习题1.1：（A）5，9，16（4），20，（B）5，7，

习题1.2：1，6，7（2），10（2、3、6），11（4），12，14，（B），2（2），3，5

习题1.1 (A)

5.

无界:

$\forall M>0, \exists X \in A,$ 使得$|X|>M$

上无界:

$\forall M \in \mathbf{R}, \exists X \in A,$ 使得$X>M$

下无界:

$\forall M \in \mathbf{R}, \exists X \in A,$ 使得$X<M$

9.

(1) 满射

(2) 满射

(3) 单射

16.(4)

是由 $y=\sqrt{x}$, $y=\ln x, y=x^2$ 和 $y=\arcsin x$ 复合而成的.

$\because \sqrt{1+\ln^2(\arcsin x)}$

$\therefore 1+\ln^2(\arcsin x) \geq 0$

$\therefore \arcsin x>0$

$\therefore 0 < x \le 1$

20.

设底面半径为r.

底面半径$C = 2\pi r = \theta R$

$\therefore r = \dfrac{\theta R}{2\pi}$

$\therefore$ 底面面积$S = \pi r^2 = \dfrac{\theta^2 R^2}{4\pi}$

$\therefore$ 高$h = \sqrt{R^2 - r^2} = \sqrt{R^2 - \dfrac{\theta^2 R^2}{4\pi^2}}$

$\therefore$ 容积$V(\theta) = \dfrac{1}{3} Sh = \dfrac{\theta^2 R^2}{12\pi}\sqrt{R^2 - \dfrac{\theta^2 R^2}{4\pi^2}}$

