第七次作业
习题3.1：（A）4（1、3），7，9，11，12，14，（B）3，4，6，习题3.2：（A）3（4、6），4（1、3、6、7），6，7，9（1），14，（B）1，3，5，习题3.3：（A）1（3、6、10、14、18、），3（5、8、11、12、13），7（3、7、12），8（1），9（5、11、15）

3.1 (A)

4.

(1)

$\because \sin x$是奇函数,对应几何图形关于原点对称

$\therefore \int_{-\pi}^{\pi} \sin x \mathrm{d}x = 0$

(3)

令$y = \sqrt{a^2 - x^2}$,则$x^2 + y^2 = a^2$,且$0 \leq x \leq a, y \geq 0$

$\therefore$ 对应的几何图形为以a为半径的四分之一圆

$\therefore \int_0^a \sqrt{a^2 - x^2} \mathrm{d}x = \frac{\pi a^2}{4}$

7.

设原来的f为g

$\therefore g \in C[a,b]$,则在$[a,b]$上一致连续

$\therefore$ 有$\forall \varepsilon > 0, \exists \delta > 0$,使得$\forall x', x'' \in [a,b]$,当$|x' - x''| < \delta$,必有

$$|g(x') - g(x'')| < \varepsilon$$

设f改变的有限个点为$x = a_1, a_2, \cdots, a_s$

$\therefore$ 由区间套定理可知,可以划分闭区间$[x_{i-1}, x_i]$
使得$\Delta x = x_i - x_{i-1} \to 0$,有$x \to a_i$

此时对于$x \in [x_{i-1}, x_i], \omega_i = |f(a_i) - g(a_i)|$

$\therefore \sum_{i=1}^{s} \omega_i \Delta x < \varepsilon'$

分割$[a,b]$为n个子区间$[x_{k-1},x_k](k=1,2,\cdots,n)$,并且$a_i \notin [x_{k-1},x_k]$

根据闭区间上连续函数的性质,$\exists \xi_k', \xi_k'' \in [x_{k-1},x_k]$,使得

$$f(\xi_k') = M_k, f(\xi_k'') = m_k$$

$\therefore \omega_k = f(\xi_k') - f(\xi_k'') < \varepsilon$

$\therefore \sum_{k=1}^{n} \Delta x_k < \epsilon \sum_{k=1}^{n} \Delta x_k = \varepsilon(b-a)$

$\therefore \sum_{i=1}^{s} \omega_i \Delta x + \sum_{k=1}^{n} \Delta x_k < \varepsilon(b-a) + \varepsilon'$

$\therefore$ 改变f的有限个点不影响可积性和积分值

9.

(1)

不正确.

对于$a=-\frac{\pi}{2}, b=\frac{\pi}{2}, f(x)=\sin x$,有$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \mathrm{d}x = 0$

但是$f(-\frac{\pi}{2}) = -1$

(2)

不正确.

f可能没有间断点

(3)

不正确.

令$f(x) = \begin{cases} 1, & x为有理数 \\ -1, & x为无理数 \end{cases}$

此时$|f|$在$[a,b]$上可积, f在$[a,b]$上不可积

(4)

不正确.

令$f(x)=\begin{cases}1, & x\text{为有理数}\\ -1, & x\text{为无理数}\end{cases}$

令$g(x)=\begin{cases}-1, & x\text{为有理数}\\ 1, & x\text{为无理数}\end{cases}$

此时f与g在$[a,b]$上都不可积, 但$f+g$在$[a,b]$上可积

(5)

正确.

$\because f$在$[a,b]$上黎曼可积

$\therefore \sum \omega_i \Delta x_i < \varepsilon$, 其中$\omega_i = |f(x_i)-f(x_{i-1})|$

因为f可积, f有上确界, 设为M

$\therefore |f^2(x_i)-f^2(x_{i-1})| = |f(x_i)+f(x_{i-1})||f(x_i)-f(x_{i-1})| \leq 2Mf(x_i)-f(x_{i-1})$

$\therefore \sum |f^2(x_i)-f^2(x_{i-1})|\Delta x_i < 2M\varepsilon$

$\therefore f^2$在$[a,b]$上黎曼可积

(6)

正确.

