习题2.4： （A）16（5、10），18，19，（B）1，3，5，习题2.5：（A）3（2、4），7（2、3），8，（B）1，4，
习题2.6：（A）9（3），12（2），22（3），24（2），26（2） （B）2（3），5

2.4 (A)

16.

(5)

$$\begin{aligned}
\lim_{x\to 0}\cot x\ln\frac{1+x}{1-x} &= \lim_{x\to 0}\frac{\ln(1+\frac{2x}{1-x})}{\tan x} \\
&= \lim_{x\to 0}\frac{2x}{(1-x)\tan x} \\
&= \lim_{x\to 0}\frac{2x}{x} \\
&= 2
\end{aligned}$$

(10)

$$\begin{aligned}
\lim_{x\to 0}\frac{e-(1+x)^{\frac{1}{x}}}{x} &= \lim_{x\to 0}\frac{(e-(1+x)^{\frac{1}{x}})'}{x'} \\
&= \lim_{x\to 0}-((1+x)^{\frac{1}{x}})' \\
&= \lim_{x\to 0}-(\exp(\frac{1}{x}\ln(1+x)))' \\
&= \lim_{x\to 0}-\exp[\frac{1}{x}\ln(1+x)](\frac{1}{x}\ln(1+x))' \\
&= \lim_{x\to 0}-e^{\frac{1}{x}\ln(1+x)}(\frac{1}{x(1+x)}-\frac{1}{x^2}\ln(1+x)) \\
&= \lim_{x\to 0}-e^{\frac{1}{x}\ln(1+x)}\frac{x-\ln(1+x)}{x^2} \\
&= \lim_{x\to 0}-e^{\frac{1}{x}\ln(1+x)}\frac{x-x+\frac{1}{2}x^2}{x^2} \\
&= \lim_{x\to 0}-\frac{1}{2}e^{\frac{1}{x}\ln(1+x)} \\
&= -\frac{e}{2}
\end{aligned}$$

18.

$$\lim_{x\to 0}\frac{1+a\cos 2x+b\cos 4x}{x^4}$$
$$=\lim_{x\to 0}\frac{1+a(1-\frac{(2x)^2}{2}+\frac{(2x)^4}{24}+o(x^5))+b(1-\frac{(4x)^2}{2}+\frac{(4x)^4}{24}+o(x^5))}{x^4}$$
$$=\lim_{x\to 0}\frac{(1+a+b)-2(a+4b)x^2+\frac{a(2x)^4}{24}+\frac{b(4x)^4}{24}+o(x^5)}{x^4}$$

$\because$ 极限存在

$$\therefore \begin{cases} a+b+1=0 \\ a+4b=0 \end{cases} \Rightarrow \begin{cases} a=-\frac{4}{3} \\ b=\frac{1}{3} \end{cases}$$

$$\therefore \lim_{x\to 0}\frac{\frac{a(2x)^4}{24}+\frac{b(4x)^4}{24}+o(x^5)}{x^4}=\frac{2a}{3}+\frac{32b}{3}=\frac{24}{9}$$

19.

(1)

要使$g(x)$处处连续

$$\therefore \lim_{x\to 0}g(x)=\lim_{x\to 0}\frac{f(x)}{x}=\lim_{x\to 0}f'(x)=f'(0)=0=g(0)=a$$

$\therefore a=0$

(2)

当$x\neq 0$时,

$$\therefore g'(x)=(\frac{f(x)}{x})'=\frac{xf'(x)-f(x)}{x^2}$$

$\therefore g(x)$在$x\neq 0$处连续

当$x=0$时,

$\therefore g'(0)=\lim\limits_{x\to 0}\dfrac{g(x)-g(0)}{x}=\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}=\lim\limits_{x\to 0}\dfrac{f'(x)}{2x}=\dfrac{1}{2}f''(0)$

$\therefore \lim\limits_{x\to 0}g'(x)=\dfrac{xf'(x)-f(x)}{x^2}=\dfrac{f'(x)+xf''(x)-f'(x)}{2x}=\dfrac{1}{2}f''(0)$

$\therefore g'(0)$在$x=0$处也连续

2.4 (B)

1.

