$$
\text{解方程组：}(1)\begin{cases} x_1 - x_2 - x_3 - 3x_4 = 0 \\ x_1 - x_2 + x_4 = 0 \\ 4x_1 - 4x_2 - x_3 = 0 \end{cases}
$$

$$
(2)\begin{cases} x_1 - x_2 - x_3 - 3x_4 = -2 \\ x_1 - x_2 + x_4 = 1 \\ 4x_1 - 4x_2 - x_3 = 1 \end{cases}
$$

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1(2)、（4）、（6）；

P.155 2.（2）；3.；4.；6.；9.；11.（1）；13.; 16.; 18.(1);

习题

P.157 19.(3);20.(3),(4);22.;23.;26.;

1.

(1)

$$\begin{bmatrix}1 & -1 & -1 & -3\\1 & -1 & 0 & 1\\4 & -4 & -1 & 0\end{bmatrix}=\begin{bmatrix}1 & -1 & -1 & -3\\0 & 0 & 1 & 4\\0 & 0 & 3 & 12\end{bmatrix}=\begin{bmatrix}1 & -1 & -1 & -3\\0 & 0 & 1 & 4\\0 & 0 & 0 & 0\end{bmatrix}$$
$$=\begin{bmatrix}1 & -1 & 0 & 1\\0 & 0 & 1 & 4\\0 & 0 & 0 & 0\end{bmatrix}$$

$\therefore x_1-x_2+x_4=0,x_3+4x_4=0$

(2)

$$\begin{bmatrix}1 & -1 & -1 & -3 & -2\\1 & -1 & 0 & 1 & 1\\4 & -4 & -1 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & -1 & -1 & -3 & -2\\0 & 0 & 1 & 4 & 3\\0 & 0 & 3 & 12 & 9\end{bmatrix}=\begin{bmatrix}1 & -1 & -1 & -3 & -2\\0 & 0 & 1 & 4 & 3\\0 & 0 & 1 & 4 & 3\end{bmatrix}$$
$$=\begin{bmatrix}1 & -1 & -1 & -3 & -2\\0 & 0 & 1 & 4 & 3\\0 & 0 & 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1 & -1 & 0 & 1 & 1\\0 & 0 & 1 & 4 & 3\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$\therefore x_1-x_2+x_4=1,x_3+4x_4=3$

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1.

(2)

$$\begin{bmatrix}1 & 2 & 0 & -3 & 2 & 1\\1 & -1 & -3 & 1 & -3 & 2\\2 & -3 & 4 & -5 & 2 & 7\\9 & -9 & 6 & -16 & 2 & 25\end{bmatrix}=\begin{bmatrix}1 & 2 & 0 & -3 & 2 & 1\\0 & -3 & -3 & 4 & -5 & 1\\0 & -7 & 4 & 1 & -2 & 5\\0 & -27 & 6 & 11 & -16 & 16\end{bmatrix}$$
$$=\begin{bmatrix}1 & 2 & 0 & -3 & 2 & 1\\0 & -3 & -3 & 4 & -5 & 1\\0 & 0 & 33 & -25 & 29 & 8\\0 & 0 & 33 & -25 & 29 & 7\end{bmatrix}=\begin{bmatrix}1 & 2 & 0 & -3 & 2 & 1\\0 & -3 & -3 & 4 & -5 & 1\\0 & 0 & 33 & -25 & 29 & 8\\0 & 0 & 0 & 0 & 0 & -1\end{bmatrix}$$

