Problem Set 9

Problem 1

(1) $|A| = 3$

(2) $|B| = \aleph_0$

(3) $|C| = \aleph_0$

(4) $|B \cap C| = \aleph_0$

(5) $|B \cap C| = \aleph_0$

(6) $|D| = \aleph_1$

Problem 2

(1)

令$C = B - A$,易知C也是可数集

$\therefore A \cup B = A \cup (B - A) = A \cup C, A \cap C = \emptyset$

不妨假设$|A| \leq |C|$,否则将A与C对调

由A, C均为可数集可定义$f_A : |A| \xrightarrow[onto]{1-1} A, f_C : |C| \xrightarrow[onto]{1-1} C$

定义函数$f : |A \cup B| \to A \cup B$

当$x < 2|A|$时,

$$f(x) = \begin{cases} f_A(\dfrac{x}{2}), & x = 2k, k \in \mathbb{N} \\ f_C(\dfrac{x-1}{2}), & x = 2k+1, k \in \mathbb{N} \end{cases}$$

当$x \geq 2|A|$时,

$f(x) = f_C(x - |A|)$

$\therefore$ 易知f为双射函数, $|A\cup B|\in\mathbb{N}$或$|A\cup B|=\aleph_0$

$\therefore A\cup B$是可数集

(2)

不妨定义$f_A:\mathbb{N}\xrightarrow{1-1,onto}A, f_B:\mathbb{N}\xrightarrow{1-1,onto}B$

令$f:A\times B\to\mathbb{N}, f((f_A(m),f_B(n)))=\sum\limits_{i=1}^{m+n}i+m=\dfrac{(m+n)(m+n+1)}{2}+m$

$\because$ 易知f是双射函数

$\therefore A\times B$是可列集

Problem 3

a) 可数无限的, 取$f(x)=x+11$

b) 可数无限的, 取$f(x)=-2x-1$

c) 有限的

d) 不可数的

e)可数无限的, 取$f(x)=\begin{cases}(2,\dfrac{x}{2}+1), & x=2k,k\in\mathbb{N}\\(3,\dfrac{x+1}{2}), & x=2k+1,k\in\mathbb{N}\end{cases}$

f)可数无限的, 取$f(x)=\begin{cases}5x, & x=2k,k\in\mathbb{N}\\-5x-5, & x=2k+1,k\in\mathbb{N}\end{cases}$

Problem 4

假设$A-B$是可数的

$\because B$是可数集合

$\therefore A\cap B$是可数集合

$\because A-(A\cap B)=A-B$也为可数集合

$\therefore A = A - (A \cap B) + (A \cap B)$

由Problem 2.(1)可知两个可数集合的并集也是可数集合

$\therefore A$是可数集合

$\because A$是不可数集合,产生矛盾,假设不成立

$\therefore A - B$是不可数的

Problem 5

$\because A$是可数集合

当A是有穷集时,不妨记$A \approx n$,则有$g : n \xrightarrow[onto]{1-1} A$

$\because$ 存在$f : A \xrightarrow[onto]{1-1} B$

$\therefore f \circ g : n \xrightarrow[onto]{1-1} B$

$\therefore B \approx n$

$\therefore B$是有穷集

$\therefore B$是可数的

当A是无穷可列集时,$A \approx \mathbb{N}$

$\therefore$ 同理可知$B \approx \mathbb{N}$

$\therefore B$是可数的

Problem 6

取$f_A : |A| \xrightarrow[onto]{1-1} A, f_B : |B| \xrightarrow[onto]{1-1} B$

$\because A \subset B$

$\therefore |A| < |B|$

$\therefore$ 令$f : A \to B, f(f_A(n)) = f_B(n)$

易知f是单射函数

$\therefore A \preceq \cdot B$

Problem 7

$\because A = \{a, b, c\}$

$\therefore \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

$$
\begin{aligned}
\therefore B = \{ & \\
&\qquad \{(a,0),(b,0),(c,0)\}, \\
&\qquad \{(a,0),(b,0),(c,1)\}, \\
&\qquad \{(a,0),(b,1),(c,0)\}, \\
&\qquad \{(a,0),(b,1),(c,1)\}, \\
&\qquad \{(a,1),(b,0),(c,0)\}, \\
&\qquad \{(a,1),(b,0),(c,1)\}, \\
&\qquad \{(a,1),(b,1),(c,0)\}, \\
&\qquad \{(a,1),(b,1),(c,1)\} \\
& \}
\end{aligned}
$$

