数学分析第七次作业

习题6.3(A): 2, 4(2, 6, 10, 15), 5, 6(2, 7), (B): 1(3), 5, 习题6.4: (A)1(3), 2(2, 3), 4, 5(3), 习题6.6: (A)1(2, 5, 6), 10(2, 6, 8)

6.3 (A)

2.

(1)

正确.

对于上半球体的任意一个点 $(x, y, z), z \geqslant 0$,
可以找到与其关于原点对称的点 $(-x, -y, -z)$,

并且有 $(x+y+z)^2 = (-x, -y, -z)^2$

而上半球体和下半球体关于原点对称

$\therefore \iiint_{(V)} (x+y+z)^2 \mathrm{d}V = 2\iiint_{(V_1)} (x+y+z)^2 \mathrm{d}V$

(2)

正确.

对于上半球体的任意一个点 $(x, y, z), z \geqslant 0$,
可以找到与其关于原点对称的点 $(-x, -y, -z)$,

并且有 $xyz + (-x)(-y)(-z) = 0$

而上半球体和下半球体关于原点对称

$\therefore \iiint_{(V)} xyz \mathrm{d}V = 0$

(3)

正确.

由三重积分的运算性质 $\iiint k \mathrm{d}V = k \iiint \mathrm{d}V$, 和三重积分的几何意义 $\iiint \mathrm{d}V = V$ 可知正确.

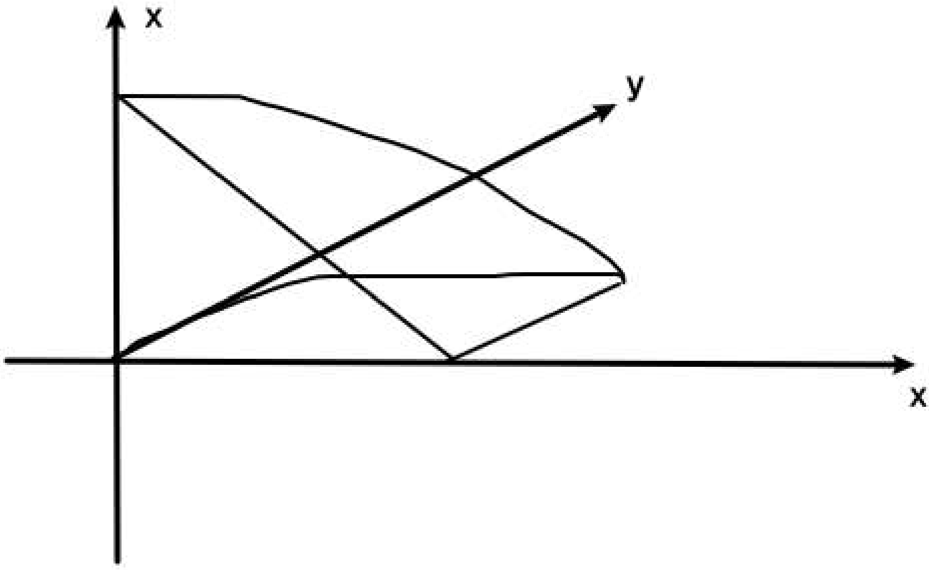
(4)

错误.

只有球面上的点才有 $x^2+y^2+z^2=4$, 球内的点无法这样代换.

4.

(2)

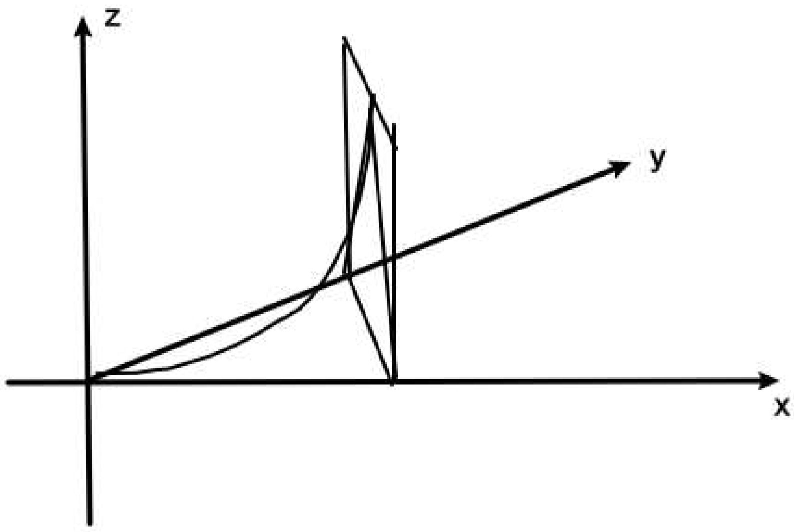


先二后一:

