Chapter 3

A Little Bit of Model Theory

Interpretations of $\mathcal{ALC}$ can be viewed as graphs
(with labeled edges and nodes).

- We introduce the notion of bisimulation between graphs/interpretations
- We show that $\mathcal{ALC}$-concepts cannot distinguish bisimular nodes
- We use this to show restrictions of the expressive power of $\mathcal{ALC}$
- We use this to show interesting properties of models for $\mathcal{ALC}$:
  - tree model property
  - closure under disjoint union
- We show the finite model property of $\mathcal{ALC}$.

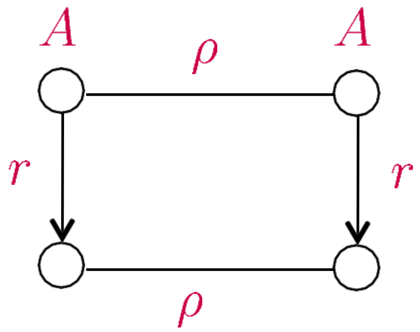
Section 3.1: Bisimulation

Definition 3.1 (bisimulation)

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations.

The relation $\rho \subseteq \Delta^{\mathcal{I}_1} \times \Delta^{\mathcal{I}_2}$ is a bisimulation between $\mathcal{I}_1$ and $\mathcal{I}_2$ iff

- $d_1 \; \rho \; d_2$ implies $d_1 \in A^{\mathcal{I}_1}$ iff $d_2 \in A^{\mathcal{I}_2}$ for all $A \in \mathbf{C}$
- $d_1 \; \rho \; d_2$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$ implies the existence of $d_2' \in \Delta^{\mathcal{I}_2}$ such that $d_1' \; \rho \; d_2'$ and $(d_2, d_2') \in r^{\mathcal{I}_2}$ for all $r \in \mathbf{R}$
- $d_1 \; \rho \; d_2$ and $(d_2, d_2') \in r^{\mathcal{I}_2}$ implies the existence of $d_1' \in \Delta^{\mathcal{I}_1}$ such that $d_1' \; \rho \; d_2'$ and $(d_1, d_1') \in r^{\mathcal{I}_1}$ for all $r \in \mathbf{R}$



Note:

- $\mathcal{I}_1 = \mathcal{I}_2$ is possible
- the empty relation $\emptyset$ is a bisimulation.

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations and $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$.

$(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$ iff there is a bisimulation ρ between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $d_1 \rho d_2$

"d_1 in $\mathcal{I}_1$ is bisimilar to d_2 in $\mathcal{I}_2$"

