高等代数作业

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P317 1.(3)(4)(5)(8) 3. 6. 7.(1)(5)(6) 9.(2) 10. 例题

1.

(3)

$\because \mathcal{A}(k\vec{x}) = \mathcal{A}(kx_1, kx_2, kx_3) = (k^2x_1^2, kx_2 + kx_3, k^2x_3^2)$

$\quad k\mathcal{A}(\vec{x}) = k\mathcal{A}(x_1, x_2, x_3) = k(x_1^2, x_2 + x_3, x_3^2) = (kx_1^2, kx_2 + kx_3, kx_3^2)$

$\therefore \mathcal{A}(k\vec{x}) \neq k\mathcal{A}(\vec{x})$

$\therefore$ 不是线性变换.

(4)

$$\begin{aligned}\because \mathcal{A}(k\vec{x} + l\vec{y}) &= (2(kx_1 + ly_1) - (kx_2 + ly_2), kx_2 + ly_2 + kx_3 + ly_3, kx_1 + ly_1) \\ &= (2kx_1 - kx_2, kx_2 + kx_3, kx_1) + (2ly_1 - ly_2, ly_2 + ly_3, ly_1) \\ &= k(2x_1 - x_2, x_2 + x_3, x_1) + l(2y_1 - y_2, y_2 + y_3, y_1) \\ &= k\mathcal{A}(\vec{x}) + l\mathcal{A}(\vec{y})\end{aligned}$$

$\therefore$ 是线性变换.

(5)

$\because \mathcal{A}(kf(x) + lg(x)) = kf(x+1) + lg(x+1) = k\mathcal{A}(f(x)) + l\mathcal{A}(g(x))$

$\therefore$ 是线性变换.

(8)

$\because \mathcal{A}(k\vec{X} + l\vec{Y}) = B(k\vec{X} + l\vec{Y})C = kB\vec{X}C + lB\vec{Y}C = k\mathcal{A}(\vec{X}) + l\mathcal{A}(\vec{Y})$

$\therefore$ 是线性变换.

3.

$$
\begin{aligned}
\therefore (\mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A})(f(x)) &= (\mathcal{A}\mathcal{B})(f(x))-(\mathcal{B}\mathcal{A})(f(x)) \\
&= \mathcal{A}(\mathcal{B}(f(x)))-\mathcal{B}(\mathcal{A}(f(x))) \\
&= \mathcal{A}(xf(x))-\mathcal{B}(f'(x)) \\
&= f(x)+xf'(x)-xf'(x) \\
&= f(x) \\
&= \mathcal{E}(f(x))
\end{aligned}
$$

$\therefore \mathcal{A}\mathcal{B}-\mathcal{B}\mathcal{A}=\mathcal{E}$

6.

设 $\mathcal{A}$ 在基 $\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n$ 下矩阵为 A, 即

$(\mathcal{A}\varepsilon_1,\mathcal{A}\varepsilon_2,\cdots,\mathcal{A}\varepsilon_n)=(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_n)A$

并且我们知 $\mathcal{A}\varepsilon_1,\mathcal{A}\varepsilon_2,\cdots,\mathcal{A}\varepsilon_n$ 线性无关的充要条件为 $\mathrm{rank}(A)=n$

而 $\mathcal{A}$ 可逆的充要条件是 A 可逆, 即也是 $\mathrm{rank}(A)=n$

因此 $\mathcal{A}\varepsilon_1,\mathcal{A}\varepsilon_2,\cdots,\mathcal{A}\varepsilon_n$ 线性无关的充要条件是 $\mathcal{A}$ 可逆.

7.

(1)

$\because \mathcal{A}(\varepsilon_1)=(2,0,1),\mathcal{A}(\varepsilon_2)=(-1,1,0),\mathcal{A}(\varepsilon_3)=(0,1,0)$

$$\therefore A=\begin{pmatrix}2&-1&0\\0&1&1\\1&0&0\end{pmatrix}$$

(5)

$$\because (\eta_1,\eta_2,\eta_3)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)X=(\varepsilon_1,\varepsilon_2,\varepsilon_3)\begin{pmatrix}-1&1&0\\1&0&1\\1&-1&1\end{pmatrix}$$

$$\therefore X=\begin{pmatrix}-1&1&0\\1&0&1\\1&-1&1\end{pmatrix}$$

$\because \mathcal{A}(\varepsilon_1,\varepsilon_2,\varepsilon_3)=(\varepsilon_1,\varepsilon_2,\varepsilon_3)A, \mathcal{A}(\eta_1,\eta_2,\eta_3)=(\eta_1,\eta_2,\eta_3)B, B=\begin{pmatrix}1&0&1\\1&1&0\\-1&2&1\end{pmatrix}$

$\therefore B=X^{-1}AX$

$\because X^{-1}=\begin{pmatrix}-1&1&0\\1&0&1\\1&-1&1\end{pmatrix}^{-1}=\begin{pmatrix}-1&1&-1\\0&1&-1\\1&0&1\end{pmatrix}$

$\therefore A=XBX^{-1}=\begin{pmatrix}-1&1&-2\\2&2&0\\3&0&2\end{pmatrix}$

(6)

