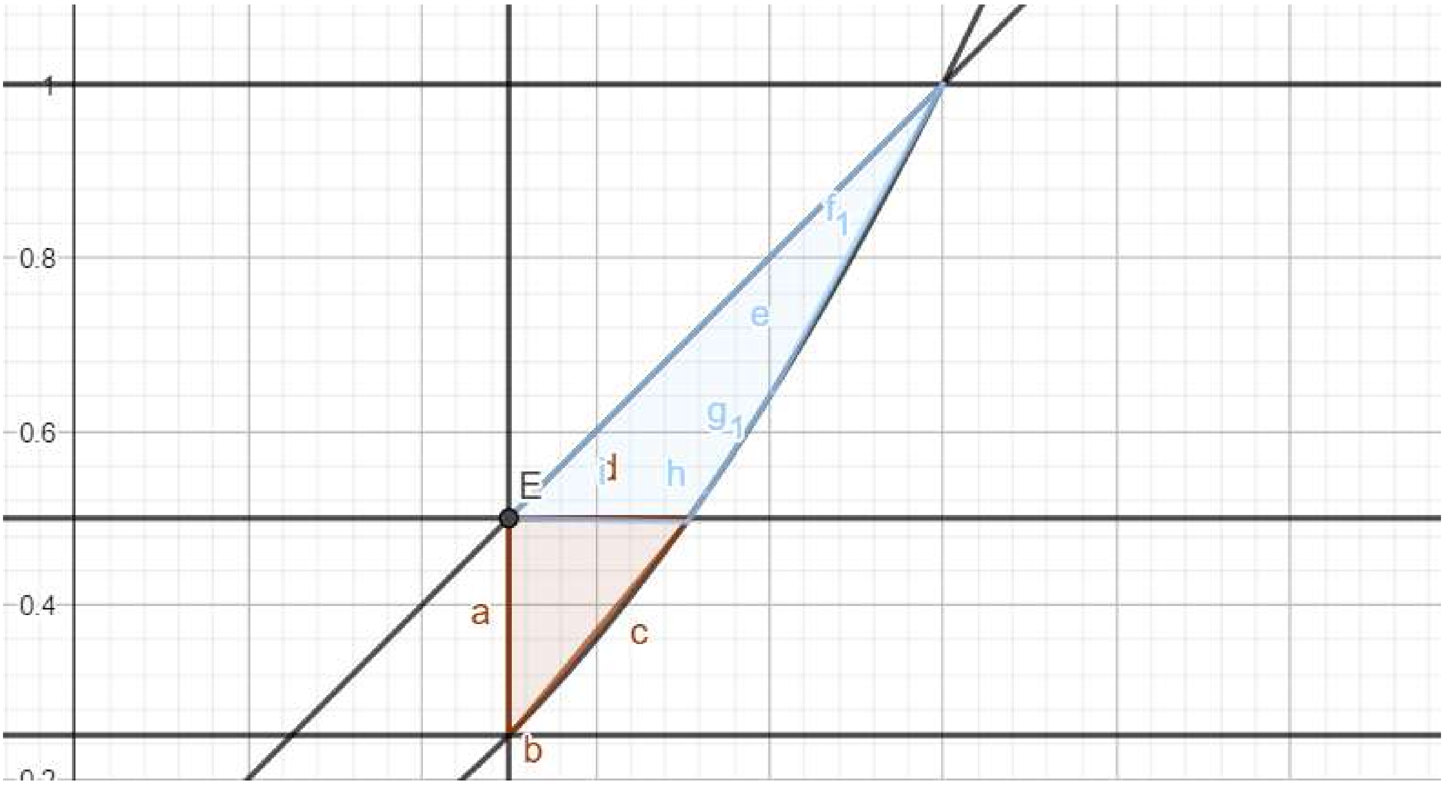
$$\begin{aligned}
\therefore I_n &= \int \sin^n x \mathrm{d}x \\
&= \int \sin^{n-1} x \mathrm{d}(-\cos x) \\
&= -\cos x \sin^{n-1} + \int \cos x \mathrm{d} \sin^{n-1} x \\
&= -\cos x \sin^{n-1} + \int \cos^2 x (n-1) \sin^{n-2} x \mathrm{d}x \\
&= -\cos x \sin^{n-1} + (n-1) \int (1-\sin^2 x) \sin^{n-2} x \mathrm{d}x \\
&= -\cos x \sin^{n-1} + (n-1)(I_{n-2} - I_n)
\end{aligned}$$