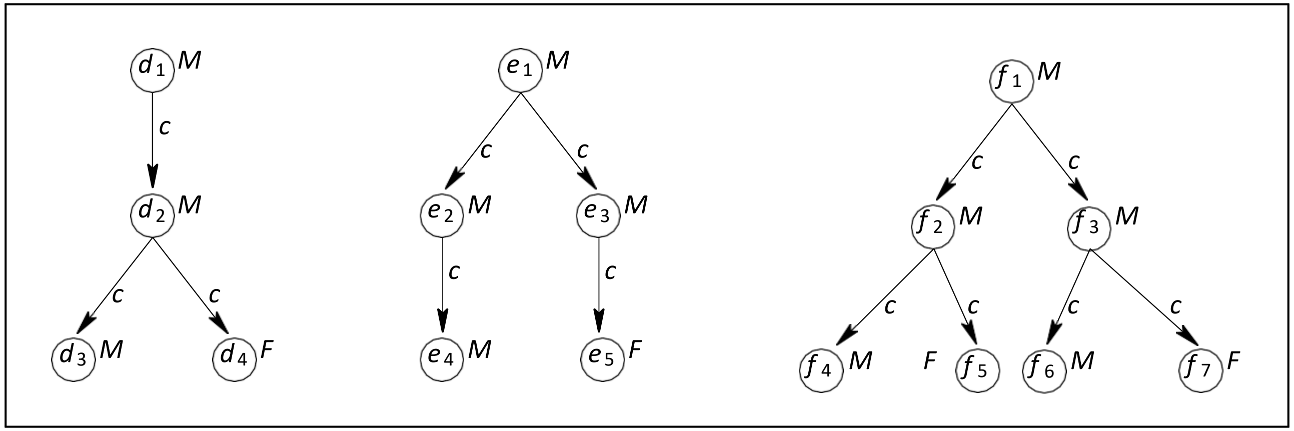


Fig. 3.1. Three interpretations I_1, I_2, I_3 represented as graphs

$(d_1, \mathcal{I}_1) \sim (f_1, \mathcal{I}_3)$ $(d_1, \mathcal{I}_1) \not\sim (e_1, \mathcal{I}_2)$

Let $\mathcal{I}_1$ and $\mathcal{I}_2$ be interpretations and $d_1 \in \Delta^{\mathcal{I}_1}$, $d_2 \in \Delta^{\mathcal{I}_2}$.

$(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$ iff there is a bisimulation ρ between $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $d_1 \rho d_2$

"d_1 in $\mathcal{I}_1$ is bisimilar to d_2 in $\mathcal{I}_2$"

Theorem 3.2 (bisimulation invariance of $\mathcal{ALC}$)

If $(\mathcal{I}_1, d_1) \sim (\mathcal{I}_2, d_2)$, then the following holds for all $\mathcal{ALC}$-concepts C:

$$\boxed{d_1 \in C^{\mathcal{I}_1} \quad \text{iff} \quad d_2 \in C^{\mathcal{I}_2}}$$

"$\mathcal{ALC}$-concepts cannot distinguish between bisimilar elements."

Proof: blackboard

Section 3.2: Expressive power

We have introduced extensions of $\mathcal{ALC}$ by the concept constructors number restrictions, nominals and the role constructor inverse role.

How can we show that these constructors really extend $\mathcal{ALC}$,
i.e., that they cannot be expressed using the constructors of $\mathcal{ALC}$?

To this purpose, we show that, using any of these constructors,
we can construct concept descriptions

- that cannot be expressed by $\mathcal{ALC}$-concept descriptions,
- i.e, there is no equivalent $\mathcal{ALC}$-concept description.

Expressive power of $\mathcal{ALC}$

Proposition 3.3 ($\mathcal{ALCN}$ is more expressive than $\mathcal{ALC}$)

> No $\mathcal{ALC}$-concept description is equivalent to the $\mathcal{ALCN}$-concept description $(\leq 1r)$.

Expressive power of $\mathcal{ALC}$

Proposition 3.4 ($\mathcal{ALCI}$ is more expressive than $\mathcal{ALC}$)

> No $\mathcal{ALC}$-concept description is equivalent to the $\mathcal{ALCI}$-concept description $\exists r^{-}.\top$.

Expressive power of $\mathcal{ALC}$

Proposition 3.5 ($\mathcal{ALCO}$ is more expressive than $\mathcal{ALC}$)

> No $\mathcal{ALC}$-concept description is equivalent to the $\mathcal{ALCO}$-concept description $\{a\}$.

Section 3.3: Closure under disjoint union

Definition 3.6

Let $\mathfrak{N}$ be an index set and $(\mathcal{I}_\nu)_{\nu\in\mathfrak{N}}$ a family of interpretations $\mathcal{I}_\nu = (\Delta^{\mathcal{I}_\nu}, \cdot^{\mathcal{I}_\nu})$.

Their disjoint union $\mathcal{J}$ is defined as follows:

$$\begin{array}{rcl}
\Delta^{\mathcal{J}} & = & \{(d,\nu) \mid \nu \in \mathfrak{N} \text{ and } d \in \Delta^{\mathcal{I}_\nu}\}; \\
A^{\mathcal{J}} & = & \{(d,\nu) \mid \nu \in \mathfrak{N} \text{ and } d \in A^{\mathcal{I}_\nu}\} \text{ for all } A \in \mathbf{C}; \\
r^{\mathcal{J}} & = & \{((d,\nu),(e,\nu)) \mid \nu \in \mathfrak{N} \text{ and } (d,e) \in r^{\mathcal{I}_\nu}\} \text{ for all } r \in \mathbf{R}.
\end{array}$$

