DRAFT: GO IMPLEMENTATION OF UP-TO TECHNIQUES FOR WEIGHTED LANGUAGE EQUIVALENCE

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ABSTRACT

Weighted automata generalize non-deterministic automata by adding a quantity expressing the weight (or probability) of the execution of each transition. In this work we propose an implementation of two algorithms for computing language equivalence in finite state weighted automata (WAs). The first, a linear partition refinement algorithm, computes the largest linear weighted bisimulation for any given LWA (Linear Weighted Automaton) through an iterative method, the second algorithm checks the language equivalence of two vectors (states) for a given weighted automata by using an additional data structure representing a congruence relation. We then compare results of the two algorithms to verify their correctness and performance on randomly generated samples. We finally provide a comparison and runtime statistics.

***K*eywords** First keyword · Second keyword · More

1 Introduction

TODO da fare

Checking whether two nondeterministic finite automata (NFA) accept the same language is important in many application domains such as compiler construction and model checking. Unfortunately, solving this problem is costly: it is PSPACE-complete (TODO cita)

In [1], up-to techniques are defined for weighted systems over arbitrary semirings, while in [2], up-to techniques are defined for Linear Weighted Automata (LWAs), under a more abstract coalgebraic perspective.

A well detailed example of the comparison of algorithms to compute language equivalence, precisely between `HKC` and an alternative algorithm called the *antichain algorithm* ([3]), was published in 2017 [4].

2 Preliminaries and Notation

Note. Given two vector spaces V_1, V_2 we write $V_1 + V_2$ to denote span $(V_1 \cup V_2)$

Definition 2.1. A *weighted automaton* over a field $\mathbb{K}$ and an alphabet A is a triple (X, o, t) such that X is a finite set of states, $t = \left\{t_a : X \to \mathbb{K}^X\right\}_{a \in A}$ is a set of transition functions indexed over the symbols of the alphabet A and $o : X \to \mathbb{K}$ is the output function. The transition functions will be represented as $X \times X$ matrices. A^* is the set of all words over A, more precisely the free monoid with string concatenation as the monoid operation and the empty word ϵ as the identity element. We denote with aw the concatenation of a symbol a to the word $w \in A^*$. A weighted language is a function $\psi : A^* \to \mathbb{K}$. A function mapping each state vector into its accepted language, $[\![\cdot]\!] : \mathbb{K}^X \to \mathbb{K}^{A^*}$ is defined as follows for every weighted automaton:

$$\forall v \in \mathbb{K}^X, a \in A, w \in A^* \qquad [\![v]\!](\epsilon) = o(v) \qquad [\![v]\!](aw) = [\![t_a(v)]\!](w)$$

Two vectors $v_1, v_2 \in \mathbb{K}^{X\times 1}$ are called weighted language equivalent, denoted with $v_1 \sim_l v_2$ if and only if $[\![v_1]\!] = [\![v_2]\!]$. One can extend the notion of language equivalence to states rather than for vectors by assigning to each state $x \in X$ the corresponding unit vector $e_x \in \mathbb{K}^X$. When given an initial state i for a weighted automaton, the language of the automaton can be defined as $[\![i]\!]$.

Definition 2.2. A binary relation $R \subseteq X \times Y$ between two sets X, Y is a subset of the cartesian product of the sets. A relation is called *homogeneous* or an *endorelation* if it is a binary relation over X and itself: $R \subseteq X \times X$. In such case, it is simply called a binary relation over X. An *equivalence relation* is a binary relation that is reflexive, symmetric and transitive.

An equivalence relation which is compatible with all the operations of the algebraic structure on which it is defined on, is called a *congruence relation*. Compatibility with the algebraic structure operations means that algebraic operations applied on equivalent elements will still yield equivalent elements.

Definition 2.3. The congruence closure $c(R)$ of a relation R is the smallest congruence relation R' such that $R \subseteq R'$

Definition 2.4. *Generating set for the congruence closure:*
Let $\mathbb{K}$ be a field, X a finite set and $R \subseteq \mathbb{K}^X \times \mathbb{K}^X$ a relation. Let $(v, v') \in \mathbb{K}^X \times \mathbb{K}^X$ be a pair of vectors. The generating set is defined as $U_R = \{u - u' \mid (u, u') \in R\}$. Then $(v, v') \in c(R) \iff (v - v') \in U_R$

We omit the coalgebraic definition for *linear weighted automata* seen in [2] and give a more intuitive definition, which fits our implementation when $\mathbb{K} = \mathbb{R}$. In this implementation, we focus only on weighted automata defined over the field of real numbers $\mathbb{R}$.

