Knights, Knaves, and Logical Reasoning

Mechanising the Laws of Thought

[1]Special thanks to Fabio Papacchini and Francis Southern

Introduction

Thinking

Formalising

Modelling

Computing

Thinking

A Puzzle

You are on a strange island where people are divided into

- Knights – always saying the truth
- Knaves – always saying lies

You meet two natives of the island Alice and Bob, and ask them

"Are you knights or knaves?"

Alice answers *"At least one of us is a knave"*

What are Alice and Bob?

Alice: *"At least one of us is a knave"*

Alice

Bob

Alice

Bob

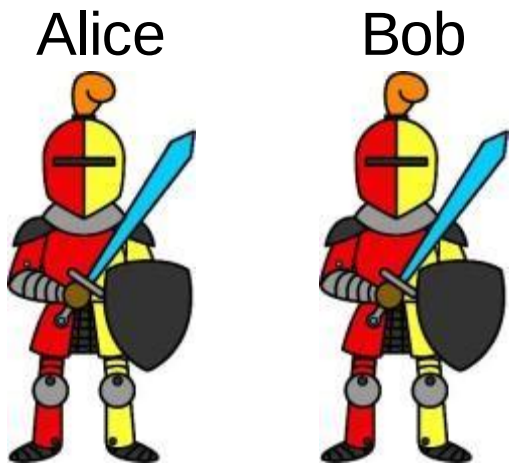
Alice

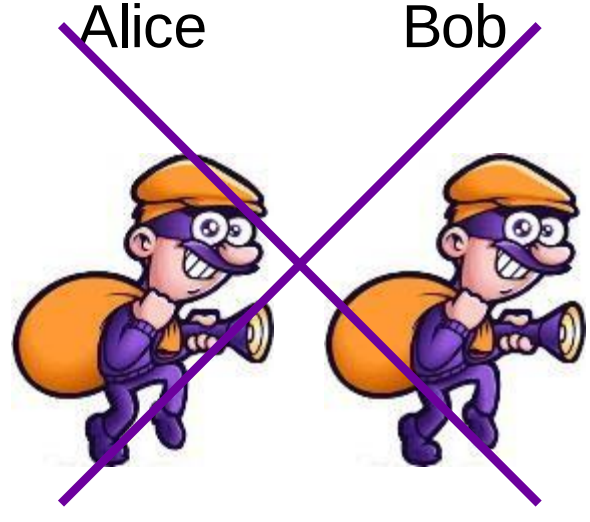
Bob

Alice

Bob

Alice: *"At least one of us is a knave"*





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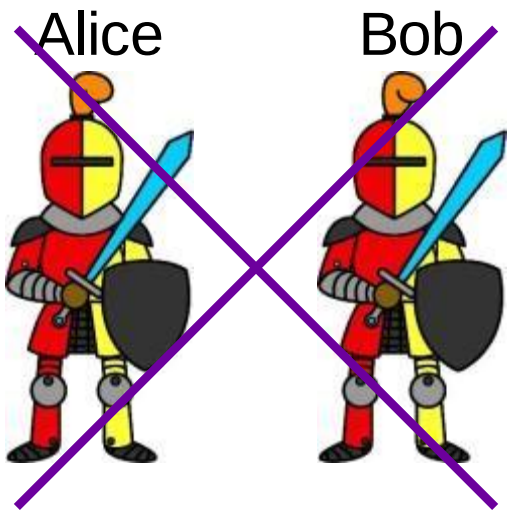