令$F(x)=x^nf(x)$,则$F(0)=0^nf(0)=0, F(1)=f'(1)=0$

$\therefore F'(x)=nx^{n-1}f(x)+x^nf'(x)$

$\because f(x)$在$[0,1]$连续,在$(0,1)$可导

$\therefore F(x)$在$[0,1]$连续,在$(0,1)$可导

$\therefore \exists x_0\in(0,1)$,使得$F'(x_0)=nx_0^{n-1}f(x_0)+x_0^nf'(x_0)=0$

$\therefore \exists x_0\in(0,1), nf(x_0)+x_0f'(x_0)=0$

3.

令$F(x)=e^{-\lambda x}f(x)$,则$F'(x)=e^{-\lambda x}f'(x)-\lambda e^{-\lambda x}f(x)$

$\because F(a)=F(b)=0, F(x)$在$[a,b]$上连续,在$(a,b)$内可微

$\therefore \exists c\in(a,b), e^{-\lambda c}f'(c)-\lambda e^{-\lambda c}f(c)=0$

$\therefore \exists c\in(a,b), f'(c)=\lambda f(c)$

5.

即证$\exists\xi\in(a,b)$,
使得$f(a)g(b)-f(b)g(a)=(b-a)[f(a)g'(\xi)-f'(\xi)g(a)]$

令$F(x)=(b-a)[f(a)g(x)-g(a)f(x)]+[f(b)g(a)-f(a)g(b)]x$

$\therefore F(a) = af(b)g(a) - af(a)g(b)$
$\quad F(b) = af(b)g(a) - af(a)g(b)$

$\therefore F(a) = F(b), F(x)$在$[a,b]$连续,在$(a,b)$可导

$\therefore \exists \xi \in (a,b), F'(\xi) = 0$

$\therefore f(a)g(b) - f(b)g(a) = (b-a)[f(a)g'(\xi) - f'(\xi)g(a)]$

2.5 (A)

3.

(2)

$$f(x) = \sum_{k=1}^{n}(-1)^{k-1}(x-1)^k + o((x-1)^n)$$

(4)

$$f(x) = \sum_{k=0}^{n}\frac{\sin^{(k)}(\frac{\pi}{4})}{k!}(x-\frac{\pi}{4})^k + o((x-x_0)^n)$$

当n为偶数时,

$$f(x) = \sum_{k=0}^{\frac{n}{2}}(-1)^k\frac{\sqrt{2}}{2}\left[\frac{(x-\frac{\pi}{4})^{2k}}{(2k)!} + \frac{(x-\frac{\pi}{4})^{2k+1}}{(2k+1)!}\right] + o((x-x_0)^{n+1})$$

当n为奇数时,

$$f(x) = \sum_{k=0}^{\frac{n-1}{2}}(-1)^k\frac{\sqrt{2}}{2}\left[\frac{(x-\frac{\pi}{4})^{2k}}{(2k)!} + \frac{(x-\frac{\pi}{4})^{2k+1}}{(2k+1)!}\right] + o((x-x_0)^n)$$

7.

(2)

$$
\begin{aligned}
&\lim_{x\to\infty}[(x^3-x^2+\frac{x}{2})e^{\frac{1}{x}}-\sqrt{x^6+1}] \\
=&\lim_{x\to\infty}[(x^3-x^2+\frac{x}{2})(1+\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3})-\sqrt{x^6+1}] \\
=&\lim_{x\to\infty}[x^3+\frac{1}{6}-\sqrt{x^6+1}] \\
=&\frac{1}{6}
\end{aligned}
$$

(3)

$$
\begin{aligned}
&\lim_{x\to\infty}[x-x^2\ln(1+\frac{1}{x})] \\
=&\lim_{x\to\infty}[x-x^2(\frac{1}{x}-\frac{1}{2x^2})] \\
=&\frac{1}{2}
\end{aligned}
$$

8.

$$
\lim_{x\to 0}\frac{f(x)-x}{x^2}=\lim_{x\to 0}\frac{f'(x)-1}{2x}=\lim_{x\to 0}\frac{1}{2}\frac{f'(x)-f'(0)}{x-0}=\frac{f''(0)}{2}=1
$$

2.5 (B)

1.

