P109~203:

1.(2) 2(3,5,6) 4.(1,2) 5. 7.(3) 10. 12. 14. 15. 16. 17. 18. 20(3,7,10) 21. 22. 23.(2,4) 24.(1) 25. 28. 29. 30.

1. (2)

$$AB = \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix} = \begin{pmatrix} a+b+c & a^2+b^2+c^2 & 2ac+b^2 \\ c+b+a & 2ac+b^2 & a^2+b^2+c^2 \\ 3 & a+b+c & a+b+c \end{pmatrix}$$

$$\begin{aligned} AB - BA &= \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix} - \begin{pmatrix} 1 & a & c \\ 1 & b & b \\ 1 & c & a \end{pmatrix} \begin{pmatrix} a & b & c \\ c & b & a \\ 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} a+b+c & a^2+b^2+c^2 & 2ac+b^2 \\ c+b+a & 2ac+b^2 & a^2+b^2+c^2 \\ 3 & a+b+c & a+b+c \end{pmatrix} \\ &- \begin{pmatrix} a+ac+c & b+ab+c & 2c+a^2 \\ a+bc+b & 2b+b^2 & c+ab+b \\ 2a+c^2 & b+bc+a & c+ac+a \end{pmatrix} \\ &= \begin{pmatrix} ac-b & b+ab+c-a^2-b^2-c^2 & 2c+a^2-b^2-2ac \\ bc-c & 2b-2ac & c+ab+b-a^2-b^2-c^2 \\ 2a+c^2-3 & bc-c & ac-b \end{pmatrix} \end{aligned}$$

2.

(3)

当$n=1$时, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

当$n=k$时,假设 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{k+1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^k \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1k+1 \\ 0 & 1 \end{pmatrix}$

$\therefore$ 综上 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

(5)

$$\begin{pmatrix}2 & 3 & -1\end{pmatrix}\begin{pmatrix}1\\-1\\-1\end{pmatrix}=2-3+1=0$$

$$\begin{pmatrix}1\\-1\\-1\end{pmatrix}\begin{pmatrix}2 & 3 & -1\end{pmatrix}=\begin{pmatrix}2 & 3 & -1\\-2 & -3 & 1\\-2 & -3 & 1\end{pmatrix}$$

(6)

$$\begin{aligned}&\begin{pmatrix}x & y & 1\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & b_1\\a_{12} & a_{22} & b_2\\b_1 & b_2 & c\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}\\=&\begin{pmatrix}a_{11}x+a_{12}x+b_1x & a_{12}x+a_{22}x+b_2x & b_1x+b_2y+c\end{pmatrix}\begin{pmatrix}x\\y\\1\end{pmatrix}\\=&a_{11}x^2+a_{12}x^2+b_1x^2+a_{12}xy+a_{22}xy+b_2xy+b_1x+b_2y+c\end{aligned}$$

4.

(1)

$$AB=\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}\begin{pmatrix}x_1 & x_2\\x_3 & x_4\end{pmatrix}=\begin{pmatrix}x_1+x_3 & x_2+x_4\\x_3 & x_4\end{pmatrix}$$

$$BA=\begin{pmatrix}x_1 & x_2\\x_3 & x_4\end{pmatrix}\begin{pmatrix}1 & 1\\0 & 1\end{pmatrix}=\begin{pmatrix}x_1 & x_1+x_2\\x_3 & x_3+x_4\end{pmatrix}$$

$$\therefore\begin{cases}x_1=x_1+x_3\\x_2+x_4=x_1+x_2\\x_3=x_3\\x_4=x_3+x_4\end{cases}\Rightarrow\begin{cases}x_3=0\\x_4=x_1\end{cases}$$

