Representing Graphs
and
Graph Traversal

Data Structures and Algorithms

Nanjing University, Fall 2021
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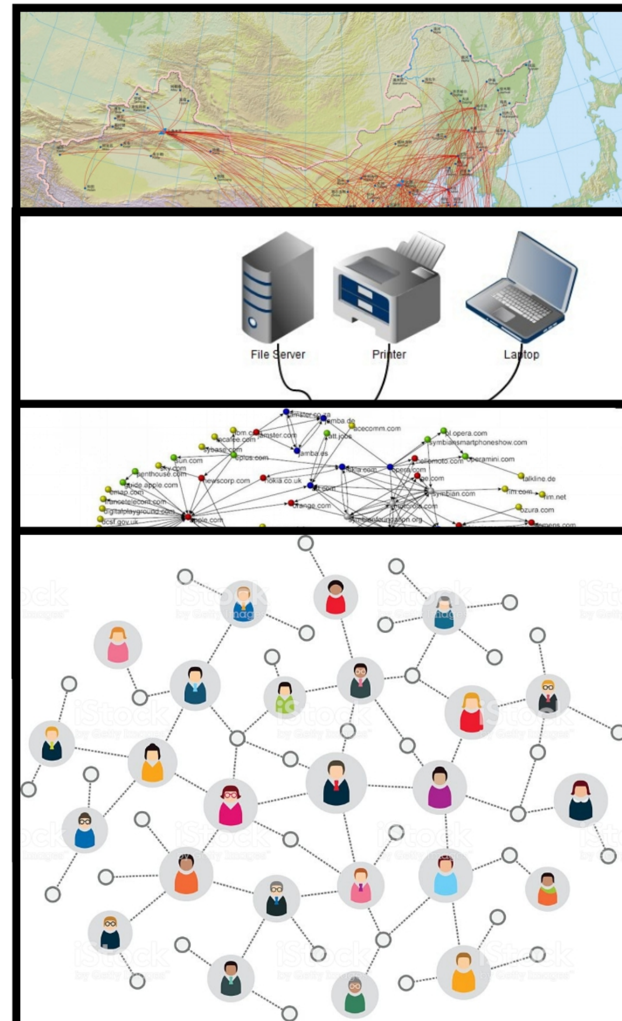
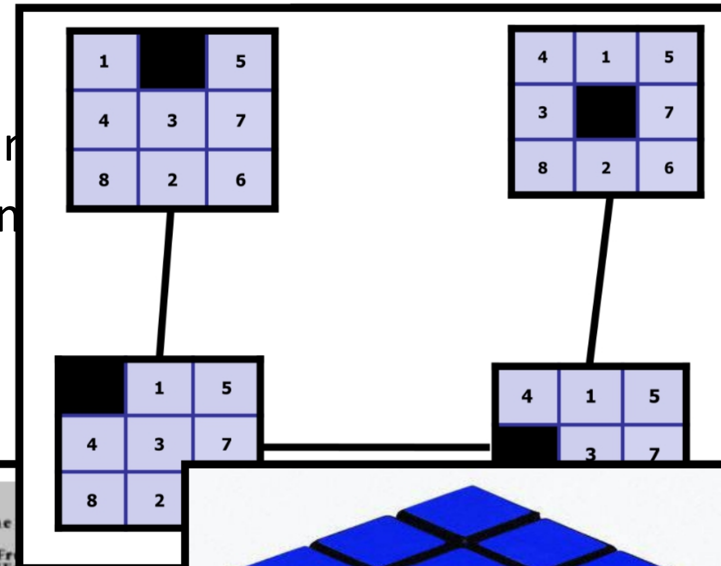
Graphs are Everywhere!

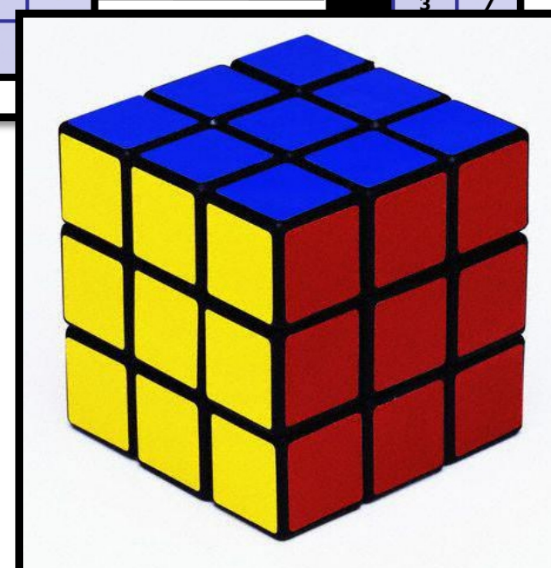
- Transportation Networks.
  - **Nodes:** Airports; **Edges:** Nonstop flights.
- Communication Networks.
  - **Nodes:** Computers; **Edges:** Physical links.
- Information Networks.
  - **Nodes:** Webpages; **Edges:** Hyperlinks.
- Social Networks.
  - **Nodes:** People; **Edges:** Friendship.
- …



Graphs are Everywhere! Really!

- Coloring Maps.
  - **Nodes:** Countries; **Edges:** Neighborir
  - **Question of Interest:** Chromatic nun
- Scheduling Exams.
  - **Nodes:** Exams; **Edges:** Conflicts.
  - **Question of Interest:**
- Solving Sliding Puz
  - **Nodes:** States; **Edges**
  - **Question of Interest:**
- Solving Rubik's Cu
  - **Nodes:** States; **Edges**
  - **Question of Interest:**





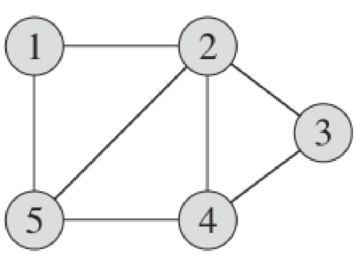
Representing graphs in computers

Adjacency Matrix

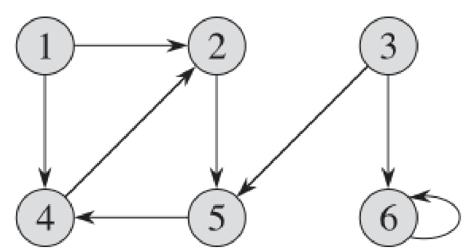
- Consider a graph $G = (V, E)$ where $|V| = n$ and $|E| = m$
- The **Adjacency Matrix** of G is an $n \times n$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

- The matrix will be symmetry if G is undirected.
- The matrix will always cost $\Theta(n^2)$ memory, regardless of m.
- **Quick Question:** What does A^2 mean, if anything?