对于每一个 z, 均有一个由 $x=y^2, y=0, x=\frac{\pi}{2}-z$ 围成的面积.

$$
\begin{aligned}
\iiint_V y\cos(x+z)\mathrm{d}V &= \int_0^{\frac{\pi}{2}}\mathrm{d}z\iint_{D_{xy}} y\cos(x+z)\mathrm{d}x\mathrm{d}y \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}z\int_0^{\sqrt{\frac{\pi}{2}-z}}\mathrm{d}y\int_{y^2}^{\frac{\pi}{2}-z} y\cos(x+z)\mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}}\mathrm{d}z\int_0^{\sqrt{\frac{\pi}{2}-z}} y\left(1-\sin\left(y^2+z\right)\right)\mathrm{d}y \\
&= \int_0^{\frac{\pi}{2}}\left(-\frac{z}{2}-\frac{\cos(z)}{2}+\frac{\pi}{4}\right)\mathrm{d}z \\
&= -\frac{1}{2}+\frac{\pi^2}{16}
\end{aligned}
$$

(6)



$$\begin{aligned}
\iiint_V xy\mathrm{d}V &= \iiint_V z\mathrm{d}V \\
&= \iint_{D_{xy}} \mathrm{d}x\mathrm{d}y \int_0^{xy} xy\mathrm{d}z \\
&= \iint_{D_{xy}} xy\mathrm{d}x\mathrm{d}y \int_0^{xy} \mathrm{d}z \\
&= \int_0^1 \mathrm{d}x \int_0^{1-x} x^2y^2\mathrm{d}y \\
&= \int_0^1 -\frac{x^2\left(x-1\right)^3}{3}\mathrm{d}x \\
&= \frac{1}{180}
\end{aligned}$$

(10)

对于上半部分的 V_1 对于高度 z 有 $S=\pi r^2=\pi(R^2-z^2)$

对于下半部分的 V_2 对于高度 z 有 $S=\pi r^2=\pi(R^2-(R-z)^2)=\pi z\left(2R-z\right)$

$$\begin{aligned}
\iiint_V z^2\mathrm{d}V &= \iiint_{V_1} z^2\mathrm{d}V + \iiint_{V_1} z^2\mathrm{d}V \\
&= \int_0^{\frac{R}{2}} z^2\mathrm{d}z \iint_{D_{xy_1}} \mathrm{d}S + \int_{\frac{R}{2}}^R z^2\mathrm{d}z \iint_{D_{xy_1}} \mathrm{d}S \\
&= \pi\int_0^{\frac{R}{2}} z^3(2R-z)\mathrm{d}z + \pi\int_{\frac{R}{2}}^R z^2\left(R^2-z^2\right)\mathrm{d}z \\
&= \frac{59\pi R^5}{480}
\end{aligned}$$

(15)

进行柱面坐标变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, 则有 $J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$