$\therefore$取x_1和x_2为自由变量

$\therefore$与A可交换的矩阵$B=\begin{pmatrix}x_1 & x_2\\0 & x_1\end{pmatrix}$

(2)

$$AB=\begin{pmatrix}1&0&0\\0&1&2\\3&1&2\end{pmatrix}\begin{pmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{pmatrix}=$$
$$\begin{pmatrix}x_1&x_2&x_3\\x_4+2x_7&x_5+2x_8&x_6+2x_9\\3x_1+x_4+2x_7&3x_2+x_5+2x_8&3x_3+x_6+2x_9\end{pmatrix}$$

$$BA=\begin{pmatrix}x_1&x_2&x_3\\x_4&x_5&x_6\\x_7&x_8&x_9\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&2\\3&1&2\end{pmatrix}=\begin{pmatrix}x_1+3x_3&x_2+x_3&2x_2+2x_3\\x_4+3x_6&x_5+x_6&2x_5+2x_6\\x_7+3x_9&x_8+x_9&2x_8+2x_9\end{pmatrix}$$

$$\begin{cases}x_1=x_1+3x_3\\x_2=x_2+x_3\\x_3=2x_2+2x_3\\x_4+2x_7=x_4+3x_6\\x_5+2x_8=x_5+x_6\\x_6+2x_9=2x_5+2x_6\\3x_1+x_4+2x_7=x_7+3x_9\\3x_2+x_5+2x_8=x_8+x_9\\3x_3+x_6+2x_9=2x_8+2x_9\end{cases}\Rightarrow\begin{cases}x_2=0\\x_3=0\\x_5=x_1+\frac{1}{3}x_4\\x_6=\frac{2}{3}x_7\\x_8=\frac{1}{3}x_7\\x_9=x_5+\frac{1}{3}x_7\end{cases}$$

取x_1,x_4,x_7为自由变量,则$B=\begin{pmatrix}x_1&0&0\\x_4&x_1+\frac{1}{3}x_4&\frac{2}{3}x_7\\x_7&\frac{1}{3}x_7&x_1+\frac{1}{3}x_4+\frac{1}{3}x_7\end{pmatrix}$

5.

$$AB=\begin{pmatrix}a_1&0&\cdots&0\\0&a_2&\cdots&0\\\vdots&\vdots&&\vdots\\0&0&\cdots&a_n\end{pmatrix}\begin{pmatrix}x_{11}&x_{12}&\cdots&x_{1n}\\x_{21}&x_{22}&\cdots&x_{2n}\\\vdots&\vdots&&\vdots\\x_{n1}&x_{n2}&\cdots&x_{nn}\end{pmatrix}=$$
$$\begin{pmatrix}a_1x_{11}&a_1x_{12}&\cdots&a_1x_{1n}\\a_2x_{21}&a_2x_{22}&\cdots&a_2x_{2n}\\\vdots&\vdots&&\vdots\\a_nx_{n1}&a_nx_{n2}&\cdots&a_nx_{nn}\end{pmatrix}$$

$$BA=\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix}\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}=$$

$$\begin{pmatrix} a_1x_{11} & a_2x_{12} & \cdots & a_nx_{1n} \\ a_1x_{21} & a_2x_{22} & \cdots & a_nx_{2n} \\ \vdots & \vdots & & \vdots \\ a_1x_{n1} & a_2x_{n2} & \cdots & a_nx_{nn} \end{pmatrix}$$

$\because a_i \neq a_j, i \neq j$

$\therefore a_i x_{pq} \neq a_j x_{pq}, i \neq j$

$\therefore x_{ij}=0, i \neq j$

$\therefore$ 与A可交换的矩阵只能是对角矩阵

7. (3)

设B是任一个n级矩阵, $B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}$

由第5.题可知, A一定是一个对角矩阵, 设为$A=\begin{pmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_n \end{pmatrix}$

$$\therefore AB=\begin{pmatrix} a_1b_{11} & a_1b_{12} & \cdots & a_1b_{1n} \\ a_2b_{21} & a_2b_{22} & \cdots & a_2b_{2n} \\ \vdots & \vdots & & \vdots \\ a_nb_{n1} & a_nb_{n2} & \cdots & a_nb_{nn} \end{pmatrix}$$

$$BA=\begin{pmatrix} a_1b_{11} & a_2b_{12} & \cdots & a_nb_{1n} \\ a_1b_{21} & a_2b_{22} & \cdots & a_nb_{2n} \\ \vdots & \vdots & & \vdots \\ a_1b_{n1} & a_2b_{n2} & \cdots & a_nb_{nn} \end{pmatrix}$$

