概率统计第六次作业

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6.1

要证 $P(X \geqslant \epsilon) = \int_{\epsilon}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x \geqslant \frac{1}{3} e^{-\frac{(\epsilon+1)^2}{2}}$

即证 $\int_{\epsilon}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x - \frac{1}{3} e^{-\frac{(\epsilon+1)^2}{2}} \geqslant 0$

令 $f(\epsilon) = \int_{\epsilon}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x - \frac{1}{3} e^{-\frac{(\epsilon+1)^2}{2}}$

$\therefore f'(\epsilon) = \frac{1}{3}(\epsilon+1) e^{-\frac{\epsilon^2+2\epsilon+1}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{\epsilon^2}{2}} = e^{-\frac{\epsilon^2}{2}} \left[\frac{1}{3}(\epsilon+1) e^{-\frac{1}{2}} \cdot e^{-\epsilon} - \frac{1}{\sqrt{2\pi}} \right]$

令 $g(\epsilon) = \frac{1}{3}(\epsilon+1) e^{-\frac{1}{2}} \cdot e^{-\epsilon} - \frac{1}{\sqrt{2\pi}}$

$\therefore g'(\epsilon) = -\frac{1}{3} \epsilon e^{-\epsilon-\frac{1}{2}} < 0$

$\therefore g(\epsilon) < g(0) = \frac{1}{3} e^{-\frac{1}{2}} - \frac{1}{\sqrt{2\pi}} < 0$

$\therefore f'(\epsilon) < 0$

$\therefore f(\epsilon) \geqslant \lim_{\epsilon \to \infty} \left[\int_{\epsilon}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \mathrm{d}x - \frac{1}{3} e^{-\frac{(\epsilon+1)^2}{2}} \right] = 0$

$\therefore P(X \geqslant \epsilon) \geqslant \frac{1}{3} e^{-\frac{(\epsilon+1)^2}{2}}$

6.2

$$
\begin{aligned}
E(X) &= \int_0^{+\infty} \frac{x}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} \, \mathrm{d}x \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\frac{x}{\beta}} \, \mathrm{d}x \\
&= \frac{-\beta}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^\alpha \mathrm{d}e^{-\frac{x}{\beta}} \\
&= \frac{-\beta}{\beta^\alpha \Gamma(\alpha)} \left[[x^\alpha e^{-\frac{x}{\beta}}]_0^{+\infty} - \int_0^{+\infty} e^{-\frac{x}{\beta}} \, \mathrm{d}x^\alpha \right] \\
&= \frac{\alpha\beta}{\Gamma(\alpha)} \int_0^{+\infty} (\frac{x}{\beta})^{\alpha-1} e^{-\frac{x}{\beta}} \, \mathrm{d}\frac{x}{\beta} \\
&= \alpha\beta
\end{aligned}
$$

$$
\begin{aligned}
E(X^2) &= \int_0^{+\infty} \frac{x^2}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}} \, \mathrm{d}x \\
&= \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} e^{-\frac{x}{\beta}} \, \mathrm{d}x \\
&= \frac{-\beta}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^{\alpha+1} \mathrm{d}e^{-\frac{x}{\beta}} \\
&= \frac{\beta}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} e^{-\frac{x}{\beta}} \, \mathrm{d}x^{\alpha+1} \\
&= \frac{\beta(\alpha+1)}{\beta^\alpha \Gamma(\alpha)} \int_0^{+\infty} x^\alpha e^{-\frac{x}{\beta}} \, \mathrm{d}x \\
&= \alpha^2\beta^2 + \alpha\beta^2
\end{aligned}
$$

$$
\mathrm{Var}(X) = E(X^2) - E(X)^2 = \alpha\beta^2
$$

6.3

26.

