# 概率统计第十次作业

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## 1.

令 $Y = X - \mu$, 而 $t > 0$, 则

$$
\begin{aligned}
P(X - \mu \leqslant -\epsilon) = P(Y \leqslant -\epsilon) = P(Y - t \leqslant -\epsilon - t) \\
&\leqslant P((Y-t)^2 \geqslant (\epsilon + t)^2) \\
&\leqslant \frac{E((Y-t)^2)}{(\epsilon+t)^2} = \frac{\mathrm{Var}(Y-t) + E(Y-t)^2}{(\epsilon+t)^2} \\
&= \frac{\sigma^2 + t^2}{(\epsilon+t)^2}
\end{aligned}
$$

为了令 $f(t) = \dfrac{\sigma^2 + t^2}{(\epsilon+t)^2}$ 取到最小值, 我们对 $t$ 求导得 $f'(t) = \dfrac{2(\epsilon t - \sigma^2)}{(\epsilon + t)^3}$

即当 $t = \dfrac{\sigma^2}{\epsilon}$ 时有最小值 $f(\dfrac{\sigma^2}{\epsilon}) = \dfrac{\sigma^2 + (\frac{\sigma^2}{\epsilon})^2}{(\epsilon + \frac{\sigma^2}{\epsilon})^2} = \dfrac{\sigma^2}{\epsilon^2 + \sigma^2}$

所以有 $P(X - \mu \leqslant -\epsilon) \leqslant \dfrac{\sigma^2}{\sigma^2 + \epsilon^2}$

## 2.

因为 $E(X+Y) = E(X) + E(Y) = -2 + 2 = 0$, 所以我们可以应用 Chebyshev 不等式 $P(|X+Y| \geqslant \epsilon) \leqslant \dfrac{\mathrm{Var}(X+Y)}{\epsilon^2}$

$\therefore P(|X+Y| \geqslant 6) \leqslant \dfrac{\mathrm{Var}(X+Y)}{6^2} = \dfrac{\mathrm{Var}(X) + \mathrm{Var}(Y) + 2\rho_{XY}\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}}{36} = \dfrac{1 + 4 + 2 \times (-\frac{1}{2}) \times \sqrt{1} \times \sqrt{4}}{36} = \dfrac{1}{12}$

## 3.

由 Chebyshev 不等式可知

$$P\left[\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right|\geqslant\epsilon\right]\leqslant\frac{1}{\epsilon^2}\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)$$

而由独立同分布和 $\mathrm{Var}(X_i)\leqslant v$ 又知

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n}X_i\right)=\frac{1}{n^2}\sum_{i=1}^{n}\mathrm{Var}(X_i)=\frac{v}{n}$$

所以有 $P\left[\left|\frac{1}{n}\sum_{i=1}^{n}X_i-\mu\right|\geqslant\epsilon\right]\leqslant\frac{v}{n\epsilon^2}$

## 4.

Chernoff 方法:

借助矩生成函数和 Markov 不等式, 给定任意随机变量和任意 $t>0$ 和 $\epsilon>0$, 有

$$P[X\geqslant E(X)+\epsilon]=P[e^{tX}\geqslant e^{tE(X)+t\epsilon}]\leqslant e^{-t\epsilon-tE(X)}E[e^{tX}]$$

特别地有

$$P[X-E(X)\geqslant\epsilon]\leqslant\min_{t>0}\{e^{-t\epsilon-tE(X)}E[e^{tX}]\}$$

## 5.

令 $\bar{X}=\sum_{i=1}^{n}X_i,\mu=\sum_{i=1}^{n}E[X_i]=\sum_{i=1}^{n}p_i$

$$\begin{aligned}P\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-E[X_i])\geqslant\epsilon\right]&=P\left[\sum_{i=1}^{n}X_i\geqslant n\epsilon+\sum_{i=1}^{n}E[X_i]\right]\\&=P\left[e^{t\bar{X}}\geqslant e^{tn\epsilon+t\mu}\right]\\&\leqslant e^{-tn\epsilon-t\mu}E[e^{t\bar{X}}]\end{aligned}$$

而用随机变量的独立性以及 $1+x\leqslant e^x$ 有

$$
\begin{aligned}
E[e^{t\bar{X}}] &= E[e^{\sum_{i=1}^{n} tX_i}] = \prod_{i=1}^{n} E[e^{tX_i}] \\
&= \prod_{i=1}^{n}[(1-p_i)\cdot 1 + p_i e^t] = \prod_{i=1}^{n}[1 + p_i(e^t - 1)] \\
&\leqslant e^{\sum_{i=1}^{n} p_i(e^t-1)} = e^{\mu(e^t-1)}
\end{aligned}
$$