其中定义域为：

$\because R^2 - r^2 = (R+r)(R-r) > 0$

$\therefore R > r = \dfrac{\theta R}{2\pi} > 0$

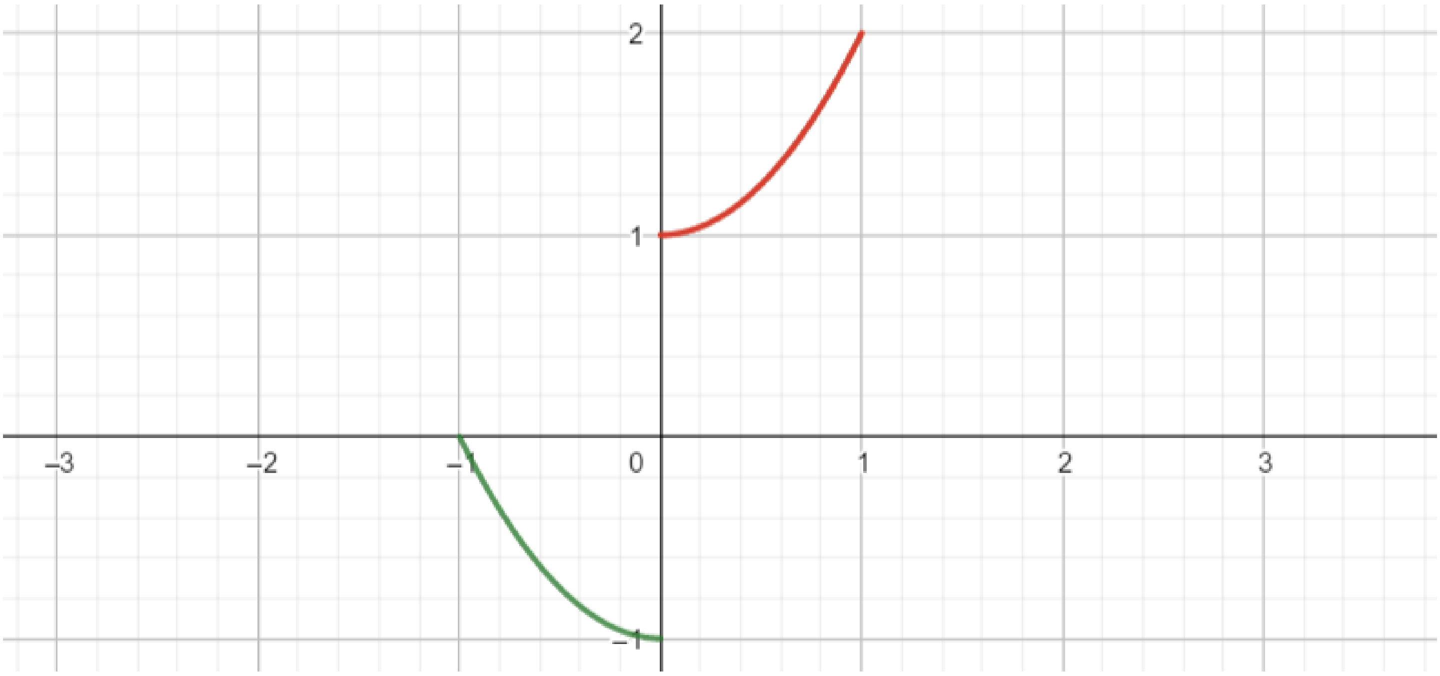
$\therefore 0 < \theta < 2\pi$

$\therefore$ 定义域为$(0, 2\pi)$

习题1.1 (B)

5.

$$f(x) = \begin{cases} x^2 - 1, x \in [-1, 0) \\ x^2 + 1, x \in [0, 1] \end{cases}$$



所以

$$f^{-1}(x)=\begin{cases}-\sqrt{x+1}, x\in(-1,0] \\ \sqrt{x-1}, x\in[1,2]\end{cases}$$

7.

$\because f(xy)=f(x)f(y)-x-y$

令$x=1,y=1.$

$\therefore f(1)=f^2(1)-2$

$\therefore f(1)=2$或$f(1)=-1$

令$x=0,y=0.$

$\therefore f(0)=f^2(0)$

$\therefore f(0)=0$或$f(0)=1$

假设$f(1)=-1$

令$y=1.$

$\therefore f(x)=-f(x)-x-1$

$\therefore f(x)=\frac{-x-1}{2}$

$\therefore f(0)=-\frac{1}{2}$不符合题意

假设$f(1)=2$

令$y=1.$

$\therefore f(x)=2f(x)-x-1$

$\therefore f(x)=x+1$

$\therefore f(0)=0+1=1$符合题意

综上$f(x)=x+1$

习题1.2 (A)

1.

(1) 不能, 无穷多个ε并不能代表对任意的ε都成立.

(2) 不能, 无穷多项a_n并不能代表对N以后的任意a_n都成立.

(3) 不能, 可能找到比ε_0还要小的正数ε使得$|a_n-a|>\varepsilon$

6.

(1)

正确.

证明:

$\because \lim\limits_{n\to\infty} a_n = A$

$\therefore \forall\varepsilon > 0, \exists N > 0,$使得$n > N$时$, |a_n - A| < \varepsilon$

当$A = 0, n > N$时$,$有$|a_n| < \varepsilon,$即有$||a_n| - |0|| < \varepsilon$成立

当$A \neq 0, n > N$时$,$由保号性可知a_n和A同号.

$\therefore ||a_n| - |A|| = |a_n - A| < \varepsilon$

综上$\lim\limits_{n\to\infty} |a_n| = |A|$

(2)

不正确.

证明:

令$\{a_n\}$的通项公式$a_n = 1, A = -1$

易知$\lim\limits_{n\to\infty} |a_n| = |A| = 1$且$\lim\limits_{n\to\infty} a_n = 1 \neq A = -1$

(3)

正确.

证明:

$\because \lim\limits_{n\to\infty} |a_n| = 0$

$\therefore \forall\varepsilon > 0, \exists N > 0,$使得$n > N$时$, ||a_n| - 0| < \varepsilon$

$\therefore$当$n > N$时$, |a_n - 0| = ||a_n|| = ||a_n| - 0| < \varepsilon$

$\therefore \lim\limits_{n\to\infty} a_n = 0$

(4)

正确.

