高等代数作业

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练习.

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练习

题目: σ 在基 $\alpha_1,\alpha_2,\alpha_3$ 下的矩阵为 $A=\begin{pmatrix}1&2&2\\2&1&2\\2&2&1\end{pmatrix}$, 证明 $W=L(-\alpha_1+\alpha_2,-\alpha_1+\alpha_3)$ 是 σ 的不变子空间.

证明:

令 $\beta_1=-\alpha_1+\alpha_2,\beta_2=-\alpha_1+\alpha_3$

则 $\begin{pmatrix}\beta_1&\beta_2\end{pmatrix}=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\end{pmatrix}\begin{pmatrix}-1&-1\\1&0\\0&1\end{pmatrix}$

$$\begin{aligned}
\sigma\begin{pmatrix}\beta_1&\beta_2\end{pmatrix}&=\sigma\left(\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\end{pmatrix}\begin{pmatrix}-1&-1\\1&0\\0&1\end{pmatrix}\right)\\
&=\sigma\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\end{pmatrix}\begin{pmatrix}-1&-1\\1&0\\0&1\end{pmatrix}\\
&=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\end{pmatrix}A\begin{pmatrix}-1&-1\\1&0\\0&1\end{pmatrix}\\
&=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\end{pmatrix}\begin{pmatrix}1&1\\-1&0\\0&-1\end{pmatrix}\\
&=\begin{pmatrix}\alpha_1-\alpha_2&\alpha_1-\alpha_3\end{pmatrix}\\
&=\begin{pmatrix}-\beta_1&-\beta_2\end{pmatrix}
\end{aligned}$$

$\because\ \beta_1,\beta_2$ 是 W 的两组基

$\therefore W = L(-\alpha_1 + \alpha_2, -\alpha_1 + \alpha_3)$ 是 σ 的不变子空间.

1.

(1)

对称性: $(\alpha, \beta) = \alpha A \beta' = (\alpha A \beta')' = \beta A' \alpha' = \beta A \alpha'$

数乘: $(k\alpha, \beta) = (k\alpha) A \beta' = k(\alpha A \beta') = k(\alpha, \beta)$

可加性: $(\alpha + \beta, \gamma) = (\alpha + \beta) A \gamma' = \alpha A \gamma' + \beta A \gamma' = (\alpha, \gamma) + (\beta, \gamma)$

正定性:

由正定矩阵的定义可知:

当 $\alpha = 0$ 时, $(\alpha, \alpha) = \alpha A \alpha' = 0$

当 $\alpha \neq 0$ 时, $(\alpha, \alpha) = \alpha A \alpha' > 0$

$\therefore$ 在这个定义之下, R^n 成一欧氏空间.

(2)

$$\begin{aligned} \because (\varepsilon_i, \varepsilon_i) &= \varepsilon_i A \varepsilon_i' = a_{ii}, \\ (\varepsilon_i, \varepsilon_j) &= \varepsilon_i A \varepsilon_j' = a_{ij} \end{aligned}$$

$\therefore$ 度量矩阵 $B = \begin{pmatrix} (\varepsilon_1, \varepsilon_1) & (\varepsilon_1, \varepsilon_2) & \cdots & (\varepsilon_1, \varepsilon_n) \\ (\varepsilon_2, \varepsilon_1) & (\varepsilon_2, \varepsilon_2) & \cdots & (\varepsilon_2, \varepsilon_n) \\ \vdots & \vdots & \ddots & \vdots \\ (\varepsilon_n, \varepsilon_1) & (\varepsilon_n, \varepsilon_2) & \cdots & (\varepsilon_n, \varepsilon_n) \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = A$

(3)

柯西 - 布涅柯夫斯基不等式为 $|(\alpha, \beta)| \leqslant |\alpha||\beta|$

即 $|\alpha A \beta'| \leqslant \sqrt{\alpha A \alpha'} \cdot \sqrt{\beta A \beta'}$

即 $\left| \sum_{i,j} a_{ij} x_i y_j \right| \leqslant \sqrt{\sum_{i,j} a_{ij} x_i x_j} \cdot \sqrt{\sum_{i,j} a_{ij} y_i y_j}$

2. (2)

$\because \cos\langle\alpha,\beta\rangle = \dfrac{(\alpha,\beta)}{|\alpha||\beta|} = \dfrac{3+2+10+3}{\sqrt{1+4+4+9}\cdot\sqrt{9+1+25+1}} = \dfrac{\sqrt{2}}{2}$

$\therefore \langle\alpha,\beta\rangle = \dfrac{\pi}{4}$

5.

