Lattice Reduction Techniques To Attack RSA

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RSA ?

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And to **decrypt** :

$$m = c^d \pmod{N}$$

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Use p and q to generate the pair **private key/public key** (d, e).


ATTACKS ?

On the implementation or on the mathematics.

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- Recover the private key d

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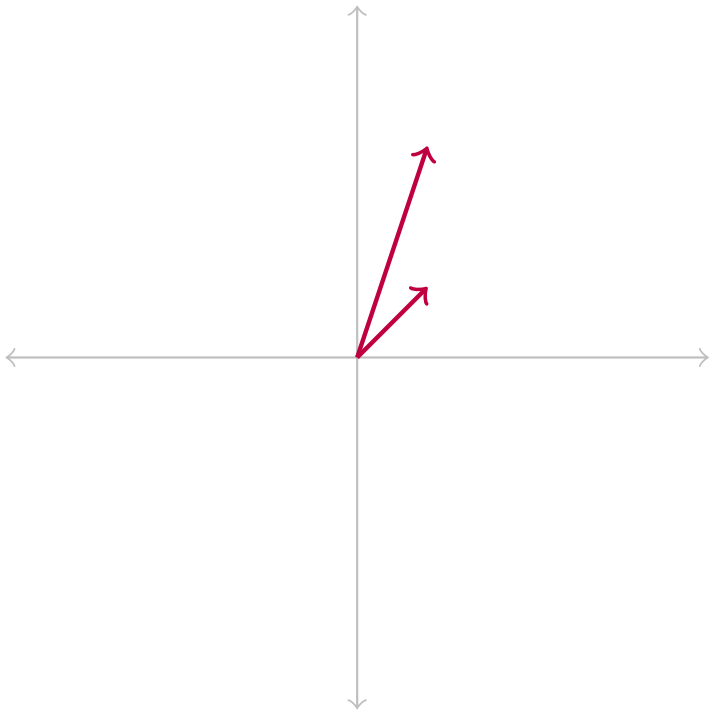
- Recover the private key d

Relaxed model :

- We know a part of the message
- We know an approximation of one of the prime
- The private exponent is too small

LATTICE ?

A bit like a **vector space**.



