P234. 4.(1) 7.(1)(2)(4) 8.(1) 11. 12. 13.

P234

4.(1)

$\Rightarrow$:

$\therefore A$ 为反称矩阵

$\therefore A=-A'$

$\therefore X'AX=(X'AX)'=X'A'X=-X'AX$

$\therefore X'AX=0$

$\Leftarrow$:

$\because \forall X, X'AX=0$

设 $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

不妨令 $X'=\begin{pmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$, 其中 1 为第 i 位

$\therefore X'AX=\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{pmatrix} X=a_{ii}=0$

$\therefore A$ 对角线上元素全为 0

再令 $X'=\begin{pmatrix} 0 & \cdots & 1 & \cdots & 1 & \cdots & 0 \end{pmatrix}$, 其中 1 为第 i 位和第 j 位

$\therefore X'AX=\begin{pmatrix} a_{i1}+a_{j1} & a_{i2}+a_{j2} & \cdots & a_{in}+a_{jn} \end{pmatrix} X=a_{ii}+a_{ji}+a_{ij}+a_{jj}=a_{ji}+a_{ij}=0$

$\therefore A$ 中对称的元素互为相反数

$\therefore A$ 为反称矩阵

7.

(1)

$\because \begin{vmatrix} 99 \end{vmatrix} = 99 > 0$

$\begin{vmatrix} 99 & -6 \\ -6 & 130 \end{vmatrix} = 12834 > 0$

$\begin{vmatrix} 99 & -6 & 24 \\ -6 & 130 & -30 \\ 24 & -30 & 71 \end{vmatrix} = 755874 > 0$

$\therefore$ 该二次型正定

(2)

$\because \begin{vmatrix} 10 \end{vmatrix} = 10 > 0$

$\begin{vmatrix} 10 & 4 \\ 4 & 2 \end{vmatrix} = 4 > 0$

$\begin{vmatrix} 10 & 4 & 12 \\ 4 & 2 & -14 \\ 12 & -14 & 1 \end{vmatrix} = -3588 < 0$

$\therefore$ 该二次型不正定

(4)

$$A_n = \begin{pmatrix} 1 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 1 \end{pmatrix}_n$$

当 $n = 1$ 时, $A_1 = |1| = 1$

当 $n = 2$ 时, $A_2 = \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{vmatrix} = \frac{3}{4}$

$$
\begin{aligned}
\because |A_n| &= \begin{vmatrix} 1 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 1 \end{vmatrix}_n \\
&= -\frac{1}{2} \begin{vmatrix} 1 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & 0 & \frac{1}{2} \end{vmatrix}_{n-1} + \begin{vmatrix} 1 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 1 \end{vmatrix}_{n-1} \\
&= -\frac{1}{4} \begin{vmatrix} 1 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 1 \end{vmatrix}_{n-2} + \begin{vmatrix} 1 & \frac{1}{2} & \cdots & 0 & 0 \\ \frac{1}{2} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & \frac{1}{2} \\ 0 & 0 & \cdots & \frac{1}{2} & 1 \end{vmatrix}_{n-1} \\
&= |A_{n-1}| - \frac{1}{4}|A_{n-2}|
\end{aligned}
$$

令 $x^2 - x + \frac{1}{4} = 0$, 解得 $x = \frac{1}{2}$

$\therefore |A_n| = (\alpha_1 + \alpha_2 n)\dfrac{1}{2^n}$ 是递推数列 $|A_n| = |A_{n-1}| - \frac{1}{4}|A_{n-2}|$ 的一个通解

带入 $|A_1| = 1$ 得 $\frac{1}{2}(\alpha_1 + \alpha_2) = 1$

带入 $|A_2| = \frac{3}{4}$ 得 $\frac{1}{4}(\alpha_1 + 2\alpha_2) = \frac{3}{4}$

$\therefore \alpha_1 = 1, \alpha_2 = 1$

$\therefore |A_n| = (n+1)\dfrac{1}{2^n}$, 带入 $n = 3, |A_3| = \frac{1}{2}$ 可知成立

$\because |A_n| = (n+1)\dfrac{1}{2^n} > 0$

$\therefore$ 该二次型正定

8.(1)

$\because |1| = 1 > 0$

$$\begin{vmatrix} 1 & t \\ t & 1 \end{vmatrix} = 1 - t^2 > 0 \Rightarrow -1 < t < 1$$

$$\begin{vmatrix} 1 & t & -1 \\ t & 1 & 2 \\ -1 & 2 & 5 \end{vmatrix} = -4t - 5t^2 > 0 \Rightarrow t(5t+4) < 0 \Rightarrow -\frac{4}{5} < t < 0$$

$\therefore -\dfrac{4}{5} < t < 0$

11.

$\because A$ 是正定矩阵

$\therefore$ 存在可逆矩阵 P, 使得 $E = P^T A P$

$\therefore (P^T A P)^{-1} = P^{-1} A^{-1} (P^T)^{-1} = ((P^{-1})^T)^T A^{-1} (P^{-1})^T$

令 $Q = (P^{-1})^T$, 则有 $Q^T A^{-1} Q = E$, 且 Q 为可逆矩阵

$\therefore A^{-1}$ 与 E 合同

$\therefore A^{-1}$ 也是正定矩阵

12.

若 A 正定, 则存在可逆矩阵 C, 使 $A = C^T C$

$\therefore |A| = |C^T C| = |C|^2 > 0$

由逆否命题可知, 若 $|A| < 0$, 则 $|A|$ 一定不是正定矩阵

$\therefore$ 必定存在实 n 维向量 $X \neq 0$ 使 $X^T A X < 0$

13.

$\because A, B$ 都是 n 级正定矩阵

$\because \forall x \in \mathbb{R}^n, x \neq 0, x^T A x > 0, x^T B x > 0$

$\therefore x^T (A+B) x = x^T A x + x^T B x > 0$

$\therefore A + B$ 也是正定矩阵