Definition 2.5. A *linear weighted automaton* (in short, LWA) over the field $\mathbb{K}$ and an alphabet A is a triple $L = (V, o, \{t_a\}_{a\in A})$ where V is a vector space representing the state space, $o : V \to \mathbb{K}$ is a linear map associating to each state its output weight, and $t = \{t_a = V \times V\}_{a\in A}$ is the set of transition functions, represented with liner maps that for each input $a \in A$ associate the next state, in this case a vector in V. As in [5], we have that $\dim(L) = \dim(V)$.

Given a weighted automaton, one can build a corresponding linear weighted automaton by considering the free vector space generated by the set of states X in the WA, and by linearizing o and t. If X is finite, as in our implementations of the algorithms, we can use the same matrices for t and o in both the WA and the corresponding LWA. We are only considering a finite number of states and therefore finite dimensional vector spaces. Let n be the number of states in an WA. We have that in the corresponding LWA, the transition functions t_a are still represented as $\mathbb{K}^{n\times n}$ matrices. $o \in \mathbb{K}^{1\times n}$ is represented as a row vector. $t_a(v)$ denotes the vector obtained by multiplying the matrix t_a by the column vector $v \in \mathbb{K}^{n\times 1}$. $o(v)$ denotes the scalar $s \in \mathbb{K}$ obtained by dot product of the row vector o with $v \in \mathbb{K}^{n\times 1}$.

Definition 2.6. The language recognized by a vector $v \in V$ of an LWA (V, o, t) is defined for all words $w \in A*$ as $[\![v]\!]^{\mathcal{L}}_V(w) = o(v_n)$ where v_n is the vector reached from v through the composition of the transition functions corresponding to the words in w.

$$[\![v]\!]^{\mathcal{L}}_V(w) = \begin{cases} o(v) & \text{if } w = \epsilon \\ [\![t_a(v)]\!]^{\mathcal{L}}_V(w') & \text{if } w = aw' \end{cases}$$

We define $\approx_{\mathcal{L}}$ as the behavioral equivalence for a given LWA (V, o, t) as

$$\forall v_1, v_2 \in V,\ v_1 \approx_{\mathcal{L}} v_2 \iff [\![v_1]\!]^{\mathcal{L}}_V = [\![v_2]\!]^{\mathcal{L}}_V \tag{1}$$

Proof is available in section 3.3 of [2]

Language equivalence can be now expressed in terms of linear weighted bisimulations (LWBs for short). Differently from weighted bisimulations, LWBs can be seen both as relations and as subspaces. The subspace representation of LWBs is used in the backwards partition refinement algorithm implemented in [2] and in this work.

Definition 2.7. *Linear Relations:*
Let U be a subspace of V. The binary relation R_U over V is defined by

$$v_1\ R_U\ v_2 \qquad \iff \qquad v_1 - v_2 \in U$$

The relation R is linear if there exists a subspace U such that $R \equiv R_U$. A linear relation is a total equivalence relation on V.

Definition 2.8. *Kernel of a Relation and Linear Extension*
Let R be a binary relation over V. The *kernel* of R, is the set $\ker(R) = \{v_1 - v_2 \mid v_1 \; R \; v_2\}$. The *linear extension* of R, written as R^ℓ, is defined by

$$v_1 \; R^\ell \; v_2 \quad \iff \quad (v_1 - v_2) \in \text{span}\,(\ker(R))$$

Lemma 2.1. *Let U be a subspace of V, then ker $(R_U) = U$*

Definition 2.9. *Linear Weighted Bisimulation:*
Let (V, o, t) be a linear weighted automaton. A linear relation $R \subseteq V \times V$ is a *linear weighted bisimulation* if $\forall (v_1, v_2) \in R$ it holds that:

1. $o(v_1) = o(v_2)$
2. $\forall a \in A, \; t_a(v_1) \; R \; t_a(v_2)$

Lemma 2.2. *Let (V, o, t) be a linear weighted automaton. A linear relation R over V is a linear weighted bisimulation if and only if*

1. $R \subseteq \ker(o)$
2. R is t_a-invariant $\forall a \in A$

Theorem 3 in section 3.3 of [2], states that $\ker\left([\![-]\!]_V^{\mathcal{L}}\right)$ is the largest linear weighted bisimulation on V. As a corollary, we obtain that $\approx_{\mathcal{L}}$ is the largest linear weighted bisimulation.