$\therefore$ 无解

(4)

$$\begin{bmatrix}3&4&-5&7\\2&-3&3&-2\\4&11&-13&16\\7&-2&1&3\end{bmatrix}=\begin{bmatrix}3&4&-5&7\\0&-17&19&-20\\0&17&-19&20\\0&-34&38&-40\end{bmatrix}=\begin{bmatrix}3&4&-5&7\\0&17&-19&20\\0&0&0&0\\0&0&0&0\end{bmatrix}$$
$$=\begin{bmatrix}1&\frac{4}{3}&-\frac{5}{3}&\frac{7}{3}\\0&1&-\frac{19}{17}&\frac{20}{17}\\0&0&0&0\\0&0&0&0\end{bmatrix}=\begin{bmatrix}1&0&-\frac{3}{17}&\frac{13}{17}\\0&1&-\frac{19}{17}&\frac{20}{17}\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$\therefore x_1--\frac{3}{17}x_3+\frac{13}{17}x_4=0,x_2-\frac{19}{17}x_3+\frac{20}{17}x_4=0$

(6)

$$\begin{bmatrix}1&2&3&-1&1\\3&2&1&-1&1\\2&3&1&1&1\\2&2&2&-1&1\\5&5&2&0&2\end{bmatrix}=\begin{bmatrix}1&2&3&-1&1\\0&-4&-8&2&-2\\0&-1&-5&3&-1\\0&-2&-4&1&-1\\0&-5&-13&5&-3\end{bmatrix}=\begin{bmatrix}1&2&3&-1&1\\0&1&5&-3&1\\0&2&4&-1&1\\0&5&13&-5&3\\0&0&0&0&0\end{bmatrix}$$
$$=\begin{bmatrix}1&2&3&-1&1\\0&1&5&-3&1\\0&0&-6&5&-1\\0&0&-12&10&-2\\0&0&0&0&0\end{bmatrix}=\begin{bmatrix}1&2&3&-1&1\\0&1&5&-3&1\\0&0&1&-\frac{5}{6}&\frac{1}{6}\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}=\begin{bmatrix}1&2&0&\frac{3}{2}&\frac{1}{2}\\0&1&0&\frac{7}{6}&\frac{1}{6}\\0&0&1&-\frac{5}{6}&\frac{1}{6}\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$
$$=\begin{bmatrix}1&0&0&-\frac{5}{6}&\frac{1}{6}\\0&1&0&\frac{7}{6}&\frac{1}{6}\\0&0&1&-\frac{5}{6}&\frac{1}{6}\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

$$\therefore x_1=\frac{1}{6}+\frac{5}{6}x_4,x_2=\frac{1}{6}-\frac{7}{6}x_4,x_3=\frac{1}{6}+\frac{5}{6}x_4$$

2.(2)

构建线性方程组：

$$
\begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 1 & 1 & 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 & 0 \\ 0 & -1 & -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

$$
= \begin{bmatrix} 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
$$

$\therefore k_1 = 1, k_3 = -1, k_2 = k_4 = 0$

$\therefore \vec{\beta} = \vec{\alpha}_1 - \vec{\alpha}_3$

3.

设有不全为零的k_i使得：

$k_1\vec{\alpha}_1 + k_2\vec{\alpha}_2 + \cdots + k_r\vec{\alpha}_r + k_0\vec{\beta} = 0$

当$k_0 = 0$时,则有$k_0\vec{\beta} = 0$

$\therefore$ 有不全为零的$k_i, i \geq 1$使得$k_1\vec{\alpha}_1 + k_2\vec{\alpha}_2 + \cdots + k_r\vec{\alpha}_r = 0$

$\therefore$ 与$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_r$线性无关矛盾,$k_0 = 0$不成立

当$k_0 \neq 0$时,

$\therefore \vec{\beta} = -\frac{k_1}{k_0}\vec{\alpha}_1 - \frac{k_2}{k_0}\vec{\alpha}_2 - \cdots - \frac{k_r}{k_0}\vec{\alpha}_r$

$\therefore$ 向量$\vec{\beta}$可由$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_r$线性表出

4.