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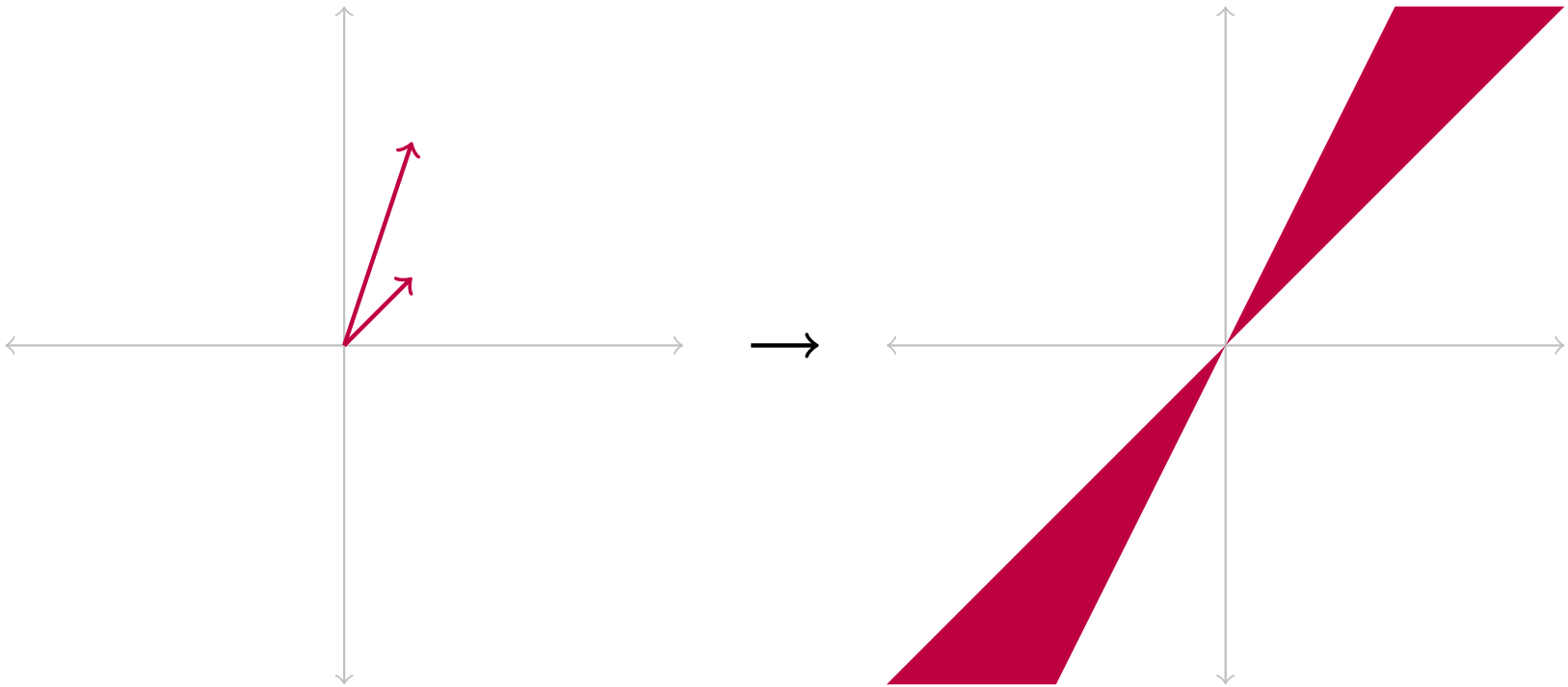