假设不存在$c\in(a,b)$使得$f(c)=0$

$\therefore$ 由零点存在性定理可知$f(x)$恒正或恒负, 不妨假设$f(x)>0$

$\therefore$ 由定积分的几何意义可知, $f(x)$总在x轴上方, 面积大于0

$\therefore \int_a^b f(x)\mathrm{d}x > 0$, 与题设矛盾

若$f(x)$恒负, 同理可知$\int_a^b f(x)\mathrm{d}x < 0$, 与题设矛盾

$\therefore \exists c\in(a,b)$, 使得$f(c)=0$

11.

(1)

对于$x \in [0,1]$

有$x \geq x^2$

$\therefore e^x \geq e^{x^2}$

$\therefore \int_0^1 e^x \mathrm{d}x > \int_0^1 e^{x^2} \mathrm{d}x$

(2)

令$f(x) = 2\sqrt{x} + \frac{1}{x} - 3$,则$f(1) = 0$

$\therefore f'(x) = \frac{1}{\sqrt{x}} - \frac{1}{x^2} = \frac{(\sqrt{x})^3 - 1}{x^2} \geq 0, 1 \leq x \leq 2$

$\therefore f(x) \geq 0$即$2\sqrt{x} \geq 3 - \frac{1}{x}, f(2) > 0$

$\therefore \int_1^2 2\sqrt{x}\mathrm{d}x > \int_1^2 (3 - \frac{1}{x})\mathrm{d}x$

(3)

令$f(x) = (1+x)\ln(1+x) - \arctan x$,则$f(0) = 0$

$\therefore f'(x) = \ln(1+x) + 1 - \frac{1}{1+x^2}, f'(0) = 0$

$\therefore f''(x) = \frac{1}{1+x} + \frac{2x}{(1+x^2)^2} > 0$

$\therefore f'(x) \geq 0, f(x) \geq 0$

$\therefore \int_0^1 \ln(1+x)\mathrm{d}x > \int_0^1 \frac{\arctan x}{1+x}\mathrm{d}x$

12.

(1)

$\because 1 \leq e^{x^2} \leq e^x$

$\therefore \int_0^1 \mathrm{d}x < \int_0^1 e^{x^2}\mathrm{d}x < \int_0^1 e^x \mathrm{d}x$

$\therefore 1 < \int_0^1 e^{x^2}\mathrm{d}x < e$

(2)

$\because 6 \leq \sqrt{100-x^2} \leq 10, -6 \leq x \leq 10$

$\therefore \int_{-6}^8 6\mathrm{d}x < \int_{-6}^8 \sqrt{100-x^2}\mathrm{d}x < \int_{-6}^8 10\mathrm{d}x$

$\therefore 84 < \int_{-6}^8 \sqrt{100-x^2}\mathrm{d}x < 140$

14.

令$F(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a)$

$\therefore F(a) = 0, F(b) = 0, F'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$

由拉格朗日中值定理知$\exists \xi \in (a,b)$,使得$f'(\xi) = \frac{f(b)-f(a)}{b-a}$

$\therefore F'(x) = f'(x) - f'(\xi)$

$\because F''(x) = f''(x) > 0, f'$递增

$\therefore F'(x)$在(a,ξ)上小于0,在(ξ,b)上大于0

$\therefore F(x)$在(a,ξ)上递减,在(ξ,b)上递增

$\therefore F(x)$在(a,ξ)上小于$F(a)=0$,在(ξ,b)上小于$F(b)=0$

$\because f'(x) > 0$

$\therefore f(a) \leq f(x) \leq \frac{f(b)-f(a)}{b-a}(x-a) + f(a), a \leq x \leq b$

$\therefore \int_a^b f(a)\mathrm{d}x < \int_a^b f(x)\mathrm{d}x < \int_a^b (\frac{f(b)-f(a)}{b-a}(x-a) + f(a))\mathrm{d}x$

$\therefore (b-a)f(a) < \int_a^b f(x)\mathrm{d}x < \frac{b-a}{2}[f(a)-f(b)]$

3.1 (B)

3.