Notation: $\mathcal{J} = \biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$

Example: $\mathfrak{N} = \{1,2\}$

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\end{array}
$$

Lemma 3.7

For $\nu \in \mathfrak{N}$, all $\mathcal{ALC}$-concept descriptions C, and all $d \in \Delta^{\mathcal{I}_\nu}$ we have

$$d \in C^{\mathcal{I}_\nu} \text{ iff } (d,\nu) \in C^{\mathcal{J}}$$

Proof: blackboard

Section 3.3: Closure under disjoint union

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Theorem 3.8

Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and $(\mathcal{I}_\nu)_{\nu\in\mathfrak{N}}$ a family of models of $\mathcal{T}$.

Then its disjoint union $\mathcal{J} = \biguplus_{\nu\in\mathfrak{N}} \mathcal{I}_\nu$ is also a model of $\mathcal{T}$.

Section 3.3: Closure under disjoint union

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\end{array}
$$

Corollary 3.9

Let $\mathcal{T}$ be an $\mathcal{ALC}$ TBox and C an $\mathcal{ALC}$ concept that is satisfiable w.r.t. $\mathcal{T}$.

Then there is a model $\mathcal{J}$ of $\mathcal{T}$ in which the extension $C^{\mathcal{J}}$ of C is infinite.

Section 3.4: Finite model property

Definition 3.10 (finite model)

The interpretation $\mathcal{I}$ is a model of a concept C w.r.t. a TBox $\mathcal{T}$ if $\mathcal{I}$ is a model of $\mathcal{T}$ such that $C^{\mathcal{I}} \neq \emptyset$.

We call this model finite if $\Delta^{\mathcal{I}}$ is finite.

Finite model property of $\mathcal{ALC}$:

> If $\mathcal{T}$ is an $\mathcal{ALC}$-TBox and C an $\mathcal{ALC}$-concept description such that C is satisfiable w.r.t. $\mathcal{T}$, then C has a finite model w.r.t. $\mathcal{T}$.

Proof first requires some definitions and auxiliary results.

Size

of $\mathcal{ALC}$-concepts

- $C = A \in \mathbf{C}$: $\mathsf{size}(C) := 1$;
- $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$: $\mathsf{size}(C) := 1 + \mathsf{size}(C_1) + \mathsf{size}(C_2)$;
- $C = \neg D$ or $C = \exists r.D$ or $C = \forall r.D$: $\mathsf{size}(C) := 1 + \mathsf{size}(D)$.

$$\mathsf{size}(A \sqcap \exists r.(A \sqcup B)) = 1 + 1 + (1 + (1 + 1 + 1)) = 6$$

Counts the occurrences of concept names, role names, and Boolean operators.

$$\boxed{\mathsf{size}(\mathcal{T}) := \sum_{C \sqsubseteq D \in \mathcal{T}} \mathsf{size}(C) + \mathsf{size}(D)}$$

Subconcepts

of $\mathcal{ALC}$-concepts

- $C = A \in \mathbf{C}$: $\mathsf{sub}(C) := \{A\}$;
- $C = C_1 \sqcap C_2$ or $C = C_1 \sqcup C_2$: $\mathsf{sub}(C) := \{C\} \cup \mathsf{sub}(C_1) \cup \mathsf{sub}(C_2)$;
- $C = \neg D$ or $C = \exists r.D$ or $C = \forall r.D$: $\mathsf{sub}(C) := \{C\} \cup \mathsf{sub}(D)$.

$\mathsf{sub}(A \sqcap \exists r.(A \sqcup B))$

$$\mathsf{sub}(\mathcal{T}) := \bigcup_{C \sqsubseteq D \in \mathcal{T}} \mathsf{sub}(C) \cup \mathsf{sub}(D)$$

Lemma 3.11

$|\,\mathsf{sub}(C)| \leq \mathsf{size}(C)$ and $|\,\mathsf{sub}(\mathcal{T})| \leq \mathsf{size}(\mathcal{T})$.