$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 & 1 & 0 \\ \hline \end{array}$$

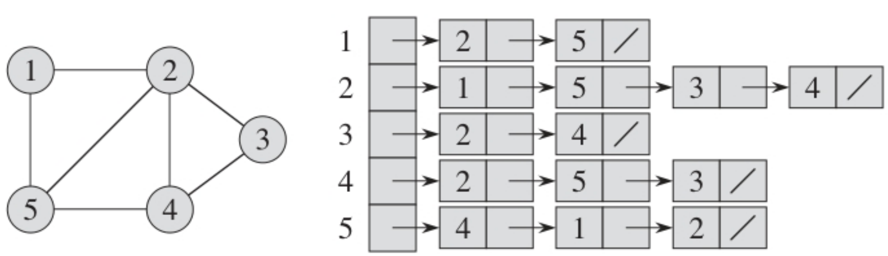


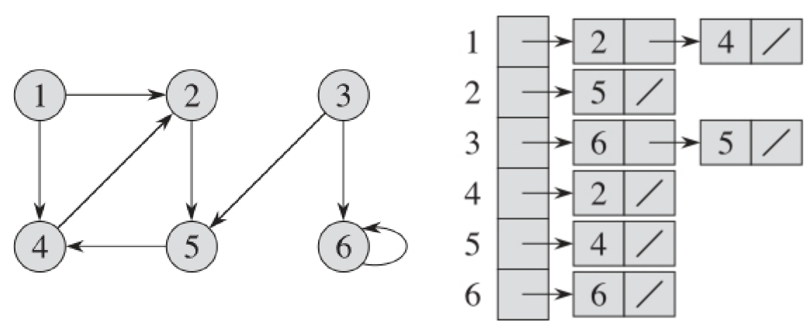
$$\begin{array}{c|cccccc|} & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 & 1 \\ 4 & 0 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 1 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline \end{array}$$

Representing graphs in computers

Adjacency List

- Consider a graph $G = (V, E)$ where $|V| = n$ and $|E| = m$
- The **Adjacency List** of G is a collection of n lists:
  - One for each vertex $u \in V$
  - In the list for u, vertex v exists iff edge $(u, v) \in E$
- Each edge appears twice if G is undirected.
- The space cost is $\Theta(n + m)$.





Adjacency Matrix and Adjacency List

Trade-offs

- Adjacency Matrix
  - **Fast Query:** Are u and v neighbors?
  - **Slow Query:** Find me any neighbor of u.
  - **Slow Query:** Enumerate all neighbors of u.
- Adjacency List
  - **Fast Query:** Find me any neighbor of u.
  - **Fast Query:** Enumerate all neighbors of u.
  - **Slow Query:** Are u and v neighbors?
- Important question to ask:
  - **Queries:** What types of queries are needed and/or frequent?
  - **Space usage:** Is the graph dense or sparse?

Searching in a Graph (or, Graph Traversal)

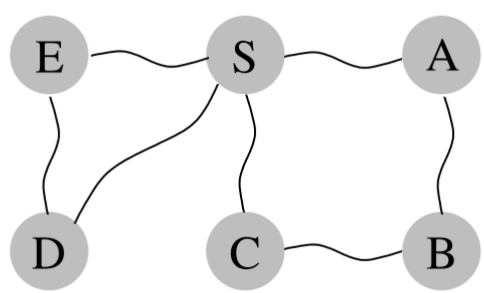
- **Goal:** Start at node s and find some node t.
- **Or:** Visit all nodes reachable from s.
- Two Basic Strategies:
  - **Breath-First Search** (**BFS**)
  - **Depth-First Search** (**DFS**)
- Many applications, beside searching and traversal!
- Usually use adjacency list when discussing BFS/DFS.
  (At least in this course...)

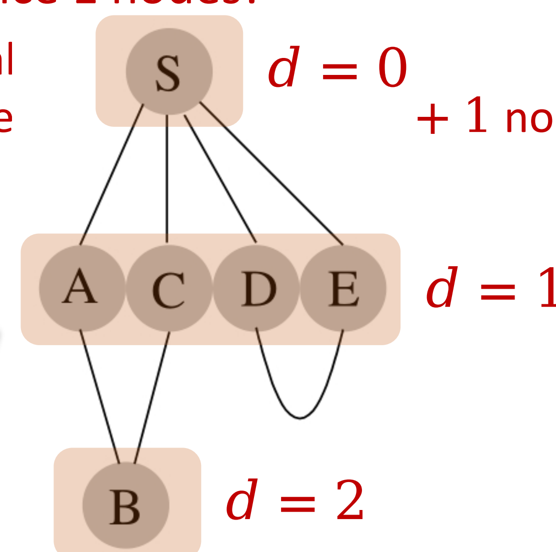
Breath-First Search (BFS)

- Basic Idea of BFS:
  - Start at the source node s;
  - Visit other nodes (reachable from s) "*layer by layer*".
- A (somewhat) more precise description:
  - Start at the source node s;
  - Visit nodes at *distance* 1 from s;
  - Visit nodes at *distance* 2 from s;
  - …

These nodes are neighbors of distance 1 nodes!

Visit al
before





$+ 1$ node.

BFS Implementa

BFSSkeleton(G,s):

```
for (each u in V)
  u.dist=INF, u.visited=false
s.dist = 0
Q.enque(s)
while (!Q.empty())
  u = Q.dequeue()
  u.visited = true
  for (each edge (u,v) in E)
    if (!v.visited)
      v.dist = u.dist+1
      Q.enque(v)
```

- How to implement BFS? (Hint: re
- Use a FIFO Queue!

- Nodes have 3 status:
  - **Undiscovered** (WHITE): Not in queue yet.
  - **Discovered but not visited** (GRAY): In queue but not processed.
  - **Visited** (BLACK): Ejected from queue and processed.

- We can "store" a shortest path, instead of only the length of the path.

BFS Implementation

<u>BFS(G,s):</u>

```
for (each u in V)
  u.c = WHITE, u.d = INF, u.p = NIL
s.c = GRAY, s.d = 0, s.p = NIL
Q.enque(s)
while (!Q.empty())
  u = Q.dequeue()
  u.c = BLACK
  for (each edge (u,v) in E)
    if (v.c == WHITE)
      v.c = GRAY
      v.d = u.d+1
      v.p = u
      Q.enque(v)
```

