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# GO IMPLEMENTATION OF UP-TO TECHNIQUES FOR EQUIVALENCE OF WEIGHTED LANGUAGES

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## ABSTRACT

Weighted automata generalize non-deterministic automata by adding a quantity expressing the weight (or probability) of the execution of each transition. In this work we propose an implementation of two algorithms for computing language equivalence in finite state weighted automata (WAs). The first, a linear partition refinement algorithm, calculates the largest linear weighted bisimulation for any given LWA (Linear Weighted Automaton) through an iterative method, the second algorithm checks the language equivalence of two vectors (states) for a given weighted automata by using an additional data structure representing a congruence relationship between states. We then compare the two algorithms results to verify their correctness on randomly generated automata samples, and provide some runtime statistics.

**Keywords** First keyword · Second keyword · More

## 1 Introduction

In [1], up-to techniques are defined for weighted systems over arbitrary semirings, while in [2], up-to techniques are defined for Linear Weighted Automata (LWAs), under a more abstract coalgebraic perspective. For ease of implementation, we focus only on weighted automata over the field of real numbers  $\mathbb{R}$ .

## 2 Preliminaries and Notation

**Definition 2.1.** A *weighted automaton* over a field  $\mathbb{K}$  and an alphabet  $A$  is a triple  $(X, o, t)$  such that  $X$  is a finite set of states,  $t = (t_a : X \rightarrow K)_{a \in A}$  is a set of transition functions indexed over the symbols of the alphabet  $A$  and  $o : X \rightarrow \mathbb{K}$  is the output function. The transition functions  $t_a$  are represented as  $\mathbb{K}^{X \times X}$  matrices.  $o \in \mathbb{K}^{1 \times X}$  is represented as a row vector.  $t_a(v)$  denotes the vector obtained by multiplying the matrix  $t_a$  by the column vector  $v \in \mathbb{K}^{X \times 1}$ .  $o(v)$  denotes the scalar  $s \in \mathbb{K}$  obtained by dot product of the row vector  $o$  with  $v \in \mathbb{K}^{X \times 1}$ .  $A^*$  is the set of all words over  $A$ .  $\epsilon$  is the empty word and  $aw$  is the concatenation of a symbol  $a$  to the word  $w \in A^*$ . A weighted language is a function  $\psi : A^* \rightarrow \mathbb{K}$ . A function mapping each state vector into its accepted language,  $\llbracket \cdot \rrbracket : \mathbb{K}^X \rightarrow K^{A^*}$  is defined as follows for every weighted automaton:

$$\forall v \in \mathbb{K}^X, a \in A, w \in A^* \quad \llbracket v \rrbracket(\epsilon) = o(v) \quad \llbracket v \rrbracket(aw) = \llbracket t_a(v) \rrbracket(w)$$

Two vectors  $v_1, v_2 \in \mathbb{K}^{X \times 1}$  are called language equivalent, denoted with  $v_1 \sim v_2$  if and only if  $\llbracket v_1 \rrbracket = \llbracket v_2 \rrbracket$ . One can extend the notion of language equivalence to states rather than for vectors by assigning to each state  $x \in X$  the

corresponding unit vector  $e_x \in \mathbb{K}^X$ . When given an initial state  $i$  for a weighted automaton, the language of the automaton can be defined as  $\llbracket i \rrbracket$ .

**Definition 2.2.** A binary relation  $R \subseteq X \times Y$  between two sets  $X, Y$  is a subset of the cartesian product of the sets. A relation is called *homogeneous* or an *endorelation* if it is a binary relation over  $X$  and itself:  $R \subseteq X \times X$ . In such case, it is simply called a binary relation over  $X$ . An *equivalence relation* is a binary relation that is reflexive, symmetric and transitive.

An equivalence relation which is compatible with all the operations of the algebraic structure on which it is defined on, is called a *congruence relation*. Compatibility with the algebraic structure operations means that algebraic operations applied on equivalent elements will still yield equivalent elements.

**Definition 2.3.** The congruence closure  $c(R)$  of a relation  $R$  is the smallest congruence relation  $R'$  such that  $R \subseteq R'$

**Definition 2.4.**

### 3 Algorithms

#### 3.1 Forward Partition Refinement Algorithm for the Largest Weighted Bisimulation

The algorithm is defined by the iterative method:

$$R_0 = \ker(o), \quad R_{i+1} = R_i \cap \bigcap_{a \in A} t_a(R_i)^{-1} \quad (1)$$

The algorithm stops when  $R_{j+1} = R_j$ . Proof is available in section 4.1 of [2]

#### 3.2 Backwards Partition Refinement Algorithm for the Largest Weighted Bisimulation

The algorithm is defined by the iterative method:

$$R_0 = \ker(o)^0, \quad R_{i+1} = R_i + \sum_{a \in A} t_a(R_i)^t \quad (2)$$

The algorithm stops when  $R_{j+1} = R_j$ . Proof is available in section 4.2 of [2]

### 4 Implementation

The algorithms and data structures for this paper are implemented in the Go programming language. This implementation makes use of the *Gonum* library for numerical computations. We only import the Gonum libraries for matrices and linear algebra and visual plotting of samples and functions.

