数学分析第八次作业

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习题 6.7: (A) 2(2, 6), 4(3), 8, 10(2), 12(2, 4, 6, 7), 15, (B)2, 5. 6
习题 6.8: (A) 1, 2(3, 5), 3, 8(2), 10, 16(1, 3, 6)

6.7 (A)

2.

(2)

对于直线 $y = x$:

$$\int_{(C)} xy\mathrm{d}x + (y - x)\mathrm{d}y = \int_0^1 x^2\mathrm{d}x + \int_0^1 (x - x)\mathrm{d}x = \frac{1}{3}$$

对于抛物线 $y^2 = x$:

$$\int_{(C)} xy\mathrm{d}x + (y - x)\mathrm{d}y = \int_0^1 y^3\mathrm{d}y^2 + \int_0^1 (y - y^2)\mathrm{d}y = \frac{2}{5} + \frac{1}{6} = \frac{17}{30}$$

对于立方抛物线 $y = x^3$:

$$\int_{(C)} xy\mathrm{d}x + (y - x)\mathrm{d}y = \int_0^1 x^4\mathrm{d}x + \int_0^1 (x^3 - x)\mathrm{d}x^3 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}$$

(6)

设 $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 2 - \cos\theta + \sin\theta \end{cases}$

则有

$$
\begin{aligned}
&\oint_{(C)}(z-y)\mathrm{d}x+(x-z)\mathrm{d}y+(x-y)\mathrm{d}z\\
=&\int_0^{-2\pi}(2-\cos\theta)\mathrm{d}\cos\theta+(2\cos\theta-\sin\theta-2)\mathrm{d}\sin\theta+(\cos\theta-\sin\theta)\mathrm{d}(2-\cos\theta+\sin\theta)\\
=&\int_0^{-2\pi}(2-\cos\theta)(-\sin\theta)\mathrm{d}\theta+(2\cos\theta-\sin\theta-2)\cos\theta\mathrm{d}\theta+(\cos\theta-\sin\theta)(\sin\theta+\cos\theta)\mathrm{d}\theta\\
=&\int_0^{-2\pi}\left(-4\sin^2(\theta)-2\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)+3\right)\mathrm{d}\theta\\
=&-2\pi
\end{aligned}
$$

4. (3)

对于 $(1,0)$ 到 (0,1) 的下半圆周 $(x-1)^2+(y-1)^2=1$

使用参数方程 $\begin{cases}x=1+\cos\theta\\ y=1+\sin\theta\end{cases}$, 则 $\vec{\tau}=\dfrac{\{x'(\theta),y'(\theta)\}}{\sqrt{x'(\theta)^2+y'(\theta)^2}}=\{-\sin\theta,\cos\theta\}$, 其中 θ 由 $\theta=\dfrac{3}{2}\pi$ 到 $\theta=\pi$, 即 π 到 $\dfrac{3}{2}\pi$ 时取反.

$$
\int_{(C)}P(x,y)\mathrm{d}x+Q(x,y)\mathrm{d}y=-\int_{(C)}\{P,Q\}\cdot\vec{\tau}\mathrm{d}s=
$$
$$
-\int_{(C)}\left[Q(1+\cos\theta,1+\sin\theta)\cos\theta-P(1+\cos\theta,1+\sin\theta)\sin\theta\right]\mathrm{d}s
$$

而又有 $\sin\theta=-\sqrt{1-\cos^2\theta}=-\sqrt{1-(x-1)^2}=-\sqrt{2x-x^2}$

所以原式转化为 $\displaystyle\int_{(C)}\left[-\sqrt{2x-x^2}P(x,y)+(1-x)Q(x,y)\right]\mathrm{d}s$

8.

(1)

进行参数变换 $\begin{cases}x=\rho\cos\theta\\ y=\rho\sin\theta\\ z=\rho\end{cases}$, 则有

$$
\frac{\partial(x,y)}{\partial(\rho,\theta)}=\begin{vmatrix}\cos\theta & -\rho\sin\theta\\ \sin\theta & \rho\cos\theta\end{vmatrix}=\rho\sin^2\theta+\rho\cos^2\theta=\rho
$$

$$
\iint_{(S)}\frac{e^z}{\sqrt{x^2+y^2}}\mathrm{d}x\wedge\mathrm{d}y=-\iint_{D_{\theta\rho}}\frac{e^\rho}{\rho}\cdot\rho\mathrm{d}\theta\mathrm{d}\rho=-\int_0^{2\pi}\mathrm{d}\theta\int_1^2 e^\rho\mathrm{d}\rho
$$