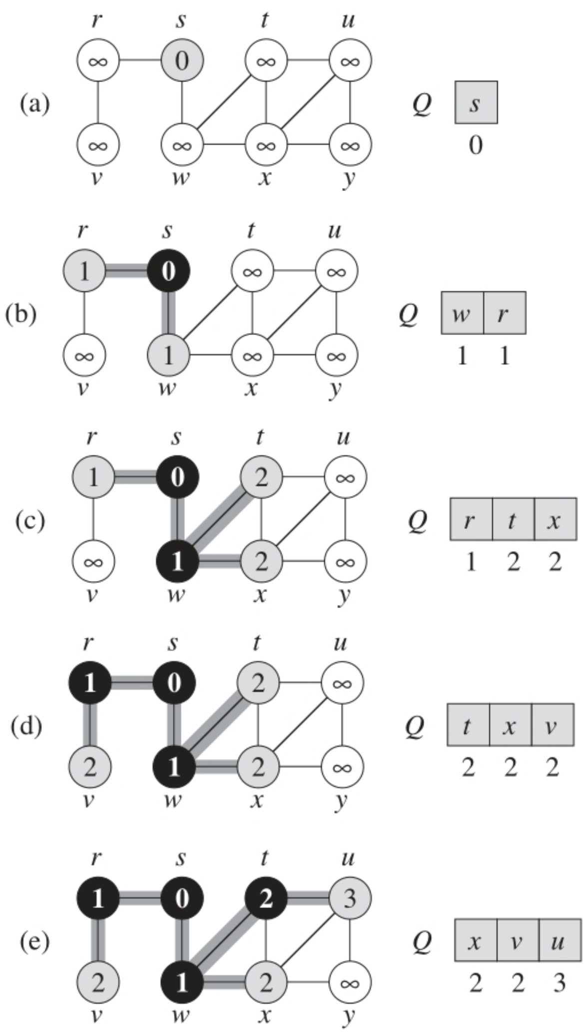
Sample Executio

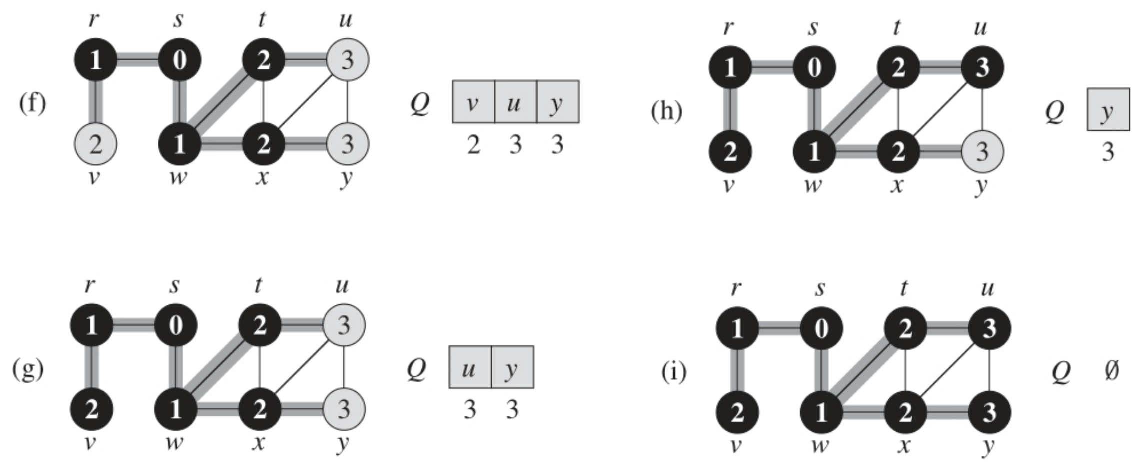
BFS(G,s):

```
for (each u in V)
  u.c=WHITE, u.d=INF, u.p=NIL
s.c=GRAY, s.d=0, s.p=NIL
Q.enque(s)
while (!Q.empty())
  u = Q.dequeue()
  u.c = BLACK
  for (each edge (u,v) in E) {
    if (v.c == WHITE)
      v.c = GRAY
      v.d = u.d+1
      v.p = u
      Q.enque(v) }
```

"else" clause?

first discovery (preprocessing)





Performance of BFS

- Runtime of BFS?
  (Assuming G is connected.)
- "while" loop $\Theta(n)$ times.
  - Each node in Q at most once.
- "for" loop $\Theta(m)$ times.
  - Each edge visited at most once or twice.
- Runtime of BFS is $\Theta(n + m)$.

BFS(G,s):

```
for (each u in V)
  u.c=WHITE, u.d=INF, u.p=NIL
s.c=GRAY, s.d=0, s.p=NIL
Q.enque(s)
while (!Q.empty())
  u = Q.dequeue()
  u.c = BLACK
  for (each edge (u,v) in E)
    if (v.c == WHITE)
      v.c = GRAY
      v.d = u.d+1
      v.p = u
      Q.enque(v)
```

- What if we use adjacency matrix instead of adjacency list?

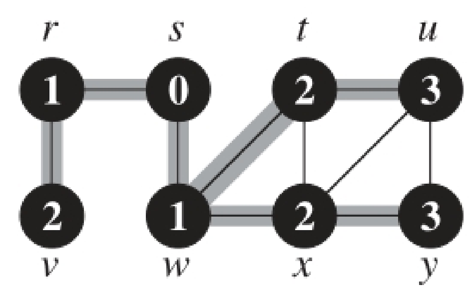
Correctness and Properties of BFS

- **Theorem:** BFS visits a node iff it is reachable from s.
- **Proof:**
- [*only if*] If a node is not reachable from s, then BFS does not visit it, since BFS only moves along edges.
- [*if*] If a node is reachable from s, then BFS visits it.
  - **Claim:** For all $k \geq 0$, all nodes within k hops of s are visited.
  - [*Basis*]: Clearly s is visited.
  - [*Hypothesis*]: All nodes within $k-1$ hops of s are visited.
  - [*Inductive Step*]: Consider a node v that is k hops away from s.
    Let u be v's neighbor on (one of) v's shortest path back to s.
    By induction hypothesis, u gets visited.
    When BFS visits u, node v is already GRAY or BLACK, or will be put in Q.
    Either way, v eventually gets visited.

Will this really happen?!

Correctness and Properties of BFS

- **Theorem:** BFS correctly computes $u.dist$, for every node u that is reachable from s.
- [*Proof Idea*] Use induction to show:
  for all $d \geq 0$, there is a moment at which:
  (**a**) every node u with $dist(s, u) \leq d$ correctly computes $u.dist$;
  (**b**) every other node v has $v.dist = \infty$;
  (**c**) Q contains exactly the nodes d hops away from s.
- **Corollary:** For any $u \neq s$ that is reachable from s, one of the shortest path from s to u is a shortest path from s to $u.p$ followed by the edge $(u.p, u)$
- $G_p = (V_p, E_p)$ is a spanning tree of the component containing s. Here:
  $V_p = \{u \in V : u.p \neq NIL\} \cup \{s\}$,
  $E_p = \{(u.p, u) : u \in V_p - \{s\}\}$.



One last note on BFS

- What if the graph is not connected?
- Easy, do a BFS for each connected component!