Real numbers are implemented with double precision floating point numbers, known as the `float64` type in the Go programming language.

See Section ??.

#### 4.1 Input file format

#### 4.2 Calculating the intersection of linear subspaces

To compute algorithm (1), we need an efficient way to calculate a basis of the intersection of an arbitrary number of linear subspaces of  $\mathbb{R}^n$ , given their spanning sets.

**Definition 4.1. Algorithm for a basis of the intersection of subspaces**

Defined in file `lin/intersect.go`. Let's consider two linear subspaces of  $\mathbb{R}^n$  defined with their spanning sets of column vectors:  $U = \langle u_1, u_2, \dots, u_p \rangle$  and  $W = \langle w_1, w_2, \dots, w_k \rangle$ . We create the block matrix  $A \in \mathbb{R}^{n \times (p+k)}$  defined as follows:

$$a_{ij} = \begin{cases} (u_j)_i & \text{for } j = 1, \dots, p \quad i = 1, \dots, n \\ (w_{j-p})_i & \text{for } j = p+1, \dots, p+k \quad i = 1, \dots, n \end{cases}$$

The matrix will have the form

$$A = \left[ \begin{pmatrix} u_1 \end{pmatrix}, \dots, \begin{pmatrix} u_p \end{pmatrix}, \begin{pmatrix} w_1 \end{pmatrix}, \dots, \begin{pmatrix} w_k \end{pmatrix} \right]$$

It follows that  $\ker(A)$  is a basis for  $U \cap W$ . This can be done with an arbitrary finite number of subspaces of  $\mathbb{R}^n$ , as long as their spanning set is *finite*.

**Definition 4.2. Algorithm for computing the nullspace**

The algorithm implementation can be found in file `lin/nullspace.go`; it is adapted from [3]. Let's consider the singular value decomposition of a matrix  $A \in \mathbb{R}^{m \times n}$ :

$$A = U \Sigma V^T \quad \Sigma = \text{diag}(\sigma_1, \sigma_2, \dots, \sigma_{\min(m,n)}) \quad U \in \mathbb{R}^{m \times m} \quad V \in \mathbb{R}^{n \times n}$$

Where  $V$  and  $U$  are orthogonal and the singular values are ordered:  $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_{\min(m,n)} \geq 0$ . It follows that  $\text{rank}(A)$  is equal to the number of nonzero singular values and a basis of the (right) null space of  $A$  is the spanning set of the columns of  $V$  corresponding to singular values equal to zero.

**Example 4.1.** The actual Go implementation is quite verbose, we can show a shorter Python implementation of the algorithm to compute the nullspace, using the *SciPy* library [3]:

```
def nullspace(A, atol=1e-13, rtol=0):
    A = np.atleast_2d(A)
    u, s, vh = svd(A)
    tol = max(atol, rtol * s[0])
    nnz = (s >= tol).sum()
    ns = vh[nnz:].conj().T
    return ns
```

#### 4.2.1 Headings: third level

##### Paragraph

## 5 Examples of citations, figures, tables, references

[?, ?] and see [?].

The documentation for `natbib` may be found at

<http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf>

Of note is the command `\citet`, which produces citations appropriate for use in inline text. For example,

`\citet{hasselmo}` investigated\dots

produces

Hasselmo, et al. (1995) investigated...

<https://www.ctan.org/pkg/booktabs>

### 5.1 Figures

See Figure 1. Here is how you add footnotes. <sup>1</sup>

### 5.2 Tables

See awesome Table 1.

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<sup>1</sup>Sample of the first footnote.

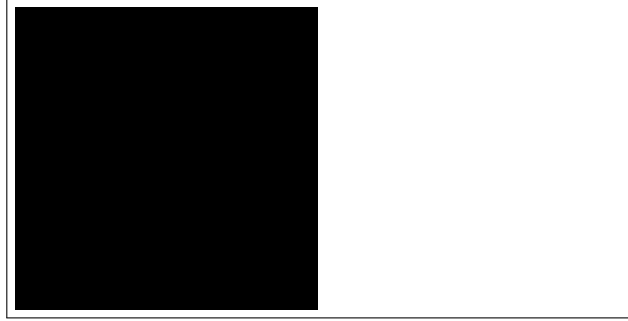


Figure 1: Sample figure caption.

Table 1: Sample table title

| Part     |                 |                        |
|----------|-----------------|------------------------|
| Name     | Description     | Size ( $\mu\text{m}$ ) |
| Dendrite | Input terminal  | $\sim 100$             |
| Axon     | Output terminal | $\sim 10$              |
| Soma     | Cell body       | up to $10^6$           |

## References

- [1] Filippo Bonchi, Barbara König, and Sebastian Küpper. Up-to techniques for weighted systems (extended version). *CoRR*, abs/1701.05001, 2017.
- [2] Filippo Bonchi, Marcello Bonsangue, Michele Boreale, Jan Rutten, and Alexandra Silva. A coalgebraic perspective on linear weighted automata. *Information and Computation*, 211:77 – 105, 2012.
- [3] Open Source Community. Scipy cookbook: Rank and nullspace of a matrix, 2011. Available online at <https://scipy-cookbook.readthedocs.io/items/RankNullspace.html>.