$$\therefore I_n = \frac{1}{n} \cos x \sin^{n-1} x + \frac{n-1}{n} I_{n-2}, \quad n \geq 2$$

$$\begin{aligned}
\therefore I_8 &= \frac{1}{8}\cos x \sin^7 x + \frac{7}{8}(\frac{1}{6}\cos x \sin^5 x + \frac{5}{6}(\frac{1}{4}\cos x \sin^3 x + \frac{3}{4}(\frac{1}{2}\cos x \sin x + \frac{1}{2}x))) + C \\
&= \frac{1}{8}\cos x \sin^7 x + \frac{7}{48}\cos x \sin^5 x + \frac{35}{192}\cos x \sin^3 x + \frac{105}{384}\cos x \sin x + \frac{105}{384}x + C
\end{aligned}$$

$$\therefore \iint_{(\sigma)} y^2 \mathrm{d}\sigma = \frac{32}{3}a^4 \int_0^{\pi} \sin^8 u \mathrm{d}u = \frac{32}{3}a^4 \cdot \frac{105}{384}\pi = \frac{35}{12}\pi a^4$$

2.



联解 $x=\sqrt{y}$ 和 $y=\dfrac{1}{2}$ 得 $x=\dfrac{\sqrt{2}}{2}$

$$\int_{\frac{1}{4}}^{\frac{1}{2}} \mathrm{d}y \int_{\frac{1}{2}}^{\sqrt{y}} e^{\frac{y}{x}} \mathrm{d}x+\int_{\frac{1}{2}}^{1} \mathrm{d}y \int_{y}^{\sqrt{y}} e^{\frac{y}{x}} \mathrm{d}x$$

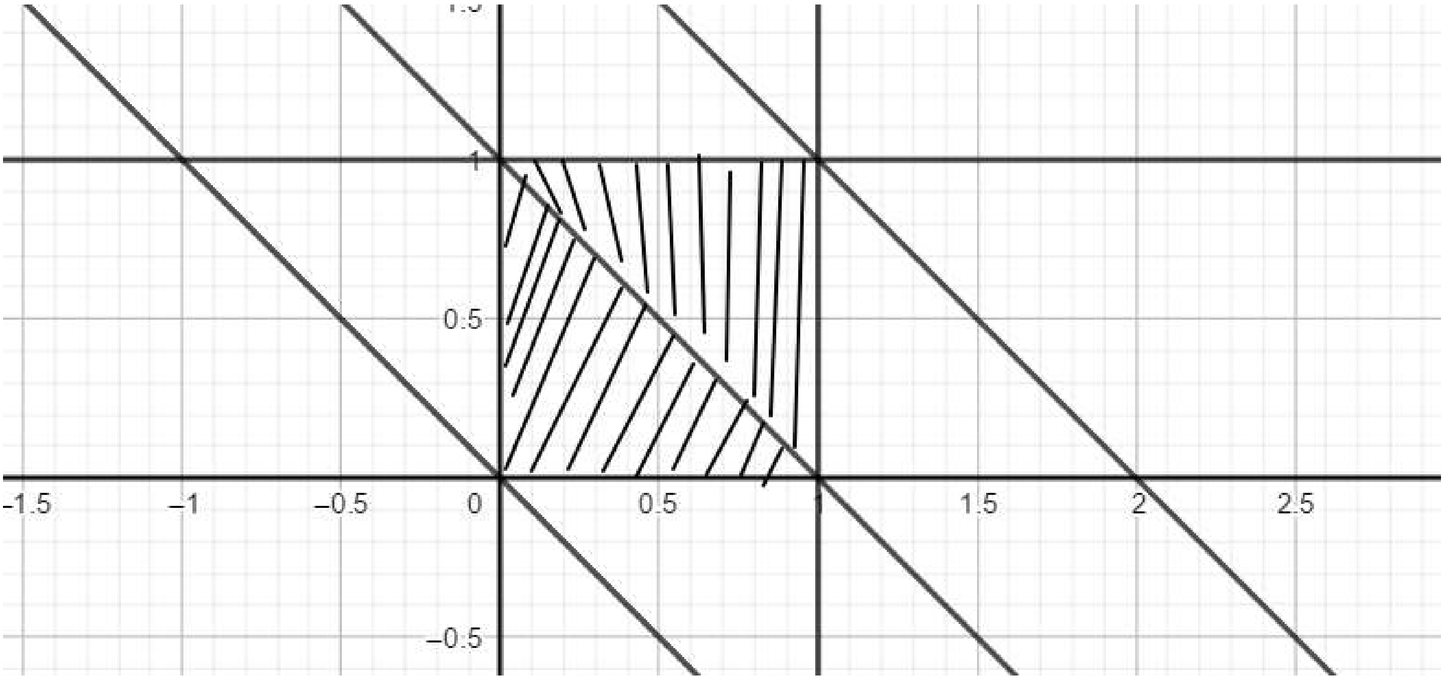
$$=\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}x \int_{x^{2}}^{\frac{1}{2}} e^{\frac{y}{x}} \mathrm{d}y+\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \mathrm{d}x \int_{\frac{1}{2}}^{x} e^{\frac{y}{x}} \mathrm{d}y+\int_{\frac{\sqrt{2}}{2}}^{1} \mathrm{d}x \int_{x^{2}}^{x} e^{\frac{y}{x}} \mathrm{d}y$$