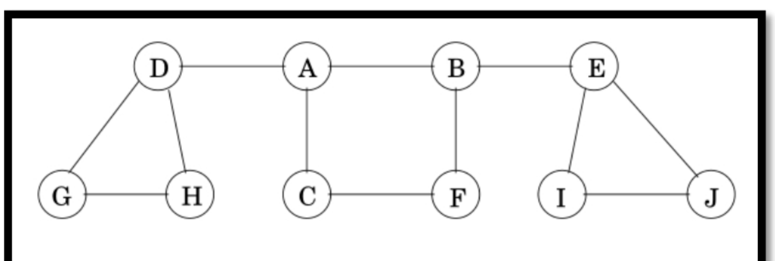
BFS(G):

```
for (each u in V)
  u.c = WHITE, u.d = INF, u.p = NIL
for (each u in V)
  if (u.c == WHITE)
    u.c = GRAY, u.d = 0, u.p = NIL
    Q.enque(u)
    while (!Q.empty())
      v = Q.dequeue()
      v.c = BLACK
      for (each edge (v,w) in E)
        if (w.c == WHITE)
          w.c = GRAY
          w.d = v.d+1
          w.p = v
          Q.enque(w)
```

Runtime of this procedure?

Depth-First Search (DFS)

- Much like exploring a maze:
  - Use a ball of string and a piece of chalk.
  - Follow path (unwind string and mark at intersections), until stuck (reach dead-end or already-visited place).
  - Backtrack (rewind string), until find unexplored neighbo
  - Repeat above two steps.
- How to do this for a graph, in
  - Chalk: boolean variables.
  - String: a stack.

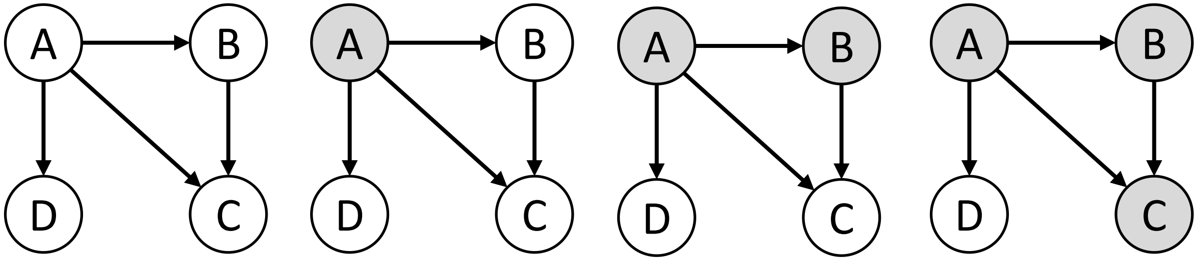


DFSSkeleton(G,s):

```
s.visited = true
for (each edge (s,v) in E)
  if (!v.visited)
    DFSSkelecton(G,v)
```

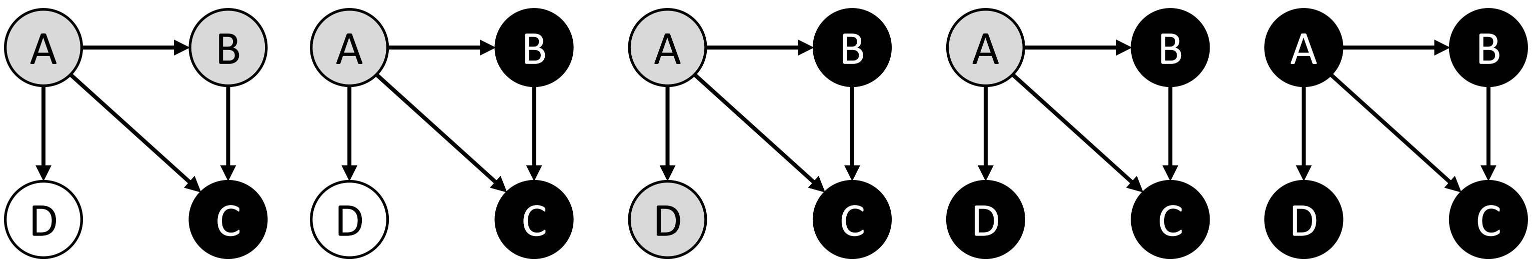
DFSIterSkeleton(G,s):

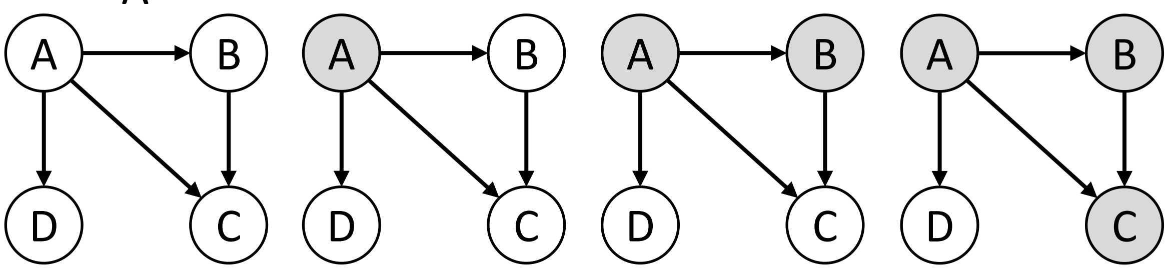
```
Stack Q
Q.push(s)
while (!Q.empty())
  u = Q.pop()
  if (!u.visited)
    u.visited = true
    for (each edge (u,v) in E)
      Q.push(v)
```



DFSSkeleton(G,s):

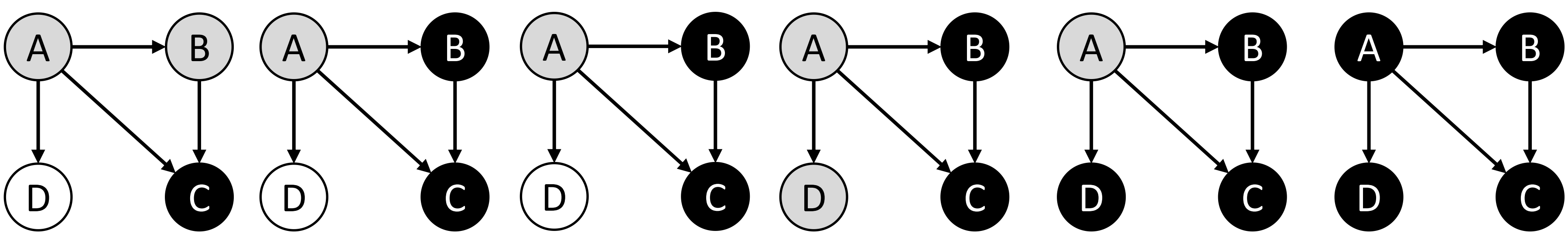
```
s.visited = true
for (each edge (s,v) in E)
  if (!v.visited)
    DFSSkelecton(G,v)
```





DFSIterSkeleton(G,s):

```
Stack Q
Q.push(s)
while (!Q.empty())
  u = Q.pop()
  if (!u.visited)
    u.visited = true
    for (each edge (u,v) in E)
      Q.push(v)
```



Depth-First Search (DFS)

- **Q**: What if the graph is not (strongly) connected?
- **A**: Do DFS from multiple sources.