(1)

$\because \gamma \in V, \alpha_1, \alpha_2, \cdots, \alpha_n$ 是 V 的一组基

$\therefore \gamma = k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n$

$\because (\gamma, \alpha_i) = 0, i = 1, 2, \cdots, n$

$$
\begin{aligned}
\therefore (\gamma,\gamma) &= (\gamma, k_1\alpha_1 + k_2\alpha_2 + \cdots + k_n\alpha_n) \\
&= k_1(\gamma,\alpha_1) + k_2(\gamma,\alpha_2) + \cdots + k_n(\gamma,\alpha_n) \\
&= 0 + 0 + \cdots + 0 \\
&= 0
\end{aligned}
$$

由内积的正定性可知

$\therefore \gamma = 0$

(2)

$\because (\gamma_1, \alpha) = (\gamma_2, \alpha)$

$\therefore (\gamma_1 - \gamma_2, \alpha) = (\gamma_1, \alpha) - (\gamma_2, \alpha) = 0$

$\because \alpha \in V$, 自然也包括 $\alpha_i, i = 1, 2, \cdots, n$

由 (1) 可知

$\therefore \gamma_1 - \gamma_2 = 0$

$\therefore \gamma_1 = \gamma_2$

6.

$\because \varepsilon_1, \varepsilon_2, \varepsilon_3$ 是三维欧氏空间中一组标准正交基

$\therefore (\alpha_1, \alpha_1) = \dfrac{1}{9}(4\varepsilon_1^2 + 4\varepsilon_2^2 + \varepsilon_3^2) = 1$

$$(\alpha_2,\alpha_2)=\frac{1}{9}(4\varepsilon_1^2+\varepsilon_2^2+4\varepsilon_3^2)=1$$

$$(\alpha_3,\alpha_3)=\frac{1}{9}(\varepsilon_1^2+4\varepsilon_2^2+4\varepsilon_3^2)=1$$

$$(\alpha_1,\alpha_2)=\frac{1}{9}(4\varepsilon_1^2-2\varepsilon_2^2-2\varepsilon_3^2)=0$$

$$(\alpha_1,\alpha_3)=\frac{1}{9}(2\varepsilon_1^2-4\varepsilon_2^2+2\varepsilon_3^2)=0$$

$$(\alpha_2,\alpha_3)=\frac{1}{9}(2\varepsilon_1^2+2\varepsilon_2^2-4\varepsilon_3^2)=0$$

$\therefore \alpha_1,\alpha_2,\alpha_3$ 也是一组标准正交基

7.

对于线性无关的向量组 $\alpha_1,\alpha_2,\alpha_3$, 化成正交向量组 β_1,β_2,β_3

$$\beta_1=\alpha_1=\varepsilon_1+\varepsilon_5$$

$$\beta_2=\alpha_2-\frac{(\alpha_2,\beta_1)}{(\beta_1,\beta_1)}\beta_1=\varepsilon_1-\varepsilon_2+\varepsilon_4-\frac{1}{1+1}(\varepsilon_1+\varepsilon_5)=\frac{1}{2}\varepsilon_1-\varepsilon_2+\varepsilon_4-\frac{1}{2}\varepsilon_5$$

$$\begin{aligned}\beta_3&=\alpha_3-\frac{(\alpha_3,\beta_1)}{(\beta_1,\beta_1)}\beta_1-\frac{(\alpha_3,\beta_2)}{(\beta_2,\beta_2)}\beta_2\\&=2\varepsilon_1+\varepsilon_2+\varepsilon_3-\frac{2}{1+1}(\varepsilon_1+\varepsilon_5)+\frac{1-1}{\frac{1}{4}+1+1+\frac{1}{4}}(\frac{1}{2}\varepsilon_1-\varepsilon_2+\varepsilon_4-\frac{1}{2}\varepsilon_5)\\&=\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_5\end{aligned}$$