由积分中值定理可知

$\exists \xi' \in [\frac{2a}{3}, a]$,使得$3\int_{\frac{2a}{3}}^{a} f(x)\mathrm{d}x = f(\xi')\int_{\frac{2a}{3}}^{a} 3\mathrm{d}x = f(\xi')a$

$\because 3\int_{\frac{2a}{3}}^{a} f(x)\mathrm{d}x = f(0)a$

$\therefore f(\xi') = f(0)$

$\therefore$ 由$Rolle$中值定理可知$\exists \xi \in (0, \xi') \subseteq (0, a)$,使得$f'(\xi) = 0$

4.

$\because \forall \lambda \in R, \int_a^b [\lambda f(x) - g(x)]^2 \mathrm{d}x \geq 0$

$\therefore$ 对于方程$\lambda^2 \int_a^b f^2(x)\mathrm{d}x + 2\lambda \int_a^b f(x)g(x)\mathrm{d}x + \int_a^b g^2(x)\mathrm{d}x = 0$必定无解或只有相同解

$\therefore \Delta = (2\int_a^b f(x)g(x)\mathrm{d}x)^2 - 4(\int_a^b f^2(x)\mathrm{d}x)(\int_a^b g^2(x)\mathrm{d}x) \leq 0$

$\therefore \int_a^b f(x)g(x)\mathrm{d}x \leq (\int_a^b f^2(x)\mathrm{d}x)^{\frac{1}{2}}(\int_a^b g^2(x)\mathrm{d}x)^{\frac{1}{2}}$

6.

由柯西不等式可知

$(\int_a^b (e^{\frac{f(x)}{2}})(e^{-\frac{f(x)}{2}})\mathrm{d}x)^2 \leq (\int_a^b (e^{\frac{f(x)}{2}})^2\mathrm{d}x)(\int_a^b (e^{-\frac{f(x)}{2}})^2\mathrm{d}x)$

即$\int_a^b e^{f(x)}\mathrm{d}x \int_a^b e^{-f(x)}\mathrm{d}x \geq (b-a)^2$

3.2 (A)

3.

(4)

$$\therefore \int_{-1}^{1} |x| \mathrm{d}x = \int_{-1}^{0} -x \mathrm{d}x + \int_{0}^{1} x \mathrm{d}x = 2\int_{0}^{1} x \mathrm{d}x = 1$$

(6)

$$\begin{aligned}
\therefore \int_{-1}^{1} f(x) \mathrm{d}x &= \int_{-1}^{0} f(x) \mathrm{d}x + \int_{0}^{1} f(x) \mathrm{d}x \\
&= \int_{-1}^{0} x \mathrm{d}x + \int_{0}^{1} x^2 \mathrm{d}x \\
&= \frac{1}{2} x^2 |_{-1}^{0} + \frac{1}{3} x^3 |_{0}^{1} \\
&= -\frac{1}{2} + \frac{1}{3} \\
&= -\frac{1}{6}
\end{aligned}$$

4.