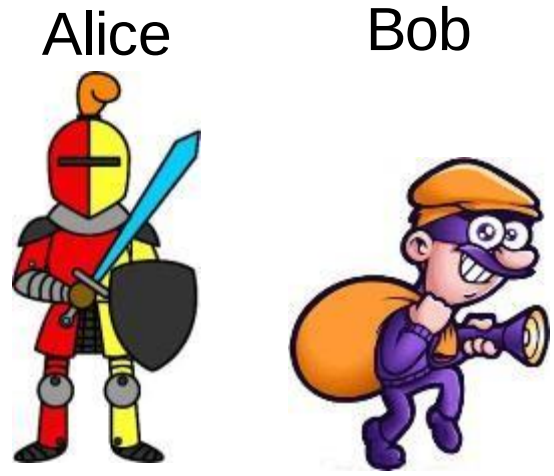

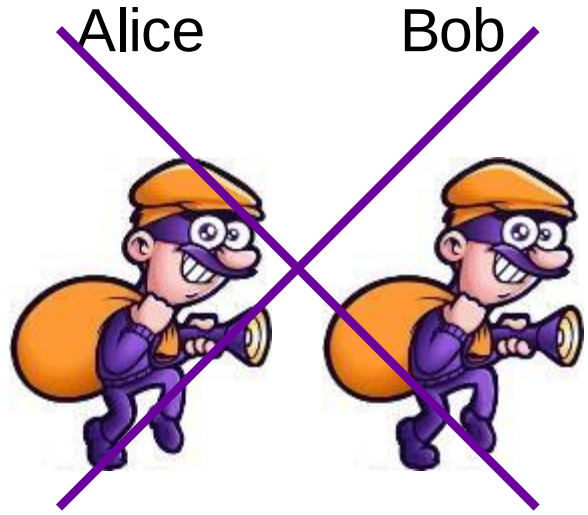

Formalizing

Formalizing Correct Reasoning

A: Socrates is a man

B: All men are mortal

C: All men are Socrates

C: Socrates is mortal

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Woody Allen - Love and Death

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Aristotle

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Aristotle

Linguistic, philosophical, or mathematical approaches to formalisation

Propositional Logic

Propositions

An expression which is either true or false.

Propositions

An expression which is either true or false.

Proposition test: Is it true that. . . ?

- $2 + 2 = 5$
- Nanjing
- Grass is green
- We're in Nanjing
- What's your name?
- It's raining

Not – ¬,And – &

Not

p	$\neg p$
F	T
T	F

It's *not* raining

Grass is *not* green.

Not – ¬, And – &

Not

p	$\neg p$
F	T
T	F

It's *not* raining

Grass is *not* green.

And

p	q	p & q
F	F	F
F	T	F
T	F	F
T	T	T

Grass is green *and* it's raining.

We're in Nanjing *and* we're in NJU.

Or – |, Implication (If, then) – →

Or

p	q	$p \mid q$
F	F	F
F	T	T
T	F	T
T	T	T

Take an aspirin *or* lie down.

You can have milk *or* sugar in your tea.

Or – |, Implication (If, then) – →

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F	F	F
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Implication

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

If you get 90% on this assignment, *then* you'll pass the course.

If you're late, *then* you'll give me a fiver.

Biimplication (If and only if) – ↔

Biimplication

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

I'll buy you a new wallet *if (and only if)* you need one.

He studies *if (and only if)* he can.

An Example: $(p \,\&\, q) \rightarrow r$

p	q	r	$(p \,\&\, q)$	$(p \,\&\, q) \rightarrow r$
F	F	F		
F	F	T		
F	T	F		
F	T	T		
T	F	F		
T	F	T		
T	T	F		
T	T	T		

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p	q	r	$(p \ \& \ q)$	$(p \ \& \ q) \rightarrow r$
F	F	F	F	
F	F	T	F	
F	T	F	F	
F	T	T	F	
T	F	F	F	
T	F	T	F	
T	T	F		
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F	F	T	F	T
F	T	F	F	T
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Modeling

The Trick

k_A = Alice is a knight $\neg k_A$ = Alice is a knave

"Alice says X" is the same as $k_A \leftrightarrow X$

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Alice says

- "at least one of us is a knave"
- "I'm a knave or Bob is a knave"
- $\neg k_A \mid \neg k_B$

$$\Rightarrow\ k_A \leftrightarrow (\neg k_A \mid \neg k_B)$$

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F	F	T	T	T	
F	T	T	F	T	
T	F	F	T	T	
T	T	F	F	F	