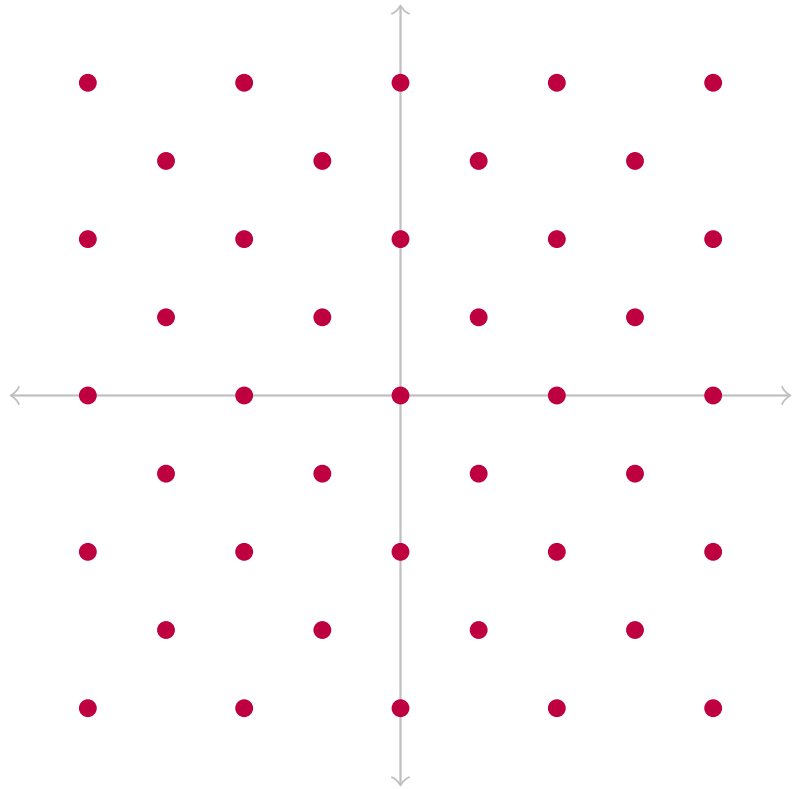
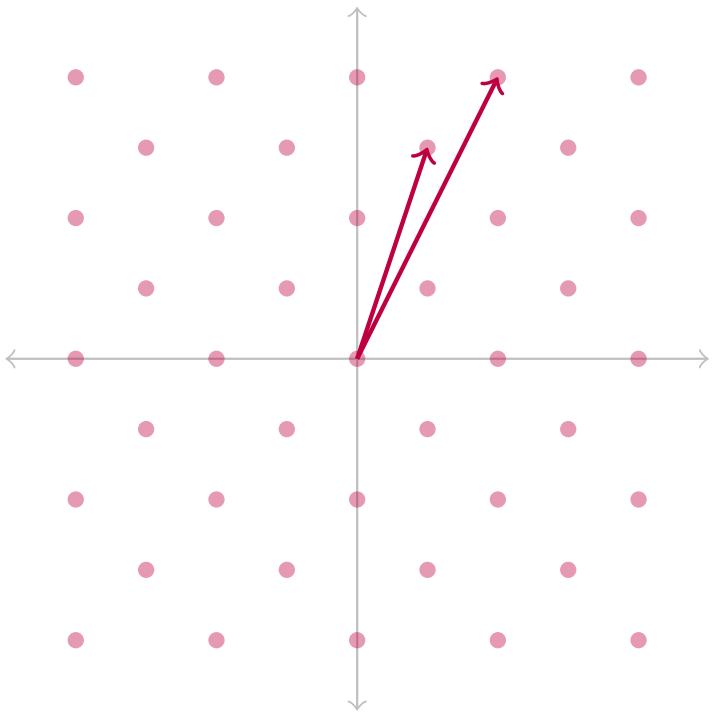




