Greedy Algorithms

Data Structures and Algorithms

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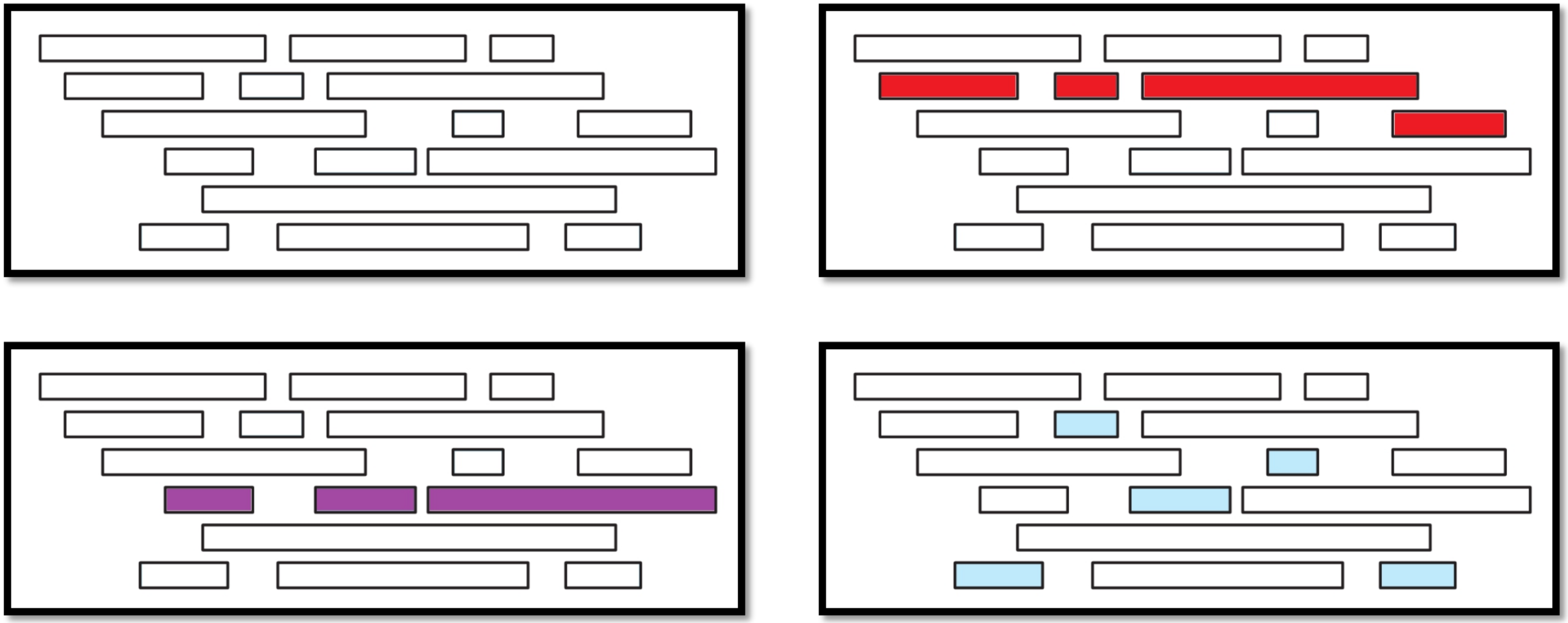
The Greedy Strategy

- For many games, you should *think ahead*, a strategy which focuses on immediate advantage could easily lead to defeat.
  - Such as playing chess.
- But for many other games, you can do quite well by simply making whichever move *seems best at the moment*, without worrying too much about future consequences.
  - Such as building an MST.
- **The Greedy Algorithmic Strategy**: given a problem, build up a solution piece by piece, always choosing the next piece that offers the most obvious and immediate benefit.
  - Sometimes it gives optimal solution.
  - Sometimes it gives near-optimal solution.
  - Or, it simply fails…

An Activity-Selection Problem

- Assume we have one hall and n activities $S = \{a_1, \cdots, a_n\}$.
- Each activity has a start time s_i and a finish time f_i.
- Two activities cannot happen simultaneously in the hall.
- Maximum number of activities we can schedule?



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- Let's start with "divide-and-conquer"
- Define S_i to be the set of activities start after a_i finishes;
  Define F_i to be the set of activities finish before a_i starts.
- $OPT(S) = \max_{1 \le i \le n} \{OPT(F_i) + 1 + OPT(S_i)\}$

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- Let's start with "divide-and-conquer"
- Define S_i to be the set of activities start after a_i finishes;
  Define F_i to be the set of activities finish before a_i starts.
- In any solution, some activity is the first to finish.
- $OPT(S) = \max_{1 \le i \le n} \{1 + OPT(S_i)\}$
- **Observation**: To make $OPT(S)$ as large as possible, the activity that finishes first should finish as early as possible!

An Activity-Selection Problem

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- Two activities cannot happen simultaneously in the hall.
- Maximum number of activities we can schedule?