$\because AB=BA$

$\therefore a_i b_{ij}=a_j b_{ij}$

$\therefore a_i = a_j = a$

$\therefore A$是一个数量矩阵, $A = aE$

10.

$$\therefore A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$\therefore A^2 = \begin{pmatrix} \sum a_{1j}^2 & \cdots & \cdots & \cdots \\ \cdots & \sum a_{2j}^2 & \cdots & \cdots \\ \vdots & \vdots & & \vdots \\ \cdots & \cdots & \cdots & \sum a_{nj}^2 \end{pmatrix}$$

$\because A^2 = O$

$\therefore \sum a_{1j}^2 = 0, \sum a_{2j}^2 = 0, \cdots, \sum a_{nj}^2 = 0$

$\therefore a_{ij}^2 = 0$

$\therefore a_{ij} = 0$

$\therefore A = O$

12.

设A是任意一个$n \times n$矩阵

令$B = \frac{1}{2}(A + A^T), C = \frac{1}{2}(A - A^T)$

易知$A = B + C = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$

$\because B^T = [\frac{1}{2}(A + A^T)]^T = \frac{1}{2}(A + A^T)^T = \frac{1}{2}(A + A^T) = B$

$\therefore B$是对称矩阵

$\because C^T = [\frac{1}{2}(A - A^T)]^T = \frac{1}{2}(A^T - A) = -\frac{1}{2}(A - A^T) = -C$

$\therefore C$是反称矩阵

$\therefore$ 原命题得证

14.

对于必要性：

$\because |A| = 0$

$\therefore A$必有两个或以上的列向量线性相关

$\therefore A$可以写成$\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$\because$ 令$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

$\therefore AB = O,$存在一个非零矩阵B成立

对于充分性：

假设$|A| \neq 0,$则A可逆,设A的逆矩阵为A^{-1}

$\because AB = O$

$\therefore B = A^{-1}AB = A^{-1}O = O,$与$B$为非零矩阵矛盾

$\therefore |A| = 0$

原命题得证

15.

设$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

$$\therefore AX=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{pmatrix}\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}\sum a_{1i}x_i\\\sum a_{2i}x_i\\\vdots\\\sum a_{ni}x_i\end{pmatrix}=\begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$$

不妨假设$a_{11}\neq 0$

当$x_i=1$时,则$a_{11}x_1=a_{11}=-\sum_{i=2}^{n}a_{1i}\neq 0$

再取$x_1=-1,x_i=1,i=1,2,\cdots,n$

此时$\sum_{i=1}^{n}a_{1i}x_i=-a_{11}+\sum_{i=2}^{n}a_{1i}=-2a_{11}\neq 0$,产生矛盾

因此$a_{11}=0$

同理可知$a_{ij}=0$

$\therefore A=O$

16.

(1)

$\because rank(C)=r$

$\therefore r\leq n$

$\therefore$ 可知C可以改写成$\begin{pmatrix}c_{11}&c_{12}&\cdots&c_{1r}&0&\cdots&0\\c_{21}&c_{22}&\cdots&c_{2r}&0&\cdots&0\\\vdots&\vdots&&\vdots&\vdots&&\vdots\\c_{r1}&c_{r2}&\cdots&c_{rr}&0&\cdots&0\end{pmatrix}$ 即记为$(D\quad 0\quad\cdots\quad 0)$,其中$D$为$r\times r$矩阵

$\therefore BC=B(D\quad 0\quad\cdots\quad 0)=(BD\quad 0\quad\cdots\quad 0)$

其中BD为$r\times r$矩阵相乘

$\because rank(C)=r$

$\therefore rank(D)=r$,即$|D|\neq 0$,D存在逆矩阵,记为D^{-1}

$\because BC=O$

$\therefore BD = O$

$\therefore B = BDD^{-1} = OD^{-1} = O$

(2)

同理可设$C = (D \quad 0 \quad \cdots \quad 0)$

$\because BC = C$

$\therefore BD = D$

$\therefore B = BDD^{-1} = DD^{-1} = E$

17.