Type

of an element of a model

Definition 3.12 (S-type)

Let S be a finite set of concept descriptions, and $\mathcal{I}$ an interpretation.

The S-type of $d \in \Delta^{\mathcal{I}}$ is defined as

$$t_S(d) := \{C \in S \mid d \in C^{\mathcal{I}}\}.$$

Lemma 3.13 (number of S-types)

$$\boxed{|\{t_S(d) \mid d \in \Delta^{\mathcal{I}}\}| \leq 2^{|S|}}$$

Proof: obvious

Filtration

create a model in which every S-type is realized by at most one element

Definition 3.14 (S-filtration)

Let S be a finite set of concept descriptions, and $\mathcal{I}$ an interpretation.

We define an equivalence relation $\simeq$ on $\Delta^{\mathcal{I}}$ as follows:

$$d \simeq e \text{ iff } t_S(d) = t_S(e)$$

The $\simeq$-equivalence class of $d \in \Delta^{\mathcal{I}}$ is denoted by $[d]$.

The S-filtration of $\mathcal{I}$ is the following interpretation $\mathcal{J}$:

- $\Delta^{\mathcal{J}} := \{[d] \mid d \in \Delta^{\mathcal{I}}\}$
- $A^{\mathcal{J}} := \{[d] \mid \exists d' \in [d].\ d' \in A^{\mathcal{I}}\}$ for all $A \in \mathbf{C}$
- $r^{\mathcal{J}} := \{([d],[e]) \mid \exists d' \in [d], e' \in [e].\ (d', e') \in r^{\mathcal{I}}\}$ for all $r \in \mathbf{R}$

By Lemma 3.13, $|\Delta^{\mathcal{J}}| \leq 2^{|S|}$.

Filtration

important property

We say that the finite set S of concept descriptions is closed iff

$$\bigcup\{\mathsf{sub}(C) \mid C \in S\} \subseteq S$$

Lemma 3.15

Let S be a finite, closed set of $\mathcal{ALC}$-concept descriptions, $\mathcal{I}$ an interpretation, and $\mathcal{J}$ the S-filtration of $\mathcal{I}$. Then we have

$$\boxed{d \in C^{\mathcal{I}} \quad \text{iff} \quad [d] \in C^{\mathcal{J}}}$$

for all $d \in \Delta^{\mathcal{I}}$ and $C \in S$.

The following proposition shows that $\mathcal{ALC}$ satisfies a property that is even stronger than the finite model property.

Theorem 3.16 (bounded model property)

Let $\mathcal{T}$ be an $\mathcal{ALC}$-TBox, C an $\mathcal{ALC}$-concept description, and $n = \mathsf{size}(\mathcal{T}) + \mathsf{size}(C)$.

> If C has a model w.r.t. $\mathcal{T}$, then it has a model $\widehat{\mathcal{I}}$ such that $|\Delta^{\widehat{\mathcal{I}}}| \leq 2^n$.

Proof: let $\mathcal{I}$ be a model of $\mathcal{T}$ with $C^{\mathcal{I}} \neq \emptyset$, and $\widehat{\mathcal{I}}$ be the S-filtration of $\mathcal{I}$, where $S := \mathsf{sub}(C) \cup \mathsf{sub}(\mathcal{T})$.

We must show:

- $|\Delta^{\widehat{\mathcal{I}}}| \leq 2^n$ — Lemma 3.11 and Lemma 3.13
- $C^{\widehat{\mathcal{I}}} \neq \emptyset$
- $\widehat{\mathcal{I}}$ is a model of $\mathcal{T}$

(the last two) follow from Lemma 3.15

The following proposition shows that $\mathcal{ALC}$ satisfies a property that is even stronger than the finite model property.

Theorem 3.16 (bounded model property)

Let $\mathcal{T}$ be an $\mathcal{ALC}$-TBox, C an $\mathcal{ALC}$-concept description, and $n = \mathsf{size}(\mathcal{T}) + \mathsf{size}(C)$.

> If C has a model w.r.t. $\mathcal{T}$, then it has a model $\widehat{\mathcal{I}}$ such that $|\Delta^{\widehat{\mathcal{I}}}| \leq 2^n$.