$$\begin{aligned}
&\iiint_V z(x^2+y^2)\mathrm{d}V \\
=&\iint_{D_{xy}} (x^2+y^2)\mathrm{d}x\mathrm{d}y \int z\mathrm{d}z \\
=&\iint_{D_{r\theta}} r^3\mathrm{d}\theta\mathrm{d}r \int z\mathrm{d}z \\
=&\int_0^{2\pi} \mathrm{d}\theta \int_0^{\frac{\sqrt{2}}{2}} r^3\mathrm{d}r \int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} z\mathrm{d}z + \int_0^{2\pi}\mathrm{d}\theta \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} r^3\mathrm{d}r \int_r^{\sqrt{4-r^2}} z\mathrm{d}z \\
=&2\pi \int_0^{\frac{\sqrt{2}}{2}} r^3\mathrm{d}r \int_{\sqrt{1-r^2}}^{\sqrt{4-r^2}} z\mathrm{d}z + 2\pi \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} r^3\mathrm{d}r \int_r^{\sqrt{4-r^2}} z\mathrm{d}z \\
=&2\pi \int_0^{\frac{\sqrt{2}}{2}} \frac{3}{2}x^3\mathrm{d}x + 2\pi \int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} x^3(2-x^2)\mathrm{d}x \\
=&\frac{21\pi}{16}
\end{aligned}$$

5.

(1)

进行柱面坐标变换 $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$, 则有 $J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$

$$\begin{aligned}
\int_{-1}^1 \mathrm{d}x \int_0^{\sqrt{1-x^2}} \mathrm{d}y \int_{\sqrt{x^2+y^2}}^1 z^3\mathrm{d}z &= \int_0^1 z^3\mathrm{d}z \iint_{D_{xy}} \mathrm{d}x\mathrm{d}y \\
&= \int_0^1 z^3\mathrm{d}z \iint_{D_{r\theta}} r\mathrm{d}\theta\mathrm{d}r \\
&= \int_0^1 z^3\mathrm{d}z \int_0^{\pi}\mathrm{d}\theta \int_0^1 r\mathrm{d}r \\
&= \int_0^1 \frac{1}{2}\pi z^5\mathrm{d}z \\
&= \frac{\pi}{12}
\end{aligned}$$

(2)

进行球面坐标变换 $\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases}$, 则有 $J=\rho^2\sin\varphi$

$0\leqslant\rho\cos\varphi\leqslant 3, \rho\sin\varphi+\rho\cos\varphi=3$

$$
\begin{aligned}
&\int_{-3}^{3}\mathrm{d}x\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\mathrm{d}y\int_{0}^{\sqrt{9-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\mathrm{d}z \\
&=\iiint_{V_{\rho\varphi\theta}}\rho^2\cos\varphi\cdot\rho^2\sin\varphi\mathrm{d}V \\
&=\int_{0}^{\frac{\pi}{2}}\mathrm{d}\varphi\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{3}\rho^2\cos\varphi\cdot\rho^2\sin\varphi\mathrm{d}\rho \\
&=\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{\frac{\pi}{2}}\frac{243}{5}\sin\varphi\cos\varphi\mathrm{d}\varphi \\
&=\frac{243\pi}{5}
\end{aligned}
$$

6.

(2)

进行柱面坐标变换 $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z \end{cases}$, 则有 $J=\begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=\rho$

则转化为 $z=6-\rho^2, z=\rho$ 围成的立体.

令 $6-\rho^2=\rho$ 得 $\rho^2+\rho-6=(\rho+3)(\rho-2)=0$ 即 $\rho=2, \rho=-3$ (舍去)

$$
\begin{aligned}
V&=\iiint_V\mathrm{d}V \\
&=\iint_{D_{\rho\theta}}\rho\mathrm{d}\theta\mathrm{d}\rho\int_{\rho}^{6-\rho^2}\mathrm{d}z \\
&=\int_{0}^{2\pi}\mathrm{d}\theta\int_{0}^{2}\rho(-\rho^2-\rho+6)\mathrm{d}\theta\mathrm{d}\rho \\
&=\frac{32\pi}{3}
\end{aligned}
$$

(7)

进行类似柱面坐标变换的变换 $\begin{cases} x = a\rho\cos\theta \\ y = b\rho\sin\theta \\ z = ct \end{cases}$ ，

则有 $J = \begin{vmatrix} a\cos\theta & -a\rho\sin\theta & 0 \\ b\sin\theta & b\rho\cos\theta & 0 \\ 0 & 0 & c \end{vmatrix} = abc\rho$