证明:

$\because \lim\limits_{n\to\infty} a_n = A$

$\therefore \forall\varepsilon > 0, \exists N > 0,$使得$n > N$时$, |a_n - A| < \varepsilon$

$\because n + 1 > n > N$

$\therefore |a_{n+1} - A| < \varepsilon$

$\therefore \lim\limits_{n\to\infty} a_{n+1} = A$

(5)

不正确.

证明:

令$A = 0,$则a_n可能为$0.$

$\because$ 分母不能为$0.$

$\therefore \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = A$不成立

(6)

不正确.

证明:

令$\alpha = 0, \{a_n\}$的通项公式$a_n = 1, A = 2$

$\therefore \lim\limits_{n\to\infty} \alpha a_n = \alpha A = 0$

但是此时 $\lim\limits_{n\to\infty} a_n = 1 \neq A = 2$

7.(2)

对任意$\varepsilon > 0$,要使

$$|n-\sqrt{n^2-n}-\frac{1}{2}|<\varepsilon$$

成立

$$\begin{aligned}|\sqrt{n^2}-\sqrt{n^2-n}-\frac{1}{2}| &= |\frac{n}{\sqrt{n^2}+\sqrt{n^2-n}}-\frac{1}{2}| \\ &\leq |\frac{n}{\sqrt{n^2}+\sqrt{n^2-2n+1}}-\frac{1}{2}| \\ &= |\frac{n}{2n-1}-\frac{1}{2}| \\ &= |\frac{1}{4n-2}| \\ &\leq \varepsilon\end{aligned}$$

$\therefore 4n-2 \geq \dfrac{1}{\varepsilon}$

$\therefore n \geq \dfrac{1}{4\varepsilon}+\dfrac{1}{2}$

$\therefore n > N = \min\{1, [\dfrac{1}{4\varepsilon}+\dfrac{1}{2}]\}$

10.

(2)

$$\begin{aligned}&\lim_{n\to\infty}\frac{3^n+(-2)^n}{3^{n+1}+(-2)^{n+1}} \\ =&\lim_{n\to\infty}\frac{1+(\frac{-2}{3})^n}{3-2(\frac{-2}{3})^n} \\ =&\frac{\lim\limits_{n\to\infty}1+(\frac{-2}{3})^n}{\lim\limits_{n\to\infty}3-2(\frac{-2}{3})^n} \\ =&\frac{1}{3}\end{aligned}$$

(3)

$$
\begin{aligned}
&\lim_{n\to\infty}\left(\frac{1+2+\cdots+n}{n+2}-\frac{n}{2}\right)\\
=&\lim_{n\to\infty}\left(\frac{n(1+n)}{2n+4}-\frac{n(n+2)}{2n+4}\right)\\
=&\lim_{n\to\infty}\frac{-n}{2n+4}\\
=&\lim_{n\to\infty}\frac{-1}{2+\frac{4}{n}}\\
=&\frac{-1}{\lim\limits_{n\to\infty}(2+\frac{4}{n})}\\
=&-\frac{1}{2}
\end{aligned}
$$

(6)

易知

$$\sqrt[n]{2}\le\sqrt[n]{2+\sin^2 n}\le\sqrt[n]{3}$$

又有

$$\lim_{n\to\infty}\sqrt[n]{2}=1,\lim_{n\to\infty}\sqrt[n]{3}=1$$

由夹逼定理可知

$$\lim_{n\to\infty}\sqrt[n]{2+\sin^2 n}=1$$

11.(4)