Corollary 3.17 (Finite model property)

Let $\mathcal{T}$ be an $\mathcal{ALC}$-TBox and C an $\mathcal{ALC}$-concept description

If C has a model w.r.t. $\mathcal{T}$, then it has a finite model.

Corollary 3.18 (Decidability)

In $\mathcal{ALC}$, satisfiability of a concept description w.r.t. a TBox is decidable.

No finite model property

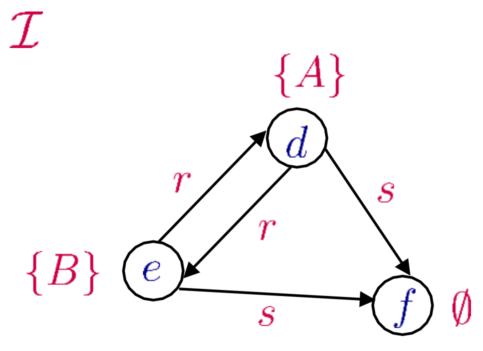
Theorem 3.19 (no finite model property)

$\mathcal{ALCIN}$ does not have the finite model property.

Section 3.5: Tree model property

Recall that interpretations can be viewed as graphs:

- nodes are the elements of $\Delta^{\mathcal{I}}$;
- interpretation of role names yields edges;
- interpretation of concept names yields node labels.



model of

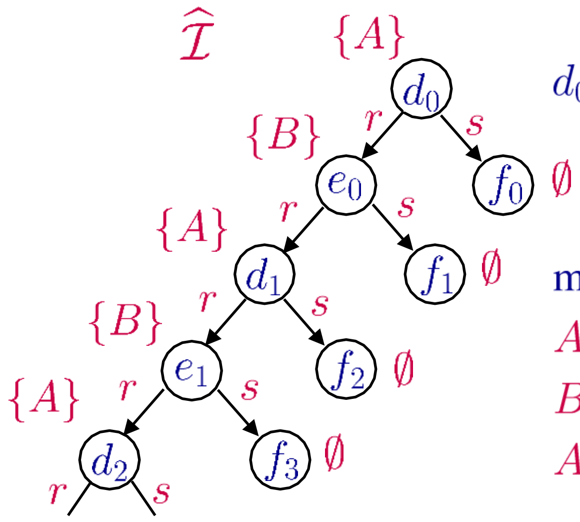
$A \sqsubseteq \exists r.B$

$B \sqsubseteq \exists r.A$

$A \sqcup B \sqsubseteq \exists s.\top$

$d \in A^{\mathcal{I}}$

Starting with a given node, the graph can be unraveled into a tree without "changing membership" in concepts.



$d_0 \in A^{\widehat{\mathcal{I}}}$

model of

$A \sqsubseteq \exists r.B$

$B \sqsubseteq \exists r.A$

$A \sqcup B \sqsubseteq \exists s.\top$

Definition 3.20 (Tree model)

Let $\mathcal{T}$ be a TBox and C a concept description.

The interpretation $\mathcal{I}$ is a tree model of C w.r.t. $\mathcal{T}$ iff $\mathcal{I}$ is a model of $\mathcal{T}$, and the graph

$$\mathcal{G}_{\mathcal{I}} = (\Delta^{\mathcal{I}}, \bigcup_{r \in \mathbf{R}} r^{\mathcal{I}})$$

is a tree whose root belongs to $C^{\mathcal{I}}$.

Goal: Show that every $\mathcal{ALC}$-concept that is satisfiable w.r.t. $\mathcal{T}$ has a tree model w.r.t. $\mathcal{T}$.

Unraveling

more formally

Let $\mathcal{I}$ be an interpretation and $d \in \Delta^{\mathcal{I}}$.

A d-path in $\mathcal{I}$ is a finite sequence $p = d_0, d_1, \ldots, d_{n-1}$ of $n \geq 1$ elements of $\Delta^{\mathcal{I}}$ such that

- $d_0 = d$,
- for all $i, 1 \leq i < n$, there is a role $r_i \in \mathbf{R}$ such that $(d_{i-1}, d_i) \in r_i^{\mathcal{I}}$.