We now introduce a lemma that will be fundamental in the next sections of this work.

Lemma 2.3. *$\approx_{\mathcal{L}}$ coincides with $\sim_l$:*

Let (X, o, t) be a WA and $(\mathbb{K}^X, o^\sharp, t^\sharp)$ the corresponding linear weighted automaton. Then $\forall x \in X, \;\; [\![x]\!] = [\![x]\!]_{\mathbb{K}^X}^{\mathcal{L}}$

3 Algorithms

The first algorithm we implement to compute language equivalence, called `HKC`, is adapted from [1]. The algorithm returns `true` $\iff$ $[\![v_1]\!] - [\![v_2]\!]$. It was first introduced by Bonchi and Pous in [6]. The algorithm, extending the Hopcroft and Karp procedure [7] with *congruence closure*, is proven to be sound and complete [1]. It is defined for WAs over semirings, but in this implementation we are only considering fields, in particular the field of real numbers ($\mathbb{K} = \mathbb{R}$). The problem of checking language equivalence has been proven undecidable for an arbitrary semiring, so termination may not always be guaranteed. However, it has been shown to be decidable for a broad range of semirings, for example, all the complete and distributive lattices. `HKC` computes $v_1 \sim_l v_2$ for a given weighted automaton $W = (X, t, o)$ and two vectors $v_1, v_2 \in \mathbb{K}^X$. by computing a congruence closure, and it does so without linearizing the state space.

We compare `HKC` with an algorithm called *Backwards Partition Refinement*, that we will call BPR for short.

(TODO chiedi a filippo: nelle conclusioni di [2] dice che è diverso dall'algoritmo visto nel paper di boreale [5]. è il caso anche per i field e i numeri reali???? o sono lo equivalenti?).

Adapted from [2], `BPR` is a fixed-point iterative method that, given an LWA $L = (V, t, o)$, it computes a basis of the subspace of V representing the complementary relation of $\approx_{\mathcal{L}}$ (we later show it to be the orthogonal complement in case V is an inner product space). Another version of the algorithm is defined in the same work, called *Forward Partition Refinement*, which directly computes a basis for $\approx_{\mathcal{L}}$ but is shown to be way less efficient than the backwards version.
Note. Recall from section 2 that $\approx_{\mathcal{L}}$ is a linear relation, therefore $v_1 \approx_{\mathcal{L}} v_2 \iff (v_1 - v_2) \in \ker(\approx_{\mathcal{L}})$

The `BPR` algorithm starts from a relation R_0, that is the complement of the relation identifying vectors with equal weights. It then incrementally computes the space of all states that are reachable from R_0 in a *backwards* direction. Intuitively, "going backwards" means working with the transpose transitions functions t_a^T.

In the next sections we will compare execution of our implementation of the algorithms `BPR` and `HKC` to verify correctness, and to provide insight on runtime results. Lemma 2.3, introduced above, is key to our work. By stating that $\approx_{\mathcal{L}}$ coincides with $\sim_l$, we can confidently say that the two algorithms compute an answer for same the decision problem:

Are two vectors v_1 and v_2 language-equivalent for a given weighted automata?

TODO costo computazionale HKC.

`BPR` has a cost of $O(n^4)$ operations to initially compute the largest linear weighted bisimulation, which can be eventually reduced to $O(n^3)$ [2]. In our implementation, by initially computing a basis of the orthogonal complement of $\approx_{\mathcal{L}}$, the cost of checking if two vectors are language equivalent is then reduced to the cost of matrix multiplication ($O(n^2)$). `BPR` is a great choice when we have to decide if a large number of vectors in a WA are language equivalent.

