数学分析作业

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习题 5.3: (A) 26(5), 28, 33, (B) 2, 7,
习题 5.4: (A) 2, 4(4), 5(3), 6, 10

5.3 (A)

26. (5)

$\therefore \dfrac{\partial z}{\partial x} = f_1 y + \dfrac{f_2}{y} - \dfrac{g_1 y}{x^2}$

$\therefore \dfrac{\partial^2 z}{\partial x \partial y} = f_{11} y x - \dfrac{f_{12} y x}{y^2} + f_1 + \dfrac{f_{21} x}{y} - \dfrac{f_{22} x}{y^3} - \dfrac{f_2}{y^2} - \dfrac{g_{11} y}{x^3} - \dfrac{g_1}{x^2}$

28.

令 $\cos\alpha = \dfrac{1}{\sqrt{a^2+1}}, \sin\alpha = \dfrac{a}{\sqrt{a^2+1}}, \cos\beta = \dfrac{1}{\sqrt{b^2+1}}, \sin\beta = \dfrac{b}{\sqrt{b^2+1}}$

$\therefore \dfrac{\partial u}{\partial \xi} = \dfrac{\partial u}{\partial x}\cos\alpha + \dfrac{\partial u}{\partial y}\sin\alpha$

$\therefore \dfrac{\partial^2 u}{\partial \xi \partial \eta} = \dfrac{\partial^2 u}{\partial x^2}\cos\alpha\cos\beta + \dfrac{\partial^2 u}{\partial x \partial y}(\sin\alpha\cos\beta + \cos\alpha\sin\beta) + \dfrac{\partial^2 u}{\partial^2 y}\sin\alpha\sin\beta$

$\therefore \begin{cases} \cos\alpha\cos\beta = \dfrac{1}{\sqrt{a^2+1}\sqrt{b^2+1}} = k \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta = \dfrac{a+b}{\sqrt{a^2+1}\sqrt{b^2+1}} = 4k \\ \sin\alpha\sin\beta = \dfrac{ab}{\sqrt{a^2+1}\sqrt{b^2+1}} = 3k \end{cases}$

$\therefore \begin{cases} a+b=4 \\ ab=3 \end{cases} \Rightarrow \begin{cases} a=1 \\ b=3 \end{cases} \text{or} \begin{cases} a=3 \\ b=1 \end{cases}$

33.

令 $u(x,y,z) = x + \dfrac{z}{y}, v(x,y,z) = y + \dfrac{z}{x}$

$\because F(x+\dfrac{z}{y}, y+\dfrac{z}{x}) = F(u,v) = 0$

$\therefore F_u \mathrm{d}(x+\dfrac{z}{y}) + F_v \mathrm{d}(y+\dfrac{z}{x}) = 0$

$\therefore F_u \mathrm{d}x + \dfrac{F_u}{y}\mathrm{d}z - \dfrac{F_u z}{y^2}\mathrm{d}y + F_v \mathrm{d}y + \dfrac{F_v}{x}\mathrm{d}z - \dfrac{F_v z}{x^2}\mathrm{d}x = 0$

$\therefore (\dfrac{F_u}{y}+\dfrac{F_v}{x})\mathrm{d}z = (\dfrac{F_v z}{x^2} - F_u)\mathrm{d}x + (\dfrac{F_u z}{y^2} - F_v)\mathrm{d}y$

$\therefore \mathrm{d}z = \dfrac{\dfrac{F_v z}{x^2} - F_u}{\dfrac{F_u}{y}+\dfrac{F_v}{x}}\mathrm{d}x + \dfrac{\dfrac{F_u z}{y^2} - F_v}{\dfrac{F_u}{y}+\dfrac{F_v}{x}}\mathrm{d}y$

$\because \mathrm{d}z = \dfrac{\partial z}{\partial x}\mathrm{d}x + \dfrac{\partial z}{\partial y}\mathrm{d}y$

$\therefore \dfrac{\partial z}{\partial x} = \dfrac{\dfrac{F_v z}{x^2} - F_u}{\dfrac{F_u}{y}+\dfrac{F_v}{x}}, \dfrac{\partial z}{\partial y} = \dfrac{\dfrac{F_u z}{y^2} - F_v}{\dfrac{F_u}{y}+\dfrac{F_v}{x}}$