DFSAll(G):

```
for (each node u)
  u.visited = false
for (each node u)
  if (u.visited == false)
    DFSSkeleton(G,u)
```

DFSSkeleton(G,s):

```
s.visited = true
for (each edge (s,v) in E)
  if (!v.visited)
    DFSSkelecton(G,v)
```

DFSAll(G):

```
for (each node u)
  u.visited = false
for (each node u)
  if (u.visited == false)
    DFSIterSkeleton(G,u)
```

DFSIterSkeleton(G,s):

```
Stack Q
Q.push(s)
while (!Q.empty())
  u = Q.pop()
  if (!u.visited)
    u.visited = true
    for (each edge (u,v) in E)
      Q.push(v)
```

Depth-First Search (DFS)

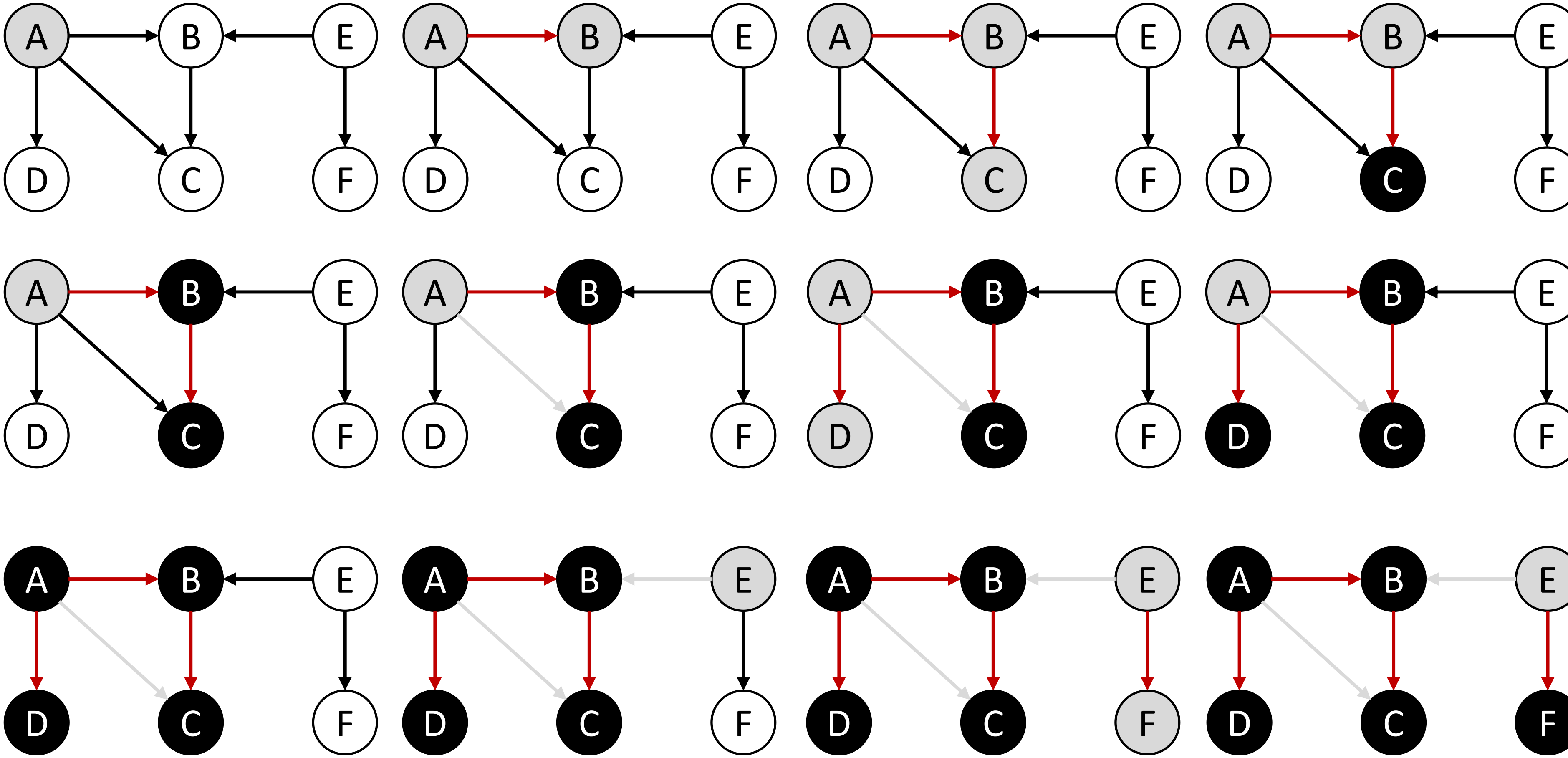
- Each node u have 3 status during DFS:
  - **Undiscovered [**WHITE**]**: before calling `DFSSkeleton(G,u)`
  - **Discovered [**GRAY**]**: during execution of `DFSSkeleton(G,u)`
  - **Finished [**BLACK**]**: `DFSSkeleton(G,u)` returned
- `DFS(G,u)` builds a tree among nodes reachable from u:
  - Root of this tree is u.
  - For each non-root, its parent is the node that makes it turn GRAY.
- DFS on entire graph builds a forest.

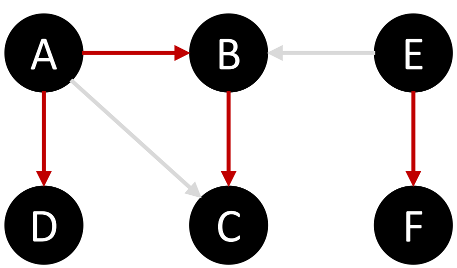
DFSAll(G):

```
for (each node u)
  u.color = WHITE
  u.parent = NIL
for (each node u)
  if (u.color == WHITE)
    DFS(G,u)
```

DFS(G,s):

```
s.color = GRAY
for (each edge (s,v) in E)
  if (v.color == WHITE)
    v.parent = s
    DFS(G,v)
s.color = BLACK
```





DFSAll(G):

```
for (each node u)
  u.color = WHITE
  u.parent = NIL
for (each node u)
  if (u.color == WHITE)
    DFS(G,u)
```

DFS(G,s):

```
s.color = GRAY
for (each edge (s,v) in E)
  if (v.color == WHITE)
    v.parent = s
    DFS(G,v)
s.color = BLACK
```

Depth-First Search (DFS)

- DFS provides (at least) two chances to process each node:
  - **Pre-Visit**: WHITE -> GRAY
  - **Post-Visit**: GRAY -> BLACK
- Sample application: Track active intervals of nodes
  - Clock ticks whenever some node's color changes.
  - **Discovery time**: when the node turn GRAY.
  - **Finish time**: when the node turn BLACK.