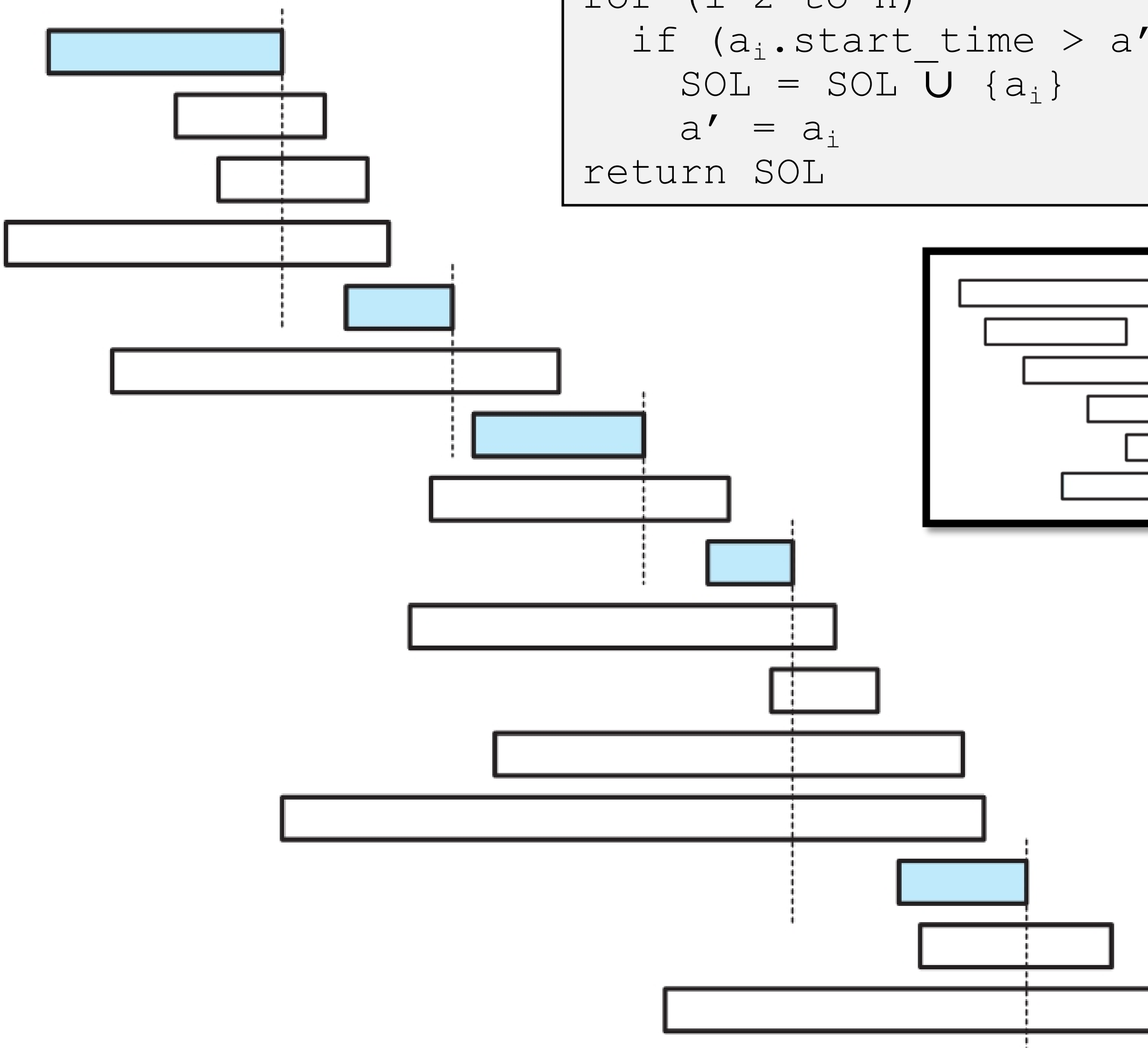
ActivitySelection(S):

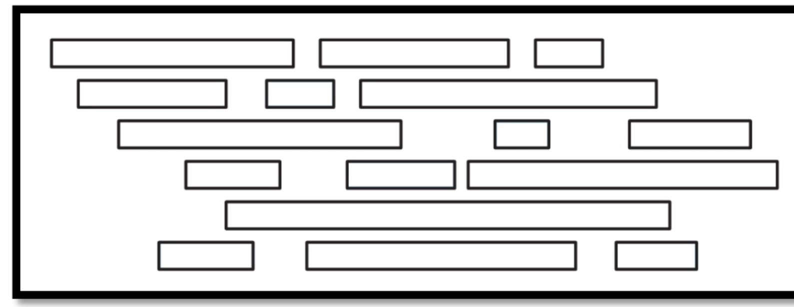
```
Sort S into increasing order of finish time
SOL = {a1}, a' = a1
for (i=2 to n)
  if (ai.start_time > a'.finish_time)
    SOL = SOL ∪ {ai}
    a' = ai
return SOL
```

- But we can have a better implementation!

ActivitySelection(S):

```
Sort S into increasing order of finish time
SOL = {a₁}, a′ = a₁
for (i=2 to n)
  if (aᵢ.start_time > a′.finish_time)
    SOL = SOL ∪ {aᵢ}
    a′ = aᵢ
return SOL
```





Greedy strategy for the activity-selection problem

Correctness

- The Greedy Algorithm for the Activity-Selection Problem:
  - Add earliest finish activity a' to solution, remove ones overlapping with a'.
  - Repeat until all activities are processed.
- How to formally prove this algorithm is correct?
- **Lemma 1:** let a' be the earliest finishing activity in S, then a' is in some optimal solution of the problem.
- **Proof:**
- Let $OPT(S)$ be an optimal solution to the problem, let a be the earliest finishing activity in $OPT(S)$.
- Assume $a' \notin OPT(S)$, otherwise we are done.
- Then $SOL(S) = OPT(S) + a' - a$ is also a feasible solution, and it has same size as $OPT(S)$.
- So $SOL(S)$ is also an optimal solution.