(2)

对于 $I=\displaystyle\oiint_{\Sigma}(x+y+z)\mathrm{d}x\wedge\mathrm{d}y+(y-z)\mathrm{d}y\wedge\mathrm{d}z$

其中 Σ 是正方体的表面, 即六个面, 边长为 a.

$$
\begin{aligned}
I &= \iint_{\Sigma_{\text{top}}} + \iint_{\Sigma_{\text{bottom}}} + \iint_{\Sigma_{\text{front}}} + \iint_{\Sigma_{\text{back}}} + \iint_{\Sigma_{\text{left}}} + \iint_{\Sigma_{\text{right}}} \\
&= \iint_{D_{xy}} -(x+y+0)\mathrm{d}x\mathrm{d}y + \iint_{D_{xy}} (x+y+1)\mathrm{d}x\mathrm{d}y + 0 + 0 + 0 + 0 \\
&= \int_0^1 \mathrm{d}x \int_0^1 (x+y+1)\mathrm{d}y - \int_0^1 \mathrm{d}x \int_0^1 (x+y)\mathrm{d}y
\end{aligned}
$$

10. (2)

坐标变换 $\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 0 \end{cases}$

则有

$$
\begin{aligned}
&\oint_{(C)} (3x^2 - 3yz + 2xz)\mathrm{d}x + (3y^2 - 3xz + z^2)\mathrm{d}y + (3z^2 - 3xy + x^2 + 2yz)\mathrm{d}z \\
=& \int_0^{2\pi} (3\cos^2\theta)\mathrm{d}\cos\theta + (3\sin^2\theta)\mathrm{d}\sin\theta \\
=& \int_0^{2\pi} (-3\sin\theta\cos^2\theta + 3\sin^2\theta\cos\theta)\mathrm{d}\theta \\
=& \, 0
\end{aligned}
$$

12.

(2)

对于 $S_1 : x = 0, S_2 : y = 0, S_3 : z = 0, S_4 : x + y + z = 1$

$$
\begin{aligned}
&\oiint_{(S)} xy\mathrm{d}y \wedge \mathrm{d}z + yz\mathrm{d}z \wedge \mathrm{d}x + zx\mathrm{d}x \wedge \mathrm{d}y \\
=& \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} \\
=& \, 0 + 0 + 0 + \iint_{S_4} (1-y-z)y\mathrm{d}y\mathrm{d}z + \iint_{S_4} (1-x-z)z\mathrm{d}z\mathrm{d}x + \iint_{S_4} (1-x-y)x\mathrm{d}x\mathrm{d}y \\
=& \int_0^1 \mathrm{d}y \int_0^{1-y} (1-y-z)y\mathrm{d}z + \int_0^1 \mathrm{d}z \int_0^{1-z} (1-x-z)z\mathrm{d}x + \int_0^1 \mathrm{d}x \int_0^{1-x} (1-x-y)x\mathrm{d}y \\
=& \int_0^1 \frac{y\left(y-1\right)^2}{2}\mathrm{d}y + \int_0^1 \frac{z\left(z-1\right)^2}{2}\mathrm{d}z + \int_0^1 \frac{x\left(x-1\right)^2}{2}\mathrm{d}x \\
=& \, 3\int_0^1 \frac{y\left(y-1\right)^2}{2}\mathrm{d}y \\
=& \, \frac{1}{8}
\end{aligned}
$$

(4)

$$
\begin{aligned}
&\iint_{(S)} -y\mathrm{d}z \wedge \mathrm{d}x + (z+1)\mathrm{d}x \wedge \mathrm{d}y \\
=& \iint_{S_1} (-\sqrt{4-x^2})\mathrm{d}z\mathrm{d}x - \iint_{S_1} (-\sqrt{4-x^2})\mathrm{d}z\mathrm{d}x + 0 \\
=& -2\int_{-2}^{2} \mathrm{d}x \int_{0}^{2-x} \sqrt{4-x^2}\mathrm{d}z \\
=& -2\int_{-2}^{2} (2-x)\sqrt{4-x^2}\mathrm{d}x \\
=& -2\int_{-2}^{2} (2-x)\sqrt{4-x^2}\mathrm{d}x \\
=& -2\int_{-2}^{2} 2\sqrt{4-x^2}\mathrm{d}x + 2\int_{-2}^{2} x\sqrt{4-x^2}\mathrm{d}x \\
=& -8\pi
\end{aligned}
$$