(1)

$P(2 < X \leqslant 5) = P(\frac{2-3}{2} < \frac{X-3}{2} \leqslant \frac{5-3}{2}) = \Phi(1) - \Phi(-\frac{1}{2}) = \Phi(1) + \Phi(\frac{1}{2}) - 1 = 0.5328$

$P(-4 < X \leqslant 10) = P(\frac{-4-3}{2} < \frac{X-3}{2} \leqslant \frac{10-3}{2}) = \Phi(\frac{7}{2}) - \Phi(-\frac{7}{2}) = 2\Phi(\frac{7}{2}) - 1 = 0.9996$

$P(|X|>2)=P(\frac{X-3}{2}<\frac{-2-3}{2}\text{ or }\frac{X-3}{2}>\frac{2-3}{2})=1-\Phi(-\frac{1}{2})+\Phi(-\frac{5}{2})=1+\Phi(\frac{1}{2})-\Phi(\frac{5}{2})=0.6977$

$P(X>3)=P(\frac{X-3}{2}>\frac{3-3}{2})=1-\Phi(0)=\Phi(0)=0.5$

(2)

由 (1) 知 $P(X>3)=\Phi(0)$, 且有 $P(X\leqslant 3)=P(\frac{X-3}{2}\leqslant\frac{3-3}{2})=\Phi(0)$

$\therefore c=3$

(3)

$\because P(X>d)\geqslant 0.9$

$\therefore 1-\Phi(\frac{d-3}{2})=\Phi(-\frac{d-3}{2})\geqslant 0.9=\Phi(1.282)$

$\because \Phi(x)$ 单调递增

$\therefore -\frac{d-3}{2}\geqslant 1.282$

$\therefore d\leqslant 0.436$

32.

$\because f(x),g(x)$ 都是概率密度函数

$\therefore f(x)\geqslant 0, g(x)\geqslant 0, \int_{-\infty}^{+\infty} f(x)\mathrm{d}x=1, \int_{-\infty}^{+\infty} g(x)\mathrm{d}x=1$

$\therefore h(x)=\alpha f(x)+(1-\alpha)g(x)\geqslant 0$

$\therefore \int_{-\infty}^{+\infty} h(x)\mathrm{d}x=\int_{-\infty}^{+\infty}[\alpha f(x)+(1-\alpha)g(x)]\mathrm{d}x=\alpha\int_{-\infty}^{+\infty} f(x)\mathrm{d}x+(1-\alpha)\int_{-\infty}^{+\infty} g(x)\mathrm{d}x=\alpha+(1-\alpha)=1$

$\therefore h(x)$ 是概率密度函数

34.

(1)

$\because X$ 在 $(0,1)$ 服从均匀分布

$\therefore f_x(x)=1, 0\leqslant x\leqslant 1$

$\therefore F_y(y)=P(Y\leqslant y)=P(e^X\leqslant y)=P(X\leqslant \ln y)=F_x(\ln y), 1\leqslant y\leqslant e$

$\therefore f_y(y)=f_x(\ln y)\cdot \dfrac{1}{y}=\dfrac{1}{y}, 1\leqslant y\leqslant e$

$\therefore f_y(y)=\begin{cases}\frac{1}{y}, & 1\leqslant y\leqslant e\\ 0, & \text{otherwise}\end{cases}$

(2)

$\therefore F_y(y)=P(Y\leqslant y)=P(-2\ln X\leqslant y)=P(X\geqslant e^{-\frac{1}{2}y})=1-F_x(e^{-\frac{1}{2}y}), y\geqslant 0$

$\therefore f_y(y)=-f_x(e^{-\frac{1}{2}y})\cdot e^{-\frac{1}{2}y}\cdot(-\dfrac{1}{2})=\dfrac{1}{2}e^{-\frac{1}{2}y}, y\geqslant 0$

$\therefore f_y(y)=\begin{cases}\frac{1}{2}e^{-\frac{1}{2}y}, & y\geqslant 0\\ 0, & \text{otherwise}\end{cases}$

35.

(1)

$\because X\sim N(0,1)$

$\therefore f_x(x)=\dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$

$\therefore F_y(y)=P(Y\leqslant y)=P(e^X\leqslant y)=P(X\leqslant \ln y)=F_x(\ln y), y>0$

$\therefore f_y(y)=f_x(\ln y)\cdot\dfrac{1}{y}=\dfrac{1}{\sqrt{2\pi}y}e^{-\frac{(\ln y)^2}{2}}, y>0$

$\therefore f_y(y)=\begin{cases}\frac{1}{\sqrt{2\pi}y}e^{-\frac{(\ln y)^2}{2}}, & y>0\\ 0, & y\leqslant 0\end{cases}$

(2)

$\therefore F_y(y)=P(Y\leqslant y)=P(2X^2+1\leqslant y)=P(-\sqrt{\dfrac{y-1}{2}}\leqslant X\leqslant\sqrt{\dfrac{y-1}{2}})=F_x(\sqrt{\dfrac{y-1}{2}})-F_x(-\sqrt{\dfrac{y-1}{2}}), y\geqslant 1$