$n =$ length of this path $\qquad$ $\mathsf{end}(p) = d_{n-1}$ end node of this path

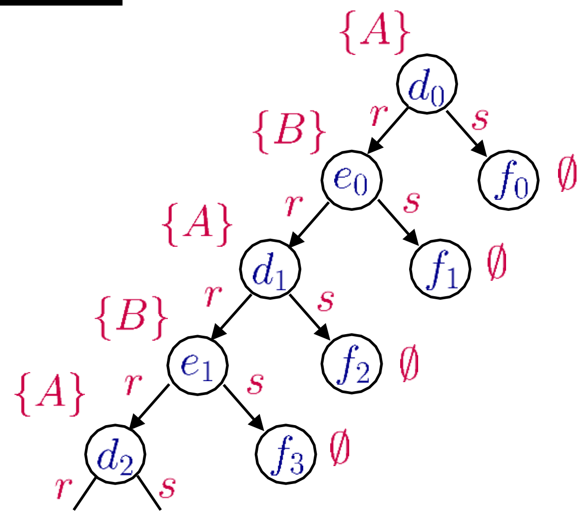
Definition 3.21 (Unraveling)

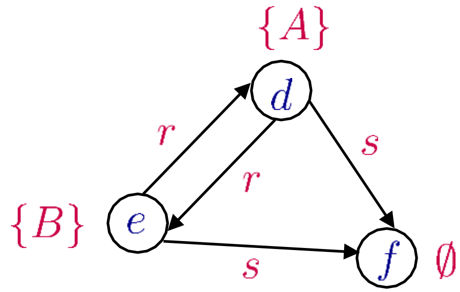
The unravelling of $\mathcal{I}$ at d is the following interpretation $\mathcal{J}$:

$$
\begin{aligned}
\Delta^{\mathcal{J}} &= \{p \mid p \text{ is a } d\text{-path in } \mathcal{I}\}, \\
A^{\mathcal{J}} &= \{p \in \Delta^{\mathcal{J}} \mid \mathsf{end}(p) \in A^{\mathcal{I}}\} \text{ for all } A \in \mathbf{C}, \\
r^{\mathcal{J}} &= \{(p, p') \in \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}} \mid p' = (p, \mathsf{end}(p')) \text{ and } (\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}\} \\
&\quad\ \text{for all } r \in \mathbf{R}.
\end{aligned}
$$

Unraveling

example





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The unravelling of $\mathcal{I}$ at d is the following interpretation $\mathcal{J}$:

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r^{\mathcal{J}} &= \{(p, p') \in \Delta^{\mathcal{J}} \times \Delta^{\mathcal{J}} \mid p' = (p, \mathsf{end}(p')) \text{ and } (\mathsf{end}(p), \mathsf{end}(p')) \in r^{\mathcal{I}}\} \\
&\quad \text{for all } r \in \mathbf{R}.
\end{aligned}
$$

Lemma 3.22

The relation

$$\rho = \{(p, \mathsf{end}(p)) \mid p \in \Delta^{\mathcal{J}}\}$$

is a bisimulation between $\mathcal{J}$ and $\mathcal{I}$.

Proposition 3.23

For all $\mathcal{ALC}$ concepts C and all $p \in \Delta^{\mathcal{J}}$ we have

$$p \in C^{\mathcal{J}} \text{ iff } \mathsf{end}(p) \in C^{\mathcal{I}}.$$

Theorem 3.24 (tree model property)

$\mathcal{ALC}$ has the tree model property,

i.e., if $\mathcal{T}$ is an $\mathcal{ALC}$-TBox and C an $\mathcal{ALC}$-concept description such that C is satisfiable w.r.t. $\mathcal{T}$, then C has a tree model w.r.t. $\mathcal{T}$.

Proposition 3.25 (no tree model property)

$\mathcal{ALCO}$ does not have the tree model property.

Proof:

The concept $\{a\}$ does not have a tree model w.r.t. $\{\{a\} \sqsubseteq \exists r.\{a\}\}$.