$\because \mathcal{A}(\eta_1,\eta_2,\eta_3)=(\mathcal{A}\eta_1,\mathcal{A}\eta_2,\mathcal{A}\eta_3)=(\eta_1,\eta_2,\eta_3)B$

$\therefore B=(\eta_1,\eta_2,\eta_3)^{-1}(\mathcal{A}\eta_1,\mathcal{A}\eta_2,\mathcal{A}\eta_3)=\begin{pmatrix}-1&0&3\\0&1&-1\\2&1&0\end{pmatrix}^{-1}\begin{pmatrix}-5&0&-5\\0&-1&-1\\3&6&9\end{pmatrix}=$
$\begin{pmatrix}2&3&5\\-1&0&-1\\-1&1&0\end{pmatrix}$

$\because X=\begin{pmatrix}-1&0&3\\0&1&-1\\2&1&0\end{pmatrix}$

$\therefore A=XBX^{-1}=\begin{pmatrix}-5&0&-5\\0&-1&-1\\3&6&9\end{pmatrix}\begin{pmatrix}-1&0&3\\0&1&-1\\2&1&0\end{pmatrix}^{-1}=\begin{pmatrix}-\frac{5}{7}&-\frac{20}{7}&-\frac{20}{7}\\-\frac{4}{7}&-\frac{5}{7}&-\frac{2}{7}\\\frac{27}{7}&\frac{18}{7}&\frac{24}{7}\end{pmatrix}$

9. (2)

$\because X=\begin{pmatrix}1&0&0\\0&k&0\\0&0&1\end{pmatrix}$

$\therefore X^{-1}=\begin{pmatrix}1&0&0\\0&\frac{1}{k}&0\\0&0&1\end{pmatrix}$

$$\therefore B=X^{-1}AX=\begin{pmatrix}1&0&0\\0&\frac{1}{k}&0\\0&0&1\end{pmatrix}\begin{pmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{pmatrix}\begin{pmatrix}1&0&0\\0&k&0\\0&0&1\end{pmatrix}=\begin{pmatrix}a_{11}&ka_{12}&a_{13}\\\frac{1}{k}a_{21}&a_{22}&\frac{1}{k}a_{23}\\a_{31}&ka_{32}&a_{33}\end{pmatrix}$$

10.

设 $a_1\mathcal{A}^0\xi+a_2\mathcal{A}^1\xi+\cdots+a_k\mathcal{A}^{k-1}\xi=0$

使用 $\mathcal{A}^{k-1}$ 作用于等式两端, 可得

$a_1\mathcal{A}^{k-1}\xi+a_2\mathcal{A}^{k}\xi+\cdots+a_2\mathcal{A}^{2k-2}\xi=0$

$\because \mathcal{A}^k\xi=\mathcal{A}^{k+1}\xi=\cdots=0$

$\therefore a_1\mathcal{A}^{k-1}\xi=0$

$\because \mathcal{A}^{k-1}\xi\neq 0$

$\therefore a_1=0$

剩余 $a_2\mathcal{A}^1\xi+\cdots+a_2\mathcal{A}^{k-1}\xi=0$

同理使用 $\mathcal{A}^{k-2}$ 作用于等式两端, 依次类推, 可得

$a_1=a_2=\cdots=a_k=0$

$\therefore \xi,\mathcal{A}\xi,\cdots\mathcal{A}^{k-1}\xi$ 线性无关.

例题

由 $f\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}x_1-2x_2+3x_3\\2x_1-x_3\end{pmatrix}$ 定义 $f:\mathbb{R}^3\to\mathbb{R}^2$, 求下列基下的矩阵 A.

$\mathbb{R}^3:\left\{\begin{pmatrix}1\\2\\0\end{pmatrix}\begin{pmatrix}-3\\1\\1\end{pmatrix}\begin{pmatrix}2\\0\\3\end{pmatrix}\right\},\mathbb{R}^2:\left\{\begin{pmatrix}5\\3\end{pmatrix}\begin{pmatrix}2\\1\end{pmatrix}\right\}$

解:

令 $\alpha_1=\begin{pmatrix}a_{11}\\a_{21}\end{pmatrix},\alpha_2=\begin{pmatrix}a_{12}\\a_{22}\end{pmatrix},\alpha_3=\begin{pmatrix}a_{13}\\a_{23}\end{pmatrix},A=\begin{pmatrix}\alpha_1&\alpha_2&\alpha_3\end{pmatrix}$

$$\therefore f\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}-3\\2\end{pmatrix}=a_{11}\begin{pmatrix}5\\3\end{pmatrix}+a_{21}\begin{pmatrix}2\\1\end{pmatrix}$$

$$f\begin{pmatrix}-3\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\-7\end{pmatrix}=a_{12}\begin{pmatrix}5\\3\end{pmatrix}+a_{22}\begin{pmatrix}2\\1\end{pmatrix}$$

$$f\begin{pmatrix}2\\0\\3\end{pmatrix}=\begin{pmatrix}11\\1\end{pmatrix}=a_{13}\begin{pmatrix}5\\3\end{pmatrix}+a_{23}\begin{pmatrix}2\\1\end{pmatrix}$$

解方程可得 $A=\begin{pmatrix}7 & -12 & -9\\-19 & 29 & 28\end{pmatrix}$