再进行单位化, 得到一组新的标准正交基 η_1,η_2,η_3

$$\eta_1=\frac{1}{|\beta_1|}\beta_1=\frac{1}{\sqrt{1+1}}(\varepsilon_1+\varepsilon_5)=\frac{\sqrt{2}}{2}(\varepsilon_1+\varepsilon_5)$$

$$\eta_2=\frac{1}{|\beta_2|}\beta_2=\frac{1}{\sqrt{\frac{1}{4}+1+1+\frac{1}{4}}}(\frac{1}{2}\varepsilon_1-\varepsilon_2+\varepsilon_4-\frac{1}{2}\varepsilon_5)=\frac{\sqrt{10}}{10}\left(\varepsilon_1-2\varepsilon_2+2\varepsilon_4-\varepsilon_5\right)$$

$$\eta_3=\frac{1}{|\beta_3|}\beta_3=\frac{1}{\sqrt{1+1+1+1}}(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_5)=\frac{1}{2}\left(\varepsilon_1+\varepsilon_2+\varepsilon_3-\varepsilon_5\right)$$

8.

$$\begin{bmatrix}2&1&-1&1&-3\\1&1&-1&0&1\end{bmatrix}\xrightarrow[r_2-r_1]{2r_2}\begin{bmatrix}2&1&-1&1&-3\\0&1&-1&-1&5\end{bmatrix}\xrightarrow[\frac{1}{2}r_1]{r_1-r_2}\begin{bmatrix}1&0&0&1&-4\\0&1&-1&-1&5\end{bmatrix}$$

$\therefore \begin{cases} x_1 = -x_4 + 4x_5 \\ x_2 = x_3 + x_4 - 5x_5 \end{cases}$

令 $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, 可得 $\alpha_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, 令 $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, 可得 $\alpha_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, 令 $\begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, 可得 $\alpha_3 = \begin{pmatrix} 4 \\ -5 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

对于线性无关的向量组 $\alpha_1, \alpha_2, \alpha_3$, 化成正交向量组 $\beta_1, \beta_2, \beta_3$

$\beta_1 = \alpha_1 = (0, 1, 1, 0, 0)$

$\beta_2 = \alpha_2 - \frac{(\alpha_2, \beta_1)}{(\beta_1, \beta_1)}\beta_1 = (-1, 1, 0, 1, 0) - \frac{1}{1+1}(0, 1, 1, 0, 0) = (-1, \frac{1}{2}, -\frac{1}{2}, 1, 0)$

$$\begin{aligned} \beta_3 &= \alpha_3 - \frac{(\alpha_3, \beta_1)}{(\beta_1, \beta_1)}\beta_1 - \frac{(\alpha_3, \beta_2)}{(\beta_2, \beta_2)}\beta_2 \\ &= (4, -5, 0, 0, 1) - \frac{-5}{1+1}(0, 1, 1, 0, 0) - \frac{-4 - \frac{5}{2}}{1 + \frac{1}{4} + \frac{1}{4} + 1}(-1, \frac{1}{2}, -\frac{1}{2}, 1, 0) \\ &= (4, -5, 0, 0, 1) + (0, \frac{5}{2}, \frac{5}{2}, 0, 0) + (-\frac{13}{5}, \frac{13}{10}, -\frac{13}{10}, \frac{13}{5}, 0) \\ &= (\frac{7}{5}, -\frac{6}{5}, \frac{6}{5}, \frac{13}{5}, 1) \end{aligned}$$

再进行单位化, 得到一组标准正交基 η_1, η_2, η_3

$\eta_1 = \frac{1}{|\beta_1|}\beta_1 = \frac{1}{\sqrt{1+1}}(0, 1, 1, 0, 0) = (0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0, 0)$

$\eta_2 = \frac{1}{|\beta_2|}\beta_2 = \frac{1}{\sqrt{1 + \frac{1}{4} + \frac{1}{4} + 1}}(-1, \frac{1}{2}, -\frac{1}{2}, 1, 0) = (-\frac{\sqrt{10}}{5}, \frac{\sqrt{10}}{10}, -\frac{\sqrt{10}}{10}, \frac{\sqrt{10}}{5}, 0)$

$\eta_3 = \frac{1}{|\beta_3|}\beta_3 = \frac{1}{\sqrt{(\frac{7}{5})^2 + (-\frac{6}{5})^2 + (\frac{6}{5})^2 + (\frac{13}{5})^2 + 1}}(\frac{7}{5}, -\frac{6}{5}, \frac{6}{5}, \frac{13}{5}, 1) =$
$(\frac{\sqrt{35}}{15}, -\frac{2\sqrt{35}}{35}, \frac{2\sqrt{35}}{35}, \frac{13\sqrt{35}}{105}, \frac{\sqrt{35}}{21})$

10.