LLL, a lattice basis reduction algorithm

random basis



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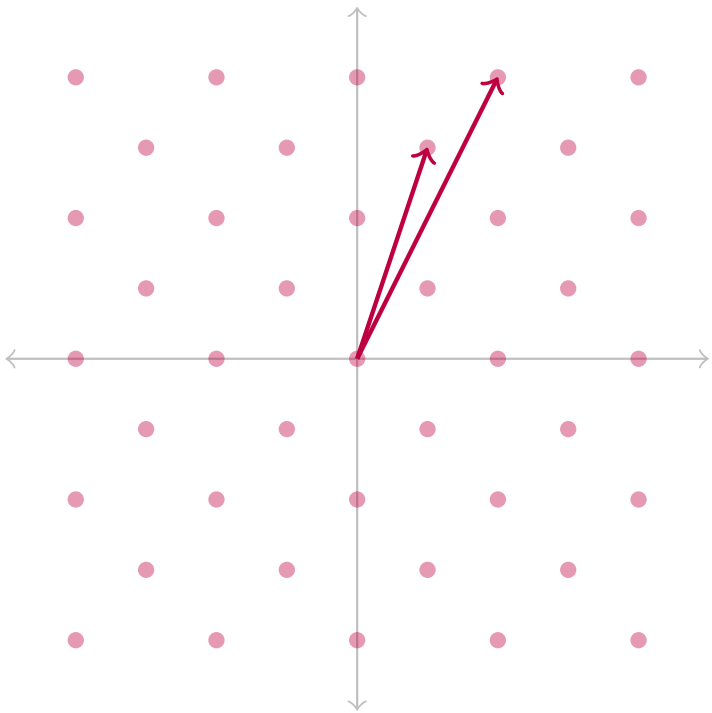
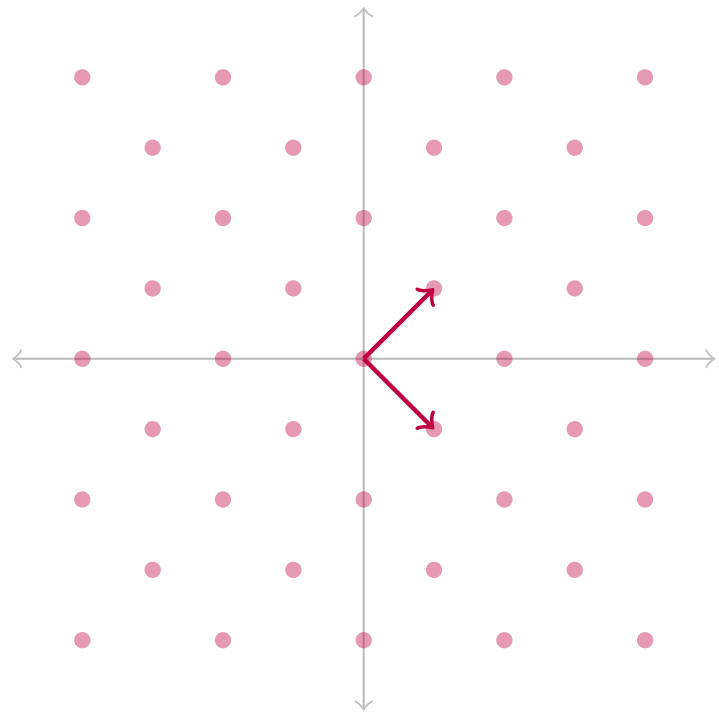