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F	F	T	T	T	F
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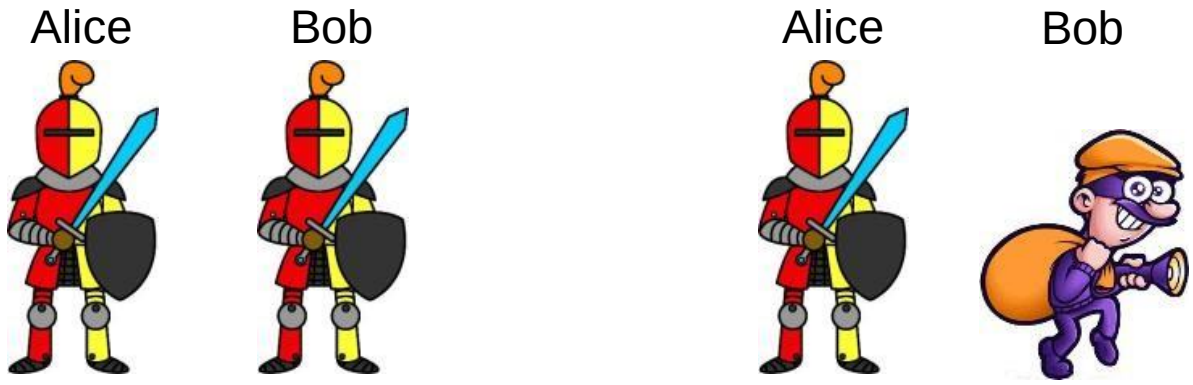
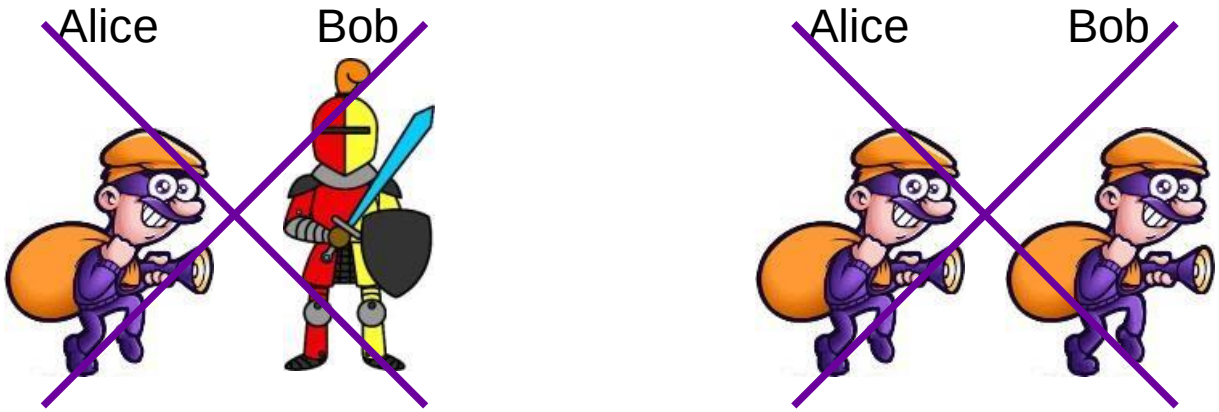
T	T	F	F	F	F

From Solving to Modeling

Alice: *"At least one of us is a knave"*





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Alice says "At least one of us is a knave" = $k_A \leftrightarrow (\neg k_A \mid \neg k_B)$

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It can be (really) hard, but you only have to do it once!

Modeling a Sudoku

			7			4	1	
		3		2				6
1		7	4			5	2	3
4		1	6				8	
	2	9		7		6	3	
	7				4	2		1
7	5	2			6	3		9
3				4		1		
	1	4			3			

What propositions do we need?

Modelling a Sudoku

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Number n is in row i and column j

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- number 2 is in row 6 and column 7

$$p_{1,1,1}, p_{1,1,2}, \ldots, p_{9,9,8}, p_{9,9,9}$$

729 propositions!

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What do we have to model?

- at least one number per cell $(p_{1,1,4} \mid \ldots \mid p_{9,1,4})$
- at most one number per cell $(p_{7,1,4} \rightarrow \neg p_{1,1,4}, p_{7,1,4} \rightarrow \neg p_{2,1,4})$

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- no number can be repeated in a row

Modeling a Sudoku

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- no number can be repeated in a row/column

Modeling a Sudoku

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		3		2				6
1		7	4			5	2	3
4		1	6				8	
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- at most one number per cell $(p_{7,1,4} \rightarrow \neg p_{1,1,4}, p_{7,1,4} \rightarrow \neg p_{2,1,4})$
- no number can be repeated in a row/column/region

Computing

Automating the Process

Truth table

- mechanical
- time consuming (2^n rows!)
- tedious

Automating the Process

Truth table

- mechanical
- time consuming (2^n rows!)
- tedious

Let a computer do it for you!