3.1 HKC Algorithm

We give a pseudocode definition of the `HKC` procedure:

Figure 1: The `HKC`(v_1, v_2) procedure

```
1  HKC(v_1, v_2):
2  R := ∅; todo := ∅
3  while todo is not empty do
4      extract (v'_1, v'_2) from todo
5      if (v'_1, v'_2) ∈ c(R) then continue
6      if o(v'_1) ≠ o(v'_2) then return false
7      for all  a ∈ A
8          insert (t_a(v'_1), t_a(v'_2)) into todo
9      insert (v'_1, v'_2) into R
10 return true
```

3.2 Backwards Partition Refinement Algorithm for the Largest Weighted Bisimulation

We now recall the backwards algorithm for computing $\approx_{\mathcal{L}}$ defined in [2]. The algorithm is defined by the iterative method:

$$R_0 = \ker(o)^0, \qquad R_{i+1} = R_i + \sum_{a \in A} t_a^T(R_i) \tag{2}$$

Where $\ker(o)^0$ is an annihilator. The algorithm stops when $R_{j+1} = R_j$. An index $j \leq \dim(V)$ is guaranteed to exist, such that the algorithms terminates at step j. It follows that $\approx_{\mathcal{L}} = R_j^0$. Proof is available in section 4.2 of [2]

4 Implementation

The algorithms and data structures are implemented in the Go programming language. Real numbers are implemented with double precision floating point numbers, precisely of `float64` type.
This implementation makes use of the Gonum library, an excellent toolkit for high-performance numerical computations. We only import the Gonum libraries for matrices, linear algebra and visual plotting. Although not GPU accelerated, Gonum matrix operations are run on the CPU and accelerated with BLAS and LAPACK.

4.1 Data Structures

In this implementation, the data structure for representing weighted automata is a `struct` containing the following attributes:

1. An integer `Dim` representing the number of states n. (note: states are finite and indexed on natural numbers $0, \ldots, n-1$)
2. A slice of strings representing the *alphabet* A.
3. A map of strings as keys (the alphabet symbols) and dense $n \times n$ `float64` matrices as values, representing the set of transition functions.
4. A dense `float64` vector, representing the linearization of the output function.
5. A `float64` value representing the tolerance to be used in numerical computations.
6. An optionally `nil`, dense floating point matrix providing a basis for the orthogonal complement of $\approx_{\mathcal{L}}$

Note. Slices in Go are a convenient and efficient extension of the concept of arrays: they provide an abstraction for indexed, variable length sequences of typed data, and provide useful helper functions for creating, appending and selecting elements.
The definition can be found in file `automata/automata.go`. We then provide methods for reading an automaton from a text stream, applying transition and output functions to vectors, and generating random automata with real and natural valued weights.

4.2 Implementation of HKC

Instead of creating dedicated structs, we exploit the `mat.Dense` data type in Gonum to efficiently represent:

- Sets of vectors with $n \times k$ dense matrices (k is the number of vectors in the set)
- Pairs of vectors with $n \times 2$ dense matrices
- Sets and the "`todo`" stack of pairs with $n \times 2k$ dense matrices (k is the number of pairs in the set or stack)

To increase efficiency of the methods for inclusion checking and insertion in sets of vectors, one could keep the columns ordered (by vector norm) in the corresponding matrix.
To represent the congruence relation R, we introduce a `struct` containing:

- A dense matrix `s` of size $n \times 2k$ containing the set of pairs in the relation
- A dense matrix `u` to represent the generating set U_R for the congruence closure (see definition 2.4).
- Two integers representing the number of pairs in the set, and the size of vectors.
- A tolerance value to be used in equivalence checks. Best results were obtained with 10^{-14}.

When adding a pair of vectors to the congruence relation, we extend the columns of the matrix `s` with the pair (v_1, v_2), if and only if $(v_1 - v_2)$ is not already in U. To check if the pair is in the congruence closure $c(R)$, we check if $(v_1 - v_2)$ is contained in U.