代入原式可得 $\dfrac{\dfrac{F_v z}{x} - F_u x}{\dfrac{F_u}{y}+\dfrac{F_v}{x}} + \dfrac{\dfrac{F_u z}{y} - F_v y}{\dfrac{F_u}{y}+\dfrac{F_v}{x}} = z - xy$

$\therefore \dfrac{F_v z}{x} - F_u x + \dfrac{F_u z}{y} - F_v y = (z-xy)(\dfrac{F_u}{y}+\dfrac{F_v}{x})$

$\therefore x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z - xy$ 成立

5.3 (B)

2.

$$\begin{aligned}\frac{\partial f}{\partial \boldsymbol{l}} &= \frac{3}{\sqrt{13}}\cdot\frac{\partial f}{\partial x} + \frac{2}{\sqrt{13}}\cdot\frac{\partial f}{\partial y} \\ &= a(\frac{2}{2\sqrt{2}}\cdot\frac{\partial f}{\partial x} - \frac{2}{2\sqrt{2}}\cdot\frac{\partial f}{\partial y}) + b(-\frac{\partial f}{\partial x}) \\ &= (\frac{2a}{2\sqrt{2}} - b)\cdot\frac{\partial f}{\partial x} - \frac{2a}{2\sqrt{2}}\cdot\frac{\partial f}{\partial y}\end{aligned}$$

$$\begin{cases} -\dfrac{2a}{2\sqrt{2}} = \dfrac{2}{\sqrt{13}} \\ \dfrac{2a}{2\sqrt{2}} - b = \dfrac{3}{\sqrt{13}} \end{cases} \Rightarrow \begin{cases} a = -\dfrac{2\sqrt{26}}{13} \\ b = -\dfrac{5\sqrt{13}}{13} \end{cases}$$

$$\therefore \frac{\partial f}{\partial \boldsymbol{l}} = -\frac{2\sqrt{26}}{13} \cdot \frac{\partial f}{\partial \boldsymbol{l}_1} - \frac{5\sqrt{13}}{13} \cdot \frac{\partial f}{\partial \boldsymbol{l}_2} = \frac{15\sqrt{13} - 2\sqrt{26}}{13}$$

7.

令 $z = \sqrt{x^2 + y^2}$

$$\therefore \frac{\partial u}{\partial x} = \frac{\mathrm{d}u}{\mathrm{d}z}\frac{\partial z}{\partial x} = \frac{\mathrm{d}u}{\mathrm{d}z}\frac{x}{\sqrt{x^2+y^2}}, \frac{\partial u}{\partial y} = \frac{\mathrm{d}u}{\mathrm{d}z}\frac{\partial z}{\partial y} = \frac{\mathrm{d}u}{\mathrm{d}z}\frac{y}{\sqrt{x^2+y^2}}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}\frac{x^2}{x^2+y^2} + \frac{\mathrm{d}u}{\mathrm{d}z}\frac{\sqrt{x^2+y^2} - \dfrac{x^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}\frac{x^2}{z^2} + \frac{\mathrm{d}u}{\mathrm{d}z}\frac{z - \dfrac{x^2}{z}}{z^2}$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}\frac{y^2}{x^2+y^2} + \frac{\mathrm{d}u}{\mathrm{d}z}\frac{\sqrt{x^2+y^2} - \dfrac{y^2}{\sqrt{x^2+y^2}}}{x^2+y^2} = \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}\frac{y^2}{z^2} + \frac{\mathrm{d}u}{\mathrm{d}z}\frac{z - \dfrac{y^2}{z}}{z^2}$$

$$\because \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{x} \cdot \frac{\partial u}{\partial x} + u = x^2 + y^2$$

$$\therefore \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}\frac{x^2}{z^2} + \frac{\mathrm{d}u}{\mathrm{d}z}\frac{z - \dfrac{x^2}{z}}{z^2} + \frac{\mathrm{d}^2 u}{\mathrm{d}z^2}\frac{y^2}{z^2} + \frac{\mathrm{d}u}{\mathrm{d}z}\frac{z - \dfrac{y^2}{z}}{z^2} - \frac{1}{x}\frac{\mathrm{d}u}{\mathrm{d}z}\frac{x}{z} + u = z^2$$

$$\therefore \frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + u = z^2$$

$\therefore u = u_h + u_p$, 其中 u_h 是 $\dfrac{\mathrm{d}^2 u}{\mathrm{d}z^2} + u = 0$ 的解, u_p 是满足原式的特解

$$\therefore u_h = C_1 \cos z + C_2 \sin z, h_p = z^2 - 2$$

$$\therefore u = C_1 \cos z + C_2 \sin z + z^2 - 2$$

$$\therefore u = C_1 \cos\sqrt{x^2+y^2} + C_2 \sin\sqrt{x^2+y^2} + x^2 + y^2 - 2$$

