概率统计第一周作业

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1.1

1. **频率与概率的关系:** 频率是随机事件在相同条件下重复试验出现次数和试验总次数的比值 f, 频率会根据重复试验的次数发生变化, 一般随着试验次数的增加而在一个常数 p 附近摆动, 并且摆幅越来越小, 这个常数就是概率, 它是恒定的, 不会随着试验次数的变化而变化.
2. **随机现象中的二重性:** 频率 f 会随着试验次数的增加而发生一定变化, 这是 **随机性**; 但随着试验次数增多, 频率也就越来越接近概率 p, 这是 **稳定性**.
3. **对立与互不相容事件的关系:** 对立事件是互不相容事件, 但互不相容事件不一定是对立事件.

1.2

(i)

化简 $(A-AB)\cup B$:

根据事件的差公式可以将 $(A-AB)\cup B$ 化简为 $(A\cap\overline{B})\cup B$, 再根据分配律展开成 $(A\cup B)\cap(\overline{B}\cup B)$, 最后化简成 $A\cup B$.

化简 $\overline{\overline{A}\cup B}$:

根据德摩根律可将 $\overline{\overline{A}\cup B}$ 化简成 $A\cap\overline{B}$.

(ii)

$\because A,B,C$ 互不相容, 即 $A\cap B=B\cap C=A\cap C=\emptyset$

$$\begin{aligned}\therefore (A\cup B)-C =&(A\cup B)\cap\overline{C}\\ =&(A\cap\overline{C})\cup(B\cap\overline{C})\\ =&(A-A\cap C)\cup(B-B\cap C)\\ =&(A-\emptyset)\cup(B-\emptyset)\\ =&A\cup B\end{aligned}$$

即化简成 $A\cup B$.

1.3

(i) $\overline{A_1}\cap\left(\bigcap\limits_{i=2}^{n}A_i\right)$

(ii) $\bigcup\limits_{i=1}^{n}\overline{A_i}$

(iii) $\bigcup\limits_{i=1}^{n}\left[\overline{A_i}\cap\left(\bigcap\limits_{1\leqslant j\leqslant n,i\neq j}A_j\right)\right]$

(iv) $\bigcup\limits_{1\leqslant i<j\leqslant n}\overline{A_i}\cap\overline{A_j}$

(v) $\overline{\bigcup\limits_{1\leqslant i<j<k\leqslant n}\overline{A_i}\cap\overline{A_j}\cap\overline{A_k}}$

(vi) $\bigcap\limits_{i=1}^{n}A_i$

1.4

使用数学归纳法, 先证 $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$.

奠基 (Basis): 当 $n=2$ 时, 已有 $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ 即 $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$ 成立.

归纳假设 (I.H.): 对于 $n \geqslant 2$ 有 $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$ 成立.

归纳步骤 (I.S.):

要证明 $\overline{\bigcup_{i=1}^{n+1} A_i} = \bigcap_{i=1}^{n+1} \overline{A_i}$.

即证明 $\overline{\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{i+1}} = \left(\bigcap_{i=1}^{n} \overline{A_i}\right) \cap \overline{A_{i+1}}$.

套入 $n=2$ 时的对偶律 $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$ 可知

$\overline{\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{i+1}} = \overline{\left(\bigcup_{i=1}^{n} A_i\right)} \cap \overline{A_{i+1}}$

且由归纳假设有 $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$.

即有 $\overline{\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{i+1}} = \left(\bigcap_{i=1}^{n} \overline{A_i}\right) \cap \overline{A_{i+1}}$ 成立.

归纳结束.

综上可证 $\overline{\bigcup_{i=1}^{n} A_i} = \bigcap_{i=1}^{n} \overline{A_i}$ 成立.

同理可证 $\overline{\bigcap_{i=1}^{n} A_i} = \bigcup_{i=1}^{n} \overline{A_i}$ 成立.

1.5

$P(\overline{A}B) = P(B-A) = P(B) - P(AB) = \frac{1}{5} - \frac{1}{20} = \frac{3}{20}$

$P(\overline{A} \cup \overline{B}) = P(\overline{AB}) = 1 - P(AB) = 1 - \frac{1}{20} = \frac{19}{20}$

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC) = \frac{1}{3} + \frac{1}{5} + \frac{1}{6} - \frac{1}{20} - \frac{1}{20} - \frac{1}{60} + \frac{1}{100} = \frac{89}{150}$

$P(\overline{A}\,\overline{B}\,\overline{C}) = P(\overline{A \cup B \cup C}) = 1 - P(A \cup B \cup C) = 1 - \frac{89}{150} = \frac{61}{150}$

$P(\overline{A}\,\overline{B}C) = P(\overline{A \cup B}C) = P(C - (A \cup B)) = P(C) - P(C(A \cup B)) = P(C) - P(AC \cup BC) = P(C) - [P(AC) + P(BC) - P(ABC)] = \frac{1}{6} - \left[\frac{1}{20} + \frac{1}{60} - \frac{1}{100}\right] = \frac{11}{100}$

$P((\overline{A}\,\overline{B}) \cup C) = P(\overline{A}\,\overline{B}) + P(C) - P(\overline{A}\,\overline{B}C) = P(\overline{A \cup B}) + P(C) - P(\overline{A}\,\overline{B}C) = 1 - P(A \cup B) + P(C) - P(\overline{A}\,\overline{B}C) = 1 - [P(A) + P(B) - P(AB)] + P(C) - P(\overline{A}\,\overline{B}C) = 1 - (\frac{1}{3} + \frac{1}{5} - \frac{1}{20}) + \frac{1}{6} - \frac{11}{100} = \frac{43}{75}$

1.6

由容斥原理可知 $P(A \cup B) = P(A) + P(B) - P(AB)$ 即有 $P(AB) = P(A) + P(B) - P(A \cup B)$.

