Go Implementation of Up-To Techniques for Equivalence of Weighted Languages

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Abstract

Weighted automata generalize non-deterministic automata by adding a quantity expressing the weight (or probability) of the execution of each transition. In this work we propose an implementation of two algorithms for computing language equivalence in finite state weighted automata (WAs). The first, a linear partition refinement algorithm, calculates the largest linear weighted bisimulation for any given LWA (Linear Weighted Automaton) through an iterative method, the second algorithm checks the language equivalence of two vectors (states) for a given weighted automata by using an additional data structure representing a congruence relationship between states. We then compare the two algorithms results to verify their correctness on randomly generated automata samples, and provide some runtime statistics.

***K*eywords** First keyword · Second keyword · More

1 Introduction

In [1], up-to techniques are defined for weighted systems over arbitrary semirings, while in [2], up-to techniques are defined for Linear Weighted Automata (LWAs), under a more abstract coalgebraic perspective.

2 Preliminaries and Notation

Note. Given two vector spaces V_1, V_2 we write $V_1 + V_2$ to denote span $(V_1 \cup V_2)$

Definition 2.1. A *weighted automaton* over a field $\mathbb{K}$ and an alphabet A is a triple (X, o, t) such that X is a finite set of states, $t = \{t_a : X \to \mathbb{K}\}_{a \in A}$ is a set of transition functions indexed over the symbols of the alphabet A and $o : X \to \mathbb{K}$ is the output function. A^* is the set of all words over A. ϵ is the empty word and aw is the concatenation of a symbol a to the word $w \in A^*$. A weighted language is a function $\psi : A^* \to \mathbb{K}$. A function mapping each state vector into its accepted language, $[\![\cdot]\!] : \mathbb{K}^X \to \mathbb{K}^{A^*}$ is defined as follows for every weighted automaton:

$$\forall v \in \mathbb{K}^X, a \in A, w \in A^* \qquad [\![v]\!](\epsilon) = o(v) \qquad [\![v]\!](aw) = [\![t_a(v)]\!](w)$$

Two vectors $v_1, v_2 \in \mathbb{K}^{X \times 1}$ are weighted called language equivalent, denoted with $v_1 \sim_l v_2$ if and only if $[\![v_1]\!] = [\![v_2]\!]$. One can extend the notion of language equivalence to states rather than for vectors by assigning to each state $x \in X$ the corresponding unit vector $e_x \in \mathbb{K}^X$. When given an initial state i for a weighted automaton, the language of the automaton can be defined as $[\![i]\!]$.

Definition 2.2. A binary relation $R \subseteq X \times Y$ between two sets X, Y is a subset of the cartesian product of the sets. A relation is called *homogeneous* or an *endorelation* if it is a binary relation over X and itself: $R \subseteq X \times X$. In such case, it is simply called a binary relation over X. An *equivalence relation* is a binary relation that is reflexive, symmetric and transitive.

Definition 2.3. The congruence closure $c(R)$ of a relation R is the smallest congruence relation R' such that $R \subseteq R'$

An equivalence relation which is compatible with all the operations of the algebraic structure on which it is defined on, is called a *congruence relation*. Compatibility with the algebraic structure operations means that algebraic operations applied on equivalent elements will still yield equivalent elements.

We omit the coalgebraic definition for *linear weighted automata* seen in [2] and give a more intuitive definition. In this implementation, we focus only on weighted automata defined over the field of real numbers $\mathbb{R}$.

Definition 2.4. A *linear weighted automaton* (in short, LWA) over the field $\mathbb{K}$ and an alphabet A is a triple $L = (V, o, \{t_a\}_{a\in A})$ where V is a vector space representing the state space, $o : V \to \mathbb{K}$ is a linear map associating to each state its output weight, and $t = \{t_a = V \times V\}_{a\in A}$ is the set of transition functions, represented with liner maps that for each input $a \in A$ associate the next state, in this case a vector in V. As in [3], we have that $\dim(L) = \dim(V)$.

Given a weighted automaton, one can build an isomorphic linear weighted automaton by considering the free vector space generated by the set of states X in the WA, and by linearizing o and t. If X is finite, we can use the same matrices for t and o in both the WA and the corresponding LWA. We are only considering a finite number of states and therefore finite dimensional vector spaces. Let n be the (finite) number of states in an WA. We have that in the corresponding LWA, the transition functions t_a are still represented as $\mathbb{K}^{n\times n}$ matrices. $o \in \mathbb{K}^{1\times n}$ is represented as a row vector. $t_a(v)$ denotes the vector obtained by multiplying the matrix t_a by the column vector $v \in \mathbb{K}^{n\times 1}$. $o(v)$ denotes the scalar $s \in \mathbb{K}$ obtained by dot product of the row vector o with $v \in \mathbb{K}^{n\times 1}$.