$\therefore f_y(y) = f_x(\sqrt{\frac{y-1}{2}}) \cdot \frac{1}{4} \cdot (\frac{y-1}{2})^{-\frac{1}{2}} + f_x(-\sqrt{\frac{y-1}{2}}) \cdot \frac{1}{4} \cdot (\frac{y-1}{2})^{-\frac{1}{2}} = \frac{1}{4}(\frac{y-1}{2})^{-\frac{1}{2}} \left[f_x(\sqrt{\frac{y-1}{2}}) + f_x(-\sqrt{\frac{y-1}{2}}) \right] = \frac{1}{2\sqrt{2\pi}}(\frac{y-1}{2})^{-\frac{1}{2}} e^{-\frac{y-1}{4}}, y \geqslant 0$

$$\therefore f_y(y) = \begin{cases} \frac{1}{2\sqrt{2\pi}}(\frac{y-1}{2})^{-\frac{1}{2}} e^{-\frac{y-1}{4}}, & y \geqslant 0 \\ 0, & y < 0 \end{cases}$$

(3)

$\therefore F_y(y) = P(Y \leqslant y) = P(|X| \leqslant y) = P(-y \leqslant X \leqslant y) = F_x(y) - F_x(-y), y \geqslant 0$

$$\therefore f_y(y) = f_x(y) + f_x(-y) = \frac{\sqrt{2\pi}}{\pi} e^{-\frac{y^2}{2}}, y \geqslant 0$$

$$\therefore f_y(y) = \begin{cases} \frac{\sqrt{2\pi}}{\pi} e^{-\frac{y^2}{2}}, & y \geqslant 0 \\ 0, & y < 0 \end{cases}$$

36.

(1)

$$\therefore F_y(y) = P(Y \leqslant y) = P(X^3 \leqslant y) = P(X \leqslant \sqrt[3]{y}) = F_x(\sqrt[3]{y})$$

$$\therefore f_y(y) = \frac{1}{3} y^{-\frac{2}{3}} f(\sqrt[3]{y})$$

(2)

$\therefore F_y(y) = P(Y \leqslant y) = P(X^2 \leqslant y) = P(-\sqrt{y} \leqslant X \leqslant \sqrt{y}) = F_x(\sqrt{y}) - F_x(-\sqrt{y}) = F_x(\sqrt{y}), y \geqslant 0$

$$\therefore f_y(y) = \frac{1}{2} y^{-\frac{1}{2}} f(\sqrt{y}) = \frac{1}{2} y^{-\frac{1}{2}} e^{-\sqrt{y}}, y \geqslant 0$$

$$\therefore f_y(y) = \begin{cases} \frac{1}{2} y^{-\frac{1}{2}} e^{-\sqrt{y}}, & y \geqslant 0 \\ 0, & y < 0 \end{cases}$$

37.

$\therefore F_y(y) = P(Y \leqslant y) = P(\sin X \leqslant y) = P(X \leqslant \arcsin y \text{ or } X \geqslant \pi - \arcsin y) = F_x(\arcsin y) + 1 - F_x(\pi - \arcsin y), 0 < y < 1$

$$\therefore f_y(y) = \frac{2\arcsin y}{\pi^2\sqrt{1-y^2}} + \frac{2(\pi - \arcsin y)}{\pi^2\sqrt{1-y^2}} = \frac{2}{\pi\sqrt{1-y^2}}$$

$$\therefore f_y(y) = \begin{cases} \frac{2}{\pi\sqrt{1-y^2}}, & 0 < y < 0 \\ 0, & \text{otherwise} \end{cases}$$

6.4

$\because P(x_1 < X \leqslant x_2, y_1 < Y \leqslant y_2) = F(x_2, y_2) - F(x_2, y_1) - F(x_1, y_2) + F(x_1, y_1)$

$\therefore P(X > x, Y > y) = P(x < X \leqslant +\infty, y < Y \leqslant +\infty) = F(+\infty, +\infty) - F(+\infty, y_1) - F(x_1, +\infty) + F(x_1, y_1) = 1 - F(+\infty, y_1) - F(x_1, +\infty) + F(x_1, y_1)$

$\therefore P(X > x, Y > y) = 1 - F(+\infty, y_1) - F(x_1, +\infty) + F(x_1, y_1)$