PreProcess(G):

```
time = 0
```

DFSAll(G):

```
PreProcess(G)
for (each node u)
  u.color = WHITE
  u.parent = NIL
for (each node u)
  if (u.color == WHITE)
    DFS(G,u)
```

DFS(G,s):

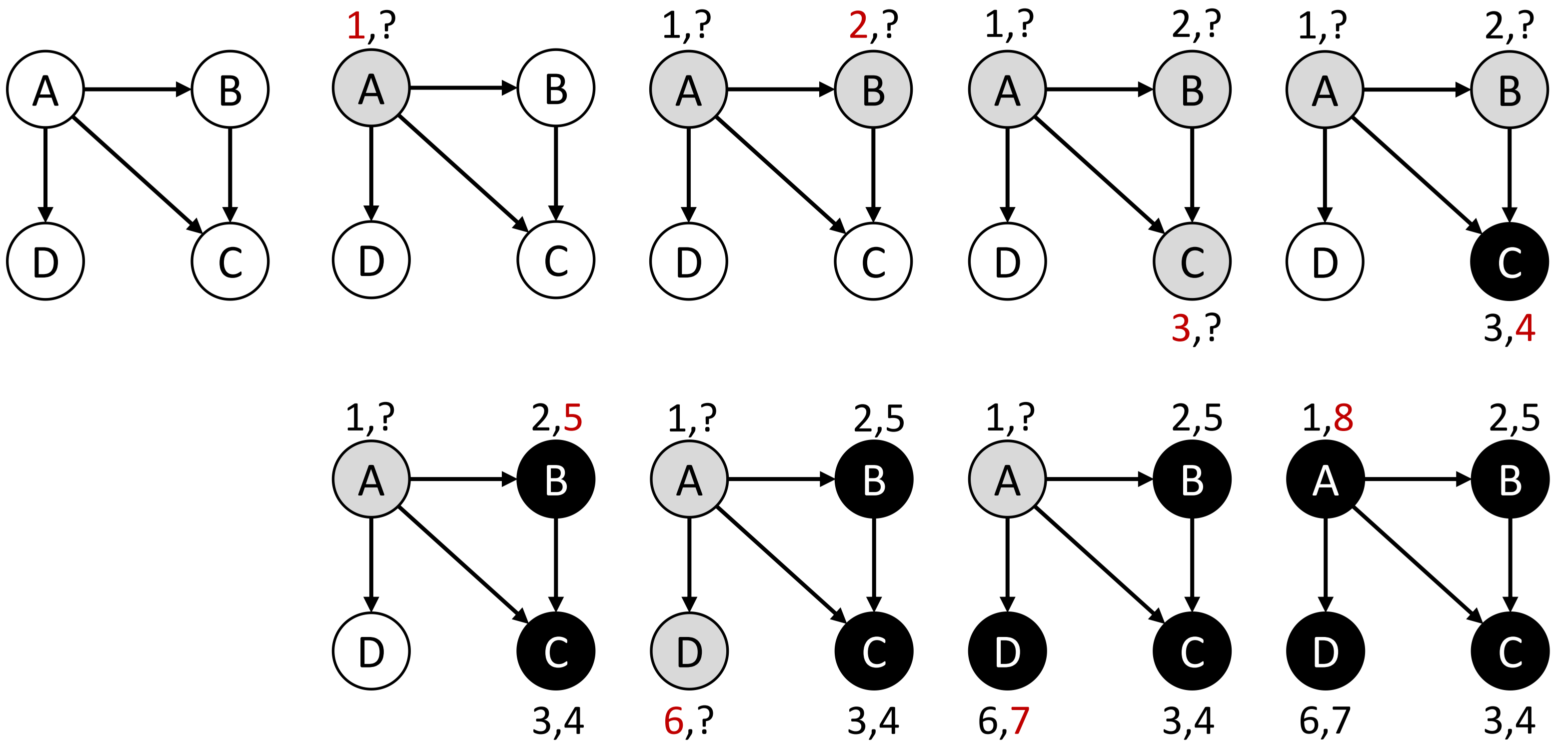
```
PreVisit(s)
s.color = GRAY
for (each edge (s,v) in E)
  if (v.color == WHITE)
    v.parent = s
    DFS(G,v)
s.color = BLACK
PostVisit(s)
```

PreVisit(s):

```
time = time+1
s.d = time
```

PostVisit(s):

```
time = time+1
s.f = time
```



PreProcess(G):

```
time = 0
```

DFSAll(G):

```
PreProcess(G)
for (each node u)
  u.color = WHITE
  u.parent = NIL
for (each node u)
  if (u.color == WHITE)
    DFS(G,u)
```

DFS(G,s):

```
PreVisit(s)
s.color = GRAY
for (each edge (s,v) in E)
  if (v.color == WHITE)
    v.parent = s
    DFS(G,v)
s.color = BLACK
PostVisit(s)
```

PreVisit(s):

```
time = time+1
s.d = time
```

PostVisit(s):

```
time = time+1
s.f = time
```

Runtime of DFS

- Time spent on each node: $O(1)$
  - `DFS(G,u)` is called once for each node u.
- Time spent on each edge: $O(1)$
  - Each edge is examined $O(1)$ times.
- Total runtime: $O(n+m)$

PreProcess(G):

```
time = 0
```

DFSAll(G):

```
PreProcess(G)
for (each node u)
  u.color = WHITE
  u.parent = NIL
for (each node u)
  if (u.color == WHITE)
    DFS(G,u)
```

DFS(G,s):

```
PreVisit(s)
s.color = GRAY
for (each edge (s,v) in E)
  if (v.color == WHITE)
    v.parent = s
    DFS(G,v)
s.color = BLACK
PostVisit(s)
```

PreVisit(s):

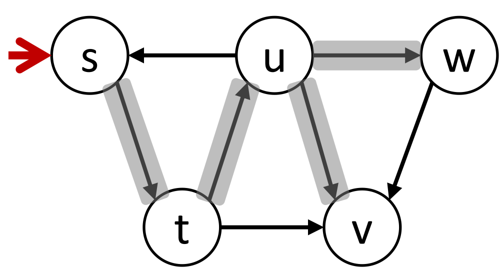
```
time = time+1
s.d = time
```

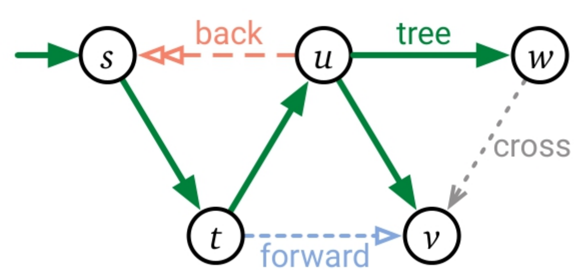
PostVisit(s):