Definition 2.5. The language recognized by a vector $v \in V$ of an LWA (V, o, t) is defined for all words $w \in A*$ as $[\![v]\!]^{\mathcal{L}}_V(w) = o(v_n)$ where v_n is the vector reached from v through the composition of the transition functions corresponding to the words in w.

$$[\![v]\!]^{\mathcal{L}}_V(w) = \begin{cases} o(v) & \text{if } w = \epsilon \\ [\![t_a(v)]\!]^{\mathcal{L}}_V(w') & \text{if } w = aw' \end{cases}$$

We define $\approx_{\mathcal{L}}$ as the behavioral equivalence for a given LWA (V, o, t) as

$$\forall v_1, v_2 \in V,\ v_1 \approx_{\mathcal{L}} v_2 \iff [\![v_1]\!]^{\mathcal{L}}_V = [\![v_2]\!]^{\mathcal{L}}_V \tag{1}$$

Lemma 2.1. *$\approx_{\mathcal{L}}$ coincides with $\sim_l$:*
Let (X, o, t) be a WA and $(\mathbb{K}^X, o^\sharp, t^\sharp)$ the corresponding linear weighted automaton. Then $\forall x \in X,\ \ [\![x]\!] = [\![x]\!]^{\mathcal{L}}_{\mathbb{K}^X}$

Proof. Proved in section 3.2 of [2] ☐

3 The Problem

In this work, we

4 Algorithms

We now recall the backwards algorithm for computing $\approx_{\mathcal{L}}$ defined in [2].

4.1 Backwards Partition Refinement Algorithm for the Largest Weighted Bisimulation

The algorithm is defined by the iterative method:

$$R_0 = \ker(o)^0, \qquad R_{i+1} = R_i + \sum_{a\in A} t_a(R_i)^t \tag{2}$$

Where $\ker(o)^0$ is an annihilator. The algorithm stops when $R_{j+1} = R_j$. Proof is available in section 4.2 of [2]

5 Implementation

The algorithms and data structures for this paper are implemented in the Go programming language. This implementation makes use of the Gonum library for numerical computations. We only import the Gonum libraries for matrices and linear algebra and visual plotting of samples and functions. Real numbers are implemented with double precision floating point numbers, known as the `float64` type in the Go programming language.

See Section **??**.

Definition 5.1. Algorithm for computing the null space of a vector subspace
The algorithm implementation can be found in file `lin/nullspace.go`; it is adapted from [4]. Let's consider the singular value decomposition of a matrix $A \in \mathbb{R}^{m \times n}$:

$$A = U \Sigma V^T \qquad \Sigma = \text{diag}\left(\sigma_1, \sigma_2, \ldots, \sigma_{\min(m,n)}\right) \qquad U \in \mathbb{R}^{m \times m} \qquad V \in \mathbb{R}^{n \times n}$$

Where V and U are orthogonal and the singular values are ordered: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$. It follows that rank (A) is equal to the number of nonzero singular values and a basis of the (right) null space of A is the spanning set of the columns of V corresponding to singular values equal to zero.

Example 5.1. First, we show a shorter Python implementation of the algorithm to compute the nullspace, using the *SciPy* library [4]:

```
def nullspace(A, atol=1e-13, rtol=0):
    A = np.atleast_2d(A)
    u, s, vh = svd(A)
    tol = max(atol, rtol * s[0])
    nnz = (s >= tol).sum()
    ns = vh[nnz:].conj().T
    return ns
```

The Go implementation is quite longer:

```
package lin

import (
	"log"
	"math"

	"gonum.org/v1/gonum/mat"
)

const tol = 10e-13

// columns of the returned matrix form an orthonormal basis
// for the nullspace of matrix a, computed
// through svd decomposition. also returns the maximum residual
func Nullspace(a mat.Matrix) (mat.Matrix, float64) {
	// compute svd decomposition
	var svd mat.SVD
	if ok := svd.Factorize(a, mat.SVDFullV); !ok {
		log.Fatal("failed to factorize A")
	}
	vt := mat.NewDense(1, 1, nil)
	vt.Reset()
	svd.VTo(vt)

	// residual
	res := 0.0

	// the (right) null space of A is the columns of vt corresponding to
	// singular values equal to zero.
	j := 0
```

```
        for _,  := range svd.Values(nil) {
                if  <= tol {
                        break
                }
                j++
        }

        // compute the residuum
        for k := j; k < vt.RawMatrix().Cols; k++ {
                v := mat.NewVecDense(1, nil)
                v.Reset()
                v.MulVec(a, vt.ColView(k))
                // current residual
                currRes := mat.Norm(v, math.Inf(1))
                if currRes > res {
                        res = currRes
                }
        }

        m, n := vt.Dims()
        ker := vt.Slice(0, m, j, n)
        return ker, res
}
```