(1)

对于任意两个向量 $\beta, \gamma \in V_1$, 即有 $(\beta, \alpha) = 0, (\gamma, \alpha) = 0$

对于向量加法: $(\beta + \gamma, \alpha) = (\beta, \alpha) + (\gamma, \alpha) = 0 + 0 = 0$

即有 $\beta + \gamma \in V_1$

对于标量乘法: $(k\beta, \alpha) = k(\beta, \alpha) = 0$

即有 $k\beta \in V_1$

$\therefore V_1 = \{x|(x, \alpha) = 0, x \in V\}$ 是 V 的一个子空间

(2)

将 α 扩充为 V 上的一组正交基 $\alpha, \beta_1, \beta_2, \cdots, \beta_{n-1}$

$\because \alpha \neq 0$

$\therefore (\alpha, \alpha) \neq 0, (\beta_i, \alpha) = 0, i = 1, 2, \cdots, n-1$

$\therefore \beta_1, \beta_2, \cdots, \beta_i$ 是 V_1 的一组基

$\therefore V_1$ 的维数等于 $n-1$

12.

设 $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m, x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix}$

"⇒":

当 $|\Delta| \neq 0$ 时, 令 $\sum\limits_{i=1}^{m} x_i\alpha_i = 0$, 要证 $\alpha_1, \alpha_2, \cdots, \alpha_m$ 线性无关, 只需证 $x_i = 0, i = 1, 2, \cdots, m$

$\because (\sum\limits_{i=1}^{m} x_i\alpha_i, \alpha_j) = \sum\limits_{i=1}^{m} x_i(\alpha_j, \alpha_i), j = 1, 2, \cdots, m$

合并成一个方程组, 即有

$$\therefore \Delta \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = \Delta x = 0$$

$\because |\Delta| \neq 0$, 由克拉默法则可知

$$\therefore x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = 0$$

$\therefore \alpha_1, \alpha_2, \cdots, \alpha_m$ 线性无关

"$\Leftarrow$":

当 $\alpha_1, \alpha_2, \cdots, \alpha_m$ 线性无关时, 即有

当 $x \neq 0$, 有 $\alpha = x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m \neq 0$

由内积的正定性可知

$$\therefore (\alpha, \alpha) = (\sum_{i=1}^{m} x_i\alpha_i, \sum_{i=1}^{m} x_i\alpha_i) = \begin{pmatrix} x_1 & x_2 & \cdots & x_m \end{pmatrix} \Delta \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{pmatrix} = x^T \Delta x > 0$$

即 Δ 是正定二次型 $x^T \Delta x$ 的正定矩阵

由正定矩阵的性质可知

$\therefore |\Delta| \neq 0$

13.

假设有上三角的正交矩阵 $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$

$$\therefore A^TA=\begin{pmatrix}a_{11} & 0 & \cdots & 0\\ a_{12} & a_{22} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ a_{1n} & a_{2n} & \cdots & a_{nn}\end{pmatrix}\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ 0 & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & a_{nn}\end{pmatrix}=$$
$$\begin{pmatrix}a_{11}^2 & a_{11}a_{12} & \cdots & a_{11}a_{1n}\\ a_{12}a_{11} & a_{12}^2+a_{22}^2 & \cdots & a_{12}a_{1n}+a_{22}a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{1n}a_{11} & a_{1n}a_{12}+a_{2n}a_{22} & \cdots & \sum_{i=1}^n a_{in}^2\end{pmatrix}=E$$

数学归纳法:

奠基: 对于第 1 行, 我们有 $a_{11}^2=1$, 即 $a_{11}=\pm1$,

紧接着由 $a_{11}a_{1i}=0$ 可以推出 $a_{1i}=0,i=1,2,\cdots,n$

归纳假设: 假设有 $a_{(k-1)(k-1)}=\pm1,a_{(k-1)j}=0,j=k+1,\cdots,n$

归纳步骤:

由归纳假设可知,

对于第 k 行的对角线上的元素 $\displaystyle\sum_{i=1}^k a_{ik}^2$ 可以推出 $a_{kk}=\pm1$

并对第 k 行对角线元素之后的元素 $\displaystyle\sum_{i=1}^k a_{ik}a_{ij}$ 可以推出 $a_{ij}=0,j=k+1,\cdots,n$

可知归纳成立.

$\therefore$ 上三角的正交矩阵必为对角矩阵, 且对角线上的元素为 $+1$ 或 -1