(1)

$$f'(x) = \arctan x$$

(3)

$$F'(x) = \frac{e^x}{2\sqrt{x}}$$

(6)

$$y' = -\sin x \cos(\pi \cos^2 x) - \cos x \cos(\pi \sin^2 x)$$

(7)

$$y = x\int_{x^2}^{x^3}\varphi(t)\mathrm{d}t + \int_{x^2}^{x^3} t\varphi(t)\mathrm{d}t$$

$$\begin{aligned} y' &= 3x^3\varphi(x^3) - 2x^2\varphi(x^2) + 3x^5\varphi(x^3) - 2x^3\varphi(x^2) \\ &= (3x^3 + 3x^5)\varphi(x^3) - (2x^2 + 2x^3)\varphi(x^2) \end{aligned}$$

6.

$$\therefore x' = \sin^2 t, y' = 2t\cos|t|$$

$$\therefore f'(x) = \frac{2t\cos|t|}{\sin^2 t}, t \neq k\pi, k = \mathbb{Z}$$

7.

$$\because \int_0^y e^{t^2}\mathrm{d}t + \int_0^{x^2} te^t\mathrm{d}t = 0$$

$$\therefore y'e^{y^2} + 2x^3e^{x^2} = 0$$

$$\therefore y' = -\frac{2x^3e^{x^2}}{e^{y^2}}$$

9. (1)

$$\begin{aligned} &\lim_{x\to 0}\frac{\int_0^x \sin t^2\mathrm{d}t}{\sin^3 x} \\ =&\lim_{x\to 0}\frac{\sin^2 x}{3\sin^2 x\cos x} \\ =&\frac{1}{3} \end{aligned}$$

14.

要证$F(x)$在(a,b)内递减

即证$F'(x)$在(a,b)小于或等于0

$$\therefore F'(x)=\frac{(x-a)f(x)-\int_a^x f(t)\mathrm{d}t}{(x-a)^2}$$

令$G(x)=(x-a)f(x)-\int_a^x f(t)\mathrm{d}t$

$$\therefore \lim_{x\to a} F'(x)=\frac{f(x)+(x-a)f'(x)-f(x)}{2(x-a)}=\frac{f'(x)}{2}\le 0$$

$\because G'(x)=(x-a)f'(x)\le 0, a\le x\le b$

$\therefore G(x)\le G(a)=0$

$\therefore F'(x)\le 0$

$\therefore F(x)$在(a,b)内单调递减

3.2 (B)

1.

$\forall x_0\in[a,b]$

当$\Delta x\to 0$,由微分中值定理可知$\exists\mu\in[\inf\{f(x)\},\sup\{f(x)\}]$使得

$$\begin{aligned}\Delta y&=F(x_0+\Delta x)-F(x_0)\\&=\int_a^{x_0+\Delta x} f(t)\mathrm{d}t-\int_a^{x_0} f(t)\mathrm{d}t\\&=\int_{x_0}^{x_0+\Delta x} f(t)\mathrm{d}t\\&=\mu\int_{x_0}^{x_0+\Delta x}\mathrm{d}t\\&=\mu\Delta x\\&=0\end{aligned}$$

$\therefore F(x)$在$[a,b]$上可积

3.

$$
\begin{aligned}
&\lim_{x\to1}\frac{\displaystyle\int_1^x(t\int_t^1 f(u)\mathrm{d}u)\mathrm{d}t}{(1-x)^3}\\
=&\lim_{x\to1}\frac{\displaystyle x\int_x^1 f(u)\mathrm{d}u}{-3(1-x)^2}\\
=&\lim_{x\to1}\frac{\displaystyle x\int_1^x f(u)\mathrm{d}u}{3(1-x)^2}\\
=&\lim_{x\to1}\frac{xf(x)}{6(x-1)}\\
=&\lim_{x\to1}\frac{f(x)+xf'(x)}{6}\\
=&\frac{1}{6}
\end{aligned}
$$

5.

$\because f$在$[a,c]$上连续, $\displaystyle\int_a^b f(x)\mathrm{d}x=\int_b^c f(x)\mathrm{d}x=0$

假设不存在$x_1\in(a,b)$使得$f(x_1)=0$,
则由零点存在性定理可知$f(x)$在(a,b)上同号, 不妨设为大于0

$\therefore$ 由定积分的几何意义可知, 此时$\displaystyle\int_a^b f(x)\mathrm{d}x>0$, 产生矛盾
同理$f(x)<0$也会产生矛盾

$\therefore \exists x_1\in(a,b),x_2\in(b,c)$使得$f(x_1)=f(x_2)=0$

$\therefore$ 由$Rolle$中值定理可知$\exists\xi\in(x_1,x_2)\subseteq(a,c)$使得$f'(\xi)=0$

3.3 (A)

1.