- ideal for mechanical tasks
- only needs an input formula
- more reliable than us
- much faster than us
- the output is easily customisable

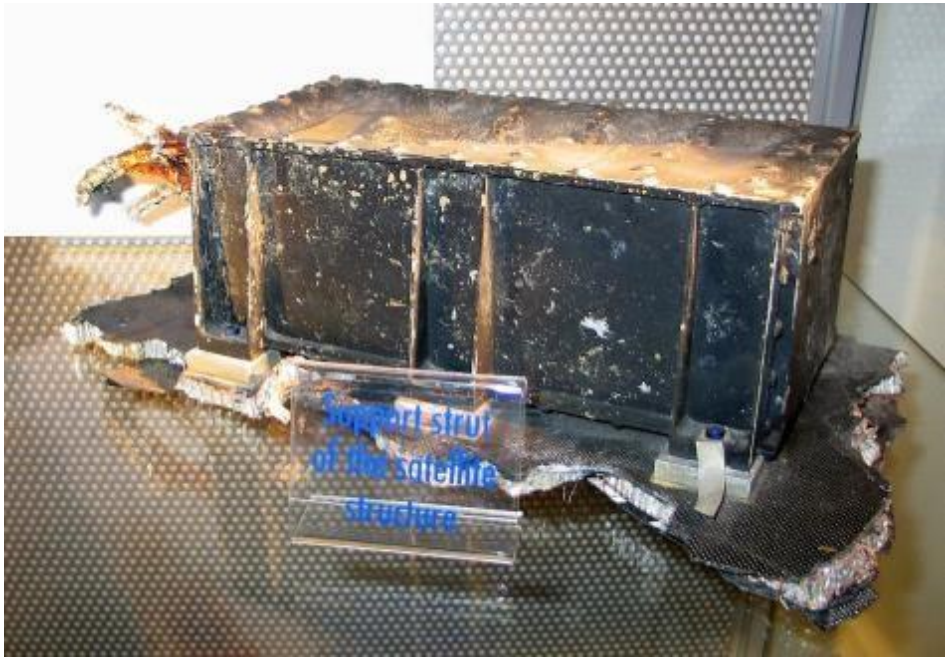
Automated Reasoning

Much more than solving puzzles!

- software and hardware verification
  - Intel and Microsoft
- information management
  - biomedical ontologies, Semantic Web, databases
- combinatorial reasoning
  - constraint satisfaction, planning, scheduling
- Internet security
- theorem proving in mathematics

