概率统计第七次作业

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2.

$\because F(x_i, y_i) = F_x(x_i)F_y(y_j)$

$$
\begin{aligned}
\therefore p_{ij} &= F(x_i, y_i) - F(x_i - 1, y_i) - F(x_i, y_i - 1) + F(x_i - 1, y_i - 1) \\
&= F_x(x_i)F_y(y_i) - F_x(x_i - 1)F_y(y_i) - F_x(x_i)F_y(y_i - 1) + F_x(x_i - 1)F_y(y_i - 1) \\
&= [F_x(x_i) - F_x(x_i - 1)]F_y(y_i) - [F_x(x_i) - F_x(x_i - 1)]F_y(y_i - 1) \\
&= [F_x(x_i) - F_x(x_i - 1)][F_y(y_i) - F_y(y_i - 1)] \\
&= p_{i\cdot}p_{\cdot j}
\end{aligned}
$$

3.

$\therefore f_x(x) =$
$$\frac{\exp(-\frac{1}{2}(\frac{x-\mu_x}{\sigma_x})^2)}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \int_{-\infty}^{+\infty} \exp(-\frac{1}{2(1-\rho^2)}\left[(\frac{y-\mu_y}{\sigma_y})^2 - 2\rho\frac{x-\mu_x}{\sigma_x}\frac{y-\mu_y}{\sigma_y} + \rho^2(\frac{y-\mu_y}{\sigma_y})^2\right])\mathrm{d}y$$

令 $t = \dfrac{y-\mu_y}{\sigma_y}$, 可得

$$\therefore f_x(x) = \frac{\exp(-\frac{1}{2}(\frac{x-\mu_x}{\sigma_x})^2)}{2\pi\sqrt{1-\rho^2}\sigma_x} \int_{-\infty}^{+\infty} \exp(-\frac{(t-\frac{\rho(x-\mu_x)}{\sigma_x})^2}{2(1-\rho^2)})\mathrm{d}t$$

$\because$ 由正态分布可知 $\dfrac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \displaystyle\int_{-\infty}^{+\infty} \exp(-\frac{(t-\frac{\rho(x-\mu_x)}{\sigma_x})^2}{2(1-\rho^2)})\mathrm{d}t = 1$

$$\therefore f_x(x) = \frac{\exp(-\frac{1}{2}(\frac{x-\mu_x}{\sigma_x})^2)}{\sqrt{2\pi}\sigma_x}$$

$\therefore X \sim N(\mu_x, \sigma_x^2)$

4.

3.

(1)

$$\because 1 = \int_2^4 \mathrm{d}y \int_0^2 k(6-x-y)\mathrm{d}x = \int_2^4 2k\,(5-y)\,\mathrm{d}y = 8k$$

$$\therefore k = \frac{1}{8}$$

(2)

$$\therefore P(X<1,Y<3)=\int_2^3 \mathrm{d}y\int_0^1 \frac{1}{8}(6-x-y)\mathrm{d}x=\int_2^3(\frac{11}{16}-\frac{y}{8})\mathrm{d}y=\frac{3}{8}$$

(3)

$$\therefore P(X<1.5)=\int_2^4 \mathrm{d}y\int_0^{1.5} \frac{1}{8}(6-x-y)\mathrm{d}x=\int_2^4(\frac{63}{64}-\frac{3y}{16})\mathrm{d}y=\frac{27}{32}$$

(4)

$$\therefore P(X+Y\leqslant 4)=\int_2^4 \mathrm{d}y\int_0^{4-y} \frac{1}{8}(6-x-y)\mathrm{d}x=\int_2^4 \frac{(y-8)\,(y-4)}{16}\mathrm{d}y=\frac{2}{3}$$

4.

(1)

$$P(X<Y)=\int_0^{+\infty} \mathrm{d}y\int_0^y f_x(x)f_y(y)\mathrm{d}x=\int_0^{+\infty} F_x(y)f_y(y)\mathrm{d}y=\int_0^{+\infty} F_x(x)f_y(x)\mathrm{d}x$$

(2)

$$\therefore F_x(x)=\int_0^x \lambda_1 e^{-\lambda 1x}\mathrm{d}x=1-e^{-\lambda 1x}$$

$$\therefore P(X<Y)=\int_0^{+\infty}(1-e^{-\lambda 1x})\lambda_2 e^{-\lambda 2x}\mathrm{d}x=\frac{\lambda_1}{\lambda_1+\lambda_2}$$

5.

$$\therefore F_x(x)=\lim_{y\to+\infty}F(x,y)=1-e^{-x},x>0$$

$$F_y(y)=\lim_{x\to+\infty}F(x,y)=1-e^{-y},y>0$$

$$\therefore F_x(x)=\begin{cases}1-e^{-x}, & x>0\\ 0, & x\leqslant 0\end{cases}$$

$$F_y(y)=\begin{cases}1-e^{-y}, & x>0\\ 0, & x\leqslant 0\end{cases}$$

6.