(3)

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1-16x^2}} &= \frac{1}{4}\int \frac{\mathrm{d}4x}{\sqrt{1-(4x)^2}} \\
&= \frac{1}{4}\arcsin 4x + C
\end{aligned}
$$

(6)

$$
\begin{aligned}
\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\mathrm{d}x &= 2\int \sqrt{1+\sqrt{x}}\mathrm{d}(1+\sqrt{x}) \\
&= \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} + C
\end{aligned}
$$

(10)

$$
\begin{aligned}
\int \cos^4 x\mathrm{d}x &= \int \cos^3 x\mathrm{d}\sin x \\
&= \sin x\cos^3 - \int \sin x\mathrm{d}\cos^3 x \\
&= \sin x\cos^3 + 3\int \sin^2 x\cos^2 x\mathrm{d}x \\
&= \sin x\cos^3 + 3\int (1-\cos^2 x)\cos^2 x\mathrm{d}x \\
&= \sin x\cos^3 + 3\int \cos^2 x\mathrm{d}x - 3\int \cos^4 x\mathrm{d}x \\
&= \sin x\cos^3 + 3\int \frac{1+\cos 2x}{2}\mathrm{d}x - 3\int \cos^4 x\mathrm{d}x \\
&= \sin x\cos^3 + \frac{3}{2}x + \frac{3}{4}\int \cos 2x\mathrm{d}2x - 3\int \cos^4 x\mathrm{d}x \\
&= \sin x\cos^3 + \frac{3}{2}x + \frac{3}{4}\sin 2x - 3\int \cos^4 x\mathrm{d}x
\end{aligned}
$$

$$
\therefore \int \cos^4 x\mathrm{d}x = \frac{1}{4}\sin x\cos^3 + \frac{3}{8}x + \frac{3}{16}\sin 2x + C
$$

(14)

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{e^x+1} &= \int \frac{\mathrm{d}e^x}{e^x(e^x+1)} \\
&= \int (\frac{1}{e^x} - \frac{1}{e^x+1})\mathrm{d}e^x \\
&= x - \ln(e^x+1) + C
\end{aligned}
$$

(18)

$$\begin{aligned}\int \frac{\mathrm{d}x}{\sqrt{4-x^2}\arccos\frac{x}{2}} &= \int \frac{\mathrm{d}x}{2\sqrt{1-(\frac{x}{2})^2}\arccos\frac{x}{2}} \\ &= 2\int \frac{\mathrm{d}\arctan\frac{x}{2}}{\arctan\frac{x}{2}} \\ &= 2\ln\arctan\frac{x}{2}+C\end{aligned}$$

3.

(5)

令$x=3\sec t, 0<t<\frac{\pi}{2}$

$$\begin{aligned}\int \frac{\mathrm{d}x}{x^2\sqrt{x^2-9}} &= \int \frac{\mathrm{d}\sec t}{27\sec^2 t\tan t} \\ &= \frac{1}{27}\int \frac{\mathrm{d}x}{\sec t} \\ &= \frac{1}{27}\sin t+C \\ &= \frac{1}{27}\sin\arccos\frac{3}{x}+C\end{aligned}$$

(8)