$\because |a_{ij}| \neq 0$

$\therefore |a_{ij}|$对应的方程组$k_1\vec{\alpha}_1 + k_2\vec{\alpha}_2 + \cdots + k_r\vec{\alpha}_n = 0$,即

$$
\begin{cases} a_{11}k_1 + a_{12}k_2 + \cdots + a_{1n}k_n = 0 \\ a_{21}k_1 + a_{22}k_2 + \cdots + a_{2n}k_n = 0 \\ \cdots \\ a_{n1}k_1 + a_{n2}k_2 + \cdots + a_{nn}k_n = 0 \end{cases}
$$

的齐次线性方程组的根仅有唯一的零解

$\therefore$ 使得该式成立的k_i一定全都为0

$\therefore \vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$线性无关

6.

设$k_1(\vec{\alpha}_1 + \vec{\alpha}_2) + k_2(\vec{\alpha}_2 + \vec{\alpha}_3) + k_3(\vec{\alpha}_3 + \vec{\alpha}_1) = 0$

$\therefore (k_1 + k_3)\vec{\alpha}_1 + (k_1 + k_2)\vec{\alpha}_2 + (k_2 + k_3)\vec{\alpha}_3 = 0$

$\because \vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$线性无关

$\therefore k_1 + k_3 = 0, k_2 + k_3 = 0, k_1 + k_2 = 0$

$\therefore k_1 = k_2 = k_3 = 0$

$\therefore \vec{\alpha}_1 + \vec{\alpha}_2, \vec{\alpha}_2 + \vec{\alpha}_3, \vec{\alpha}_3 + \vec{\alpha}_1$也线性无关

9.

设一个向量组有一个线性无关组$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_r$,
我们取向量组的任意一个极大无关组$\vec{\beta}_1, \vec{\beta}_2, \cdots, \vec{\beta}_n$

易知$\vec{\alpha}_i$可用$\vec{\beta}_j$表示,即

$$\begin{cases} k_{11}\vec{\beta}_1 + k_{12}\vec{\beta}_2 + \cdots + k_{1n}\vec{\beta}_n = \vec{\alpha}_1 \\ k_{21}\vec{\beta}_1 + k_{22}\vec{\beta}_2 + \cdots + k_{2n}\vec{\beta}_n = \vec{\alpha}_2 \\ \cdots \\ k_{n1}\vec{\beta}_1 + k_{n2}\vec{\beta}_2 + \cdots + k_{rn}\vec{\beta}_n = \vec{\alpha}_r \end{cases}$$

若$r = n, \vec{\beta}_j$可由$\vec{\alpha}_i$唯一地表示

此时$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_r$已是极大无关组

若$r < n$,则可知

$$\vec{\beta}_j = \sum_{i=1}^{r} d_{ji}\vec{\alpha}_i + \sum_{i=r+1}^{n} d_{ji}\vec{\beta}_i$$

$\therefore$ 只需给$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_r$加入$\vec{\beta}_{r+1}, \vec{\beta}_{r+2}, \cdots, \vec{\beta}_n$即可成为极大无关组

13.

由题目易知

$$\begin{cases} k_{11}\vec{\alpha}_1 + k_{12}\vec{\alpha}_2 + \cdots + k_{1n}\vec{\alpha}_n = \vec{\varepsilon}_1 \\ k_{21}\vec{\alpha}_1 + k_{22}\vec{\alpha}_2 + \cdots + k_{2n}\vec{\alpha}_n = \vec{\varepsilon}_2 \\ \cdots \\ k_{n1}\vec{\alpha}_1 + k_{n2}\vec{\alpha}_2 + \cdots + k_{nn}\vec{\alpha}_n = \vec{\varepsilon}_n \end{cases}$$