所有情况如下:

样本	HHH	HHT	HTH	THH	HTT	THT	TTH	TTT
X	2	2	1	1	1	1	0	0
Y	3	2	2	2	1	1	1	0

最终可得联合分布律和边缘分布律如下:

Y \ X	0	1	2	P(Y=j)
0	1/8	0	0	1/8
1	1/8	2/8	0	3/8

Y \ X	0	1	2	P(Y=j)
0	0	2/8	1/8	3/8
0	0	0	1/8	1/8
P(X=i)	1/4	2/4	1/4	1

7.

$$\therefore f_x = \int_0^x 4.8y(2-x)\mathrm{d}y = 2.4(2-x)x^2, 0 \leqslant x \leqslant 1$$

$$f_y = \int_y^1 4.8y(2-x)\mathrm{d}x = 2.4(3-4y+y^2), 0 \leqslant y \leqslant 1$$

$$\therefore f_x = \begin{cases} 2.4(2-x)x^2, & 0 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_y = \begin{cases} 2.4(3-4y+y^2), & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

8.

$$\therefore f_x(x) = \begin{cases} \int_x^{+\infty} e^{-y}\mathrm{d}y = e^{-x}, & x > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_x(x) = \begin{cases} \int_0^y e^{-y}\mathrm{d}x = ye^{-y}, & y > 0 \\ 0, & \text{otherwise} \end{cases}$$

9.

(1)

$$\therefore 1 = \int_{-1}^1 \mathrm{d}x \int_{x^2}^1 cx^2y\mathrm{d}y = c\int_{-1}^1 x^2\mathrm{d}x \int_{x^2}^1 y\mathrm{d}y = c\int_{-1}^1 x^2(\frac{1}{2} - \frac{x^4}{2})\mathrm{d}x = \frac{4c}{21}$$

$$\therefore c = \frac{21}{4}$$

(2)

$$\therefore f_x(x) = \begin{cases} \int_{x^2}^1 \frac{21}{4}x^2y\mathrm{d}y = \frac{21}{8}x^2\left(1-x^4\right), & -1 \leqslant x \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

$$f_x(x) = \begin{cases} \int_{-\sqrt{y}}^{\sqrt{y}} \frac{21}{4}x^2y\mathrm{d}x = \frac{7}{2}y^{\frac{5}{2}}, & 0 \leqslant y \leqslant 1 \\ 0, & \text{otherwise} \end{cases}$$

17.

(1)

$$\therefore F_x(x) = F(x, +\infty) = \begin{cases} 1 - e^{-ax}, & x \geqslant 0 \\ 0, & \text{otherwise} \end{cases}$$

$$F_y(y) = F(+\infty, y) = \begin{cases} y, & 0 \leqslant y \leqslant 1 \\ 1, & y > 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore F_x(x)F_y(y) = F(x, y) = \begin{cases} (1 - e^{-ax})y, & 0 \leqslant y \leqslant 1 \\ 1 - e^{-ax}, & y > 1 \\ 0, & \text{otherwise} \end{cases}$$

$\therefore X, Y$ 相互独立.

(2)

$$\therefore P(X = x) = \sum_{y=1}^{+\infty} p^2(1-p)^{x+y-2} = p^2(1-p)^{x-1}\sum_{y=1}^{+\infty}(1-p)^{y-1} = p(1-p)^{x-1}$$

$$P(Y = y) = p(1-p)^{y-1}$$

$$\therefore P(X = x, Y = y) = P(X = x)P(Y = y) = p^2(1-p)^{x+y-2}$$

$\therefore X, Y$ 相互独立.

18.

(1)

$$\therefore f(x, y) = \begin{cases} \frac{1}{2}e^{-\frac{y}{2}}, & 0 < x < 1, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

(2)

$\therefore \Delta = 4X^2 - 4Y \geqslant 0$, 即 $X^2 \geqslant Y$

$$\therefore P(X^2 \geqslant Y) = \int_0^1 \mathrm{d}x \int_0^{x^2} \frac{1}{2}e^{-\frac{y}{2}}\mathrm{d}y = 1 - \int_0^1 e^{-\frac{x^2}{2}}\mathrm{d} = 1 - \sqrt{2}\pi(\Phi(1) - \frac{1}{2})$$

6.