$$=\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}}\left(e x-x e^{x}\right) \mathrm{d}x+\int_{\frac{\sqrt{2}}{2}}^{1}\left(e x-x e^{x}\right) \mathrm{d}x$$

$$=\int_{\frac{1}{2}}^{1} e x \mathrm{d}x-\int_{\frac{1}{2}}^{1} x e^{x} \mathrm{d}x$$

$$=\frac{3}{8} e-\frac{1}{2} e^{\frac{1}{2}}$$

3.



当 $t \leq 0$ 时,

易知 $f(x,y)=0$ 在 $x+y\leq t$ 恒成立

$$\therefore F(t)=\iint_{x+y\leq t} f(x,y)\mathrm{d}\sigma=0$$

当 $0<t\leq 1$ 时,

$$\therefore F(t)=\iint_{x+y\leq t} f(x,y)\mathrm{d}\sigma=\int_0^t \mathrm{d}x\int_0^{t-x} 2x\mathrm{d}y=\frac{1}{3}t^3$$

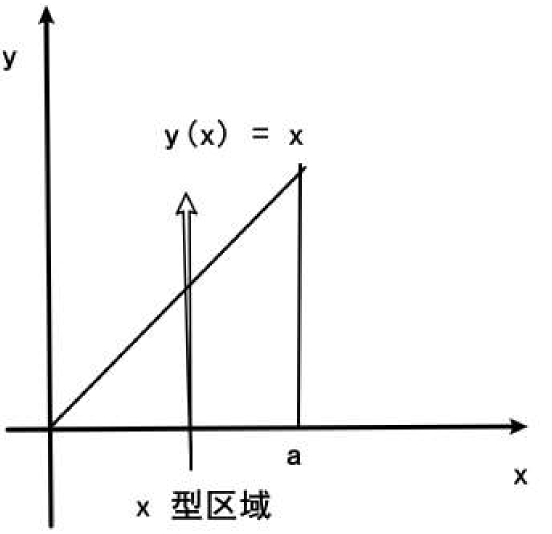
当 $1<t\leq 2$ 时,

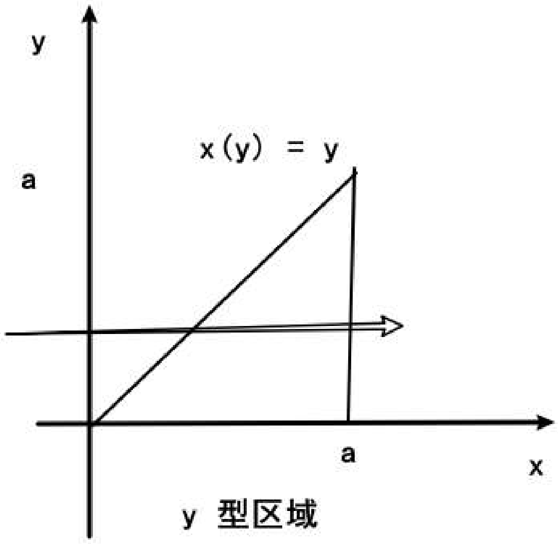
$$\begin{aligned}\therefore F(t)&=\iint_{x+y\leq t} f(x,y)\mathrm{d}\sigma\\&=\frac{1}{3}+\int_0^{t-1}\mathrm{d}x\int_{1-x}^{1}2x\mathrm{d}y+\int_{t-1}^{1}\mathrm{d}x\int_{1-x}^{t-x}2x\mathrm{d}y\\&=\frac{1}{3}+\frac{2}{3}(t-1)^3+[(t-1)-(t-1)^3]\\&=-\frac{1}{3}t^3+t^2-\frac{1}{3}\end{aligned}$$

当 $t>2$ 时,

$$\therefore F(t)=F(2)=1$$

6.





如图所示, $\int_0^a \mathrm{d}x \int_0^x f(x,y)\mathrm{d}y$ 对应的 x 型区域为 $y=x, x=0, x=1, y=0$ 围成的区域.

而 $\int_0^a \mathrm{d}x \int_y^a f(x,y)\mathrm{d}y$ 对应的 y 型区域为 $y=x, x=0, x=1, y=0$ 围成的区域.