5.1 Implementing the backwards algorithm

In [2], for the first step of the iterative backwards algorithm to compute the largest linear weighted bisimulation, we need to compute $\ker(o)^0$. If V is a vector space and W is a subspace of W, the annihilator of W, respectively W^0 is a subspace of the space V^* of linear functionals on V. W^0 are the functionals that annihilate on W. Since we are working on subspaces of $\mathbb{R}^n$, we can directly compute the orthogonal complement in our implementation instead of the annihilator.

Proposition 5.1. *If V is a finite dimensional vector space defined with an inner product $\langle\cdot,\cdot\rangle$ and W is a subspace of V then the image of the annhilitaor W^0 through the linear isomorphism $\varphi : V^* \to V$ induced by the inner product, is the orthogonal of W with respect to the said inner product.*

Proof. Let V be an inner product space over the field $\mathbb{K}$ with an inner product defined as $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{K}$. Every linear functional can be represented with a vector. Let $\xi : V \to \mathbb{K}$ be a functional, $\xi \in W^0$. Because $\xi(w) = 0 \quad \forall w \in W$, if v represents ξ we have that $(v, w) = \xi(w) = 0$ for all $w \in W$. We obtain that $\varphi(W^0) \subseteq W^\perp$. If $v \in W^\perp$ then the functional $x \mapsto (v, x)$ cancels over W (by the definition of orthogonality). $\square$

To compute the orthogonal complement of a vector subspace W, we compute $W^\perp = \ker\left(A^T\right)$, where A is the matrix with column vectors in the spanning set of W as its columns. Precisely, W is represented as the column space of A. Proof is available in [5].

5.1.1 Headings: third level

Paragraph

6 Examples of citations, figures, tables, references

[**?**, **?**] and see [**?**].

The documentation for `natbib` may be found at

`http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf`

Of note is the command `\citet`, which produces citations appropriate for use in inline text. For example,

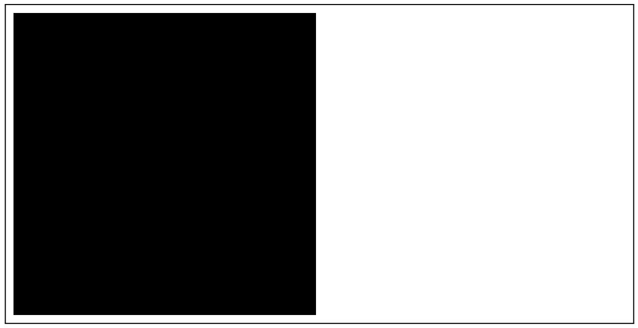

Figure 1: Sample figure caption.

Table 1: Sample table title

Part		
Name	Description	Size (μm)
Dendrite	Input terminal	$\sim$100
Axon	Output terminal	$\sim$10
Soma	Cell body	up to 10^6

```
\citet{hasselmo} investigated\dots
```

produces

Hasselmo, et al. (1995) investigated…

`https://www.ctan.org/pkg/booktabs`

6.1 Figures

See Figure 1. Here is how you add footnotes. [1]

6.2 Tables

See awesome Table 1.

References

[1] Filippo Bonchi, Barbara König, and Sebastian Küpper. Up-to techniques for weighted systems (extended version). *CoRR*, abs/1701.05001, 2017.

[2] Filippo Bonchi, Marcello Bonsangue, Michele Boreale, Jan Rutten, and Alexandra Silva. A coalgebraic perspective on linear weighted automata. *Information and Computation*, 211:77 – 105, 2012.

[3] Michele Boreale. Weighted bisimulation in linear algebraic form. In *International Conference on Concurrency Theory*, pages 163–177. Springer, 2009.

[4] Open Source Community. Scipy cookbook: Rank and nullspace of a matrix, 2011. Available online at `https://scipy-cookbook.readthedocs.io/items/RankNullspace.html`.

[5] Dan Margalit and Joseph Rabinoff. *Interactive Linear Algebra*.

[1] Sample of the first footnote.