设该正态分布的随机变量被划分成 $\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$, 其中 y 是一个向量. 则有概率密度函数

$$f(x, y) = (2\pi)^{-\frac{n}{2}}|\Sigma|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}^T \begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{yx} & \Sigma_{yy} \end{pmatrix}^{-1} \begin{pmatrix} x - \mu_x \\ y - \mu_y \end{pmatrix}\right)$$

由 Schur 补可知

$$\begin{pmatrix} \Sigma_{xx} & \Sigma_{xy} \\ \Sigma_{xy} & \Sigma_{yy} \end{pmatrix} = \begin{pmatrix} I & 0 \\ \Sigma_{yx}\Sigma_{xx}^{-1} & I \end{pmatrix} \begin{pmatrix} \Sigma_{xx} & 0 \\ 0 & \Sigma_{yy} - \Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy} \end{pmatrix} \begin{pmatrix} I & \Sigma_{xx}^{-1}\Sigma_{xy} \\ 0 & I \end{pmatrix}$$

并且我们有

$$\begin{pmatrix} I & 0 \\ \Sigma_{yx}\Sigma_{xx}^{-1} & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -\Sigma_{yx}\Sigma_{xx}^{-1} & I \end{pmatrix}, \begin{pmatrix} I & \Sigma_{xx}^{-1}\Sigma_{xy} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} I & -\Sigma_{xx}^{-1}\Sigma_{xy} \\ 0 & I \end{pmatrix}$$

令 $\Theta_{yy}=\Sigma_{yy}-\Sigma_{yx}\Sigma_{xx}^{-1}\Sigma_{xy}$

将其带入 $f(x,y)$ 可得

$$
\begin{aligned}
f(x,y)&=(2\pi)^{-\frac{n}{2}}\left|\Sigma\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\begin{pmatrix}x-\mu_x\\y-\mu_y\end{pmatrix}^T\begin{pmatrix}\Sigma_{xx}&\Sigma_{xy}\\\Sigma_{yx}&\Sigma_{yy}\end{pmatrix}^{-1}\begin{pmatrix}x-\mu_x\\y-\mu_y\end{pmatrix}\right)\\
&=(2\pi)^{-\frac{n}{2}}\left|\Sigma\right|^{-\frac{1}{2}}\exp(-\frac{1}{2}(x-\mu_x)^T\Sigma_{xx}^{-1}(x-\mu_x)\\
&-\frac{1}{2}(y-(\mu_y+\Sigma_{yx}\Sigma_{xx}^{-1}(x-\mu_x)))^T\Theta_{yy}(y-(\mu_y+\Sigma_{yx}\Sigma_{xx}^{-1}(x-\mu_x))))\\
&=(2\pi)^{-\frac{1}{2}}\left|\Sigma_{xx}\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(x-\mu_x)^T\Sigma_{xx}^{-1}(x-\mu_x)\right)\\
&\cdot(2\pi)^{-\frac{n-1}{2}}\left|\Theta_{yy}\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(y-(\mu_y+\Sigma_{yx}\Sigma_{xx}^{-1}(x-\mu_x)))^T\Theta_{yy}(y-(\mu_y+\Sigma_{yx}\Sigma_{xx}^{-1}(x-\mu_x)))\right)
\end{aligned}
$$

因为 $f(x,y)=p(x,y)=p(x)p(y|x)=f_x(x)p(y|x)$, 有

$$
f_x(x)=(2\pi)^{-\frac{1}{2}}\left|\Sigma_{xx}\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(x-\mu_x)^T\Sigma_{xx}^{-1}(x-\mu_x)\right)
$$

$$
p(y|x)=\\
(2\pi)^{-\frac{n-1}{2}}\left|\Theta_{yy}\right|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(y-(\mu_y+\Sigma_{yx}\Sigma_{xx}^{-1}(x-\mu_x)))^T\Theta_{yy}(y-(\mu_y+\Sigma_{yx}\Sigma_{xx}^{-1}(x-\mu_x)))\right)
$$

可以看出, x 的概率密度函数 $f_x(x)$ 依然满足正态分布

所以 $X\sim N(\mu_x,\Sigma_{xx})$