Greedy strategy for the activity-selection problem

Correctness

- The Greedy Algorithm for the Activity-Selection Problem:
  - Add earliest finish activity a' to solution, remove ones overlapping with a'.
  - Repeat until all activities are processed.
- How to formally prove this algorithm is correct?
- **Lemma 2:** let a' be the earliest finishing activity in S, let S' be the activities starting after a', then $OPT(S') \cup \{a'\}$ is an optimal solution of the problem.
- **Proof:**
- Let $OPT(S)$ be an optimal solution to the original problem, and $a' \in OPT(S)$. (Lemma 1 ensures such solution exists.)
- Thus, $OPT(S) = SOL(S') \cup \{a'\}$.
- If $OPT(S') \cup \{a'\}$ is not an optimal solution to the original problem, then it must be the case that $|SOL(S')| > |OPT(S')|$.
- But this contradicts that $OPT(S')$ is an optimal solution for problem S'.

Greedy strategy for the activity-selection problem

Correctness

- The Greedy Algorithm for the Activity-Selection Problem:
  - Add earliest finish activity a' to solution, remove ones overlapping with a'.
  - Repeat until all activities are processed.
- How to formally prove this algorithm is correct?
- **Lemma 1:** let a' be the earliest finishing activity in S, then a' is in some optimal solution of the problem.
- **Lemma 2:** let a' be the earliest finishing activity in S, let S' be the activities starting after a', then $OPT(S') \cup \{a'\}$ is an optimal solution of the problem.
- **Theorem**: The greedy algorithm is correct.
- **Proof:**
- By induction on size of S.
- When $|S| = 1$, the algorithm clearly is correct.
- When $|S| = n$. Due to Lemma 2, $OPT(S) = OPT(S') \cup \{a'\}$. By induction hypothesis, the algorithm correctly finds $OPT(S')$. So we are done.

Elements of the Greedy Strategy

- If an (optimization) problem has following two properties, then the greedy strategy usually works for it:
- **Optimal substructure;**
- **Greedy property.**

Elements of the Greedy Strategy

Optimal Substructure

- A problem exhibits **optimal substructure** if an optimal solution to the problem contains within it optimal solution(s) to subproblem(s).

- Size n problem $P(n)$, and optimal solution of $P(n)$ is $OPT_{P(n)}$
- Solving $P(n)$ needs to solve size $n' < n$ subproblem $P(n')$
- Optimal solution of $P(n')$: $OPT_{P(n')}$
- $OPT_{P(n)}$ contains a solution of $P(n')$: $SOL_{P(n')}$
- **Optimal Substructure Property**: $SOL_{P(n')} = OPT_{P(n')}$
  (Or these two solutions provide same "utility" under certain metric.)

- **Example:** Lemma 2 in activity selection: let a' be the earliest finishing activity in S, let S' be the activities starting after a', then $OPT(S') \cup \{a'\}$ is some $OPT(S)$.
- There are problems that do not exhibit optimal substructure property!

Elements of the Greedy Strategy

Greedy-Choice Property

- At each step when building a solution, make the choice that looks best for the *current* problem, *without* considering results from subproblems. That is, make local **greedy choice** at each step.

- To solve $P(n)$, currently have k choices a_1 to a_k. If we choose a_i, the problem is reduced to a smaller size n_i subproblem $P(n_i)$.
- If the problem only admits optimal structure:
  - Find i that maximize, $\text{Utility}(a_i + OPT_{P(n_i)})$.
  - We have to compute $OPT_{P(n_i)}$ for all i first.
- With greedy choice:
  - We *have a way* to pick correct i, without knowing any $OPT_{P(n_i)}$.

Identifying a greedy-choice property is the challenging part!

- **Example:** Lemma 1 in activity selection: let a' be the earliest finishing activity in S, then a' is in some optimal solution of the problem.

Fractional Knapsack Problem

- A thief robbing a house finds n items $A = \{a_1, \ldots, a_n\}$.
- Item a_i is worth v_i dollars and weighs w_i pounds.
- The thief can carry at most W pounds in his knapsack.
- The thief can carry fraction of items.
- What should the thief take to maximize his profit?

- **A greedy strategy:** keep taking the most cost efficient item (i.e., $\max\{v_i/w_i\}$) until the knapsack is full.
- The greedy solution is optimal!

Fractional Knapsack Problem

Correctness of the greedy algorithm

- **Lemma 1 [greedy-choice]:** let a_m be a most cost efficient item, then in some optimal solution, at least $w'_m = \max\{w_m, W\}$ pounds of a_m are taken.
- **Proof:**
- Consider an optimal solution, assume $w' < w'_m$ pounds of a_m are taken.
- Now, substitute $w'_m - w'$ pounds of other items with a_m.
- Since a_m is the most cost-efficient, the new solution cannot be worse.
- **Lemma 2 [optimal substructure]:** let a_m be a most cost efficient item in A, then "$OPT_{W-\max\{w_m,W\}}(A - a_m)$ with $\max\{w_m, W\}$ pounds of a_m" is an optimal solution of the problem.
- **Proof:**
- Consider some $OPT_W(A)$ containing $\max\{w_m, W\}$ pounds of a_m.
- If optimal substructure does not hold, then $OPT_W(A)$ gives $SOL_{W-\max\{w_m,W\}}(A - a_m) > OPT_{W-\max\{w_m,W\}}(A - a_m)$.
- But this contradicts the optimality of $OPT_{W-\max\{w_m,W\}}(A - a_m)$.

