P233: 1.(1)(2)(5) 2.

1.

(I)

(1)

$\because f(x_1,x_2,x_3) = -4x_1x_2 + 2x_1x_3 + 2x_2x_3$

$\therefore A = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$$\therefore \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_1-r_2]{c_1-c_2} \begin{pmatrix} 4 & -2 & 0 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[r_2+\frac{1}{2}r_1]{c_2+\frac{1}{2}c_1} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & \frac{1}{2} & 0 \\ -1 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3+r_2]{c_3+c_2} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} D \\ C \end{pmatrix}$$

$\therefore C = \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix}$

$\therefore XC = Y, f(x_1,x_2,x_3) = 4y_1^2 - y_2^2 + y_3^2$

(2)

$\because f(x_1,x_2,x_3) = x_1^2 + 2x_1x_2 + 2x_2^2 + 4x_2x_3 + 4x_3^2$

$\therefore A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \end{pmatrix}$

$$\therefore \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 2 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_2 - r_1]{c_2 - c_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_3 - 2r_2]{c_3 - 2c_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore C = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\therefore XC = Y, f(x_1, x_2, x_3) = y_1^2 + y_2^2$$

(5)

$$\because f(x_1, x_2, x_3, x_4) = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4$$

$$\therefore \begin{pmatrix} A \\ E \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow[r_1 + r_2]{c_1 + c_2} \begin{pmatrix} 1 & \frac{1}{2} & 1 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[r_2 - \frac{1}{2}r_1, r_3 - r_1, r_4 - r_1]{c_2 - \frac{1}{2}c_1, c_3 - c_1, c_4 - c_1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -1 & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & -1 \\ 1 & -\frac{1}{2} & -1 & -1 \\ 1 & \frac{1}{2} & -1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[r_4 - \frac{1}{2}r_3]{c_4 - \frac{1}{2}c_3} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\frac{3}{4} \\ 1 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 1 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\therefore C=\begin{pmatrix}1 & -\frac{1}{2} & -1 & -\frac{1}{2}\\ 1 & \frac{1}{2} & -1 & -\frac{1}{2}\\ 0 & 0 & 1 & -\frac{1}{2}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\therefore XC=Y, f(x_1,x_2,x_3)=y_1^2-\frac{1}{4}y_2^2-y_3^2-\frac{3}{4}y_4^2$$

(II)

(1)

对于实数域:

$$令\ D=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\therefore D^TC^TACD=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}4 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

对于虚数域:

$$令\ D=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & i & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\therefore D^TC^TACD=\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & i & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}4 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{2} & 0 & 0\\ 0 & i & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

(2)

对于实数域与虚数域均有:

$$令\ D=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\therefore D^TC^TACD=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

(5)

对于实数域:

令 $D=\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&\frac{2\sqrt{3}}{3}\end{pmatrix}$

$$\begin{aligned}\therefore D^TC^TACD&=\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&\frac{2\sqrt{3}}{3}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-\frac{1}{4}&0&0\\0&0&-1&0\\0&0&0&-\frac{3}{4}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&1&0\\0&0&0&\frac{2\sqrt{3}}{3}\end{pmatrix}\\&=\begin{pmatrix}1&0&0&0\\0&-1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\end{aligned}$$

对于虚数域:

令 $D=\begin{pmatrix}1&0&0&0\\0&2i&0&0\\0&0&i&0\\0&0&0&\frac{2\sqrt{3}}{3}i\end{pmatrix}$

$$\begin{aligned}\therefore D^TC^TACD&=\begin{pmatrix}1&0&0&0\\0&2i&0&0\\0&0&i&0\\0&0&0&\frac{2\sqrt{3}}{3}i\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&-\frac{1}{4}&0&0\\0&0&-1&0\\0&0&0&-\frac{3}{4}\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&2i&0&0\\0&0&i&0\\0&0&0&\frac{2\sqrt{3}}{3}i\end{pmatrix}\\&=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\end{aligned}$$

2.

设有对称矩阵 A, 则我们能找到非退化矩阵 C 使得 $B=(C^{-1})^TAC^{-1}$ 为对角矩阵, 且 B 的秩等于 A 的秩 r, $A=C^TBC$.

不妨令:

$$
\begin{aligned}
B &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \\
&= \overbrace{\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots}^{r} \\
&= D_1 + D_2 + \cdots + D_r
\end{aligned}
$$

其中 D_i 中只在对角线有一个元素 1, 即 D_i 的秩为 1.

带入得

$$
A = C^T(D_1 + D_2 + \cdots + D_r)C = C^T D_1 C + C^T D_2 C + \cdots + C^T D_r C
$$

其中对称矩阵 D_i 经过合同变换后的 $C^T D_i C$ 仍然是秩为 1 的对称矩阵.

得证: 秩等于 r 的对称矩阵可以表示成 r 格秩等于 1 的对称矩阵之和.