令$x+1=\sqrt{2}\tan t, t=\arctan\frac{\sqrt{2}}{2}(x+1)$

$$
\begin{aligned}
&\int \frac{\mathrm{d}x}{(x+1)\sqrt{x^2+2x+3}} \\
&= \int \frac{\mathrm{d}(x+1)}{(x+1)\sqrt{(x+1)^2+2}} \\
&= \int \frac{\mathrm{d}\sqrt{2}\tan t}{\sqrt{2}\tan t\sqrt{(\sqrt{2}\tan t)^2+2}} \\
&= \int \frac{\sqrt{2}\mathrm{d}t}{2\sin t} \\
&= -\frac{\sqrt{2}}{4}\int (\frac{1}{1+\cos t}+\frac{1}{1-\cos t})\mathrm{d}\cos t \\
&= -\frac{\sqrt{2}}{4}\ln(1+\cos t)+\frac{\sqrt{2}}{4}\ln(1-\cos t)+C \\
&= -\frac{\sqrt{2}}{4}\ln(1+\cos(\arctan\frac{\sqrt{2}}{2}(x+1)))+\frac{\sqrt{2}}{4}\ln(1-\cos(\arctan\frac{\sqrt{2}}{2}(x+1)))+C
\end{aligned}
$$

(11)

令$x = 2\arctan t$

$$
\begin{aligned}
\int \frac{\mathrm{d}x}{1+\sin x+\cos x} &= \frac{1}{2}\int \frac{\mathrm{d}x}{\sin\frac{x}{2}\cos\frac{x}{2}+\cos^2\frac{x}{2}} \\
&= \frac{1}{2}\int \frac{\tan^2\frac{x}{2}+1}{\tan\frac{x}{2}+1}\mathrm{d}x \\
&= \frac{1}{2}\int (\tan\frac{x}{2}-1+\frac{2}{\tan\frac{x}{2}+1})\mathrm{d}x \\
&= -\ln|\cos\frac{x}{2}|-\frac{1}{2}x+\int \frac{\mathrm{d}2\arctan t}{t+1} \\
&= -\ln|\cos\frac{x}{2}|-\frac{1}{2}x+2\int \frac{\mathrm{d}t}{(t+1)(t^2+1)}
\end{aligned}
$$

$$
\because \frac{A}{t+1}+\frac{Bt+C}{t^2+1} = \frac{At^2+A+Bt^2+Bt+Ct+C}{(t+1)(t^2+1)} = \frac{1}{(t+1)(t^2+1)}
$$

$$
\therefore \begin{cases} A+B=0 \\ B+C=0 \\ A+C=1 \end{cases} \Rightarrow \begin{cases} A=\frac{1}{2} \\ B=-\frac{1}{2} \\ C=\frac{1}{2} \end{cases}
$$

$$
\begin{aligned}
2\int\frac{\mathrm{d}t}{(t+1)(t^2+1)}&=\int\frac{\mathrm{d}(t+1)}{t+1}-\int\frac{t-1}{t^2+1}\mathrm{d}t\\
&=\ln|t+1|-\int\frac{\frac{1}{2}(t^2+1)'-1}{t^2+1}\mathrm{d}t\\
&=\ln|t+1|-\frac{1}{2}\int\frac{\mathrm{d}(t^2+1)}{t^2+1}+\int\frac{\mathrm{d}t}{t^2+1}\\
&=\ln|t+1|-\frac{1}{2}\ln|t^2+1|+\arctan t+C\\
&=\ln|\tan\frac{x}{2}+1|-\frac{1}{2}\ln|\tan^2\frac{x}{2}+1|+\frac{x}{2}+C
\end{aligned}
$$

$$
\therefore\int\frac{\mathrm{d}x}{1+\sin x+\cos x}=-\ln|\cos\frac{x}{2}|-\frac{1}{2}x+\ln|\tan\frac{x}{2}+1|-\frac{1}{2}\ln|\tan^2\frac{x}{2}+1|+\frac{x}{2}+C
$$

(12)