则原式转化为 $t = \sqrt{\rho^2+1}, \rho = 1$

$$
\begin{aligned}
V &= \iiint_V \mathrm{d}V \\
&= 2\iint_{D_{\rho\theta}} abc\rho\mathrm{d}\theta\mathrm{d}\rho \int_0^{\sqrt{\rho^2+1}} \mathrm{d}t \\
&= 2\int_0^{2\pi} \mathrm{d}\theta \int_0^1 abc\rho\mathrm{d}\rho \int_0^{\sqrt{\rho^2+1}} \mathrm{d}t \\
&= 2\int_0^{2\pi} \mathrm{d}\theta \int_0^1 abc\rho\sqrt{\rho^2+1}\mathrm{d}\rho \\
&= 4\pi abc\int_0^1 \rho\sqrt{\rho^2+1}\mathrm{d}\rho \\
&= \frac{4}{3}\pi abc(2\sqrt{2}-1)
\end{aligned}
$$

6.3 (B)

1. (3)

进行球面坐标变换 $\begin{cases} x = a\rho\sin\varphi\cos\theta \\ y = b\rho\sin\varphi\sin\theta \\ z = c\rho\cos\varphi \end{cases}$, 则有 $J = abc\rho^2\sin\varphi$

$$
\begin{aligned}
&\iiint_V \sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}}\mathrm{d}V \\
=&\iiint_V \sqrt{1-\rho^2}\cdot abc\rho^2\sin\varphi\mathrm{d}\rho \\
=&abc\int_0^{2\pi}\mathrm{d}\theta\int_0^{\pi}\sin\varphi\mathrm{d}\varphi\int_0^1\rho^2\sqrt{1-\rho^2}\mathrm{d}\rho \\
=&4abc\pi\int_0^1\rho^2\sqrt{1-\rho^2}\mathrm{d}\rho \\
=&2abc\pi\int_0^1\sqrt{\rho^2(1-\rho^2)}\mathrm{d}\rho^2 \\
=&2abc\pi\int_0^1\sqrt{t(1-t)}\mathrm{d}t \\
=&2abc\pi\int_0^1\frac{\sqrt{1-(2t-1)^2}}{2}\mathrm{d}t \\
=&\frac{\pi^2abc}{4}
\end{aligned}
$$

5.

进行柱面坐标变换 $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z \end{cases}$, 则有 $J=\begin{vmatrix} \cos\theta & -\rho\sin\theta & 0 \\ \sin\theta & \rho\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}=\rho$

即 $0\leqslant z\leqslant h,\rho\leqslant t$ 围成的区域

$$
\begin{aligned}
F(t)&=\iiint_{V_{xyz}}\left[z^2+f(x^2+y^2)\right]\mathrm{d}V \\
&=\iiint_{V_{\rho\theta z}}\rho\left[z^2+f(\rho^2)\right]\mathrm{d}V \\
&=\int_0^{2\pi}\mathrm{d}\theta\int_0^t\rho\mathrm{d}\rho\int_0^h\left[z^2+f(\rho^2)\right]\mathrm{d}z \\
&=2\pi\int_0^t\rho\mathrm{d}\rho\left[\int_0^h z^2\mathrm{d}z+\int_0^1 f(\rho^2)\mathrm{d}z\right] \\
&=2\pi\int_0^t\rho\left[\frac{h^3}{3}+f(\rho^2)\right]\mathrm{d}\rho
\end{aligned}
$$

$$
\therefore\frac{\mathrm{d}F}{\mathrm{d}t}=2\pi t\left[\frac{h^3}{3}+f(t^2)\right]
$$

$$
\begin{aligned}
\therefore \lim_{t\to0^+}\frac{F(t)}{t^2} &= \lim_{t\to0^+}\frac{2\pi}{t^2}\int_0^t \rho\left[\frac{h^3}{3}+f(\rho^2)\right]\mathrm{d}\rho \\
&= \lim_{t\to0^+}\frac{1}{t}\cdot\frac{2\pi\int_0^t \rho\left[\frac{h^3}{3}+f(\rho^2)\right]\mathrm{d}\rho-0}{t-0} \\
&= \lim_{t\to0^+}\frac{1}{t}\cdot\frac{\mathrm{d}F}{\mathrm{d}t}|_0 \\
&= 2\pi\left[\frac{h^3}{3}+f(t^2)\right]
\end{aligned}
$$

6.4 (A)

1. (3)

$$
\begin{aligned}
\lim_{\alpha\to0}\int_0^1\sqrt{1+\alpha^2-x^2}\mathrm{d}x &= \int_0^1\lim_{\alpha\to0}\sqrt{1+\alpha^2-x^2}\mathrm{d}x \\
&= \int_0^1\sqrt{1-x^2}\mathrm{d}x \\
&= \frac{\pi}{4}
\end{aligned}
$$

2.