而我们又有 $P(A \cup B)$ 的取值范围 $0.9 = P(B) \leqslant P(A \cup B) \leqslant 1$

当 $P(A \cup B)$ 取得最小值 0.9 时, $P(AB)$ 取得最大值 $P(AB) = 0.6 + 0.9 - 0.9 = 0.6$

当 $P(A \cup B)$ 取得最小值 0.9 时, $P(AB)$ 取得最小值 $P(AB) = 0.6 + 0.9 - 1 = 0.5$

1.7

$\therefore P(\overline{A}\,\overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(AB)$

$\because P(AB) = P(\overline{A}\,\overline{B})$

$\therefore 0 = 1 - P(A) - P(B)$

$\because P(B) = \frac{1}{4}$

$\therefore P(A) = 1 - P(B) = \frac{3}{4}$

1.8

$\because P(\overline{A}\,\overline{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - P(A) - P(B) + P(AB)$

$\therefore P(AB) = P(\overline{A}\,\overline{B}) - 1 + P(A) + P(B)$

$\therefore P(B - A) = P(B) - P(AB) = P(B) - [P(\overline{A}\,\overline{B}) - 1 + P(A) + P(B)] = 1 - P(\overline{A}\,\overline{B}) - P(A) = 1 - 0.7 - 0.1 = 0.2$

1.9

使用数学归纳法.

奠基 (Basis):

当 $n = 2$ 时,

$\because A_1 \cup A_2 = (A_1 - A_2) \cup (A_1A_2) \cup (A_2 - A_1)$, 且 $A_1 - A_2$, A_1A_2, $A_2 - A_1$ 两两互不相容, 由有限可加性可知

$\therefore P(\bigcup_{i=1}^{n} A_i) = P(A_1 \cup A_2) = P(A_1 - A_2) + P(A_1A_2) + P(A_2 - A_1)$

$\because P(A_1 - A_2) = P(A_1) - P(A_1A_2), P(A_2 - A_1) = P(A_2) - P(A_1A_2)$

$\therefore P(\bigcup_{i=1}^{n} A_i) = P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1A_2)$

满足式子 $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) + \cdots + (-1)^{n-1}P(A_1A_2\cdots A_n)$

归纳假设 (I.H.):

当 $n \geqslant 2$ 时有 $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) + \cdots + (-1)^{n-1}P(A_1A_2\cdots A_n)$ 成立.

即有 $P(\bigcup_{i=1}^{n} A_i) = \sum_{r=1}^{n}(-1)^{r+1} \sum_{i_1<\cdots<i_r} P(A_{i_1}\cdots A_{i_r})$ 成立.

归纳步骤 (I.H.):

对于 $n+1$ 的情况, 要证明 $P(\bigcup_{i=1}^{n+1} A_i) = \sum_{i=1}^{n+1} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) + \cdots + (-1)^{n}P(A_1A_2\cdots A_nA_{n+1})$

即 $P(\bigcup_{i=1}^{n+1} A_i) = \sum_{r=1}^{n+1}(-1)^{r+1} \sum_{i_1<\cdots<i_r} P(A_{i_1}\cdots A_{i_r})$

根据 $n = 2$ 情况的式子 $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1A_2)$, 我们可知

$$
\begin{aligned}
P(\bigcup_{i=1}^{n+1} A_i) =& P((\bigcup_{i=1}^{n} A_i) \cup A_{n+1}) \\
=& P(\bigcup_{i=1}^{n} A_i) + P(A_{n+1}) - P((\bigcup_{i=1}^{n} A_i) \cap A_{n+1}) \\
=& P(\bigcup_{i=1}^{n} A_i) + P(A_{n+1}) - P(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})) \\
=& \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r}) - \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r} A_{n+1}) + P(A_{n+1}) \\
=& \sum_{i=1}^{n} P(A_i) + \sum_{r=2}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r}) - \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r} A_{n+1}) + P(A_{n+1}) \\
=& \sum_{i=1}^{n+1} P(A_i) + \sum_{r=2}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r}) - \sum_{r=1}^{n-1} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r} A_{n+1}) + (-1)^{n+2} P(A_1\cdots A_{n+1})
\end{aligned}
$$

而我们又可以展开

$$
\begin{aligned}
& \sum_{r=1}^{n+1} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n+1} P(A_{i_1}\cdots A_{i_r}) \\
=& \sum_{i=1}^{n+1} P(A_i) + \sum_{r=2}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n+1} P(A_{i_1}\cdots A_{i_r}) + (-1)^{n+2} P(A_1\cdots A_{n+1}) \\
=& \sum_{i=1}^{n+1} P(A_i) + \sum_{r=2}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r}) - \sum_{r=1}^{n-1} (-1)^{r+1} \sum_{i_1<\cdots<i_r\leqslant n} P(A_{i_1}\cdots A_{i_r} A_{n+1}) + (-1)^{n+2} P(A_1\cdots A_{n+1})
\end{aligned}
$$

即有 $P(\bigcup_{i=1}^{n} A_i) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{i_1<\cdots<i_r} P(A_{i_1}\cdots A_{i_r})$ 成立.

归纳完毕.

综上可证, 对于任意 n 个事件 $A_1, A_2, \cdots, A_n$, 有 $P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) + \cdots + (-1)^{n-1} P(A_1A_2\cdots A_n)$ 成立.