概率统计第四次作业

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3.1

证明 $E(X) = np$:

$\because X \sim B(n,p)$

$\therefore P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\therefore E(X) = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k} = (1-p)^n \sum_{k=1}^{\infty} k \binom{n}{k} \left(\frac{p}{1-p}\right)^k$

令 $x = \frac{p}{1-p}$, 则

$\because \sum_{k=0}^{\infty} \binom{n}{k} x^k = (1+x)^n$

两边求导得

$\therefore \sum_{k=1}^{\infty} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$

$\therefore \sum_{k=1}^{\infty} k \binom{n}{k} x^{k} = nx(1+x)^{n-1}$

$\therefore E(X) = (1-p)^n \cdot n \cdot \frac{p}{1-p} \cdot \left(1 + \frac{p}{1-p}\right)^{n-1} = np$

证明 $\mathrm{Var}(X) = np(1-p)$:

$$\begin{aligned}
\therefore \mathrm{E}(X^2) &= \sum_{k=0}^{\infty} k^2 \cdot P(X=k) \\
&= \sum_{k=0}^{\infty} k^2 \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{\infty} k(k-1) \binom{n}{k} p^k (1-p)^{n-k} + \sum_{k=0}^{\infty} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= (1-p)^n \sum_{k=1}^{\infty} k(k-1) \binom{n}{k} \left(\frac{p}{1-p}\right)^k + (1-p)^n \sum_{k=1}^{\infty} k \binom{n}{k} \left(\frac{p}{1-p}\right)^k
\end{aligned}$$

$\because \sum_{k=1}^{\infty} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$

$\therefore \sum_{k=2}^{\infty} k(k-1) \binom{n}{k} x^{k-2} = n(n-1)(1+x)^{n-2}$

$$\therefore \sum_{k=2}^{\infty} k(k-1)\binom{n}{k}x^k = n(n-1)x^2(1+x)^{n-2}$$

$$\therefore E(X^2) = (1-p)^n \cdot n(n-1)\left(\frac{p}{1-p}\right)^2\left(1+\frac{p}{1-p}\right)^{n-2} + np = n(n-1)p^2 + np = np(1-p) + n^2p^2$$

$$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = np(1-p)$$

3.2

证明 $E(X) = \dfrac{1}{p}$**:**

$$\because P(X=k) = (1-p)^{k-1}p$$

$$\therefore E(X) = \sum_{k=1}^{\infty} k \cdot P(X=k) = \sum_{k=1}^{\infty} k(1-p)^{k-1}p = p\sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$\because \sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

$$\therefore \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$\therefore \sum_{k=1}^{\infty} k(1-p)^{k-1} = \frac{1}{p^2}$$

$$\therefore E(X) = p \cdot \frac{1}{p^2} = \frac{1}{p}$$

证明 $\mathrm{Var}(X) = \dfrac{1-p}{p^2}$**:**

$$\therefore E(X^2) = \sum_{k=1}^{\infty} k^2(1-p)^{k-1}p = p\sum_{k=2}^{\infty} k(k-1)(1-p)^{k-1} + p\sum_{k=1}^{\infty} k(1-p)^{k-1}$$

$$\because \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2}$$

$$\therefore \sum_{k=2}^{\infty} k(k-1)x^{k-2} = \frac{2}{(1-x)^3}$$

$$\therefore \sum_{k=2}^{\infty} k(k-1)x^{k-1} = \frac{2x}{(1-x)^3}$$

$$\therefore E(X^2) = p \cdot \frac{2(1-p)}{p^3} + \frac{1}{p} = \frac{1}{p} + \frac{2(1-p)}{p^2} = \frac{2-p}{p^2}$$

$$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}$$

3.3

证明 $E(X) = \dfrac{r}{p}$**:**

$$\because P(X=k) = \binom{k-1}{r-1}p^r(1-p)^{k-r}$$

$\therefore E(X)=\sum_{k=r}^{\infty} k \cdot \binom{k-1}{r-1} p^r (1-p)^{k-r} = rp^r \sum_{k=r}^{\infty} \binom{k}{r} (1-p)^{k-r}$

$\because \sum_{k=r}^{\infty} \binom{k}{r} (1-p)^{k-r} = \sum_{k=r}^{\infty} \binom{k+1-1}{r+1-1} (1-p)^{k-r} = p^{-(r+1)}$

$\therefore E(X) = rp^r \cdot p^{-(r+1)} = \dfrac{r}{p}$

证明 $\mathrm{Var}(X) = \dfrac{r(1-p)}{p^2}$:

$\therefore E(X^2)=\sum_{k=r}^{\infty} k^2 \cdot \binom{k-1}{r-1} p^r (1-p)^{k-r} = r(r+1)p^r \sum_{k=r}^{\infty} \binom{k+1}{r+1} (1-p)^{k-r} - rp^r \sum_{k=r}^{\infty} \binom{k}{r} (1-p)^{k-r}$

$\because \sum_{k=r}^{\infty} \binom{k+1}{r+1} (1-p)^{k-r} = \sum_{k=r}^{\infty} \binom{k+2-1}{r+2-1} (1-p)^{k-r} = p^{-(r+2)}$

$\therefore E(X^2) = r(r+1)p^r \cdot p^{-(r+2)} - \dfrac{r}{p}$

$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = r(r+1)p^r \cdot p^{-(r+2)} - \dfrac{r}{p} - \dfrac{r^2}{p^2} = \dfrac{r\,(1-p)}{p^2}$

3.4

证明 $E(X) = \lambda$:

$\because P(X=k) = \dfrac{\lambda^k}{k!} e^{-\lambda}$

$\therefore E(X) = \sum_{k=0}^{\infty} k \cdot \dfrac{\lambda^k}{k!} e^{-\lambda} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} \dfrac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$

证明 $\mathrm{Var}(X) = \lambda$:

$\therefore E(X^2) = \sum_{k=0}^{\infty} k^2 \cdot \dfrac{\lambda^k}{k!} e^{-\lambda} = \sum_{k=2}^{\infty} k(k-1) \cdot \dfrac{\lambda^k}{k!} e^{-\lambda} + \sum_{k=1}^{\infty} k \cdot \dfrac{\lambda^k}{k!} e^{-\lambda} = \lambda^2 e^{-\lambda} \cdot e^{\lambda} + \lambda = \lambda^2 + \lambda$

$\therefore \mathrm{Var}(X) = E(X^2) - E(X)^2 = \lambda$

3.5

设 Y 表示一个的叶节点的高度.

设 X_i 表示第 i 轮该叶节点是否被选中, 选中时 $X_i = 1$, 未选中时 $X_i = 0$.

因为在第 i 轮一共有 i 个节点, 因此该节点被选中的概率 $P(X_i = 1) = \dfrac{1}{i}$, 则 $E(X_i) = 1 \cdot P(X_i = 1) = \dfrac{1}{i}$

$\therefore E(Y) = E\left(\sum_{i=1}^{k} X_i\right) = \sum_{i=1}^{k} E(X_i) = \sum_{i=1}^{k} \dfrac{1}{i} \approx \ln k$

3.6

有放回:

$\because P(X=k)=\dfrac{\sum\limits_{i=1}^{5}\binom{5}{i}(k-1)^{5-i}}{10^5}$

$\therefore P(X=1)=0.00001, P(X=2)=0.00031, P(X=3)=0.00211, P(X=4)=0.00781, P(X=5)=0.02101, P(X=6)=0.04651, P(X=7)=0.09031, P(X=8)=0.15961, P(X=9)=0.26281, P(X=10)=0.40951$

X	1	2	3	4	5	6	7	8	9	10
P	0.00001	0.00031	0.00211	0.00781	0.02101	0.04651	0.09031	0.15961	0.26281	0.40951

无放回:

$\because P(X=k)=\frac{\binom{k-1}{4}}{\binom{10}{5}}$

X	5	6	7	8	9	10
P	$\frac{1}{252}$	$\frac{5}{252}$	$\frac{5}{84}$	$\frac{5}{36}$	$\frac{5}{18}$	$\frac{1}{2}$

3.7

令 $X\sim B(n,0.99)$, 则 X 服从参数为 n 和 0.99 的二项分布.

$\therefore P(X\geqslant k)=\sum\limits_{k=k}^{n}\binom{n}{k}\times 0.99^k\times 0.01^{n-k}$

令 $n=102$, 则

$\therefore P(X\geqslant 100)=\sum\limits_{k=100}^{102}\binom{102}{k}\times 0.99^k\times 0.01^{102-k}=0.916911014889440$

令 $n=103$, 则

$\therefore P(X=100)=\sum\limits_{k=100}^{103}\binom{103}{k}\times 0.99^k\times 0.01^{103-k}=0.979758767886053$

所以 $x=103-100=3$. 即 x 最小值是 3.