0-1 Knapsack Problem

- A thief robbing a house finds n items $A = \{a_1, \ldots, a_n\}$.
- Item a_i is worth v_i dollars and weighs w_i pounds.
- The thief can carry at most W pounds in his knapsack.
- The thief **CANNOT** carry fraction of items!
- What should the thief take to maximize his profit?

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- The greedy solution is **NOT** optimal!

0-1 Knapsack Problem

- **A greedy strategy:** keep taking the most cost efficient item (i.e., $\max\{v_i/w_i\}$) until the knapsack is full.
- The greedy solution is **NOT** optimal!
- **A simple counterexample:**
  - There are only two items.
  - Item One has value 2 and weighs 1 pound.
  - Item Two has value W and weighs W pounds.
- The greedy solution can be **arbitrarily bad**!

0-1 Knapsack Problem

Why greedy strategy fail?

- ~~**Lemma 1 [greedy choice]**: let a_m be a most cost efficient item that can fit into the bag, then in some optimal solution, this item is taken.~~
- **Proof:**
- Consider an optimal solution, assume a_m is NOT taken.
- Now, substitute $w' \geq w_m$ pounds of other items with a_m.
- ~~Since a_m is the most cost efficient, the new solution cannot be worse.~~

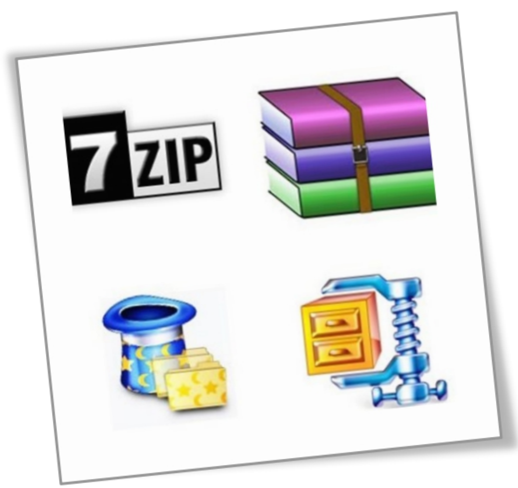
- These w' pounds of items may have aggregate value larger than v_m.
- Let v' be the total value of these w' pounds of items.
- Indeed, $v'/w' \leq v_m/w_m$;
  but it could happen that $v' > v_m$ when $w' > w_m$.

- The optimal substructure property still holds.

A data compression problem

- Assume we have a data file containing 100k characters.
- Further assume the file only uses 6 characters. (Huh?!)
- How to store this file to save space?

- Simplest way: use 3 bits to encode each char.
  - `a=000,b=001,…,f=101`
- This costs 300k bits in total.
- Can we do better?



A data compression problem

- Assume we have a data file containing 100k characters.
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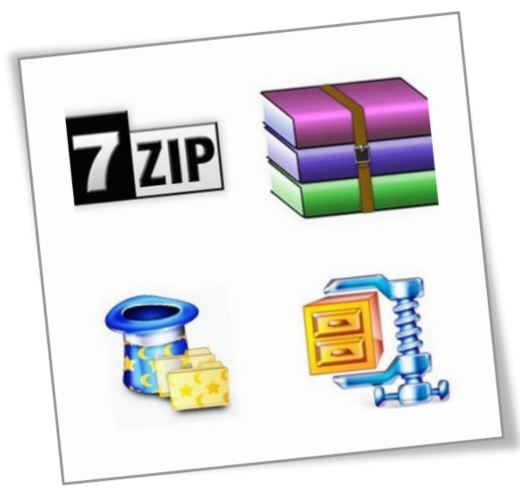
- Instead of using fixed-length codeword for each char, we should **let frequent chars use shorter codewords**. That is, use a *variable-length code*.

	a	b	c	d	e	f
Frequency	45k	13k	12k	16k	9k	5k
Fixed-length Code	000	001	010	011	100	101
Var-length Code	0	00	01	1	10	11

How to decode bit string `000`?



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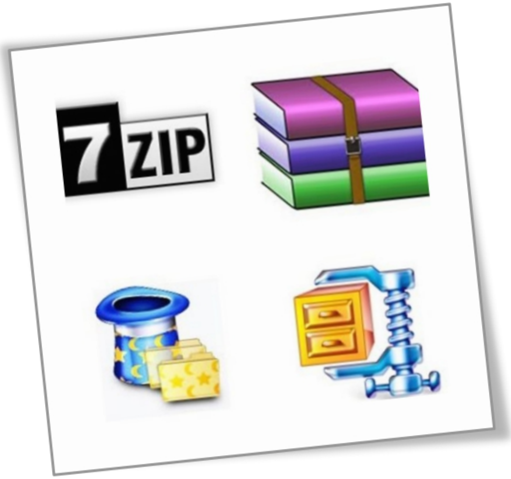
- How to store this file to save space?
- Instead of using fixed-length codeword for each char, we should **let frequent chars use shorter codewords**. That is, use a *variable-length code*.
- To avoid ambiguity in decoding, variable-length code should be **prefix-free**: no codeword is also a prefix of some other codeword.