5.4 (A)

2.

$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + f_x\Delta x + f_y\Delta y + \frac{1}{2}f_{xx}(\Delta x)^2 + f_{xy}\Delta x\Delta y + f_{yy}(\Delta y)^2$

$\because f(x,y) = \sin x \sin y$

$\therefore f_x = \cos x \sin y, f_y = \sin x \cos y$

$\therefore f_{xx} = -\sin x \sin y, f_{xy} = \cos x \cos y, f_{yy} = -\sin x \sin y$

$\therefore f_x(\frac{\pi}{4},\frac{\pi}{4}) = \frac{1}{2}, f_y(\frac{\pi}{4},\frac{\pi}{4}) = \frac{1}{2}, f_{xx}(\frac{\pi}{4},\frac{\pi}{4}) = -\frac{1}{2}, f_{xy}(\frac{\pi}{4},\frac{\pi}{4}) = \frac{1}{2}, f_{yy}(\frac{\pi}{4},\frac{\pi}{4}) = -\frac{1}{2}$

$\therefore f(x_0+\Delta x, y_0+\Delta y) = \frac{1}{2} + \frac{1}{2}\Delta x + \frac{1}{2}\Delta y - \frac{1}{4}f_{xx} + \frac{1}{2}f_{xy} - \frac{1}{4}f_{yy} + o(\Delta x^2 + \Delta y^2)$

4. (4)

令 $\dfrac{\partial z}{\partial x} = 2e^{2x}(x+2y+y^2) + e^{2x} = 2e^{2x}(x+2y+y^2+\dfrac{1}{2}) = 0$

$\dfrac{\partial z}{\partial y} = 2e^{2x}(1+y) = 0$

$\therefore \begin{cases} x = \frac{1}{2} \\ y = -1 \end{cases}$

$\because f_{xx} = 4e^{2x}(x+2y+y^2+1) = 2e, f_{xy} = 4e(1+y) = 0, f_{yy} = 2e^{2x} = 2e$

$\therefore$ Hesse 矩阵 $H = \begin{pmatrix} 2e & 0 \\ 0 & 2e \end{pmatrix}$

$\because \begin{vmatrix} 2e & 0 \\ 0 & 2e \end{vmatrix} = 4e^2 > 0, |2e| > 0$

$\therefore$ Hesse 矩阵正定

$\therefore$ 函数 $z = e^{2x}(x+2y+y^2)$ 仅在点 $(\frac{1}{2},-1)$ 有极小值 $z_{min} = -\dfrac{e}{2}$

5. (3)

令 $f_x = 2x-12 = 0, f_y = 2y+16$

$\therefore x = 6, y = -8$, 而 $6^2 + (-8)^2 = 100 > 25$, 即驻点不在区域 D 内

令 $x^2+y^2 = 25$, 则有 $y^2 = 25-x^2, y = \sqrt{25-x^2}$

$\therefore z = x^2 + 25 - x^2 - 12x + 16\sqrt{25-x^2} = 25 - 12x + 16\sqrt{25-x^2}$

$\therefore z' = -12 - \dfrac{16x}{\sqrt{25-x^2}} = 0 \Rightarrow x = -3$

$$\therefore z'' = -\frac{16\sqrt{25-x^2}+16\frac{x^2}{\sqrt{25-x^2}}}{25-x^2} = -\frac{25}{4} < 0$$

$\therefore z(-3) = 125, z(-5) = 85, z(5) = -35$

$\therefore$ 在 $x=-3$ 有最大值 125, 在 $x=5$ 有最小值 -35

6.