为了证明$\{a_n\}$收敛,只需证明它满足Chauchy条件.由于$\forall n,p\in\mathbf{N}_+$

$$
\begin{aligned}
|a_{n+p}-a_n| &= \frac{1}{(n+1)!}+\frac{1}{(n+2)!}+\cdots+\frac{1}{(n+p)!} \\
&< \frac{1}{n(n+1)}+\frac{1}{(n+1)(n+2)}+\cdots+\frac{1}{(n+p-1)(n+p)} \\
&= \frac{1}{n}-\frac{1}{n+1}+\frac{1}{n+1}-\frac{1}{n+2}+\cdots+\frac{1}{n+p-1}-\frac{1}{n+p} \\
&= \frac{1}{n}-\frac{1}{n+p} \\
&< \frac{1}{n}
\end{aligned}
$$

所以, $\forall\varepsilon>0$, 只要取$N=[\frac{1}{\varepsilon}]$, 则$\forall n>N$以及$p\in\mathbf{N}_+$, 恒有$|a_{n+p}-a_n|<\varepsilon$, 故$\{a_n\}$满足Cauchy条件, 所以收敛.

12.

证明:

$\because$ 该数列是Cauchy数列

$\therefore$ 该数列收敛

$\therefore$ 该数列有界

$\therefore$ 由Weierstrass定理知该数列存在收敛子列

14.

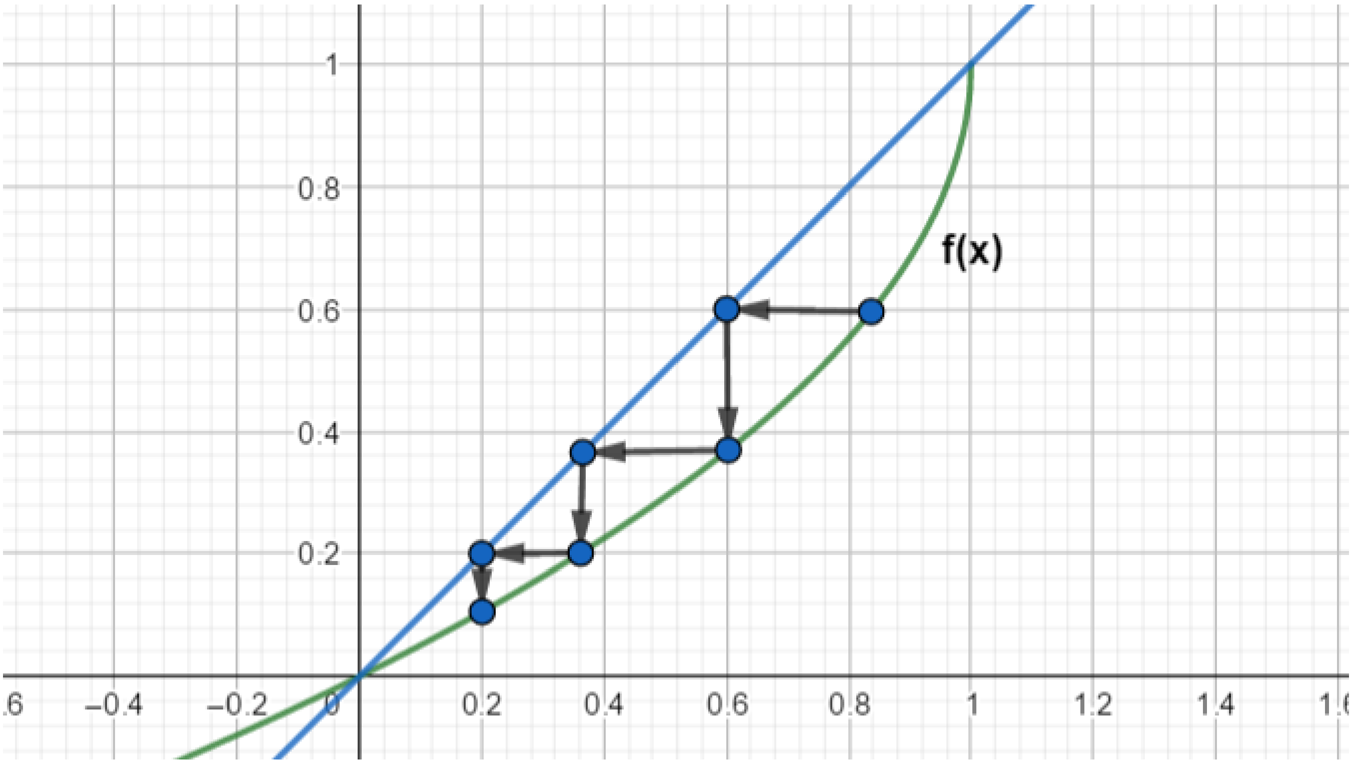
证明数列$\{x_n\}$收敛:

