9.(3) 12. 13. 17. 18. (2)

9.(3)

设 $\mu_1=(1,0,0,0),\mu_2=(0,1,0,0),\mu_3=(0,0,1,0),\mu_4=(0,0,0,1)$

$$\therefore(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)=(\mu_1,\mu_2,\mu_3,\mu_4)\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}$$

$$\therefore(\eta_1,\eta_2,\eta_3,\eta_4)=(\mu_1,\mu_2,\mu_3,\mu_4)\begin{pmatrix}1&2&1&0\\1&1&1&1\\0&3&0&-1\\1&1&0&-1\end{pmatrix}$$

$$=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\begin{pmatrix}1&1&1&1\\1&1&-1&-1\\1&-1&1&-1\\1&-1&-1&1\end{pmatrix}^{-1}\begin{pmatrix}1&2&1&0\\1&1&1&1\\0&3&0&-1\\1&1&0&-1\end{pmatrix}$$

$$=(\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4)\begin{pmatrix}\frac{3}{4}&\frac{7}{4}&\frac{1}{2}&-\frac{1}{4}\\\frac{1}{4}&-\frac{1}{4}&\frac{1}{2}&\frac{3}{4}\\-\frac{1}{4}&\frac{3}{4}&0&-\frac{1}{4}\\\frac{1}{4}&-\frac{1}{4}&0&-\frac{1}{4}\end{pmatrix}$$

$\therefore$ 过渡矩阵为 $\begin{pmatrix}\frac{3}{4}&\frac{7}{4}&\frac{1}{2}&-\frac{1}{4}\\\frac{1}{4}&-\frac{1}{4}&\frac{1}{2}&\frac{3}{4}\\-\frac{1}{4}&\frac{3}{4}&0&-\frac{1}{4}\\\frac{1}{4}&-\frac{1}{4}&0&-\frac{1}{4}\end{pmatrix}$

设 $\xi=(1,0,0,-1)$ 在基 $\eta_1,\eta_2,\eta_3,\eta_4$ 下的坐标为 (x_1,x_2,x_3,x_4)

$$\therefore(x_1,x_2,x_3,x_4)=\begin{pmatrix}1&2&1&0\\1&1&1&1\\0&3&0&-1\\1&1&0&-1\end{pmatrix}^{-1}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}=\begin{pmatrix}-2\\-\frac{1}{2}\\4\\-\frac{3}{2}\end{pmatrix}$$

12.

设 $\alpha_1,\alpha_2,\cdots,\alpha_r$ 是 V_2 的一组基, 则 $\alpha_1,\alpha_2,\cdots,\alpha_r$ 是 V_2 的生成元

$\therefore V_2=L(\alpha_1,\alpha_2,\cdots,\alpha_r)$

$\because V_1 \subseteq V_2$

$\therefore \alpha_1, \alpha_2, \cdots, \alpha_r$ 也是 V_2 的生成元

$\therefore V_1 = L(\alpha_1, \alpha_2, \cdots, \alpha_r)$

$\because V_1$ 的维数和 V_2 的维数相等

$\therefore \alpha_1, \alpha_2, \cdots, \alpha_r$ 也是 V_1 的基

$\therefore V_1 = V_2$

13.

(1)

假设有矩阵 B, C, 满足 $A \cdot B = B \cdot A, A \cdot C = C \cdot A$

对于任意 $\alpha, \beta \in P$

$\therefore A \cdot (\alpha B + \beta C) = \alpha A \cdot B + \beta A \cdot C = \alpha B \cdot A + \beta C \cdot A = (\alpha B + \beta C) \cdot A$

$\therefore$ 全体与 A 可交换的矩阵组成了 $P^{n \times n}$ 的一个子空间 $C(A)$

(2)

对任意 $B \in C(A)$, 有 $E \cdot B = B \cdot E$

即只需满足 $B = B$, 易见对任意 $P^{n \times n}$ 上的矩阵均满足该条件

$\therefore C(A) = P^{n \times n}$

(3)