两者所对应的区域一模一样, 并且我们知道 f 在该区域内连续.

所以我们有 $\int_0^a \mathrm{d}x \int_0^x f(x,y)\mathrm{d}y = \int_0^a \mathrm{d}x \int_y^a f(x,y)\mathrm{d}y$.

同理有 $\int_0^a \mathrm{d}x \int_x^a f(x,y)\mathrm{d}y = \int_0^a \mathrm{d}x \int_0^y f(x,y)\mathrm{d}y$.

$$\therefore \int_0^a \mathrm{d}y \int_0^y f(x)\mathrm{d}x = \int_0^a \mathrm{d}x \int_x^a f(x)\mathrm{d}y = \int_0^a \mathrm{d}x[yf(x)]|_x^a = \int_0^a (a-x)f(x)\mathrm{d}x$$

13.

$$\begin{aligned}\therefore f(t) &= e^{4\pi t^2} + \iint_{x^2+y^2\le 4t^2} f(\frac{1}{2}\sqrt{x^2+y^2})\mathrm{d}\sigma \\ &= e^{4\pi t^2} + 4\int_0^{\frac{\pi}{2}} \mathrm{d}\theta \int_0^{2t} f(\frac{1}{2}\rho)\rho\mathrm{d}\rho \\ &= e^{4\pi t^2} + 2\pi \int_0^{2t} f(\frac{1}{2}\rho)\rho\mathrm{d}\rho\end{aligned}$$

$$\therefore f'(t) - 8\pi t f(t) = 8\pi t e^{4\pi t^2}$$

将其变为齐次线性微分方程 $f'(t) - 8\pi t f(t) = 0$

对这个方程解得 $y = C_1 e^{\int 8\pi t \mathrm{d}t} = C_1 e^{4\pi t^2}$

对于原方程的解, 解得

$$\therefore f(t)=e^{4\pi t^2}(\int(8\pi te^{4\pi t^2})e^{-4\pi t^2}\mathrm{d}t+C)=e^{4\pi t^2}(4\pi t^2+C)$$

当 $t=0$ 时, $f(t)=1$, 带入可得 $f(0)=C=1$

解得 $f(t)=(4\pi t^2+1)e^{4\pi t^2}$