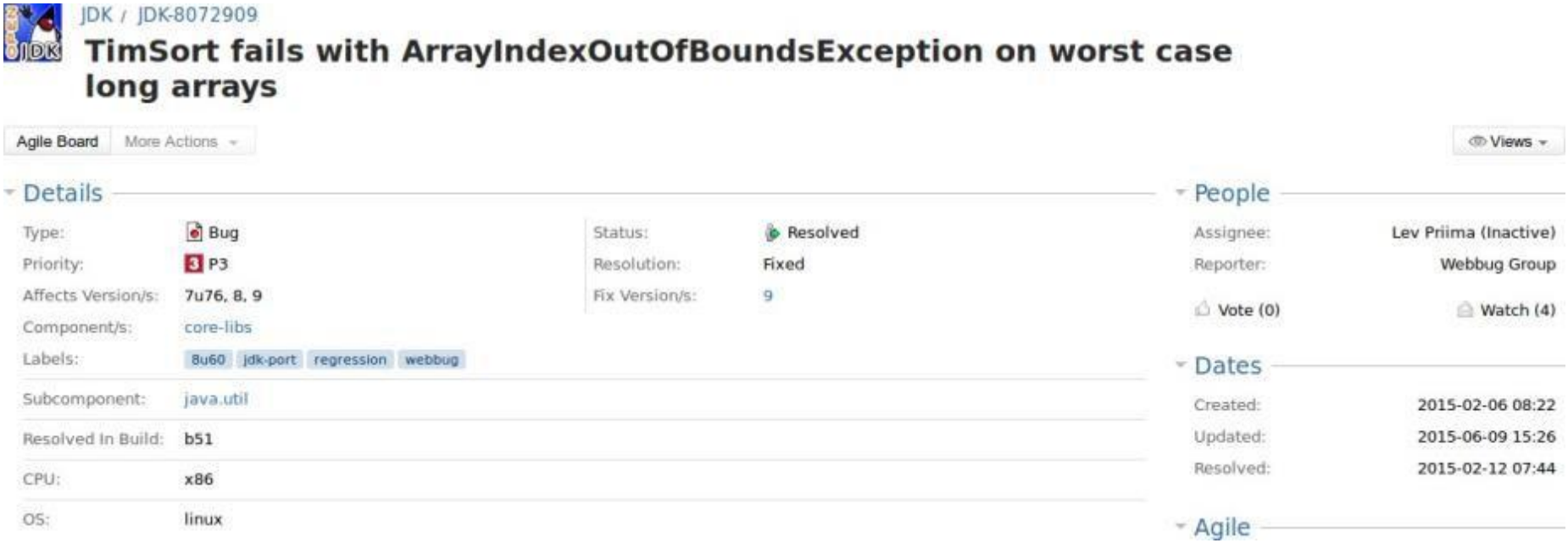
Where Could Have Been Used

Ariane 5 rocket failure due to a software bug, cost $370 million.



Where Has Been Used

To find and fix a bug in a widely used sorting algorithm!



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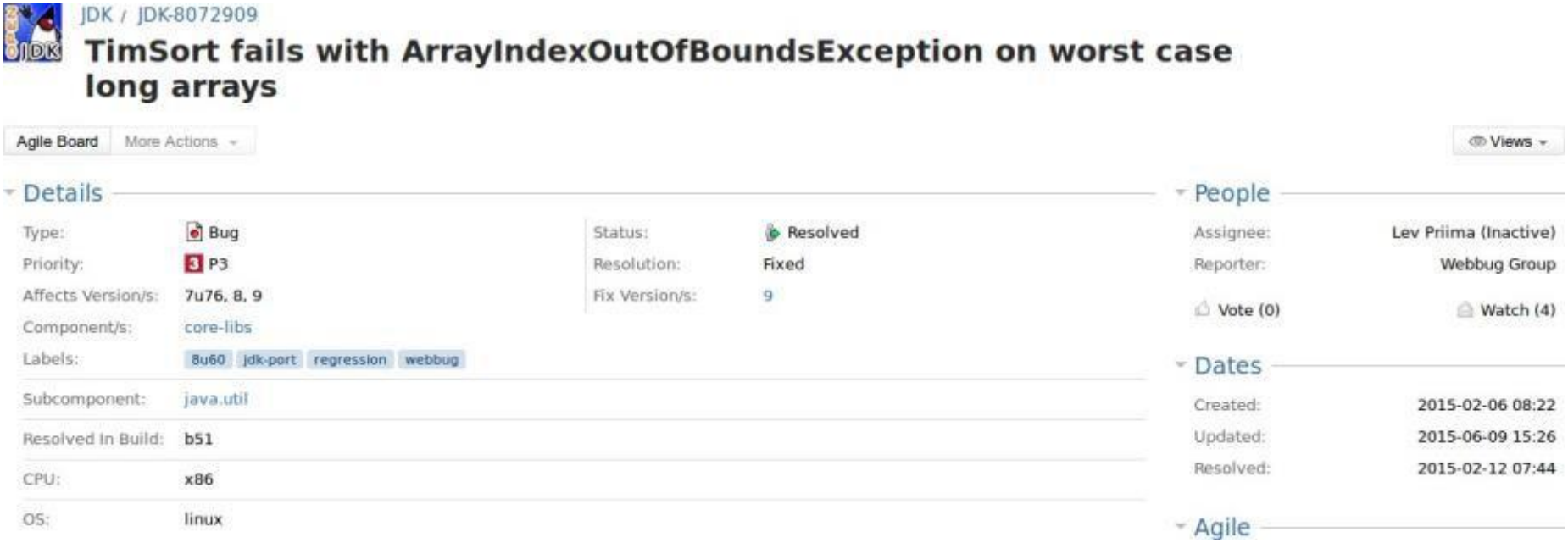

Even Amazon and Facebook use automated reasoning techniques!

Do You Want to Know More?

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