因此 $P\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) \geqslant \epsilon\right] \leqslant e^{-tn\epsilon - t\mu + \mu(e^t-1)}$

令 $f(t) = -tn\epsilon - t\mu + \mu(e^t - 1)$, 则 $f'(t) = \mu e^t - \epsilon n - \mu$

则当 $t = \ln\frac{\epsilon n + \mu}{\mu}$ 时有最小值 $f(\ln\frac{\epsilon n + \mu}{\mu}) = -(n\epsilon + \mu)(\ln(\epsilon n + \mu) - \ln\mu) + \mu(\frac{\epsilon n + \mu}{\mu} - 1) = \epsilon n + (\epsilon n + \mu)\left[\ln\left(\mu\right) - \ln\left(\epsilon n + \mu\right)\right]$

因此 $P\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) \geqslant \epsilon\right] \leqslant e^{\epsilon n + (\epsilon n + \mu)[\ln(\mu) - \ln(\epsilon n + \mu)]}$

当 $t < 0$ 时, 同理有

$$
\begin{aligned}
P\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) \leqslant -\epsilon\right] &= P\left[\sum_{i=1}^{n} X_i \leqslant -n\epsilon + \sum_{i=1}^{n} E[X_i]\right] \\
&= P\left[e^{t\bar{X}} \geqslant e^{-tn\epsilon + t\mu}\right] \\
&\leqslant e^{tn\epsilon - t\mu} E[e^{t\bar{X}}]
\end{aligned}
$$

因此 $P\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) \geqslant \epsilon\right] \leqslant e^{tn\epsilon - t\mu + \mu(e^t-1)}$

令 $f(t) = tn\epsilon - t\mu + \mu(e^t - 1)$, 则 $f'(t) = \mu e^t + \epsilon n - \mu$

则当 $t = \ln\frac{\mu - \epsilon n}{\mu}$ 时有最小值 $f(\ln\frac{\mu - \epsilon n}{\mu}) = (n\epsilon - \mu)(\ln(\mu - \epsilon n) - \ln\mu) + \mu(\frac{\mu - \epsilon n}{\mu} - 1) = -\epsilon n - (\epsilon n - \mu)\left(\ln\left(\mu\right) - \ln\left(-\epsilon n + \mu\right)\right)$

因此 $P\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - E[X_i]) \geqslant \epsilon\right] \leqslant e^{-\epsilon n - (\epsilon n - \mu)[\ln(\mu) - \ln(-\epsilon n + \mu)]}$

# 6.

令 $c=\dfrac{b-a}{2}, Y=X_i-\dfrac{a+b}{2}$, 则 $P(Y_i=-c)=P(Y_i=c)=\dfrac{1}{2}$

则 $E[e^{tY}]=\dfrac{1}{2}e^{ct}+\dfrac{1}{2}e^{-ct}=e^c(\dfrac{1}{2}e^t+\dfrac{1}{2}e^{-t})\leqslant e^{\frac{ct^2}{2}}$

使用随机变量的独立同分布, 根据 Chernoff 方法有

$$
\begin{aligned}
P\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\frac{a+b}{2}\right)\geqslant\epsilon\right]=P\left[\frac{1}{n}\sum_{i=1}^{n}Y_i\geqslant\epsilon\right]\leqslant e^{-nt\epsilon}E\left[\exp\left(\sum_{i=1}^{n}tY_i\right)\right] \\
=e^{-nt\epsilon}\prod_{i=1}^{n}E[e^{tY_i}]\leqslant e^{-nt\epsilon+\frac{cnt^2}{2}}
\end{aligned}
$$

对上式右边求最小值解得 $t=\dfrac{\epsilon}{c}$, 带入上式可得

$$
P\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\frac{a+b}{2}\right)\geqslant\epsilon\right]\leqslant e^{-\frac{n\epsilon^2}{2c}}=e^2
$$

同理可有

$$
P\left[\frac{1}{n}\sum_{i=1}^{n}\left(X_i-\frac{a+b}{2}\right)\leqslant-\epsilon\right]\leqslant e^{-\frac{n\epsilon^2}{2c}}
$$