(2)

$$
\begin{aligned}
\frac{\mathrm{d}F(y)}{\mathrm{d}y} &= \int_{a+y}^{b+y}\frac{\partial\frac{\sin(xy)}{x}}{\partial y}\mathrm{d}x+\frac{\sin(by+y^2)}{b+y}-\frac{\sin(ay+y^2)}{a+y} \\
&= \int_{a+y}^{b+y}-\cos(xy)\mathrm{d}x+\frac{\sin(by+y^2)}{b+y}-\frac{\sin(ay+y^2)}{a+y} \\
&= \frac{\sin\left(ay+y^2\right)}{y}-\frac{\sin\left(by+y^2\right)}{y}+\frac{\sin(by+y^2)}{b+y}-\frac{\sin(ay+y^2)}{a+y}
\end{aligned}
$$

(3)

$$
\begin{aligned}
\frac{\mathrm{d}F(y)}{\mathrm{d}x} &= \int_0^x\frac{\partial[(x+y)f(y)]}{\partial x}\mathrm{d}y+2xf(x) \\
&= \int_0^x f(y)\mathrm{d}y+2xf(x)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}^2 F(y)}{\mathrm{d}x^2} &= \int_0^x \frac{\partial f(y)}{\partial x}\mathrm{d}y + f(x) + 2f(x) + 2xf'(x) \\
&= 3f(x) + 2xf'(x)
\end{aligned}
$$

4.

$$
\because a>0, b>0, \frac{e^{-ax}-e^{-bx}}{x} = \int_a^b e^{-xy}\mathrm{d}y
$$

$$
\begin{aligned}
\therefore \int_0^{+\infty} \frac{e^{-ax}-e^{-bx}}{x}\mathrm{d}x &= \int_0^{+\infty}\left(\int_a^b e^{-xy}\mathrm{d}y\right)\mathrm{d}x \\
&= \int_a^b \left(\int_0^{+\infty} e^{-xy}\mathrm{d}x\right)\mathrm{d}y \\
&= \int_a^b \frac{1}{y}\mathrm{d}y \\
&= \ln\frac{b}{a}
\end{aligned}
$$

5. (3)

令 $x=\rho\cos\theta, y=\rho\sin\theta$

$x^2+y^2\leqslant x \Rightarrow \rho\leqslant\cos\theta$

挖掉 $(0,0)$ 点.

$$
\begin{aligned}
\iint_D \frac{\mathrm{d}\sigma}{\sqrt{x^2+y^2}} &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\cos\theta}\frac{1}{\rho}\cdot\rho\mathrm{d}\rho \\
&= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\mathrm{d}\theta \\
&= 2
\end{aligned}
$$

6.6 (A)

1.

(2)

$$
\int_C (x^2+y^2)^n\mathrm{d}s = \int_C (a^2)^n\mathrm{d}s = a^{2n}\int_C\mathrm{d}s = 2\pi a^{2n+1}
$$

(5)

$\because x^2+y^2+z^2=4, z=\sqrt{3}$

$\therefore x^2+y^2=1$

令 $x=\cos t, y=\sin t, 0\leqslant t\leqslant 2\pi$

$$\oint_C x^2\mathrm{d}s=\int_0^{2\pi}\cos^2 t\sqrt{\sin^2 t+\cos^2 t}\mathrm{d}t=\pi$$

(6)

$$x^2+y^2+z^2=2, x=y\Rightarrow x^2+\frac{z^2}{2}=1$$

令 $x=y=\cos t, z=\sqrt{2}\sin t$

$$\oint|y|\mathrm{d}s=2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t\sqrt{\sin^2 t+\sin^2 t+2\cos^2 t}\mathrm{d}t=2\sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t\mathrm{d}t=4\sqrt{2}$$

10.