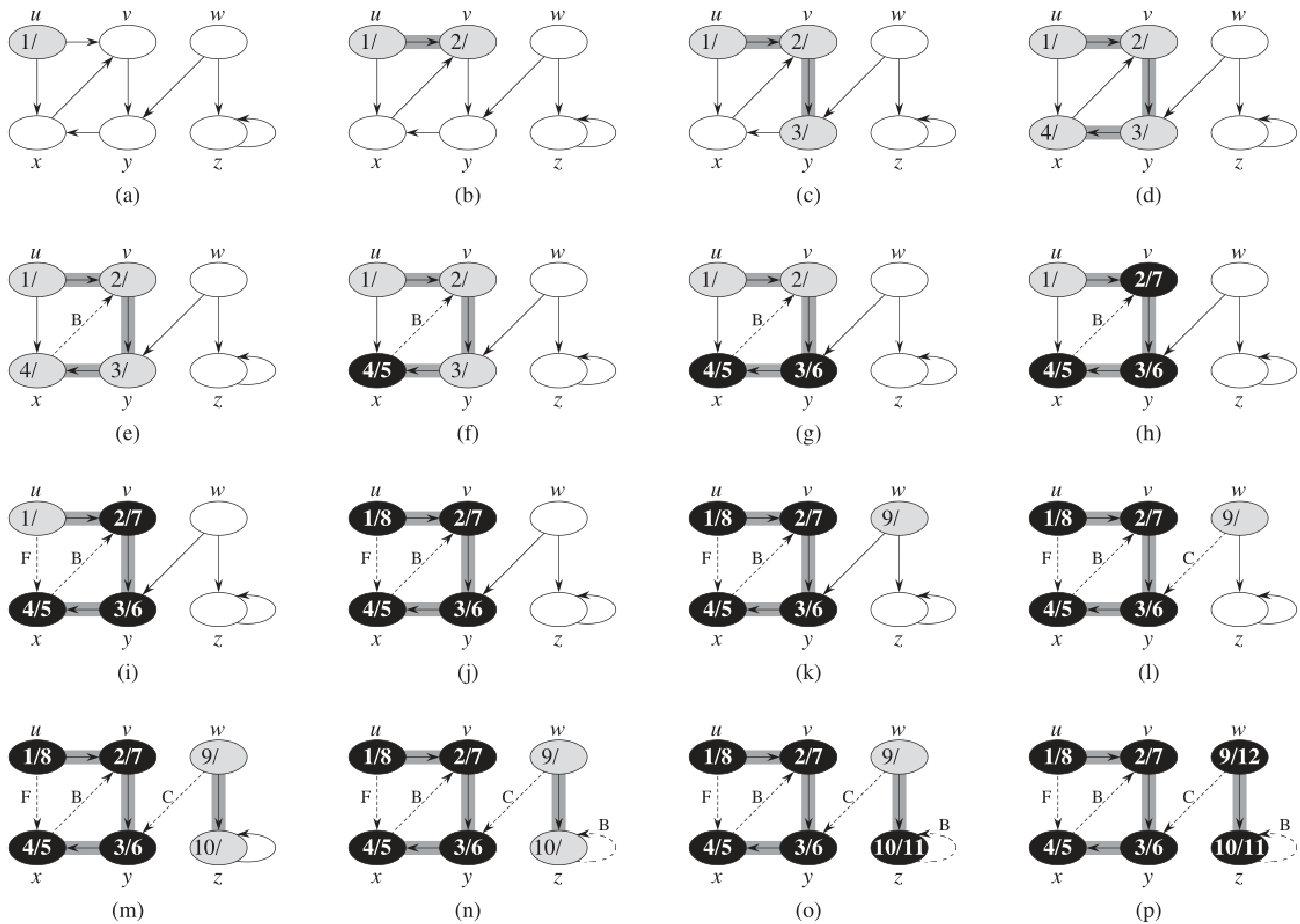
```
time = time+1
s.f = time
```

Classification of edges

- DFS process classify edges of input graph into **four** types.
- **Tree Edges**: Edges in the DFS forest.
- **Back Edges**: Edges (u, v) connecting u to an ancestor v in a DFS tree.
- **Forward Edges**:
  Non-tree edges (u, v) connecting u to a descendant v in a DFS tree.
- **Cross Edges:**
  Other edges. (Connecting nodes in same DFS tree with no ancestor-descendant relation, or connecting nodes in different DFS trees.)







u
v
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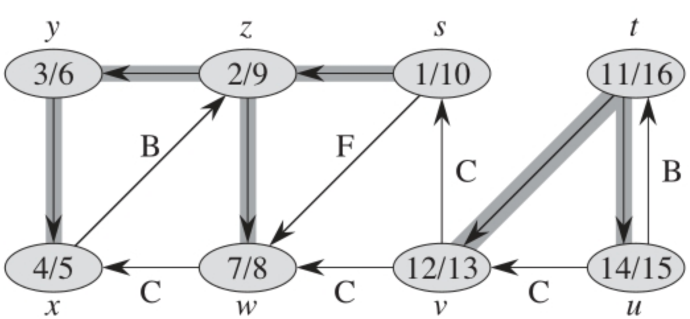
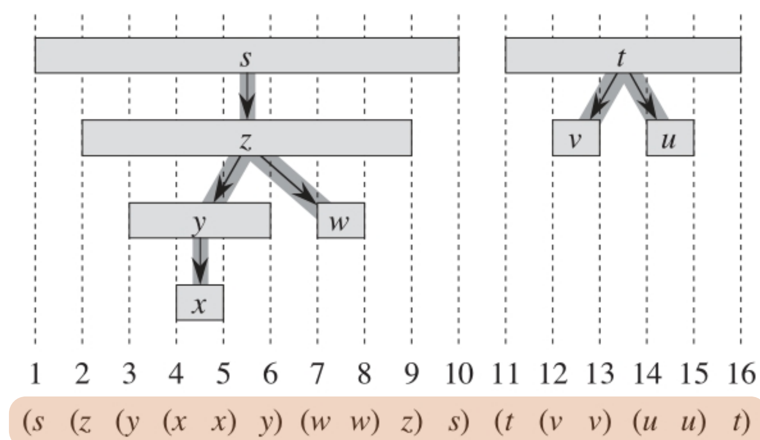
(n)
(o)
(p)

Parenthesis Theorem

- Active intervals of two nodes are either: (**a**) entirely disjoint; or (**b**) one is entirely contained within another.
- For any two nodes u and v, _exactly_ one of following holds:
  (**a**) $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint, and u, v have no ancestor-descendant relation in the DFS forest;
  (**b**) $[u.d, u.f] \subset [v.d, v.f]$, and u is a descendant of v in a DFS tree;
  (**c**) $[v.d, v.f] \subset [u.d, u.f]$, and