$\xrightarrow{LLL}$

reduced basis



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$$\|b'_1\| \leq \|b'_2\| \leq \ldots \leq \|b'_i\| \leq 2^{\frac{n(n-1)}{4(n+1-i)}} \cdot det(L)^{\frac{1}{n+1-i}}$$


COPPERSMITH ?

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$$f(x) = c - (m_0 + x)^e \pmod{N}$$

$$f(x) = 0 \pmod{N} \text{ with } |x| < X$$

$$g(x) = 0 \text{ over } \mathbb{Z}$$

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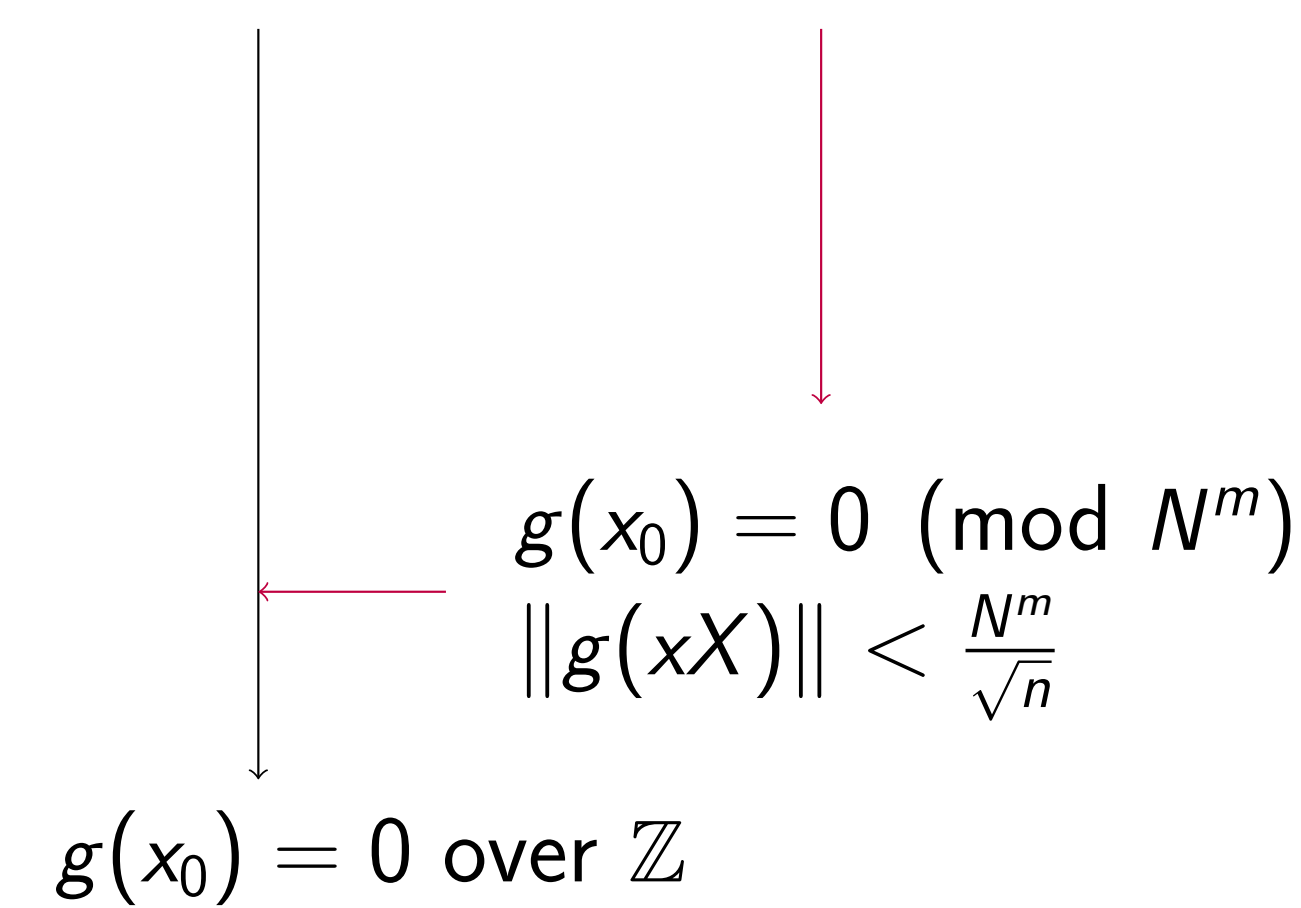
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LLL reduction :

- It only does **integer linear operations** on the basis vectors
- The **shortest vector of the output basis is bound** (as seen in **Property 1**)