(2)

进行柱坐标系变换, 即 $\begin{cases}x=\rho\cos\theta\\y=\rho\sin\theta\\z=z\end{cases}$, 原曲面则转为 $\rho\leqslant z\leqslant 1$

对于上曲面 $z=1$, 有 $||\vec{r}_\rho\times\vec{r}_\theta||=\rho$

对于上曲面 $z=\rho$, 有 $||\vec{r}_\rho\times\vec{r}_\theta||=\sqrt{(\rho\cos\theta)^2+(\rho\sin\theta)^2+\rho^2}=\sqrt{2}\rho$

$$\iint_S(x^2+y^2)\mathrm{d}S=\int_0^{2\pi}\mathrm{d}\theta\int_0^1(1+\sqrt{2})\rho^3\mathrm{d}\rho=\frac{\pi\left(1+\sqrt{2}\right)}{2}$$

(6)

对于 xOy 面:

$$\iint_{S_1}\frac{1}{(1+x+y)^2}\mathrm{d}S=\int_0^1\mathrm{d}x\int_0^{1-x}\frac{\mathrm{d}y}{(1+x+y)^2}=-\frac{1}{2}+\ln 2$$

对于 xOz 面:

$$\iint_{S_2}\frac{1}{(1+x+y)^2}\mathrm{d}S=\int_0^1\mathrm{d}x\int_0^{1-x}\frac{1}{(1+x)^2}\mathrm{d}z=1-\ln 2$$

对于 yOz 面, 同 xOz 可得:

$$\iint_{S_3} \frac{1}{(1+x+y)^2}\mathrm{d}S = 1-\ln 2$$

对于斜面:

该面为 $x+y+z=1$, 那么有 $z=1-x-y$

$$\iint_{S_4} \frac{1}{(1+x+y)^2}\mathrm{d}S = \iint_{S_4} \frac{\sqrt{1+(-1)^2+(-1)^2}}{(1+x+y)^2}\mathrm{d}x\mathrm{d}y = \sqrt{3}(-\frac{1}{2}+\ln 2)$$

那么最终的结果即为四个面加起来

$$\iint_{S} \frac{1}{(1+x+y)^2}\mathrm{d}S = (-\frac{1}{2}+\ln 2)+2(1-\ln 2)+\sqrt{3}(-\frac{1}{2}+\ln 2) = \frac{3}{2}-\frac{\sqrt{3}}{2}+\sqrt{3}\ln 2-\ln 2$$

(8)

进行柱坐标系变换, 即 $\begin{cases} x=\rho\cos\theta \\ y=\rho\sin\theta \\ z=z \end{cases}$, 原曲面则转为 $z=\rho$ 被 $\rho=2a\cos\theta$ 截取的部分.

对于 $z=\rho$, 有 $||\vec{r}_\rho\times\vec{r}_\theta|| = \sqrt{(\rho\cos\theta)^2+(\rho\sin\theta)^2+\rho^2} = \sqrt{2}\rho$

$$\begin{aligned}
&\iint_{S_{xyz}} (xy+yz+zx)\mathrm{d}S \\
&= \iint_{S_{\rho\theta}} (\rho^2\sin\theta\cos\theta+\rho^2(\sin\theta+\cos\theta))\sqrt{2}\rho\mathrm{d}S \\
&= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\sin\theta\cos\theta+\sin\theta+\cos\theta)\mathrm{d}\theta\int_0^{2a\cos\theta}\rho^3\mathrm{d}\rho \\
&= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 4a^4\cos^4\theta(\sin\theta\cos\theta+\sin\theta+\cos\theta)\mathrm{d}\theta \\
&= 4\sqrt{2}a^4\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\cos^5\theta\mathrm{d}\theta+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin\theta\cos^4\theta\mathrm{d}\theta+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^5\theta\mathrm{d}\theta\right] \\
&= 4\sqrt{2}a^4\left[0+0+\frac{16}{15}\right] \\
&= \frac{64\sqrt{2}a^4}{15}
\end{aligned}$$