解法一:

$$\therefore \begin{cases} a = xyz \\ x > 0 \\ y > 0 \\ z > 0 \end{cases}$$

$$\therefore f = \frac{1}{x}+\frac{1}{y}+\frac{1}{z} = \frac{1}{x}+\frac{1}{y}+\frac{xy}{a}$$

令 $f_x = -\frac{1}{x^2}+\frac{y}{a} = 0, f_y = -\frac{1}{y^2}+\frac{x}{a} = 0$

$\therefore x = y = \sqrt[3]{a}$

$$\because f_{xx} = \frac{2}{x^3} = \frac{2}{a}, f_{xy} = \frac{1}{a}, f_{yy} = \frac{2}{y^3} = \frac{2}{a}$$

$\therefore$ Hesse 矩阵 $H = \begin{pmatrix} \frac{2}{a} & \frac{1}{a} \\ \frac{1}{a} & \frac{2}{a} \end{pmatrix}$

$$\therefore |\frac{2}{a}| > 0, \begin{vmatrix} \frac{2}{a} & \frac{1}{a} \\ \frac{1}{a} & \frac{2}{a} \end{vmatrix} = \frac{3}{a^2} > 0$$

$\therefore$ 在 $(\frac{1}{\sqrt[3]{a}}, \frac{1}{\sqrt[3]{a}})$ 有最小值

$\therefore x = y = z = \sqrt[3]{a}$

解法二:

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq 3\sqrt[3]{\frac{1}{xyz}} = \frac{3}{\sqrt[3]{a}}$$

当且仅当 $x = y = z = \sqrt[3]{a}$ 时取到最小值.

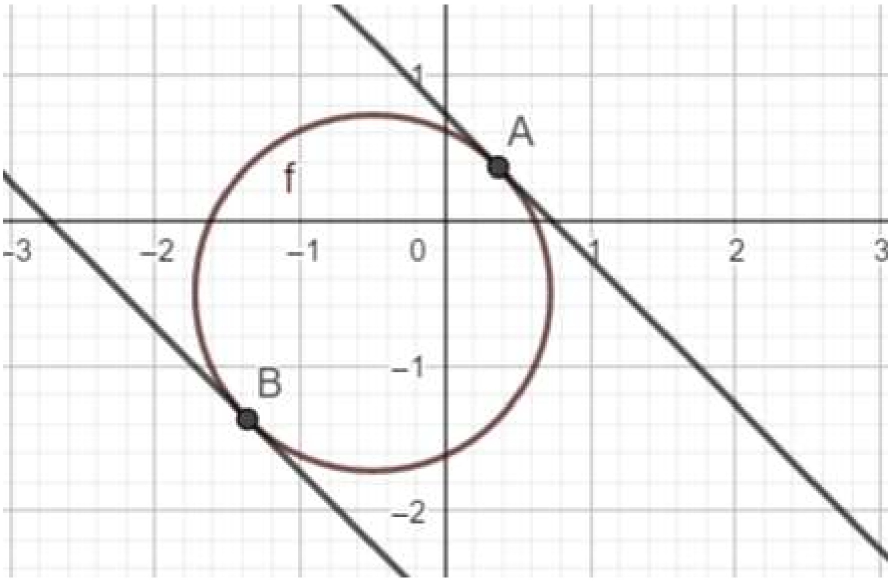
10.

$\therefore \begin{cases} x^2+y^2=z \\ x+y+z=1 \end{cases}$

$\therefore (x+\frac{1}{2})^2+(y+\frac{1}{2})^2=\frac{3}{2}$

$\therefore x,y$ 的范围为以 $(-\frac{1}{2},-\frac{1}{2})$ 为圆心, $\frac{\sqrt{6}}{2}$ 为半径的圆上点

$\therefore z=1-(x+y) \Rightarrow y=-x+1-z$



$\therefore (x+y)_{min}=\sqrt{2}(-\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}),(x+y)_{max}=\sqrt{2}(-\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2})$

$\therefore z_{max}=2+\sqrt{3},z_{min}=2-\sqrt{3}$

$\because d^2=x^2+y^2+z^2=z+z^2=(z+\dfrac{1}{2})^2-\dfrac{1}{4}$

$\therefore d_{max}=\sqrt{9+5\sqrt{3}},d_{min}=\sqrt{9-5\sqrt{3}}$