可知方程组解唯一,即$\vec{\alpha}_j$可由$\vec{\varepsilon}_i$唯一线性表示

$\therefore \vec{\alpha}_j = \sum\limits_{i=1}^{n} d_{ji}\vec{\varepsilon}_i$

设$k_1\vec{\alpha}_1 + k_2\vec{\alpha}_2 + \cdots + k_n\vec{\alpha}_r = 0$

$\therefore k_1\sum\limits_{i=1}^{n} d_{1i}\vec{\varepsilon}_i + k_2\sum\limits_{i=1}^{n} d_{2i}\vec{\varepsilon}_i + \cdots + k_n\sum\limits_{i=1}^{n} d_{ni}\vec{\varepsilon}_i = 0$

$\therefore \vec{\varepsilon}_1\sum\limits_{i=1}^{n} k_i d_{i1} + \vec{\varepsilon}_2\sum\limits_{i=1}^{n} k_i d_{i2} + \cdots + \vec{\varepsilon}_n\sum\limits_{i=1}^{n} k_i d_{in} = 0$

$\because \vec{\varepsilon}_1, \vec{\varepsilon}_2, \cdots, \vec{\varepsilon}_n$线性无关

$\therefore$ 可知存在线性方程组

$$\begin{cases} d_{11}k_1 + d_{21}k_2 + \cdots + d_{n1}k_n = 0 \\ d_{12}k_1 + d_{22}k_2 + \cdots + d_{n2}k_n = 0 \\ \cdots \\ d_{1n}k_1 + d_{2n}k_2 + \cdots + d_{nn}k_n = 0 \end{cases}$$

$\because |d_{ij}| \neq 0$

$\therefore k_1 = k_2 = \cdots = k_n = 0$

$\therefore \vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_n$线性无关

16.

设向量组$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_r$和$\vec{\alpha}_1, \vec{\alpha}_2, \cdots, \vec{\alpha}_r, \vec{\alpha}_r, \cdots, \vec{\alpha}_s$的秩为$n$

则不妨设$\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_r$的一个极大无关组为$\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_n$,
其中$n\le r$

$\because$ 该极大线性无关组可以线性表示$\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_r$,且知有相同的秩$n$

$\therefore$ $\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_n$也是$\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_r,\vec{\alpha}_r,\cdots,\vec{\alpha}_s$的极大无关组

$\because$ $\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_r$与该极大无关组等价,该极大无关组与$\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_s$等价

$\therefore$ 由传递性可知$\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_r$和$\vec{\alpha}_1,\vec{\alpha}_2,\cdots,\vec{\alpha}_s$等价

18.(1)

19.(3)

$$\begin{bmatrix} a & 1 & 1 & 4 \\ 1 & b & 1 & 3 \\ 1 & 2b & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & b & 1 & 3 \\ 0 & b & 0 & 1 \\ 0 & 1-ab & 1-a & 4-3a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & b & 0 & 1 \\ 0 & 1-ab & 1-a & 4-3a \end{bmatrix}$$

当$b=0$时,易知无解,舍去

当$b\neq 0$时,

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & b & 0 & 1 \\ 0 & 1-ab & 1-a & 4-3a \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{b} \\ 0 & 0 & 1-a & 4-3a+b-\frac{1}{b} \end{bmatrix}$$

若$a=1$,

则需$4-3a+b-\dfrac{1}{b}=1+b-\dfrac{1}{b}\neq 0$

即$b^2+b-1\neq 0, b\neq \dfrac{-1\pm\sqrt{5}}{2}$

此时$x_1+x_3=2, x_2=\dfrac{1}{b}$

若$a\neq 1$,

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 0 & \frac{1}{b} \\ 0 & 0 & 1-a & 4-3a+b-\frac{1}{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2-\frac{4b-3ab+b^2-1}{b-ab} \\ 0 & 1 & 0 & \frac{1}{b} \\ 0 & 0 & 1 & \frac{4b-3ab+b^2-1}{b-ab} \end{bmatrix}$$

$$\therefore x_1 = 2 - \frac{4b - 3ab + b^2 - 1}{b - ab}, x_2 = \frac{1}{b}, x_3 = \frac{4b - 3ab + b^2 - 1}{b - ab}$$