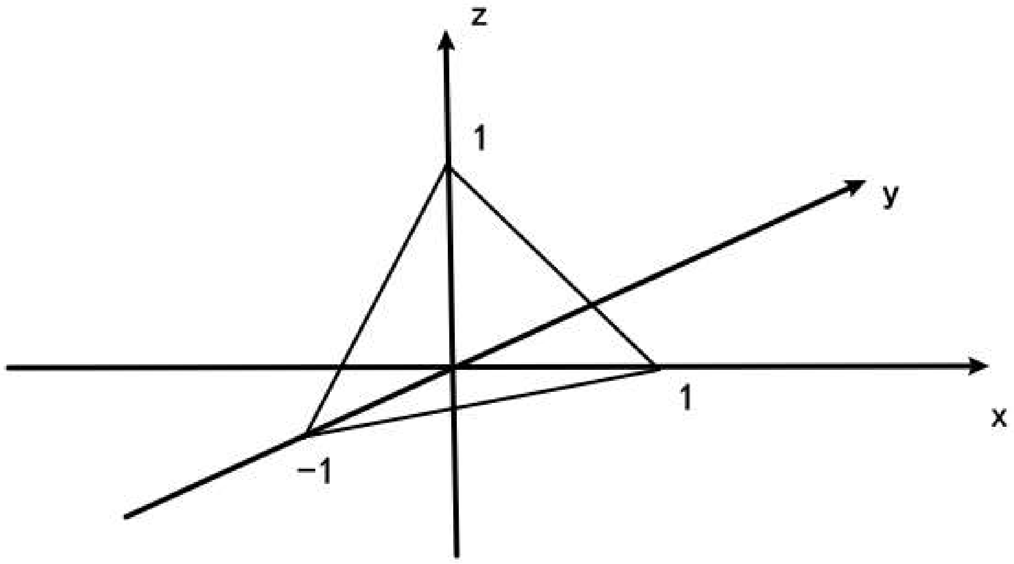
(6)

令 $\begin{cases} x = \sin\varphi\cos\theta \\ y = \sin\varphi\sin\theta \\ z = 1+\cos\varphi \end{cases}$

则 $\dfrac{\partial(x,y)}{\partial(\varphi,\theta)} = \begin{vmatrix} \cos\varphi\cos\theta & -\sin\varphi\sin\theta \\ \cos\varphi\sin\theta & \sin\varphi\cos\theta \end{vmatrix} = \dfrac{1}{2}\sin(2\varphi)$

$$
\begin{aligned}
&\iint_{(S)} z^2\mathrm{d}x \wedge \mathrm{d}y \\
=& \iint_{D_{\varphi,\theta}} (1+\cos\varphi)^2 \cdot \frac{1}{2}\sin(2\varphi)\mathrm{d}\varphi\mathrm{d}\theta \\
=& \int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{\pi} (1+\cos\varphi)^2 \cdot \frac{1}{2}\sin(2\varphi)\mathrm{d}\varphi \\
=& \int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{\pi} (\sin\varphi\cos\varphi + 2\sin\varphi\cos^2\varphi + \sin\varphi\cos^3\varphi)\mathrm{d}\varphi \\
=& \int_{0}^{2\pi} \mathrm{d}\theta \int_{0}^{\pi} (\sin\varphi\cos\varphi + 2\sin\varphi\cos^2\varphi + \sin\varphi\cos^3\varphi)\mathrm{d}\varphi \\
=& \frac{8\pi}{3}
\end{aligned}
$$

(7)