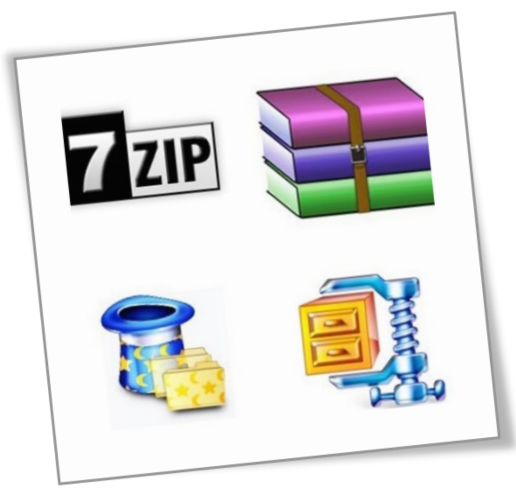
A data compression problem

- Assume we have a data file containing 100k characters.
- Further assume the file only uses 6 characters. (Huh?!)
- How to store this file to save space?
- **Use (prefix-free) variable-length code.**

	a	b	c	d	e	f
Frequency	45k	13k	12k	16k	9k	5k
Fixed-length Code	000	001	010	011	100	101
Var-length Code	0	101	100	111	1101	1100

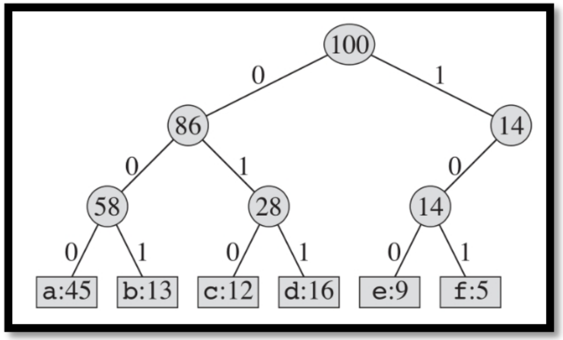
Fixed-len code vs Var-len code: 300k vs 224k.
This is $\approx 25\%$ saving.

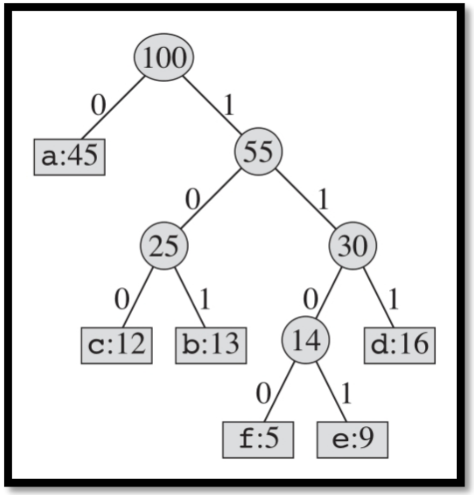
Given data file, how to generate optimal code?



Properties of prefix-free code

- Use a binary tree to visualize a prefix-free code.
  - Each leaf denotes a char.
  - Each internal node: left branch is 0, right branch is 1.
  - Path from root to leaf is the codeword of that char.
- Optimal code must be represented by a *full binary tree*: a tree each node having zero or two children. (WHY?)





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Length of encoded message

- Consider a file using a size n alphabet $C = \{c_1, \ldots, c_n\}$. For each character, let f_i be the frequency of char c_i.
- Let T be a full binary tree representing a prefix-free code. For each character c_i, let $d_T(i)$ be the depth of c_i in T.
- Length of encoded msg is $\sum_{i=1}^{n} f_i \cdot d_T(i)$
- Alternatively, recursively (bottom-up) define each internal node's frequency to be sum of its two children.
- Length of encoded msg is $\sum_{u \text{ in tree}\backslash\text{root}} f_u$

