高等代数作业

作业

在 P^4 中, $V_1 = L(\alpha_1, \alpha_2, \alpha_3), V_2 = L(\beta_1, \beta_2, \beta_3)$

其中 $\alpha_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ 3 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ -2 \end{pmatrix}, \beta_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -6 \end{pmatrix}, \beta_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \\ 4 \end{pmatrix}, \beta_3 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ -5 \end{pmatrix}$

分别求 $V_1 + V_2, V_1 \cap V_2$ 的一个基和维数.

解:

对交空间:

任取 $\gamma \in L(\alpha_1, \alpha_2, \alpha_3) \cap L(\beta_1, \beta_2, \beta_3)$, 设 $\gamma = x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = -y_1\beta_1 - y_2\beta_2 - y_3\beta_3$

则有 $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 + y_1\beta_1 + y_2\beta_2 + y_3\beta_3 = 0$

$$\therefore \begin{cases} x_1 + x_2 + x_3 + y_1 + y_2 + 2y_3 = 0 \\ x_1 + x_2 + 2x_3 + 2y_1 - 2y_2 + 3y_3 = 0 \\ -x_2 + x_3 + 2y_2 + y_3 = 0 \\ 2x_1 + 3x_2 - 2x_3 - 6y_1 + 4y_2 - 5y_3 = 0 \end{cases}$$

对和空间:

$$A=\begin{pmatrix}1&1&1&1&1&2\\1&1&2&2&-2&3\\0&-1&1&0&2&1\\2&3&-2&-6&4&-5\end{pmatrix}\xrightarrow[r_4-2r_1]{r_2-r_1}\begin{pmatrix}1&1&1&1&1&2\\0&0&1&1&-3&1\\0&-1&1&0&2&1\\0&1&-4&-8&2&-9\end{pmatrix}$$

$$\xrightarrow[r_3+r_2]{r_2\leftrightarrow r_4}\begin{pmatrix}1&1&1&1&1&2\\0&1&-4&-8&2&-9\\0&0&-3&-8&4&-8\\0&0&1&1&-3&1\end{pmatrix}\xrightarrow[r_4+3r_3]{r_3\leftrightarrow r_4}\begin{pmatrix}1&1&1&1&1&2\\0&1&-4&-8&2&-9\\0&0&1&1&-3&1\\0&0&0&-5&-5&-5\end{pmatrix}$$

$$\xrightarrow[r_1-r_2]{-\frac{1}{5}r_4}\begin{pmatrix}1&0&5&9&-1&11\\0&1&-4&-8&2&-9\\0&0&1&1&-3&1\\0&0&0&1&1&1\end{pmatrix}\xrightarrow[r_2+4r_3]{r_1-5r_3}\begin{pmatrix}1&0&0&4&14&6\\0&1&0&-4&-10&-5\\0&0&1&1&-3&1\\0&0&0&1&1&1\end{pmatrix}$$

$$\xrightarrow[r_2+4r_4]{r_1-4r_4}\begin{pmatrix}1&0&0&0&10&2\\0&1&0&0&-6&-1\\0&0&1&1&-3&1\\0&0&0&1&1&1\end{pmatrix}\xrightarrow{r_3-r_4}\begin{pmatrix}1&0&0&0&10&2\\0&1&0&0&-6&-1\\0&0&1&0&-4&0\\0&0&0&1&1&1\end{pmatrix}$$

$$\therefore\begin{cases}x_1=-10u-2v\\x_2=6u+v\\x_3=4u\\y_1=-u-v\\y_2=u\\y_3=v\end{cases}$$

$\therefore \gamma=(-10u-2v)\alpha_1+(6u+v)\alpha_2+4u\alpha_3=-(-u-v)\beta_1-u\beta_2-v\beta_3$

取 $\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$ 和 $\begin{pmatrix}u\\v\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}$ 得 $V_1\cap V_2$ 的一组基为

$$-10\alpha_1+6\alpha_2+4\alpha_3=\begin{pmatrix}0\\4\\1\\-2\\-10\end{pmatrix},-2\alpha_1+\alpha_2=\begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix}$$

$\therefore V_1\cap V_2=L(\gamma)$ 是二维的.

对和空间:

由交空间部分推导有

$$A=\begin{pmatrix}1&1&1&1&1&2\\1&1&2&2&-2&3\\0&-1&1&0&2&1\\2&3&-2&-6&4&-5\end{pmatrix}\to\begin{pmatrix}1&0&0&0&10&2\\0&1&0&0&-6&-1\\0&0&1&0&-4&0\\0&0&0&1&1&1\end{pmatrix}$$

可知 $\alpha_1,\alpha_2,\alpha_3,\beta_1$ 为 $\alpha_1,\alpha_2,\alpha_3,\beta_1,\beta_2,\beta_3$ 的一个极大无关组.

$\therefore V_1+V_2=L(\alpha_1,\alpha_2,\alpha_3,\beta_1)$ 为 4 维的, $\alpha_1,\alpha_2,\alpha_3,\beta_1$ 为其的一组基.

20.

设 $(\alpha_{11}+\alpha_{12})+\alpha_2=0$, 其中 $\alpha_{11}+\alpha_{12}\in V_1,\alpha_{11}\in V_{11},\alpha_{12}\in V_{12},\alpha_2\in V_2$

$\because V=V_1\oplus V_2$

$\therefore \alpha_{11}+\alpha_{12}=0,\alpha_2=0$

$\because V_1=V_{11}\oplus V_{12},\alpha_{11}\in V_{11},\alpha_{12}\in V_{12}$

$\therefore \alpha_{11}=\alpha_{12}=0$

$\therefore \alpha_{11}=\alpha_{12}=\alpha_2=0$

$\therefore V=V_{11}\oplus V_{12}\oplus V_2$

22.

必要性:

$$\because V_i\cap\sum_{j=1}^{i-1}V_j\subseteq V_i\cap\sum_{j\neq i}V_j=\{0\}$$

故必要性成立.

充分性:

设有零向量的一个分解 $(\alpha_1+\alpha_2+\cdots+\alpha_{s-1})+\alpha_s=0$,

其中 $\alpha_1+\alpha_2+\cdots+\alpha_{s-1}\in\sum_{j=1}^{s-1},\alpha_s\in V_s$.

$$\because V_s\cap\sum_{j=1}^{s-1}V_j=\{0\}$$

$\therefore V_s + \sum_{j=1}^{s-1} V_j$ 直和

$\therefore \alpha_1 + \alpha_2 + \cdots + \alpha_{s-1} = 0, \alpha_s = 0$

$\because V_i \cap \sum_{j=1}^{i-1} V_j = \{0\}, \quad (i = 2, \cdots, n)$

同理可依次推得 $\alpha_{s-1} = 0, \cdots, \alpha_2 = 0, \alpha_1 = 0$

$\therefore \sum_{i=1}^{s} V_i$ 是直和