$\because f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{1}{2}f''(\xi)(x-x_0)^2$

$\therefore f(2)=f(x_0)+f'(x_0)(2-x_0)+\frac{1}{2}f''(\xi_1)(2-x_0)^2$
$\quad f(0)=f(x_0)-x_0f'(x_0)-x_0^2\frac{1}{2}f''(\xi_2)$

$\therefore f(2)-f(0)=2f'(x_0)+\frac{1}{2}(2-x_0)^2f''(\xi_1)+\frac{1}{2}x_0^2f''(\xi_2)$

$$
\begin{aligned}
\therefore 2f'(x_0) &= f(2)-f(0)-\frac{1}{2}(2-x_0)^2f''(\xi_1)-\frac{1}{2}x_0^2f''(\xi_2) \\
&\le |f(2)|+|f(0)|+\frac{1}{2}(2-x_0)^2|f''(\xi_1)|+\frac{1}{2}x_0^2|f''(\xi_2)| \\
&\le 2+\frac{1}{2}(2-x_0)^2+\frac{1}{2}x_0^2 \\
&\le 4
\end{aligned}
$$

$\therefore f'(x) < 2$

4.

$\because$ 极限存在

$\therefore$ 应为1^∞的不定型, $f(0)=0$

$$
\because \lim_{x\to 0}(1+x+\frac{f(x)}{x})^{\frac{1}{x}}
$$
$$
=\lim_{x\to 0}\exp(\frac{1}{x}\ln(1+x+\frac{f(x)}{x}))
$$
$$
=\lim_{x\to 0}\exp(1+\frac{f(x)}{x^2})
$$
$$
=e^3
$$

$$
\therefore \lim_{x\to 0}\frac{f(x)}{x^2}=\lim_{x\to 0}\frac{f'(x)}{2x}=\frac{f''(0)}{2}=3-1=2
$$

$\therefore f'(0)=0, f''(0)=4$

2.6 (A)

9.(3)

$$
f(x)=\frac{(x+1)^{\frac{2}{3}}}{x-1}
$$

$\therefore f(x)$定义域为$(-\infty,1)\cup(1,+\infty)$

$$
\therefore f'(x)=\frac{\frac{2}{3}(x+1)^{-\frac{1}{3}}(x-1)+(x+1)^{\frac{2}{3}}}{(x-1)^2}=\frac{(5x+1)(x+1)^{\frac{2}{3}}}{3(x-1)^2}
$$

令$f'(x)=0$,得$x=-\frac{1}{5}$或$x=-1$

$\therefore f'(x)$在$(-\infty,-\frac{1}{5})$小于或等于0,在$f(-\frac{1}{5},1)\cup(1,+\infty)$大于0

$\therefore f(x)$在$(-\infty,-\frac{1}{5})$递减,在$f(-\frac{1}{5},1)\cup(1,+\infty)$递增,
在$x=-\frac{1}{5}$处有极小值

12. (2)

$\because f(x)=\sin^3 x+\cos^3 x, x\in[\frac{\pi}{6},\frac{3\pi}{4}]$

$$\begin{aligned}\therefore f'(x)&=3\sin^2 x\cos x-3\cos^2 x\sin x\\&=3(\sin x-\cos x)\sin x\cos x\\&=\frac{3\sqrt{2}}{4}\sin(x-\frac{\pi}{4})\sin 2x\end{aligned}$$