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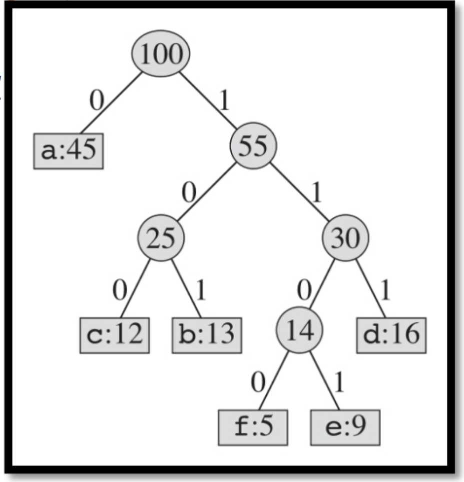
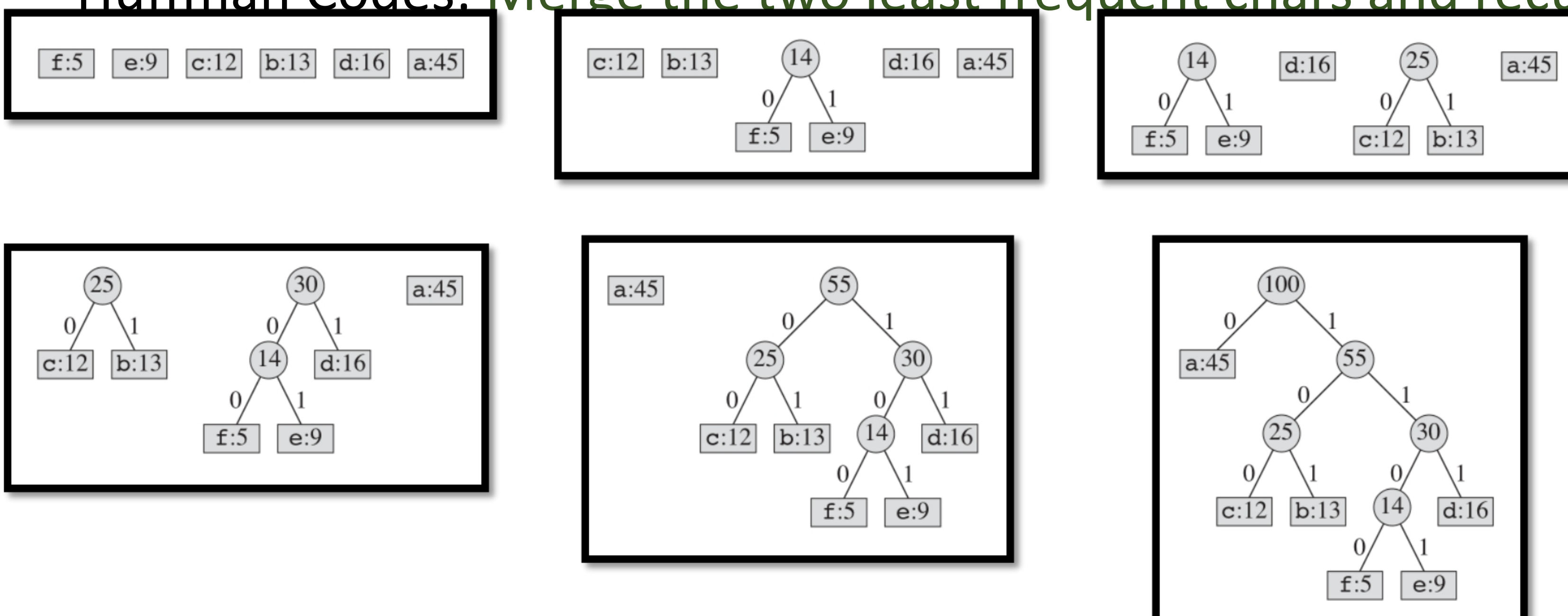


Huffman Codes

- Length of encoded msg is $\sum_{i=1}^{n} f_i \cdot d_T(i)$
- Length of encoded msg is $\sum_{u \text{ in tree}\setminus\text{root}} f_u$
- How to construct optimal prefix-free code?
- Huffman Codes: Merge the two least frequent chars and recurse.



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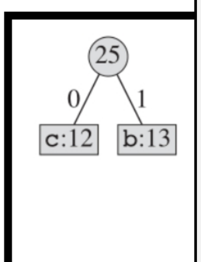
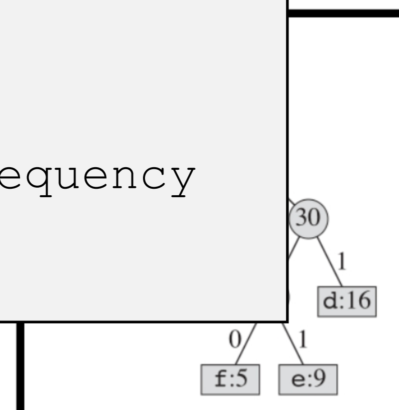


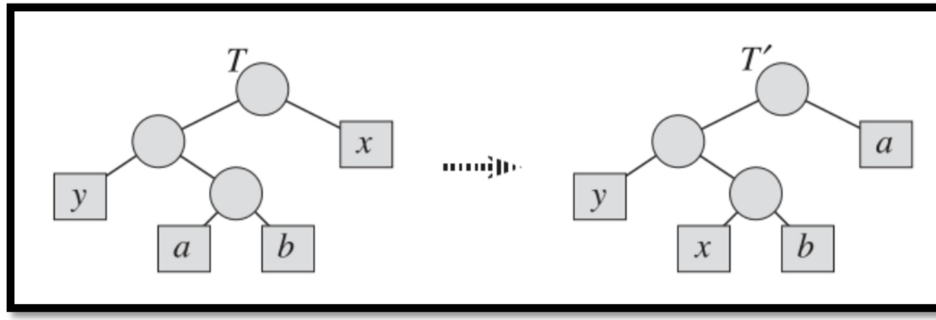
Huffman(C):

Time complexity is $O(n \log n)$

```
Build a priority queue Q based on frequency
for (i=1 to n-1)
  Allocate new node z
  x = z.left = Q.ExtractMin()
  y = z.right = Q.ExtractMin()
  z.frequency = x.frequency + y.frequency
  Q.Insert(z)
return Q.ExtractMin()
```



Huffman Codes
Correctness



- Length of encoded msg is $\sum_{i=1}^{n} f_i \cdot d_T(i) = \sum_{u \text{ in tree}\backslash\text{root}} f_u$
- Huffman Codes: Merge the two least frequent chars and recurse.
- **Lemma 1 [greedy choice]:** Let x and y be two least frequent chars, then in some optimal code tree, x and y are siblings and have largest depth.
- **Proof sketch:**
- Let T be an optimal code tree with depth d.
- Let a and b be siblings with depth d. (Recall T is a full binary tree.)
- Assume a and b are *not* x and y. (Otherwise we are done.)
- Let T' be the code tree obtained by swapping a and x.
- $$\begin{aligned} cost(T') &= cost(T) + (d - d_T(x)) \cdot f_x - (d - d_T(x)) \cdot f_a \\ &= cost(T) + (d - d_T(x)) \cdot (f_x - f_a) \le cost(T) \end{aligned}$$
- Swapping b and y, obtaining T'', further reduces the total cost.
- So T'' must also be an optimal code tree.