设$A = \begin{pmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{pmatrix}, B = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vdots \\ \vec{b}_n \end{pmatrix}$

设向量组$V = \{\vec{a}_1, \vec{a}_2, \cdots, \vec{a}_n, \vec{b}_1, \vec{b}_2, \cdots, \vec{b}_n\}$

易知向量组V的秩小于等于$rank(A) + rank(B)$

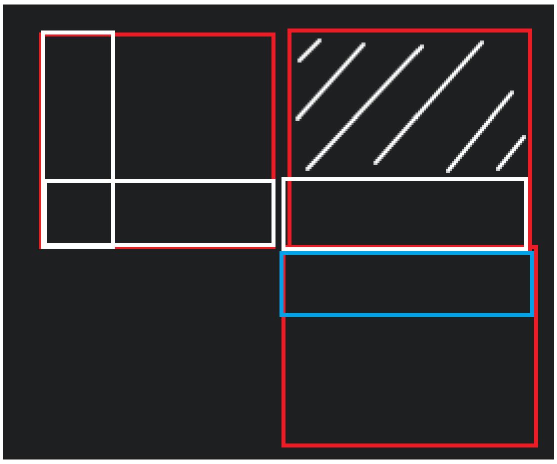
$\because A + B = \begin{pmatrix} \vec{a}_1 + \vec{b}_1 \\ \vec{a}_2 + \vec{b}_2 \\ \vdots \\ \vec{a}_n + \vec{b}_n \end{pmatrix},$

即$A + B$中每个向量都能由V中的向量线性表示

$\therefore rank(A + B) \leq rank(V) \leq rank(A) + rank(B)$

$\therefore rank(A + B) \leq rank(A) + rank(B)$

18.



$\because AB=O$

$\therefore |AB|=|A||B|=|O|=0$

$\therefore A$和B中必定有一个行列式为0,不妨设$|A|=0, rank(A)=r$

$$\therefore A可以表示成A=\begin{pmatrix} 0 & \cdots & 0 & a_{11} & a_{12} & \cdots & a_{1r} \\ 0 & \cdots & 0 & a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & a_{r1} & a_{r2} & \cdots & a_{rr} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$简写为A=\begin{pmatrix} 0 & A_r \\ 0 & 0 \end{pmatrix}$$

$$B可以写为B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{(n-r)1} & b_{(n-r)2} & \cdots & b_{(n-r)n} \\ c_{11} & c_{12} & \cdots & c_{1n} \\ \vdots & \vdots & & \vdots \\ c_{r1} & c_{r2} & \cdots & c_{rn} \end{pmatrix}$$

$$简写为B=\begin{pmatrix} B_r \\ C \end{pmatrix}$$

其中B的前$n-r$个行向量都确保尽量线性无关

$$\therefore AB=\begin{pmatrix} A_rC \\ 0 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\therefore$ 可知$A_rC=O$,且$|A_r|\neq 0$,存在逆矩阵A_r^T

$\therefore C=A_r^TA_rC=A_r^TO=O$

$\therefore rank(B) \leq n - r$

$\therefore rank(A) + rank(B) \leq r + n - r = n$

20.

(3)

$$\therefore A^* = \begin{pmatrix} -1 & 4 & 3 \\ -1 & 5 & 3 \\ 1 & -6 & -4 \end{pmatrix}, |A| = -2 + 6 - 3 - 2 = -1$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -4 & -3 \\ 1 & -5 & -3 \\ -1 & 6 & 4 \end{pmatrix}$$

(7)

$$\therefore A^* = \begin{pmatrix} 1 & -3 & 11 & -38 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, |A| = 1$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -3 & 11 & -38 \\ 0 & 1 & -2 & 7 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(10)

$$\therefore A^* = \begin{pmatrix} 16 & -8 & 4 & -2 & 1 \\ 0 & 16 & -8 & 4 & -2 \\ 0 & 0 & 16 & -8 & 4 \\ 0 & 0 & 0 & 16 & -8 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix}, |A| = 32$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} \\ 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 0 & 0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

21.