# 7.

令 $\bar{X}=\sum_{i=1}^{n}X_i, \mu=\sum_{i=1}^{n}E[X_i]=\sum_{i=1}^{n}p_i$, 由 Chernoff 方法有

$$
P\left[\bar{X}\geqslant(1+\epsilon)\mu\right]=P[e^{t\bar{X}}\geqslant e^{t(1+\epsilon)\mu}]\leqslant e^{-t(1+\epsilon)\mu}E[e^{t\bar{X}}]
$$

利用随机变量的独立性和 $1+x\leqslant e^x$ 有

$$
\begin{aligned}
E[e^{t\bar{X}}]&=E[e^{\sum_{i=1}^{n}tX_i}]=\prod_{i=1}^{n}E[e^{tX_i}] \\
&=\prod_{i=1}^{n}[(1-p_i)\cdot 1+p_ie^t]=\prod_{i=1}^{n}[1+p_i(e^t-1)] \\
&\leqslant e^{\sum_{i=1}^{n}p_i(e^t-1)}=e^{\mu(e^t-1)}
\end{aligned}
$$

由此可得

$$
P[\bar{X}\geqslant(1+\epsilon)\mu]\leqslant e^{-t(1+\epsilon)\mu+\mu(e^t-1)}
$$

对上式求最小值解可得 $t=\ln(1+\epsilon)$, 带入得

$$P[\bar{X}\geqslant(1+\epsilon)\mu]\leqslant\left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}\right)^{\mu}$$

则只需证明当 $\epsilon\in(0,1)$ 时, 有

$$f(\epsilon)=\ln\left(\frac{e^{\epsilon}}{(1+\epsilon)^{(1+\epsilon)}}\right)+\frac{\epsilon^2}{3}=\epsilon-(1+\epsilon)\ln(1+\epsilon)+\frac{\epsilon^2}{3}\leqslant 0$$

因此 $f'(\epsilon)=-\ln(1+\epsilon)+\dfrac{2\epsilon}{3}, f''(\epsilon)=-\dfrac{1}{1+\epsilon}+\dfrac{2}{3}$

可知 $f'(\epsilon)$ 在 $(0,\dfrac{1}{2})$ 递减, 在 $(\dfrac{1}{2},1)$ 递增, 则有 $f'(\epsilon)<f(1)<f(0)=0$

因此 $f(\epsilon)$ 单调递减, $f(\epsilon)<f(0)=0$ 成立.

综上 $P[\bar{X}\geqslant(1+\epsilon)\mu]\leqslant e^{-\frac{\mu\epsilon^2}{3}}$ 成立.

# 8.

令 $Y=\dfrac{X-a}{b-a},\mu'=E[Y]$, 则有 $Y\in[0,1]$, 并且 $X=(b-a)Y+a,\mu=E[X]=(b-a)\mu'+a$

所以 $E[e^{tY}]=E[e^{t((b-a)Y+a)}]=e^{at}E[e^{(b-a)tY}]$

由凸函数性质可知

$$E[e^{(b-a)tY}]=e^{(b-a)tY+(b-a)(1-Y)0}\leqslant Ye^{(b-a)t}+(1-Y)$$

两边同时取期望有

$$E(e^{tY})\leqslant 1-\mu'+\mu'e^{(b-a)t}=\exp(\ln(1-\mu'+\mu'e^{(b-a)t}))$$

令 $f(t)=\ln(1-\mu'+\mu'e^{(b-a)t})$, 我们有 $f(0)=0$ 以及

$$f'(t)=\frac{(b-a)\mu'e^{(b-a)t}}{1-\mu'+\mu'e^{(b-a)t}},f'(0)=(b-a)\mu'$$

进一步有

$$f''(t)=\frac{(b-a)^2\mu'e^{(b-a)t}}{1-\mu'+\mu'e^{(b-a)t}}-\frac{(b-a)^2\mu'^2e^{2(b-a)t}}{(1-\mu'+\mu'e^{(b-a)t})^2}\leqslant\frac{1}{4}(b-a)^2$$

根据泰勒中值定理有

$$f(t)=f(0)+tf'(0)+f''(\xi)\frac{t^2}{2}\leqslant(b-a)t\mu'+\frac{t^2(b-a)^2}{8}$$

因此 $E[X]\leqslant e^{at}\exp\left((b-a)t\mu'+\frac{t^2(b-a)^2}{8}\right)=\exp\left(\mu t+\frac{t^2(b-a)^2}{8}\right)$ 成立