Huffman Codes

Correctness

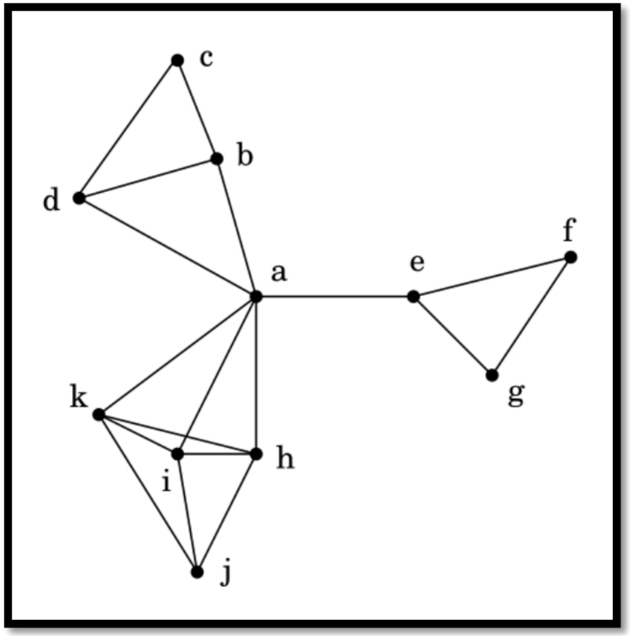
- Length of encoded msg is $\sum_{i=1}^{n} f_i \cdot d_T(i) = \sum_{u \text{ in tree}\backslash\text{root}} f_u$
- Huffman Codes: Merge the two least frequent chars and recurse.
- **Lemma 2 [optimal substructure]:** Let x and y be two least frequent chars in C. Let $C_z = C - \{x, y\} + \{z\}$ with $f_z = f_x + f_y$. Let T_z be an optimal code tree for C_z. Let T be a code tree obtained from T_z by replacing leaf node z with an internal node having x and y as children. Then, T is an optimal code tree for C.
- **Proof sketch:**
- Let T' be an optimal code tree for C, with x and y being sibling leaves.
- $cost(T') = f_x + f_y + \sum_{u \in T'\backslash\text{root and } u \notin \{x,y\}} f_u = f_x + f_y + cost(T'_z)$

  $\geq f_x + f_y + cost(T_z) = cost(T)$
- So T must be an optimal code tree for C.

Set Cover

- Suppose we need to build schools for n towns.
- Each school must be in a town,
  no child should travel more than 30km to reach a school.
- Minimum number of schools we need to build?

- **The Set Cover Problem:**
- **Input:** a universe U of n elements; and $\mathcal{S} = \{S_1, \ldots, S_m\}$ where each $S_i \subseteq U$.
- **Output:** $\mathcal{C} \subseteq \mathcal{S}$ such that $\bigcup_{S_i \in \mathcal{C}} S_i = U$.
  (I.e., a subset of $\mathcal{S}$ that "covers" U.)
- **Goal:** minimize $|\mathcal{C}|$.



Set Cover

- Suppose we need to build schools for n towns.
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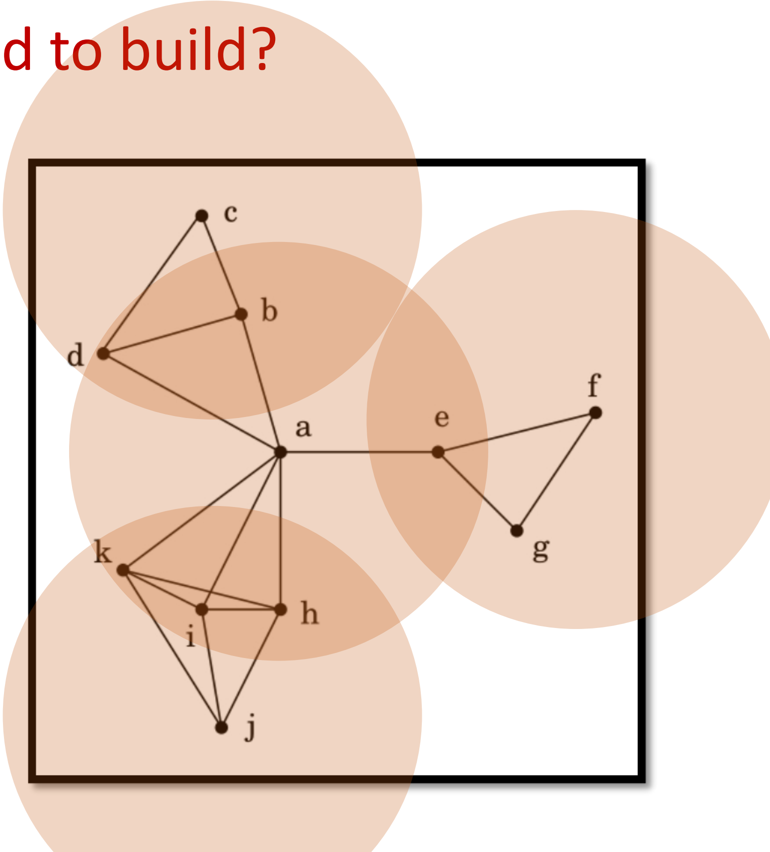
- Minimum number of schools we need to build?
- **Simple greedy strategy:**
- Keep picking the town that covers most remaining uncovered towns, until we are done.
  (Pick the set that covers most uncovered elements, until all elements are covered.)
- Greedy solution: a, f, c, j
- Can we do better?