$S: x-y+z=1$ 的法向量 $\vec{n}=\{1,-1,1\}$

$\because \{f(x,y,z),2f(x,y,z),f(x,y,z)\}\cdot\{1,-1,1\}=0$

$$
\begin{aligned}
\therefore I &= \iint_{(S)} x\mathrm{d}y\wedge\mathrm{d}z+y\mathrm{d}z\wedge\mathrm{d}x+z\mathrm{d}x\wedge\mathrm{d}y \\
&= \iint_{D_{yz}}(y-z+1)\mathrm{d}y\mathrm{d}z-\iint_{D_{zx}}(x+z-1)\mathrm{d}z\mathrm{d}x+\iint_{D_{xy}}(y-x+1)\mathrm{d}x\mathrm{d}y \\
&= \int_{-1}^{0}\mathrm{d}y\int_{0}^{y+1}(y-z+1)\mathrm{d}z-\int_{0}^{1}\mathrm{d}x\int_{0}^{1-x}(x+z-1)\mathrm{d}z+\int_{-1}^{0}\mathrm{d}y\int_{0}^{y+1}(y-x+1)\mathrm{d}x \\
&= 2\int_{-1}^{0}\frac{(y+1)^2}{2}\mathrm{d}y+\int_{0}^{1}\frac{(x-1)^2}{2}\mathrm{d}x \\
&= \frac{1}{2}
\end{aligned}
$$

15.

(1)

$\therefore y=3-\frac{3}{2}x-\sqrt{3}z$

$\therefore \frac{\partial(y,z)}{\partial(z,x)}=\begin{vmatrix}-\sqrt{3} & -\frac{3}{2}\\ 1 & 0\end{vmatrix}=\frac{3}{2},\frac{\partial(z,x)}{\partial(z,x)}=1,\frac{\partial(x,y)}{\partial(z,x)}=\begin{vmatrix}0 & 1\\ -\sqrt{3} & -\frac{3}{2}\end{vmatrix}=\sqrt{3}$

且 $\sqrt{1+y_z^2+y_x^2}=\frac{5}{2}$

$$\begin{aligned}
&\iint_{(S)} P\mathrm{d}y\wedge\mathrm{d}z+Q\mathrm{d}z\wedge\mathrm{d}x+R\mathrm{d}x\wedge\mathrm{d}y\\
=&\iint_{D_{zx}}\left(\frac{3}{2}P+Q+\sqrt{3}R\right)\mathrm{d}z\mathrm{d}x\\
=&\iint_{D_{zx}}\left(\frac{3}{5}P+\frac{2}{5}Q+\frac{2\sqrt{3}}{5}R\right)\sqrt{1+y_z^2+y_x^2}\mathrm{d}z\mathrm{d}x\\
=&\iint_{(S)}\left(\frac{3}{5}P+\frac{2}{5}Q+\frac{2\sqrt{3}}{5}R\right)\mathrm{d}S
\end{aligned}$$

(2)

令 $\begin{cases}x=\rho\cos\theta\\y=\rho\sin\theta\\z=8-\rho^2\end{cases}$

$\therefore \dfrac{\partial(y,z)}{\partial(\rho,\theta)}=\begin{vmatrix}\sin\theta & \rho\cos\theta\\-2\rho & 0\end{vmatrix}=2\rho^2\cos\theta,\dfrac{\partial(z,x)}{\partial(\rho,\theta)}=\begin{vmatrix}-2\rho & 0\\\cos\theta & -\rho\sin\theta\end{vmatrix}=2\rho^2\sin\theta,\dfrac{\partial(x,y)}{\partial(\rho,\theta)}=\begin{vmatrix}\cos\theta & -\rho\sin\theta\\\sin\theta & \rho\cos\theta\end{vmatrix}=\rho$

且 $|\vec{r}_\rho\times\vec{r}_\theta|=\sqrt{(2\rho^2\cos\theta)^2+(2\rho^2\sin\theta)^2+\rho^2}=\sqrt{4\rho^2+1}\rho$

$$\begin{aligned}
&\iint_{(S)} P\mathrm{d}y\wedge\mathrm{d}z+Q\mathrm{d}z\wedge\mathrm{d}x+R\mathrm{d}x\wedge\mathrm{d}y\\
=&-\iint_{D_{\rho\theta}}\left(P\cdot2\rho^2\cos\theta+Q\cdot2\rho^2\sin\theta+R\rho\right)\mathrm{d}\rho\mathrm{d}\theta\\
=&-\iint_{D_{\rho\theta}}\frac{1}{\sqrt{4\rho^2+1}}(2P\rho\cos\theta+2Q\rho\sin\theta+R)\,|\vec{r}_\rho\times\vec{r}_\theta|\mathrm{d}\rho\mathrm{d}\theta\\
=&-\iint_{D_{\rho\theta}}\frac{2xP+2yQ+R}{\sqrt{4x^2+4y^2+1}}\mathrm{d}S
\end{aligned}$$

6.7 (B)

2.