Figure 2: Implementation of the `HKC`(v_1, v_2) procedure

```
1  // this file contains the implementation of the HKC procedure
2
3  package automata
4
5  import (
6          "gonum.org/v1/gonum/mat"
7  )
8
9  // checks the language equivalence of two state vectors for a given weighted automaton
10 func (a Automaton) HKC(v1, v2 *mat.VecDense) (bool, error) {
11         rel := NewRelation(0, a.Dim)
12         todo := NewPairStack()
13
14         p, err := NewPair(v1, v2)
15         if err != nil {
16                 return false, err
17         }
18
19         // insert (v1, v2) into the todo list
20         todo = PairStackPush(todo, p)
21
22         for !todo.IsEmpty() {
23                 // extract (v1', v2') from todo
24                 q, err := PairStackPop(todo)
25                 if err != nil {
26                         return false, err
27                 }
28
29                 if rel.PairIsInCongruenceClosure(q) {
30                         continue
31                 }
32
33                 o1 := a.GetOutput(PairLeft(q))
34                 o2 := a.GetOutput(PairRight(q))
35                 if o1 != o2 {
36                         return false, nil
37                 }
38
39                 for _, sym := range a.A {
```

```
40
41                      w1 := a.ApplyTransition(sym, PairLeft(q))
42                      w2 := a.ApplyTransition(sym, PairRight(q))
43                      wp, err := NewPair(w1, w2)
44                      if err != nil {
45                              return false, err
46                      }
47
48                      PairStackPush(todo, wp)
49              }
50
51              // insert (v1', v2') into R
52              rel.Add(q)
53      }
54
55      return true, nil
56  }
```

4.3 Implementation of BPR

Given a WA L, at the first step of BPR, we set $R_0 = o$, with o being the dense vector representing the output function of L, as seen in [2]. For each step $i = 0, \ldots, j+1$, with $j \leq \dim(L)$, the implemented algorithm:

1. For each $a \in A$ computes $t_a^T(R_i)$ through matrix multiplication
2. Concatenates those matrices $t_a^T(R_i)$ to R_i in a resulting matrix G
3. Instead of checking linear independence of the columns through Gaussian elimination, it computes R_{i+1} as the orthonormal basis of the column space of G, through singular value decomposition.

At the end of BPR, we store the basis for the orthogonal complement of $\approx_{\mathcal{L}}$ as an attribute of the automaton. To check if two vectors $v_1 \approx_{\mathcal{L}} v_2$, we check that $R_j^T(v_1 - v_2) = \vec{0}$, with a prefixed tolerance.

To compute a basis for $\approx_{\mathcal{L}}$, at the last step of the algorithm, we would need to compute R_j^0. If V is a vector space and W is a subspace of W, the annihilator of W, respectively W^0 is a subspace of the space V^* of linear functionals on V. W^0 are the functionals that annihilate on W. Since we are working on subspaces of $\mathbb{R}^n$, we can directly compute the orthogonal complement in our implementation instead of the annihilator.

Proposition 4.1. *If V is a finite dimensional vector space defined with an inner product $\langle \cdot, \cdot \rangle$ and W is a subspace of V then the image of the annhilator W^0 through the linear isomorphism $\varphi : V^* \to V$ induced by the inner product, is the orthogonal of W with respect to the said inner product.*

Proof. Let V be an inner product space over the field $\mathbb{K}$ with an inner product defined as $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{K}$. Every linear functional can be represented with a vector. Let $\xi : V \to \mathbb{K}$ be a functional, $\xi \in W^0$. Because $\xi(w) = 0 \quad \forall w \in W$, if v represents ξ we have that $(v, w) = \xi(w) = 0$ for all $w \in W$. We obtain that $\varphi(W^0) \subseteq W^\perp$. If $v \in W^\perp$ then the functional $x \mapsto (v, x)$ cancels over W (by the definition of orthogonality). □

To compute the orthogonal complement of a vector subspace W, we compute $W^\perp = \ker\left(A^T\right)$, where A is the matrix whose column space is W. Proof is available in [8].