令$x_{n+1}=x_n$, 带入$x_{n+1}=1-\sqrt{1-x_n}$, 解得 :

$x_n=0$或$x_n=1$

令$f(x)=1-\sqrt{1-x}$

即有$f(0)=0, f(1)=1$



由图可知, $f(x)$在$[0,1]$单调递增, 数列$\{x_n\}$是单调减数列.

由数学归纳法:

(1) 当$x_n=x_1$时, $0<x_1<1$成立

(2)

假设当$x_n=x_k$时, $0<x_k<1$成立

当$x_n=x_{k+1}$时,

$\because f(x)$在$[0,1]$单调递增

$\therefore f(0)<f(x_k)<f(1)$

$\therefore 0<x_{k+1}<1$

综上可知$\{x_n\}$是有界的, 又因$\{x_n\}$是单调递减数列

$\therefore \{x_n\}$收敛

求 $\lim\limits_{n\to\infty} x_n$:

设 $\lim\limits_{n\to\infty} x_n=A, f(x)=1-\sqrt{1-x}$

$\because 0<x_n<1$

$\therefore A\geq 0$

假设$A > 0$

$\therefore \forall \varepsilon > 0, \exists N, \forall n > N,$使得$0 \leq |x_n - A| = x_n - A < \varepsilon$

$\therefore A \leq x_n < A + \varepsilon$

令$A = 1 - \sqrt{1-x}$

$\therefore 1 - x = (1-A)^2$

$\therefore x = 1 - (1-A)^2 > A$

若$x_{n-2} = 1 - (1-A)^2,$则$x_{n-1} = f(x_{n-2}) = A$

$\therefore x_n = f(x_{n-1}) < x_{n-1} = A,$与$A \leq x_n$矛盾

若$A < x_{n-2} < 1 - (1-A)^2,$则$x_{n-1} < f(x_{n-2}) = A$

$\therefore x_n = f(x_{n-1}) < x_{n-1} < A,$与$A \leq x_n$矛盾

综上

$A = 0$

求$\lim\limits_{n\to\infty} \dfrac{x_{n+1}}{x_n}$:

证明 $\lim\limits_{n\to\infty}(n - \sqrt{n^2 - n}) = \dfrac{1}{2}$

对任意$\varepsilon > 0,$要使

$$|n - \sqrt{n^2 - n} - \frac{1}{2}| < \varepsilon$$

成立

$$\begin{aligned} |\sqrt{n^2} - \sqrt{n^2 - n} - \frac{1}{2}| &= |\frac{n}{\sqrt{n^2} + \sqrt{n^2 - n}} - \frac{1}{2}| \\ &\leq |\frac{n}{\sqrt{n^2} + \sqrt{n^2 - 2n + 1}} - \frac{1}{2}| \\ &= |\frac{n}{2n-1} - \frac{1}{2}| \\ &= |\frac{1}{4n-2}| \\ &\leq \varepsilon \end{aligned}$$