3.8

2. (1)

$\because P(X=k)=\frac{\binom{k-1}{2}}{\binom{5}{3}}$

$\therefore P(X=3)=0.1, P(X=4)=0.3, P(X=5)=0.6$

X	3	4	5
P	0.1	0.3	0.6

2. (2)

$\because P(X=k)=\frac{\binom{6-k}{1}}{\binom{6}{2}}$

$\therefore P(X=1)=\frac{1}{3}, P(X=2)=\frac{4}{15}, P(X=3)=\frac{1}{5}, P(X=4)=\frac{2}{15}, P(X=5)=\frac{1}{15}$

X	1	2	3	4	5
P	$\frac{1}{3}$	$\frac{4}{15}$	$\frac{1}{5}$	$\frac{2}{15}$	$\frac{1}{15}$

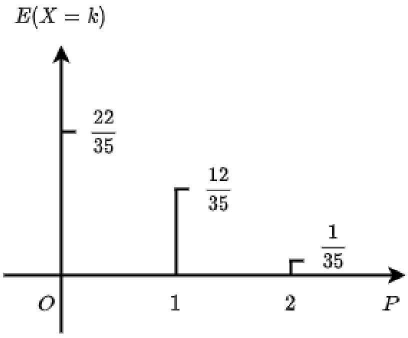
3. (1)

$\because P(X=k)=\frac{\binom{2}{k}\binom{13}{3-k}}{\binom{15}{3}}$

$\therefore P(X=0)=\frac{22}{35}, P(X=1)=\frac{12}{35}, P(X=2)=\frac{1}{35}$

X	0	1	2
P	$\frac{22}{35}$	$\frac{12}{35}$	$\frac{1}{35}$

3. (2)



3.9

2.

设 Y 为每次抽取 10 件产品进行检验时发现的次品数. 则 $Y\sim B(10,0.1)$

$\therefore P(Y=0)=\binom{10}{0}\times 0.1^0\times 0.9^{10}=0.9^{10}$

$\therefore P(Y=1)=\binom{10}{1}\times 0.1^1\times 0.9^{9}=0.9^{9}$

$\therefore P(Y>1)=1-0.9^{10}-0.9^9=1-1.9\times 0.9^9$

即发现超过 1 件次品的概率为 $p=1-1.9\times 0.9^9$

则我们有 $X\sim(4,p)$, 即 $X\sim(4,1-1.9\times 0.9^9)$

$\therefore E(X)=4p=4-7.6\times 0.9^9$

3.

$$\because P(X=k)=\frac{\sum_{i=1}^{3}\binom{3}{i}(4-k)^{3-i}}{4^3}$$

$$\therefore P(X=1)=\frac{37}{64},P(X=2)=\frac{19}{64},P(X=3)=\frac{7}{64},P(X=4)=\frac{1}{64}$$

X	1	2	3	4
P	$\frac{37}{64}$	$\frac{19}{64}$	$\frac{7}{64}$	$\frac{1}{64}$

$$\therefore E(X)=1\times\frac{37}{64}+2\times\frac{19}{64}+3\times\frac{7}{64}+4\times\frac{1}{64}=\frac{25}{16}$$

3.10

4. (1)

因为 $\sum_{j=1}^{\infty}\left|(-1)^{j+1}\frac{3^j}{j}\cdot\frac{2}{3^j}\right|=\sum_{j=1}^{\infty}\frac{2}{j}$ 调和级数发散, 即 $E(X)$ 并不绝对收敛.

所以 X 的数学期望并不存在.

4. (2)

$$\because P(X=k)=\left(\prod_{i=1}^{k-1}\frac{i}{i+1}\right)\cdot\frac{1}{k+1}$$

$\therefore E(X)=\sum_{k=1}^{\infty}k\cdot\left(\prod_{i=1}^{k-1}\frac{i}{i+1}\right)\cdot\frac{1}{k+1}=\sum_{k=1}^{\infty}\frac{1}{k+1}$ 调和级数发散.

6. (1)

$$E(X)=-2\times0.4+0\times0.3+2\times0.3=-0.2$$

$$E(X^2)=4\times0.4+0\times0.3+4\times0.3=2.8$$

$$E(3X^2+5)=3E(X^2)+5=3\times2.8+5=13.4$$

6. (2)

$$\because X\sim\pi(\lambda)$$

$$\therefore P(X=k)=\frac{\lambda^k}{k!}e^{-\lambda}$$

$$\therefore E\left(\frac{1}{X+1}\right)=\sum_{k=0}^{\infty}\frac{1}{k+1}\cdot\frac{\lambda^k}{k!}e^{-\lambda}=\frac{e^{-\lambda}}{\lambda}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{(k+1)!}=\frac{\left(e^{\lambda}-1\right)e^{-\lambda}}{\lambda}$$