$$\because \begin{pmatrix} O & A \\ C & O \end{pmatrix} \begin{pmatrix} O & C^{-1} \\ A^{-1} & O \end{pmatrix} = \begin{pmatrix} AA^{-1} & O \\ O & CC^{-1} \end{pmatrix} = E$$

$$\therefore X^{-1} = \begin{pmatrix} O & C^{-1} \\ A^{-1} & O \end{pmatrix}$$

22.

$$\because \begin{pmatrix} 0 & a_1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1} \\ a_n & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \frac{1}{a_n} \\ \frac{1}{a_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{n-1}} & 0 \end{pmatrix} = E$$

$$\therefore X^{-1} = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & \frac{1}{a_n} \\ \frac{1}{a_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \frac{1}{a_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \frac{1}{a_{n-1}} & 0 \end{pmatrix}$$

23.

(2)

$$设A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 2 \\ 1 & -1 & 0 \end{pmatrix}$$

$$\therefore A^* = \begin{pmatrix} 2 & 1 & 4 \\ 2 & 1 & -2 \\ -2 & 2 & 2 \end{pmatrix}, |A| = 6$$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\therefore X=\begin{pmatrix}\frac{1}{3}&\frac{1}{6}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{6}&-\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix}\begin{pmatrix}1&-1&1\\1&1&0\\2&1&1\end{pmatrix}=\begin{pmatrix}\frac{11}{6}&\frac{1}{2}&1\\-\frac{1}{6}&-\frac{1}{2}&0\\\frac{2}{3}&1&0\end{pmatrix}$$

(4)

$$令A=\begin{pmatrix}1&1&-1\\0&2&2\\1&-1&0\end{pmatrix}$$

$$\therefore A^*=\begin{pmatrix}2&1&4\\2&1&-2\\-2&2&2\end{pmatrix},|A|=6$$

$$\therefore A^{-1}=\begin{pmatrix}\frac{1}{3}&\frac{1}{6}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{6}&-\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix}$$

$$\therefore X=\begin{pmatrix}1&-1&1\\1&1&0\\2&1&1\end{pmatrix}\begin{pmatrix}\frac{1}{3}&\frac{1}{6}&\frac{2}{3}\\\frac{1}{3}&\frac{1}{6}&-\frac{1}{3}\\-\frac{1}{3}&\frac{1}{3}&\frac{1}{3}\end{pmatrix}=\begin{pmatrix}-\frac{1}{3}&\frac{1}{3}&\frac{4}{3}\\\frac{2}{3}&\frac{1}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{5}{6}&\frac{4}{3}\end{pmatrix}$$

24. (1)

假设A为对称矩阵,则$A=A^T$

$\therefore A^{-1}=(A^T)^{-1}=(A^{-1})^T$

即A^{-1}也为对称矩阵

假设A为反称矩阵,则$A=-A^T$

$\therefore A^{-1}=(-A^T)^{-1}=-(A^T)^{-1}=-(A^{-1})^T=(-A^{-1})^T$

即A^{-1}也为对称矩阵

25.

(1)

设$A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, B=\begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ 0 & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b_{nn} \end{pmatrix}$

令$C=AB$,则有

当$i\le j$时,

$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}=\sum_{k=i}^{n}a_{ik}b_{kj}$不一定等于0

当$i>j$时,

$c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj}$

当$1\le k<i, a_{ik}=0,$当$j<k\le n, b_{kj}=0$

$\therefore c_{ij}=0$

$\therefore$ 两个三角形矩阵的乘积仍是三角形矩阵

(2)

设$A^*=(A_{ji})$

当$j=1$或$i=1$且$i\ne j$时,易知$A_{ji}=0$

当$i>j$且$i\ge 2$且$j\ge 2$时

$A_{ji}=a_{11}\cdots\times 0\times\cdots a_{nn}=0$

$\therefore$ 可逆的三角矩阵的逆依然是三角矩阵

28. (1)

$A^*=\begin{pmatrix} 4 & 4 & 4 & 4 \\ 4 & -4 & 4 & -4 \\ 4 & 4 & -4 & -4 \\ 4 & -4 & -4 & 4 \end{pmatrix}, |A|=16$

$$\therefore A^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{1}{4} \end{pmatrix}$$