令$x=\dfrac{2}{t^2+1}-1,-1<x\le 1,t\ge 0$

$$
\begin{aligned}
\int x\sqrt{\frac{1-x}{1+x}}\mathrm{d}x&=\int x\sqrt{\frac{2}{1+x}-1}\mathrm{d}x\\
&=\int(\frac{2t}{t^2+1}-t)\mathrm{d}\frac{2}{t^2+1}\\
&=4\int\frac{(t^2+1-1)^2-(t^2+1)+1}{(t^2+1)^3}\mathrm{d}t\\
&=4\int\frac{(t^2+1)^2-3(t^2+1)+2}{(t^2+1)^3}\mathrm{d}t\\
&=4\int(\frac{1}{t^2+1}-\frac{3}{(t^2+1)^2}+\frac{2}{(t^2+1)^3})\mathrm{d}t\\
&=4\arctan t-4\int(\frac{3}{(t^2+1)^2}-\frac{2}{(t^2+1)^3})\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
\int\frac{1}{(t^2+1)^2}\mathrm{d}t&=\frac{t}{(t^2+1)^2}-\int t\mathrm{d}\frac{1}{(t^2+1)^2}\\
&=\frac{t}{(t^2+1)^2}+4\int\frac{t^2+1-1}{(t^2+1)^3}\mathrm{d}t\\
&=\frac{t}{(t^2+1)^2}+4\int\frac{1}{(t^2+1)^2}\mathrm{d}t-4\int\frac{1}{(t^2+1)^3}\mathrm{d}t
\end{aligned}
$$

$$
\int\frac{1}{(t^2+1)^2}\mathrm{d}t=\frac{t}{2(1+t^2)}+\frac{1}{2}\arctan t+C
$$

$$\begin{aligned}\int \frac{1}{(t^2+1)^3}\mathrm{d}t &= \frac{t}{4(1+t^2)^2}+\frac{3}{4}(\frac{t}{2(1+t^2)}+\frac{1}{2}\arctan t)\\ &= \frac{t}{4(1+t^2)^2}+\frac{3t}{8(1+t^2)}+\frac{3}{8}\arctan t + C\end{aligned}$$

$$\begin{aligned}\int x\sqrt{\frac{1-x}{1+x}}\mathrm{d}x &= 4\arctan t - 4\int(\frac{3}{(t^2+1)^2}-\frac{2}{(t^2+1)^3})\mathrm{d}t\\ &= \arctan t - \frac{3t}{1+t^2}+\frac{2t}{(1+t^2)^2}+C\\ &= \arctan\sqrt{\frac{2}{x+1}-1}-\frac{3x+3}{2}\sqrt{\frac{2}{x+1}-1}+2(\frac{x+1}{2})^2\sqrt{\frac{2}{x+1}-1}+C\end{aligned}$$

(13)

令$x=\dfrac{\ln t}{2}, t=u^2-5, u>\sqrt{5}$

$\therefore u=\sqrt{e^{2x}+5}$

$$\begin{aligned}\int \sqrt{e^{2x}+5}\mathrm{d}x &= \int \sqrt{t+5}\mathrm{d}\frac{\ln t}{2}\\ &= \frac{1}{2}\int\frac{\sqrt{t+5}}{t}\mathrm{d}t\\ &= \frac{1}{2}\int\frac{u}{u^2-5}\mathrm{d}(u^2-5)\\ &= \int\frac{u^2}{u^2-5}\mathrm{d}u\\ &= x+5\int\frac{1}{(u-\sqrt{5})(u+\sqrt{5})}\mathrm{d}u\\ &= x+\frac{\sqrt{5}}{2}\int\frac{1}{u-\sqrt{5}}\mathrm{d}u-\frac{\sqrt{5}}{2}\int\frac{1}{u+\sqrt{5}}\mathrm{d}u\\ &= x+\frac{\sqrt{5}}{2}\ln|u-\sqrt{5}|-\frac{\sqrt{5}}{2}\ln|u+\sqrt{5}|+C\\ &= x+\frac{\sqrt{5}}{2}\ln|\sqrt{e^{2x}+5}-\sqrt{5}|-\frac{\sqrt{5}}{2}\ln|\sqrt{e^{2x}+5}+\sqrt{5}|+C\end{aligned}$$

7.

