第四次作业

5.5 (A) 1.(2) 2.(3) 5.(2) 7. 8.

5.5 (A)

1. (2)

$\because \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{a}, \boldsymbol{A} = (a_{ij})_{m\times n}, \boldsymbol{a} = (a_1, a_2, \cdots, a_n)$

$\therefore f_i(x) = a_{i1}x_1 + a_{i2}x_1 + \cdots + a_{in}x_n + a_i$

$\therefore \dfrac{\partial f_i}{\partial x_j} = a_{ij}$

$$\therefore \mathrm{D}\boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix} = (a_{ij})_{m\times n} = \boldsymbol{A}$$

所以导数即为 $\boldsymbol{A}$

2. (3)

$$\therefore \mathrm{D}\boldsymbol{f}(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix} = \begin{pmatrix} \cos y & -x\sin y & 0 \\ ye^x & e^x & 0 \\ z\cos(xz) & 0 & x\cos(xz) \end{pmatrix}$$

5. (2)

对某个自变量求偏导, 例如对 x 求偏导, 可得

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} & \frac{\partial F_1}{\partial w} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} & \frac{\partial F_2}{\partial w} \\ \frac{\partial F_3}{\partial u} & \frac{\partial F_3}{\partial v} & \frac{\partial F_3}{\partial w} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial x} \\ \frac{\partial F_3}{\partial x} \end{pmatrix}$$

即

$$\begin{pmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

进行初等变换可得

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ v+w & u+w & u+v & 0 \\ vw & uw & uv & 0 \end{pmatrix} \xrightarrow[r_3-vwr_1]{r_2-(v+w)r_1} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & u-v & u-w & 0 \\ 0 & (u-v)w & (u-w)v & 0 \end{pmatrix} \xrightarrow[r_1-\frac{1}{u-v}r_2]{r_3-wr_2} \begin{pmatrix} 1 & 0 & 1-\frac{u-w}{u-v} & 1 \\ 0 & u-v & u-w & 0 \\ 0 & 0 & (u-w)(v-w) & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & u-v & u-w & 0 \\ 0 & 0 & (u-w)(v-w) & 0 \end{pmatrix}$$

$$\therefore \frac{\partial u}{\partial x} = 1$$

对 y 同理有

$$\begin{pmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

进行初等变换可得

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ v+w & u+w & u+v & 1 \\ vw & uw & uv & 0 \end{pmatrix} \xrightarrow[r_3-vwr_1]{r_2-(v+w)r_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & u-v & u-w & 1 \\ 0 & (u-v)w & (u-w)v & 0 \end{pmatrix} \xrightarrow[r_1-\frac{1}{u-v}r_2]{r_3-wr_2} \begin{pmatrix} 1 & 0 & -\frac{v-w}{u-v} & -\frac{1}{u-v} \\ 0 & u-v & u-w & 1 \\ 0 & 0 & (u-w)(v-w) & -w \end{pmatrix} \xrightarrow{r_1+\frac{1}{(u-w)(u-v)}r_3} \begin{pmatrix} 1 & 0 & 0 & \frac{1-u+w}{(u-w)(u-v)} \\ 0 & u-v & u-w & 1 \\ 0 & 0 & (u-w)(v-w) & -w \end{pmatrix}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{1-u+w}{(u-w)(u-v)}$$

对 z 同理有

$$\begin{pmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial z} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

进行初等变换可得

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ v+w & u+w & u+v & 0 \\ vw & uw & uv & 1 \end{pmatrix} \xrightarrow[r_3-vwr_1]{r_2-(v+w)r_1} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & u-v & u-w & 0 \\ 0 & (u-v)w & (u-w)v & 1 \end{pmatrix} \xrightarrow[r_1-\frac{1}{u-v}r_2]{r_3-wr_2} \begin{pmatrix} 1 & 0 & -\frac{v-w}{u-v} & 0 \\ 0 & u-v & u-w & 0 \\ 0 & 0 & (u-w)(v-w) & 1 \end{pmatrix} \xrightarrow{r_1+\frac{1}{(u-w)(u-v)}r_3}$$

$$\begin{pmatrix} 1 & 0 & 0 & \frac{1}{(u-w)(u-v)} \\ 0 & u-v & u-w & 0 \\ 0 & 0 & (u-w)(v-w) & 1 \end{pmatrix}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{1}{(u-w)(u-v)}$$

综上可得

$$\therefore \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = \frac{1-u+w}{(u-w)(u-v)}, \frac{\partial u}{\partial z} = \frac{1}{(u-w)(u-v)}$$

7.

令 $G(x,y,z) = xf(x+y) - z$

$$\therefore \begin{pmatrix} \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial y}{\partial x} \\ \frac{\partial z}{\partial x} \end{pmatrix} = \begin{pmatrix} \frac{\partial G}{\partial x} \\ \frac{\partial F}{\partial x} \end{pmatrix}$$

$$\therefore \begin{pmatrix} xf_y(x+y) & -1 \\ F_y & F_z \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}y}{\mathrm{d}x} \\ \frac{\mathrm{d}z}{\mathrm{d}x} \end{pmatrix} = \begin{pmatrix} f(x+y) + xf_x(x+y) \\ F_x \end{pmatrix}$$

$$\therefore [F_y + xF_z f_y(x+y)] \frac{\mathrm{d}y}{\mathrm{d}x} = F_z f(x+y) + xF_z f_x(x+y)$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{F_z f(x+y) + xF_z f_x(x+y)}{F_y + xF_z f_y(x+y)}$$

$$\therefore \frac{\mathrm{d}z}{\mathrm{d}x} = \frac{F_x}{F_z} - \frac{F_y}{F_z} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{F_x}{F_z} - \frac{F_y F_z f(x+y) + xF_y F_z f_x(x+y)}{F_y F_z + xF_z^2 f_y(x+y)}$$

8.

$$\therefore \begin{pmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial z} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial y} \\ \frac{\partial z}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial y} \end{pmatrix}$$

$$\therefore \begin{pmatrix} -F_1 & -F_2 \\ G_1 y & \frac{G_2}{y} \end{pmatrix} \begin{pmatrix} \frac{\mathrm{d}x}{\mathrm{d}y} \\ \frac{\mathrm{d}z}{\mathrm{d}y} \end{pmatrix} = \begin{pmatrix} F_1 + F_2 \\ G_1 x - \frac{G_2 z}{y^2} \end{pmatrix}$$

$$\therefore [G_1 y + \frac{G_2}{F_2 y}(-F_1)] \frac{\mathrm{d}x}{\mathrm{d}y} = G_1 x - \frac{G_2 z}{y^2} + \frac{G_2}{F_2 y}(F_1 + F_2)$$

$$\therefore \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{G_1 F_2 x y^2 - G_2 F_2 z + G_2 F_1 y + G_2 F_2 y}{G_1 F_2 y^3 - G_2 F_1 y}$$