$\therefore 4n-2 \geq \frac{1}{\varepsilon}$

$\therefore n \geq \frac{1}{4\varepsilon}+\frac{1}{2}$

$\therefore n > N=\min\{1,[\frac{1}{4\varepsilon}+\frac{1}{2}]\}$

$\therefore \lim\limits_{n\to\infty}(n-\sqrt{n^2-n})=\frac{1}{2}$

对原式:

$$
\begin{aligned}
\lim_{n\to\infty}\frac{x_{n+1}}{x_n} &= \frac{1-\sqrt{1-x_n}}{x_n} \\
&= \frac{1}{x_n}-\sqrt{\frac{1}{x_n^2}-\frac{1}{x_n}} \\
&= \frac{1}{2}
\end{aligned}
$$

1.2 (B)

2.(2)

$$
\begin{aligned}
&\lim_{n\to\infty}[1-\frac{1}{3}+\frac{1}{9}-\cdots+\frac{(-1)^{n-1}}{3^{n-1}}] \\
=&\lim_{n\to\infty}\frac{1-(-\frac{1}{3})^n}{1+\frac{1}{3}} \\
=&\lim_{n\to\infty}\frac{3}{4}-\frac{3}{4}(-\frac{1}{3})^n \\
=&\frac{3}{4}
\end{aligned}
$$

3.

(1)

$\because \lim\limits_{n\to\infty} a_n = 0$

$\therefore \forall\varepsilon > 0, \exists N > 0, \forall n > N$,使得$|a_n| < \varepsilon$

令$A = a_1 + \cdots + a_N$,易知A是一个存在确切值的实数

$$
\begin{aligned}
\therefore |\frac{a_1 + a_2 + \cdots + a_n}{n}| &= |\frac{a_1 + \cdots + a_N}{n} + \frac{a_{N+1} + \cdots + a_n}{n}| \\
&= |\frac{A}{n} + \frac{0 \times (n-N)}{n}| \\
&= |\frac{A}{n}| \\
&< \varepsilon
\end{aligned}
$$

$\therefore \exists N = \left[|\frac{A}{\varepsilon}|\right]$,使得$|\frac{a_1 + a_2 + \cdots + a_n}{n} - 0| < \varepsilon$成立

$\therefore \lim\limits_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0$

(2)

$\because \lim\limits_{n\to\infty} a_n = a$

$\therefore \forall\varepsilon > 0, \exists N > 0, \forall n > N$,使得$|a_n - a| < \varepsilon$

令$A = a_1 + \cdots + a_N$,易知A是一个存在确切值的实数

$$
\begin{aligned}
\therefore |\frac{a_1 + a_2 + \cdots + a_n}{n} - a| &= |\frac{a_1 + \cdots + a_N}{n} + \frac{a_{N+1} + \cdots + a_n}{n} - a| \\
&= |\frac{A}{n} + \frac{(n-N)a}{n} - a| \\
&= |\frac{A - Na}{n}| \\
&< \varepsilon
\end{aligned}
$$

$\therefore \exists N = \left[|\frac{A - Na}{\varepsilon}|\right]$,使得$|\frac{a_1 + a_2 + \cdots + a_n}{n} - a| < \varepsilon$成立

$\therefore \lim\limits_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = a$

5.

设$\{b_n\}$是单调增数列$\{a_n\}$的一个收敛子列

$\therefore \forall \varepsilon > 0, \exists M > 0, \forall n > M,$使得$|b_n - a| < \varepsilon$

$\because \{a_n\}$是单调递增数列$, b_M \in \{a_n\}$

当$n > N, N$满足$a_N = b_M$时,

对于每个a_n,总能找到$c_n < a_n < d_n$,其中$c_n, d_n \in \{b_n\}$

$\therefore$ 构造出新单调增数列$\{c_n\}$和$\{d_n\}$

$\because$ 单调增数列$\{c_n\}$和$\{d_n\}$都是收敛数列$\{b_n\}$的子列

$\therefore \lim\limits_{n \to \infty} c_n = a, \lim\limits_{n \to \infty} d_n = a$

由夹逼定理得 $\lim\limits_{n \to \infty} a_n = a$