(3)

$$
\begin{aligned}
\int x^2 \arctan x \mathrm{d}x &= \frac{1}{3}\int \arctan x \mathrm{d}x^3 \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int x^3 \mathrm{d} \arctan x \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{3}\int \frac{x^3}{1+x^2}\mathrm{d}x \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{6}\int (1-\frac{1}{1+x^2})\mathrm{d}x^2 \\
&= \frac{1}{3}x^3 \arctan x - \frac{1}{6}x + \ln(1+x^2) + C
\end{aligned}
$$

(7)

$$
\begin{aligned}
\int \frac{x}{\cos^2 x}\mathrm{d}x &= \int x \mathrm{d} \tan x \\
&= x \tan x - \int \tan x \mathrm{d}x \\
&= x \tan x + \ln \cos x + C
\end{aligned}
$$

(12)

$$
\begin{aligned}
\int \sin(\ln x)\mathrm{d}x &= x\sin(\ln x) - \int x \mathrm{d} \sin(\ln x) \\
&= x\sin(\ln x) - \int \cos(\ln x)\mathrm{d}x \\
&= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\mathrm{d}x
\end{aligned}
$$

$$
\int \sin(\ln x)\mathrm{d}x = \frac{1}{2}x\sin(\ln x) - \frac{1}{2}x\cos(\ln x) + C
$$

8. (1)

$$
\begin{aligned}
\because I_n = \int \tan^n x \mathrm{d}x &= \int \tan^{n-2} x (\sec^2 -1)\mathrm{d}x \\
&= \int \tan^{n-2} x \mathrm{d} \tan x - \int \tan^{n-2} x \mathrm{d}x \\
&= \frac{1}{n-1}\tan^{n-1} x - I_{n-2}
\end{aligned}
$$

9.

(5)

令$\tan\frac{x}{2}=t$,则$x=2\arctan t,\mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t$

$$\begin{aligned}\int\frac{\mathrm{d}x}{3+2\cos x}&=\int\frac{\frac{2}{1+t^2}\mathrm{d}t}{3+2\frac{1-t^2}{1+t^2}}\\&=\frac{2\sqrt{5}}{5}\int\frac{\mathrm{d}\frac{\sqrt{5}}{5}t}{1+(\frac{\sqrt{5}}{5}t)^2}\\&=\frac{2\sqrt{5}}{5}\arctan\frac{\sqrt{5}}{5}t\\&=\frac{2\sqrt{5}}{5}\arctan(\frac{\sqrt{5}}{5}\tan\frac{x}{2})+C\end{aligned}$$

(11)

$$\begin{aligned}\int\frac{x+\sin x}{1+\cos x}\mathrm{d}x&=\int\frac{x+\sin x}{2\cos^2\frac{x}{2}}\mathrm{d}x\\&=\int(x+\sin x)\mathrm{d}\tan\frac{x}{2}\\&=(x+\sin x)\tan\frac{x}{2}-\int\tan\frac{x}{2}\mathrm{d}(x+\sin x)\\&=(x+\sin x)\tan\frac{x}{2}-\int\sin x\mathrm{d}x\\&=(x+\sin x)\tan\frac{x}{2}+\cos x+C\end{aligned}$$

(15)

令$t=x-1$,则

$$\begin{aligned}\int\frac{x^2+2}{(x-1)^4}\mathrm{d}x&=\int\frac{(t+1)^2+2}{t^4}\mathrm{d}t\\&=\int(\frac{1}{t^2}+\frac{2}{t^3}+\frac{3}{t^4})\mathrm{d}t\\&=-\frac{1}{t}-\frac{1}{t^2}-\frac{1}{t^3}+C\\&=-\frac{1}{x-1}-\frac{1}{(x-1)^2}-\frac{1}{(x-1)^3}+C\end{aligned}$$