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$$g_{i,j}(x) = x^j \cdot N^i \cdot f^{m-i}(x)$$
$$\text{for } i = 0, \ldots, m-1, \; j = 0, \ldots, \delta - 1$$
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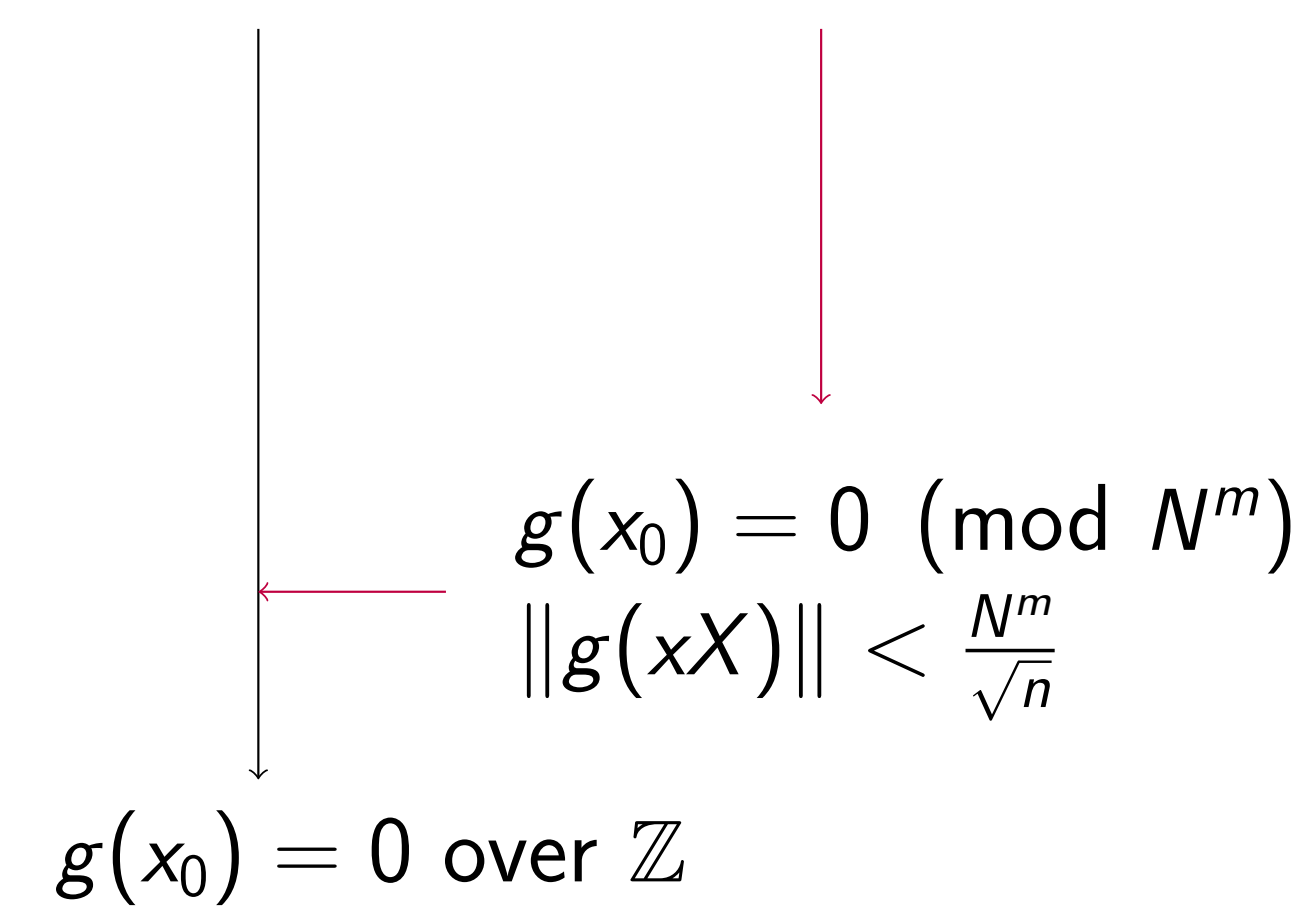
- each iteration introduce a new polynomial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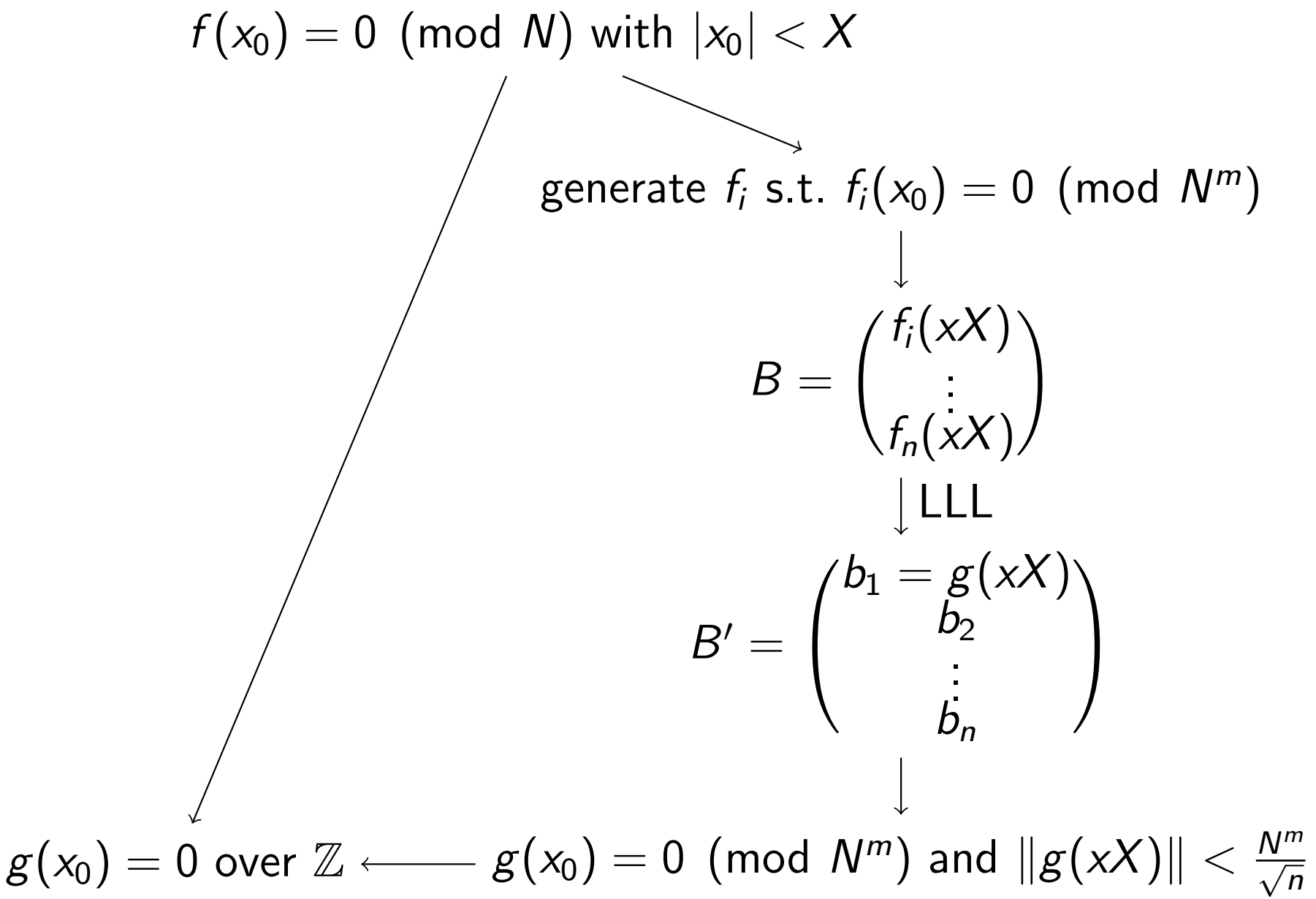
$f(x_0) = 0 \pmod N$ with $|x_0| < X$
generate f_i s.t. $f_i(x_0) = 0 \pmod{N^m}$
$B = \begin{pmatrix} f_i(xX) \\ \vdots \\ f_n(xX) \end{pmatrix}$
LLL
$B' = \begin{pmatrix} b_1 = g(xX) \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$
$g(x_0) = 0$ over $\mathbb{Z}$
$g(x_0) = 0 \pmod{N^m}$ and $\|g(xX)\| < \frac{N^m}{\sqrt{n}}$

Let $g(x)$ be an univariate polynomial with n monomials. Further, let m be a positive integer. Suppose that

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Then $g(x_0) = 0$ holds over the integers.

$$|\tilde{p} - p| < N^{\frac{1}{4}}$$

Now we have an equation with one unknown, modulo another unknown :

$$\tilde{p} = x_0 \pmod{p}$$

Coppersmith Theorem

Let N be an integer of unknown factorization, which has a divisor $b \geq N^{\beta}$, $0 < \beta \leq 1$. Let $f(x)$ be a univariate monic polynomial of degree δ and let $c \geq 1$.

Then we can find in time $\mathcal{O}(c\delta^5 log^9(N))$ all solutions x_0 of the equation

$$f(x) = 0 \pmod{b} \quad \text{with} \quad |x_0| \leq c \cdot N^{\frac{\beta^2}{\delta}}$$

BONEH-DURFEE ?