令 $\begin{cases}x=R\sin\varphi\cos\theta\\y=R\sin\varphi\sin\theta\\z=R\cos\varphi\end{cases}$，则柱面转化为 $R^2\sin^2\varphi=R^2\sin\varphi\cos\theta,0\leqslant\varphi\leqslant\frac{\pi}{2}$

即 $\sin\varphi=\cos\theta$, 原式化为 $\begin{cases}x=R\cos^2\theta\\y=R\sin\theta\cos\theta\\z=R|\sin\theta|\end{cases}$

$$
\begin{aligned}
&\oint_{(C)} y^2\mathrm{d}x + z^2\mathrm{d}y + x^2\mathrm{d}z \\
=& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} R^2\sin^2\theta\cos^2\theta\mathrm{d}R\cos^2\theta + R^2\sin^2\theta\mathrm{d}R\sin\theta\cos\theta + R^2\cos^4\theta\mathrm{d}R|\sin\theta| \\
=& \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (-2R^3\sin^3\theta\cos^3\theta + R^3\sin^2\theta\cos 2\theta)\mathrm{d}\theta + 0 \\
=& -\frac{\pi}{4}R^3
\end{aligned}
$$

5.

令 $\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\sin\theta \\ z=\rho\cos\varphi \end{cases}$，则柱面转化为 $\rho^2\sin^2\varphi = R^2$, 即 $\rho = \dfrac{R}{\sin\varphi}$

则 $\begin{cases} x=R\cos\theta \\ y=R\sin\theta \\ z=\dfrac{R}{\tan\varphi} \end{cases}$，$\dfrac{\partial(y,z)}{\partial(\varphi,\theta)} = \begin{vmatrix} 0 & R\cos\theta \\ -\frac{R}{\sin^2\varphi} & 0 \end{vmatrix} = \dfrac{R^2\cos\theta}{\sin^2\varphi}$

其中 $-R \leqslant z = \dfrac{R}{\tan\varphi} \leqslant R$, 即 $-\frac{\pi}{2} \leqslant \varphi \leqslant -\frac{\pi}{4}$ 或 $\frac{\pi}{4} \leqslant \varphi \leqslant \frac{\pi}{2}$

$$
\begin{aligned}
&\iint_{(S)} \frac{x\mathrm{d}y\wedge\mathrm{d}z + z^2\mathrm{d}x\wedge\mathrm{d}y}{x^2+y^2+z^2} \\
=& \iint_{S_{\text{bottom}}} + \iint_{S_{\text{top}}} + \iint_{S_{\text{side}}} \\
=& 0+0+\iint_S \frac{x\mathrm{d}y\wedge\mathrm{d}z}{x^2+y^2+z^2} \\
=& 0+0+\iint_{S_{\varphi\theta}} \frac{R\cos\theta}{\frac{R^2}{\sin^2\varphi}}\cdot\frac{R^2\cos\theta}{\sin^2\varphi}\mathrm{d}\theta\mathrm{d}\varphi \\
=& \left(\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}}\mathrm{d}\varphi + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\mathrm{d}\varphi\right)\int_0^{2\pi} R\cos^2\theta\mathrm{d}\theta \\
=& \frac{1}{2}\pi^2 R
\end{aligned}
$$

6.

令 $\begin{cases} x=R\sin\varphi\cos\theta \\ y=R\sin\varphi\sin\theta \\ z=R\cos\varphi \end{cases}$，其中 $0 \leqslant \varphi \leqslant \frac{\pi}{2}$

则有 $\dfrac{\partial(y,z)}{\partial(\varphi,\theta)} = \begin{vmatrix} R\cos\varphi\sin\theta & R\sin\varphi\cos\theta \\ -R\sin\varphi & 0 \end{vmatrix} = R^2\sin^2\varphi\cos\theta$

$$\frac{\partial(z,x)}{\partial(\varphi,\theta)}=\begin{vmatrix}-R\sin\varphi & 0\\ R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta\end{vmatrix}=R^2\sin^2\varphi\sin\theta$$

$$\frac{\partial(x,y)}{\partial(\varphi,\theta)}=\begin{vmatrix}R\cos\varphi\cos\theta & -R\sin\varphi\sin\theta\\ R\cos\varphi\sin\theta & R\sin\varphi\cos\theta\end{vmatrix}=R^2\sin\varphi\cos\varphi$$