$\because \sin(x-\frac{\pi}{4})$在$[\frac{\pi}{6},\frac{\pi}{4}]$上$\leq 0$,在$[\frac{\pi}{4},\frac{3\pi}{4}]$上$\geq 0$,
$\sin 2x$在$[\frac{\pi}{6},\frac{\pi}{2}]$上$\geq 0$,在$[\frac{\pi}{2},\frac{3\pi}{4}]$上$\leq 0$

$\therefore f(x)$在$[\frac{\pi}{6},\frac{\pi}{4}]$和$[\frac{\pi}{2},\frac{3\pi}{4}]$递减,在$[\frac{\pi}{4},\frac{\pi}{2}]$递增

$f(x)$在$x=\frac{\pi}{4}$有极小值,在$x=\frac{\pi}{2}$有极大值

22. (3)

$\because f(x)=\frac{x}{1+x^2}$

$\therefore f'(x)=\frac{1-x^2}{1+x^2}=\frac{2}{1+x^2}-1$,
$f''(x)=-\frac{4x}{(1+x^2)^2}$

$\therefore f''(x)$在$(-\infty,0)$上>0,在$(0,+\infty)$上<0, $f''(0)=0$

$\therefore f(x)$在$(-\infty,0)$上是凸函数,在$(0,+\infty)$上是凹函数,拐点为$x=0$

24. (2)

$\because y = \dfrac{4(x+1)}{x^2} - 2$

$\therefore f(x)$的定义域为$(-\infty, 0) \cup (0, +\infty)$

$\because \lim\limits_{x \to 0} \dfrac{4(x+1)}{x^2} - 2 = +\infty$

$\therefore x = 0$是其中的一条垂直渐近线

$\because \lim\limits_{x \to \infty} \dfrac{4(x+1)}{x^2} - 2 = -2$

$\therefore y = -2$是其中的一条水平渐近线,无斜渐近线

26. (2)

$\because f(x) = \dfrac{2x-1}{(x-1)^2}$

$\therefore f(x)$的定义域为$(-\infty, 1) \cup (1, +\infty)$

$\therefore f'(x) = \dfrac{2(x-1)^2 - 2(x-1)(2x-1)}{(x-1)^4} = \dfrac{-2x}{(x-1)^3}$

$\therefore f''(x) = \dfrac{4x+2}{(x-1)^4}$

$\therefore f(x)$在$(-\infty, 0)$和$(1, +\infty)$递减,在$(0, 1)$递增,
在$x = 0$有极小值$f(0) = -1$