Properties of DFS

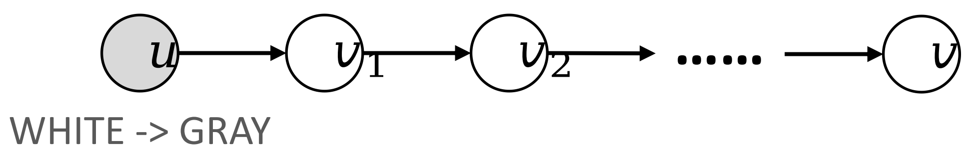
Parenthesis Theorem

- For any two nodes u and v, *exactly* one of following holds:
  (**a**) $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint, and u, v have no ancestor-descendant relation in the DFS forest;
  (**b**) $[u.d, u.f] \subset [v.d, v.f]$, and u is a descendant of v in a DFS tree;
  (**c**) $[v.d, v.f] \subset [u.d, u.f]$, and u is an ancestor of v in a DFS tree.
- **Proof**: Consider two nodes u and v. W.l.o.g., assume $u.d < v.d$.
- If $v.d < u.f$, then v is discovered (WHITE->GRAY) while u is being processed (GRAY); and DFS will finish v first, before returning to u.
- In this case, $[v.d, v.f] \subset [u.d, u.f]$, and u is an ancestor of v.
- If $v.d > u.f$, then obviously $u.d < u.f < v.d < v.f$; and DFS has finished exploring u (BLACK), before v is discovered (WHITE->GRAY).
- In this case, $[u.d, u.f]$ and $[v.d, v.f]$ are disjoint, and u, v have no ancestor-descendant relation.

Properties of DFS

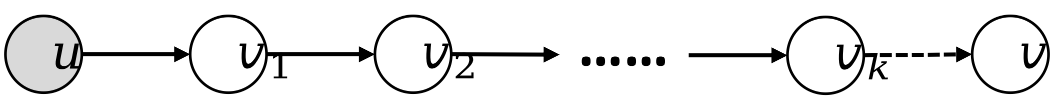
White-path Theorem

- **Thm**: In the DFS forest, v is a descendant of u iff when u is discovered, there is a path from u to v containing only WHITE nodes.
- **Proof of [==>]:**
- If $v = u$, then [==>] direction trivially holds.
- If v is a proper descendant of u, then $u.d < v.d$.
  Claim: If v is a proper descendant of u, then v is WHITE when u is discovered.
- For any node along the path from u to v in the DFS forest, above claim holds.
- Therefore, [==>] direction of theorem holds.



Properties of DFS
White-path Theorem

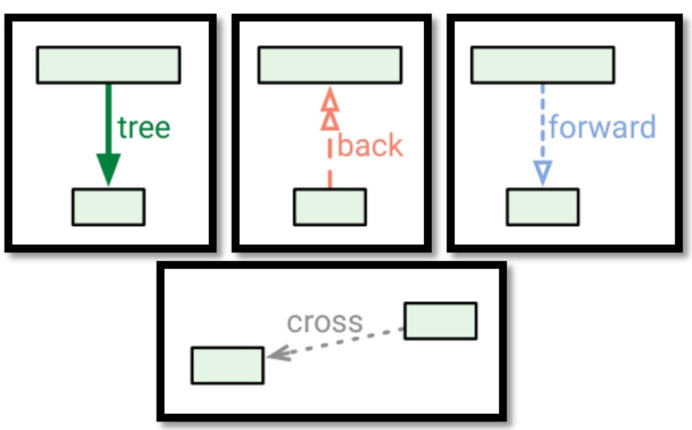
- **Thm**: In the DFS forest, v is a descendant of u iff when u is discovered, there is a path from u to v containing only WHITE nodes.
- **Proof of [<==]:**
- W.l.o.g., assume v is the *first* node along the path that does *not* become a descendant of u.
- So we have $[v_k.d, v_k.f] \subset [u.d, u.f]$.
- But v is discovered after u is discovered, and before v_k is finished.
- So we have $u.d < v.d < v_k.f \leq u.f$.
- Then it must be $[v.d, v.f] \subset [u.d, u.f]$, implying v is a descendant of u.

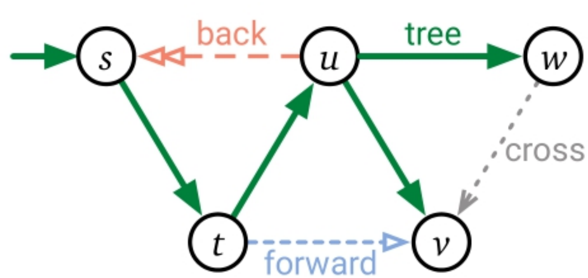


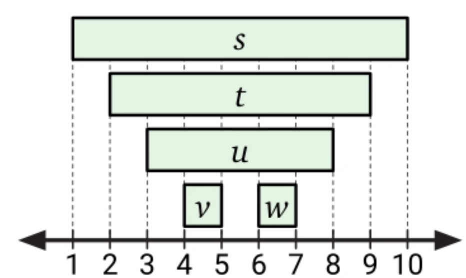
Properties of DFS

Classification of edges

- Determine (u, v) type by color of v during DFS execution.
- **Tree Edges**: Node v is WHITE
  Edges in the DFS forest.
- **Back Edges**: Node v is GRAY
  Edges (u, v) connecting u to an ancestor v in a DFS tree.
- **Forward Edges**: Node v is BLACK
  Non-tree edges (u, v) connecting u to a descendant v in a DFS tree.
- **Cross Edges**: Node v is BLACK
  Other edges. (Connecting nodes in same DFS tree with no ancestor-descendant relation, or connecting nodes in different DFS trees.)







Properties of DFS

Types of edges in undirected graphs

- Will all four types of edges appear in DFS of undirected graphs?
- **Thm:** In DFS of an *undirected* graph G, every edge of G is either a *tree edge* or a *back edge*.
- **Proof:**
- Consider an arbitrary edge (u, v). W.l.o.g., assume $u.d < v.d$.
- Edge (u, v) must be explored while u is GRAY.
- Consider the first time the edge (u, v) is explored.
- If the direction is $u \to v$. Then, v must be WHITE by then, for otherwise the edge would have been explored from direction $v \to u$ earlier.
- In such case, the edge (u, v) becomes a **tree edge**.
- If the direction is $v \to u$. Then, the edge is “GRAY -> GRAY”.
- In such case, the edge (u, v) becomes a **back edge**.

DFS, BFS, and others...

DFSIterSkeleton(G,s):

```
Stack Q
Q.push(s)
while (!Q.empty())
  u = Q.pop()
  if (!u.visited)
    u.visited = true
    for (each edge (u,v) in E)
      Q.push(v)
```

BFSSkeletonAlt(G,s):

```
FIFOQueue Q
Q.enque(s)
while (!Q.empty())
  u = Q.dequeue()
  if (!u.visited)
    u.visited = true
    for (each edge (u,v) in E)
      Q.enque(v)
```

GraphExploreSkeleton(G,s):

```
GenericQueue Q
Q.add(s)
while (!Q.empty())
  u = Q.remove()
  if (!u.visited)
    u.visited = true
    for (each edge (u,v) in E)
      Q.add(v)
```

Other queuing disciplines lead to more interesting algorithms!

Reading

- [CLRS] Ch.22 (22.1-22.3)