$$\begin{aligned}&\iint_{(S)}\vec{F}\cdot\mathrm{d}\vec{S}\\ &=-\iint_{S_{\varphi\theta}}\frac{1}{R^2}\left(R^3\sin^3\varphi\cos^2\theta+R^3\sin^3\varphi\sin^2\theta+R^3\sin\varphi\cos^2\varphi\right)\mathrm{d}\varphi\mathrm{d}\theta\\ &=-R\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\left(\sin^3\varphi+\sin\varphi\cos^2\varphi\right)\mathrm{d}\varphi\\ &=-2\pi R\end{aligned}$$

6.8 (A)

1.

(1)

错误. 封闭曲线并非正向.

正确解法:

$$\int_{\overparen{OB}\cup\overline{BA}\cup\overline{AO}}y\mathrm{d}x=-\iint_{(\sigma)}-1\mathrm{d}\sigma=\frac{\pi}{4}$$

由于 $\int_{\overline{BA}}y\mathrm{d}x=0,\int_{\overline{AO}}y\mathrm{d}x=0$

所以 $\int_{(C)}y\mathrm{d}x=\frac{\pi}{4}$

(2)

解法一错误, 解法二正确.

函数在点 $(0,0)$ 处并不可导, 不能使用 Green 公式, 解法一错误.

而解法二没有包括 $(0,0)$ 点, 所以正确.

对于解法一:

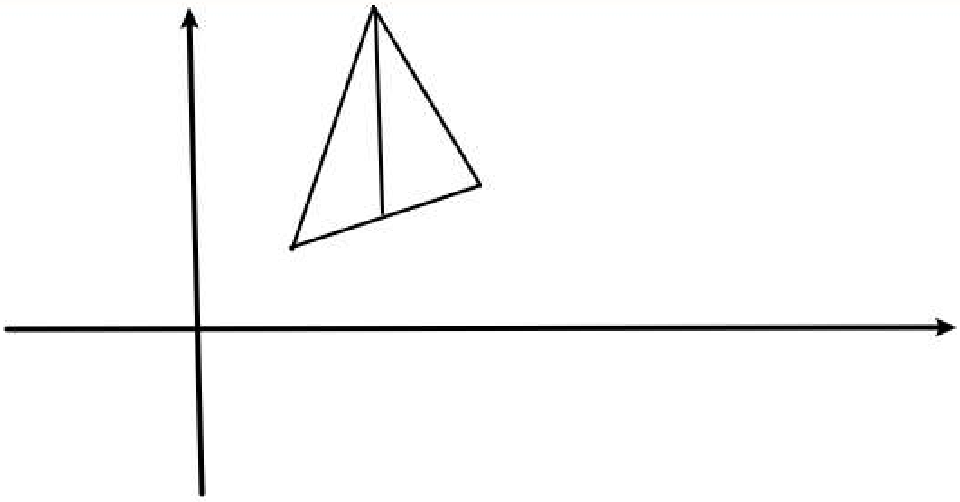
设 $\Gamma: x^2+y^2=\delta^2$

则 $\oint_{(C)\cup\overparen{DB'A}}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}=\oint_{(C)\cup\overparen{DB'A}-\Gamma}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}+\oint_{\Gamma}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}=$
$\oint_{\Gamma}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}=\oint_{\Gamma}\frac{-y\mathrm{d}x+x\mathrm{d}y}{\delta^2}=-2\pi$

则 $I=\oint_{\overparen{AB'D}}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}+\oint_{(C)\cup\overparen{DB'A}}\frac{-y\mathrm{d}x+x\mathrm{d}y}{x^2+y^2}=-\pi$

2.