对任意 $B \in C(A)$, 有 $A \cdot B = B \cdot A$

设 $B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$

$\therefore AB = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & n \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 2b_{21} & 2b_{22} & \cdots & 2b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ nb_{n1} & nb_{n2} & \cdots & nb_{nn} \end{pmatrix}$

$$BA=\begin{pmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&\ddots&\vdots\\b_{n1}&b_{n2}&\cdots&b_{nn}\end{pmatrix}\begin{pmatrix}1&0&\cdots&0\\0&2&\cdots&0\\\vdots&\vdots&\ddots&\vdots\\0&0&\cdots&n\end{pmatrix}=\begin{pmatrix}b_{11}&2b_{12}&\cdots&nb_{1n}\\b_{21}&2b_{22}&\cdots&nb_{2n}\\\vdots&\vdots&\ddots&\vdots\\b_{n1}&2b_{n2}&\cdots&nb_{nn}\end{pmatrix}$$

令 $AB=BA$

$\therefore ib_{ij}=jb_{ij}\Rightarrow(i-j)b_{ij}=0$

当 $i\neq j$ 时, $i-j\neq 0\Rightarrow b_{ij}=0$

$\therefore B$ 是对角矩阵

令 $\varepsilon_i=\begin{pmatrix}0&\cdots&0&\cdots&0\\\vdots&\ddots&\vdots&\ddots&\vdots\\0&\cdots&1_{(ii)}&\cdots&0\\\vdots&\ddots&\vdots&\ddots&\vdots\\0&\cdots&0&\cdots&0\end{pmatrix},i=1,2,\cdots,n$

$\because$ 对角矩阵可以由 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 线性表示

$\therefore \varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 是 $C(A)$ 的一组基, $C(A)$ 的维数是 n

17.

$$\begin{pmatrix}3&2&-5&4\\3&-1&3&-3\\3&5&-13&11\end{pmatrix}\xrightarrow[r_3-r_1]{r_2-r_1}\begin{pmatrix}3&2&-5&4\\0&-3&8&-7\\0&3&-8&7\end{pmatrix}\xrightarrow{r_3+r_2}$$
$$\begin{pmatrix}3&2&-5&4\\0&-3&8&-7\\0&0&0&0\end{pmatrix}\xrightarrow[\frac{1}{3}r_1,-\frac{1}{3}r_2]{r_1+\frac{2}{3}r_2}\begin{pmatrix}1&0&\frac{1}{9}&-\frac{2}{9}\\0&1&-\frac{8}{3}&\frac{7}{3}\\0&0&0&0\end{pmatrix}$$

$\therefore\begin{cases}x_3=-\frac{1}{9}x_3+\frac{2}{9}x_4\\x_4=\frac{8}{3}x_3-\frac{7}{3}x_4\end{cases}$

分别令 $\begin{pmatrix}x_3\\x_4\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}x_3\\x_4\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$

基础解系为 $\eta_1=\begin{pmatrix}-\frac{1}{9}\\\frac{8}{3}\\1\\0\end{pmatrix},\eta_2=\begin{pmatrix}\frac{2}{9}\\-\frac{7}{3}\\1\\0\end{pmatrix}$

$\therefore \eta_1,\eta_2$ 是解空间的基, 维数是 2

18. (2)

任取 $\gamma \in L(\alpha_1, \alpha_2) \cap L(\beta_1, \beta_2)$

设 $\gamma = x_1\alpha_1 + x_2\alpha_2 = y_1\beta_1 + y_2\beta_2$

则有 $x_1\alpha_1 + x_2\alpha_2 - y_1\beta_1 - y_2\beta_2 = 0$

即 $\begin{cases} x_1 + x_2 = 0 \\ x_1 - y_2 = 0 \\ x_2 - y_1 - y_2 = 0 \\ x_2 - y_1 = 0 \end{cases}$

解得 $\begin{cases} x_1 = 0 \\ x_2 = 0 \\ y_1 = 0 \\ y_2 = 0 \end{cases}$

$\therefore$ 基为 $(0, 0, 0, 0)$, 维度为 0.