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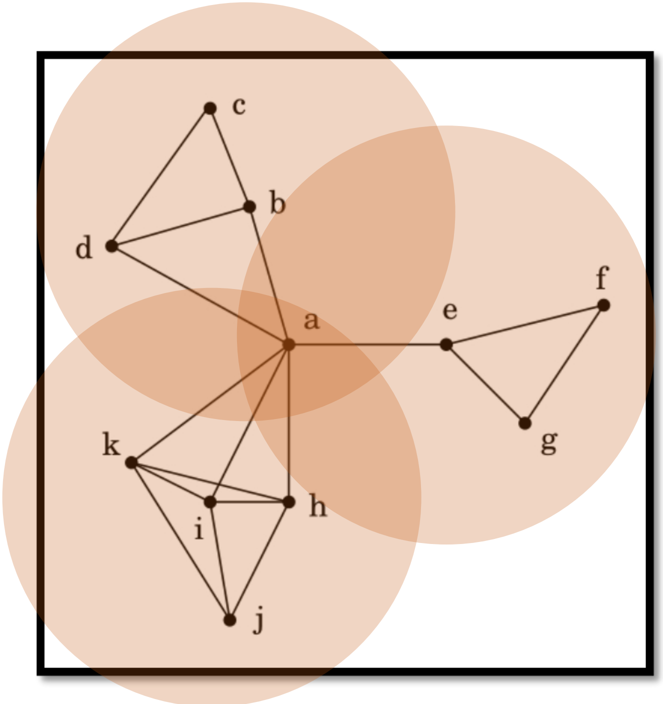
- Optimal solution: b, e, i



Set Cover

Greedy solution is close to optimal

- The Set Cover Problem:
  - Given a set U of n elements, and $\mathcal{S} = \{S_1, \dots, S_m\}$ where each $S_i \subseteq U$.
  - Output $\mathcal{C} \subseteq \mathcal{S}$ such that $\bigcup_{S_i \in \mathcal{C}} S_i = U$. (I.e., subsets of $\mathcal{S}$ that "cover" U.)
  - Goal is to minimize $|\mathcal{C}|$.
- Simple greedy strategy: Keep picking the set that covers most uncovered elements, until all elements are covered.
- **Theorem**: Suppose the optimal solution uses k sets, then the greedy strategy will use at most $k \ln n$ sets.
- **Proof:** Let n_t be number of uncovered elements after t iterations. (Thus $n_0 = n$.)
- These n_t elements can be covered by some k sets. (The optimal solution will do.)
- So one of the remaining sets will cover at least n_t/k of these uncovered elements.
- Thus $n_{t+1} \le n_t - n_t/k = n_t(1 - 1/k)$
- $n_t \le n_0(1 - 1/k)^t < n_0\left(e^{-1/k}\right)^t = n \cdot e^{-t/k}$ $1 - x < e^{-x}$ when $x \neq 0$
- With $t = k \ln n$ we have $n_t < 1$, by then we must have done!

Set Cover

Greedy solution is close to optimal

- Simple greedy strategy: Keep picking the set the covers most uncovered elements, until all elements are covered.
- **Theorem**: Suppose the optimal solution uses k sets, then the greedy strategy will use at most $k \ln n$ sets.

- So the greedy strategy gives a $\ln n$ approximation algorithm, and it has efficient implementation. (Polynomial runtime.)

- **Can we do better?**
- Most likely, **NO**! If we only care about efficient algorithms. ([*Dinur & Steuer 14*] There is no poly-time $(1 - o(1)) \ln n$ approx. alg. unless **P** = **NP**.)

Summary

- **Basic idea of greedy strategy:** At each step when building a solution, make the choice that looks best at that moment, based on some metric.
- **Properties that make greedy strategy work:**
  - **Optimal substructure [usually easy to prove]:** optimal solution to the problem contains within it optimal solution(s) to subproblem(s).
  - **Greedy choice** [could be hard to identify and prove]: the greedy choice is contained within some optimal solution.

- Greed gives <u>optimal</u> solutions: MST, Huffman codes, …
- Greed gives <u>near-optimal</u> solutions: Set cover, …
- Greed gives <u>arbitrarily bad</u> solutions: 0-1 knapsack, …

Reading

- [CLRS] Ch.16 (16.1-16.3)
- Optional reading:
  - Ch.35 (35.3) of [CLRS] discusses the set cover problem.
  - [Vazirani] and [Williamson & Shmoys] are two standard textbooks on approximation algorithms. Both of them introduce several hard problems that have efficient approximation algorithms using the greedy strategy.
  - Greedy strategy can also be used to solve non-optimization problems, such as the “stable matching” problem. See [Erickson v1] Ch.4 (4.5).