(3)



由 Green 公式可知

$$
\begin{aligned}
&\oint_{(+C)}(x+y)^2\mathrm{d}x-(x^2+y^2)\mathrm{d}y\\
=&\iint_{D_{xy}}\left(-2x-2(x+y)\right)\mathrm{d}x\mathrm{d}y\\
=&\int_1^2\mathrm{d}x\int_{\frac{1}{2}x+\frac{1}{2}}^{4x-3}\left(-4x-2y\right)\mathrm{d}y+\int_2^3\mathrm{d}x\int_{\frac{1}{2}x+\frac{1}{2}}^{-3x+11}\left(-4x-2y\right)\mathrm{d}y\\
=&\int_1^2(-\frac{119x^2}{4}+\frac{77x}{2}-\frac{35}{4})\mathrm{d}x+\int_2^3(\frac{21x^2}{4}+\frac{49x}{2}-\frac{483}{4})\mathrm{d}x\\
=&-\frac{245}{12}-\frac{105}{4}\\
=&-\frac{140}{3}
\end{aligned}
$$

(5)

令 $L: y=0$, 从点 $O(0,0)$ 到 $A(a,0)$

则 $\int_L(e^x\sin y-my)\mathrm{d}x+(e^x\cos y-m)\mathrm{d}y=0$

那么

$$
\begin{aligned}
&\int_{(C)}(e^x\sin y-my)\mathrm{d}x+(e^x\cos y-m)\mathrm{d}y\\
=&\oint_{C\cup L}(e^x\sin y-my)\mathrm{d}x+(e^x\cos y-m)\mathrm{d}y-\int_L\\
=&\iint_{C\cup L}\left(e^x\cos y-e^x\cos y+m\right)\mathrm{d}x\mathrm{d}y\\
=&m\iint_{C\cup L}\mathrm{d}x\mathrm{d}y\\
=&m\cdot\frac{1}{2}\pi\left(\frac{a}{2}\right)^2\\
=&\frac{m\pi a^2}{8}
\end{aligned}
$$

3.

取 $P=-y,Q=x$

则 $S=\iint_{D_{xy}}\mathrm{d}x\mathrm{d}y=\frac{1}{2}\oint_\Gamma -y\mathrm{d}x+x\mathrm{d}y$

其中 $\Gamma: x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

进行参数方程变换 $\begin{cases}x=a\cos^3\theta\\ y=a\sin^3\theta\end{cases},0\leqslant\theta\leqslant 2\pi$

$$
\begin{aligned}
S=&\frac{1}{2}\oint_\Gamma -y\mathrm{d}x+x\mathrm{d}y\\
=&\frac{1}{2}\int_0^{2\pi}\left(-a\sin^3\theta\mathrm{d}a\cos^3\theta+a\cos^3\theta\mathrm{d}a\sin^3\theta\right)\\
=&\frac{1}{2}\int_0^{2\pi}\left(3a^2\sin^4\theta\cos^2\theta+3a^2\sin^2\theta\cos^4\theta\right)\mathrm{d}\theta\\
=&\frac{3}{2}a^2\int_0^{2\pi}\sin^2\theta\cos^2\theta\mathrm{d}\theta\\
=&\frac{3\pi}{8}a^2
\end{aligned}
$$

8. (2)

由 Green 公式可知

$$
\begin{aligned}
&\oint\frac{x\mathrm{d}x+y\mathrm{d}y}{\sqrt{x^2+y^2}}\\
=&\iint_{D_{xy}}\left(-\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}+\frac{xy}{(x^2+y^2)^{\frac{3}{2}}}\right)\mathrm{d}x\mathrm{d}y\\
=&0
\end{aligned}
$$

所以与路径无关

从点 $(1,0)$ 到点 $(6,0)$, 再到点 $(6,8)$

$$
\begin{aligned}
&\int_{(1,0)}^{(6,8)} \frac{x\mathrm{d}x+y\mathrm{d}y}{\sqrt{x^2+y^2}} \\
&=\int_1^6 \frac{x}{\sqrt{x^2+0}}\mathrm{d}x+\int_0^8 \frac{y}{\sqrt{36+y^2}}\mathrm{d}y \\
&=5+4 \\
&=9
\end{aligned}
$$

10.

由 Stokes 公式可知:

$$
\begin{aligned}
&\oint_{(C)} y\mathrm{d}x+\mathrm{d}y+x\mathrm{d}z \\
&=\iint_{\Sigma}(0-1)\mathrm{d}y\wedge\mathrm{d}z+(0-1)\mathrm{d}z\wedge\mathrm{d}x+(0-1)\mathrm{d}x\wedge\mathrm{d}y \\
&=-\sqrt{3}\iint_{\Sigma}\frac{\sqrt{3}}{3}\mathrm{d}y\wedge\mathrm{d}z+\frac{\sqrt{3}}{3}\mathrm{d}z\wedge\mathrm{d}x+\frac{\sqrt{3}}{3}\mathrm{d}x\wedge\mathrm{d}y \\
&=-\sqrt{3}\pi a^2
\end{aligned}
$$

16.