Figure 3: Implementation of Backwards Partition Refinement

```
1  // this file contains definitions for the backwards algorithm for
2  // computing the largest linear weighted bisimulation
3
4  package automata
5
6  import (
7          "log"
8
9          "github.com/0x0f0f0f/lwa-techniques/lin"
10         "gonum.org/v1/gonum/mat"
11 )
12
```

```
13 // compute and store a basis for the largest linear weighted bisimulation of
14 // the linear weighted automaton
15 func (a *Automaton) BackwardsPartitionRefinement() {
16         // i = 0
17         lastBasis := mat.NewDense(a.Dim, 1, a.O.RawVector().Data)
18         currBasis := lastBasis
19         // index of the column of last basis that has already been computed
20         // lastIndex := 0
21
22         for i := 1; i <= a.Dim; i++ {
23                 // \sum_{a \in A} T_a^T(R_i)
24                 for _, sym := range a.A {
25                         newBasis := a.ApplyTransposeTransitionBasis(sym, lastBasis)
26                         currBasis = lin.Union(currBasis, newBasis)
27                 }
28                 currBasis = lin.OrthonormalColumnSpaceBasis(currBasis, a.Tol).(*mat.Dense)
29
30                 lastBasis = currBasis
31         }
32
33         a.LLWBperp = currBasis
34         // we could compute the orthogonal complement to find a basis of LLWB:
35         // a.LLWB = lin.Complement(currBasis).(*mat.Dense)
36 }
37
38 // method that, after an LLWB is computed through BPR,
39 // checks the equivalence of 2 vectors
40 func (a Automaton) BPREquivalence(v1, v2 *mat.VecDense) bool {
41         if a.LLWBperp == nil {
42                 log.Fatalln("largest linear weighted bisimulation not computed for
                      automaton")
43                 return false
44         }
45
46         sub := mat.VecDenseCopyOf(v1)
47         sub.SubVec(v1, v2)
48
49         mul := mat.VecDenseCopyOf(sub)
50         mul.Reset()
51         mul.MulVec(a.LLWBperp.T(), sub)
52
53         return lin.IsZeroTol(mul, a.Tol)
54 }
```

Note. **Applications of SVD**

Let's consider the singular value decomposition of a matrix $A \in \mathbb{R}^{m \times n}$:

$$A = U\Sigma V^T \qquad \Sigma = \text{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_r) \qquad U \in \mathbb{R}^{m \times m} \qquad V \in \mathbb{R}^{n \times n}$$

Where V and U are orthogonal and the singular values are ordered: $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r \geq 0$. It follows that $\text{rank}\,(A)$ is equal to the number of nonzero singular values, and as explained in [9]:

1. $\text{rank}\,(A) = \text{rank}\,(\Sigma) = r$
2. The column space of A is spanned by the first r columns of U.
3. The null space of A is spanned by the last nr columns of V.
4. The row space of A is spanned by the first r columns of V.
5. The null space of A^T is spanned by the last mr columns of U.

Of our interest, are only the computation of the null space and column space. The implementation can be found in files `lin/colspace.go` and `lin/nullspace.go`.

5 Runtime Results

TODO da fare e sistemare.
The program was executed on an x86_64 AMD Ryzen 5 2600X Six Core CPU, running at 3.6GHz with 32GB RAM, running Void Linux. For 3 test runs on 100000 automata composed of 4 states, 2 alphabet symbols and transition matrices with weights on natural numbers ≤ 3, with a tolerance of 10e-14, we found a non empty kernel of $\approx_{\mathcal{L}}$ for an average of 1718 automata.
(TODO spiega: più aumentano i simboli/stati più si riduce la probabilità di trovare $\approx_{\mathcal{L}}$ non vuoto?? investiga, trova un modo di generare automi con LLWB non vuoto)
By generating 1000 random couples of language equivalent vectors (v_1, v_2) such that $(v_1 - v_2) \in \ker(\approx_{\mathcal{L}})$ and 1000 couples of completely random vectors (w_1, w_2), for each automata, we verified with 100% confidence that:

$$v_1 \; R_j \; v_2 \iff \texttt{HKC}(v_1, v_2) = \texttt{true}$$

Where R_j is the largest linear weighted bisimulation $\approx_{\mathcal{L}}$ computed by `BPR`.
Sample output (19992 KBs in memory):

```
$ time go run main.go
tests completed            100000...

1000 samples per automaton
100000 automata
LLWP is not empty for 1726 automata
language equivalence HKC = BPR for 3452000 samples
BPR equivalence verified for 1726000 samples
HKC equivalence verified for 1726000 samples
confidence is 100%

real    0m21.322s
user    0m23.614s
sys     0m0.961s
```

6 Conclusion and Future Work

TODO da fare

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