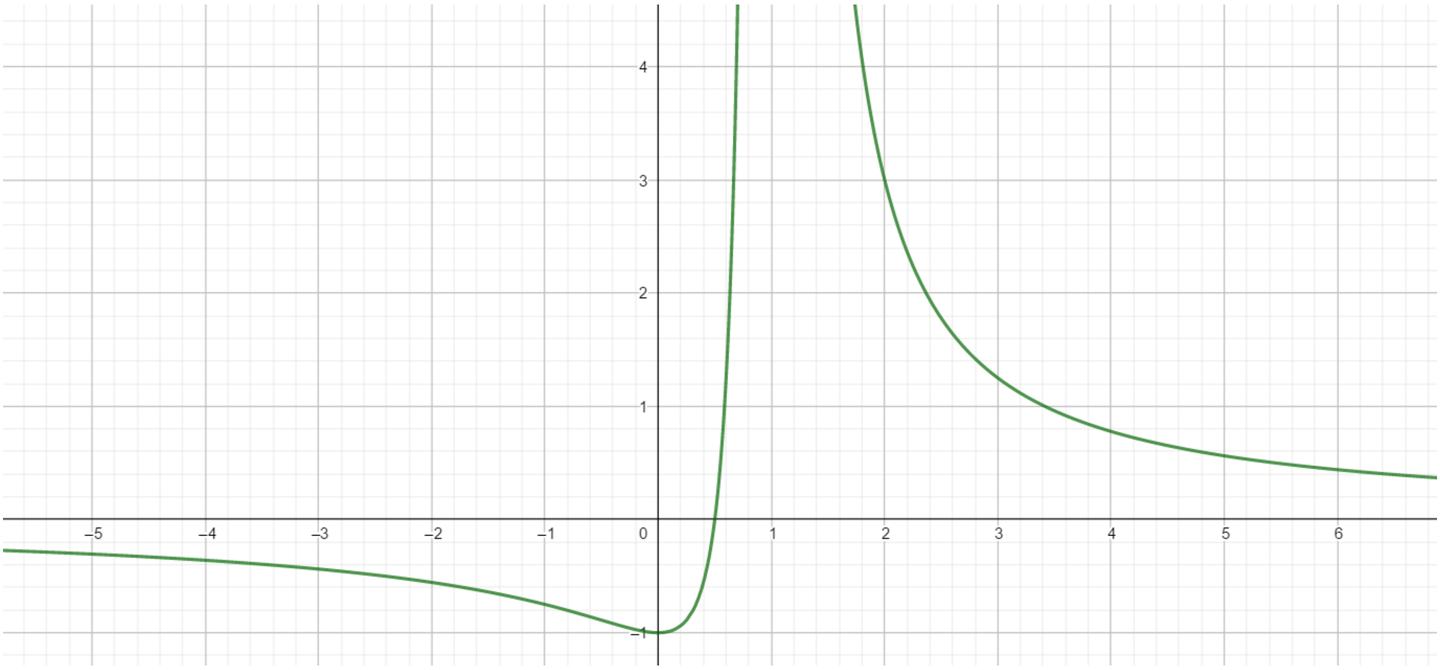
$\therefore f(x)$在$(-\frac{1}{2}, 1) \cup (1, +\infty)$为凸函数,在$(-\infty, -\frac{1}{2})$为凹函数

$\because \lim\limits_{x \to 1} f(x) = \lim\limits_{x \to 1} \dfrac{2x-1}{(x-1)^2} = +\infty$

$\therefore$有一条垂直渐近线$x = 0$

$\because \lim\limits_{x \to \infty} \dfrac{2x-1}{(x-1)^2} = 0$

$\therefore$有一条水平渐近线$y = 0$,无斜渐近线



2.6 (B)

2. (3)

要证$\sin x+\tan x>2x,(0<x<\frac{\pi}{2})$

即证$\dfrac{\sin x+\tan x}{x}>2$

令$f(x)=\dfrac{\sin x+\tan x}{x}$

$\therefore \lim\limits_{x\to 0}\dfrac{\sin x+\tan x}{x}=2$

$\because f(x)=\dfrac{\cos x+\sec^2 x+x\sin x+x\tan x}{x^2}>0$

$\therefore f(x)>2$

$\therefore \sin x+\tan x>2x,(0<x<\frac{\pi}{2})$

5.

当$n=1$时,

$\therefore f(x)=(x-x_0)g(x),f'(x)=g(x)+(x-x_0)g'(x)$

$\therefore f'(x_0) = g(x_0) \neq 0$

$\therefore$ 此时$f(x)$在x_0处无极值

当$n = 2$时,

$\therefore f'(x) = 2(x - x_0)g(x) + (x - x_0)^2 g'(x)$

$\therefore f'(x_0) = 0$

$\therefore f''(x) = 2g(x) + 2(x - x_0)g'(x) + [(x - x_0)^2 g'(x)]'$

$\therefore f''(x_0) = 2g(x_0) \neq 0$

$\therefore f(x)$在x_0处有极值

同理可知, $f^{(n-1)} = 0, f^{(n)} \neq 0$

$\therefore$ 当n为奇数时, 在x_0处无极值, 当n为偶数时, 在x_0处有极值