(1)

由 Gauss 公式可知

$$
\begin{aligned}
&\oiint_{(S)} x^2\mathrm{d}y\wedge\mathrm{d}z+y^2\mathrm{d}z\wedge\mathrm{d}x+z^2\mathrm{d}x\wedge\mathrm{d}y \\
&=\iiint_{\Omega}(2x+2y+2z)\mathrm{d}V \\
&=2\int_0^a\mathrm{d}x\int_0^a\mathrm{d}y\int_0^a(x+y+z)\mathrm{d}z \\
&=3a^4
\end{aligned}
$$

(3)

进行球面坐标代换 $\begin{cases} x=\rho\sin\varphi\cos\theta \\ y=\rho\sin\varphi\cos\theta \\ z=\rho\cos\varphi \end{cases}$, 其中 $0\leqslant\varphi\leqslant\frac{\pi}{2}$

则 $J=\rho^2\sin\varphi$

由 Gauss 公式可知

$$
\begin{aligned}
&\oiint_{(S)}(x^2-2xy)\mathrm{d}y\wedge\mathrm{d}z+(y^2-2yz)\mathrm{d}z\wedge\mathrm{d}x+(1-2xz)\mathrm{d}x\wedge\mathrm{d}y\\
&=\iiint_\Omega(2x-2y+2y-2z-2x)\mathrm{d}V\\
&=-2\iiint_\Omega z\mathrm{d}V\\
&=-2\int_0^{2\pi}\mathrm{d}\theta\int_0^{\frac{\pi}{2}}\mathrm{d}\varphi\int_0^a\rho\cos\varphi\cdot\rho^2\sin\varphi\mathrm{d}\rho\\
&=-\pi a^4\int_0^{\frac{\pi}{2}}\sin\varphi\cos\varphi\mathrm{d}\varphi\\
&=-\frac{\pi}{2}a^4
\end{aligned}
$$

$$
\because\iint_{S_{\text{bottom}}}(1-2x\cdot 0)\mathrm{d}x\wedge\mathrm{d}y=\pi a^2
$$

$$
\begin{aligned}
&\iint_{(S)}(x^2-2xy)\mathrm{d}y\wedge\mathrm{d}z+(y^2-2yz)\mathrm{d}z\wedge\mathrm{d}x+(1-2xz)\mathrm{d}x\wedge\mathrm{d}y\\
&=\oiint_{(S)}(x^2-2xy)\mathrm{d}y\wedge\mathrm{d}z+(y^2-2yz)\mathrm{d}z\wedge\mathrm{d}x+(1-2xz)\mathrm{d}x\wedge\mathrm{d}y-\iint_{S_{bottom}}\\
&=\frac{\pi a^2\left(2-a^2\right)}{2}
\end{aligned}
$$

(6)

由 Gauss 公式可知

令 $\Gamma: z=e^a$

$$
\begin{aligned}
&\oiint_{S\cup\Gamma}4xz\mathrm{d}y\wedge\mathrm{d}z-2yz\mathrm{d}z\wedge\mathrm{d}x+(1-z^2)\mathrm{d}x\wedge\mathrm{d}y\\
&=\iiint_\Omega(4z-2z-2z)\mathrm{d}V\\
&=0
\end{aligned}
$$

$$
\begin{aligned}
&\iint_S 4xz\mathrm{d}y\wedge\mathrm{d}z-2yz\mathrm{d}z\wedge\mathrm{d}x+(1-z^2)\mathrm{d}x\wedge\mathrm{d}y\\
&=\oiint_{S\cup\Gamma}-\iint_\Gamma\\
&=-\iint_\Gamma 4xz\mathrm{d}y\wedge\mathrm{d}z-2yz\mathrm{d}z\wedge\mathrm{d}x+(1-z^2)\mathrm{d}x\wedge\mathrm{d}y\\
&=-\iint_{D_{xy}}(1-(e^a)^2)\mathrm{d}x\wedge\mathrm{d}y\\
&=(e^{2a}-1)\iint_{D_{xy}}\mathrm{d}x\wedge\mathrm{d}y\\
&=(e^{2a}-1)\pi a^2
\end{aligned}
$$