$\therefore |\mathcal{P}(A)| = |B| = 8$

$\therefore \mathcal{P}(A) \approx B$

Problem 8

(1) 是, 取$f(x) = \begin{cases} x, & x = 2k, k \in \mathbb{N} \\ -x-1, & x = 2k+1, k \in \mathbb{N} \end{cases}$

(2) 不是, $(0, 0.5) \approx \mathbb{R} \not\approx \mathbb{N}$

(3) 是, 取$f(x) = \begin{cases} \frac{7x}{2}, & x = 2k, k \in \mathbb{N} \\ \frac{-7x-7}{2}, & x = 2k+1, k \in \mathbb{N} \end{cases}$

(4) 是, 取$f(x) = \begin{cases} \frac{3x}{2}+1, & x = 2k, k \in \mathbb{N} \\ \frac{x-1}{2}+2, & x = 2k+1, k \in \mathbb{N} \end{cases}$

Problem 9

a) $A=\mathbb{R},B=\mathbb{R}$

b) $A=\mathbb{R},B=\mathbb{R}-\mathbb{N}$

c) $A=P(\mathbb{R}),B=\mathbb{R}$

Problem 10

设这些可数集合为$C_i,\quad (i=1,2,\cdots,m)$

令$S_1=C_1,S_i=C_j-(C_1\cup\cdots\cup C_{j-1}),\quad (j=2,3,\cdots,m)$
易知S_i也是可数集,且互不相同的两个S_i的交集为空

不妨假定$|S_1|<|S_2|<\cdots<|S_m|$,否则交换它们的位置

$\therefore C_1\cup C_2\cup\cdots\cup C_i=S_1\cup S_2\cup\cdots\cup S_i,\quad (i=1,2,\cdots,m)$

由S_i均为可数集可定义$f_{S_i}:|S_i|\xrightarrow[onto]{1-1}A$

定义函数$f:|C_1\cup C_2\cup\cdots\cup C_i|\to C_1\cup C_2\cup\cdots\cup C_i$

当$0\le x<m|S_1|$时,

$$f(x)=\begin{cases}f_{S_1}(\frac{x}{m}), & x=mk,k\in\mathbb{N}\\ f_{S_2}(\frac{x-1}{m}), & x=mk+1,k\in\mathbb{N}\\ \cdots \\ f_{S_m}(\frac{x-m+1}{m}), & x=mk+m-1,k\in\mathbb{N}\end{cases}$$

当$\sum_{i=1}^{n}(m-i+1)|S_i|\le x<\sum_{i=1}^{n+1}(m-i+1)|S_i|$时,

$$f(x)=$$
$$\begin{cases}f_{S_n}(\frac{x}{m-n}), & x=(m-n)(\sum_{i=1}^{n}(m-i+1)|S_i|+k),k\in\mathbb{N}\\ f_{S_{n+1}}(\frac{x-1}{m-n}), & x=(m-n)(\sum_{i=1}^{n}(m-i+1)|S_i|+k)+1,k\in\mathbb{N}\\ \cdots \\ f_{S_m}(\frac{x-m+n+1}{m-n}), & x=(m-n)(\sum_{i=1}^{n}(m-i+1)|S_i|+k)+m-n-1,k\in\mathbb{N}\end{cases}$$

其中$n=1,2,\cdots,m-1$

$\therefore$ 易知f为双射函数, $|C_1 \cup C_2 \cup \cdots \cup C_i| \in \mathbb{N}$或$|C_1 \cup C_2 \cup \cdots \cup C_i| = \aleph_0$

$\therefore C_1 \cup C_2 \cup \cdots \cup C_i$是可数集